QUANTUM FIELD THEORY AND STATISTICAL MECHANICS

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1. Introduction

In the last decade the ideas of Migdal and Polyakov[1] on one hand and Kadanoffs[2] work on the other hand led to Wilsons[3] renormalization group approach and the Wilson-Fischer ε-expansion[4]. These two steps renewed the interest of many physicists in critical phenomena and second order phase transitions[5]. The origin of the methods which contributed so much to the understanding of the statistical mechanics of phase transitions is quantum field theory, which on the other hand benefited a lot from the applications to phase transitions. Many of the relevant new ideas in the area of quantum field theory and the theory of phase transitions developed parallel in the two fields so the ideas of scaling of operator product expansions etc. The aim of the present lectures is a discussion of critical behaviour directly in renormalized field theory. But first I will briefly discuss some characteristic properties of second order phase transitions and give a heuristic understanding how the relation between quantum field theory and statistical mechanics near criticality comes about.
We will then turn to renormalized quantum field theory in $4-\varepsilon$ dimensions, calculate critical indices, and introduce a (pre -) scaling parametrization which will turn out to be most appropriate for a discussion of the scaling behaviour. We then investigate the structure of corrections to scaling for the thermodynamical quantities and the correlation functions.

2. Critical Phenomena[2] [5]

a. The thermodynamic quantities

We first introduce the relevant thermodynamic quantities for the study of second order phase transitions. For definiteness we start from a ferromagnetic Lenz-Ising system.

On a D dimensional lattice G_a in configuration space, with lattice spacing a, there is associated a discrete classical spin $\sigma_{\vec{n}} = \pm 1$ to each lattice point $\vec{n}$ (labeled by integers). The spins interact with its nearest neighbours (n.n.) only. Parallel spins are attractive with energy -K, a spin parallel to an external magnetic field H has energy -H; for antiparallel spins the energy is K and H. Accordingly the Lenz-Ising Hamiltonian reads

$$\mathcal{H} = -K \sum_{\substack{\vec{n}\,\vec{m} \\ n.n.}} \sigma_{\vec{n}} \sigma_{\vec{m}} - H \sum_{\vec{n}} \sigma_{\vec{n}} \tag{2.1}$$

For a finite system with N spins we obtain in the usual way thermodynamical quantities from the partition function

$$Z_N = \sum_{\substack{all \\ conf.}} \exp(-\beta \mathcal{H}) \quad ; \quad \beta = \frac{1}{kT} \tag{2.2}$$

For the free energy density

$$f(K, H) = \frac{F_N}{N} \quad ; \quad F_N = -\ln Z_N \tag{2.3}$$

the spin correlation functions

$$\langle \sigma_{\vec{n}} \rangle = Z_N^{-1} \sum_{conf.} \sigma_{\vec{n}} \, e^{-\beta \mathcal{H}}$$
$$\langle \sigma_{\vec{n}} \sigma_{\vec{m}} \rangle = Z_N^{-1} \sum_{conf.} \sigma_{\vec{n}} \sigma_{\vec{m}} \, e^{-\beta \mathcal{H}} \qquad etc. \tag{2.4}$$

and the energy correlation functions

$$\langle E_{\vec{n}} \rangle = Z_N^{-1} \sum_{conf.} E_{\vec{n}} \, e^{-\beta \mathcal{H}}$$

$$\langle E_{\vec{n}} E_{\vec{m}} \rangle = Z_N^{-1} \sum_{conf.} E_{\vec{n}} E_{\vec{m}} \, e^{-\beta \mathcal{H}} \qquad etc.$$

$$E_{\vec{n}} \doteq \sum_{\substack{\vec{m} \\ n.n.}} \sigma_{\vec{n}} \sigma_{\vec{m}}$$

we will always take the thermodynamic limit $N \to \infty$.
From the free energy density f we define the thermodynamical quantities ($k = \beta K$; $h = \beta H$):

$$M = \frac{\partial f}{\partial h} = \langle \sigma_{\vec{n}} \rangle = \langle \sigma_0 \rangle$$ magnetization density

$$E = \frac{\partial f}{\partial k} = \langle E_{\vec{n}} \rangle = \langle E_0 \rangle$$ energy density

$$\chi = -\frac{\partial^2 f}{\partial h^2} = \sum_{\vec{n}} \langle \sigma_{\vec{n}} \sigma_0 \rangle^{conn}$$ susceptibility (2.5)

$$= \sum_{\vec{n}} \{ \langle \sigma_{\vec{n}} \sigma_0 \rangle - \langle \sigma_{\vec{n}} \rangle \langle \sigma_0 \rangle \}$$

$$C = -\frac{\partial^2 f}{\partial k^2} = \sum_{\vec{n}} \langle E_{\vec{n}} E_0 \rangle^{conn}$$ specific heat.

$$= \sum_{\vec{n}} \{ \langle E_{\vec{n}} E_0 \rangle - \langle E_{\vec{n}} \rangle \langle E_0 \rangle \}$$

Here we have taken into account translation invariance.

b) Second order phase transitions

The 2nd order phase transitions are related to the spin fluctuations in the system. For large distances the spin correlations away from the critical point show the Ornstein-Zernicke exponential fall off

$$\langle \sigma_{\vec{n}} \sigma_0 \rangle \underset{|\vec{x}| \gg a}{\simeq} \frac{e^{-|\vec{x}|/\xi}}{|\vec{x}|} \quad ; \quad \vec{x} = \vec{n}\, a \tag{2.6}$$

This relation defines the correlation length ξ (the most important parameter in the study of 2. order phase transitions).
The phase diagram for a ferromagnetic system is depicted in Fig. 1

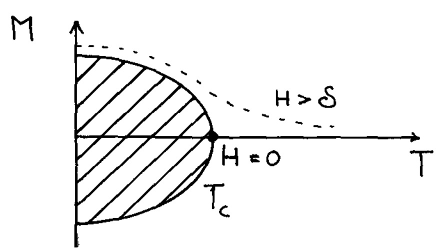

Fig. 1

For $H \neq 0$ all T and $H = 0$, $T > T_c$ the thermodynamic functions are analytic functions of T and H due to finite ξ, which means that the physics actually takes place in a finite box of size $L \gtrsim \xi$.

For H = 0, T $<$ T_c the spins are aligned and a spontaneous magnetization

$$<\sigma> = \pm M(T)$$

occurs. The state is not longer uniquely defined as a function of T. A first order phase-transition takes place as H changes sign and the system jumps from M to -M. As T $\to$ T_c from below the first order transition disappears. This is the critical point of a second order phase transition. It is necessarily a point of non analyticity as M(H,T) = 0 for T $>$ T_c and $\neq$ 0 for T $<$ T_c.
What happens is that for H = 0, T $<$ T_c there is a net magnetization, in the z direction say, and clusters of spins pointing in the wrong direction of maximal size ξ (with a clustering down to microscopic scale). To turn the spins it costs energy and therefore macroscopically the system is in a stable state.
As $\xi \to \infty$ (T $\to$ T_c, T $\leq$ T_c) criticality is approached; the difference in magnetization approaches zero and together with it the energy cost per area of producing a region of wrong phase. This is the region of large scale weak fluctuations in magnetization. The physics is then no longer determined by what's happening in a finite box. We are faced with a system of infinitly many degrees of freedom. With the critical behaviour there are associated characteristic singularities which are caused by correlations over infinite distances in space i.e. the fundamental reason is the divergence of ξ at the critical point

$$\xi \propto t^{-\nu} \qquad at \quad H=0 \quad as \quad t' - \frac{T-T_c}{T_c} \to 0 \tag{2.7}$$

The correlation functions then behave as

$$\left.\begin{aligned} <\sigma_{\vec{x}}\,\sigma_0> &\simeq \frac{const.}{|\vec{x}|^{2d_\sigma}} \\ <E_{\vec{x}}\,E_0> &\simeq \frac{const.}{|\vec{x}|^{2d_E}} \end{aligned}\right. \quad ; \quad |\vec{x}| \to \infty \tag{2.8}$$

causing divergent thermodynamic quantities (infinite sums over the densities). The singularities may be parametrized by power laws as calculations from the Lenz-Ising model and the mean field theory (qualitatively correct picture) as well as experiments confirm. The exponents are the critical indices defined by:

$$M|_H \simeq \begin{cases} 0 \\ (-t)^{\beta} \end{cases}$$

$$\chi|_H \simeq \begin{cases} t^{-\gamma} \\ (-t)^{-\gamma'} \end{cases} \qquad t \to \begin{cases} +0 \\ -0 \end{cases} \tag{2.9}$$

$$C|_H \simeq \begin{cases} t^{-\alpha} \\ (-t)^{-\alpha'} \end{cases}$$

$$M|_{t=0} \simeq \pm |H|^{1/\delta} \quad ; \quad H \to \pm 0$$

c. The Kadanoff picture of critical behaviour[2)]

What is an appropriate theory of critical phenomena? Kadanoff had the idea that the critical system can be reduced to the consideration of the physics contained in a finite box. Kadanoff's block spin picture can be roughly described as follows: The microscopic theory is described by cells of size a^D As $T \simeq T_c$ (i.e. $\xi \gg a$) a coarser division of the system into cells should give a good approximation to the macroscopic properties of the system. Hence one obtaines a new description of the system by forming block spins i.e. cells of size L^D ($a < L \ll \xi$); within these cells the spins are strongly correlated and behave essentially as one big spin with nearest neighbour interaction.
By forming the big spin

$$\tilde{\sigma}_i = \sum_{cell_i} \sigma_{i'} \tag{2.10}$$

one actually averages out the non relevant degrees of freedom."Renormalizing" the big spins to $\pm$ 1 one gets an equivalent description of the system

$$\mathbb{H}_a \doteq \beta \mathcal{H}_a = -k \sum_{n.n.} \sigma_{\vec{n}} \sigma_{\vec{m}} - h \sum \sigma_{\vec{n}} \qquad ; \vec{n} \in G_a$$

by the Hamiltonian

$$\mathbb{H}_L \doteq \beta \mathcal{H}_L = -\tilde{k} \sum_{n.n.} \sigma_{\vec{n}} \sigma_{\vec{m}} - \tilde{h} \sum \sigma_{\vec{n}} \qquad ; \vec{n} \in G_L \tag{2.11}$$

with $\tilde{k} = (L/a)^y k$ and $\tilde{h} = (L/a)^x h$ (2.12)

For exactly aligned spins in each cell

$$y = y_0 = D - 1 \qquad \text{and} \qquad x = x_0 = D \tag{2.13}$$

The crucial point is that the spins are not exactly lined up due to fluctuations down to microscopic scale and therefore the coefficients x and y have not the values (2.13), they merely have to be considered as unknown parameters*. It will be one of the main goals of a theory of critical phenomena to explain and calculate these indices. In a precise formulation of the block spin picture the "average" (2.10) has to be done actually in the partition function. This will be discussed in detail by Wegner in his lectures. The transformation

$$\mathbb{H}_a \longrightarrow \mathbb{H}_L = T_{L/a}\, \mathbb{H}_a \tag{2.14}$$

is called a renormalization group (RG) transformation[3]. It has the semigroup property.
For $\xi \gg L > a$ we expect the physics described by $\mathbb{H}_L$ to be essentially unchanged

$$\mathbb{H}_L \simeq \mathbb{H}_a$$

At the critical point the physics is expected to be independent of the cell size such that

$$\mathbb{H}_L^{(o)} = \mathbb{H}_a^{(o)} = \mathbb{H}^* \qquad \forall L \quad \text{as} \quad H=0\,;\ T=T_c \tag{2.15}$$

i. e. we have a fixed point of the above transformation $T_{L/a}$.

d. Fixed point properties[3)6)]

As the system deviates from criticality $\xi \neq \infty$ the Hamiltonian may be viewed as consisting of a critical part $\mathbb{H}^*$ and a remainder

$$\mathbb{H} = \mathbb{H}^* + \delta\mathbb{H}$$

i.e.

$$\mathbb{H} = -k_c \sum \sigma_{\vec{n}} \sigma_{\vec{m}} - (k-k_c) \sum \sigma_{\vec{n}} \sigma_{\vec{m}} - h \sum \sigma_{\vec{n}} = \mathbb{H}^* + \sum_i h_i\, O_i \tag{2.16}$$

*(Note that the homogeneous Ansatz (2.12) is assumed to make sense only near criticality.)

with $k - k_c$ proportional to the reduced temperatur t. Hence h_i are the parameters ("fields") which describe the deviation from criticality and the O_i's are the conjugate operators. For infinitesimal

$$\delta \mathbb{H} = \sum_i h_i O_i$$

and the O_i's choosen (if possible) diagonal under $T_{L/a}$ it follows:

$$T_{L/a} O_i = (L/a)^{y_i} O_i$$
$$\delta \mathbb{H}_L = T_{L/a} \, \delta \mathbb{H}_a = \sum_i h_i (L/a)^{y_i} O_i = \sum_i \tilde{h}_i O_i \qquad (2.17)$$

According to whether $h_i \to \tilde{h}_i$ is increasing or decreasing the eigen-operators (and the conjugate fields) are classified:

$$\begin{array}{ll} y_i > 0 & \text{relevant} \\ y_i < 0 & \text{irrelevant} \\ y_i = 0 & \text{marginal} \end{array} \qquad (2.18)$$

If the relevant fields are zero we call $\mathbb{H} = \mathbb{H}^{(0)}$ critical. We have

$$T_{L/a} \mathbb{H}^{(0)} \longrightarrow \mathbb{H}^{*}; \quad L \to \infty \qquad (2.19)$$

under suitable behaviour of the marginal fields. As we will see in our field theoretical treatment the marginal operators alone determine the fixed point (if any) properties of $\mathbb{H}$.
When (2.17) can be realized globally ($\forall L$) by a suitable choice of the parametrization of non marginal fields we call this a parametrization in terms of global <u>scaling fields</u>. These fields have been introduced by Wegner[6].

e. <u>Scaling</u>[2]

What follows from this intuitive block spin picture for the thermodynamic properties in the critical region?
As a cell of size L contains $(\frac{L}{a})^D$ spins it follows in view of (2.10-12) that as we increase the cell size from a to L:

$$f \longrightarrow f(\tilde{t},\tilde{h}) = (L/a)^{D} f(t,h)$$

Hence

$$\langle\sigma\rangle \to \langle\tilde{\sigma}\rangle = \langle\sigma\rangle(\tilde{t},\tilde{h}) = \frac{\partial f}{\partial \tilde{h}} = (L/a)^{D-x}\frac{\partial f}{\partial h} = (L/a)^{D-x}\langle\sigma\rangle(t,h)$$

$$\langle E\rangle \to \langle\tilde{E}\rangle = \langle E\rangle(\tilde{t},\tilde{h}) = \frac{\partial f}{\partial \tilde{k}} = (L/a)^{D-y}\frac{\partial f}{\partial k} = (L/a)^{D-y}\langle E\rangle(t,h) \tag{2.20}$$

Thus we are able to express the functions $\langle\sigma\rangle$, $\langle E\rangle$ etc. we are interested in through functions $\langle\tilde{\sigma}\rangle$, $\langle\tilde{E}\rangle$ etc. referring to a system with a reduced number of degrees of freedom.
With $\varkappa = L/a$

$$\langle\sigma\rangle(t,h) = \varkappa^{x-D}\langle\sigma\rangle(\varkappa^{y}t, \varkappa^{x}h) \quad .$$

In order the cell size L to cancel the function on the r.h.s. can only depend on the invariant product : $h|t|^{-x/y}$ hence (set $\varkappa^{y}|t|=1$ or $\varkappa^{x}|h|=1$)

$$\begin{aligned}\langle\sigma\rangle(t,h) &= \operatorname{sign} h\, |t|^{(D-x)/y}\ \varphi_\sigma(h|t|^{-x/y}) \\ &= \operatorname{sign} h\, |h|^{(D-x)/x}\ \chi_\sigma(t|h|^{-y/x})\end{aligned} \tag{2.21}$$

(with $\varphi_\sigma(0)$, $\chi_\sigma(0)$ finite) where we used in addition the symmetry properties of the system.
Similarly

$$\begin{aligned}\langle\sigma\sigma\rangle(|\vec{x}|,t,h) &= \varkappa^{2(x-D)}\langle\sigma\sigma\rangle(\varkappa^{-1}|\vec{x}|, \varkappa^{y}t, \varkappa^{x}h) \\ &= |t|^{2\frac{D-x}{y}}\ \varphi_{\sigma\sigma}(|\vec{x}||t|^{1/y};\ h|t|^{-x/y}) \\ &= |\vec{x}|^{2(x-D)}\ \psi_{\sigma\sigma}(t|\vec{x}|^{-y};\ h|\vec{x}|^{-x})\end{aligned} \tag{2.22}$$

$$\begin{aligned}\langle EE\rangle(|\vec{x}|,t,h) &= \varkappa^{2(y-D)}\langle EE\rangle(\varkappa^{-1}|\vec{x}|, \varkappa^{y}t, \varkappa^{x}h) \\ &= |t|^{2\frac{D-y}{y}}\ \varphi_{EE}(|\vec{x}||t|^{1/y},\ h|t|^{-x/y}) \\ &= |\vec{x}|^{2(y-D)}\ \psi_{EE}(t|\vec{x}|^{-y},\ h|\vec{x}|^{-x})\end{aligned}$$

The functions $\varphi_{..}$ and $\psi_{..}$ are expected to be finite at zero* and integrable (summable) on $\mathbb{R}^D$ such that at the critical point the behaviour (2.8) for the spin and energy correlation can be read off i.e.

$$x = D - d_\sigma \quad ; \quad y = D - d_E \tag{2.23}$$

Similarly we get for the thermodynamical quantities:

$$\alpha = 2 - D/y \; ; \; \beta = \frac{D-x}{y} \; ; \; \gamma = \frac{2x-D}{y} \; ; \; \delta = \frac{x}{D-x} \tag{2.24}$$

From the comparison of the Ornstein-Zernicke form (2.6) and (2.22) we have

$$\nu = 1/y \tag{2.25}$$

As all critical indices are related to x and y we have the following scaling relations among them:

$$\begin{gathered} \alpha = \alpha' \; ; \; \gamma = \gamma' \; ; \; \nu = \nu' \\ \beta(\delta+1) = 2\beta + \gamma = 2 - \alpha \end{gathered} \tag{2.26}$$

To summarize the Kadanoff scaling picture leads to the following results:

(1) Second order phase transitions are described by homogeneous functions. More refined arguments show that at the critical point physics is governed by a scale invariant theory (powerlaws, in exceptional cases also logarithms).

(2) The scaling assumption relates all critical indices to two independent coefficients determined from the knowledge of the spin and the energy twopoint functions.

f. Kadanoff Universality[2)7)]

Our discussion makes plausible, and it is supported by experiments and from model calculations, that the critical behaviour of systems with short range forces is independent on the

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* (as they refer to a system with a finite number of degrees of freedom by the elimination process)

a) lattice structure and the discretness
b) details of interactions.

Accordingly one expects universality classes of critical theories with identical critical properties. Within an universality class one can perform transformations on the physical parameters such that different systems are described by the same functions. It is well known that e.g. the ferromagnetic transitions and the liquid-gas transitions have the same critical indices.
On the other hand it is found that critical behaviour is differentiated by

a) the dimension of the system;
as Wilson pointed out critical indices seem to depend analytically on D as $D \leq 4$ (This suggested the Wilson-Fisher ε-expansion)
b) Symmetry of the system;
e.g. Lenz-Ising, Heisenberg, Spherical model.
c) ev. other unknown parameters.

This closes our phenomenological discussion of critical phenomena. What has to be done is to make Kadanoff's ideas quantitative. In particular one has to explain the scaling and universality properties and to calculate the critical indices. It was the main benefit from Wilson's RG approach relating Kadanoff's picture to field theory and the ε-expansion and the 1/n - expansion that one has approximate solutions for a considerable range of universality classes which also cover many systems realized in nature[2)]. In the next section we will discuss how field theory is related to statistical mechanics.

3. The Lenz-Ising System and Euclidean Field Theory

a. The Lenz-Ising system

Migdal, Polyakov and in most powerful manner Wilson used the Kadanoff ideas in order to relate the lattice systems of classical statistical mechanics to euclidean field theory by disgarding details (short range fluctuations) such that one stays in the original universality class, i. e. not changing the critical behaviour. We briefly discuss Wilsons instructive argumentation to manufacture a field theory which is in the universality class of the L. I. model[3)].

Let us consider a lattice system of classical spins $\sigma_{\vec{n}}$, with spin distribution $\rho(\sigma_{\vec{n}}^2)$. The generating functional for the spin correlation functions is

$$Z\{J\} = C \int \prod_{\vec{n}} d\sigma_{\vec{n}}\, \rho(\sigma_{\vec{n}})\, e^{\frac{1}{2}\sigma K \sigma + J\sigma} \tag{3.1}$$
$$= C \int \prod_{\vec{n}} d\sigma_{\vec{n}}\, e^{-\mathcal{H}[\sigma] + J\sigma}$$

$$\left.\frac{\delta^m Z\{J\}}{\delta J_{\vec{n}_1} \cdots \delta J_{\vec{n}_m}}\right|_{J=0} = \langle \sigma_{\vec{n}_1} \cdots \sigma_{\vec{n}_m} \rangle \tag{3.2}$$

with $\sigma K \sigma = \sum_{\vec{n}\vec{m}} K_{\vec{n}\vec{m}} \sigma_{\vec{n}} \sigma_{\vec{m}}$ and $J\sigma = \sum_{\vec{n}} J_{\vec{n}} \sigma_{\vec{n}}$

C serves to normalize Z to Z $\{o\}$ = 1. For the ferromagnetic Lenz-Ising model $K_{\vec{n}\vec{m}} \leq 0$; $K_{\vec{n}\vec{n}} = 0$; $K_{\vec{n}\vec{m}} = K_{\vec{n}-\vec{m}}$

$$\text{with} \quad K_{\vec{n}} = \begin{cases} k < 0 & \text{for} \quad |\vec{n}| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

i.e. in Fourierspace

$$\tilde{K}(\vec{q}) = (2\pi)^{-D/2} \sum_{\vec{n}} K_{\vec{n}}\, e^{-i\vec{q}\vec{n}} = 2k \sum_{i=1}^{D} \cos q_i \tag{3.4}$$

The spin values are fixed to $\sigma = \pm 1$ with

$$\rho(\sigma_{\vec{n}}^2) = \delta(\sigma_{\vec{n}}^2 - 1) = \lim_{u_0 \to \infty} \frac{u_0}{\pi}\, e^{-u_0(\sigma_{\vec{n}}^2 - 1)} \tag{3.5}$$

The approximate Lenz-Ising model we are interested in we obtain for finite u_0($u_0 \gg 1$); the L.I. system will be recovered as $u_0 \to \infty$

The bilinear part of $\mathcal{H}[\sigma]$ then reads

$$\mathcal{H}_0 = \frac{1}{2} \int_{-\pi}^{+\pi} d^D q\, |\tilde{\sigma}(\vec{q})|^2\, G_0^{-1}(\vec{q}) \tag{3.6}$$

with "propagator"

$$G_0(\vec{q}) = \frac{1}{-2u_0 - 2k\sum_{1}^{D}\cos q_i} \tag{3.7}$$

The "interaction" part is:

$$\mathcal{H}_I = \mathcal{H} - \mathcal{H}_0 = u_0 \sum_{\vec{n}} (\sigma_{\vec{n}}^2)^2 \tag{3.8}$$

The generating functional (3.1) may then be written in the form

$$\begin{aligned} Z\{J\} &= C \int \prod_{\vec{n}} d\sigma_{\vec{n}}\, e^{-\mathcal{H}_I[\sigma]}\, e^{-\mathcal{H}_0[\sigma] + J\sigma} \\ &= \hat{C}\, e^{-\mathcal{H}_I[\frac{\delta}{\delta J}]}\; Z_0\{J\} \end{aligned} \tag{3.1'}$$

with

$$Z_0\{J\} = e^{\frac{1}{2} J G_0 J}$$

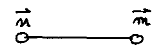
the free generating functional.

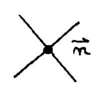
A formal power expansion in u_0 gives rise to a Feynman graph expansion for the correlation functions (3.2):

$$\begin{aligned} \langle \sigma_{\vec{n}_1} \cdots \sigma_{\vec{n}_m} \rangle = \sum_j \frac{(-u_0)^j}{j!} \sum_{\vec{m}_1,\ldots,\vec{m}_j} \int_{-\infty}^{+\infty} \prod_{\vec{n}} d\sigma_{\vec{n}} \;\times \\ \times\; \sigma_{\vec{n}_1} \cdots \sigma_{\vec{n}_m}\, \sigma_{\vec{m}_1}^4 \cdots \sigma_{\vec{m}_j}^4\, e^{-\mathcal{H}_0[\sigma]} \end{aligned} \tag{3.2'}$$

This expression equals by (3.1') and (3.2) to the sum over all total contractions of pairs of σ 's in $\sigma_{\vec{n}_1} \ldots \sigma_{\vec{m}_j}^4$

The Feynman-rules are:

Contractions: $\sigma_{\vec{n}}\, \sigma_{\vec{m}} = G_0(\vec{n} - \vec{m})$: 

vertices: $\sum_{\vec{n}} \sigma_{\vec{n}}^4$: $\sum_{\vec{n}}$ 

As we will see under the RG-transformation u_0 transforms in analogy to (2.12) to small effective couplings and perturbation theory becomes applicable near criticality. (In renormalized field theory the renormalized coupling will turn out to be small whereas the bare coupling $\mu_0 \to \infty$ as $\Lambda \to \infty$).

For small momenta $p_i = \frac{q_i}{a}$ we observe that

$$G_0(\vec{p}) \simeq a^{-2} k^{-1} \frac{1}{\vec{p}^{\,2} + m^2} \quad \text{with} \quad m^2 = -2a^{-2}(u_0 k^{-1} + D) \tag{3.9}$$

is apart from a factor, which is eliminated by rescaling the field $\sigma \to \hat{\sigma} = a\sqrt{k}\,\sigma$ an euclidean scalar propagator i.e. in the long range region

$$\sigma_{\vec{n}} \to \hat{\sigma}(\vec{x}) \tag{3.9a}$$

behaves as a continuous euclidean scalar field. The energy density (2.4) in a similar way

$$E_{\vec{n}} = \sum_{\substack{\vec{m} \\ n.n.}} \sigma_{\vec{n}} \sigma_{\vec{m}} \to \hat{E}(\vec{x}) = \tfrac{1}{2} \{ \partial_i \hat{\sigma}(\vec{x}) \partial^i \hat{\sigma}(\vec{x}) - m_0^2 \hat{\sigma}^2(\vec{x}) \} \tag{3.9b}$$

behaves as a field.

If one change according to Wilson (3.6) to

$$\mathcal{H}_0 = \tfrac{1}{2} \int_{|\vec{p}|<\Lambda} d^D p \, |\tilde{\hat{\sigma}}(\vec{p})|^2 (\vec{p}^{\,2} + m^2) \tag{3.10}$$

one expects not to change the critical behaviour as the small momentum behaviour (long range) is kept exactly. The rotational invariant cut-off here represents a substitute for the lattice cut-off a^{-1}. The difficulty is that the classical functional (3.1) with the replaced $\mathcal{H}_0$ is illdefined and $\hat{\sigma}(\vec{x})$ has to be replaced by a box field

$$\varphi^{(L)}(\vec{x}) = \frac{1}{L^{D/2}} \sum_{|\vec{p}|<\Lambda} e^{i\vec{p}\vec{x}} \, \tilde{\varphi}(\vec{p}) \, ; \quad \vec{p} = \frac{2\pi}{L} \vec{n}$$

For the correlation functions the "thermodynamic limit" $L \to \infty$ may then be carried out:

$$\langle \varphi(\vec{x}_1) \ldots \varphi(\vec{x}_n) \rangle = \lim_{L\to\infty} \langle \varphi^{(L)}(\vec{x}_1) \ldots \varphi^{(L)}(\vec{x}_n) \rangle$$

We prefer however to construct directly an euclidean cut-off field theory with $\mathcal{H}_0$ of the form (3.10) avoiding the difficulty of the functional formulation.

b. Euclidean Field Theory[8)]

An euclidean cut-off theory may be constructed as follows: Let A(k) and $A^+(k)$ be annihilation and creation operators subject to the commutation relations

$$[A(k), A(k')] = [A^+(k), A^+(k')] = 0 \tag{3.11}$$
$$[A(k), A^+(k')] = \delta^{(D)}(k-k')$$

From the cyclic euclidean free vacuum $|\Phi_o>_E$

$$A(k)\,|\Phi_o>_E = 0 \tag{3.12}$$

we generate the euclidean Fock-space

$$\mathcal{H}_E = \overline{\{\mathcal{P}\{A^+\}\,|\Phi_o>_E}\,\} \tag{3.13}$$

Then the free field

$$A_o(x) = (2\pi)^{-D/2} \int_{|k|<\Lambda} \frac{d^D k}{\sqrt{k^2+m^2}} \{ e^{-ikx} A^+(k) + h.c. \} \tag{3.14}$$

leads to the propagator (3.10)

$$G_o(x-y) = {}_E< \Phi_o|\, A_o(x)\, A_o(y)\, |\Phi_o>_E \tag{3.15}$$

The commuting fields $A_o(x)$ generate from the euclidean vacuum a cut-off Hilbert-space $\mathcal{H}_{E,\Lambda} \subset \mathcal{H}_E$. In order to obtain a complete set of operators one introduces the canonical conjugate field

$$\Pi_o(x) = \frac{i}{2} \int_{|k|<\Lambda} d^D k \sqrt{k^2+m^2} \{ e^{-ikx} A^+(k) - h.c. \} \tag{3.16}$$

with $$[\Pi_o(x), A_o(y)] = -i\delta(x-y)\,;\ [\Pi_o(x), \Pi_o(y)] = 0 \tag{3.17}$$

Contrary to the relativistic case (non commuting fields) the euclidean generators of symmetry transformations cannot be represented in terms of the now commuting A's e. g. the euclidean Hamiltonian, generating time translations in $\mathcal{H}_{E,\Lambda}$ is:

$$H = -\frac{i}{2}\int d^{\mathfrak{J}}x \;\; :\Pi_o(x)\overleftrightarrow{\partial_t}A_o(x): \tag{3.18}$$

In the interacting case, with $\mathcal{H}_I[A_o]$ an integral over a local polynomial in A_o, the euclidean Green functions

$$_E\langle o|A(x_1)\ldots A(x_n)|o\rangle_E = {}_E\langle\Phi_o|A_o(x_1)\ldots A_o(x_n)e^{-\mathcal{H}_I[A_o]}|\Phi_o\rangle_E \tag{3.19}$$

are identical with the probabilistic correlation functions (3.2) for

$$\mathcal{H}_I[A_o] = u_o\int d^{\mathfrak{J}}x : A_o^4(x): \tag{3.20}$$

There are some peculiar features to euclidean fields: Due to

$$i\int d^{\mathfrak{J}}k\, e^{-ikx}A_o(k)\sqrt{k^2+m^2} = i\,\Pi_o(x) + \tfrac{1}{2}(-\Delta+m^2)A_o(x) \tag{3.21}$$

there exist <u>short range</u> fields as

$$\Psi_o(x) = (-\Delta+m^2)A_o(x) \tag{3.22}$$

with

$$_E\langle\Phi_o|\Psi_o(x)A_o(y)|\Phi_o\rangle_E = \delta_\Lambda(x-y) \tag{3.23}$$

In the relativistic case of course $\Psi_o(x) \equiv o$. The set of euclidean local fields therefore consists of the usual Wick ordered fields

$$:A_o^n(x):$$

<u>and</u> short range composite fields like

$$:A_o^n(x)\,\Psi_o(x):$$

This situation of course persists in the interacting case. We believe that the so called "redundant" operators introduced by Wegner are related to the short range fields discussed here.

A further serious difference is the following: After renormalization relativistic composite fields remain in the class of operator-valued distribution as $\Lambda\to\infty$. This is not true for euclidean composite fields as e. g. in D = 4

$$\|A^2_{o,\Lambda}(f)|\Phi_o\rangle_E\| \to \infty\,,\quad(\Lambda\to\infty) \tag{3.24}$$

i.e. : $A_o^2(x)$: can only have a meaning as a bilinear form and does not exist as an operator. For non-overlapping test-functions $f_1,\dots,f_n$ composite correlation functions however exist in the limit $\Lambda \to \infty$:

$$_E\langle \Phi_o | A^2_{o,\Lambda}(f_1) \ . \ . \ . \ A^2_{o,\Lambda}(f_n) | \Phi_o \rangle_E \quad \longrightarrow \quad \text{finite} \qquad (3.25)$$

in D = 2 dimensions: $A^n_{o,\Lambda}$: exists for $\Lambda \to \infty$, for D = 3 only : A^2_o,: and A_o in D = 4 only A_o exists as an operator in the limit .
This situation is a handicap for the Kadanoff-Wilson operator product expansion. Either one has to consider it as a statement on correlation functions only or one has to go to the relativistic theory.
The only thing we should learn from the above discussion is that the L. I. model and the A^4-field theory are likely to belong to the same universality class.

4. Construction of Critical Theories

In the construction of critical theories there are two different possibilities. The more ambitious one is to study critical behaviour and deviations from it directly within the global physical theory (e.g. for certain physical systems the Lenz-Ising model in D = 3 dimensions). In this case also non universal properties of the system may be calculated. Recent progress in this approach has been made by Nauenberg and Nienhuis[9)] for the LI system.

The other attempt in the spirit of Kadanoff, is to take care only of the universal properties i.e. to construct critical theories lying in a particular universality class where one hopes to find a single scale invariant (and hence conformal invariant)[10)] theory. The most reasonable approach in constructing critical theories therefore seems to be the direct construction of conformal invariant theories and to determine the spectrum of e. g. anomalous dimensions of conformal theories (classification of critical theories). This actually was the first attempt in the construction of critical theories by Polyakov[1)] in his bootstrap approach as developed further by Parisi-Peliti[11)], Mack[12)] and others. This ambitious program unfortunately did not yet succeed but we believe that this is the way to construct non trivial critical theories beyond the present approximation schemes.

It was Wilson who succeeded first in the construction of nontrivial critical theories by his RG approach. A quantitative realization of Kadanoff's idea of eliminating irrelevant degrees of freedom (i.e. the short range fluctuations) for the A^4 cut-off theory in functional form (3.1) led him to a study of the RG transformation

$$e^{-\mathcal{H}' + \text{const.}} = \int \prod_{\frac{\Lambda}{s} < |p| < \Lambda} d[\varphi^{(L)}] \; e^{-\mathcal{H}[\varphi^{(L)}]} \Big|_{\varphi^{(L)} \to \alpha_s \varphi^{(sL)}}$$

with $\alpha_s = s^{1-n}/2$. In this procedure all internal lines in Feynman diagrams are integrated out over the short range part $s^{-1}\Lambda < |p| < \Lambda$ giving rise to new effective mass and coupling (renormalized) and new (nonrenormalizable) vertices which however should be irrelevant in the critical region. The external lines have momenta restricted to $|p| < \Lambda s^{-1}$. In an approximate form Wilson was able to determine fixed points of the transformation from computer calculations. Under further approximation using perturbation theory in $\varepsilon = 4-D$ dimensions analytic calculations for nontrivial fixed points have been done by Wilson and Fisher[4)] and Wegner and Houghton[4)] and others.

According to our philosophy only the universal scaling properties of models (which differ from the global physical model) can be taken seriously. These models have to be choosen within one universality class from the point of view of simplicity and computability. Concerning the universality class of the L.I. system we presented the arguments which suggest that A^4-field theory models are in the same class and we are faced simply with the problem of constructing scale invariant A^4 models. As the direct conformal construction was not yet successfull the next step would be to use renormalized perturbation theory. This is what we will do in the following. We will eliminate the cut-off Λ in the euclidean A^4-model from the beginning and use renormalized field theory[13)] for the study of scale invariant theories by looking at the fixed point properties of the dilatation Ward-identity (Callan-Symanzik equation).

We will go one step further and consider the relativistic local A^4-theory avoiding thereby the peculiarities of euclidean theories mentioned in the last section. In doing so we refer to the equivalence statement of Osterwalder and Schrader[8)]. By the spectrum condition the time ordered relativistic Green functions in D space-time dimensions $x = (x^0, x^1, \ldots, x^{D-1})$ are analytic in x^0 and are identical with the euclidean Green functions for

$$x_i^0 = i\, x_i^D \,; \qquad x_i^D \;\; \text{real}$$

Criteria on the validity of the Landau mean field approximation as confirmed by model calculations show that the critical theory is a free field theory in D $\geq$ 4 dimensions and critical indices are likely to depend analytically on D below D = 4. This suggested Wilson and Fisher[4] to compute critical theories starting from D = 4 by analytic continuation in $1 \gg \varepsilon = D-4 \geq 0$ (perturbation around free theory: ε-expansion). Low order ε calculations are in remarkable agreement with LI-calculations and experiments for $\varepsilon = 1$ and even for $\varepsilon = 2$ (see Tab. 2). In our approach we will use the ε - expansion for the construction of critical theories. A direct approach to critical theories in D = 3 and D = 2 was given by Parisi[14]; see also the investigations of Symanzik[14] and Schroer[29].

To summarize: What we will do in our further discussion within the framework of renormalized perturbation theory, is to

a) construct a long range scale invariant theory
b) calculate critical indices and prove the relations among them
c) formulate field theoretical Kadanoff scale transformations
d) calculate corrections to scaling

5. Renormalized Perturbation Theory and ε-Expansion

a. Parametrizations of Green Functions

We briefly discuss renormalized quantum field theory as used in our further considerations. We start from a Lagrangian cut-off Λ theory with $(\mathcal{L}_{int} = -\mathcal{H}_I)$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} = \frac{1}{2}(\partial A)^2 - \frac{m_0^2}{2} A^2 - \frac{u_0}{4!} A^4 \tag{5.1}$$

The correlation functions (time ordered Green functions) are obtained as a formal power series expansion (Feynman graph expansion) in u_0 from the Gell-Mann-Low formula[16]

$$\langle T \prod_{k=1}^{N} A(x_k) \rangle = \langle \Phi_0 | T \prod_{k=1}^{N} A^{(0)}(x_k)\, e^{i\int \mathcal{L}_{int}^{(0)} dx} | \Phi_0 \rangle_{\otimes} \tag{5.2}$$

$|\Phi_o\rangle$ denotes the free Fock vacuum; $A^{(o)}(x)$ is the free scalar field of mass m_o and $\mathcal{L}^{(o)}_{int} = \mathcal{L}_{int}(A^{(o)})$. $\otimes$ denotes the omission of vacuum diagrams i.e. the division by $\langle\Phi_o| e^{i\int \mathcal{L}^{(o)}_{int} dx} |\Phi_o\rangle$.
The generating functional for the disconnected Green functions (partition functional) is

$$Z\{J\} = \langle \Phi_o | e^{i\int(\mathcal{L}^{(o)}_{int} + J(x)A^{(o)}(x))dx} | \Phi_o \rangle_{\otimes} \tag{5.3}$$

The generating functional of the connected Green functions (Gibbs potential, free entalpy functional) is given by $G\{J\} = \ln Z\{J\}$

$$G^{(N)}(x_1,\ldots,x_N) = \langle T \prod_{k=1}^{N} A(x_k)\rangle^{conn} = (-i)^N \frac{\delta^N G\{J\}}{\delta J(x_1)\ldots\delta J(x_N)}\bigg|_{J=0} = \sum_{conn} \tag{5.4}$$

The parametrization in terms of the bare parameters u_o and m_o is not convenient for the purpose of statistical mechanics. At criticality not the bare mass m_o but the renormalized mass m($\xi = m^{-1}$ correlation length) defined by the momentum space location of the propagator pole has to vanish (see (2.6) respectively (2.8)).
Like in particle physics it is therefore much more convenient to use a parametrization in terms of renormalized quantities. To this end a multiplicative renormalization of fields $A \to \hat{A} = Z^{-1/2} A$ and subtractions (by adding appropriate counterterms to the bare Lagrangian (5.1)) are performed to the correlation functions in such a way that certain normalization conditions (defining the physical interpretation of the new parameters) are satisfied.

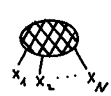
The re-normalization (re-parametrization) is most conveniently done for the vertex functions, the Legendre-transforms (with respect to the source J(x)) of the connected Green functions. The generating functional (Helmholz potential, free energy functional)[17] reads:

$$\Gamma\{K\} = G\{J\} - i\int K(x)\, J(x)\, dx \tag{5.5}$$

with $$K(x) = -i\,\frac{\delta G\{J\}}{\delta J(x)}$$

The vertex functions

$$\Gamma^{(N)}(x_1,\ldots,x_N) = \frac{\delta^N \Gamma\{K\}}{\delta K(x_1)\ldots\delta K(x_N)}\bigg|_{K=0} = \sum_{prop} \tag{5.6}$$



are represented by a sum over the proper i.e. the connected one-particle irreducible (i.e. connected after cutting a single line) amputated (i.e. no external legs) diagrams and

$$\Gamma^{(2)} = -\{G^{(2)}\}^{-1} \tag{5.7}$$

The $G^{(N)}$'s which are then trees in the $\Gamma^{(N)}$'s (no extra loops!) are given by the inverse Legendre transformation. The renormalization problem is completly solved by the renormalization of the $\Gamma^{(N)}$'s. The Fourier transforms of $\Gamma^{(N)}$ may be written as:

$$\Gamma^{(N)}(p_1,\ldots,p_N) = \langle T A(0) \tilde{A}(p_2) \ldots \tilde{A}(p_N) \rangle^{prop} \tag{5.8}$$

We now consider the different parametrizations of the correlation functions. The parametrization standard in particle physics (mass shell normalization) is defined through the normalization conditions (see e.g. [18] [19]):

$$\begin{aligned} &\Gamma^{(2)}\big|_{p^2=m^2} = 0 \\ &\frac{\partial \Gamma^{(2)}}{\partial p^2}\Big|_{p^2=m^2} = i \\ &\Gamma^{(4)}\big|_{s.p.\,m^2} = -ig\,m^{\varepsilon} \qquad with \;\; s.p.\,m^2:\; p_i p_j = \tfrac{1}{4}(3\delta_{ij}-1)\,m^2 \end{aligned} \tag{5.9}$$

This parametrization is not suitable to our aim of constructing a critical theory $\xi^{-1} = m = 0$ (zero mass theory) as the Green functions are not continuous at m = 0 (diverging residue of the propagator pole). A parametrization with a continuous zero mass limit was given by Gell-Mann and Low[16] [20]

$$\begin{aligned} &\tilde{\Gamma}^{(2)}\big|_{p^2=\tilde{m}^2} = 0 \\ &\tilde{\Gamma}^{(2)}\big|_{p^2=-\mu^2} = -i\mu^2 \qquad \left(or \;\; \frac{\partial \tilde{\Gamma}^{(2)}}{\partial p^2}\Big|_{p^2=-\mu^2} = i \right) \\ &\tilde{\Gamma}^{(4)}\big|_{s.p.\,-\mu^2} = -i\tilde{g}\,\mu^{\varepsilon} \end{aligned} \tag{5.10}$$

Here the critical theory is obtained for $\xi^{-1} = \tilde{m} = 0$ where the $\tilde{\Gamma}$'s exist (finite residue of the propagator pole).

There is an other parametrization (<u>soft</u> or <u>pre-scaling</u> parametrization) which will be most adequate for our purpose. It is defined by [21]:

$$
\begin{aligned}
&\hat{\Gamma}^{(2)}\Big|_{\substack{p^2=0\\ \hat{m}^2=0}} = 0\\
&\hat{\Gamma}^{(2)}\Big|_{\substack{p^2=-\mu^2\\ \hat{m}^2=0}} = -i\mu^2 \quad \left(\text{or} \quad \frac{\partial \hat{\Gamma}^{(2)}}{\partial p^2}\Big|_{\substack{p^2=-\mu^2\\ \hat{m}^2=0}} = i\ \right) \qquad (5.11)\\
&\hat{\Gamma}^{(4)}\Big|_{\substack{s.p.-\mu^2\\ \hat{m}^2=0}} = -i\hat{g}\mu^{\varepsilon}
\end{aligned}
$$

and

$$
\frac{\partial \hat{\Gamma}^{(2)}}{\partial \hat{m}^2}\Big|_{\substack{p=0\\ \hat{m}^2=\mu^2}} = i
$$

This parametrization will be used in the following. The properties we will discuss in section 8. We only mention here that the critical theory again is obtained for $\hat{m} = o$ (however now $\hat{m} \neq \xi^{-1}$) where the $\hat{\Gamma}$'s exist. Equivalent parametrizations have been discussed in Ref. 22) and 23) in a different context in Ref. 24). We will see that $\hat{m}^2 = t$ is a parameter proportional to the reduced temperatur $(T-T_c)/T_c$ in the critical region. t will simply be called temperature in the following. All the parametrizations mentioned above have a limit $\Lambda \to \infty$ and we are dealing hence with a renormalized local quantum field theory.

Also our model is superrenormalizable in D = 4- ε ($\varepsilon > o$) dimensions we will keep the normalization conditions as for D = 4 in order to have a continuous transition $\varepsilon \to o$ where the critical theory will turn out to be mean field (free theory).

b. Composite fields

For the study of energy fluctuations (2.8), (3.9b) we will also need correlation functions involving composite fields $\mathcal{O}_i(x)$ = local monomial in A and derivatives of A. Composite correlation functions are defined from a corresponding Gell-Mann-Low formula

$$
\langle T \prod_{j=1}^{K} \mathcal{O}_i(y_j) \prod_{k=1}^{N} A(x_k) \rangle = \langle \Phi_o | T \prod_{j=1}^{K} \mathcal{O}_i^{(o)}(y_j) \prod_{k=1}^{N} A^{(o)}(x_k)\, e^{i\int \mathcal{L}_{int}^{(o)} dx} | \Phi_o \rangle_{\otimes} \qquad (5.12)
$$

with $\mathcal{O}_i^{(o)}(x)$ the monomical $\mathcal{O}_i$ in terms of free fields.
The generating functional is

$$
Z\{J, h_i\} = \langle \Phi_o | e^{i\int (\mathcal{L}_{int}^{(o)} + J(x) A^{(o)}(x) + h_i(x)\, \mathcal{O}_i^{(o)}(x)\, dx} | \Phi_o \rangle_{\otimes} \qquad (5.13)
$$

The connected correlation functions are generated by $G\{J,h_i\} = \ln Z\{J,h_i\}$

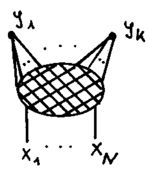
$$\langle T \prod_j O_i(y_j) \prod_k A(x_k) \rangle^{conn} = (-i)^{N+K} \; \frac{\delta^K \delta^N G\{J,h_i\}}{\delta h_i(y_1)\ldots\delta h_i(y_K)\,\delta J(x_1)\ldots\delta J(x_N)}\bigg|_{J=h_i=0} \tag{5.14}$$

$$= \sum_{conn}$$

The Legendre transform (5.5) of G $\{J, h_i\}$ with respect to J generates the composite vertex functions

$$\Gamma^{(N,K)}(x_1,\ldots,x_N;\, y_1,\ldots,y_K) = \frac{\delta^K \delta^N \Gamma\{K,h_i\}}{\delta h_i(y_1)\ldots\delta h_i(y_K)\,\delta K(x_1)\ldots\delta K(x_N)}\bigg|_{K=h_i=0} \tag{5.15}$$

$$= \sum_{prop}$$



where proper (prop) graphs are connected and one-particle irreducible with respect to all cuts not separating y-vertices. The Fourier transforms may be written as

$$\Gamma^{(N,K)}(p_1,\ldots,p_N;\, q_1,\ldots,q_K) = \langle T\, A(0)\, \tilde{A}(p_1)\ldots\tilde{A}(p_N)\, \tilde{O}_i(q_1)\ldots\tilde{O}_i(q_K) \rangle^{prop} \tag{5.16}$$

$$= \langle T\, \tilde{A}(p_1)\ldots\tilde{A}(p_N)\, O_i(0)\, \tilde{O}_i(q_2)\ldots\tilde{O}_i(q_K) \rangle^{prop}$$

The composite fields have to be normalized according to the assigned physical interpretation. Composite fields which have an interpretation directly in the critical theory must be renormalized such that the limit $\Lambda \to \infty$ as well as $\hat{m}$ (or $\tilde{m}$ or m) $\to$ o is finite for D $\leq$ 4. Corresponding composite fields (normal products) are denoted by

$$\hat{N}[O_i](y)$$

Note that composite fields need apart from multiplicative renormalization

$$O_i(y) \longrightarrow \hat{N}[O_i](y) = Z_i\, O_i(y) \tag{5.17}$$

also additive renormalizations (depending on K and N)

$$\hat{N}[O_i](y_1)\ldots\hat{N}[O_i](y_K)\,A(x_1)\ldots A(x_N) \quad — \quad \text{local distribution} \tag{5.18}$$

Examples will be given in section 6.

For statistical mechanics the euclidean correlation functions are obtained by analytic continuation in x^o to euclidean points:

$$\langle T\,\Pi\,O_i(y_j)\,\Pi\,A(x_k)\rangle\Big|_{\substack{x^o = i x^D \\ x^D \text{ real}}} = \langle \Pi\,O_i(\vec{y}_j)\,\Pi\,A(\vec{x}_k)\rangle_E \tag{5.19}$$

For structural investigations and proofs of the existence of various limits to all orders of perturbation theory one most conveniently uses the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ)-renormalization scheme[26) 27)]. In this approach the correlation functions ((5.2),(5.12)) are defined directly by a finite part prescription to the Feynman integrands avoiding a cut-off or other regularizations. For technical details we refer to Ref. 21).

c. ε-Expansion

The continuation of a scalar field theory from D integer to non-integer dimensions is possible only via the continuation of Feynman integrals.

Let

$$I(p_j, m, \varepsilon) = \int \prod_{a=1}^{\ell} d^D k_a \left\{ \prod_1^{\ell} \frac{i}{k_a^2 - m^2 + i0} \prod_{j=1}^{n} \delta^{(D)}(p_j - \varepsilon_{ja} k_a) - \text{subtr.} \right\} \tag{5.20}$$

be a Feynman integrand in momentum space to a connected Feynman-diagram with ℓ internal lines and n vertices. D denotes the number of space-time (with metric (+,(D-1)-) dimensions. p_j is the external momentum at vertex j and

$$\varepsilon_{ja} = \begin{cases} +1 & \text{for a line ending at vertex } j \\ -1 & \text{for a line originating at vertex } j \\ 0 & \text{otherwise} \end{cases}$$

The Schwinger-parametric representation of (5.20) (which is defined for D = integer) is obtained with

$$\frac{i}{k^2 - m^2 + i0} = \int_0^\infty d\alpha \, e^{i\alpha(k^2 - m^2 + i0)}$$

$$\delta^{(D)}(k) = \frac{1}{(2\pi)^D} \int_{-\infty}^{+\infty} d^D x \, e^{ikx} \tag{5.21}$$

The four momentum integrals are then all of the Gaussian type

$$\int d^D k \, e^{ia(k^2 + 2bpk)} = \left(\frac{\pi}{ia}\right)^{D/2} e^{-iab^2p^2} \tag{5.22}$$

and lead to (see e. g. 25))

$$I(p_j, m, \varepsilon) = i^L (-i\pi)^{L\frac{D}{2}} \delta^{(D)}(\Sigma p_j) \times$$

$$\times \int_0^\infty d\alpha_1 \dots d\alpha_\ell \, e^{-i\sum_1^\ell \alpha_a (m_a^2 - i0)} \left\{ \frac{e^{ip_i d_{ij}^{-1} p_j}}{P^{D/2}} - \text{subtr. terms} \right\} \tag{5.23}$$

Here L is the number of loops of the graph,
P is a homogeneous polynomial in the α's of degree L
$d_{ij}^{-1} = N_{ij} / P$ with N_{ij} a homogeneous polynomial in the α's of degree L + 1.

The representation (5.23) may now be analytically continued to complex D. $I(p_j, m, \varepsilon)$ is for $m > 0$ a meromorphic function in D with poles at some negative rational $\varepsilon = 4-D$. Hence $I(p_j, m, \varepsilon)$ has a power expansion in ε for $\varepsilon \geqslant 0$. In this way the correlation functions are obtained as double (formal) power series in g and ε.

For a treatment of field theory in $D = 4-\varepsilon$ dimensions not using the ε-expansion see Parisi and Symanzik[14].

Footnote:

For $m = 0$ there are infrared poles at some positive rational values of ε in the region $\varepsilon > \frac{2}{n}$, n the perturbation theoretic order of $I(p_j, 0, \varepsilon)$. Due to these IR divergences the Green functions to all orders in g do not exist at $m = 0$ in $4-\varepsilon$ ($\varepsilon > 0$) dimensions in an usual perturbation theory. Symanzik[14] has given a new expansion exhibiting terms non analytic in g which is free of the IR singularities.

6. <u>Critical Theory</u> (Preasymptotic Zero-Mass Theory)

We will first construct the critical theory in order to understand and calculate the behaviour (2.8) field theoretically

$$\langle A(\vec{x})\, A(0)\rangle_E \simeq \frac{\text{const.}}{|\vec{x}|^{2d_A}} \tag{6.1a}$$

$$\langle A^2(\vec{x})\, A^2(0)\rangle_E \simeq \frac{\text{const.}}{|\vec{x}|^{2d_{A^2}}} \tag{6.1b}$$

To this end we have to look for a scale invariant (for long distances) A^4-theory. The only candidate for a Lagrangian that can lead to a scale invariant field theory is (no dimensional parameters!):

$$\text{"}\mathcal{L}^{(0)} = \frac{1}{2}(\partial A)^2 - \frac{g\mu^\varepsilon}{4!} A^4\text{"} \tag{6.2}$$

The Lagrangian (6.2) however only makes sense either in a UV-cut-off (Λ) theory where Λ destroys scale invariance or (as UV-subtractions at zero momenta cause infrared divergencies) after performing UV-subtractions at some spacelike normalization spot μ where μ destroys scale invariance. Hence in perturbation theory there is no scale invariance (nonexistence of a zero theory without scale parameter!).

We consider in the following the preasymptotic zero mass theory normalized by

$$\Gamma_0^{(2)}\Big|_{p=0} = 0\ ;\quad \Gamma_0^{(2)}\Big|_{p^2=-\mu^2} = -i\mu^2\ ;\quad \Gamma_0^{(4)}\Big|_{s.p.\,-\mu^2} = -ig\mu^{4-D} \tag{6.3}$$

The μ -dependence is governed by the Gell-Mann-Low renormalization group (RG) equation

$$\left\{\mu\frac{\partial}{\partial\mu} + \sigma(g)\frac{\partial}{\partial g} - N\tau(g)\right\}\Gamma_0^{(N)} = 0 \tag{6.4}$$

Here $\mu\,\partial_\mu$ acts as the dilatation operator in the parameter space and (6.4) represents the <u>dilatation Ward-identity</u>.

If we assume (6.4) to be true <u>beyond</u> perturbation theory the vertex functions Γ_0 scale (i. e. are homogeneous functions) provided $\sigma(g) = 0$ for some value $g = g^*$. Hence scale invariance is found in the (by the differential equation) summed up perturbation theory. Expanding the scale invariant solution in g leads back to the leading perturbation terms being individually non scale invariant.

When g^* is small we can use the perturbation theory to calculate $\sigma(g)$ and $\tau(g)$. In this case scaling is in an approximate sense <u>computable</u>.

The global solution of (6.4) is

$$\Gamma_0^{(N)}(\{\varkappa p_i\}; \mu, g) = \varkappa^{D-Nd_A}\, r_\tau^{-N}\, \Gamma_0^{(N)}(\{p_i\}; \mu, g(\varkappa)) \tag{6.5}$$

with $\quad g(\varkappa)$: $\quad \ln \varkappa = \int_g^{g(\varkappa)} \sigma^{-1}(g')\, dg' \quad (6.6)$

$$r_\tau(\varkappa, g) = \exp \int_g^{g(\varkappa)} dg' \, \frac{\tau(g') - \tau(g^*)}{\sigma(g')} \tag{6.7}$$

with $d_A = d + \gamma_A$ the dynamical dimension of A;
$d = \frac{D-2}{2}$ the canonical dimension of A, and $\gamma_A = \tau(g^*)$ the anomalous dimension of A relative to g^*.
At $g = g^*$ we have

$$\hat{\Gamma}_0^{(N)}(\{\varkappa p_i\}; \mu, g^*) = \varkappa^{D-Nd_A}\, \hat{\Gamma}_0^{(N)}(\{p_i\}; \mu, g^*) \tag{6.8}$$

i.e. the $\hat{\Gamma}_0$'s are homogeneous functions in the momenta, μ as a completely passive scale can be eliminated.

There is no reason that in nature $g = g^*$, hence we have to study the case $g \neq g^*$.
If $g \neq g^*$ we distinguish two cases

1) As $\varkappa \to 0$, $g(\varkappa)$ has to go to a value (if any) g_0 where $\sigma(g_0) = 0$ and σ increasing at g_0 (Fig. 2).

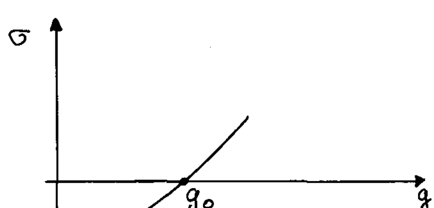


Fig. 2

If $\sigma'(g_o) > o$ then $r_\tau(\varkappa) \to r_o$ finite and

$$\Gamma_o^{(N)}(\{\varkappa p_i\}; \mu, g)_{\varkappa \ll 1} \simeq \varkappa^{D-Nd_A^o}\, r_o^{-N}\, \Gamma_o^{(N)}(\{p_i\}; \mu, g_o) \tag{6.9}$$

Hence if there is a zero g_o of σ with $\sigma'(g_o) > 0$ the long range part of the preasymptotic zero mass theory approaches a scale invariant limit (long range scaling). g_o is called an infrared stable scaling fixed point. This limit is the one relevant for statistical mechanics (i. e. agrees with the critical regime of the L. I. System) where scaling is expected too only in the long range region. The relation is a special case of (6.9) for N = 2. The Lagrangian (6.2) $\mathcal{L}^{(o)}$ can be identified as a critical one.

2) As $\varkappa \to \infty$ by (6.6) $g(\varkappa)$ has to approach a value (if any) g_∞ with $\sigma(g_\infty) = o$ and σ decreasing at g_∞ (Fig. 3).

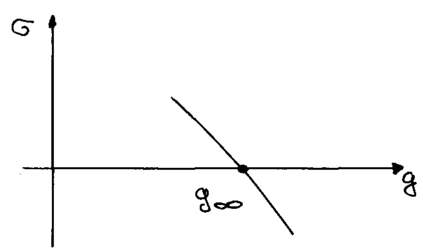


Fig. 3

When $\sigma'(g_\infty) < 0$ then $r_\tau(\varkappa) \to r_\infty$ finite and

$$\Gamma_o^{(N)}(\{\varkappa p_i\}; \mu, g)_{\varkappa \gg 1} \simeq \varkappa^{D-Nd_A^\infty}\, r_\infty^{-N}\, \Gamma_o^{(N)}(\{p_i\}; \mu, g_\infty) \tag{6.10}$$

Thus if there is a zero g_∞ of σ with $\sigma'(g_\infty) < 0$ the short range part of the preasymptotic zero mass theory shows scale invariance (short distance scaling). g_∞ is an ultraviolet stable fixed point. This limit might be relevant for high energy physics. This scaling limit is present only in our renormalized embbeding theory not in the L. I. model or the cut-off field theory which exhibit smooth ultraviolet behaviour.

Note that if $\sigma' = 0$, $\tau' \neq 0$ at g^* then

$$\Gamma_\tau(\varkappa) \to \Gamma_\tau^{(as)}(\varkappa) - |\ln\varkappa|^{-2\tau'/\sigma''} + const \to \left\{ \begin{matrix} 0 \\ \infty \end{matrix} \right. \tag{6.11}$$

for $\sigma'' \neq 0$ i. e. in this case one has logarithmic modifications and no scaling in the strict sense (see also section 8).

From our consideration we see that the preasymptotic theory contains all information about the scaling structure of A^4-theory whether long range or short range. The question of computable scaling we will discuss below.

We turn now to the consideration of composite fields in the preasymptotic theory, in order to derive (6.1b).
The energy density by (3.9b) is of the form $E(x) \propto \frac{1}{2}(\partial A)^2 + \frac{1}{2} m_0^2 A^2$
In the long range region (relevant for statistical mechanics) however the term of lowest dimension is dominant and hence

$$E(x) \propto A^2$$

We thus consider the field

$$\frac{1}{2}\hat{N}[A^2](x)$$

The composite vertex-functions are

$$\Gamma_0^{(N,K)}(p_i, q_j; \mu, g) = 2^{-K} \langle T A(0)\tilde{A}(p_2) \ldots \tilde{\hat{N}}[A^2](q_1) \ldots \rangle^{prop} \tag{6.12}$$

normalized by (6.3) and

$$\Gamma_0^{(2,1)}\left(\tfrac{p}{2}, \tfrac{p}{2}; -p\right)\Big|_{p^2=-\mu^2} = 1 \quad ; \quad \Gamma_0^{(0,2)}(\,; q, -q)\Big|_{q^2=-\mu^2} = 0 \tag{6.13}$$

They obey the RG-equation

$$\left\{\mu\frac{\partial}{\partial\mu} + \sigma(g)\frac{\partial}{\partial g} - N\tau(g) + K\delta(g)\right\}\Gamma_0^{(N,K)} = -i\mu^{D-4}\omega(g)\,\delta_{N0}\,\delta_{K2} \tag{6.14}$$

The term $\delta(g)$ is due to multiplicative renormalization of A^2 whereas the inhomogeneous term occurs from the additive renormalization of the "energy fluctuation" $\langle T \hat{N}[A^2](x)\,\hat{N}[A^2](0)\rangle$ which is already present in the free field case.

In A^4-theory there are no other dynamically independent composite fields with $d_{\mathcal{O}} < D$; the A^3 field is connected by the equation of motion to A and A^2. Non renormalizable fields as

$$\frac{1}{6!}\ \hat{N}[A^6](x)$$

can be included in a similar manner[21].

From the normalization conditions (6.2) and (6.13) the coefficients in (6.14) are given by:

$$\begin{aligned}
\tau(g,\varepsilon) &= i\ \frac{\partial \Gamma_0^{(2)}}{\partial \mu^2}\Big|_{p^2=-\mu^2} \\
\sigma(g,\varepsilon) &= -2i\mu^{2-\varepsilon}\ \frac{\partial \Gamma_0^{(4)}}{\partial \mu^2}\Big|_{s.p.-\mu^2} + 4g\tau \\
\delta(g,\varepsilon) &= -2i\mu^2\ \frac{\partial \Gamma_0^{(2,1)}}{\partial \mu^2}\Big|_{p^2=-\mu^2} + 2\tau \\
\omega(g,\varepsilon) &= i\mu^{1-\varepsilon}\ \frac{\partial \Gamma_0^{(0,2)}}{\partial \mu}\Big|_{q^2=-\mu^2}
\end{aligned} \tag{6.15}$$

and may be calculated in perturbation theory (see Appendix A). In perturbation theory to n-th order these functions are holomorphic in ε for $\frac{2}{n} > \mathrm{Re}\,\varepsilon \geqslant 0$ (see however Appendix C). In $D = 4-\varepsilon$ dimensions the leading terms in the ε-expansion are [20]

$$\begin{aligned}
\sigma &= -\varepsilon g + \frac{3}{(4\pi)^2}\, g^2 + O(\varepsilon g^2, g^3) \\
\tau &= \frac{1}{12}\ \frac{1}{(4\pi)^4}\, g^2 + O(\varepsilon g^2, g^3) \\
\delta &= \frac{1}{(4\pi)^2}\, g + O(\varepsilon g, g^2) \\
\omega &= \frac{1}{(4\pi)^2} + O(\varepsilon, g)
\end{aligned} \tag{6.16}$$

The fixed point condition $\sigma(g^*, \varepsilon) = 0$ can now be solved explicitly for g^* (= power series in ε) in an approximate sense (computable scaling) (Fig. 4).

For $\varepsilon > 0$ there is a long range scaling fixed point ($\sigma' > 0$)

$$g_o = \frac{(4\pi)^2}{3}\varepsilon + O(\varepsilon^2) \qquad (6.17)$$

and a Gaussian (free field) short range fixed point ($\sigma' < o$) $g_\infty = o$

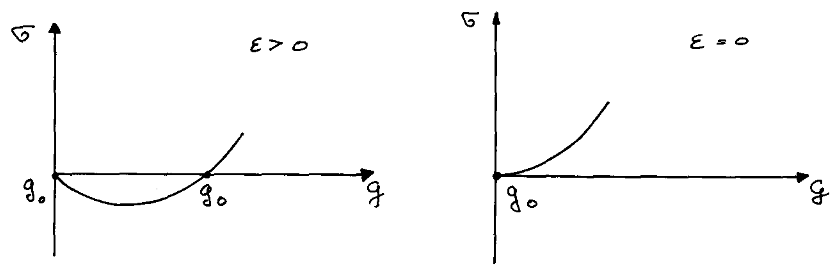


Fig.4

Hence in <u>D = 4-ε</u> (ε > o) dimensions we have a <u>non-trivial</u> critical scaling theory with anomalous dimensions

$$\tau(g_o) = \gamma_A = \frac{\varepsilon^2}{12\,g} + O(\varepsilon^3)$$

$$\delta(g_o) = \gamma_{A^2} = \frac{\varepsilon}{3} + \frac{19}{2\,g^2}\,\varepsilon^3 + O(\varepsilon^3) \qquad (6.18)$$

For calculations to order $O(\varepsilon^4)$ see Ref. 19).
We have actually calculated the critical indices

$$d_\sigma = d_A = d + \gamma_A \qquad (6.19)$$

and

$$d_E = d_{A^2} = 2d + \gamma_{A^2}$$

appearing in formula (2.8) and by (2.23) we have calculated the two independent Kadanoff coefficients.
The short range (high energy) asymptote is a canonical theory.

In <u>D = 4</u> dimensions there is (in perturbation theory) only a second order infrared scaling fixed point at $g_o = o$. We thus have reproduced the well known result that in D = 4 dimensions the critical theory (associated with a ferromagnetic Lenz-Ising system) is a <u>mean field</u>

theory (mean field critical indices). The ε -expansion appears hence as a perturbation expansion around a free field theory.
As $\tau' = o$ in view of (6.11) the $\Gamma_0^{(N)}$'s scale. However as $\delta' \neq o$ (an analogue of (6.11) holds for Γ_δ) the $\Gamma_0^{(N,K)}$'s (K ≠ o) show up logarithms in the leading terms[26].

In order to study the singular behaviour (2.9) we have now to consider the non critical (t,H) ≠ (o,o) theory.

7. Non Critical Theory (Linearly broken massive A^4-theory)

a) Scaling parametrization[20) - 24)]

We will now perturb the preasymptotic (critical) theory by the relevant fields (the temperature and the magnetization) in the sense of Kadanoff-Wegner (2.16) (remember $E(x) \propto A^2(x)$):

$$\mathcal{L} = \mathcal{L}^{(o)} + \delta\mathcal{L} = \mathcal{L}^{(o)}(x) - \frac{t}{2} N[A^2](x) + H A(x) \tag{7.1}$$

in order to study the singular behaviour (2.9). By a translation of the field A

$$A \to \bar{A} = A - M \; ; \quad M = \langle A \rangle \; ; \quad \langle \bar{A} \rangle = 0 \tag{7.2}$$

our Lagrangian takes the form $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

$$\begin{aligned} \mathcal{L}_0 &= \tfrac{1}{2}(\partial\bar{A})^2 - \tfrac{1}{2}\left(t + \tfrac{\bar{g}M^2}{2}\right)\bar{A}^2 \\ \mathcal{L}_{int} &= -\tfrac{\bar{g}}{4!}\bar{A}^4 - \tfrac{\bar{g}M}{3!}\bar{A}^3 + C\bar{A} \end{aligned} \quad ; \; \bar{g} = g\mu^{\varepsilon} \tag{7.1'}$$

with $C = H - M\left(t + \frac{\bar{g}M^2}{3!}\right)$ determined by $\langle\bar{A}\rangle = o$

As independent parameters we choose

t, M, g and μ .

In the perturbation expansion g M^2 is (as a mass term) treated as O (1). The equation of state reads

$$H = H(\mu, t, M, g) = M\left(t + \frac{\bar{g}M^2}{3!}\right) + C \tag{7.3}$$

All technical details are given in Ref. 21).

In o.th order we see that the phase diagram is of the correct form (Fig. 1).
The theory is normalized by (5.11)

a) the preasymptotic normalizations

$$\Gamma^{(2)}\Big|_{\substack{p=0\\ t,M=0}} = 0 \;; \quad \Gamma^{(2)}\Big|_{\substack{p^2=-\mu^2\\ t,M=0}} = -i\mu^2 \;; \quad \Gamma^{(4)}\Big|_{\substack{s.p.\,-\mu^2\\ t,M=0}} = -ig\mu^{4-D} \tag{7.4}$$

i. e. at (t,M) = (o,o) μ and g are the parameters of the preasymptotic theory.

b) The normalizations of the "perturbation" terms

$$\frac{\partial\Gamma^{(2)}}{\partial t}\Big|_{\substack{p=0\\ t=\mu^2\\ M=0}} = -i \;; \qquad \Gamma^{(1)} \equiv 0 \tag{7.5}$$

these conditions define the "temperature" t and the "magnetization" M.

There are three <u>independent</u> (linear) parametric differential equations (PDE's)

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \tau(g)\left(N + M\frac{\partial}{\partial M}\right) + \delta(g)K + \hat{\delta}(g)\,t\frac{\partial}{\partial t}\right\}\Gamma^{(N,K)} =$$

$$= -i\mu^{D-4}\omega(g)\,\delta_{N0}\,\delta_{K2} \tag{7.6}$$

$$\partial_t\,\Gamma^{(N,K)} = -\tilde{\Delta}_t\,\Gamma^{(N,K)}$$

$$\tilde{\Delta}_t = \frac{i}{2}\int dx\,\tilde{N}[A^2](x) - \frac{\partial C}{\partial t}\,i\int dx\,A(x) \tag{7.7}$$

$$\partial_M\,\Gamma^{(N,K)} = -\tilde{\Delta}_M\,\Gamma^{(N,K)}$$

$$\tilde{\Delta}_M = g\mu^{\varepsilon}\frac{i}{3!}\int dx\,\bar{N}[A^3](x) + g\mu^{\varepsilon}M\,\frac{i}{2}\int dx\,\bar{N}[A^2](x) - \frac{\partial C}{\partial M}\,i\int dx\,A(x) \tag{7.8}$$

$\tilde{\Delta}_t$ and $\hat{\Delta}_M$ are <u>soft insertions</u> (in the high energy sense) i.e. $\hat{\Delta}_{t,M}\,\Gamma^{(N,K)}$ falls off relative to $\Gamma^{(N,K)}$ for large nonexceptional euclidean momenta by powers (up to logarithms) to all orders of perturbation theory. This implies that the t and M dependence of Green functions drops out for large nonexceptional momenta. We therefore call this parametrization soft.

For M = o C ≡ o and (7.7) tells us that t is actually the parameter conjugate to $N[A^2]$ i. e. the temperature. That M is the magnetization is guaranteed by $M = \langle A \rangle$. For comparison the PDE's for the parametrizations (5.9) and (5.10) are given in Appendix B.

We will see below that the (pre)-scaling equation (7.6) (replacing the usual RG equation) is nothing but a differential form of Kadanoff scaling (scaling substitution law). Actually our parametrization is a global (pre)-scaling parametrization in the sense of Wegner. We observe that the hard (in the high energy sense) dilatation symmetry breaking terms are exactly those already present in the preasymptotic theory.

The dilatation-Ward-identity (Callan-Symanzik) (CS) equation) follows from (7.6) and (7.7,8):

$$\{ D + \sigma(g)\frac{\partial}{\partial g} - \tau(g) N + \delta(g) K \} \Gamma^{(N,K)} = \\ = \{ (1+\tau)\tilde{\Delta}_M + (2-\hat{\delta})\tilde{\Delta}_t \} \Gamma^{(N,K)} \tag{7.9}$$

where $D = \mu\frac{\partial}{\partial\mu} + 2t\frac{\partial}{\partial t} + M\frac{\partial}{\partial M}$ is the dilatation operator in the parameter space. Our parametrization has the particular property that the two limits:

(i) large nonexceptional momenta
(ii) preasymptotic $(t,M) \rightarrow (o,o)$

are identical to $\Gamma_0^{(N,K)}$. In both limits the RG equation (7.6) and the CS-equation (7.9) coincide.

In the soft parametrization t only appears in the propagators not however in (the symmetric) counterterms; this explains our observations of $\Gamma_{as}^{(N)} = \Gamma_0^{(N)}$ (Γ_{as} the nonexceptional large momentum asymptote).

The main feature of the pre-scaling parametrization is that the hard dilatation symmetry breaking is completely controlled by a globally solvable pre-scaling equation. At the same time it is the appropriate parametrization (as we will see) for the study of statistical mechanics aspects of the model.

From the normalization condition (7.5) we have

$$\hat{\delta}(g) = 2\tau(g)f(g) - \sigma(g)\frac{\partial f}{\partial g} - 2i\left.\frac{\partial \Gamma^{(2)}}{\partial \mu^2}\right|_{\substack{p=0\\ t=\mu^2\\ M=0}} \tag{7.10}$$

with

$$f(g) = i\mu^{-2}\,\Gamma^{(2)}\Big|_{\substack{p=0\\ t=\mu^2\\ M=0}}$$

In D = 4- ε dimensions

$$\hat{\delta}(g) = \frac{g}{(4\pi)^2} + O(\varepsilon g, g^2) \qquad (= \delta(g) \text{ to this order}) \tag{7.11}$$

Generally $\hat{\delta}(g) \neq \delta(g)$, at the fixed point g^* ($\sigma(g^*) = 0$) however
$\hat{\delta}(g^*) = \delta(g^*)$
This is shown in Appendix C (see (C.7)).

b) Global solution of the pre-scaling equation

The global solution of (7.6) reads

$$\Gamma^{(N,K)}(\{\varkappa p_i\}; \mu, t, M, g) =$$

$$= \varkappa^{D-Nd_A+K(d_{A^2}-D)}\, r_\tau^{-N} r_\delta^{K}\, \Gamma^{(N,K)}(\{p_i\}; \mu, \varkappa^{d_{A^2}-D} r_\delta t, \varkappa^{-d_A} r_\tau^{-1} M, g^*+h) \tag{7.12}$$

$$+ i\mu^{D-4} E_\omega\, \delta_{N0}\, \delta_{K2}$$

where $g(\varkappa)$ and r_τ are defined in (6.6,7) and

$$\overset{(\wedge)}{r}_\delta = \exp \int_g^{g(\varkappa)} dg' \, \frac{\overset{(\wedge)}{\delta}(g') - \delta(g^*)}{\sigma(g')} \tag{7.13}$$

$$E_\omega = \int_g^{g(\varkappa)} dg'\, \omega(g')\, \sigma^{-1}(g') \exp \int_g^{g'} dg'' \, \frac{\delta(g'') - \varepsilon/2}{\sigma(g'')} \tag{7.14}$$

Apart from the E_ω term (7.12) represents a global substitution law (analogue of (2.11,12) and (2.17) under momentum dilatations.

$$\Gamma \to f_\Gamma(g,\varkappa)\Gamma\,;\; t \to f_t(g,\varkappa)\, t\,;\; M \to f_M(g,\varkappa) M\,;\; g \to g(g,\varkappa) \tag{7.15}$$

this is a generalization of Kadanoff's scaling.

Now if there is a scaling fixed point g^* : $\sigma(g^*) = 0$ and if τ, $\hat{\delta}$ and ω are continuous (at least one side) then for $g = g^*$ (where no hard breaking of dilatation symmetry is present) (7.12) takes a homogeneous substitution form (Kadanoff in narrow sense). We have then strict global Kadanoff scaling as

$$\hat{\Gamma}^{(N,K)} = \lim_{g \to g^*} \Gamma^{(N,K)} \tag{7.16}$$

satisfies

$$\begin{aligned} &\hat{\Gamma}^{(N,K)}(\{\varkappa p_i\}; \mu, t, M, g^*) = \\ &= \varkappa^{D-Nd_A+K(d_{A^2}-D)}\,\hat{\Gamma}^{(N,K)}(\{p_i\}; \mu, \varkappa^{d_{A^2}-D} t, \varkappa^{-d_A} M, g^*) \\ &+ i\,\delta_{N0}\,\delta_{K2}\,\omega(g^*)\,\mu^{D-4} \times \begin{cases} (2\gamma_{A^2}-\varepsilon)^{-1}[\varkappa^{2\gamma_{A^2}-\varepsilon}-1]\,; & 2\gamma_{A^2} \neq \varepsilon \\ \ln \varkappa\,; & 2\gamma_{A^2} = \varepsilon \end{cases} \end{aligned} \tag{7.17}$$

μ may be eliminated completely by introducing quantities of canonical dimension zero:

$$\bar{\hat{\Gamma}} = \frac{\hat{\Gamma}}{\mu^{D-Nd+2Kd}}\;;\quad \bar{p}_i = \frac{p_i}{\mu}\;;\quad \bar{t} = \frac{t}{\mu^2}\;;\quad \bar{M} = \frac{M}{\mu^d} \tag{7.18}$$

From positivity the dynamical dimensions of the fields A and A^2 are larger than d.

We <u>assume</u> (always true in the region where perturbation theory applies i.e. for small anomalous dimensions) d_A and d_{A^2} to be smaller than D. Hence

$$0 \leq d \leq d_A\,;\;\; d_{A^2} < D \tag{7.19}$$

In view of (2.17,18) we may then classify the fields. In order to have also an example of an irrelevant field (in the long range region) we add to the Lagrangian (7.1) a non renormalizable perturbation term

$$\mathcal{L} \longrightarrow \mathcal{L} - \frac{u}{6!}\,N[A^6](x)$$

and assume $d_{A^6} > D$[21]. We than have

(i) As $\varkappa \to \infty$

$$\left.\begin{aligned} t' &= \varkappa^{d_{A^2} - D}\, t \to 0 \\ M' &= \varkappa^{-d_A} M \to 0 \end{aligned}\right\}$$ short range irrelevant

$$g' - g_\infty = g(g,\varkappa) - g_\infty \simeq \begin{cases} (\ln \varkappa)^{-1}; & D = 4, \\ \varkappa^{\sigma'(g_\infty)}; & D < 4, \end{cases}$$ marginal / irrelevant

$$u' = \varkappa^{d_{A^6} - D} u \to \infty$$ short range relevant

(ii) As $\varkappa \to 0$

$$\left.\begin{aligned} t' &\to \infty \\ M' &\to \infty \end{aligned}\right\}$$ long range relevant

$$g' - g_0 = g(g,\varkappa) - g_0 \simeq \begin{cases} (\ln \varkappa)^{-1}; & D = 4 \\ \varkappa^{\sigma'(g_0)}; & D < 4 \end{cases}$$ marginal / irrelevant

$$u' \to 0$$ long range irrelevant

The marginal variables lying at the boundary of UV and IR-criticality are those determining the fixed point structure of the theory.
As we will see below for $g \neq g^*$ the power laws appearing in (7.17) are (under certain conditions) at most modified by logarithms and they do not change the character of the fields.

The critical surfaces and trajectories under momentum dilatations for a A^4-theory with A^2 and A^6 perturbations, normalized such that we have a global scaling parametrization

$$\left\{ \mu \frac{\partial}{\partial \mu} + \sigma(g) \frac{\partial}{\partial g} - N\tau(g) + \hat{\delta}(g)\, t \frac{\partial}{\partial t} + 6\,\tau(g)\, u \frac{\partial}{\partial u} \right\} \Gamma^{(N)} = 0$$

are depicted in Fig. 5

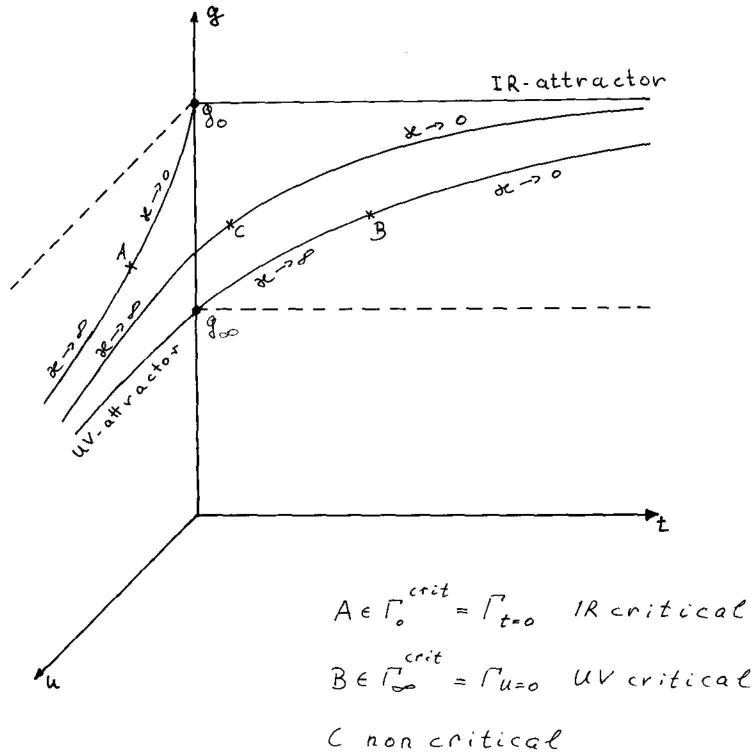


Fig. 5

8. <u>Marginal Corrections to Kadanoff Scaling</u>

As shown in the last section the homogeneous substitution laws (7.17) are violated by the non vanishing marginal variable $\Delta = g - g^*$. If we <u>assume</u> (ev. beyond perturbation theory)

1. the existence of a fixed point g^*
2. $\sigma, \tau, (\hat{\delta})$ and ω have asymptotic expansion in g at g^*

$$\alpha(g) = \sum_{n=0}^{N} \alpha_n(g^*) \frac{\Delta^n}{n!} + R_\alpha^N \; ; \quad \alpha = \sigma, \tau, \delta, \hat{\delta}, \omega$$

$$R_\alpha^N = O(\Delta^{N+1}) \; ; \quad \Delta \to 0 \tag{8.1}$$

we may expand the "correction" terms appearing in formula (7.12) in the region

$$|\varepsilon_0| < |\Delta| \ll 1 \tag{8.2}$$

with ε_0 leading term of $g(\varkappa) - g^*$.

Leading corrections:

a)

$$\sigma_1 \neq 0 ; \quad \varepsilon_0 = \varkappa^{\sigma_1} \Delta ; \quad \sigma_1 \ln \varkappa < 0$$

$$\begin{aligned} h &= \varepsilon_0 \{ 1 - \tfrac{\sigma_2}{2\sigma_1} (\Delta - \varepsilon_0) + O(\Delta^2) \} \\ r_\alpha &= 1 - \tfrac{\alpha_1}{\sigma_1} (\Delta - \varepsilon_0) + O(\Delta^2) ; \quad \alpha = \tau, \delta, \hat{\delta} \end{aligned} \tag{8.3}$$

$$r_\alpha^{(as)} = r_\alpha(\varepsilon_0 = 0, \Delta) \qquad \text{finite.}$$

In this case scaling holds in the strict sense.
We find this situation for the A^4 theory in $D = 4 - \varepsilon$ $(\varepsilon > 0)$ dimensions in the infrared and the ultraviolet region.

b)

$$\sigma_1 = 0, \sigma_2 \neq 0 ; \quad \varepsilon_0 = \frac{\Delta}{1 - \Delta \frac{\sigma_2}{2} \ln \varkappa} \quad ; \quad \sigma_2 \ln \varkappa < 0$$

$$\begin{aligned} h &= \varepsilon_0 \{ 1 - \tfrac{\sigma_3}{3\sigma_2} \varepsilon_0 \ln \tfrac{\varepsilon_0}{\Delta} + O(\Delta^2) \} \\ r_\alpha &= \left(\tfrac{\varepsilon_0}{\Delta}\right)^{2\frac{\alpha_1}{\sigma_2}} \{ 1 - \frac{3 a_2 \sigma_2 - 2\alpha_1 \sigma_3}{3 \sigma_2^2} \Delta + O(\varepsilon_0 \ln \tfrac{\varepsilon_0}{\Delta}) \} \\ r_\alpha^{(as)} &= |\ln \varkappa|^{-\frac{2\alpha_1}{\sigma_2}} \ const. \end{aligned} \tag{8.4}$$

Hence if $\tau_1 \neq 0 ; \delta_1^{(1)} = 0$ there are logarithmic corrections to scaling

if $\tau_1 = 0$ the $\Gamma^{(N)}$'s scale
if $\delta_1 = 0$ the $\Gamma^{(q,k)}$'s scale

This situation happens in $D = 4$ dimensions for the A^4 - theory at the infrared fixed point $g_0 = 0$, where $\sigma_1 = 0 ; \tau_1 = 0 ; \sigma_2 \neq 0 ; \delta_1 \neq 0$

c)

$$\sigma_1 = \sigma_2 = 0 \; ; \; \sigma_3 \neq 0 \; ; \; \varepsilon_0 = \frac{\Delta}{(1 - \Delta^2 \frac{\sigma_3}{3} \ln \varkappa)^{1/2}} \; ; \qquad \sigma_3 \ln \varkappa < 0$$

$$h = \varepsilon_0 \{ 1 - \frac{\sigma_4}{4\sigma_3} \varepsilon_0 + \frac{\sigma_4}{4\sigma_3} \varepsilon_0 (\frac{\varepsilon_0}{\Delta}) + O(\Delta^2 \ln \frac{\varepsilon_0}{\Delta} \}$$

$$r_\alpha = \exp \{ \frac{6\alpha_1}{\sigma_3} (\frac{1}{\Delta} - \frac{1}{\varepsilon_0}) - \frac{3\sigma_4 \alpha_1}{2\sigma_3^2} (1 - \frac{\varepsilon_0}{\Delta}) \} (\frac{\varepsilon_0}{\Delta})^{\frac{3(2\alpha_2\sigma_3 - \alpha_1\sigma_4)}{2\sigma_3^2}} \{ 1 + O(\Delta^2) \}$$

$$r_\alpha^{(as)} = \exp \{ \frac{6\alpha_1}{\sigma_3} \frac{1}{\Delta} (1 - \sqrt{|\ln \varkappa|}) - 3 \frac{\alpha_1 \sigma_4}{2\sigma_3^2} (1 - \frac{1}{\sqrt{|\ln \varkappa|}}) \} |\ln \varkappa|^{-\frac{3(2\alpha_2\sigma_3 - \alpha_1\sigma_4)}{4\sigma_3}} \cdot \text{const.}$$

This situation holds for the A^3-theory in D = 6 dimensions and for a class of non-abelian gauge theories (with $\tau_1 = 0, \tau_2 \neq 0, \delta_1 \neq 0$). Note that the structure of the marginal corrections are completely determined from the universal preasymptotic theory.

9. Thermodynamical quantities

From the field theoretical analogues of the definitions (2.5) and the Kadanoff relation (7.17) we obtain the singularities of the thermodynamical quantities (2.9 or 2.21,22). Using the expansion (8.1) in the region (8.2) we immediately get the corrections to scaling by expanding the r.h.s. of the Kadanoff relations. The corrections are given below for $\varepsilon > 0$; they are by (8.3) powers in $g-g^*$ and $\varkappa^\omega$ with $\omega = \sigma_1$ 30).

a) $M = 0$: $\quad \varkappa = (\frac{t}{\mu^2})^{\frac{1}{D - d_{A^2}}}$

Field susceptibility χ_A : With $\quad \gamma = \frac{D - 2\, d_A}{D - d_{A^2}}$

$$\chi_A^{-1} = \Gamma^{(2)}(0; \mu, t, 0, g) = (\frac{t}{\mu^2})^\gamma r_\tau^{-2} \Gamma^{(2)}(0, \mu, \mu^2 \hat{r}_\delta, 0, g^* + h)$$

$$= -i \mu^2 (\frac{t}{\mu^2})^\gamma \{ C_0(g^*) + (\frac{t}{\mu^2})^{\nu\omega} (g - g^*) C_1(g^*) + (g - g^*) \hat{C}_1(g^*) + O(\Delta^2) \} \qquad (9.1)$$

The numbers $C_i(g^*)$ are given by

$$C_0(g^*) = i\mu^{-2}\,\Gamma^{(2)}(0;\mu,\mu^2,0,g^*)$$

$$\hat{C}_1(g^*) = 2i\mu^{-2}\left\{\frac{\hat{\delta}_1}{\sigma_1}\mu^2\frac{\partial\Gamma^{(2)}}{\partial t} - \frac{\tau_1}{\sigma_1}\Gamma^{(2)}\right\}(0;\mu,\mu^2,0,g^*) \qquad (9.2)$$

$$C_1(g^*) = i\mu^{-2}\,\frac{\partial\Gamma^{(2)}}{\partial g}(0;\mu,\mu^2,0,g^*) - \hat{C}_1(g^*)$$

For $M \neq 0$ a similar expansion can be obtained from:

$$\chi_A^{-1} = \Gamma^{(2)}(0;\mu,t,M,g) = \left(\frac{t}{\mu^2}\right)^{\gamma} r_\tau^{-2}\,\Gamma^{(2)}(0;\mu,\mu^2\hat{r}_\delta,\mu^d x\, r_\tau^{-1},g^*+h) \quad (9.3)$$

where
$$x = \frac{M}{\mu^d}\left(\frac{t}{\mu^2}\right)^{-\beta} \qquad (9.4)$$

Equation (9.3) has a power expansion in x.

<u>Energy susceptibility</u> $\chi_{A^2} = C$ (specific heat): with $\alpha = \frac{D-2d_{A^2}}{D-d_{A^2}}$

$$\chi_{A^2} = \Gamma^{(0,2)}(0;\mu,t,0,g) = \left(\frac{t}{\mu^2}\right)^{-\alpha} r_\delta^2\,\Gamma^{(0,2)}(0,\mu,\mu^2\hat{r}_\delta,0,g^*+h) + i\mu^{-\varepsilon}E_\omega$$

$$= -i\mu^{-\varepsilon}\Big[\left(\frac{t}{\mu^2}\right)^{-\alpha}\Big\{B_0(g^*) + \left(\frac{t}{\mu^2}\right)^{\nu\omega}(g-g^*)B_1(g^*) + (g-g^*)\hat{B}_1(g^*) + O(\Delta^2)\Big\} \qquad (9.5)$$

$$- \left\{b_0(g^*) + (g-g^*)\,b_1(g^*) + O(\Delta^2)\right\}\Big] \qquad ;\ \alpha \neq 0$$

where

$$B_0(g^*) = i\mu^{+\varepsilon}\,\Gamma^{(0,2)}(0;\mu,\mu^2,0,g^*) - i\,\frac{\omega_0\nu}{\alpha}$$

$$\hat{B}_1(g^*) = 2i\mu^{+\varepsilon}\left\{\frac{\hat{\delta}_1}{\sigma_1}\mu^2\frac{\partial\Gamma^{(0,2)}}{\partial t} + \frac{\delta_1}{\sigma_1}\Gamma^{(0,2)}\right\}(0;\mu,\mu^2,0,g^*) - 2i\,\frac{\delta_1}{\sigma_1}\,\frac{\omega_0\nu}{\alpha}$$

$$B_1(g^*) = i\mu^{+\varepsilon}\left\{\frac{\partial\Gamma^{(0,2)}}{\partial g} + \frac{2\hat{\delta}_1}{\sigma_1}\mu^2\frac{\partial\Gamma^{(0,2)}}{\partial t} + \frac{2\delta_1}{\sigma_1}\Gamma^{(0,2)}\right\}(0;\mu,\mu^2,0,g^*) - i\,\frac{\omega_1\sigma_1 + 2\omega_0\delta_2}{2\delta_0+\sigma_1-\varepsilon} \qquad (9.6)$$

$$b_0(g^*) = \frac{\omega_0\nu}{\alpha}\ ;\quad b_1(g^*) = -\frac{2\delta_1}{\sigma_1}\,\frac{\omega_0\nu}{\alpha} + \frac{\omega_1\sigma_1 + 2\omega_0\delta_1}{2\delta_0+\sigma_1-\varepsilon}$$

The leading terms for t → o are:

$$\alpha > 0 \quad : \chi_{A^2}(t) \propto t^{-\alpha}$$

$$\alpha = 0 \quad : \chi_{A^2}(t) \propto \ln\left(\frac{t}{\mu^2}\right)^{\nu} \tag{9.7}$$

$$\alpha < 0 \quad : \chi_{A^2}(t) \propto const.$$

As from (7.19) $-(1-\varepsilon/2) < \delta_0 < 2$, the critical index $\alpha = -\nu(2\delta_0 - \varepsilon)$ actually can take positive and negative values.

In D = 4 we have $\alpha = 0$ and C behaves logarithmically due to the additive renormalization term E_ω ! In D = $4-\varepsilon$ ($\varepsilon > 0$) dimensions $\alpha = \nu\frac{\varepsilon}{3} + O(\varepsilon^2) > 0$ the power singularity is present.

For M ≠ o one may again expand the expression

$$\begin{aligned}\chi_{A^2} &= \Gamma^{(0,2)}(0;\mu,t,M,g)\\ &= \left(\frac{t}{\mu^2}\right)^{-\alpha} r_\delta^2\, \Gamma^{(0,2)}(0;\mu,\mu^2\hat{r}_\delta,\mu^d x\, r_\tau^{-1}, g^*+h) + i\mu^{-\varepsilon}E_\omega\end{aligned} \tag{9.8}$$

in Δ and x .

Correlation length ξ : with $\nu = \dfrac{1}{D - d_{A^2}}$

$$\begin{aligned}\xi^{-2} = m^2 &= -\frac{\Gamma^{(2)}}{\frac{\partial\Gamma^{(2)}}{\partial p^2}}(0;\mu,t,0,g) =\\ &= -\left(\frac{t}{\mu^2}\right)^{2\nu}\frac{\Gamma^{(2)}}{\frac{\partial\Gamma^{(2)}}{\partial p^2}}(0;\mu,\mu^2\hat{r}_\delta,0,g^*+h)\\ &= -\mu^2\left(\frac{t}{\mu^2}\right)^{2\nu}\left\{A_0(g^*) + \left(\frac{t}{\mu^2}\right)^{\nu\omega}(g-g^*)A_1(g^*) + (g-g^*)\hat{A}_1(g^*) + O(\Delta^2)\right\}\end{aligned} \tag{9.9}$$

with $\quad \left(\dot{\Gamma} \doteq \frac{\partial\Gamma}{\partial p^2}\right)$

$$A_0(g^*) = \mu^{-2}\,\Gamma^{(2)}\dot{\Gamma}^{(2)^{-1}}(0;\mu,\mu^2,0,g^*)$$

$$\hat{A}_1(g^*) = -\mu^{-2}\left\{\frac{\hat{\delta}_1}{\sigma_1}\mu^2\frac{\partial\Gamma^{(2)}}{\partial t} - \frac{\hat{\delta}_1}{\sigma_1}\mu^2\frac{\partial\dot{\Gamma}^{(2)}}{\partial t}\mu^2 A_0(g^*)\right\}\dot{\Gamma}^{(2)^{-1}}(0;\mu,\mu^2,0,g^*)$$

$$A_1(g^*) = \mu^{-2}\left\{\frac{\partial\Gamma^{(2)}}{\partial g} - \frac{\partial\dot{\Gamma}^{(2)}}{\partial g}\mu^2 A_0(g^*)\right\}\dot{\Gamma}^{(2)^{-1}}(0;\mu,\mu^2,0,g^*) - \hat{A}_1(g^*) \tag{9.10}$$

Again for M $\neq$ o we may expand

$$\xi^{-2} = - \frac{\Gamma^{(2)}}{\frac{\partial \Gamma^{(2)}}{\partial p^2}} (o; \mu, t, M, g) \tag{9.11}$$

$$= - \left(\frac{t}{\mu^2}\right)^{2\nu} \frac{\Gamma^{(2)}}{\frac{\partial \Gamma^{(2)}}{\partial p^2}} (o; \mu, \mu^2 \hat{r}_\delta, \mu^d x \, r_\tau^{-1}, g^* + h)$$

in Δ and x .

b) <u>Equation of state:</u> $\qquad \varkappa = \left(\frac{M}{\mu^d}\right)^{1/d_A}$

The equation of state (7.3)

$$H = M \hat{m}^2 (\mu, t, M, g)$$

satisfies the scaling equation [21)]

$$\left\{ \mu \frac{\partial}{\partial \mu} + \sigma \frac{\partial}{\partial g} - \tau \left(1 + M \frac{\partial}{\partial M}\right) + \hat{\delta} t \frac{\partial}{\partial t} \right\} H = 0 \tag{9.12}$$

and hence with $y = \frac{t}{\mu^2} \left(\frac{M}{\mu^d}\right)^{1/\beta}$ and $\delta = \frac{\mathcal{D} - d_A}{d_A}$

$$H(\mu, t, M, g) = \left(\frac{M}{\mu^d}\right)^{\delta} r_\tau^{-1} H(\mu, \mu^2 y \hat{r}_\delta, \mu^d r_\tau^{-1}, g^* + h) \tag{9.13}$$

On the <u>critical isotherme</u> y=o:

$$H(\mu, 0, M, g) = \left(\frac{M}{\mu^d}\right)^{\delta} r_\tau^{-1} H(\mu, 0, \mu^d r_\tau^{-1}, g^* + h)$$

$$= \mu^{\mathcal{D}-d} \left(\frac{M}{\mu^d}\right)^{\delta} \left\{ c_0(g^*) + \left(\frac{M}{\mu^d}\right)^{\frac{\nu\omega}{\beta}} (g - g^*) c_1(g^*) + (g - g^*) \hat{c}_1(g^*) + O(\Delta^2) \right\} \tag{9.14}$$

Here the numbers $c_i(g^*)$ are given by

$$c_0(g^*) = \mu^{d-\mathcal{D}} H(\mu, 0, \mu^d, g^*)$$

$$\hat{c}_1(g^*) = \mu^{d-\mathcal{D}} \frac{\tau_1}{\sigma_1} \left\{ H + \mu^d \frac{\partial H}{\partial M} \right\} (\mu, 0, \mu^d, g^*) \tag{9.15}$$

$$c_1(g^*) = \mu^{d-\mathcal{D}} \frac{\partial H}{\partial g} (\mu, 0, \mu^d, g^*) - \hat{c}_1(g^*)$$

On the other hand on the coexistence curve H=o[21]:

$$\left\{ \mu \frac{\partial}{\partial \mu} + \sigma \frac{\partial}{\partial g} - \tau M \frac{\partial}{\partial M} - \hat{\delta} \right\} t(\mu, M, g) = 0 \tag{9.16}$$

With $\beta = \frac{d_A}{D - d_{A^2}}$ the equation of state takes the form:

$$t(\mu, M, g) = \left(\frac{M}{\mu^d}\right)^{1/\beta} \hat{r}_\delta^{-1}\, t(\mu, \mu^d r_\tau^{-1}, g^* + h) \tag{9.17}$$

$$= \mu^2 \left(\frac{M}{\mu^d}\right)^{1/\beta} \left\{ a_0(g^*) + \left(\frac{M}{\mu^d}\right)^{\frac{\nu\omega}{\beta}} (g-g^*)\, a_1(g^*) + (g-g^*)\, \hat{a}_1(g^*) + O(\Delta^2) \right\}$$

with

$$a_0(g^*) = \mu^{-2}\, t(\mu, \mu^d, g^*)$$

$$\hat{a}_1(g^*) = \mu^{-2} \left\{ \frac{\hat{d}_1}{\sigma_1} t + \frac{\tau_1}{\sigma_1} \mu^d \frac{\partial t}{\partial M} \right\}(\mu, \mu^d, g^*)$$

$$a_1(g^*) = \mu^{-2} \frac{\partial t}{\partial g}(\mu, \mu^d, g^*) - \hat{a}_1(g^*) \tag{9.18}$$

We have now determined all the critical indices from (2.9) and we may check the scaling relations (2.26) to be satisfied exactly. For at the fixed point (6.17) the corrections to scaling for the <u>thermodynamical quantities</u> are <u>completely governed by the exponent</u>.

$$\omega = \sigma_1 = \sigma'(g^*) = \varepsilon + O(\varepsilon^2) \tag{9.19}$$

These corrections are due to non vanishing marginal variable $\Delta = g - g^*$
In the "relevant" variables x and y of the equations (9.3, 8, 11, 13) the thermodynamical quantities are analytic (to any order of perturbation theory). For the correlation functions in contrast non vanishing relevant fields give rise to corrections of non analytic type.

10. Correlation Functions

At the critical point the long range parts of the correlation functions scale according to (7.12) and (8.3)

$$\Gamma^{(N,K)}(\{\varkappa p_i\};\mu,t,M,g) \simeq \varkappa^{D-Nd_A+K(d_{A^2}-D)}\{\Gamma_0^{(N,K)}(\{p_i\};\mu,0,0,g^*) \quad (10.1)$$

+ correction terms proportional to (g-g*) and

$$(g-g^*)\varkappa^{\omega}+\dots\} + i\,\delta_{N0}\,\delta_{K2}\,\mu^{-\varepsilon}E_{\omega}$$

In particular with $\varkappa^2\mu^2 = -p^2 = \vec{p}^{\,2}$

$$\Gamma^{(2)}(-\vec{p}^{\,2};\mu,0,0,g) \simeq \left(\frac{\vec{p}^{\,2}}{\mu^2}\right)^{1-\eta/2}\Gamma^{(2)}(\mu^2;0,0,g^*) \;;\; \eta = 2\gamma_A \quad (10.2)$$

and

$$\Gamma^{(0,2)}(-\vec{p}^{\,2};\mu,0,0,g) \simeq \left(\frac{\vec{p}^{\,2}}{\mu^2}\right)^{-\frac{\alpha}{2\nu}}\Gamma^{(0,2)}(\mu^2;\mu,0,0,g^*) - i\mu^{-\varepsilon}a(g^*)\frac{\nu}{\alpha}\left[\left(\frac{\vec{p}^{\,2}}{\mu^2}\right)^{-\frac{\alpha}{2\nu}} - 1\right] \quad (10.3)$$

For (t,M) ≠ (o,o) relevant corrections to scaling occur. In the region

$$\vec{p}_i^{\,2},\, t,\, M^2 \ll \mu^2 \quad (10.4)$$

we have

$$\Gamma^{(N,K)}(\{\varkappa p_i\};\mu,\varkappa^2 t,\varkappa^d M,g) =$$

$$= \varkappa^{D-Nd_{A^2}+K(d_{A^2}-D)}\{1+\varepsilon_0\frac{\partial}{\partial g}+(\Delta-\varepsilon_0)\left[\left(N\frac{\tau_1}{\sigma_1}-K\frac{\delta_1}{\sigma_1}\right)-\frac{\delta_1}{\sigma_1}\varkappa^{d_{A^2}-D}t\frac{\partial}{\partial t}+\frac{\tau_1}{\sigma_1}\varkappa^{-d_A}M\frac{\partial}{\partial M}\right]$$

$$+O(\Delta^2)\}\,\Gamma^{(N,K)}(\{p_i\},\mu,\varkappa^{\delta_{A^2}}t,\varkappa^{-\delta_A}M,g^*)$$

Now we further may calculate the corrections for

$$t,\, M^2 \ll \vec{p}_i^{\,2} \;(\ll \mu^2) \quad (10.6)$$

using the inhomogeneous PDE's (7.7) and (7.8) for the vertex functions on the r.h.s. of (10.5) together with short distance expansions (SDE), for the r.h.s. of these PDE's.

For simplicity we only consider the leading relevant corrections for vanishing marginal field Δ = o i.e. of

$$\Gamma^{(N,K)}(\{p_i\};\mu,t,M,g^*)$$

Integration of the PDE's (7.7) and (7.8) leads to:

$$
\begin{aligned}
\Gamma^{(N,K)}(\{p_i\};\mu,t,M,g) = {} & \Gamma_0^{(N,K)}(\{p_i\};\mu,0,0,g) \\
& - \int_0^t dt'\,\hat{\Delta}_t\,\Gamma^{(N,K)}(\{p_i\};\mu,t',0,g) \\
& - \int_0^M dM'\,\hat{\Delta}_M\,\Gamma^{(N,K)}(\{p_i\};\mu,0,M',g) \\
& + \int_0^t\int_0^M dt'\,dM'\,\hat{\Delta}_t\,\hat{\Delta}_M\,\Gamma^{(N,K)}(\{p_i\};\mu,t',M',g) \qquad ; \text{ N even.}
\end{aligned}
\tag{10.7}
$$

As $\Gamma^{(N,K)}(\{p_i\};\mu,t,0,g)=0$ for N odd and

$\partial_M\Gamma^{(N,K)}(\{p_i\};\mu,0,0,g)$ finite (10.7) holds for N odd also with $\Gamma^{(N,K)}$ replaced by $\partial_M\Gamma^{(N,K)}$.

The small t, M behaviour may be obtained (using homogeneity and the fact that the vertex functions depend on μ only logarithmically) from the large momentum expansion (SDE):

N even:

$$
\hat{\Delta}_t\,\Gamma^{(N,K)}(\{p_i\};\mu,t,M,g) = \mathcal{F}^{(N+2,K)}(\{p_i,0,0\};\mu,\mu^2,0,g)\,\hat{\Delta}_t^2\,\Gamma^{(0,0)}(\mu,t,M,g) + R_t^{(N,K)}
\tag{10.8}
$$

$$
\hat{\Delta}_M\,\Gamma^{(N,K)}(\{p_i\};\mu,t,M,g) = \mathcal{F}^{(N+2,K)}(\{p_i,0,0\};\mu,\mu^2,0,g)\,\hat{\Delta}_M\,\hat{\Delta}_t\,\Gamma^{(0,0)}(\mu,t,M,g) + R_M^{(N,K)}
\tag{10.9}
$$

N odd:

$$
\hat{\Delta}_t\,\hat{\Delta}_M\,\Gamma^{(N,K)}(\{p_i\};\mu,t,M,g) = \begin{cases} \mathcal{F}^{(N+1,K)}(\{p_i,0\};\mu,\mu^2,0,g)\,\hat{\Delta}_t\,\hat{\Delta}_M\,G^{(1)}(\mu,t,M,g) + R_1\,; & (K\neq 0) \\ \mathcal{F}^{(N+3)}(\{p_i,0,0,0\};\mu,\mu^2,0,g)\,\hat{\Delta}_t\,\hat{\Delta}_M^2\,\Gamma^{(0,0)}(\mu,t,M,g) + R_2\,; & (K=0) \end{cases}
\tag{10.10}
$$

$$
\hat{\Delta}_M\,\hat{\Delta}_M\,\Gamma^{(N,K)}(\{p_i\};\mu,t,M,g) = \begin{cases} \mathcal{F}^{(N+1,K)}(\{p_i,0\},\mu,\mu^2,0,g)\,\hat{\Delta}_M^2\,G^{(1)}(\mu,t,M,g) + R_3\,; & (K\neq 0) \\ \mathcal{F}^{(N+3)}(\{p_i,0,0,0\};\mu,\mu^2,0,g)\,\hat{\Delta}_M^3\,\Gamma^{(0,0)}(\mu,t,M,g) + R_4\,; & (K=0) \end{cases}
\tag{10.11}
$$

The remainders $R^{(N,K)}_{..}$ drop out to each order of perturbation theory by powers up to logarithms. The coefficient functions $\Gamma^{(N+..,K)}$ are represented by graphs which get one particle irreducible with respect to cuts not separating the K-vertices after connecting the N-external legs to a point. The singular parts on the r.h.s. of (10.8 - 11) obey the scaling equations:

$$\{\mu\frac{\partial}{\partial\mu}+\sigma\frac{\partial}{\partial g}-\tau(m+M\frac{\partial}{\partial M})+\hat{\delta}(n+t\frac{\partial}{\partial t})\}\Gamma_{mn}=-i\,Q_{mn} \qquad (10.12)$$

with Q_{mn} a polynomial in t and M of degree $\delta=4-2n-m\geqslant o$, the coefficients depending on g, and

$$\{\mu\frac{\partial}{\partial\mu}+\sigma\frac{\partial}{\partial g}-\tau(m+M\frac{\partial}{\partial M}-1)+\hat{\delta}(n+t\frac{\partial}{\partial t})\}G^{(1)}_{mn}=-i\,P_{mn} \qquad (10.13)$$

where P_{mn} is a polynomial of degree $\delta=3-2n-m\geqslant o$.

Here we used the notation

$$\Gamma_{mn}(\mu,t,M,g)=\hat{\Delta}^m_M\,\hat{\Delta}^n_t\,\Gamma^{(0,0)}(\mu,t,M,g)$$

and

$$G^{(1)}_{mn}(\mu,t,M,g)=\hat{\Delta}^m_M\,\hat{\Delta}^n_t\,G^{(1)}(\mu,t,M,g)$$

The inhomogeneous terms are due to the additive renormalizations of Γ_{mn} and $G^{(1)}_{mn}$. In particular:

$$Q_{02}=\mu^{-\varepsilon}\omega_{02}(g)\,;\quad Q_{21}=\omega_{21}(g)\,;\quad Q_{11}=\omega_{21}(g)\,M$$

$$Q_{40}=\mu^{\varepsilon}\omega_{40}(g)\quad\text{and}\quad Q_{30}=\mu^{\varepsilon}\omega_{40}(g)\,M$$

are determined from:

$$\Gamma_{02}\Big|_{\substack{t=\mu^2\\M=0}}=0\quad;\quad\Gamma_{11}\Big|_{\substack{t=\mu^2\\M=0}}=0\quad;\quad\Gamma_{40}\Big|_{\substack{t=\mu^2\\M=0}}=0 \qquad (10.14)$$

as

$$\omega_{02}=i\mu^{\varepsilon}(\mu\partial_\mu+\hat{\delta}t\partial_t)\Gamma_{02}\Big|_{\substack{t=\mu^2\\M=0}}\,;\quad\omega_{21}=i(\mu\partial_\mu+\hat{\delta}t\partial_t)\Gamma_{11}\Big|_{\substack{t=\mu^2\\M=0}}$$

$$\omega_{40}=i\mu^{-\varepsilon}(\mu\partial_\mu+\hat{\delta}t\partial_t)\Gamma_{40}\Big|_{\substack{t=\mu^2\\M=0}}$$

With the solutions of (10.12) at $g=g^*$:

$$\Gamma_{02}(\mu,t,0,g^*) = \left(\frac{t}{\mu^2}\right)^{-\alpha}\Gamma_{02}(\mu,\mu^2,0,g^*) + i\mu^{-\varepsilon}\omega_{02}(g^*)\,\nu\alpha^{-1}\left[\left(\frac{t}{\mu^2}\right)^{-\alpha}-1\right]$$

$$\Gamma_{11}(\mu,0,M,g^*) = \left(\frac{M}{\mu^d}\right)^{\beta^{-1}(1-\alpha)-1}\Gamma_{11}(\mu,0,\mu^d,g^*) + i\mu^d\omega_{21}(g^*)\,\nu(\gamma-1)^{-1}\left[\left(\frac{M}{\mu^d}\right)^{\beta^{-1}(1-\alpha)-1} - \left(\frac{M}{\mu^d}\right)\right]$$

$$\Gamma_{21}(\mu,t,0,g^*) = \left(\frac{t}{\mu^2}\right)^{\gamma-1}\Gamma_{21}(\mu,\mu^2,0,g^*) + i\,\omega_{21}(g^*)\,\nu(\gamma-1)^{-1}\left[\left(\frac{t}{\mu^2}\right)^{\gamma-1}-1\right] \tag{10.15}$$

$$\Gamma_{30}(\mu,0,M,g^*) = \left(\frac{M}{\mu^d}\right)^{\delta-2}\Gamma_{30}(\mu,0,\mu^d,g^*) + i\mu^{d+\varepsilon}\omega_{40}(g^*)\,\nu(\gamma-2\beta)^{-1}\left[\left(\frac{M}{\mu^d}\right)^{\delta-2} - \left(\frac{M}{\mu^d}\right)\right]$$

we obtain from (10.5 - 15) the leading corrections to $\Gamma_0^{(N,K)}$ and $\partial_M\Gamma_0^{(N)}$ in the region

$$|t|,\ M^2 \ll p^2 \ll \mu^2$$

With $p_i = pn_i$, $p > 0$, $|n_i| = 1$; n_i euclidean, nonexceptional using the notation (7.18) we have:

For <u>N even</u>:

$$\bar\Gamma^{(N,K)}(\{pn_i\};\mu,t,M,g) \simeq \tag{10.16}$$

$$\simeq \bar p^{\beta/\nu[\delta-(K+N-1)]}\left[A_0(g^*) - A_1(g^*)\begin{cases}(\bar t/\bar p^{1/\nu})^{1-\alpha}D_1(g^*) - (\bar t/\bar p^{1/\nu})D_2(g^*)\,; & (M=0)\\ (\bar M/\bar p^{\beta/\nu})^{\frac{1-\alpha}{\beta}}D_3(g^*) - (\bar M/\bar p^{\beta/\nu})^2 D_4(g^*)\,; & (t=0)\end{cases} + \dots\right]$$

with

$$A_0(g^*) = \bar\Gamma_0^{(N,K)}(\{\mu n_i\};\mu,0,0,g^*)\ ;\ A_1(g^*) = \bar F^{(N+2,K)}(\{\mu n_i,0,0\};\mu,\mu^2,0,g^*)$$

$$D_1(g^*) = (1-\alpha)^{-1}\{\mu^{\varepsilon}\Gamma_{02}(\mu,\mu^2,0,g^*) + D_2(g^*)\}\ ;\ D_2(g^*) = i\,\omega_{02}(g^*)\,\nu\,\alpha^{-1}$$

$$D_3(g^*) = (1-\alpha)^{-1}\beta\{\mu^{-d}\Gamma_{11}(\mu,0,\mu^d,g^*) + 2D_4(g^*)\}\ ;\ D_4(g^*) = \frac{i}{2}\,\omega_{21}(g^*)\,\nu(\gamma-1)^{-1}$$

For <u>N odd:</u>

$$\partial_M\bar\Gamma^{(N)}(\{pn_i\};\mu,t,M,g) \simeq$$

$$\simeq \bar p^{\beta/\nu[\delta-(K+N)]}\left[B_0(g^*) - B_1(g^*)\begin{cases}(\bar t/\bar p^{1/\nu})^{\gamma}E_1(g^*) - (\bar t/\bar p^{1/\nu})E_2(g^*)\,; & (M=0)\\ (\bar M/\bar p^{\beta/\nu})^{\gamma/\beta}E_3(g^*) - (\bar M/\bar p^{\beta/\nu})^2 E_4(g^*)\,; & (t=0)\end{cases} + \dots\right]$$

with

$$B_0(g^*) = \partial_M\bar\Gamma_0^{(N)}(\{\mu n_i\};\mu,0,0,g^*)\ ;\ B_1(g^*) = \bar F^{(N+3)}(\{\mu n_i,0,0,0\};\mu,\mu^2,0,g^*)$$

with

$$E_1(g^*) = \gamma^{-1}\{ \vec{\Gamma}_{21}(\mu,\mu^2;0,g^*) + E_2(g^*)\}; \quad E_2(g^*) = i\,\omega_{21}(g^*)\,\nu(\gamma-1)^{-1}$$

(10.19)

$$E_3(g^*) = \gamma^{-1}\beta\{\mu^{-(d+\epsilon)}\vec{\Gamma}_{30}(\mu,0,\mu^d;g^*) + 2E_4(g^*)\}; \quad E_4(g^*) = \frac{1}{2}\,\omega_{40}(g^*)\,\nu(\gamma-2\beta)^{-1}$$

Higher correction terms may be calculated by taking into account further terms in the SDE's (10.8 - 11) and by applying SDE to the non leading terms in (10.5).

In view of (9.14) and (9.17) we easily obtain the relevant corrections on the coexistence curve from (10.16).

11. Conclusions

Within our field theoretical framework we are able to give a precise meaning to many of Kadanoff's considerations and we have a model matching Wegner's phenomenological scheme[6)].

Our discussion shows that the soft parametrization is most transparent for the discussion of scaling behaviour. We want to point out that using the soft renormalization technique[21)] all perturbation calculations can be performed in a usual sense (no loopwise-summation) and that all perturbation theoretical statements have been proved to all orders using PDE's.

What we have shown is that the A^4-model exhibits:

1. Long range scaling for

   $t, M^2 \ll \vec{p}_i^{\,2} \ll \mu^2$ euclidean nonexceptional

2. There are two independent critical indices and the scaling relations among the critical indices are exactly valid (calculable in $4-\varepsilon$ dim. $0 \leq \varepsilon \ll 1$).

3. In D = $4-\varepsilon$ ($\varepsilon > 0$) dimensions strict homogeneous Kadanoff substitution laws (Kadanoff scaling) are valid in the long range asymptote.

4. There is a global (pre-)scaling parametrization in the sense of Wegner.

5. The scaling structure (singular behaviour) and the structure of

corrections are universal in the sense that they are intrinsic to the preasymptotic theory (i.e. known from $\sigma(g)$, $\tau(g)$ and $\delta(g)$). Whereas thermodynamical quantities besides the "marginal" corrections are analytic in the relevant variables, the correlation functions exhibit non analytic "relevant" corrections.

In Tab. 1 we have listed the values for the critical exponents obtained from the ε-expansion to order ε^2 and ε^3 in comparison to the experimental values, the mean field values (MFA) and results from Lenz-Ising model calculations for D=3. To order ε^2 the agreement is stricking for D=3 and even for D=2 the results have the right orders of magnitude. There is of course (at present) no explanation why starting from an asymptotic expansion for small ε by setting ε = 1 (or 2) one gets reasonable answers.

Concerning the structural investigations our results immediatly generalize to n-component scalar models:

$$\hat{\varphi} = (\varphi_1, \ldots, \varphi_n)$$

$$\mathcal{L} = \frac{1}{2}(\partial\hat{\varphi})^2 - \frac{t}{2}\hat{\varphi}^2 - \frac{g}{4!}(\hat{\varphi}^2)^2 + H\varphi_n$$

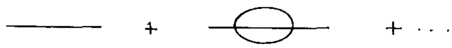
The functions σ, τ, δ and ω now depend on n and so do the corresponding critical indices[19)]. For n $>$ 1 the Goldstone phenomenon takes place in the spontaneous limit otherwise there is no principal structural change.

For similar investigations of other models we refer to the review articles Ref. 33).

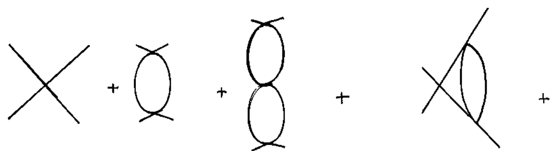
Appendix A: Graphical Representation of Green Functions

a) Zero mass

$\Gamma_0^{(2)} =$

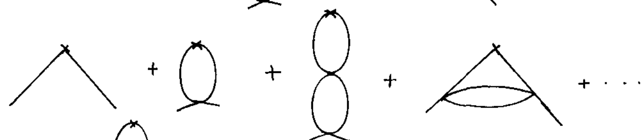


$\Gamma_0^{(4)} =$

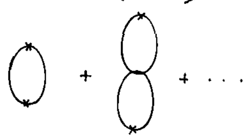


\+ crossed terms + ...

$\Gamma_0^{(2,1)} =$



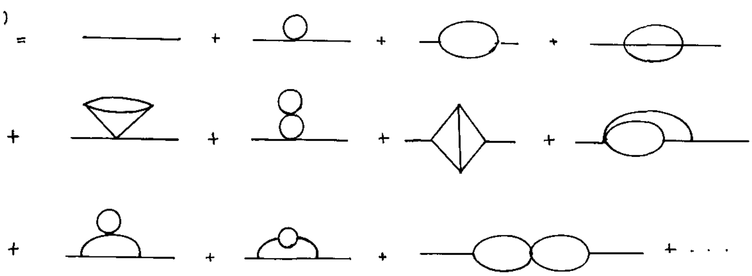
$\Gamma_0^{(0,2)} =$



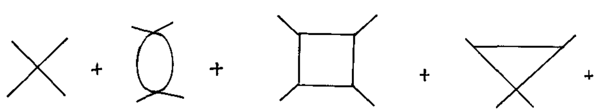
For a evaluation of the integrals see e.g. Ref. 20).

b) $(t, M) \neq (o,o)$

$\Gamma^{(2)} =$



$\Gamma^{(4)} =$



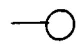
\+ crossed terms + ...

In D=4 to lowest order we have for M = o:

$$\Gamma^{(2)} = i \left\{ -t - \vec{p}^{\,2} - (4\pi)^{-2} \frac{g}{2} \left(t \ln \frac{t}{\mu^2} - t \right) + O(g^2) \right\}$$

$$\Gamma^{(4)} = i \left\{ -g - (4\pi)^{-2} \frac{g^2}{2} \int_0^1 d\alpha \ln \frac{\alpha(1-\alpha)(\vec{p}_1 + \vec{p}_2)^2 + t}{\alpha(1-\alpha)\, 4/3\, \mu^2} + \text{crossed terms} + O(g^3) \right\}$$

Equation of state to lowest order:

$$H = i \sum_{prop}$$  $$+ M\left(t + g\frac{M^2}{3!}\right) = M\left(t + g\frac{M^2}{3!}\right) + i$$  $$+ \cdots$$

$$= M\Big\{ t + g\frac{M^2}{3!} - (4\pi)^{-2}\frac{g}{2}\Big[t + g\frac{M^2}{2} + t \ln \mu^2 + g\frac{M^2}{2}\ln \tfrac{4}{3}\mu^2$$

$$- \left(t + g\frac{M^2}{2}\right)\ln\left(t + g\frac{M^2}{2}\right) + g\frac{M^2}{2}\int_0^1 d\alpha \ln \alpha(1-\alpha)\Big] + O(g^{3/2})\Big\}$$

Appendix B: Structure of PDE's for the Parametrizations (5.9) and (5.10)

In the symmetric (M = o) mass shell normalized theory (m = physical mass) there is one PDE, the standard CS-equation (Dilatation Ward-identity)[29]

$$\left\{ m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} - \gamma(g) N \right\} \bar{\Gamma}^{(N)} = - \bar{\Delta}_m \bar{\Gamma}^{(N)} \tag{B.1}$$

i.e. the physical mass breaks dilatation symmetry necessarily in a hard way as $\beta \neq o$ for generic g and the large momentum (nonexceptional) asymptote $\bar{\Gamma}_{as}^{(N)}$ differs from $\bar{\Gamma}_{o}^{(N)}$ by a complicated wave-function - and coupling-constant - renormalization. Also the $\bar{\Gamma}_{as}^{(N)}$ are vertex-functions of a zero mass theory (mass in propagators dropped out) the $\bar{\Gamma}_{as}^{(N)}$'s still depend on m (through the m-dependent counter terms). The $\bar{\Gamma}_{as}^{(N)}$ are solutions to the homgeneous CS-equation.

$$\left\{ m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} - \gamma(g) N \right\} \bar{\Gamma}_{as}^{(N)} = 0 \tag{B.2}$$

Note that the functions β (g) and γ (g) are expressed in terms of massive vertex functions and are hence holomorphic in ε for Re $\varepsilon \geqslant o$. The widely used μ -normalization of Gell-Mann and Low with mass shell normalization of the propagator pole (m physical mass) has similar properties, however with Green-functions continuous at m = o[16]. The PDE's are (see e. g. 22))

$$\left\{ \mu\frac{\partial}{\partial \mu} + \sigma\left(g, \frac{m}{\mu}\right)\frac{\partial}{\partial g} - \tau\left(g, \frac{m}{\mu}\right) N \right\} \tilde{\Gamma}^{(N)} = 0 \tag{B.3}$$

(RG-equation)

$$\{ m \frac{\partial}{\partial m} + \mu \frac{\partial}{\partial \mu} + \beta(g, \tfrac{m}{\mu}) \frac{\partial}{\partial g} - \gamma(g, \tfrac{m}{\mu}) N \} \hat{\Gamma}^{(N)} = - \hat{\Delta}_m \hat{\Gamma}^{(N)} \tag{B.4}$$

(CS-equation)

In this parametrization the RG-equation is not globally integrable for $m \neq o$. There are two regimes:

(i) Large nonexceptional momenta: $\tilde{\Delta}_m \hat{\Gamma}$ drops, however: CS $\tilde{\hat{\Gamma}}_{os}$ - RG $\tilde{\hat{\Gamma}}_{os}$ $\neq$ o still m dependence!

(ii) zero mass: $\hat{\Delta}_m \hat{\Gamma}$ drops and σ, τ, β and γ simplify such that CS $\tilde{\hat{\Gamma}}_o$ = RG $\tilde{\hat{\Gamma}}_o$.
Again $\tilde{\hat{\Gamma}}_{os}$ and $\tilde{\hat{\Gamma}}_o$ are related by complicated wave function and coupling constant renormalization.

Appendix C: Universality Properties of Γ_o.

Two zero mass A^4-theories Γ_o and $\hat{\Gamma}_o$ with length scales μ and $\tilde{\mu}$ can differ at $\mu = \tilde{\mu}$ only by a finite wave-function-and coupling constant renormalization:

$$\tilde{\Gamma}_o^{(N)}(p; \mu, V(g)) = Z(g)^{N/2} \, \Gamma_o(p; \mu, g) \tag{C.1}$$

As Γ_o and $\tilde{\Gamma}_o$ obey the PDE's

$$\{ \mu \frac{\partial}{\partial \mu} + \sigma(g) \frac{\partial}{\partial g} - \tau(g) N \} \Gamma_o(p; \mu, g) = 0$$

and

$$\{ \mu \frac{\partial}{\partial \mu} + \beta(V) \frac{\partial}{\partial V} - \gamma(V) N \} \tilde{\Gamma}_o(p; \mu, V) = 0 \tag{C.2}$$

we have

$$\beta(V) = \sigma(g) \frac{\partial V(g)}{\partial g}$$

$$\gamma(V) = \tau(g) + \tfrac{1}{2} \sigma(g) \frac{\partial}{\partial g} \ln Z(g) \tag{C.3}$$

As in perturbation theory $V(g) = g\,P(g)$ and $Z(g) = 1 + g\,Q(g)$ with P and Q polynomials in g we see that

$$\beta(V^*) = 0 \iff \sigma(g^*) = 0 \; ; \quad V^* = V(g^*)$$

$$\gamma(V^*) = \tau(g^*) \tag{C.4}$$

and $\frac{d\beta}{dV} = \frac{d\sigma}{dg} = \omega$; $\frac{d^2\beta}{dV^2} = \frac{d^2\sigma}{dg^2} \left(\frac{dV}{dg} \right)^{-1}$; etc.

i.e. fixed points and their nature as well as anomalous dimensions are universal.

Note that the composite field $\tilde{N} [A^2]$ in (7.7) normalized by

$$\frac{1}{2} < T \, \tilde{N} [A^2](0) \, \hat{A}(p_1) \, \hat{A}(p_2) >^{prop} \Big|_{\substack{p_i = 0 \\ t = \mu^2}} = 1$$

and the field $\hat{N} [A^2]$ in (6.12) normalized by

$$\frac{1}{2} < T \, \hat{N} [A^2](0) \, \hat{A}(p_1) \, \hat{A}(p_2) >^{prop} \Big|_{\substack{p_1 = p_2 = \frac{p}{2} \\ p^2 = -\mu^2 \\ t = 0}} = 1$$

are similarly related by

$$\tilde{N} [A^2] = Z_2(g) \, \hat{N} [A^2]$$

i.e.

$$\tilde{\Gamma}^{(N,1)}(p_1, \dots, p_N, q) = Z_2(g) \, \Gamma^{(N,1)}(p_1, \dots, p_N, q) \tag{C.5}$$

As $\partial_t \Gamma^{(N)} = \tilde{\Gamma}^{(N,1)}(p_1, \dots, p_N, 0)$ we have from (7.6) for non-exceptional momenta $q \neq 0$ as $t \to 0$:

$$\left\{ \mu \frac{\partial}{\partial \mu} + \sigma \frac{\partial}{\partial g} - \tau N + \hat{\delta} \right\} \tilde{\Gamma}_0^{(N,1)}(p_1, \dots, p_N, q) = 0 \tag{C.6}$$

whereas

$$\left\{ \mu \frac{\partial}{\partial \mu} + \sigma \frac{\partial}{\partial g} - \tau N + \delta \right\} \Gamma_0^{(N,1)}(p_1, \dots, p_N, q) = 0$$

Hence it follows

$$\delta(g) = \hat{\delta}(g) + \sigma(g) \frac{\partial}{\partial g} \ln Z_2(g) \tag{C.7}$$

and $\delta(g^*) = \hat{\delta}(g^*)$ at any fixed point g^* .

We see that if $\tilde{\Gamma}_o$ above is identified with the vertex functions $\bar{\Gamma}_{as}$ in (B.2) the functions β(g) and γ(g) as well as $\hat{\delta}$ (g) are expressed in terms of massive vertex functions. They are hence holomorphic in the dimension ε = D-4 for Re $\varepsilon \gg$ o. On the other hand the functions V(g) and Z(g) may show up infrared singularities (see Symanzik[14]) and so do the massless functions τ(g), σ (g) and δ(g). From (C.4) and (C.7) however we see that these infrared sigularities do not give troubles at the fixed points.

Table 1. 32)

Critical Exponents

D = 3 (i.e. ε = 1)

Exp.	Exp	MFA	L. - I.	ε^2	ε^3
α	small	0	0.125 ±.015	0.077	0.196
β	0.3÷0.4	1/2	0.312 ±.003	0.340	0.304
γ	1.2÷1.4	1	1.250 ±.003 (1.250 HTE)	1.244	1.195
δ	-	3	5.15 ± .02	4.463	-
ν	0.6÷0.7	1/2	0.642 ±.003	0.627	-
η	small	0	0.056±.01 (0.041 HTE)	0.037	0.029

D = 2 (i.e. ε = 2)

	α	β	γ	δ	ν	η
L.I.	ln	0.125	1.75	15.0	1	0.25
ε^2	-0.025	0.191	1.642	6.852	0.840	0.235

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