# A beyond-mean-field model for $\Lambda$ hypernuclei with Skyrme-type $N\Lambda$ interaction

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A beyond-mean-field approach based on the Skyrme-Hartree-Fock model is derived and implemented for  ${}^{13}_{\Lambda}$ C and  ${}^{21}_{\Lambda}$ Ne. The Skyrme force with the SLy4 parameter set is used for NN interaction, and a recently updated Skyrme-type parameter set, SLL4, is employed for NA interaction. The hyperonic spin-orbit force is also included to reproduce the spin-orbit splitting of single-particle energy levels and the no-crossing rule. Properties of different configurations, such as  ${}^{12}C \otimes \Lambda[000]1/2^+$ ,  ${}^{12}C \otimes \Lambda[110]1/2^-$ ,  ${}^{12}C \otimes \Lambda[101]3/2^-$ , and  ${}^{12}C \otimes \Lambda[101]1/2^-$  are discussed. Low-lying energy spectra of <sup>13</sup><sub>A</sub>C, including both positive- and negative-parity levels, are presented and compared to the observed data. It is shown that this current model precisely reproduces the observed positive-parity energy level with the configuration  $[{}^{12}C_{g.s.}(2^+) \otimes s_{\Lambda}]$ . The two observed negative-parity levels,  $[{}^{12}C_{g.s.}(0^+) \otimes (p_{1/2})_{\Lambda}]$  and  $[{}^{12}C_{g.s.}(0^+) \otimes (p_{3/2})_{\Lambda}]$ , are also successfully reproduced but with reversed order. Energy spectra of  ${}^{21}_{\Lambda}$ Ne are also presented and compared to the results of a beyond relativistic mean-field model; except for some details, the two models give similar results. .....

Subject Index D06, D10, D14

#### Introduction 1.

Experimental investigations of  $\Lambda$  hypernuclear systems have achieved great progress since their first discovery [1-3], and observations of detailed level schemes, such as the fine structure [4,5]and spin-orbit splitting [6,7] of light hypernuclei, are helpful in revealing the nature of the  $N\Lambda$ interaction. In very recent years, charge symmetry breaking (CSB) has been investigated for s- and *p*-shell hypernuclei at J-PARC [8] and JLab [9], respectively, and there are also experimental plans for medium and heavy  $\Lambda$  hypernuclei [10].

On the other hand, and in parallel efforts, theoretical hypernuclear models, most of which come from nuclear theory counterparts, have followed the experimental developments closely and give very useful explanations about the observed data [3]. Focusing on light hypernuclei, both the shell model [11-14] and few-body calculations [15-18] reproduce the low-lying energy spectra precisely, and the *ab initio* method has also been extended to the hypernuclear regime [19]. For sd-shell  $\Lambda$  hypernuclei, antisymmetrized molecular dynamics (AMD) gives very complete predictions for energy spectra and density distributions [20–24] that await testing in future experiments. And it must be emphasized that there have also been some successful calculations of *p*-shell hypernuclei based on the AMD model [25]. Increasing the mass number of the hypernuclei brings difficulties to almost all the theoretical models except for mean-field calculations including the Skyrme–Hartree–Fock (SHF) model [26–33] and the relativistic mean-field (RMF) model [34–39], which have the advantage of

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facilitating global investigation across the hypernuclear chart. It is worth mentioning that the recently updated Skyrme-type  $N\Lambda$  interaction, SLL4, gives a nice fit for the  $\Lambda$  binding energies in a wide range of hypernuclear mass numbers [40].

However, broken symmetries built into the mean-field SHF and RMF calculations make it difficult to give the low-lying energy spectrum of a hypernucleus, and the mixing of configurations with different shapes is not taken into account. There are generally two ways to overcome these drawbacks. The first is combination with macroscopic models such as the five-dimensional collective model (5DCH) [41,42] or the particle–rotor model (PRM) [43–45]. The second way is the beyond-mean-field calculation including angular momentum projection (AMP) and the generator coordinate method (GCM), both of which have already been implemented in the SHF [46] and RMF models [47,48]. In Ref. [47], the positive- and negative-parity rotational bands formed by different configurations are predicted for  $^{21}_{\Lambda}$ Ne, but there are no experimental values for comparison.

In  ${}^{13}_{\Lambda}$ C there are observed energy levels, including the positive-parity one,  $[{}^{12}C_{g.s.}(2^+) \otimes s_{\Lambda}]$ , and two negative-parity ones,  $[{}^{12}C_{g.s.}(0^+) \otimes (p_{1/2})_{\Lambda}]$  and  $[{}^{12}C_{g.s.}(0^+) \otimes (p_{3/2})_{\Lambda}]$  [7], which provide good benchmarks for testing the theoretical models and  $N\Lambda$  interactions. In Ref. [46], the beyond-mean-field SHF model gives positive-parity energy levels of  ${}^{13}_{\Lambda}$ C that agree with the observed data, but the negative-parity ones are not taken into account. In this paper, employing the SLy4+SLL4 parameter set [40], the beyond-mean-field SHF model is extended to negative-parity levels to give a detailed investigation of the low-lying energy spectra of  ${}^{13}_{\Lambda}$ C and  ${}^{21}_{\Lambda}$ Ne. It is also a test for the validity of the SLy4+SLL4 parameter set to see whether it is suitable to describe the energy spectra of hypernuclear systems.

This paper is organized as follows. In Sect. 2, the formalism of the beyond-mean-field calculation is introduced. Section 3 presents the results and discussions. In Sect. 4, we summarize the work and give some remarks.

### 2. Formula

The wave function of a  $\Lambda$  hypernuclear system, in a body-fixed frame of reference, takes the form of a direct product:

$$\left|\Phi^{(N\Lambda)}(\beta)\right\rangle = \left|\Phi^{N}(\beta)\right\rangle \otimes \left|\Phi^{\Lambda}\right\rangle,\tag{1}$$

where  $|\Phi^{N}(\beta)\rangle$  and  $|\Phi^{\Lambda}\rangle$  are intrinsic wave functions of the nuclear core with quadrupole deformation  $\beta$  and the  $\Lambda$  hyperon, respectively. In the hypernuclear SHF model, the intrinsic wave function  $|\Phi^{(N\Lambda)}(\beta)\rangle$  is derived by the variation of an energy density functional (EDF) [31],

$$\varepsilon = \varepsilon_N^{\text{Skyrme}} + \varepsilon_\Lambda + \varepsilon_{\text{s.o.}}^\Lambda \,, \tag{2}$$

where  $\varepsilon_N^{\text{Skyrme}}$  takes the Skyrme form for the nuclear core and  $\varepsilon_{\Lambda}$  is the Skyrme-type  $N\Lambda$  interaction. In almost all the Skyrme-type  $N\Lambda$  interactions, the hyperonic spin–orbit interaction,  $\varepsilon_{\text{s.o.}}^{\Lambda}$ , is excluded for its smallness [6,7], but in the current calculation it is included as [31,40]

$$\varepsilon_{\text{s.o.}}^{\Lambda} = -\frac{1}{2} W_{\Lambda}(\rho_{\Lambda} \nabla \cdot \boldsymbol{J}_{N} + \rho_{N} \nabla \cdot \boldsymbol{J}_{\Lambda}) .$$
(3)

It must be emphasized that an additional quadrupole constraint is included in the variational procedure, ensuring axial symmetry of the whole system around the intrinsic *z*-axis. The pairing

force of the nuclear core takes the form of a density-dependent delta interaction (DDDI), as in Ref. [49]:

$$G(\mathbf{r}) = V_0 \left[ 1 - \frac{\rho(\mathbf{r})}{\rho_0} \right],\tag{4}$$

where the saturation density  $\rho_0$  equals 0.16 fm<sup>-3</sup>.

In Ref. [38], single-particle orbitals of the  $\Lambda$  hyperon are labeled by the Nilsson quantum numbers  $\Omega^{\pi}[Nn_3m_l]$ . And in the current paper, we use a similar notation with some changes to describe the configuration of a  $\Lambda$  hypernucleus with mass number A as  $^{A-1}Z \otimes \Lambda[Nn_3m_l]\Omega^{\pi}$ , in which  $^{A-1}Z$  is the nuclear core and  $\Lambda[Nn_3m_l]\Omega^{\pi}$  represents a  $\Lambda$  hyperon occupying the orbital  $\Omega^{\pi}[Nn_3m_l]$ . However, when it refers to the configurations of observed energy levels, the notations in Ref. [7], such as  $[^{12}C_{g.s.}(2^+)\otimes s_{\Lambda}]$ ,  $[^{12}C_{g.s.}(0^+)\otimes (p_{1/2})_{\Lambda}]$ , and  $[^{12}C_{g.s.}(0^+)\otimes (p_{3/2})_{\Lambda}]$ , are adopted.

In the lab-fixed frame of reference, the eigenstate of a  $\Lambda$  hypernuclear system is represented by a superposition of different configurations,

$$\left|\Psi_{\alpha}^{JM}\right\rangle = \sum_{\beta} F_{\alpha}^{J}(\beta) \hat{P}_{MK}^{J} \left|\Phi^{(N\Lambda)}(\beta)\right\rangle,\tag{5}$$

where  $\hat{P}_{MK}^{J}$  is the AMP operator restoring the rotational symmetry, and the sum goes through  $\beta$  for configuration mixing. The third component of angular momentum, *K*, is a sum of two parts for the nuclear core and the  $\Lambda$  hyperon, respectively, as

$$K = K_{\rm c} + K_{\Lambda}.\tag{6}$$

On the assumption that time-reversal symmetry is preserved for the nuclear core,  $K_c = 0$  and K is totally determined by  $K_{\Lambda}$ .

The GCM collective state,  $F_{\alpha}^{J}(\beta)$  in Eq. (5), is determined by the Hill–Wheeler–Griffin (HWG) equation [50],

$$\sum_{\beta'} \left[ H_{KK}^{\prime J}(\beta,\beta') - E_{\alpha}^{J} N_{KK}^{J}(\beta,\beta') \right] F_{\alpha}^{J}(\beta') = 0, \qquad (7)$$

where the Hamiltonian and norm elements are given by

$$H_{KK}^{\prime J}(\beta,\beta') = \left\langle \Phi^{(N\Lambda)}(\beta') \left| \hat{H}' \hat{P}_{KK}^{J} \right| \Phi^{(N\Lambda)}(\beta) \right\rangle,\tag{8}$$

$$N_{KK}^{J}(\beta,\beta') = \left\langle \Phi^{(N\Lambda)}(\beta') \left| \hat{P}_{KK}^{J} \right| \Phi^{(N\Lambda)}(\beta) \right\rangle.$$
(9)

The corrected Hamiltonian  $\hat{H}'$  is

$$\hat{H}' = \hat{H} - \lambda_p (\hat{N}_p - Z) - \lambda_n (\hat{N}_n - N) , \qquad (10)$$

where the last two terms on the right-hand side account for the fact that the projected wave function does not provide the correct number of particles on average [51–53], and  $\hat{H}$  is determined by the EDF in Eq. (2).

When the GCM states  $F^J_{\alpha}(\beta)$  are determined, the reduced E2 transition rates are obtained by

$$B(E2, J\alpha \to J'\alpha') = \frac{1}{2J+1} \left| \langle \alpha'; J' || \hat{Q}_2 || \alpha; J \rangle \right|^2 , \qquad (11)$$

where

$$\langle \alpha'; J' || \hat{Q}_2 || \alpha; J \rangle = \sqrt{2J' + 1} \sum_{M \mu \beta \beta'} F_{\alpha'}^{J'*}(\beta') F_{\alpha}^J(\beta) C_{JM2\mu}^{J'K'} \left\langle \Phi^{(N\Lambda)}(\beta') \left| \hat{Q}_{2\mu} \hat{P}_{MK}^J \right| \Phi^{(N\Lambda)}(\beta) \right\rangle, \quad (12)$$

in which  $\hat{Q}_{2\mu} = r^2 Y_{2\mu}(\varphi, \theta)$  is the electric quadrupole transition operator [54], and  $C_{JM2\mu}^{J'K'}$  denotes the Clebsh–Gordon coefficients.

# 2.1. Parameters

In the current calculation, the parameter sets SLy4 and SLL4 are employed for the *NN* interaction and  $N\Lambda$  interaction, respectively [40]. The strength of the hyperonic spin–orbit force is adopted as  $W_{\Lambda} = 5.0 \text{ MeV fm}^5$ , which can reproduce the observed spin–orbit splitting [7] through mean-field calculation with a spherical configuration. For <sup>12</sup>C and <sup>13</sup><sub>\Lambda</sub>C, the strength of the nuclear spin–orbit term,  $W_N$ , is reduced by multiplying by a factor of  $\mu = 0.8$ , which successfully reproduces the first 2<sup>+</sup> state of <sup>12</sup>C; similar manipulations were also used in Refs. [31,55,56]. As in Refs. [31,55–57], the strength of the density-dependent pairing interaction is taken as  $V_0 = -410 \text{ MeV fm}^3$  for both protons and neutrons, and a smooth pairing energy cutoff of 5 MeV around the Fermi level is used.

# 2.2. Model space

Because intrinsic wave functions are kept axially symmetric, the GCM states are superpositions of basis functions that are characterized by the only quadrupole deformation parameter  $\beta$  of the nuclear core. Therefore, the range of  $\beta$  and the number of basis functions determine the model space. For <sup>12</sup>C and <sup>13</sup><sub>A</sub>C,  $\beta$  is from -2.3 to 5.5, and 80 basis functions are evenly spaced in this range. For <sup>20</sup>Ne and <sup>21</sup><sub>A</sub>Ne,  $\beta$  is from -1.3 to 2.4, and 100 evenly spaced basis functions are kept. It has been checked that, for low-lying energy levels, basis functions outside the chosen range are negligible, and increasing the dimension of the basis within the range makes almost no difference.

# 3. Results and discussions

In this section,  ${}^{13}_{\Lambda}$ C,  ${}^{21}_{\Lambda}$ Ne, and the corresponding core nuclei, i.e.,  ${}^{12}$ C and  ${}^{20}$ Ne, are investigated using the beyond-mean-field SHF model introduced in the previous section. Properties of potential energy surfaces (PESs), energy spectra, reduced *E*2 transition rates, and the dependence of energy spectra on the shape of the nuclear core are presented and discussed.

# 3.1. Study of ${}^{12}C$ and ${}^{13}_{\wedge}C$

Figure 1 shows the PESs for different configurations of <sup>12</sup>C and <sup>13</sup><sub>A</sub>C, respectively. For <sup>12</sup>C, the energy minimum is located at  $\beta = 0$ , and this is the same for <sup>13</sup><sub>A</sub>C with the configuration <sup>12</sup>C $\otimes \Lambda[000]1/2^+$ , because the  $\Lambda$  hyperon occupying the  $s_{\Lambda}$  orbital favors a spherical nuclear core. For the configuration <sup>12</sup>C $\otimes \Lambda[110]1/2^-$ , the PES minimum is located on the prolate side, while for both <sup>12</sup>C $\otimes \Lambda[101]3/2^-$  and <sup>12</sup>C $\otimes \Lambda[101]1/2^-$ , PES minima are located on the oblate side. This phenomenon is caused by the density distribution of the  $\Lambda$  hyperon, since the energy of the  $N\Lambda$  interaction is mainly determined by the overlap of the density distributions of the nuclear core and the additional  $\Lambda$  hyperon. So, the single  $\Lambda$  state  $\Lambda[000]1/2^+$ , which has a spherical distribution, favors a spherical nuclear core. On the other hand, in a normally deformed potential well,  $\Lambda[110]1/2^-$  has a prolate distribution while  $\Lambda[101]3/2^-$  and  $\Lambda[101]1/2^-$  are oblately distributed, and nuclear cores with corresponding shapes are favored.



**Fig. 1.** PESs as functions of deformation parameter  $\beta$ , derived from mean-field calculation, for <sup>12</sup>C and <sup>13</sup><sub>A</sub>C.  $\Lambda_1$ ,  $\Lambda_2$ ,  $\Lambda_3$ , and  $\Lambda_4$  represent different hyperonic orbitals and correspond to  $\Lambda[000]1/2^+$ ,  $\Lambda[110]1/2^-$ ,  $\Lambda[101]3/2^-$ , and  $\Lambda[101]1/2^-$ , respectively. The details near  $\beta = 0$  are shown in the inserted graph.

The inserted graph of Fig. 1 shows PESs of different configurations around  $\beta = 0$ . Similar to the nuclear mean-field models, the addition of hyperonic spin–orbit terms, Eq. (3), leads to energy splitting between  $[{}^{12}C\otimes(p_{1/2})_{\Lambda}]$  and  $[{}^{12}C\otimes(p_{3/2})_{\Lambda}]$ . In our calculation, with  $W_{\Lambda}$  being 5.0 MeV fm<sup>5</sup>, the energy splitting between these two configurations is 174 keV, compared to the observed one of  $152\pm54(\text{stat})\pm36(\text{syst}) \text{ keV}$  [7], which indicates that the value of  $W_{\Lambda}$  is reasonable. Furthermore, the hyperonic spin–orbit force also reproduces the no-crossing rule at  $\beta = 0.^1$  It is shown that, at the point of  $\beta = 0$ , the PESs of configurations  ${}^{12}C\otimes\Lambda[110]1/2^-$  and  ${}^{12}C\otimes\Lambda[101]1/2^-$  are non-continuous. The oblate part of  ${}^{12}C\otimes\Lambda[110]1/2^-$  and prolate part of  ${}^{12}C\otimes\Lambda[101]1/2^-$  coincide at  $\beta = 0$  and vice versa.

Figure 2 shows the angular momentum projected energy curves for <sup>12</sup>C and <sup>13</sup><sub>A</sub>C, respectively, and some low-lying GCM levels are also given at the positions of average deformations  $\bar{\beta}$ . In Fig. 2(a), the most obvious effect is the energy gained by AMP, which changes the mean-field PES with one spherical minimum into the  $J^{\pi} = 0^+$  curve with two pronounced minima on the prolate and oblate sides, respectively. The energy of the ground state is -90.60 MeV. The second  $0^+$  state is an oblate one which is 8.73 MeV higher than the ground state, but it is not the Hoyle state (7.65 MeV [59]). In fact, it is difficult for the current model to reproduce the Hoyle state since the dilute cluster structure [60,61] is not included in this model space. Figures 2(b), (c), (d), and (e) show the AMP energy curves and GCM levels of  ${}^{13}_{\Lambda}$ C with different configurations. In panel (b), the  $\Lambda$  occupying the  $s_{\Lambda}$  orbital makes the two minima of the  $J^{\pi} = 1/2^+$  energy curve closer to  $\beta = 0$  compared to the case of panel (a). Panel (b) also shows that the ground-state energy of  ${}^{13}_{\Lambda}$ C is -102.26 MeV, giving a  $\Lambda$  binding energy of  $B_{\Lambda} = 11.66$  MeV, which is nearly identical to the observed one,  $11.69 \pm 0.12$  MeV [62], and indicates the validity of this model. From panels (c), (d), and (e) we notice that the  $\Lambda$  occupying  $\Lambda[110]1/2^-$  deepens the energy minimum on the prolate side, while the one occupying  $\Lambda[101]3/2^$ or  $[101]1/2^-$  deepens the energy minimum on the oblate side.

<sup>&</sup>lt;sup>1</sup> The no-crossing rule means that two  $\Lambda$  single-particle energy levels with the same quantum number  $\Omega^{\pi} = 1/2^{-}$ , as functions of  $\beta$ , never intersect. It is the origin of the discontinuities of the PESs corresponding to  ${}^{12}C \otimes \Lambda [110]1/2^{-}$  and  ${}^{12}C \otimes \Lambda [101]1/2^{-}$ . For more details, please refer to Ref. [50, pp. 76–77], or Ref. [58, pp. 259–260].



**Fig. 2.** Angular momentum projected energy curves for <sup>12</sup>C and <sup>13</sup><sub>A</sub>C. Panel (a) is for <sup>12</sup>C while panels (b), (c), (d), and (e) are for <sup>13</sup><sub>A</sub>C corresponding to the configurations <sup>12</sup>C $\otimes \Lambda$ [000]1/2<sup>+</sup>, <sup>12</sup>C $\otimes \Lambda$ [110]1/2<sup>-</sup>, <sup>12</sup>C $\otimes \Lambda$ [101]3/2<sup>-</sup>, and <sup>12</sup>C $\otimes \Lambda$ [101]1/2<sup>-</sup>, respectively. Some low-lying GCM levels are shown at average deformations  $\bar{\beta}$ . For convenience of comparison, panels (b), (c), (d), and (e) are offset upward by 12 MeV, 0 MeV, -1 MeV, and -1 MeV, respectively.



**Fig. 3.** Energy spectra for <sup>12</sup>C and <sup>13</sup><sub> $\Lambda$ </sub>C. The observed data and calculated ones are labeled by "exp" and "cal," respectively. The reduced *E*2 transition rates are indicated by solid arrows, and the configurations of the calculated levels are indicated by dashed lines. The experimental data are taken from Ref. [63] for <sup>12</sup>C and Ref. [7] for <sup>13</sup><sub> $\Lambda$ </sub>C.

The energy spectra of <sup>12</sup>C and <sup>13</sup><sub> $\Lambda$ </sub>C are presented in Fig. 3, and compared to the observed ones. We can see that for <sup>12</sup>C the calculated energy levels of the ground band agree with the experimental data, although there are some deviations between the calculated and observed B(E2) values. For <sup>13</sup><sub> $\Lambda$ </sub>C, a positive-parity level is observed at 4.88 MeV, while this current work reproduces it at 4.96 MeV with the configuration <sup>12</sup>C $\otimes \Lambda$ [000]1/2<sup>+</sup>. For this configuration, the energies of the spin doublet  $(J^{\pi} = 3/2^+, J^{\pi} = 5/2^+)$  are about 0.56 MeV higher than the energy of the  $J^{\pi} = 2^+$  state of <sup>12</sup>C, while  $B(E2, 3/2^+ \rightarrow 1/2^+)$  or  $B(E2, 5/2^+ \rightarrow 1/2^+)$  is reduced by about 8%, both of which indicate the shrinkage effect of the  $\Lambda$  hyperon occupying the  $s_{\Lambda}$  orbital.

The negative-parity levels, with the  $\Lambda$  hyperon occupying the  $p_{\Lambda}$  orbitals, are also shown in Fig. 3. In experiments [6,7] there are two observed levels, at 10.830 MeV and 10.982 MeV, which



**Fig. 4.** Left panel: Mean-field PESs of <sup>12</sup>C for different values of the reduction factor  $\mu$ . Right panel: Energies as functions of  $\mu$  for the first  $J^{\pi} = 2^+$  state of <sup>12</sup>C (black solid square), for the first  $J^{\pi} = 3/2^+$  state of <sup>13</sup><sub>A</sub>C (red solid circle), and for the band heads of three negative-parity configurations (blue solid diamond, pink down triangle, and navy up triangle, respectively). The calculated static quadrupole moment of the first  $2^+$  state of <sup>12</sup>C is also given as Q(b). The relevant experimental data [7,63] are shown in the right panel as green stars for <sup>12</sup>C and green crosses for <sup>13</sup><sub>A</sub>C.

are assigned to the configurations  $[{}^{12}C_{g.s.}(0^+) \otimes (p_{3/2})_{\Lambda}]$  and  $[{}^{12}C_{g.s.}(0^+) \otimes (p_{1/2})_{\Lambda}]$ , respectively. In our calculation, there are three band-head levels with angular momenta  $J^{\pi} = 3/2^-$ ,  $3/2^-$ , and  $1/2^-$  at 12.55 MeV, 11.20 MeV, and 10.88 MeV, respectively, corresponding to the configurations  ${}^{12}C \otimes \Lambda [110]1/2^-$ ,  ${}^{12}C \otimes \Lambda [101]3/2^-$ , and  ${}^{12}C \otimes \Lambda [101]1/2^-$ . Generally speaking, the positions of the two calculated negative-parity band heads with configurations  ${}^{12}C \otimes \Lambda [101]3/2^-$  and  ${}^{12}C \otimes \Lambda [101]1/2^-$  approximately agree with the observed ones, but there are still two problems. The first is the fact that in our current calculation the first  $J^{\pi} = 3/2^-$  state is higher than the  $J^{\pi} = 1/2^-$  one, which contradicts the observed data. The second problem is the redundant quantum number K for the Nilsson notation. More details about these two problems are discussed in the remarks at the end of this paper.

In Ref. [55], it is shown that the shape of <sup>12</sup>C depends on the strength of the nuclear spin-orbit force, and this is tested by reducing the standard nuclear spin-orbit force by a factor of  $\mu$ , i.e.,  $W'_N = \mu W_N$ . In our calculation, as shown in the left panel of Fig. 4, similar results to Ref. [55] are reproduced for the SLy4 force. Different values of  $\mu$ , from 1.1 to 0.6, turn the PES minimum from a spherical one to a clear oblate one.

To investigate the dependence of the energy levels on the shape of the nuclear core, different values of the reduction factor  $\mu$  are employed to give different shapes. The right panel of Fig. 4 gives the positions of some important energy levels versus the reduction factor  $\mu$ . For <sup>12</sup>C, it is shown that the  $J^{\pi} = 2^+$  level of the ground band increases rapidly when  $\mu$  is increased, and this is caused by the decreasing moment of inertia when its shape becomes more spherical. To make the shape change of <sup>12</sup>C more clear, the static quadrupole moment Q(b) of this  $J^{\pi} = 2^+$  state is calculated for each value of  $\mu$ . We can see that the calculated Q(b) decreases with increasing  $\mu$ . When  $\mu$  is 0.8, Q(b)equals +0.08, compared to the observed one of +0.06(3) [64].

For  ${}^{13}_{\Lambda}$ C, the first  $J^{\pi} = 3/2^+$  state with configuration  ${}^{12}C \otimes \Lambda[000]1/2^+$  has the same tendency as the first  $J^{\pi} = 2^+$  state of  ${}^{12}$ C discussed above, which is shown in the right panel of Fig. 4. In the same figure, the energies of band heads with negative-parity configurations are also given as functions of  $\mu$ . Increasing  $\mu$ , which leads to a more spherical nuclear core, makes the band head of  ${}^{12}C \otimes \Lambda[110]1/2^-$  lower and the band heads of  ${}^{12}C \otimes \Lambda[101]3/2^-$  and  ${}^{12}C \otimes \Lambda[101]1/2^-$  higher.



**Fig. 5.** As Fig. 1, but for  ${}^{20}$ Ne and  ${}^{21}_{\Lambda}$ Ne.

This is due to the fact that  $\Lambda[110]1/2^-$  favors a prolate core, while  $\Lambda[101]3/2^-$  and  $\Lambda[101]1/2^-$  favor oblate ones. The observed energy levels are also shown in the right panel of Fig. 4, and the comparison indicates that  $\mu = 0.8$  makes the best predictions for both <sup>12</sup>C and <sup>13</sup><sub>A</sub>C.

# 3.2. Study of ${}^{20}Ne$ and ${}^{21}_{\wedge}Ne$

As a test of the current model for *sd*-shell hypernuclei, <sup>20</sup>Ne and <sup>21</sup><sub> $\Lambda$ </sub>Ne are investigated. Figure 5 gives the mean-field calculation for different configurations. The SLy4 parameter set gives the PES of <sup>20</sup>Ne with two minima on the prolate side and oblate side, with  $\beta = 0.5$  and -0.17, respectively, and the prolate one is much deeper than the oblate one. The addition of one  $\Lambda$ , with the configuration <sup>20</sup>Ne $\otimes \Lambda$ [000]1/2<sup>+</sup>, makes the two minima slightly closer to the spherical shape. The PES of <sup>20</sup>Ne $\otimes \Lambda$ [100]1/2<sup>-</sup> is also shown with an enhanced prolate minimum, since the  $\Lambda$  hyperon occupying the  $\Lambda$ [110]1/2<sup>-</sup> orbital favors a prolate nuclear core, while for the PESs of <sup>20</sup>Ne $\otimes \Lambda$ [101]3/2<sup>-</sup> and <sup>20</sup>Ne $\otimes \Lambda$ [101]1/2<sup>-</sup> the oblate minima are enhanced. With the hyperonic spin–orbit force, the splitting of [<sup>20</sup>Ne $\otimes (p_{1/2})_{\Lambda}$ ] and [<sup>20</sup>Ne $\otimes (p_{3/2})_{\Lambda}$ ] is 150 keV, as shown in the inserted graph, and the no-crossing rule is also reproduced in the same graph.

Figure 6 shows the energy curves from mean-field and beyond-mean-field calculations for <sup>20</sup>Ne and <sup>21</sup><sub> $\Lambda$ </sub>Ne, and the low-lying GCM levels are also given at the average deformations  $\bar{\beta}$ . From Fig. 6(a) we can see that the AMP makes both the oblate and prolate energy minima deeper than those from the mean-field calculation. Panel (b) indicates that the addition of  $\Lambda$  to the  $s_{\Lambda}$  orbital makes the deformations of the two minima smaller, as a shrinkage effect, than those of the core nucleus. From panels (c), (d), and (e), we notice that the  $\Lambda$  hyperons occupying three different  $p_{\Lambda}$  orbitals, respectively, change the projected energy curves in similar ways to the cases of <sup>13</sup><sub> $\Lambda$ </sub>C. GCM calculations show that the energies of the ground states of <sup>20</sup>Ne and <sup>21</sup><sub> $\Lambda$ </sub>Ne are -175.87 MeV and -160.62 MeV, respectively, which gives the  $\Lambda$  binding energy  $B_{\Lambda}$  as 15.25 MeV, compared to 14.11 MeV from the beyond-RMF model [47] and 16.9 MeV from AMD [21,22].

Figure 7 shows the energy spectra and reduced *E*2 transition rates for both <sup>20</sup>Ne and <sup>21</sup><sub> $\Delta$ </sub>Ne from the beyond-mean-field calculation. For <sup>20</sup>Ne, the calculated ground state band agrees with the observed one and the calculated value of  $B(E2, 2^+ \rightarrow 0^+)$  coincides precisely with the observed data. For  $B(E2, 4^+ \rightarrow 2^+)$  and  $B(E2, 6^+ \rightarrow 4^+)$ , there are some deviations between the calculated and



**Fig. 6.** As Fig. 2, but for <sup>20</sup>Ne and  $^{21}_{\Lambda}$ Ne. Panels (b), (c), (d), and (e) are offset upward by 15 MeV, 7 MeV, 3 MeV, and 3 MeV, respectively.



**Fig. 7.** As Fig. 3, but for  ${}^{20}$ Ne and  ${}^{21}_{\Lambda}$ Ne.

observed data. The static quadrupole moment Q(b) for the first 2<sup>+</sup> state is also calculated and is -0.21, compared to the observed one of -0.23(3) [64].

For  ${}^{21}_{\Lambda}$ Ne, there is a positive-parity rotational band based on the configuration  ${}^{20}$ Ne $\otimes \Lambda[000]1/2^+$ . The value of  $B(E2, 5/2^+ \rightarrow 1/2^+)$  or  $B(E2, 3/2^+ \rightarrow 1/2^+)$  is about  $56 \, e^2 \, \text{fm}^4$ , which is 14.6% smaller than that of  ${}^{20}$ Ne, indicating the shrinkage effect of the  $\Lambda$  hyperon. Figure 7 also gives the negative-parity bands with configurations  ${}^{20}$ Ne $\otimes \Lambda[110]1/2^-$ ,  ${}^{20}$ Ne $\otimes \Lambda[101]3/2^-$ , and  ${}^{20}$ Ne $\otimes \Lambda[101]1/2^-$ , respectively. The corresponding band heads are located at 8.80 MeV, 12.84 MeV and 12.33 MeV, respectively. The band head of  ${}^{20}$ Ne $\otimes \Lambda[110]1/2^-$  is obviously lower than those of the other two negative-parity ones, since the core nucleus with a prolate shape produces a larger binding energy with a prolate-distributed  $\Lambda$  single-particle state, i.e.,  $\Lambda[110]1/2^-$ , while  $\Lambda[101]3/2^-$  and  $\Lambda[101]1/2^-$  are oblately distributed.

In Ref. [47], beyond-RMF calculations are implemented for <sup>20</sup>Ne and <sup>21</sup><sub> $\Lambda$ </sub>Ne. Generally speaking, many results given in the current model, including the energy spectra, *B*(*E*2) values, and even the shrinkage effect of  $\Lambda$ , are very similar to the ones in Ref. [47]. Furthermore, for the rotational bands with configurations <sup>20</sup>Ne $\Delta$ [000]1/2<sup>+</sup> and <sup>20</sup>Ne $\Delta$ [110]1/2<sup>-</sup>, both Ref. [47] and this current work

reproduce the strong coupling limit with decoupling factors a = 1 and -1, respectively. However, some details of the energy spectra in the current calculation are different from those in Ref. [47]. For example, the energy difference of the spin doublet  $(3/2^+, 5/2^+)$ , i.e.,  $\delta E = E(3/2^+) - E(5/2^+)$ , in our calculation is -135 keV, while it is 41.5 keV in Ref. [47]. Another example is the energy difference of the negative-parity spin doublet  $(1/2^-, 3/2^-)$  of configuration  ${}^{20}\text{Ne} \otimes \Lambda [110]1/2^-$ , which is nearly zero in the current calculation but 270 keV in Ref. [47]. These contradictions may be caused by some reasons discussed in the next section.

# 4. Conclusions and remarks

In this paper, *p*- and *sd*-shell  $\Lambda$  hypernuclei,  ${}^{13}_{\Lambda}$ C and  ${}^{21}_{\Lambda}$ Ne, have been studied by the beyondmean-field SHF model with SLy4 and SLL4 parameter sets for *NN* interaction and *N* $\Lambda$  interaction, respectively.

The mean-field calculation shows that the PES of a  $\Lambda$  hypernuclear system is obviously influenced by the orbital that the  $\Lambda$  hyperon occupies. For  $^{A-1}Z \otimes \Lambda[000]1/2^+$  a spherical shape is favored, as a kind of shrinkage effect. On the other hand, for  $^{A-1}Z \otimes \Lambda[110]1/2^-$  the energy minimum on the prolate side is enhanced, while for  $^{A-1}Z \otimes \Lambda[101]3/2^-$  and  $^{A-1}Z \otimes \Lambda[101]1/2^-$  oblate ones are favored. The spin–orbit force, which was always neglected in previous SHF calculations, makes the single-particle energies of  $(p_{1/2})_{\Lambda}$  and  $(p_{3/2})_{\Lambda}$  separated, and also reproduces the no-crossing rule. The strength of the hyperonic spin–orbit force is taken as 5.0 MeV fm<sup>5</sup> in the mean-field calculation, and kept in the beyond-mean-field calculation.

From the perspective of the beyond-mean-field calculations, the current model gives the positiveand negative-parity rotational bands for various configurations separately. For  ${}^{13}_{\Lambda}$ C, the observed positive-parity energy level with  $J^{\pi} = 3/2^+$  is reproduced by the current model with configuration  ${}^{12}C \otimes \Lambda [000]1/2^+$ . The other two observed ones with  $J^{\pi} = 1/2^-$  and  $J^{\pi} = 3/2^-$  are also successfully reproduced, but with reversed order. For  ${}^{21}_{\Lambda}$ Ne, there are no observed data, but through the comparison with the beyond-RMF model, it has been found that the two models give similar results for the general positions of energy levels. However, the two models contradict each other for the order of the spin doublets  $(3/2^+, 5/2^+)$  and  $(1/2^-, 3/2^-)$ . Therefore, the beyond-mean-field SHF model, together with Skyrme-type  $N\Lambda$  interaction SLL4, gives a generally good fit for the energy spectra, but it still needs to be refined to describe the details.

# 4.1. Remarks

In Ref. [40], the Skyrme-type  $N\Lambda$  interaction, SLL4, gives a very nice global fit to  $\Lambda$  binding energies with spherical configurations in mean-field calculations. Once extended to beyond-meanfield calculations, this  $N\Lambda$  interaction also successfully reproduces the positions of observed energy levels, but some details of the calculated energy spectra contradict the experimental ones and also those from other models, such as the energy differences of the spin doublet  $(3/2^+, 5/2^+)$  and the spin– orbit doublet  $(1/2^-, 3/2^-)$ . These problems may be caused for two reasons. The first is the model itself, and the second is the Skyrme-type  $N\Lambda$  interaction. In the current model, configuration mixing of different hypernuclear channels (mixing of the configurations with  $\Lambda$  occupying different orbitals) is not taken into consideration. The mixing of, for example,  ${}^{12}C\otimes\Lambda[110]1/2^-$  and  ${}^{12}C\otimes\Lambda[101]3/2^$ would push the first  $J^{\pi} = 3/2^-$  state downward and may change the order of the calculated spin– orbit doublet  $(1/2^-, 3/2^-)$ . Such mixing would also eliminate the redundant quantum number K. The  $N\Lambda$  interaction used in the current work has no clear term for spin–spin interaction, and the timeodd terms of the spin–orbit interaction are not included either, which may also cause the problems mentioned above. And furthermore, the antisymmetric spin–orbit term of the  $N\Lambda$  interaction that is proved to be non-negligible in Ref. [15] is absent in the current model, and this may be one of the origins of the spin–orbit splitting problem.

In the current calculation, the configurations with  $\Lambda$  occupying  $s_{\Lambda}d_{\Lambda}$ -shell orbitals do not give any bound states for  ${}^{13}_{\Lambda}$ C or  ${}^{21}_{\Lambda}$ Ne. But for  ${}^{21}_{\Lambda}$ Ne, a  $J^{\pi} = 1/2^+$  state with the configuration  ${}^{20}$ Ne $\otimes \Lambda$ [211]1/2<sup>+</sup> is just 0.7 MeV above the  ${}^{20}$ Ne+ $\Lambda$  threshold, and it may be bound in a refined model.

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