The Coleman-Weinberg mechanism in a conformal (Weyl) invariant theory: application to a magnetic monopole

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Abstract. We consider a massless conformally (Weyl) invariant classical action consisting of a magnetic monopole coupled to gravity in an anti-de Sitter background spacetime. We implement quantum corrections and this breaks the conformal (Weyl) symmetry, introduces a length scale via the process of renormalization and leads to the trace anomaly. We calculate the one-loop effective potential and determine from it the vacuum expectation value (VEV). Spontaneous symmetry breaking is radiatively induced *a la* Coleman-Weinberg and the scalar coupling constant is exchanged for the dimensionful VEV via dimensional transmutation. An important result is that the Ricci scalar of the AdS background spacetime is determined entirely by the value of the VEV.

1. Introduction

There has been a general interest in scale and conformal (Weyl) symmmetry as a possible fundamental principle in physics and cosmology, as pointed out for example in [1]. These authors point out that the classical action of the standard model is already consistent with global scale symmetry if the Higgs mass is dropped. Furthermore, on cosmic scales, the nearly-scale invariant spectrum of primordial fluctuations seems to demand an explanation based on a fundamental symmetry in nature. The authors therefore argue that scale and conformal symmetry could be the clue to fitting observations simultaneously on both small and large scales. In [1], authors discuss a model containing a Higgs field H, a dilaton field ϕ, standard model fields as well as gravity. However, their theory is classical and the authors mention that in a conformally (Weyl) invariant theory there is no mechanism at the classical level to set the scale of ϕ_0, the minimum of the dilaton field in their model. They then mention that adding quantum corrections could alleviate this problem via the Coleman-Weinberg mechanism [2]. This mechanism was recently implemented in the context of the standard model [3] where the Higgs mass was induced quantum mechanically starting with a scale invariant standard model. In our work, we implement the Coleman Weinberg mechanism starting with a classically Weyl invariant action containing a

magnetic monopole. We provide here a review of the most salient features. The reader can find details in [4].

The classical Weyl invariant theory of a magnetic monopole coupled to gravity was studied in [5]. The gravity sector does not include an Einstein-Hilbert term because that term is not Weyl invariant; instead, one includes a Weyl squared term (which is Weyl invariant). A cosmological constant is also forbidden by Weyl invariance. In this massless model, the conformal coupling term $-R\,\phi^2/6$, where R is the Ricci scalar "replaces" the usual negative mass term $a^2\phi^2$ in the symmmetry breaking potential. The vacuum expectation value (VEV) is now expressed in terms of R. As pointed out in [1], there is no mechanism at the classical level that can introduce a length scale in a Weyl invariant theory; one must therefore choose a VEV value by hand (or alternatively specify the asymptotic value of the Ricci scalar). Magnetic monopole solutions in a background anti-de Sitter (AdS) space are then found numerically [5]. The static spherically symmetric solution at large distances (outside the monopole core) is Schwarzschild-AdS. It should be stressed that the solution in [5] is not a black hole (BH) solution since the core is non-singular and static. In contrast, a BH has a horizon and there exits a region in the interior which is non-stationary, where all Killing vectors are spacelike [6].

In this work, we introduce a length scale by considering the effects of quantum fluctuations of scalar fields at one loop in a classical gravitational and gauge field background. We derive the effective potential at one loop for a massless theory with a triplet of scalar fields and extract the vacuum expectation value (VEV) from it. The coupling constant λ is defined at the renormalization scale M and the dimensionless λ is traded for the dimensionful VEV of the theory, through dimensional transmutation [2]. The asymptotic (background) spacetime is AdS and an important result is that its Ricci scalar is completely determined by the VEV.

2. Schwinger-Dewitt coefficient and trace anomaly

The effects due to vacuum fluctuations of matter fields can be included via an effective action W which can be into two parts: a divergent part $W_{\rm div}$ and a renormalized finite part $W_{\rm ren} = W - W_{\rm div}$. $W_{\rm div}$ is due to high-frequency fluctuations and is therefore local and state-independent [8]. It is expressed as an integral over the Schwinger-Dewitt coefficient $a_2(x)$, which can be calculated via a set of "curvatures" [10]. $W_{\rm div}$ is incorporated into the original action S by adding the appropriate counterterms. We label this S_{ren}, the renormalized local part of the one loop effective action. $W_{\rm ren}$ is the finite state-dependent nonlocal part of the effective action due to long wavelength fluctuations that sample the entire geometry. For example, the finite part of the vacuum expectation value of the energy-momentum tensor, $\langle T_{\mu\nu} \rangle$, that will appear on the right hand side of the gravitational equations of motion, can be viewed as a Casimir effect which also arises from long wavelength quantum fluctuations [11].

The conformally invariant action that leads to a magnetic monopole coupled to gravity contains a metric $g_{\mu\nu}$, a triplet of scalar fields ϕ^a and nonabelian gauge fields A^a_μ. It is given by [5]

$$S = \int d^4x \sqrt{-g} \left(C_{\mu\nu\sigma\tau} C^{\mu\nu\sigma\tau} - \frac{1}{4e^2} F^a_{\mu\nu} F^{\mu\nu}_a + D_\mu \phi^a \, D^\mu \phi_a + \frac{1}{6} R \, \phi^a \phi_a - \frac{\lambda}{4!} (\phi^a \, \phi_a)^2 \right)$$

$$= \int d^4x \sqrt{-g} \left(C^2 - \frac{1}{4e^2} F^2 + (D\phi)^2 + \frac{1}{6} R \, \phi^2 - \frac{\lambda}{4!} \phi^4 \right) \tag{1}$$

where $C_{\mu\nu\sigma\tau}$ is the Weyl tensor, $F^a_{\mu\nu}$ is the gauge field strength defined by $\nabla_\mu A^a_\nu - \nabla_\nu A^a_\mu + \varepsilon^a_{\ bc} A^b_\mu A^c_\nu$, and $D_\mu \phi^a$ is the covariant derivative defined by $\nabla_\mu \phi^a + \varepsilon^a_{\ bc} A^b_\mu \phi^c$. The square of

the Weyl tensor is denoted as C^2 and $F^2 \equiv F^a_{\mu\nu} F^{\mu\nu}_a$, $(D\phi)^2 \equiv D_\mu\phi^a\, D^\mu\phi_a$ and $\phi^2 \equiv \phi^a\phi_a$. The Ricci-Levi-Civita three-dimensional coefficients ε_{abc} are totally antisymmetric. Raising or lowering of the internal (latin) indices does not change the sign so that $\phi_a = \phi^a$ and implicit summation is assumed throughout e.g. $\phi_a\phi^a = \phi_1^2 + \phi_2^2 + \phi_3^2$. The action (1) is invariant under the conformal transformations $g_{\mu\nu} \rightarrow \Omega^2(x)g_{\mu\nu}$ and $\phi^a \rightarrow \phi^a/\Omega(x)$ where $\Omega(x)$ is called the conformal factor, an arbitrary positive smooth function.

We consider the vacuum fluctuations of the triplet ϕ^a in a gravitational and gauge field classical background. The Euclidean effective action is given by $W = \frac{1}{2}\,\mathrm{tr}\,(\ln\hat{H})$ where $\hat{H}$ is the hessian of the Euclidean version of the action (1). It takes the following form [10]

$$\hat{H} = g^{\mu\nu}D_\mu D_\nu + \hat{P} - \frac{1}{6}R\hat{1}\,. \tag{2}$$

The above operator is a 3×3 matrix in the internal vector space of the triplet ϕ^a and $\hat{1}$ is the identity matrix. Here $\hat{P}$ [1] arises from the self interaction term $\lambda(\phi^a\,\phi_a)^2/4!$, $\frac{1}{6}R\hat{1}$ arises from the conformally coupled scalar field term $\frac{1}{6}\,R\,\phi^a\phi_a$ and $g^{\mu\nu}D_\mu D_\nu$ arises from the covariant derivative term $D_\mu\phi^a\,D^\mu\phi_a$. The gravitational and nonabelian gauge fields are treated classically. One identifies three "curvatures" for the operator (2) that enter into the calculation of the effective action [10]. These are the Riemann curvature associated with $g_{\mu\nu}$, the commutator curvature $\hat{\mathscr{R}}_{\mu\nu}$ associated with the covariant derivative D_μ and defined by

$$[D_\mu, D_\nu]\phi^a = \mathscr{R}^a{}_{b\,\mu\,\nu}\,\phi^b \quad \text{with} \quad \hat{\mathscr{R}}_{\mu\nu} \equiv \mathscr{R}^a{}_{b\,\mu\,\nu} \tag{4}$$

and the potential $\hat{P}$ which is its own curvature. The divergent part W_{div} of the Euclidean effective action can be expressed entirely in terms of these curvatures as [10]

$$W_{\mathrm{div}} = \frac{1}{n-4}\int d^4x\,\sqrt{g}\,\,\mathrm{tr}\,\hat{a}_2(x,x) \qquad (n\rightarrow 4) \tag{5}$$

$$\hat{a}_2(x,x) = \frac{1}{180}\Big(R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - R_{\mu\nu}R^{\mu\nu} + \square\,R\Big)\hat{1} + \frac{1}{12}\hat{\mathscr{R}}_{\mu\nu}\hat{\mathscr{R}}^{\mu\nu} + \frac{1}{2}\,\hat{P}^2 + \frac{1}{6}\,\square\,\hat{P}\,. \tag{6}$$

The curvatures $\hat{P}$ and $\hat{\mathscr{R}}_{\mu\nu}$ for the Euclidean version of the action (1) are calculated in [4]. Here we simply state the result, the quantity of interest that appears in the integrand of W_{div}. This is

$$a_2(x) \equiv \mathrm{tr}\,\hat{a}_2(x,x) = \frac{1}{60}\Big(R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - R_{\mu\nu}R^{\mu\nu} + \square\,R\Big) \tag{7}$$

$$-\frac{1}{6}\,F^2 + \Big(\frac{11}{3}\Big)\frac{\lambda^2}{4!}\phi^4 - \frac{5}{36}\,\lambda\square\,\phi^2$$

where the factor of $11/3$ in front of λ^2 stems from having a triplet of scalar fields (it is equal to 3 for a single scalar field). To cancel the divergent part W_{div}, we simply add the appropriate counterterm to the original action (1). The renormalized local part of the action at one loop is simply

$$S_{\mathrm{ren}} = \int d^4x\,\sqrt{-g}\,\Big(\alpha\,R^2 + \beta\,R_{\mu\nu}R^{\mu\nu} - \frac{1}{4e^2}\,F^2 + (D\phi)^2 + \frac{1}{6}\,R\,\phi^2 - \frac{\lambda}{4!}\,\phi^4\Big)\,. \tag{8}$$

---

[1] The matrix elements of the potential $\hat{P}$ are

$$P_{ij} = -\frac{\partial}{\partial\phi_i}\frac{\partial}{\partial\phi_j}\frac{\lambda}{4!}\big(\phi_a\phi^a\big)^2 = -\frac{\lambda}{6}\big(\delta_{ij}\,\phi_a\phi^a + 2\,\phi_i\,\phi_j\big) \text{ where } i,j = 1,2,3\,. \tag{3}$$

The total derivatives $\Box R$ and $\Box(\phi_a\,\phi^a)$ appearing in (7) are not included in the above action as they lead to boundary terms that have no bearing on the equations of motion. The terms $C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau}$ and $R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}$ do not appear in (8) since they can be eliminated in favor of $R_{\mu\nu}R^{\mu\nu}$ and R^2 using the equality $C_{\mu\nu\sigma\tau}C^{\mu\nu\sigma\tau} = R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - 2\,R_{\mu\nu}R^{\mu\nu} + R^2/3$ and the fact that the integral of $R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - 4\,R_{\mu\nu}R^{\mu\nu} + R^2$, the Gauss-Bonnet integral, is a topological invariant (the Euler number) in four dimensions. The quantities e, λ, α, and β are renormalized constants. They are running coupling constants governed by a renormalization group equation which we do not need to state here for our purposes (for details see [12, 13]). At one loop, the constants α and β are related by $\alpha = -\beta/3$ and the sum of the first two terms in (8) are conformally invariant. At higher loops, non-conformally invariant terms like R^2 are generated [12, 13] and the two constants become independent. Our calculations will be at the one loop level, so we assume the relation $\alpha = -\beta/3$.

2.1. Trace Anomaly

To find the trace, we first note that under a conformal tranformation the inverse metric transforms according to $g^{\mu\nu}(x) \rightarrow e^{2\sigma(x)}g^{\mu\nu}(x)$ and the scalar field transforms according to $\phi^a(x) \rightarrow e^{\sigma(x)}\phi^a(x)$ where $\sigma(x)$ is introduced for convenience and is related to the conformal factor $\Omega(x)$ (previously introduced) by $e^{-\sigma(x)} \equiv \Omega(x)$. This yields the functional relation

$$\frac{\delta}{\delta\sigma(x)} = 2g^{\mu\nu}(x)\frac{\delta}{\delta g^{\mu\nu}(x)} + \phi^a(x)\frac{\delta}{\delta\phi^a(x)}\,. \tag{9}$$

The functional variation of the one-loop effective action with respect to $\sigma(x)$ is proportional to the Scwinger-Dewitt coefficient $a_2(x)$ [10]

$$\frac{\delta\,W_{\rm ren}}{\delta\sigma} = \frac{\sqrt{-g}\,a_2(x)}{16\,\pi^2} \tag{10}$$

and using (9) this yields

$$\frac{1}{\sqrt{-g}}\Big(2\,g^{\mu\nu}\frac{\delta W_{\rm ren}}{\delta g^{\mu\nu}} + \phi^a\frac{\delta W_{\rm ren}}{\delta\phi^a}\Big) = \frac{a_2(x)}{16\pi^2}\,. \tag{11}$$

The vacuum expectation value of the energy-momentum tensor is defined as

$$\langle T_{\mu\nu}\rangle = \frac{2}{\sqrt{-g}}\frac{\delta W_{\rm ren}}{\delta g^{\mu\nu}}\,. \tag{12}$$

¿From (11), its trace is given by

$$\langle T^\mu_\mu\rangle = \frac{a_2(x)}{16\pi^2} - [E] \tag{13}$$

where

$$[E] \equiv \frac{\phi^a}{\sqrt{-g}}\frac{\delta W_{\rm ren}}{\delta\phi^a} \tag{14}$$

and $a_2(x)$ is given by (7). The renormalized composite operator $[E]$ can be obtained readily once the effective potential has been calculated.

2.1.1. AdS space   As already discussed, the magnetic monopole solution is obtained when the spacetime is asymptotically anti-de Sitter space. This is a maximally symmetric spacetime with constant Ricci scalar R (positive in our notation). It is the submanifold obtained by embedding a hyperboloid in a flat five-dimensional spacetime of signature$(+,+,-,-,-)$. The universal covering space of AdS space has the topology of R^4 and can be represented by the following metric:

$$ds^2 = (1 + k\,r^2)\,dt^2 - \frac{dr^2}{1 + k\,r^2} - r^2\big(d\theta^2 + \sin^2\theta\,d\phi^2\big) \tag{15}$$

where (r, θ, ϕ) cover the usual range of spherical coordinates $(r \geq 0\,,\ 0 \leq \theta \leq \pi\,,\ 0 \leq \phi < 2\,\pi)$ and $-\infty < t < \infty$. The constant k is positive with the Ricci scalar given by $R = 12k$.

The symmetry group (isometry group) of AdS space is the ten parameter group SO(2,3). The only maximally form invariant rank two tensor under this group is the metric $g_{\mu\nu}$ (times a constant) so that the expectation value of the energy-momentum tensor in AdS space can be expressed in terms of its trace:

$$\langle T_{\mu\nu}\rangle_0 = \frac{1}{4}\,g_{\mu\nu}\langle T^\mu_\mu\rangle_0 \tag{16}$$

where the zero subscript means that the quantity is evaluated asymptotically in AdS space, the vacuum spacetime.

3. The effective potential

In this section we obtain the one loop effective potential by summing all the one loop one-particle irreducible (1PI) Feynman diagrams in the presence of $\lambda\phi^4/24$ and $-R\phi^2/6$ interactions. The coupling constant λ is defined at a renormalization scale M. We choose the renormalization scale M to coincide with the VEV, the minimum of the effective potential. This in turn fixes the value of λ; the dimensionless constant λ is traded for the dimensionful VEV through dimensional transmutation. The expectation value of the composite operator $[E]$, which appears both in the trace anomaly and in the equations of motion for the scalar field, can be readily obtained from the effective potential. Note that at one loop the effective potential and the expectation value of composite operators do not generate new geometrical curvature terms; they are generated starting only at the two loop level [12, 13]. In particular, for the calculation of the effective potential, the $\sqrt{-g}$ factor plays no role at one loop. The calculation proceeds in the same manner as in flat space, though of course the Ricci scalar R is non-zero and acts as a vertex for the $R\phi^2$ interaction.

The (classical) potential is given by

$$U = \frac{\lambda}{4!}\,\phi^4 - \frac{1}{6}R\,\phi^2\,. \tag{17}$$

We have a triplet of scalar fields and without loss of generality we take the vacuum expectation value to lie along the third component ϕ_3. For ϕ_3 loops, U generates two vertices: $-R/3$ and $\lambda\phi_3^2/2$. They can be combined into a single vertex given by the second derivative $U''(\phi_3) = -R/3 + \lambda\phi_3^2/2$. The vacuum expectation value of ϕ_1 and ϕ_2 are zero but they can still fluctuate in loops. The vertex for both is $U''(\phi_1) = U''(\phi_2) = -R/3 + \lambda\phi_3^2/6$ which is equivalent to replacing the coupling constant λ by $\lambda/3$ in the previous case. There are three sets of one loop 1PI Feynman diagrams; these are depicted in Fig.4.1. Let the classical field $\phi_c(x)$ be defined as the vacuum expectation value of ϕ_3 in the presence of some external source $J(x)$

$$\phi_c(x) = \left.\frac{\langle 0|\phi_3(x)|0\rangle}{\langle 0|0\rangle}\right|_J\,. \tag{18}$$

where J appears in the action in the usual fashion via the source term $J\phi$. The effective potential is obtained by summing all the diagrams in Fig.4.1. Note that the propagator is massless. For the first set of diagrams, the one-loop contribution yields

$$
\begin{aligned}
V &= i \int \frac{d^4 k}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{1}{2n} \left[\frac{-R/3 + \lambda \phi_c^2/2}{k^2 + i\epsilon} \right]^n \\
&= \frac{1}{16\pi^2} \int_0^{\Lambda} k_E^3 \ln \left[1 + \frac{-R/3 + \lambda \phi_c^2/2}{k_E^2} \right]
\end{aligned}
\tag{19}
$$

where k_E is the Euclidean momenta and Λ is a momentum cut-off. The integral in (19) can be readily evaluated but we do not write it out explicitly here. The other two sets of Feynman diagrams can be evaluated by simply replacing λ by $\lambda/3$ in (19). The one loop contribution to the potential is then

$$
V_1 = V + 2V[\lambda \to \lambda/3]. \tag{20}
$$

As it stands, the expression V_1 is divergent in the infinite Λ limit. This is handled in the usual fashion by adding the necessary counterterms and then imposing the appropriate renormalization conditions. The total potential is given by

$$
V_{tot} = \frac{\lambda}{4!}\phi_c^4 - \frac{1}{6}R\phi_c^2 + V_1 + A\phi_c^2 + B\phi_c^4 \tag{21}
$$

where the last two terms are the counterterms. The constants A and B are determined via the renormalization conditions

$$
\lambda = \left. \frac{d^4 V_{tot}}{d\phi_c^4} \right|_{\phi_c = M} \quad \text{and} \quad -\frac{R}{3} = \left. \frac{d^2 V_{tot}}{d\phi_c^2} \right|_{\phi_c = 0}. \tag{22}
$$

The renormalization scale M sets the scale for the theory. Substituting A and B back into (21), taking the infinite Λ limit and then collecting terms into compact expressions, we obtain the one loop effective potential

$$
\begin{aligned}
V_{\text{eff}} = \frac{\lambda}{4!}\phi_c^4 - \frac{1}{6}R\phi_c^2 + \frac{1}{1152\pi^2} \Bigg[& 18 \left(-\frac{R}{3} + \frac{\lambda \phi_c^2}{2} \right)^2 \ln\left[\frac{2R - 3\lambda \phi_c^2}{2R} \right] \\
& + 36 \left(-\frac{R}{3} + \frac{\lambda \phi_c^2}{6} \right)^2 \ln\left[\frac{2R - \lambda \phi_c^2}{2R} \right] + 5R\lambda\phi_c^2 \\
& + \lambda^2 \phi_c^4 \left(\frac{9}{2} \ln\left[\frac{2R}{2R - 3\lambda M^2} \right] + \ln\left[\frac{2R}{2R - \lambda M^2} \right] \right) \Bigg] \\
& + \Delta
\end{aligned}
\tag{23}
$$

where

$$
\Delta = \lambda^2 \phi_c^4 \frac{(-1584R^4 + 11904M^2 R^3 \lambda - 20360 M^4 R^2 \lambda^2 + 12480 M^6 R \lambda^3 - 2475 M^8 \lambda^4)}{13824\,\pi^2 (-2R + M^2\lambda)^2 (-2R + 3M^2\lambda)^2}. \tag{24}
$$

Let $\phi_c = v$ be the vacuum expectation value (VEV). It takes on this value in the asymptotic (background) spacetime, which is AdS space. We will see later, in section 4.1, that solving the equations of motion asymptotically yields the relation (42) between R_0, the Ricci scalar of AdS

space, and the VEV: $R_0 = \sqrt{110}\,\lambda\,v^2$. The VEV occurs at the minimum of the effective potential in the AdS background spacetime, where

$$\frac{dV_{\text{eff}}}{d\phi_c}\bigg|_{\substack{\phi_c=v \\ R=\sqrt{110}\lambda\,v^2}} = 0\,. \tag{25}$$

We set the arbitrary scale M to be equal to the VEV i.e. $\phi_c = v = M$. Equation (25) then yields a numerical value of $\lambda = C/D = 2519.926$, where the exact expressions for C and D are

$$C = 288\pi^2(197355230\sqrt{110} - 1144879587) \text{ and}$$

$$D = 299795296\sqrt{110} - 3734763307 +$$
$$(12\ln(2\sqrt{110}-1) + 18\ln(2\sqrt{110}-3) - 15\ln(440))(-96294467\sqrt{110} + 524292560)\,. \tag{26}$$

The ratio of the one loop correction to the tree (classical) result for the potential can be readily calculated to be -0.504. Such ratios are typical of one loop corrections in massless theories (e.g. in massless $\lambda\phi^4$ theory in flat space with a single scalar field the ratio is close to -1 [14]). We discuss in the conclusions how adding gauge field fluctuations can effect this scenario.

We started with a classical massless theory, a conformally invariant theory with $\lambda\phi^4$ and $R\phi^2$ interactions. After including one loop quantum corrections, the dimensionless parameter λ has been traded for the dimensionful VEV. An important result is that the Ricci scalar of AdS space is now completely determined by the value of the VEV.

3.1. Composite operator [E]

The composite operator [E(y)] is defined via (14). It appears in the trace but also as a quantum correction to the equations of motion for the scalar fields (see section 4 below). Inserting this operator into an n-point Green's function $\Gamma^n(x_1, ..., x_n)$ and integrating over all y yields n times the same Green's function [13]. The Feynman diagrams are therefore identical to those used to evaluate the effective potential V_{eff}, namely those of Fig.4.1 (the only difference is that the symmetry factor is multiplied by n). The upshot is that [E] can be obtained by taking the negative of the derivative of the one loop part of the effective potential (23), $V_{\text{loop}} = V_{\text{eff}} - U$, and then multiplying it by ϕ_c,

$$[E] = -\phi_c \frac{dV_{\text{loop}}}{d\phi_c}$$

$$= -\frac{1}{1152\pi^2}\bigg[36\Big(-\frac{R}{3} + \frac{\lambda\phi_c^2}{2}\Big)\ln\Big[\frac{2R-3\lambda\phi_c^2}{2R}\Big]\lambda\,\phi_c^2 - 108\Big(-\frac{R}{3} + \frac{\lambda\phi_c^2}{2}\Big)^2\frac{\lambda\phi_c^2}{2R-3\lambda\phi_c^2}$$

$$+ 24\Big(-\frac{R}{3} + \frac{\lambda\phi_c^2}{6}\Big)\ln\Big[\frac{2R-\lambda\phi_c^2}{2R}\Big]\lambda\,\phi_c^2 - 72\Big(-\frac{R}{3} + \frac{\lambda\phi_c^2}{6}\Big)^2\frac{\lambda\phi_c^2}{2R-\lambda\phi_c^2} \tag{27}$$

$$+ 10R\,\lambda\,\phi_c^2 + \lambda^2\phi_c^4\Big(18\ln\Big[\frac{2R}{2R-3\lambda M^2}\Big] + 4\ln\Big[\frac{2R}{2R-\lambda M^2}\Big]\Big)\bigg]$$

$$- 4\Delta\,.$$

4. Equations of motion

For the magnetic monopole, we seek static spherically symmetric solutions where the spatial symmetry (isometry) and gauge symmetry are both SO(3). These can be viewed as the lowest energy or ground state solution [5]. The metric, scalar triplet, and non-abelian gauge fields take on the following spherically symmetric form [5]:

$$\text{metric: } ds^2 = N(r)\, dt^2 - \psi(r) dr^2 - r^2\, (d\theta^2 + \sin^2(\theta)\, d\phi^2) \tag{28}$$

$$\text{scalar: } \phi^a(r) = f(r)\, \frac{r^a}{r} = f(r)[\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta] \tag{29}$$

$$\text{gauge: } A_a^\mu = q(r)\, \xi_a^\mu \text{ where } \xi_a^\mu \text{ are the Killing vectors for SO(3), namely}$$

$$\xi_1^\mu = [0, 0, \cos\varphi, -\sin\varphi\cot\theta], \xi_2^\mu = [0, 0, -\sin\varphi, -\cos\varphi\cot\theta] \text{ and } \xi_3^\mu = [0, 0, 0, 1]\,. \tag{30}$$

It will be convenient to work with $a(r) \equiv 1 + r^2\, q(r)$ instead of $q(r)$. There are four functions of r to determine: the "metric" fields $N(r)$ and $\psi(r)$, the "gauge field" $a(r)$ and the "scalar" field $f(r)$. It is convenient to obtain the equations of motion by direct variation of these functions. The Lagrangian corresponding to S_{ren} is given by

$$L = 4\pi \int_0^\infty dr \sqrt{N\psi}\, r^2 \Big(\alpha\, R^2 + \beta\, R_{\mu\nu} R^{\mu\nu} - \frac{1}{4e^2}\, F_{\mu\nu}^a F_a^{\mu\nu} + D_\mu \phi^a\, D^\mu \phi_a + \frac{1}{6}\, R\, \phi^a \phi_a - \frac{\lambda}{4!} (\phi^a \phi_a)^2 \Big)$$

$$= 4\pi \int_0^\infty \mathscr{L}\, dr\,. \tag{31}$$

The quantities that appear in (31) are evaluated using Eqs. 28-30 and are given in [4].

The equation of motion for N is given by

$$\frac{\sqrt{N\psi}\, r^2}{2N}(\alpha\, R^2 + \beta \widetilde{R}_j \widetilde{R}^j) + \sqrt{N\psi}\, r^2 \Big(2\,\alpha\, R\frac{\partial R}{\partial N} + 2\,\beta\, \widetilde{R}_j \frac{\partial \widetilde{R}^j}{\partial N} \Big)$$

$$- \Big[\sqrt{N\psi}\, r^2 \Big(2\,\alpha\, R\frac{\partial R}{\partial N'} + 2\,\beta\, \widetilde{R}_j \frac{\partial \widetilde{R}^j}{\partial N'} \Big) \Big]' + \Big[\sqrt{N\psi}\, r^2 \Big(2\,\alpha\, R\frac{\partial R}{\partial N''} + 2\,\beta\, \widetilde{R}_j \frac{\partial \widetilde{R}^j}{\partial N''} \Big) \Big]''$$

$$+ \frac{1}{6\sqrt{N\psi}} \Big(f^2 - f^2\psi - f^2 r\frac{\psi'}{\psi} - r^2 f f' \frac{\psi'}{\psi} + 4rff' + 2r^2 ff'' \Big) - \frac{\lambda\, r^2 f^4 \psi}{48\sqrt{N\psi}}$$

$$- \frac{\big((a^2-1)^2\psi + 2r^2 a'^2\big)}{4\, e^2\, r^2\sqrt{N\psi}} - \frac{a^2 f^2 \psi}{\sqrt{N\psi}} - \frac{r^2 f'^2}{6\sqrt{N\psi}} = \frac{\sqrt{N\,\psi}\, r^2}{2} \langle T^{tt} \rangle \tag{32}$$

where the explicit expressions for the Ricci scalar R and for $\widetilde{R}_j = \widetilde{R}^j$ are found in [4]. Implicit summation over $j = 1, 2, 3, 4$ is assumed. The equation of motion for ψ is given by

$$\frac{\sqrt{N\psi}\, r^2}{2\psi}(\alpha\, R^2 + \beta \widetilde{R}_j \widetilde{R}^j) + \sqrt{N\psi}\, r^2 \Big(2\,\alpha\, R\frac{\partial R}{\partial \psi} + 2\,\beta\, \widetilde{R}_j \frac{\partial \widetilde{R}^j}{\partial \psi} \Big)$$

$$- \Big[\sqrt{N\psi}\, r^2 \Big(2\,\alpha\, R\frac{\partial R}{\partial \psi'} + 2\,\beta\, \widetilde{R}_j \frac{\partial \widetilde{R}^j}{\partial \psi'} \Big) \Big]' + \Big[\sqrt{N\psi}\, r^2 \Big(2\,\alpha\, R\frac{\partial R}{\partial \psi''} + 2\,\beta\, \widetilde{R}_j \frac{\partial \widetilde{R}^j}{\partial \psi''} \Big) \Big]''$$

$$- \frac{\lambda\, r^2 f^4 N}{48\sqrt{N\psi}} - \frac{N(a^2-1)^2}{4e_R^2\, r^2\sqrt{N\psi}} + \frac{N^2\, a'^2}{2\, e^2\, (N\psi)^{3/2}} + \frac{N^2 r^2 f'^2}{2\, (N\psi)^{3/2}}$$

$$+ \frac{rff'\, (4N^2 + rN'N)}{6\, (N\psi)^{3/2}} + \frac{f^2\, (N^2 - N^2\psi - 6a^2 N^2\psi + rNN')}{6\, (N\psi)^{3/2}}$$

$$= -\frac{\sqrt{N\,\psi}\, r^2}{2} \langle T^{rr} \rangle\,. \tag{33}$$

The equation of motion for a is

$$2\, a\,(a^2 - 1) + 4\, a\, e^2\, f^2\, r^2 - \frac{2}{\psi}\, a''\, r^2 + \frac{a'\, r^2}{\psi}\left(\frac{\psi'}{\psi} - \frac{N'}{N}\right) = 0 \tag{34}$$

and finally the equation of motion for f is given by

$$-\frac{4a^2 f^2}{r^2} + \frac{f^2\, R}{3} - \frac{\lambda}{6}\, f^4 + \frac{2f\, f''}{\psi} + \frac{f\, f'}{\psi}\left(\frac{4}{r} + \frac{N'}{N} - \frac{\psi'}{\psi}\right) = -[E] \tag{35}$$

where $[E]$ is given by (27).

The above equations of motion are for static field configurations. Therefore the higher derivative terms in the metric field equations (32) and (33) pose no issues as they are spatial not time derivatives. Higher spatial derivatives appear in many branches of physics e.g. in physical acoustics the wave equation is modified by a term with four spatial derivatives when the bending stiffness of a vibrating string is included.

4.1. Relation between Ricci scalar of AdS space and the VEV: asymptotic analytical solution

We now solve the equations of motion (32),(33), (34) and (35) analytically in the asymptotic region to show that the Ricci scalar of AdS space is determined entirely by the VEV. The vacuum expectation value of the energy momentum tensor in AdS space is given by (16) and the non-zero components are

$$\langle T^{tt}\rangle = \frac{1}{4}\frac{1}{1 + k\, r^2}\langle T^\mu_{\ \mu}\rangle_0 \qquad ; \qquad \langle T^{rr}\rangle = -\frac{1}{4}(1 + k\, r^2)\langle T^\mu_{\ \mu}\rangle_0$$
$$\langle T^{\theta\theta}\rangle = -\frac{1}{4\, r^2}\langle T^\mu_{\ \mu}\rangle_0 \qquad ; \qquad \langle T^{\phi\phi}\rangle = -\frac{1}{4\, r^2 \sin^2\theta}\langle T^\mu_{\ \mu}\rangle_0 \tag{36}$$

where the AdS metric (15) was used. The trace $\langle T^\mu_{\ \mu}\rangle$ is given by

$$\langle T^\mu_{\ \mu}\rangle = \frac{a_2(x)}{16\pi^2} - [E]$$
$$= \frac{1}{16\pi^2}\Big\{\frac{1}{60}\Big(R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - R_{\mu\nu}R^{\mu\nu} + \Box R\Big) - \frac{1}{6}F^2 + \frac{11}{3}\frac{\lambda^2}{4!}\phi^4 - \frac{5}{36}\lambda\Box\,[\phi^2]\Big\} - [E], \tag{37}$$

where (13) and (7) were used and $[E]$ is given by (27). We now evaluate (37) in AdS space, the asymptotic spacetime. Asymptotically, $F^a_{\mu\nu} \to 0$, $\Box\,[\phi^2] \to 0$, $R_{\rho\sigma\mu\nu} = k\big(g_{\rho\mu}\, g_{\sigma\nu} - g_{\rho\nu}\, g_{\sigma\mu}\big)$ and $R_{\mu\nu} = 3\, k\, g_{\mu\nu}$ so that $R = 12k$, $\Box R = 0$ and $R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} - R_{\mu\nu}R^{\mu\nu} = -12\, k^2$. Substituting these values into (37), we obtain

$$\langle T^\mu_{\ \mu}\rangle_0 = -\frac{1}{80}\frac{k^2}{\pi^2} + \frac{11}{1152}\frac{\lambda^2}{\pi^2}\, v^4 - [E]_0 \tag{38}$$

where $[E]_0$ is $[E]$ evaluated in asymptotic AdS space.

The boundary conditions for the magnetic monopole [5] are that asymptotically, as $r \to \infty$, the spacetime is AdS where $N \to 1 + kr^2$, $\psi \to 1/(1 + k\, r^2)$, $f \to v$, $r\, f' \to 0$ (f' drops off faster than $1/r$), $r^2\, f'' \to 0$, $a \to 0$ and $a' \to 0$. Both v and k are positive constants. Substituting these boundary conditions into the gravity equation (32) (or (33)) and using (38) yields the following relation

$$\frac{v^2}{2} - \frac{\lambda v^4}{48\, k} = \frac{1}{8\, k}\left(-\frac{1}{80}\frac{k^2}{\pi^2} + \frac{11}{1152}\frac{\lambda^2}{\pi^2}\, v^4 - [E]_0\right). \tag{39}$$

We can eliminate $[E]_0$ above by solving the scalar equation (35) asymptotically. This yields

$$4\,k\,v^2 - \frac{\lambda v^4}{6} = -[E]_0 \tag{40}$$

where we used that $R = R_0 = 12\,k$ in AdS space. Substituting the above into (39) yields the solution

$$k = \frac{\sqrt{110}}{12}\lambda\,v^2 \tag{41}$$

so that

$$R_0 = \sqrt{110}\,\lambda\,v^2\,. \tag{42}$$

The Ricci scalar of AdS space is therefore determined solely by the VEV since the value of λ is known (it is no longer a free parameter having been traded for the dimensionful VEV). Substituting (41) into (40), yields $[E]_0 = -\frac{\lambda v^4}{6}\,(2\sqrt{110} - 1)$. This agrees with expression (27) asymptotically i.e. after substituting $R = \sqrt{110}\,\lambda\,v^2$ and $\phi_c = v = M$. The remaining unbroken U(1) is associated with $F_3^{\mu\nu}$ and the magnetic field is defined via $F_3^{ij} = \epsilon^{ijk}B^k$. Asymptotically, $a(r) \to 0$ and the function $q(r)$ appearing in (30) approaches $-1/r^2$. It is easy to verify that one obtains a radial magnetic field that varies as $1/r^2$ at large distances, corresponding to a magnetic monopole.
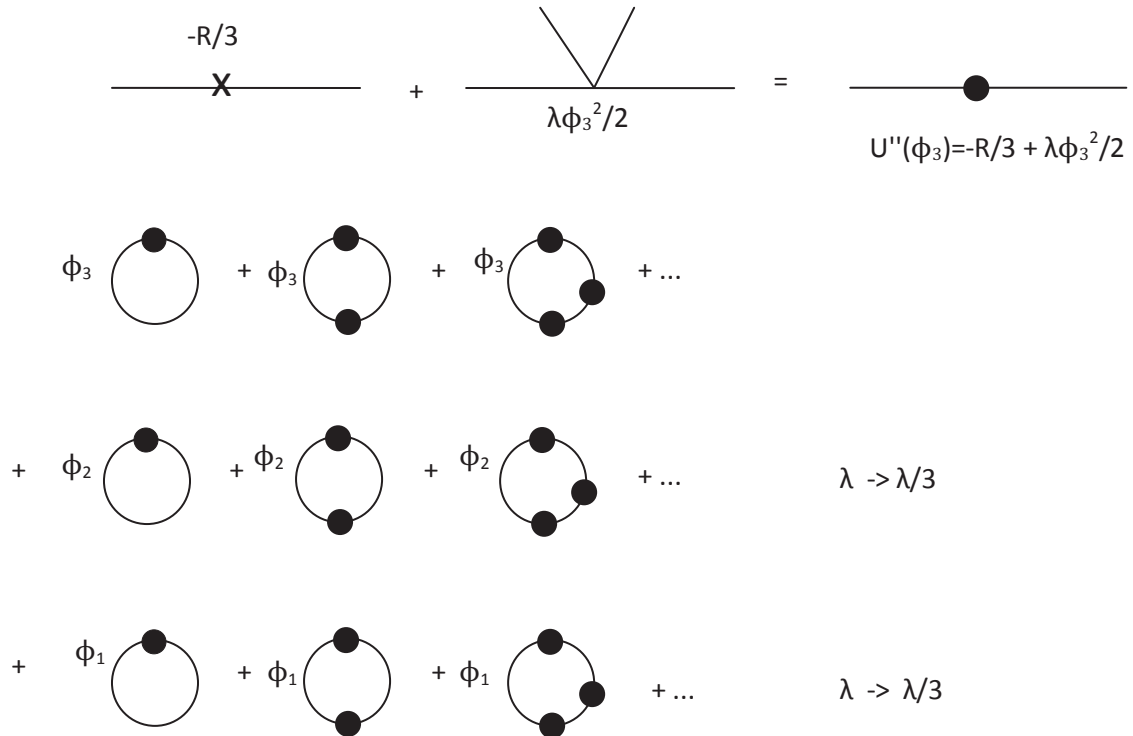

Figure 1: One loop Feynman diagrams for a triplet of scalar fields. The two vertices corresponding to $-R\phi^2/6$ and $\lambda\phi^4/24$ interactions, which are replaced by a single vertex (black dot). The last two sets of diagrams, where the components ϕ_2 and ϕ_1 run around the loop, have a vertex with $1/3$ the coupling constant λ of the first set of diagrams. The propagator is massless.

5. Conclusions

In this work, we introduced a renormalization scale into a massless theory by considering quantum corrections. Symmetry breaking was radiatively induced *a la* la Coleman-Weinberg [2]. The dimensionless λ, defined at the renormalizaton scale M, was traded for the dimensionful VEV. An important result was that the Ricci scalar of the background AdS spacetime was determined entirely by the VEV. Our work can now be naturally extended in a few ways. First, one can add gauge field fluctuations in the calculation of the effective potential. Then the coupling contant λ can be expressed in terms the electromagnetic coupling constant e as in [2]. In that scenario, the two parameters in the theory become e and the dimensionful VEV. Second, we used the symmetry of AdS space to solve for $\langle T_{\mu\nu}\rangle_0$, the asymptotic value of $\langle T_{\mu\nu}\rangle$. This allowed us to solve the equations of motion analytically in the asymptotic regime and to obtain an expression relating the Ricci scalar of AdS space to the VEV. The interior spacetime obeys spherical symmetry but not the symmetry of AdS space. Therefore, the finite and nonlocal part of $\langle T_{\mu\nu}\rangle$ for the interior would require a more elaborate calculation. One could then obtain numerical solutions of the interior. It is of interest to see how these numerical solutions containing quantum corrections in an AdS background compare with those obtained in the classical context of General Relativity (GR) [17]. Third, we worked with a background AdS spacetime because the Ricci scalar of AdS space had the right sign for classical SSB [5]. However, the VEV here is obtained from the quantum-corrected effective potential. In the presence of quantum corrections, de Sitter (dS) space could well be a viable background spacetime. The Ricci scalar of the background AdS space was determined solely by the VEV, which should apply to dS space as well. Such solutions could be of greater cosmological interest.

Acknowledgments

References

[1] I. Bars, P. Steinhardt and N. Turok, Phys. Rev. D **89**, 043515 (2014) [arXiv:1307.1848].
[2] S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973).
[3] C.T. Hill, Phys.Rev. D bf 89, 073003 (2014) [arXiv:1401.4185].
[4] A. Edery and N. Graham, JHEP **11**, 109 (2013) [arXiv:1310.7878].
[5] A. Edery, L. Fabbri and M. B. Paranjape, Class. Quant. Grav. **23**, 6409 (2006) [hep-th/0603131].
[6] A. Edery and B. Constantineau, Class. Quant. Grav. **28**, 045003 (2011)[arXiv:1010.5844].
[7] E.C. Thomas, F.R. Urban and A. R. Zhitnitsky, JHEP **08**,043 (2009) [arXiv:0904.3779]
[8] V.F. Mukhanov and S. Winitzki, *Introduction to Quantum Effects in Gravity*, Cambridge University Press, (2007).
[9] N.D. Birrell and P.C. W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, (1982).
[10] A. O. Barvisnsky, Yu. V. Gusev, G.A. Vilkovisky and V.V. Zhytnikov, Nucl. Phys. B **439**, 561 (1995)[hep-th/9404187].
[11] V.M. Mostepanenko and N.N. Trunov, *The Casimir Effect And Its Applications*, Oxford University Press, (1997)
     M. Bordag , U. Mohideen and V.M. Mostepanenko, Phys. Rep. **353**, 1 (2001)
     A. Zee, *Quantum Field Theory in a Nutshell*, Princeton University Press, (2003)
     A. Edery, J. Phys. A: Math. Gen. **39**, 685 (2006) [math-ph/0510056]

A. Edery, Phys. Rev. D **75**, 105012 (2007) [hep-th/0610173]

A. Edery, J. Stat. Mech. P06007 (2006) [hep-th/0510238].

[12] L.S. Brown and J.C. Collins, Ann. Phys. **130**, 215 (1980).

[13] S.J. Hathrell, Ann. Phys. **139**, 136 (1982).

[14] T. Cheng and L. Li, *Gauge Theory of Elementary Particle Physics*, Oxford University Press, (1984).

[15] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Westview Press, (1995).

[16] S. Caroll, *Spacetime and Geometry: An Introduction to General Relativity*, Benjamin Cummings, (2003).

[17] P. Breitenlohner, P. Forgács and D. Maison, Nucl. Phys. B **383**, 357 (1992)

K. Lee, V. P. Nair and E. J. Weinberg, Phys. Rev. D **45**, 2751 (1992)

K. Lee, V. P. Nair and E. J. Weinberg, Phys. Rev. Lett. **68**, 1100 (1992)

M.E. Ortiz, Phys. Rev. D **45**, 2586 (1992)

H. Hollmann, Phys. Lett. B **338**, 181 (1994).

[18] G. 't Hooft, Nucl. Phys. B **79**, 276, (1974).

[19] A.M. Polyakov, JETP Lett. **20**, 194 (1974).