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STABILITY OF CIRCULAR ORBITS IN NONCOMMUTATIVE SCHWARZSCHILD SPACETIME

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In this paper, we investigate the effects of space noncommutativity and the generalized uncertainty principle on the stability of circular orbits of particles in Schwarzschild spacetime. We show that, up to first order of noncommutativity parameter, an angular momentum dependent extra term will appear in effective potential which affects the stability of circular orbits. In the case of large angular momentum, the condition for stability of circular orbits will change considerably relative to commutative case.

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1. Motivations and preliminaries

An important consequence of quantum gravity scenarios such as string theory is the possible noncommutativity of spacetime structure at very short distances [1–4]. This noncommutativity leads to the modification of Heisenberg uncertainty relations in such a way that prevents one from measuring positions to better accuracies than the Planck length. In low energy limit, these quantum gravity effects can be neglected, but in circumstances such as very early universe or in the strong gravitational field of a black hole one has to consider these effects. The modifications induced by the generalized uncertainty principle on the classical orbits of particles in a central force potential firstly has been considered by Benczik *et al.* [5]. The same problem has been considered within noncommutative geometry by Mirza and Dehghani [6]. The main consequence of these two investigation is the constraint imposed on the minimal observable length and noncommutativity parameter in comparison with observational data of Mercury. Here we are going to proceed one more step in this direction. We study the effects of the space noncommutativity and the generalized uncertainty principle on the stability of circular orbits of particles in Schwarzschild geometry. We obtain a noncommutative effective potential which up to first order of noncommutativity parameter, contains an extra angular momentum dependent term and this new term affects the conditions for stability of circular orbits of particles seriously. In fact space noncommutativity shows itself by such an angular momentum dependent term. For large values of angular momentum, the effect of space noncommutativity is considerable.

1.1. Noncommutative effective potential

Recently, motivated by string theory, the effects of noncommutative geometry have been studied in various physical problems. Considering space noncommutativity, the usual quantum mechanical commutation relations should be modified in the following manner [1–4]

$$[\tilde{x_i},\tilde{x_j}] = i\theta_{ij}\,, \qquad [\tilde{p_i},\tilde{p_j}] = 0\,, \qquad [\tilde{x_i},\tilde{p_j}] = i\hbar\delta_{ij}\,, \tag{1}$$

where θ_{ij} is an anti-symmetric matrix whose elements have dimension of $(\text{length})^2$. One can show that there is a new coordinate system defined by the following transformations

$$x_i = \tilde{x_i} + \frac{1}{2}\theta_{ij}\tilde{p_j}\,, \qquad p_i = \tilde{p_i}\,, \tag{2}$$

where these new variables satisfy the following canonical commutation relations

$$[x_i,x_j] = 0\,, \qquad [x_i,p_j] = i\hbar\delta_{ij}\,, \qquad [p_i,p_j] = 0\,. \tag{3}$$

In a noncommutative space, the Hamiltonian for a particle in a central force potential has the following form

$$H = \frac{\tilde{p}^2}{2m} + V(\tilde{r})\,, \qquad \tilde{r} = \sqrt{\tilde{x_i}\tilde{x_j}}\,, \tag{4}$$

where using the coordinate transformation (2) we find

$$V(\tilde{r}) = V\left(\sqrt{\left(x_i - \frac{\theta_{ij}p_j}{2}\right)\left(x_i - \frac{\theta_{ik}p_k}{2}\right)}\right) = V(r) - \frac{\vec{\theta}\cdot\vec{L}}{2r}\frac{\partial V}{\partial r} + O(\theta^2)\,. \tag{5}$$

Up to the first order of noncommutativity parameter we find

$$V(\tilde{r}) = V(r) - \frac{\vec{\theta}\cdot\vec{L}}{2r}\frac{\partial V}{\partial r}\,, \tag{6}$$

where we have used the following definitions: $\theta_{ij} = \frac{1}{2}\epsilon_{ijk}\theta_k$, $L_k = \epsilon_{ijk}x_ip_j$, $\vec{\theta} = \Sigma_{i=1}^{3}\theta_i\hat{e}_i$, $\vec{L} = \Sigma_{i=1}^{3}L_i\hat{e}_i$ and $\epsilon_{ijr}\epsilon_{iks} = \delta_{jk}\delta_{rs} - \delta_{js}\delta_{rk}$. Equation (6) will play an important role in our forthcoming calculations.

1.2. Stability of circular orbits in commutative Schwarzschild geometry

Now consider the geometry of Schwarzschild spacetime with the following metric(with $c = 1$)

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2\Big(d\vartheta^2 + \sin^2\vartheta d\varphi^2\Big)\,. \quad (7)$$

The geodesics of this spacetime are given by the following equations [7]

$$\frac{d^2 t}{d\lambda^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0\,. \quad (8)$$

$$\frac{d^2 r}{d\lambda^2} + \frac{GM}{r^3}(r-2GM)\left(\frac{dt}{d\lambda}\right)^2 - \frac{GM}{r(r-2GM)}\left(\frac{dr}{d\lambda}\right)^2$$
$$-(r-2GM)\Big[\left(\frac{d\vartheta}{d\lambda}\right)^2 + \sin^2\vartheta\left(\frac{d\varphi}{d\lambda}\right)^2\Big] = 0\,, \quad (9)$$

$$\frac{d^2\vartheta}{d\lambda^2} + \frac{2}{r}\frac{d\vartheta}{d\lambda}\frac{dr}{d\lambda} - \sin\vartheta\cos\vartheta\left(\frac{d\varphi}{d\lambda}\right)^2 = 0\,, \quad (10)$$

$$\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} + 2\cot\vartheta\frac{d\vartheta}{d\lambda}\frac{d\varphi}{d\lambda} = 0\,, \quad (11)$$

where λ is an affine parameter. There are four conserved quantities associated with Killing vectors: magnitude of angular momentum(one component), direction of angular momentum(two components) and energy. The two Killing vectors which lead to conservation of the direction of angular momentum imply that $\vartheta = \frac{\pi}{2}$. The other two Killing vectors are corresponding to energy and the magnitude of the angular momentum. The Killing vector associated with energy is ∂_t or

$$K_\mu = \left(-\left(1 - \frac{2GM}{r}\right),\ 0,\ 0,\ 0\right) \quad (12)$$

and for the magnitude of angular momentum the Killing vector is ∂_φ or

$$L_\mu = \Big(0,\ 0,\ 0,\ r^2\sin^2\vartheta\Big). \quad (13)$$

So along the geodesics, the two corresponding conserved quantities are

$$\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda} = E\,, \quad (14)$$

and

$$r^2\frac{d\varphi}{d\lambda} = L\,, \quad (15)$$

respectively, where E and L are energy and angular momentum of the particle per its unit mass.

Along an affinely parameterized geodesic (time-like, space-like or null) the scalar quantity

$$\varepsilon = -u^\alpha u_\alpha$$

is a constant since

$$\frac{d\varepsilon}{d\lambda} = (u^\alpha u_\alpha)_{;\beta} u^\beta = (u^\alpha_{;\beta} u^\beta) u_\alpha + u^\alpha (u_{\alpha;\beta} u^\beta) = 0\,.$$

If we chose λ to be proper time or proper distance, then we find $\varepsilon = \pm 1$. For a null geodesic $\varepsilon = 0$. Generally, one can write

$$\varepsilon = -g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}\,. \tag{16}$$

We chose $\lambda = \tau$ (the proper time) and expand this equation. In equatorial plane ($\vartheta = \frac{\pi}{2}$) we find

$$-\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2 = -\varepsilon\,, \tag{17}$$

where multiplying by $(1 - \frac{2GM}{r})$ and using equations (14) and (15), we obtain

$$-E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)\left(\frac{L^2}{r^2} + \varepsilon\right) = 0\,. \tag{18}$$

This relation can be rewritten as follows

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2\,, \tag{19}$$

where we have defined

$$V(r) = \frac{1}{2}\varepsilon - \varepsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \tag{20}$$

which is the *effective potential* in Schwarzschild spacetime. Figure 1 shows the variation of this effective potential *versus* radius for different angular momentum. In terms of the Schwarzschild radius, one can rewrite this equation as follows

$$V(r) = \frac{1}{2}\varepsilon - \varepsilon\frac{GM}{r} + \frac{L^2}{2r^2}\left(1 - \frac{r_{\rm s}}{r}\right)\,, \tag{21}$$

where $r_{\rm s}$ is the Schwarzschild radius. We use this relation in our forthcoming arguments.

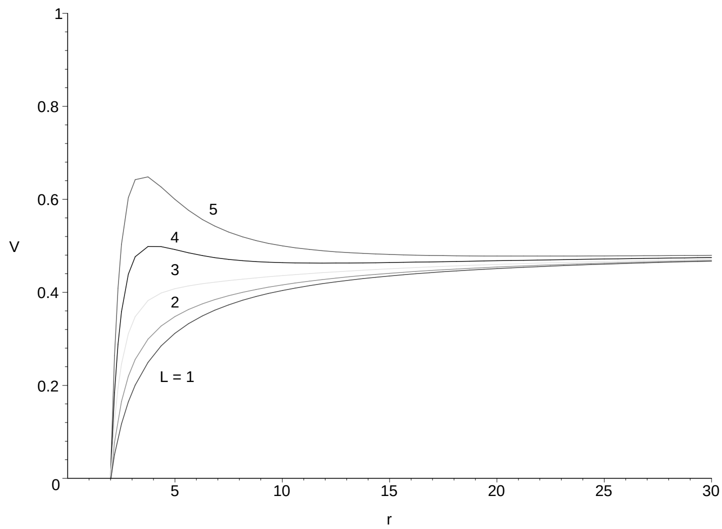

Fig. 1. Commutative effective potential of particles in Schwarzschild spacetime *versus* the radius for different values of angular momentum. This figure shows the effect of angular momentum on the values of effective potential.

In classical general relativity, the particle could have a circular orbit at r_{c} if

$$\left(\frac{dV}{dr}\right)_{r=r_{\mathrm{c}}} = 0$$

which leads to the following condition

$$\varepsilon GMr_c^2 - L^2 r_{\mathrm{c}} + 3GML^2\gamma = 0\,, \tag{22}$$

where $\gamma = 0$ in Newtonian regime and $\gamma = 1$ in general relativity [7]. Let us consider two possible cases separately. Firstly, for $\gamma = 1$ and $\varepsilon = 0$ (photons) we have

$$r_{\mathrm{c}} = 3GM\,. \tag{23}$$

Secondly, for $\gamma = 1$ and $\varepsilon = 1$ (massive particles) we find

$$r_{\mathrm{c}} = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2L^2}}{2GM}\,. \tag{24}$$

For $L \to \infty$, there exist a stable circular orbit at $\frac{L^2}{GM}$ which goes farther and farther away and an unstable one at $3GM$. For small L, at $L = \sqrt{12}GM$ two circular orbits coincide at $r_{\mathrm{c}} = 6GM$ and disappear entirely for smaller L. Therefore, $6GM$ is the smallest possible radius of a stable circular orbit in Schwarzschild metric. In brief, Schwarzschild solution in commutative space possesses stable circular orbits for $r > 6GM$ and unstable ones for $3GM < r < 6GM$. Now we consider the effects of space noncommutativity on the stability of circular orbits of particles in Schwarzschild geometry.

2. Noncommutative space considerations

In Schwarzschild spacetime, with $\varepsilon = 1$ (the case of time-like geodesics) we find from (20)

$$V(r) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}\,. \tag{25}$$

Considering the effects of space noncommutativity as given by (6), this relation changes to the following form

$$V(\tilde{r}) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} - \frac{L\theta\cos\psi}{2}\left[\frac{GM}{r^3} - \frac{L^2}{r^4} + \frac{3GML^2}{r^5}\right], \tag{26}$$

where ψ is the angle between $\vec{L}$ and $\vec{\theta}$. This noncommutative effective potential has been plotted in figures 2 and 3 for two different values of angular momentum. As these figures show, increasing the value of the particle angular momentum increases the difference between commutative and noncommutative effective potentials. In other words, large angular momentum

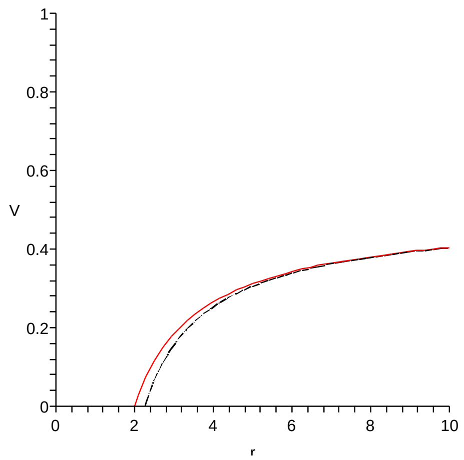

Fig. 2. The difference between commutative and noncommutative effective potential for $L = 1$ (in arbitrary units). Lower curve shows noncommutative case. For small values of L or large values of r, the difference is not considerable.

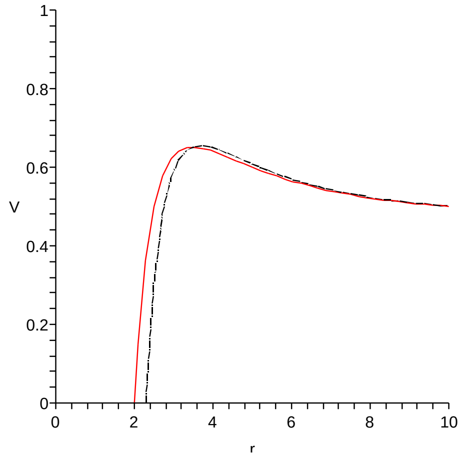

Fig. 3. The difference between commutative and noncommutative effective potential for $L = 5$ (in arbitrary units). Lower curve shows noncommutative case. For large values of L, the difference is considerable.

enhances the effect of space noncommutativity. So space noncommutativity couples with angular momentum. This point has its origin on the algebraic structure of theory on noncommutativity level.

The particle could have a circular orbit at $r_{\rm c}$ if

$$\left(\frac{dV}{dr}\right)_{r=r_{\rm c}} = 0$$

or

$$\frac{GM}{r_c^2} - \frac{L^2}{r_c^3} + \frac{3GML^2}{r_c^4} - \frac{L\theta\cos\psi}{2}\left[\frac{-3GM}{r_c^4} + \frac{4L^2}{r_c^5} - \frac{15GML^2}{r^6}\right] = 0\,. \quad (27)$$

This can be simplified to find

$$GMr_c^4 - L^2 r_{\rm c}^3 + 3GML^2 r_c^2 - \frac{L\theta\cos\psi}{2}\Big[-3GMr_c^2 + 4L^2 r_{\rm c} - 15GML^2\Big] = 0. \quad (28)$$

This is the condition for existence of circular orbits in noncommutative Schwarzschild geometry up to first order of noncommutativity parameter. Comparing this equation with equation (22) (with $\gamma = 1$ and $\epsilon = +1$) for commutative case, shows the importance of noncommutativity effect. Since noncommutativity parameter is very small, this effect can be neglected in ordinary circumstances but for large angular momentum the situation differs considerably. The dependence of this modification to angular momentum is a pure noncommutative effect which goes back to the algebraic structure of the theory.

The condition for the stability of the circular orbits is

$$\left(\frac{\partial^2 V}{\partial r^2}\right)_{r=r_c} > 0\,. \tag{29}$$

Applying this condition to the potential (26), we find

$$\frac{-2GM}{r_c^3} + \frac{3L^2}{r_c^4} - \frac{12GML^2}{r_c^5} - \frac{L\theta\cos\psi}{2}\left[\frac{12GM}{r_c^5} - \frac{20L^2}{r_c^6} + \frac{90GML^2}{r_c^7}\right] > 0\,. \tag{30}$$

Combining this equation with equation (28), we find the following condition for the stability of the circular orbits in noncommutative space

$$GMr_c^4 - 3GML^2r_c^2 - \frac{L\theta\cos(\psi)}{2}\left[3GMr_c^2 - 8L^2 r_c + 45GML^2\right] > 0\,. \tag{31}$$

Figure 4 shows the condition for stability of circular orbits of particles in Schwarzschild geometry and in the presence of space noncommutativity. In this figure we have plotted the left hand side of relation (31) *versus* $r = GM$. As this figure shows, in the case of noncommutative space, particles could

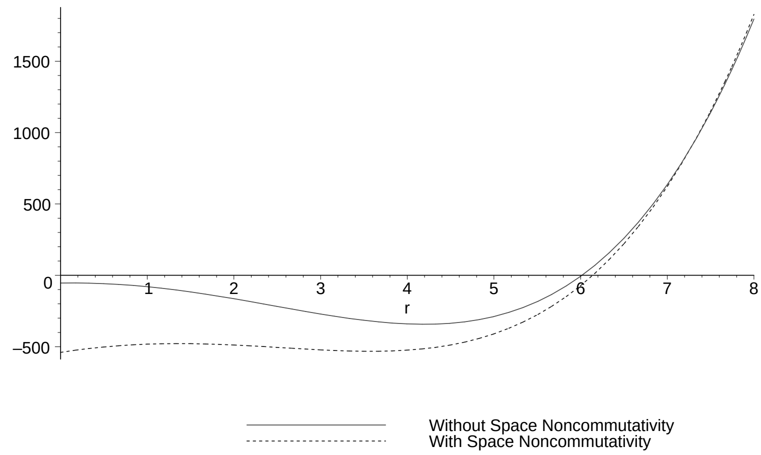

Fig. 4. The condition for stability of circular orbits for commutative and noncommutative Schwarzschild spaces. For $L = 5$, in commutative case particles could have stable circular orbits for $r \geq 6GM$ while for noncommutative case $r \geq (6.139234690)GM$. Space noncommutativity increases the radius of stable circular orbits.

have stable circular orbits for $r \geq (6.139234690)GM$ which is greater than commutative result of $r \geq 6GM$. Therefore, stable circular orbits have greater radius in noncommutative spaces.

3. The effect of GUP

Now we consider the effect of the generalized uncertainty principle on the stability of circular orbits of particles in Schwarzschild spacetime. With the generalized uncertainty principle, the standard commutation relations transform to the following general form [5, 8]

$$\begin{aligned} [x_i, p_j] &= i\hbar(\delta_{ij} + \beta p^2 \delta_{ij} + \beta' p_i p_j)\,, \\ [p_i, p_j] &= 0\,, \end{aligned} \tag{32}$$

and

$$[x_i, x_j] = i\hbar \frac{(2\beta - \beta') + (2\beta + \beta')\beta p^2}{(1 + \beta p^2)}(p_i x_j - p_j x_i)\,. \tag{33}$$

we set $\beta' = 0$ so the corresponding Poisson brackets are

$$\{x_i, p_j\} = \delta_{ij}(1 + \beta p^2)\,, \qquad \{p_i, p_j\} = 0\,, \qquad \{x_i, x_j\} = 2\beta(p_i x_j - p_j x_i)\,. \tag{34}$$

The modified canonical equations now take the following forms

$$\dot{x_i} = \{x_i, H\}\,, \qquad \dot{p_i} = \{p_i, H\}\,, \tag{35}$$

where

$$H = \frac{p^2}{2m} + V(r) \tag{36}$$

is the Hamiltonian of the system. The deformed angular momentum which is given by

$$L_{ij} = \frac{x_i p_j - x_j p_i}{(1 + \beta p^2)} \tag{37}$$

is conserved due to the rotational symmetry of the Hamiltonian. For the circular orbits of particles in a general central force problem we have the constraints $\dot{r} = 0$ and $\dot{p_r} = 0$ [8]. These conditions will lead us to

$$L = \frac{pr}{(1 + \beta p^2)}\,. \tag{38}$$

The radius of orbit r and the magnitude of the momentum $p = \sqrt{p^2}$ are now the constants of the motion.

Now in Schwarzschild spacetime the effective potential is given by equation (25). We substitute the value of L from (39) in this potential to find

$$V(\tilde{r}) = \frac{1}{2} - \frac{GM}{r} + \frac{p^2}{2(1+\beta p^2)} - \frac{GMp^2}{r(1+\beta p^2)} \tag{39}$$

so the condition for a circular orbit with radius r to be stable in the framework of GUP is

$$GMr_c^2 - \frac{p^2 r_{\rm c}^3}{(1+\beta p^2)^2} + \frac{3GMp^2 r_{\rm c}^2}{(1+\beta p^2)^2} \geq 0\,. \tag{40}$$

We emphasize that presence of β is a quantum gravitational effect which has origin on the fractal structure of spacetime at very short distances (string scale). The situation is much similar to the case presented in figure 4 with dotted curve. The effect of space noncommutativity is much similar to the effect of generalized uncertainty principle. These two concept are common features of quantum gravity era where there exists a minimal observable length of the order of Planck length.

4. Summary and conclusion

In this paper we have studied the effect of space noncommutativity and the generalized uncertainty principle on the stability of circular orbits of particles in Schwarzschild geometry. We have found the effective potential in the case of noncommutative Schwarzschild space. When the angular momentum of a particle is small, the effect of space noncommutativity can be neglected. In this case we find $r \geq 6GM$ for the stability of circular orbits in commutative Schwarzschild space. The situation differs considerably with commutative prescription when one considers space noncommutativity. For the case of large angular momentum, the condition for stability of circular orbits in noncommutative Schwarzschild space for $L = 5$ becomes $r \geq (6.139234690)GM$ which differs from commutative result, $r \geq 6GM$. Since the effect of space noncommutativity in first order approximation is the presence of angular momentum in orbital equations, we conclude that space noncommutativity changes conditions for stability of the circular orbits in an angular momentum dependent manner. This feature provides an operative approach for investigation of the spacetime geometry, although this effects are so small that current experimental devices cannot detect them. We have also calculated the effect of the generalized uncertainty principle on the stability of circular orbits of particles. GUP is a quantum gravitational effect and has effect much similar to space noncommutativity on the planetary orbits of particles. It is important to note that arguments presented in this paper are important in situations such as TeV black hole thermodynamics. Finally, we should emphasize that modification of the standard dispersion relations and therefore local Lorentz invariance violation is a direct consequence of spacetime noncommutativity which recently has been considered seriously (see [9] and references therein). Any possible detection of Lorenz invariance violation will provide an indirect test of spacetime noncommutativity. Ultra-High Energy Cosmic Rays (UHECRs) [10] and other probes such as modified Compton effect [11] provide experimental basis of testing these noncommutative effects.

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