



## PAPER

# $U(1)$ gauge symmetry free of redundancy and a generalized Byers-Yang theorem

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E-mail: [kicheon.kang@gmail.com](mailto:kicheon.kang@gmail.com)Keywords:  $U(1)$  gauge field, Andreev bound states, quantum transport, superconducting point contactOriginal content from this work may be used under the terms of the [Creative Commons Attribution 3.0 licence](https://creativecommons.org/licenses/by/4.0/).

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**Abstract**

We present a reformulation of the  $U(1)$  gauge theory by eliminating the redundancy inherent in the conventional approach. Our reformulation is constructed on the basis of local field interaction approach to electrodynamics. The gauge symmetry in our framework is associated with a physical transformation, which represents the invariance of the equation of motion of a charged scalar field under the change in the distribution of electromagnetic field at a distance. We demonstrate that all physical properties of the  $U(1)$  gauge theory are preserved with the removal of redundancy in the gauge field. In addition, our reformulation provides a generalization of the Byers-Yang theorem to open systems.

**Introduction**

Gauge theory is one of the greatest pillars in modern physics. It provides a universal framework to understand a wide range of phenomena ranging from the field theories of electromagnetism to the standard model of elementary particles and forces. Despite its great success, gauge theory consists of a disturbing feature; it is constructed based on redundancy of description rather than the physical symmetry (see e.g., p.189 of [1]). The simplest example is classical electrodynamics. A point charge  $e$  with four-velocity  $\dot{r}_\mu$  under the four-potential  $A^\mu$  is described by the Lagrangian:

$$L = L_0 + \frac{e}{c} \dot{r}_\mu A^\mu, \quad (1)$$

where  $L_0$  is the kinetic part of the particle. Gauge symmetry in this Lagrangian implies the invariance of the equation of motion for the particle under transformation

$$A^\mu \rightarrow A'^\mu = A^\mu - \partial^\mu \Lambda \quad (2)$$

with any scalar function  $\Lambda(x)$ . Notably, this transformation is not associated with the symmetry of two physical states that have the same properties. Instead, it indicates that  $A^\mu$  and  $A'^\mu$  represent the same physical state. In other words, it is impossible, even in principle, to make a gauge transformation of the system in a laboratory, unlike other physical transformations, e.g., translation, rotation, and the Lorentz transformation.

This property of gauge theory is closely related to the ‘nonlocality’ of electromagnetic interaction because the local interaction of the particle with the gauge field (equation (1)) includes a certain degree of arbitrariness (see equation (2)). Recently, it was found that this arbitrariness of the interaction can be removed by adopting the local field interaction (LFI) approach [2–5]. The LFI theory successfully reproduces the classical electrodynamics and topological Aharonov–Bohm (AB) effect [2, 3]. This implies that we can describe quantum theory involving electromagnetic interaction that does not rely on the potential, in contrast to the conventional viewpoint. However, one may inquire whether the approach without redundancy of  $A^\mu$  removes the ubiquity of gauge theory. We need to clarify the manifestation of ‘gauge symmetry’ in this redundancy-free description. In this paper, we resolve this question and reveal the gauge symmetry intrinsic in the LFI approach. The gauge symmetry in this framework is associated with a physical transformation without mathematical redundancy, while preserving other properties of  $U(1)$  gauge theory. We discuss the gauge symmetry in this framework for

both a point charge and charged scalar field. In addition, applying this gauge symmetry to a system without a closed loop, we derive a generalized Byers-Yang theorem [6, 7] and show that it can be experimentally verified in a superconducting point contact.

## Gauge invariance as a physical symmetry in classical electrodynamics

A particle with charge  $e$  and mass  $m$  under external electric ( $\mathbf{E}$ ) and magnetic ( $\mathbf{B}$ ) fields can be described by the Lagrangian:

$$L = L_0 + L_{in}, \quad (3)$$

where

$$L_0 = -mc\sqrt{c^2 - \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}} \quad (4)$$

is the kinetic part, and

$$L_{in} = \frac{1}{8\pi} \int F_{\mu\nu}^{(e)} F^{\mu\nu} d^3\mathbf{x}, \quad (5)$$

represents the interaction between the field generated by the particle and the external field (represented by the field tensors  $F_{\mu\nu}^{(e)}$  and  $F^{\mu\nu}$ , respectively). This interaction Lagrangian is derived from the Lorentz-covariant LFI approach [2, 3], and can be rewritten in the form

$$L_{in} = \dot{r}_\mu \Pi^\mu. \quad (6a)$$

In this description, redundancy of the potential  $A_\mu$  in equation (1) is eliminated and the particle's motion is coupled with the 'field-momentum four vector',  $\Pi^\mu = (\Pi^0, \mathbf{\Pi})$ , defined as

$$\Pi^0 = \frac{1}{4\pi c} \int \mathbf{E}_e \cdot \mathbf{E} d^3\mathbf{x}, \quad (6b)$$

$$\mathbf{\Pi} = \frac{1}{4\pi c} \int \mathbf{E}_e \times \mathbf{B} d^3\mathbf{x}, \quad (6c)$$

where  $\mathbf{E}_e$  is the electric field generated by charge  $e$ . Mathematically,  $\Pi^\mu$  in equation (6) plays the same role as  $A^\mu$  in the potential-based Lagrangian in equation (1). This Lagrangian successfully reproduces the classical equation of motion and the topological quantum phase. In addition, it demonstrates the locality of the interaction which can be tested in real experiments. For its details, see [2–5].

We aim to demonstrate the appearance of gauge symmetry in the absence of redundancy in the gauge field. For a given system configuration, the Lagrangian of equation (3) is unique; if  $\Pi^\mu$  is different in  $L_{in}$ , it represents a different configuration of the external  $\mathbf{E}$  and  $\mathbf{B}$ . The Lagrangian possesses a symmetry under this condition. For the transformation

$$\Pi^\mu \rightarrow \Pi'^\mu = \Pi^\mu - \partial^\mu \Lambda \quad (7)$$

the Lagrangian is transformed as

$$L \rightarrow L' = L - \frac{d\Lambda}{dt}. \quad (8)$$

This indicates that the equation of motion for the particle is invariant under the 'gauge transformation' of equation (7) because a total time derivative  $d\Lambda/dt$  does not affect the Lagrange equation of motion (see e.g., section 2 of [8]). From the Lagrangians (3), (4), and (6), we obtain the gauge-invariant equation of motion:

$$\frac{dp^\mu}{dt} = G^{\mu\nu} \frac{dr_\nu}{dt}, \quad (9a)$$

where  $p^\mu$  is the momentum 4-vector of the particle and

$$G^{\mu\nu} \equiv \partial^\mu \Pi^\nu - \partial^\nu \Pi^\mu = \frac{e}{c} F^{\mu\nu}, \quad (9b)$$

is proportional to the electromagnetic field tensor  $F^{\mu\nu}$ . Equation (9) is equivalent to the one obtained from the standard potential-based approach (see e.g., [9]) and can be rewritten as

$$\frac{d\mathcal{E}}{dt} = e\dot{\mathbf{r}} \cdot \mathbf{E}, \quad \frac{d\mathbf{p}}{dt} = e\mathbf{E} + \frac{e}{c}\dot{\mathbf{r}} \times \mathbf{B}, \quad (10)$$

where  $\mathcal{E}$ ,  $\mathbf{p}$ , and  $\dot{\mathbf{r}}$  are the energy, momentum, and velocity of the particle, respectively.

The invariance of the equation of motion under the gauge transformation of  $\Pi^\mu$  (equation (7)) is not associated with the redundancy of description because  $\Pi^\mu$  is a physical quantity without arbitrariness. Variations in  $\mathbf{E}$  and  $\mathbf{B}$  transform  $\Pi^\mu$  as one can find in equation (7). A constraint in this transformation is that the local

value of field tensor  $F^{\mu\nu}$  remains unchanged. In other words, the ‘gauge transformation’ of  $\Pi^\mu$  represents a change in distribution in the external  $\mathbf{E}$  and  $\mathbf{B}$ , while their *local* values remain unchanged at the position of the particle. Therefore, the gauge symmetry in the Lagrangian for a point charge (equations (3)–(6)) implies that *its equation of motion is invariant under any change in the external field distribution at a distance*. This invariance is evident because the equation of motion is local, although the conventional gauge theory with  $A^\mu$  does not consider the problem in this way.

## U(1) gauge invariance as a physical symmetry

Let us consider electrodynamics with charged scalar field  $\phi$ . Applying the canonical transformation of the Lagrangian (equations (3)–(6)) with the introduction of  $\phi$ , we obtain the Klein–Gordon equation:

$$\left[ -(\partial_\mu - \frac{i}{\hbar}\Pi_\mu)(\partial^\mu - \frac{i}{\hbar}\Pi^\mu) + \frac{m^2c^2}{\hbar^2} \right] \phi = 0. \quad (11)$$

It should be noted that we may work on the Dirac field for the electron, but it gives the same result for the  $U(1)$  gauge symmetry. The Klein–Gordon equation (equation (11)) for  $\phi$  is generated by the Lagrangian:

$$\mathcal{L} = -\frac{1}{m}(\hbar\partial_\mu\phi - i\Pi_\mu\phi)(\hbar\partial^\mu\phi^* + i\Pi^\mu\phi^*) - mc^2\phi^*\phi - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}. \quad (12)$$

The mathematical structure of this Lagrangian is equivalent to that given by the standard approach where  $\Pi_\mu$  is replaced by  $eA_\mu/c$ . Therefore, exploring the  $U(1)$  gauge symmetry is straightforward and it produces the following results. First, Lagrange equations for the fields  $\phi$  and  $\Pi_\mu$  lead to Klein–Gordon (11) and Maxwell equations, respectively. Second, and most importantly, the gauge symmetry is manifested in the invariance of  $\mathcal{L}$  under the transformation

$$\phi \rightarrow \phi' = \phi e^{-i\Lambda/\hbar}, \quad \Pi_\mu \rightarrow \Pi'_\mu = \Pi_\mu - \partial_\mu\Lambda, \quad (13)$$

with an arbitrary scalar function  $\Lambda$ . Similar to the case of the point particle discussed above, this transformation does not include any redundancy of description. It is a physical symmetry associated with different  $\Pi_\mu$ , or equivalently, with different distributions of external  $\mathbf{E}$  and  $\mathbf{B}$ . The gauge symmetry in  $\mathcal{L}$  indicates that *the equation of motion for  $\phi$  (equation (11)) is invariant under the change of the external electromagnetic field at a distance*. Finally, the charge conservation is derived from the symmetry via the Nöther’s theorem. In our framework, it is expressed in terms of continuity equation:

$$\partial_\mu j^\mu = 0, \quad (14)$$

for the four-charge current,

$$j^\mu = -i\frac{\hbar}{m}(\phi^*D^\mu\phi - \phi D^\mu\phi^*), \quad (15)$$

where the covariant derivative is given by

$$D^\mu\phi = (\partial^\mu - \frac{i}{\hbar}\Pi^\mu)\phi. \quad (16)$$

## Generalized Byers-Yang theorem

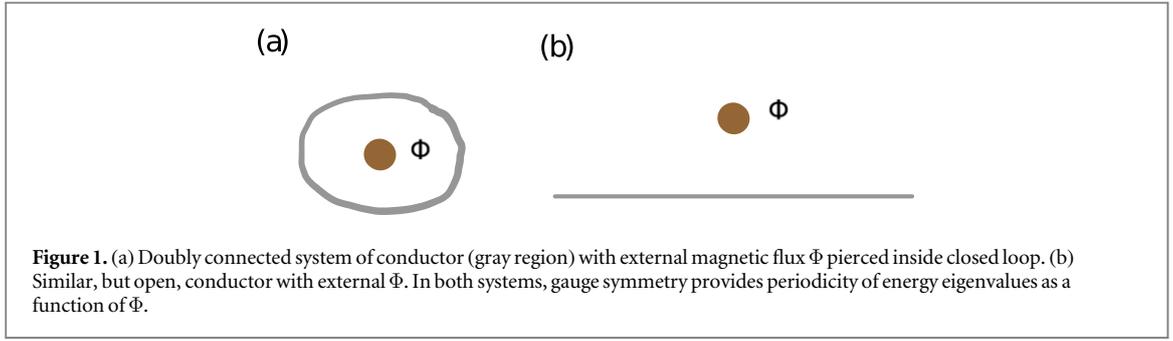
An intriguing consequence of the  $U(1)$  gauge symmetry is the Byers-Yang theorem [6]. It states that all physical properties of a doubly connected system (an annulus) enclosing a magnetic flux  $\Phi$  (see figure 1(a)) are periodic in  $\Phi$  with period  $\Phi_0 = hc/e$ . Here, we show that the theorem can be extended to an open system (figure 1(b)) with our formulation. We also propose an experimental arrangement to confirm the generalized theorem using the superconducting point contact. In our framework of the LFI approach, the eigenfunction  $\phi$  of a charged particle in both systems (figures 1(a), (b)) satisfies the wave equation

$$\frac{1}{2m}(-i\hbar\nabla - \mathbf{\Pi})^2\phi + c\Pi^0\phi = \epsilon\phi, \quad (17)$$

in the limit  $v/c \ll 1$  of equation (11). For many particles, the energy eigenfunction  $\psi$  satisfies

$$\frac{1}{2m}\sum_j(-i\hbar\nabla_j - \mathbf{\Pi}(\mathbf{r}_j))^2\psi + V\psi = E\psi. \quad (18)$$

The magnetic field vanishes in the region of nonzero  $\psi$ ,  $\mathbf{B} = (c/e)\nabla \times \mathbf{\Pi} = 0$  (Note that it is the spatial part of equation (9b)). Therefore we can write  $\mathbf{\Pi} = \nabla\Lambda$ , and  $\mathbf{\Pi}$  can be gauged away. Under the transformation



$$\begin{aligned}\psi &\rightarrow \psi' = \psi e^{-i/\hbar \sum_j \Lambda(\mathbf{r}_j)}, \\ \mathbf{\Pi} &\rightarrow \mathbf{\Pi}' = \mathbf{\Pi} - \nabla \Lambda = 0,\end{aligned}\quad (19)$$

the wave equation (18) reduces to

$$\frac{1}{2m} \sum_j (-i\hbar \nabla_j)^2 \psi' + V\psi' = E\psi', \quad (20)$$

implying that  $\mathbf{\Pi}$  is removed from the wave equation with a modified boundary condition in  $\psi'$ .

Consider the boundary condition of a doubly connected system. For any specific coordinates of a particle, say  $\mathbf{r}_i$ , that circulates around the loop once while keeping the other coordinates fixed,  $\psi'$  acquires a phase factor by the transformation (19) as

$$\psi' \rightarrow \psi' e^{-i \oint \mathbf{\Pi} \cdot d\mathbf{r} / \hbar} = \psi' e^{-i(e\Phi/\hbar c)}. \quad (21)$$

Because the wave equation (20) is independent of  $\mathbf{\Pi}$ , the  $\mathbf{\Pi}$ -dependence of  $E$  is determined by the boundary condition (21), which constitutes the original Byers-Yang theorem: all physical properties of the loop are periodic in  $\Phi$  with its period  $\Phi_0 = hc/e$ .

Our analysis on the periodicity can also be applied to an open system (figure 1(b)). For a specific coordinate of a particle,  $\mathbf{r}_i$ , let  $\psi'_L$  ( $\psi'_R$ ) be the asymptotic value of the wave function at the left (right) infinity of  $\mathbf{r}_i$  such that

$$\psi'_L \equiv \psi'(\mathbf{r}_i \rightarrow -\infty), \quad \psi'_R \equiv \psi'(\mathbf{r}_i \rightarrow \infty). \quad (22)$$

From the gauge transformation (equation (19)), we obtain

$$\frac{\psi'_L}{\psi'_R} = \alpha e^{i \int_{-\infty}^{\infty} \mathbf{\Pi} \cdot d\mathbf{r} / \hbar}, \quad (23)$$

where the constant  $\alpha$  is independent of  $\mathbf{\Pi}$ . Because the eigenvalue  $E$  is determined by the wave equation (20) and the boundary condition (23), we have:

**Theorem.** (The energy eigenvalues are periodic in  $\int_{-\infty}^{\infty} \mathbf{\Pi} \cdot d\mathbf{r} / \hbar$  with period  $2\pi$ .) Therefore all physical properties show the same periodicity, implying that the Byers-Yang theorem is extended to an open system.

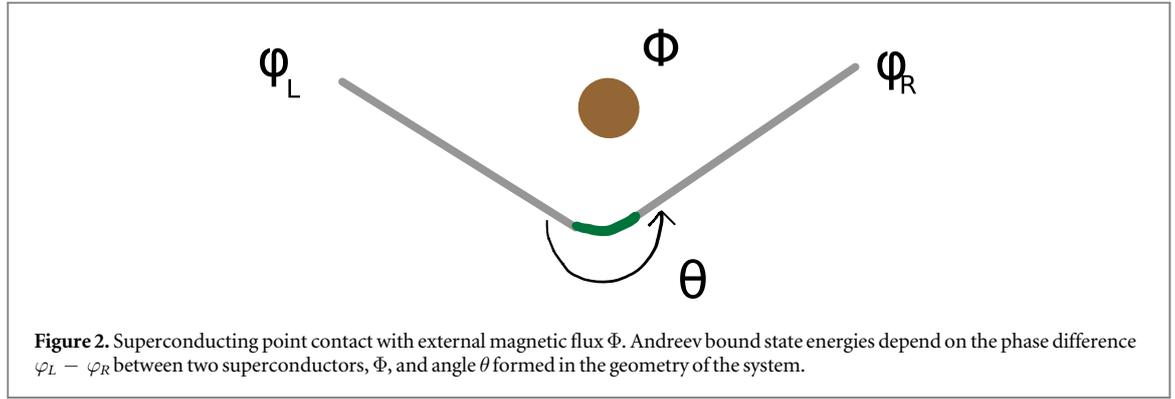
Before discussing a realistic example that demonstrates the Byers-Yang theorem for an open system, let us point out some important facts. First, there are no observable effects of the external flux in a normal conductor in the configuration of figure 1(b), because the boundary condition does not alter the physics of the open system. However, the situation is different for a superconductor with macroscopic quantum coherence. This is analyzed below in detail. Second, the standard approach with vector potential  $\mathbf{A}$  fails to describe the periodicity in the open system. In the potential-based approach, the boundary condition of equation (23) is replaced by

$$\frac{\psi'_L}{\psi'_R} = \alpha e^{i(e/\hbar c) \int_{-\infty}^{\infty} \mathbf{A} \cdot d\mathbf{r}}, \quad (24)$$

and its phase factor remains ambiguous as the integral  $\int_{-\infty}^{\infty} \mathbf{A} \cdot d\mathbf{r}$  is not a well-defined quantity for an open path.

## Andreev bound states and gauge symmetry

Now we discuss the manifestation of the Byers-Yang theorem for an open system with the boundary condition of equation (23) in a realistic system. The system under consideration is a Josephson weak link that connects the two regions of a superconductor with an external magnetic flux at a distance of the superconductor (see figure 2). The type of the junction is insignificant here. It can be described by the Bogoliubov-deGennes equation [10]:



$$\begin{pmatrix} H_e & \Delta(x) \\ \Delta^*(x) & -H_e^* \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = E \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}, \quad (25a)$$

where the components of the Hamiltonian in our framework is given by

$$\begin{aligned} H_e &= \frac{1}{2m} (-i\hbar\nabla - \mathbf{\Pi})^2 + U(x), \\ H_e^* &= \frac{1}{2m} (i\hbar\nabla - \mathbf{\Pi})^2 + U(x). \end{aligned} \quad (25b)$$

$\mathbf{\Pi}$  can be gauged away by the following transformation to the ‘primed’ functions:

$$\begin{aligned} \mathbf{\Pi}' &= \mathbf{\Pi} - \nabla\Lambda = 0 \\ u' &= ue^{-i\Lambda/\hbar}, \quad v' = ve^{i\Lambda/\hbar}, \\ \Delta' &= \Delta e^{-2i\Lambda/\hbar}, \end{aligned} \quad (26)$$

and thus, we obtain

$$\begin{pmatrix} H'_e & \Delta'(x) \\ \Delta'^*(x) & -H'_e \end{pmatrix} \begin{pmatrix} u'(x) \\ v'(x) \end{pmatrix} = E \begin{pmatrix} u'(x) \\ v'(x) \end{pmatrix}, \quad (27a)$$

where

$$H'_e = -\frac{\hbar^2}{2m} \nabla^2 + U(x). \quad (27b)$$

This transformation reveals the periodicity of the physical properties of the system. The eigenvalue  $E$  is determined by equation (27) and the boundary condition of  $\Delta'$  (represented in its phase shift)

$$\varphi \equiv \arg(\Delta'_L/\Delta'_R) = \varphi_0 + \varphi_B, \quad (28a)$$

where  $\Delta'_L \equiv \Delta'(x \rightarrow -\infty)$  and  $\Delta'_R \equiv \Delta'(x \rightarrow \infty)$  are the boundary values of  $\Delta'(x)$  at each lead.

$\varphi_0 = \varphi_L - \varphi_R$  is the intrinsic phase difference between the two sides of the superconductor, and  $\varphi_B$  is the flux dependence of the phase given by

$$\varphi_B = \frac{2}{\hbar} \int_{-\infty}^{\infty} \mathbf{\Pi} \cdot d\mathbf{r} = \frac{2e\theta}{\hbar c} \Phi, \quad (28b)$$

where  $\theta$  is the angle formed in the geometry of the system (see figure 2). Therefore, the eigenvalues are periodic functions of  $\Phi$  with period  $2\pi\hbar c/(2e\theta)$  and all physical properties display the same periodicity. Notably, for  $\theta = 2\pi$ ,  $\varphi_B$  reduces to the Aharonov–Bohm phase  $2e\Phi/(\hbar c)$  associated with the Cooper pair charge  $2e$ .

As an example, we consider a delta-function potential  $U(x) = U_0\delta(x)$  and a constant gap function  $\Delta(x) = \Delta_0$ . The latter condition gives  $\varphi_0 = 0$  in equation (28a). A solution inside the gap ( $-\Delta_0 < E < \Delta_0$ ), known as the Andreev bound state, can be determined by solving the Bogoliubov–deGennes equation (27) with the boundary condition of  $\Delta'(x)$  equation (28) [11, 12]. We obtain

$$E = \pm\Delta_0\sqrt{1 - T \sin^2(\varphi_B/2)}, \quad (29)$$

where  $T = 1/(1 + Z^2)$  is the transmission probability across the point contact with the parameter  $Z \equiv mU_0/(\hbar^2k_F)$  ( $k_F$  being the Fermi wave vector). Considering that the Andreev bound states and their phase dependence have been well confirmed in experiments with superconducting hybrid junctions, the flux dependence of the bound-state energy (equation (29)) can also be observed in real experiments. The bound-state energy may be directly probed by spectroscopic measurements (see e.g., [13, 14]) with variations in the magnetic

flux. To confirm the generalization of the Byers-Yang theorem, the superconductor should not form a closed loop that circulates around the flux to avoid observation of the ordinary AB phase  $2e\Phi/\hbar c$ .

## Conclusion

In conclusion, we have presented a reformulated  $U(1)$  gauge theory on the basis of physical symmetry. The symmetry transformation corresponds to a change in the electromagnetic field in the inaccessible region of the charged scalar field ( $\phi$ ) along with a change in the phase factor of  $\phi$ . This reformulation preserves all properties of the  $U(1)$  gauge theory but eliminates the redundancy inherent in the conventional approach. This also implies that quantum electrodynamics can be defined without relying on  $A^\mu$ . In addition, our formulation provides a generalization of the Byers-Yang theorem to an open system, which can be confirmed in an experiment for the Andreev bound states of a superconducting point contact.

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