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Striped superconductors: how spin, charge and superconducting orders intertwine in the cuprates

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Abstract. Recent transport experiments in the original cuprate high temperature superconductor, $La_{2-x}Ba_xCuO_4$, have revealed a remarkable sequence of transitions and crossovers that give rise to a form of dynamical dimensional reduction, in which a bulk crystal becomes essentially superconducting in two directions while it remains poorly metallic in the third. We identify these phenomena as arising from a distinct new superconducting state, the 'striped superconductor', in which the superconducting order is spatially modulated, so that its volume average value is zero. Here, in addition to outlining the salient experimental findings, we sketch the order parameter theory of the state, stressing some of the ways in which a striped superconductor differs fundamentally from an ordinary (uniform) superconductor, especially concerning its response to quenched randomness. We also present the results of density matrix renormalization group calculations on a model of interacting electrons in which sign oscillations of the superconducting order are established. Finally, we speculate concerning the relevance of this state to experiments in other cuprates, including recent optical studies of $La_{2-x}Sr_xCuO_4$ in a magnetic field, neutron scattering experiments in underdoped $YBa_2Cu_3O_{6+x}$ and a host of anomalies seen in STM and ARPES studies of $Bi_2Sr_2CaCu_2O_{8+\delta}$.

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Contents

1. Introduction

In this paper, we carefully characterize in terms of its broken symmetries, a novel superconducting state of matter, the ‘pair-density-wave’ (PDW), with special focus on the ‘striped superconductor’, a unidirectional PDW. We present a concrete microscopic model of interacting electrons, which we show, using density matrix renormalization group (DMRG) methods, has a striped superconducting ground state. There is an intimate relation between PDW and charge density wave (CDW) order, as a consequence of which the striped superconductor exhibits an extreme sensitivity to quenched disorder, which inevitably leads to glassy behavior. This is qualitatively different from the familiar effects of disorder in uniform superconductors.

On the experimental side, we first draw attention to a set of recently discovered transport anomalies in the high temperature superconductor, $La_{2-x}Ba_xCuO_4$, which are particularly prominent for $x = 1/8$. We will be particularly interested in the spectacular dynamical layer decoupling effects recently observed in this system [1], which indicate that the effective inter-layer Josephson coupling becomes vanishingly small with decreasing temperature. These experiments suggest that a special symmetry of the state is required to explain this previously unsuspected behavior. While no comprehensive theory of these observations currently exists, even at the phenomenological level, we show how the salient features of these observations can be straightforwardly rationalized under the assumption that $La_{2-x}Ba_xCuO_4$ is a striped superconductor. We outline some further experiments that could critically test this assumption. Finally, we speculate about the possible role of striped superconducting order as the source of a number of salient experimental anomalies in a much broader spectrum of high temperature superconductors, including recent experiments on magnetic field induced layer decoupling in $La_{2-x}Sr_xCuO_4$ [2], the notable evidence of a local gap with many characteristics of a superconducting gap in STM and ARPES experiments in $Bi_2Sr_2CaCu_2O_{8+\delta}$ [3]–[5], and, most speculatively of all, experiments indicative of time reversal symmetry breaking in the pseudo-gap regime of $YBa_2Cu_3O_{6+x}$.

The striped superconductor is a novel state of strongly correlated electronic matter in which the superconducting, charge and spin order parameters are closely intertwined with each other, rather than merely coexisting or competing. As we show here (and discussed in [6, 7]) the striped superconductor arises from the competing tendencies existing in a strongly correlated system, resulting in an inhomogeneous state in which all three forms of order are simultaneously present. The striped superconductor is thus a new type of electronic liquid crystal state [8][4]. In particular, as we shall see, the symmetry breaking pattern of the striped superconductor naturally explains the spectacular layer decoupling effects observed in experiments in $La_{2-x}Ba_xCuO_4$. In contrast, in any state with uniform superconducting order, dynamical inter-layer decoupling could only arise if a somewhat unnatural fine tuning of the inter-layer couplings led to a sliding phase [10].

The striped superconductor has an order parameter describing a paired state with non-vanishing wavevector, $\mathbf{Q}$, the ordering wavevector of the unidirectional PDW. As such, this state is closely related by symmetry to the Fulde–Ferrell (FF) [11] and Larkin–Ovchinnikov (LO) [12] states. The order parameter structure of the PDW state, involving several order parameters coupled to each other, also evokes the $SO(5)$ approach of a unified description of antiferromagnetism and a uniform d-wave superconductor [13, 14]. The relation of the present discussion to these other systems and to earlier theoretical works on the same and closely related subjects is deferred to section 8. The physics of stripe phases in the cuprate superconductors has been reviewed in [15] and more recently in [16].

It is important to stress that the macroscopically superconducting phase of the cuprates reflects the existence of a spatially uniform $\mathbf{Q} = \mathbf{0}$ component of the order parameter, whether or not there is substantial finite range superconducting order at nonzero $\mathbf{Q}$. One might therefore reasonably ask whether striped superconductivity, even if interesting in its own right, is anything but an exotic oddity, with little or no relation to the essential physics of high temperature superconductivity. The answer to this question is at present unclear, and will *not* be addressed to any great extent in the present paper. However, we wish to briefly speculate on a way in which the striped superconductor could play an essential role in the broader features of this problem. In Bardeen–Cooper–Schrieffer (BCS) theory, the superconducting

[4] Electronic liquid crystals [8] are quantum states of matter that spontaneously break some of the translation and/or rotational symmetries of an electronic system. In practice, these symmetries are typically not the continuous symmetries of free space, but rather the various discrete symmetries of the host crystal. Although an electronic smectic (stripe ordered state) has the same order parameter as a CDW or an SDW, it is a more general state that does not necessarily derive from a nesting vector of an underlying Fermi surface. The liquid crystal picture offers a broader perspective on the individual phases and on their phase transitions [9]. In particular, the way in which the PDW phase intertwines charge, spin and superconducting orders is unnatural in terms of a Fermi surface instability, but not so from the liquid crystalline perspective.

state emerges from a Fermi liquid in which the strong electronic interactions are already accounted for in the self-energy of the quasi-particles. The cuprates are different, in that the superconductivity, especially in underdoped materials, emerges from a pseudogap phase for which there is no commonly accepted model. As we will show, striped superconductivity has features in common with the pseudogap phase, such as a gapless nodal arc and antinodal gaps. We speculate that the pseudogap phase might be associated with fluctuating striped superconductivity, a state that we do not yet know how to treat. Nevertheless, analysis of the ordered PDW state and comparison to observations of stripe-ordered cuprates is a starting point. Indeed, comparison (see section 9.1) of recent photoemission results on LBCO $x = 1/8$ with transport and optical properties suggests that a 'uniform' d-wave state (i.e. one with a nonzero uniform component of the order parameter) develops on top of a striped superconductor, resulting in a fully superconducting Meissner state, albeit one with substantial coexisting short-range correlated stripe order.

The rest of this paper is organized as follows: in section 2, we give an order parameter description of the PDW state. In sections 3 and 4, we summarize the experimental evidence for this state, with section 3 focusing on the strongest case, $La_{2-x}Ba_xCuO_4$, and section 4 on other cuprates. In section 5, we discuss the microscopic mechanisms for the formation of a PDW state. In section 5.1, we implement these microscopic considerations by constructing a specific model that exhibits a PDW phase. The central conceptual ingredient is a microscopic mechanism leading to the formation of π junctions in an unidirectional PDW state, which is given in section 5.1.1 using perturbative arguments and then checked numerically using the DMRG (in section 5.1.2). A solvable microscopic model is discussed in section 5.2. The quasi-particle spectrum of the PDW state is discussed in section 5.3. Next, the Landau–Ginzburg theory of the PDW phase is discussed in section 6. In section 7, we show that the PDW state, in three-dimensional (3D) layered structures (orthorhombic and low temperature tetragonal (LTT) as well as at grain (twin) boundaries, leads to time-reversal symmetry breaking effects. In section 8, we discuss the connections that exist between the PDW state and other states discussed in the literature, particularly the Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) states. Section 9 is devoted to our conclusions.

This paper is partly a review of our recent work on the theory of the PDW state [6, 7] and of other related work, with an updated discussion of the current experimental status. However, in this paper, we have also included many new results, particularly the DMRG analysis of PDW states in strongly correlated systems of section 5.1.2, and the connection between the PDW state and non-collinear order and time-reversal symmetry breaking of section 7.

2. The order parameter of a striped superconductor

The order parameter whose nonzero expectation value defines a superconducting state is

$$\phi_{\sigma,\sigma'}(\mathbf{r},\mathbf{r}') \equiv \langle \psi^\dagger_\sigma(\mathbf{r})\psi^\dagger_{\sigma'}(\mathbf{r}')\rangle, \tag{1}$$

where $\psi^\dagger_\sigma(\mathbf{r})$ is the fermionic field operator that creates an electron with spin polarization σ at position $\mathbf{r}$. Further distinctions between different superconducting states can be drawn on the basis of the spatial and spin symmetries of ϕ. In crystalline solids, all familiar superconducting states respect the translational symmetry of the solid, $\phi(\mathbf{r}+\mathbf{R},\mathbf{r}'+\mathbf{R}) = \phi(\mathbf{r},\mathbf{r}')$, where $\mathbf{R}$ is any Bravais lattice vector. Consequently, the symmetries of the state can be classified by the irreducible representations of the point group—colloquially as s-wave, d-wave, p-wave, etc.

In the absence of spin–orbit coupling, superconducting states can be classified, as well, by their transformation under spin rotations as singlet or triplet. Finally, the superconducting state can either preserve or break time reversal symmetry (as in $p_x + \mathrm{i} p_y$).

In the presence of quenched disorder, the underlying Hamiltonian does not have any particular spatial symmetries, so the classification of distinct superconducting states by their symmetries (other than time reversal), at first seems difficult. However, there are several ways that this can be accomplished [17], of which the most obvious is to consider the symmetries of the configuration averaged order parameter

$$\bar{\phi}_{\sigma,\sigma'}(\mathbf{r}, \mathbf{r}') \equiv \overline{\langle \psi^{\dagger}_{\sigma}(\mathbf{r}) \psi^{\dagger}_{\sigma'}(\mathbf{r}') \rangle}, \tag{2}$$

where $\langle \ldots \rangle$ signifies the thermal average and $\overline{(\ldots)}$ signifies an average over realizations of the disorder configuration. It is clear, for example, that under most circumstances, a macroscopic 'phase sensitive' measurement of the symmetry of the order parameter will give [17] a result consistent with a classification based on the symmetry of the configuration averaged order parameter.

The striped superconductor is an example of a state, which has more generally called [18]–[20] a PDW, in which the translational symmetry of the crystal is spontaneously broken as well, so that $\phi(\mathbf{r}+\mathbf{R}, \mathbf{r}'+\mathbf{R})$ exhibits nontrivial dependence on $\mathbf{R}$. However, this by itself is insufficient to identify a new state of matter. In a system with coexisting CDW and superconducting order, the CDW itself introduces a new periodicity into the problem, which must generically be reflected in a spatial modulation of ϕ, as well[5]. As discussed in section 6, an analysis of the implications of a generic theory of coupled order parameters implies [7] that in a state of coexisting order, a (possibly small) modulation of the superconducting order with the same spatial period as that of the CDW will be induced. Nonetheless, in such a state, there still exists a 'dominant' uniform component to the superconducting order parameter, which we define as the spatial average of the SC order parameter:

$$\phi^{(0)}_{\sigma,\sigma'}(\mathbf{r}, \mathbf{r}') \equiv N^{-1} \sum_{\mathbf{R}} \langle \psi^{\dagger}_{\sigma}(\mathbf{r}+\mathbf{R}) \psi^{\dagger}_{\sigma'}(\mathbf{r}'+\mathbf{R}) \rangle, \tag{3}$$

where N is the number of unit cells in the system.

Instead, the pure PDW in a crystal is a state in which ϕ is nonzero, but all uniform components vanish, $\phi^{(0)}_{\sigma,\sigma'}(\mathbf{r}, \mathbf{r}') = 0$ for any $\mathbf{r}$ and $\mathbf{r}'$. Just as a CDW is often defined in terms of a fundamental harmonic, so a PDW state is characterized by the smallest value of the crystal momentum, $\mathbf{Q}$, for which

$$\phi^{(\mathbf{Q})}_{\sigma,\sigma'}(\mathbf{r}, \mathbf{r}') \equiv N^{-1} \sum_{\mathbf{R}} \exp\left[\mathrm{i}\mathbf{Q}\cdot\mathbf{R}\right] \langle \psi^{\dagger}_{\sigma}(\mathbf{r}+\mathbf{R}) \psi^{\dagger}_{\sigma'}(\mathbf{r}'+\mathbf{R}) \rangle, \tag{4}$$

has a non-vanishing expectation value[6].

[5] The problem of coexisting stripe and superconducting order in strongly correlated systems has been the focus of numerous studies in the literature. Sachdev, Vojta and co-workers have investigated this problem in detail in the context of generalized 2D t–J models in the large N approximation [21]–[23]. This problem has also been discussed in 1D systems [24].

[6] As with a uniform superconducting state, distinct PDW states with the same pattern of translation symmetry breaking can also be distinguished by different patterns of point group symmetry breaking. However, since the ordering vector (or vectors) already break the point group down to a smaller subgroup, which is then all that is left of the original symmetry for this purpose. For instance, in a tetragonal crystal, a striped superconductor with

Note that the theory of coupled order parameters [7] implies that the existence of PDW order with ordering vector $\mathbf{Q}$ generically implies the existence of CDW order with ordering vector $2\mathbf{Q}$, but so long as $\phi^0 = 0$, no CDW ordering with wavevector $\mathbf{Q}$ is expected. A 'striped superconductor' refers to the special case in which the independent ordering vectors are all parallel to each other ('unidirectional PDW').

One of the prime new characteristics of a striped superconductor that is different from a uniform superconductor is its complex sensitivity to quenched disorder. As we shall see, for much the same reasons that disorder destroys long-range CDW order, under most relevant circumstances, even weak disorder causes the configuration averaged PDW order parameter to vanish:

$$\bar{\phi}_{\sigma,\sigma'}(\mathbf{r},\mathbf{r}') = 0. \tag{5}$$

However, as in the case of an XY spin-glass, this is not the whole story: it is possible to define an analogue of the Edwards–Anderson order parameter,

$$Q_{\sigma,\sigma'}(\mathbf{r},\mathbf{r}') \equiv \overline{|\langle\psi^\dagger_\sigma(\mathbf{r})\psi^\dagger_{\sigma'}(\mathbf{r}')\rangle|^2}, \tag{6}$$

which vanishes in the normal high temperature phase, but which can be nonzero in a low temperature superconducting glass phase, where one exists. Moreover, in such a phase, as we will see, we generically expect time-reversal symmetry to be spontaneously broken. In analogy with the XY spin-glass, we expect that in 2D, the superconducting glass phase is stable only at $T = 0$ and for weak enough disorder, although in 3D it can exist below a nonzero superconducting glass transition temperature[7].

There is one more extension that is useful—we define a charge $4e$ superconducting order parameter:

$$\phi^{(4)}(1,2,3,4) \equiv \langle\psi^\dagger_{\sigma_1}(\mathbf{r}_1)\psi^\dagger_{\sigma_2}(\mathbf{r}_2)\psi^\dagger_{\sigma_3}(\mathbf{r}_3)\psi^\dagger_{\sigma_4}(\mathbf{r}_4)\rangle, \tag{7}$$

where we have introduced a compact notation in which $1 \equiv (\sigma_1, \mathbf{r}_1)$, etc. Naturally, in any state with charge $2e$ superconducting order, $\phi_{\sigma,\sigma'}(\mathbf{r},\mathbf{r}') \neq 0$, some components of the charge $4e$ order parameter will also be nonzero. This can be seen from the theory of coupled order parameters presented in section 6. At mean-field level, it can be seen by applying Wick's theorem to the expression in equation (7) to express $\phi^{(4)}$ as a sum of pairwise products of ϕs: $\phi^{(4)}(1,2,3,4) \sim \phi(12)\phi(34) + \phi(14)\phi(23) - \phi(13)\phi(24)$.

There are two reasons to consider this order parameter. In the first place, it is clear from the above that even in the PDW state, although the uniform component of ϕ vanishes, the uniform component of $\phi^4 \sim \phi^{\mathbf{Q}}\phi^{-\mathbf{Q}} \neq 0$. More importantly, $\phi^{(4)}$ ordering can be more robust than the PDW ordering. Specifically, under some circumstances [7, 25], it is possible for thermal or quantum fluctuations to destroy the PDW order by restoring translational symmetry without restoring large-scale gauge symmetry; in this case, appropriate components of $\phi^{(4)}$ remain

$\mathbf{Q}$ along the x-direction, can be classified as having s-wave or d_{xy}-wave symmetry, based on whether or not the order parameter changes sign under reflection through a symmetry plane parallel to the x-axis, but any distinction one would like to draw between a striped version of an s-wave and a $d_{x^2-y^2}$-wave superconductor are in precise, not based on broken symmetries but on quantitative differences in local pairing correlations, and so do not define distinct phases of matter. However, a checkerboard PDW, with symmetry related ordering vectors $\mathbf{Q}$ and $\mathbf{Q}'$ along the x and y axes, respectively, can be classified as s-wave or $d_{x^2-y^2}$-wave, depending on how it transforms under rotation by $\pi/2$.

[7] In [7], the possibility is discussed that in 3D there might also exist a superconducting version of a Bragg glass phase, in which ϕ exhibits quasi-long-range order. We have not further studied this potentially interesting state.

nonzero, even though ϕ vanishes identically. This is the only potentially realistic route we know of to charge $4e$ superconductivity[8].

In the absence of spin–orbit coupling, distinct phases with translationally invariant charge $4e$ ordering can be classified according to the total spin of the order parameter, which in this case can be spin 2, 1, or 0. Manifestly, any charge $4e$ superconducting state that results from the partial melting of a singlet PDW will itself have spin 0. As with paired superconductors, the charge $4e$ order parameter can also be classified according to its transformation properties under action of the point-group of the host crystal. For instance, in a crystal with a C_4 symmetry, taking the points $\mathbf{r}_j$ to lie on the vertices of a square, the transformation properties of $\phi^{(4)}$ under rotation by $\pi/2$ can be used to classify distinct spin-0 states as being d-wave or s-wave.

The definitions given here in terms of possible behaviors of the order parameter are natural from a taxonomic viewpoint. In particular, the striped superconductor seems at first to be a rather straightforward generalization of familiar uniform superconducting states. However, both at the microscopic level of the 'mechanism' of formation of such a state, and at the phenomenological level of macroscopically observable implications of the state, the problem is full of subtleties and surprises, as discussed below.

3. Striped superconductivity in $La_{2-x}Ba_xCuO_4$ and the 214 family

We now summarize some of the observations that lead to the conclusion that $La_{2-x}Ba_xCuO_4$ with $x = 1/8$ is currently the most promising candidate experimental system as a realization of a striped superconductor [1, 6].

Firstly, the existence of 'stripe order' is unambiguous. It is well known (from neutron [27]–[29] and x-ray [28, 30, 31] scattering studies) that unidirectional CDW (charge-stripe) and SDW (spin-stripe) orders exist in $La_{2-x}Ba_xCuO_4$. Such spin and charge stripe orders were originally studied in $La_{1.48}Nd_{0.4}Sr_{0.12}CuO_4$ [32]–[34], and they have now been confirmed in $La_{1.8-x}Eu_{0.2}Sr_xCuO_4$ [35]–[37]. Furthermore, substantial spin stripe order has been observed in $La_2CuO_{4.11}$ [38], Zn-doped $La_{2-x}Sr_xCuO_4$ [39] and in the spin-glass regime of $La_{2-x}Sr_xCuO_4$ $(0.02 < x < 0.055)$ [40]. While the spin-stripe order in underdoped but superconducting $La_{2-x}Sr_xCuO_4$ $(0.055 < x < 0.14)$ is weak in the absence of an applied magnetic field, it has been observed [41]–[43] that readily accessible magnetic fields (which partially suppress the superconducting order) produce well-developed and reproducible spin-stripe order. For $La_{2-x}Ba_xCuO_4$ with $x = 1/8$, the charge ordering temperature is 53 K and the spin ordering temperature is 40 K [28].

In addition to spin and charge ordering, $La_{2-x}Ba_xCuO_4$ with $x = 1/8$ exhibits transport and thermodynamic behavior that is both striking and complex. We will not rehash all of the details here (see [28]); however, there are two qualitative features of the data on which we would like to focus: (i) with the onset of spin-stripe order at 40 K,[9] there is a large (in magnitude) and strongly temperature-dependent enhancement of the anisotropy of the resistivity and other properties, such that below 40 K the in-plane charge dynamics resemble those of a superconductor, while in the c-direction the system remains poorly metallic. The most extreme illustration of this occurs

[8] It is possible to cook up models in which charge $4e$ superconductivity arises in systems in which electrons can form four-particle bound states, but do not form two-particle or many particle bound states (phase separation)—see, for example [26]. However, this involves unrealistically strong attractive interactions and an unpleasant amount of fine tuning of parameters.

[9] Static charge stripe order onsets at 54 K, at the LTT/LTO transition [30].

in the temperature range $10\,\mathrm{K} < T < 16\,\mathrm{K}$, in which the in-plane resistivity is immeasurably small, while the c-axis resistivity is in the 1–10 mΩ-cm range, so that the resistivity anisotropy ratio is consistent with infinity. (ii) Despite the fact that signatures of superconductivity onset at temperatures in excess of 40 K, and that angle resolved photoemission (ARPES) has inferred a 'gap' [44, 45] of order 20 meV, the fully superconducting state (i.e. the Meissner effect and zero resistance in all directions) only occurs below a critical temperature of 4 K. It is very difficult to imagine a scenario in which a strong conventional superconducting order develops locally on such high scales, but fully orders only at such low temperatures in a system that is 3D, non-granular in structure, and not subjected to an external magnetic field.

Evidence that similar, although somewhat less extreme transport and thermodynamic anomalies accompany stripe ordering can be recognized, in retrospect, in other materials in the 214 family. For example, in $La_{2-x}Sr_xCuO_4$ with $x = 0.08$ and 0.10, the anisotropy of the resistivity (c-axis versus in-plane) rapidly grows toward 10^4 as the superconducting T_c is approached from above [46]. In the case of $La_{1.6-x}Nd_{0.4}Sr_xCuO_4$, evidence for dynamical layer decoupling is provided by measurements of the anisotropic onset of the Meissner effect [47]. In contrast, the resistivity ratio in this material [48] only reaches 10^3; this may be limited by enhanced in-plane resistivity due to disorder [49]. Moreover, an unexpectedly strong layer decoupling in the charge dynamics produced by the application of a transverse magnetic field in $La_{2-x}Sr_xCuO_4$ has been observed [2], but only in the underdoped range of x where the magnetic field also induces spin stripe order [41, 43].

We shall see that the anomalous sensitivity of a striped superconductor to quenched disorder can account for the existence of a broad range of temperatures between the onset of strongly developed superconducting correlations on intermediate scales and the actual macroscopic transition temperature to a state of long-range coherence. Moreover, given the crystal structure of the LTT phase of $La_{2-x}Ba_xCuO_4$, there is a special symmetry of the striped superconducting state which produces interlayer decoupling. Specifically, because the stripes in alternate planes are oriented perpendicular to one another (as shown in figure 5(a)), there is no first-order Josephson coupling between neighboring planes. Indeed, analogous features of the spin-stripe order, which have gone largely unnoticed in the past, are accounted for [6] by the same geometric features of the striped state. When spin-stripe order occurs, the in-plane correlation length can be very long, in the range of 100–600 Å [28, 38], but the interplanar correlation length is never more than a few Å [38, 50], a degree of anisotropy that cannot be reasonably explained simply on the basis of the anisotropy in the magnitude of the exchange couplings [51]. Furthermore, despite the presence of long correlation lengths, true long-range spin-stripe order has never been reported. It will be made clear that the superconducting stripe order and the spin stripe order share the same periodicity and the same geometry, so both the interlayer decoupling and the suppression by quenched disorder of the transition to a long-range ordered state can be understood as arising from the same considerations applied to both the unidirectional SDW and PDW orders.

4. Experiments in other cuprates

Data that clearly reveal the existence of spin-stripe order, or that provide compelling evidence of PDW order in other families of cuprates is less extensive. However, considerable evidence of a tendency to spin-stripe order in $YBa_2Cu_3O_{6+x}$ has started to accumulate [15], and there are some persistent puzzles concerning the interpretation of various experiments in a number of cuprates that, we would like to speculate, may reflect the presence of PDW order. In this section, we will mention some of these puzzles, and will return to discuss why they may be indicative of PDW order in section 7.

$YBa_2Cu_3O_{6+x}$ is often regarded as the most ideal cuprate, having minimal structural and chemical disorder, and less tendency to stripe or any other type of charge ordering than the 214 cuprates. (Sometimes the cuprates with the highest transition temperatures, such as $HgBa_2Cu_3O_{4+\delta}$, are viewed as being similarly pristine.) However, in its underdoped regime it is well known that $YBa_2Cu_3O_{6+x}$ exhibits temperature-dependent in-plane anisotropic transport [52] as well as fluctuating spin-stripe order [15, 53, 54]. Recent neutron scattering experiments have provided strong evidence that underdoped $YBa_2Cu_3O_{6+x}$ (with $x \sim 0.45$) has nematic order below a critical temperature $T_c \sim 150$ K [55]. Even more recent neutron scattering experiments by Hinkov *et al* [56] on the same sample find that a modest c-axis magnetic field stabilizes an incommensurate static spin ordered state, detectable as a pair of peaks in the elastic scattering displaced by a distance in the crystallographic a-direction from the Neél ordering vector. Given the new found evidence of spin-stripe related structures in $YBa_2Cu_3O_{6+x}$, it is plausible that here, too, striped superconductivity may occur. However, the differences in the 3D crystal structure, and especially the weak orthorhombicity, would make the macroscopic properties of a PDW distinctly different in $YBa_2Cu_3O_{6+x}$ than in the 214 cuprates.

A remarkable recent discovery is that underdoped $YBa_2Cu_3O_{6+x}$ appears to exhibit signatures of spontaneous time-reversal symmetry breaking (at zero magnetic field) below a critical temperature comparable to that for the nematic ordering [57, 58] ($HgBa_2Cu_3O_{4+\delta}$ [59] exhibits similar signatures). Various theoretical scenarios for the existence of time-reversal symmetry breaking predated these experiments, and so in some sense predicted them [60, 61]. However, given that *both* nematic order and time-reversal symmetry breaking are seemingly present simultaneously in the same samples with comparable critical temperatures, it is reasonable to hope that both phenomena have an underlying common explanation. If we think of the superconducting order parameter as an XY pseudo-spin, then the PDW order is a form of collinear antiferromagnetism, and time-reversal symmetry breaking corresponds to non-collinear order of the pseudospins. As we will show in section 7, weak time reversal symmetry breaking can occur in a PDW state due to various patterns of geometric frustration in 3D or as a consequence of the existence of certain types of defects, such as twin boundaries (see, also [7]). There is a large body of STM and ARPES data, especially on $Bi_2Sr_2CaCu_2O_{8+\delta}$ and $Bi_2Sr_2CuO_{6+\delta}$, which has revealed a surprisingly rich and difficult to interpret set of spectral features associated with the d-wave superconducting gap and a d-wave pseudo-gap whose origin is controversial. Indeed, there is a clear 'nodal–anti-nodal dichotomy' [62, 63] in the behavior of the measured single-particle spectral functions. Some aspects of the data are suggestive that there is a single superconducting origin of all gap features, with anisotropic effects of superconducting fluctuations leading to the observed dichotomy. Other aspects suggest that there are at least two distinct origins of the near-nodal and the antinodal gaps. It is possible that PDW ordering tendencies can synthesize both aspects of the interpretation. In the presence of both uniform and PDW superconducting order, there are two distinct order parameters, both of which open gaps on portions of the Fermi surface, but they are both superconducting, and so they can smoothly evolve into one another. (Note that an early study [64] of modulated structures seen in STM [3, 65, 66] concluded that they could be understood in terms of just such a two-superconducting-gap state.)

More generally, one of the most remarkable features of the pseudo-gap phenomena is the existence of what appears to be superconducting fluctuations, detectable [67]–[69] for instance in the Nernst and magnetization signal, over a surprisingly broad range of temperatures and doping concentrations. At a broad-brush level [70], these phenomena are a consequence of a phase stiffness scale that is small compared to the pairing scale. However, it is generally difficult to understand the existence of such a broad fluctuational regime on the basis of any sensible microscopic considerations. The glassy nature of the ordering phenomena in a PDW may hold the key to this central paradox of HTC phenomenology, as it gives rise to an intrinsically broad regime in which superconducting correlations extend over large, but not infinite, distances.

5. Microscopic considerations

At a first glance, the fact that a PDW phase can arise in a microscopic model seems mysterious. Weak attractive interactions in a Fermi liquid with a generic Fermi surface are known to lead to a condensate of Cooper pairs with zero center-of-mass momentum. Moreover, in the absence of external magnetic fields, the ground state of a system of bosons is always nodeless. From these considerations, it is clear that to get a PDW ground state, strong interactions are needed, and physics in length scales smaller than the pair size ξ_0 must be involved, i.e. the fact that Cooper pairs are made out of paired fermions is important [71].

In this section, we construct a microscopic model for a unidirectional (striped) PDW. The key ingredients of the model are superconducting stripes that are weakly coupled by tunneling through insulating regions. Time reversal symmetry dictates that the effective Josephson coupling J between the superconducting regions is real; a PDW state is formed when the sign of J is negative. Negative Josephson couplings (or 'π junctions') can arise due to strong correlations in superconductor–insulator–superconductor (SIS) junctions [71]–[73] or due to the internal symmetry (e.g. d-wave) of the order parameter [74, 75]. The present model is of the former type; it differs from previously studied examples in that the Josephson junctions are extended, i.e. the Josephson coupling is proportional to the 'area' of the junction.

In this section, we build on the results of [7], where a model of an extended π junction was presented. In order to make analytic studies possible, our previous work involved considering an extreme limit in which the tunneling between the superconducting and the insulating regions was small enough to be treated by perturbation theory. Moreover, we were limited to a somewhat artificial model in which the motion of electrons in the insulating regime was entirely neglected. In this section, we first summarize the perturbative results of [7], and then present new and more extensive numerical (DMRG) results for an extended SIS junction. Under some circumstances, $J > 0$, but we also find a considerable region of parameter space where $J < 0$. Finally, we discuss how this result can be generalized to an infinite array of junctions, forming a 2D unidirectional PDW.

5.1. A solved model

Let us consider the following explicit model for a single SIS junction. The three decoupled subsystems are described by the Hamiltonian

$$H_0 = H_L + H_B + H_R, \tag{8}$$

The right (R) and left (L) superconducting regions and the barrier (B) region are 1D Hubbard models,

$$H_\alpha = \sum_{i\sigma} \left(-t c^\dagger_{\alpha,i,\sigma} c_{\alpha,i+1,\sigma} + \mathrm{h.c.} - \mu_\alpha n_{\alpha,i} \right) + U_\alpha \sum_i n_{\alpha,i,\uparrow} n_{\alpha,i,\downarrow}. \tag{9}$$

$c^\dagger_{\alpha,i+1,\sigma}$ is a creation operator of an electron on chain $\alpha = \mathrm{L}, \mathrm{R}$ or B at site i with spin σ, and we have introduced the notation $n_{\alpha,i,\sigma} = c^\dagger_{\alpha,i,\sigma} c_{\alpha,i,\sigma}$ and $n_{\alpha,i} = \sum_\sigma n_{\alpha,i,\sigma}$. The left and right superconducting chains are characterized by a negative $U_\mathrm{R} = U_\mathrm{L} = -|U_{\mathrm{L,R}}|$, while the insulating barrier has a positive $U_\mathrm{B} > 0$. The chemical potentials of the left and right superconductors are the same, $\mu_\mathrm{R} = \mu_\mathrm{L}$, but different from μ_B, which is tuned such that the barrier chain is half filled (and therefore insulating).

The three subsystems are coupled together by a single-particle hopping term,

$$H' = -t' \sum_{i,\sigma} \left[c^\dagger_{\mathrm{L},i,\sigma} c_{\mathrm{B},i,\sigma} + c^\dagger_{\mathrm{R},i,\sigma} c_{\mathrm{B},i,\sigma} + \mathrm{h.c.} \right]. \tag{10}$$

The left and right chains are characterized by a spin gap and by dominant superconducting fluctuations, as a result of their negative Us. The inter-chain hopping term H' induces a finite Josephson coupling between the local superconducting order parameters of the two chains, via virtual hopping of a Cooper pair through the barrier chain.

5.1.1. Perturbative analysis. For completeness, let us briefly review the perturbative treatment of the inter-chain hopping term (10) given in [7]. The leading (fourth order) contribution to the Josephson coupling is given by

$$J = \frac{(t')^4}{\beta} \int \mathrm{d}1\, \mathrm{d}2\, \mathrm{d}3\, \mathrm{d}4 F_\mathrm{L}(1,2) F^\star_\mathrm{R}(4,3) \Gamma(1,2;3,4) \tag{11}$$

where $1 \equiv (\tau_1, i_1)$, etc,

$$\int \mathrm{d}1 \equiv \sum_{i_1} \int_0^\beta \mathrm{d}\tau_1 \tag{12}$$

(in the limit $\beta \to \infty$) and

$$\begin{aligned} F_\alpha(1,2) &\equiv \left\langle T_\tau \left[c^\dagger_{\alpha,i_1,\uparrow}(\tau_1) c^\dagger_{\alpha,i_2,\downarrow}(\tau_2) \right] \right\rangle, \\ \Gamma(1,2;4,3) &\equiv \left\langle T_\tau \left[c^\dagger_{i_1,\uparrow}(\tau_1) c^\dagger_{i_2,\downarrow}(\tau_2) c_{i_3,\downarrow}(\tau_3) c_{i_4,\uparrow}(\tau_4) \right] \right\rangle, \end{aligned} \tag{13}$$

where we have made the identification $c^\dagger_{i,\sigma} \equiv c^\dagger_{B,i,\sigma}$. Our purpose is to determine the conditions under which $J < 0$. For the sake of simplicity, let us consider the case in which the gap to remove a particle from the barrier, Δ_h, satisfies $\Delta_\mathrm{s} \ll \Delta_\mathrm{h} \ll \Delta_\mathrm{p}$, where Δ_s is the spin gap on the superconducting chains, and Δ_p is the gap to insert a particle in the barrier. These conditions can be met by tuning appropriately the chemical potentials on the three chains and setting U_B to be sufficiently large.

In [7], it is shown that, quite generally, J can be written as a sum of two terms

$$J = J_1 + J_2, \tag{14}$$

where in terms of the spin–spin correlation function, $\langle\vec{S}(1)\cdot\vec{S}(2)\rangle$ of the barrier chain,

$$J_1 = \frac{(t')^4}{4\beta\,(\Delta_\mathrm{h})^2}\int \mathrm{d}1\,\mathrm{d}2\,|F_\mathrm{L}(1,2)|^2\,,$$
$$J_2 = -\frac{3(t')^4}{4\beta\,(\Delta_\mathrm{h})^2}\int \mathrm{d}1\,\mathrm{d}2\,|F_\mathrm{L}(1,2)|^2\,\langle\vec{S}(1)\cdot\vec{S}(2)\rangle. \tag{15}$$

Explicitly, $J_1 > 0$, while for generic circumstances one finds that $J_2 < 0$. The overall sign of J is therefore non-universal and determined by which term is bigger. We can, however, identify the conditions under which J_2 dominates. Upon a Fourier transform, $|F_\mathrm{L}(1,2)|^2$ is peaked around two values of the momentum q, at $q = 0$ and $2k_\mathrm{F}$, in which $2k_\mathrm{F} = \pi n$, where n is the number of electrons per site in the left and right chains. Since, upon Fourier transforming, $\langle\vec{S}(1)\cdot\vec{S}(2)\rangle$ is peaked at momenta $q = 0$ and π (as can be seen, e.g. from a bosonized treatment of the half-filled chain), we expect that J_2 in equation (15) is maximized when $n = 1$, i.e. when *the superconducting chains are half filled*. The requirement of proximity to half filling becomes less and less stringent when $|U_{\mathrm{R,L}}|$ is increased, since then the gap Δ_s in the superconducting chains increases and the peaks in $|F_\mathrm{L}(q)|^2$ become more and more broad. These qualitative expectations are confirmed by numerical DMRG simulations, presented in the next subsection.

5.1.2. Numerical results. We have performed DMRG simulations of the model $H = H_0 + H'$ in equations (8) and (10), with the following parameters: $t = t' = 1$, $\mu_\mathrm{B} = 6$, $U_\mathrm{B} = 10$ and variable $U_\mathrm{L} = U_\mathrm{R} \equiv -|U_{\mathrm{L,R}}|$ and $\mu_\mathrm{L} = \mu_\mathrm{R} \equiv \mu_{\mathrm{L,R}}$. Most of the calculations were done with systems of size 3×24. In a small number of parameter sets, we have verified that the results do not change when we increase the system size to 3×36. Up to $m = 1600$ states were kept in these calculations. The results (both ground state energies and local measurements) were extrapolated linearly in the truncation error [76], which is in the range 10^{-5}–10^{-6}.

In order to measure the sign of the Josephson coupling from the calculations, we have applied pairing potentials on the left and right chains, adding the following term to equation (8):

$$H_\mathrm{pair} = -\sum_{i,\alpha=L,R}\Delta_\alpha c^\dagger_{\alpha,i,\uparrow}c^\dagger_{\alpha,i,\downarrow} + \mathrm{h.c.} \tag{16}$$

In the presence of this term, the number of particles in the calculation is conserved only modulo 2. The average particle number is fixed by the overall chemical potential. Two methods were employed to determine the sign of J. (a) Pairing potentials of either the same sign, $\Delta_\mathrm{R} = \Delta_\mathrm{L}$, or of opposite signs, $\Delta_\mathrm{R} = -\Delta_\mathrm{L}$, were applied to the two chains. The ground state energies in the two cases are E_+ and E_-, respectively. Then $J = E_- - E_+$ [77]. (b) A pairing potential was applied to the left chain only, $\Delta_\mathrm{L} > 0$, while $\Delta_\mathrm{R} = 0$. The induced pair field

$$\phi_{\mathrm{R},i} \equiv \left\langle c^\dagger_{\mathrm{R},i,\uparrow}c^\dagger_{\mathrm{R},i,\downarrow}\right\rangle \tag{17}$$

on the right chain was measured. Its sign indicates the sign of J. This is the method we used in most calculations. Method (a) was applied to a small number of points in parameter space, and found to produce identical results to those of method (b) for the sign of J.

Figure 1 shows the local expectation values of the particle number, spin and pair field operators along the three chains for $|U_{\mathrm{L,R}}| = 2.5$ and various values of μ_{LR}. The density of electrons on the left and right chains increases as μ_{LR} increases, while the density on the middle

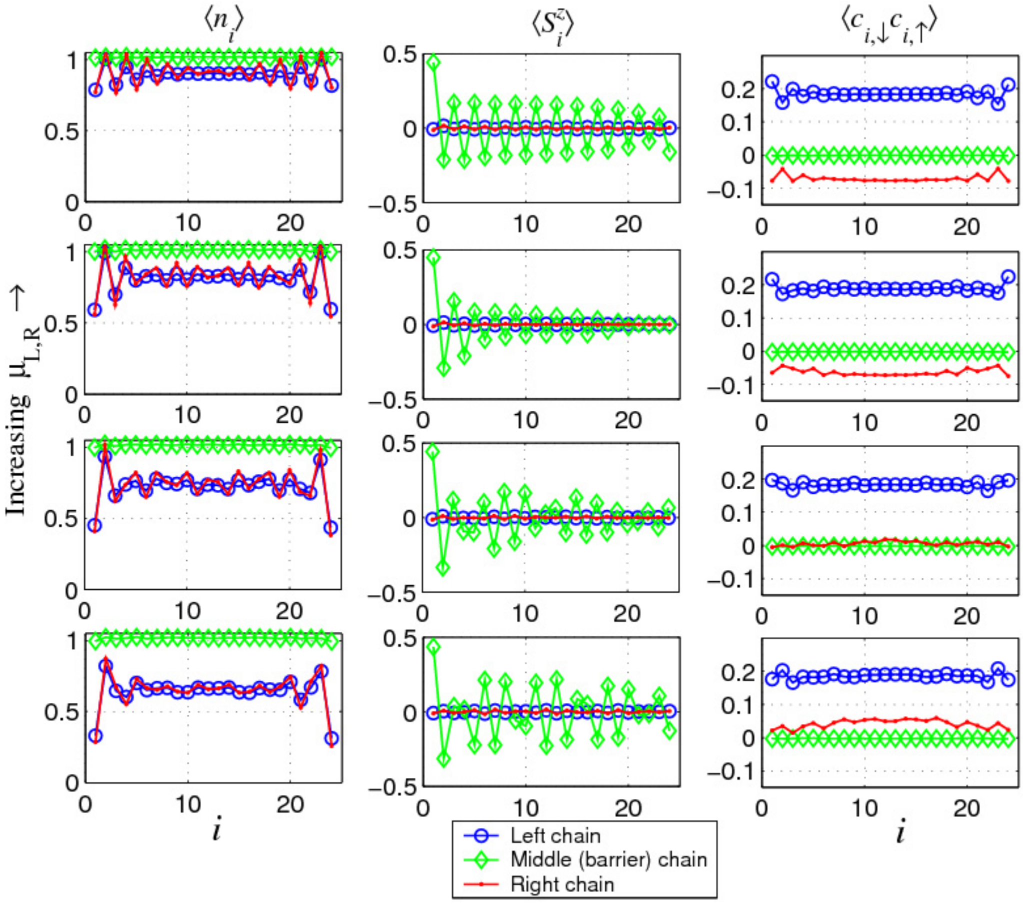

Figure 1. The left, middle and right columns show the average density $\langle n_i \rangle$, z component of the spin $\langle S_i^z \rangle$ and pair field $\langle c_{i,\downarrow} c_{i,\uparrow} \rangle$, respectively, as a function of position i along the chains, calculated by DMRG for 3×24 systems. Circles, diamonds and dots refer to the left, middle and right chains, respectively. The attractive interaction on the superconducting (left and right) chains is $|U_{\mathrm{L,R}}| = 2.5$ in all calculations. A pairing term (equation (16)) was applied with $\Delta_{\mathrm{L}} = 0.1$ and $\Delta_{\mathrm{R}} = 0$. The other model parameters are given by: $t = t' = 1$, $\mu_{\mathrm{B}} = 6$ and $U_{\mathrm{B}} = 10$. Each row corresponds to a single calculation with a specific value of the chemical potential $\mu_{\mathrm{L,R}}$ (and hence a particular particle density) on the superconducting chains.

chain is kept close to one particle per site. A positive pair potential of strength $\Delta_{\mathrm{L}} = 0.1$ was applied on the left chain, inducing a positive pair field $\phi_{\mathrm{L}} = \langle c^{\dagger}_{\mathrm{L},i,\uparrow} c^{\dagger}_{\mathrm{L},i,\downarrow} \rangle > 0$, while $\Delta_{\mathrm{R}} = 0$. A negative-induced pair field ϕ_{R} on the right chain indicates that the effective Josephson coupling J between the left and right chains is negative. Note that J is negative for the two upper rows (in which $\langle n_{\mathrm{L,R}} \rangle = 0.9$ and 0.83 respectively), while for the two lower rows (where $\langle n_{\mathrm{L,R}} \rangle = 0.75$ and 0.66) it becomes positive. This is in agreement with our expectation, based on the perturbative analysis of the previous subsection, that when the superconducting chains are close to half filling ($\langle n \rangle = 1$), the negative J_2 term dominates and the overall Josephson coupling is more likely to become negative.

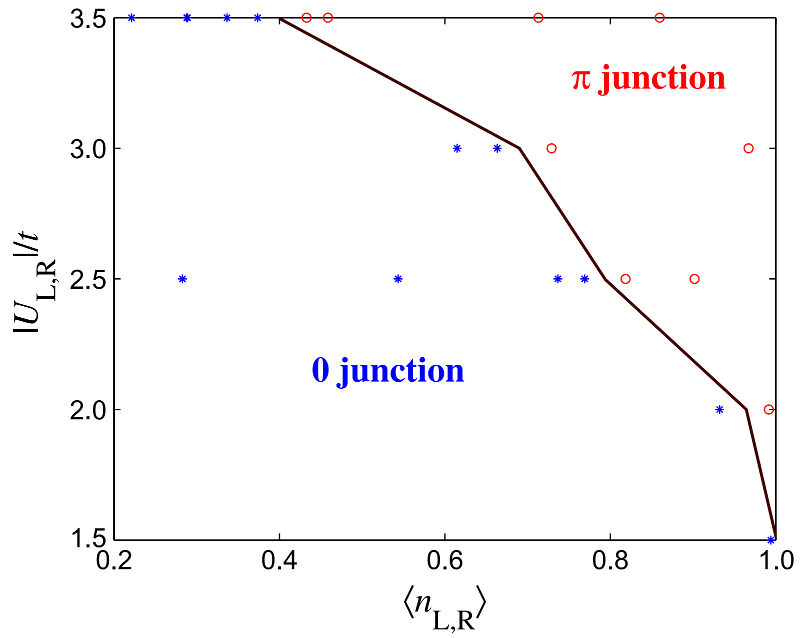

Figure 2. Phase diagram of the three chain model (equations (8) and (10)) from DMRG, as a function of $|U_{\mathrm{L,R}}|$, the attraction on the left and right chains, and $\langle n_{\mathrm{L,R}} \rangle$, the number of electrons per site on the left and right chains. The following parameters were used: $t = t' = 1$, $\mu_{\mathrm{B}} = 6$ and $U_{\mathrm{B}} = 10$. On the middle chain, $\langle n_{\mathrm{B}} \rangle \approx 1$ in all cases. The symbols show the points that were simulated.

The middle column in figure 1 shows the expectation value of the z component of the spin along the three chains. In order to visualize the spin correlations, a Zeeman field of strength $h = 0.5$ was applied to the $i = 1$ site of the middle chain. The results clearly indicate that the two outer chains have a spin gap (and therefore have a very small induced moment), while in the half filled middle chain there are strong antiferromagnetic correlations. Interestingly, as the chemical potential on the outer chains is decreased, the spin correlations along the middle chain become incommensurate. This seems to occur at the same point where the Josephson coupling changes sign (between the second and third row in figure 1). This phenomenon was observed for other values of $U_{\mathrm{L,R}}$, as well. The incommensurate correlations can be explained by the further-neighbor Ruderman–Kittel–Kasuya–Yosida (RKKY)-like interaction which are induced in the middle chain by the proximity of the outer chains. Upon decreasing the inter-chain hopping t' to 0.7, the spin correlations in the middle chain become commensurate over the entire range of $\mu_{\mathrm{L,R}}$ (and the region of negative J increases). Why $J > 0$ seems to be favored by incommensurate correlations in the middle chain is not clear at present.

Figure 2 shows the phase diagram of the three chain model as a function of the density $\langle n_{\mathrm{L,R}} \rangle$ and the attractive interaction $|U_{\mathrm{L,R}}|$ on the outer chains. In agreement with the perturbative considerations, proximity to $\langle n_{\mathrm{L,R}} \rangle = 1$ and large $|U_{\mathrm{L,R}}|$ (compared to the bandwidth $4t$) both favor a negative Josephson coupling between the outer chains.

5.2. Extension to an infinite array of coupled chains

The model presented in the previous subsection includes only a single extended π junction. However, it is straightforward to extend this model to an infinite number of coupled chains with alternating U. So long as the Josephson coupling across a single junction is small, we expect that the extension to an infinite number of chains will not change it by much. Therefore, in the appropriate parameter regime in figure 2, the superconducting order parameter changes sign from one superconducting chain to the next, forming a striped superconductor (or unidirectional PDW).

In order to demonstrate that there are no surprises in going from three chains to 2D, we have performed a simulation for a 5×12 system composed of five coupled chains with alternating $U = -3, 8, -3, 8, -3$. As before, the density of particles on the $U = 8$ chains was kept close to $\langle n \rangle = 1$, making them insulating, while the density of particles on the $U = -3$ (superconducting) chains was varied. As before, the hopping parameters are $t = t' = 1$. A pair field $\Delta = 0.1$ was applied on the bottom superconducting chain, and the induced superconducting order parameter was measured across the system. Up to $m = 2300$ states were kept. Figure 3 shows the induced pair fields and the expectation value of S^z throughout the system in two simulations, in which the average density of particles on the superconducting chains was $\langle n_{\text{sc}} \rangle = 0.7$ and 0.47. As expected according to the phase diagram in figure 2, in the $\langle n_{\text{sc}} \rangle = 0.7$ case the order parameter changes sign from one superconducting chain to the next, while in the $\langle n_{\text{sc}} \rangle = 0.47$ case the sign is uniform. It therefore seems very likely that under the right conditions, the 2D alternating chain model forms a striped superconductor.

5.3. Quasi-particle spectrum of a striped superconductor

The quasi-particle spectrum of a uniform superconductor is typically either fully gapped, or gapless only on isolated nodal points (or nodal lines in 3D). This is a consequence of the fact that, due to time reversal symmetry, the points $\mathbf{k}$ and $-\mathbf{k}$ have the same energy. Since the order parameter carries zero momentum, any point on the Fermi surface is thus perfectly nested with its time reversed counterpart, and is gapped unless the gap function $\Delta_{\mathbf{k}}$ vanishes at that point.

For a striped superconductor, the situation is different. Since the order parameter has nonzero momentum $\mathbf{Q}$, only points that satisfy the nesting condition $\varepsilon_{\mathbf{k}} = \varepsilon_{-\mathbf{k}+\mathbf{Q}}$, where $\varepsilon_{\mathbf{k}}$ is the single particle energy, are gapped for an infinitesimally weak order. Therefore, generically there are portions of the Fermi surface that remain gapless [78]. This is similar to the case of a CDW or SDW, which generically leave parts of the (reconstructed) Fermi surface gapless, until the magnitude of the order parameter reaches a certain critical value. The spectral properties of a striped superconductor were studied in detail in [79, 80].

As an illustration, we present in figure 4 the spectral function $A(\mathbf{k}, \omega = 0)$ of a superconductor with band parameters fitted to the ARPES spectrum of LSCO [45] and a striped superconducting order parameter with a single wavevector $\mathbf{Q} = (2\pi/8, 0)$ of magnitude $\Delta_{\mathbf{Q}} = 60\,\text{meV}$. Note that a portion of the Fermi surface around the nodal (diagonal) direction remains ungapped (a 'Fermi arc' [81, 82]), while both antinodal directions (around $(\pi, 0)$ and $(0, \pi)$) are gapped. The Fermi arc is in fact the back side of a reconstructed Fermi pocket, but only the back side has a sizable spectral weight [83]. Its length depends on the magnitude of the order parameter: the larger the magnitude of $\Delta_{\mathbf{Q}}$, the smaller is the arc. Note that $A(\mathbf{k}, \omega = 0)$ is not symmetric under rotation by $\pi/2$, because the striped superconducting order breaks rotational symmetry. However, in a system with an LTT symmetry (such as LBCO near

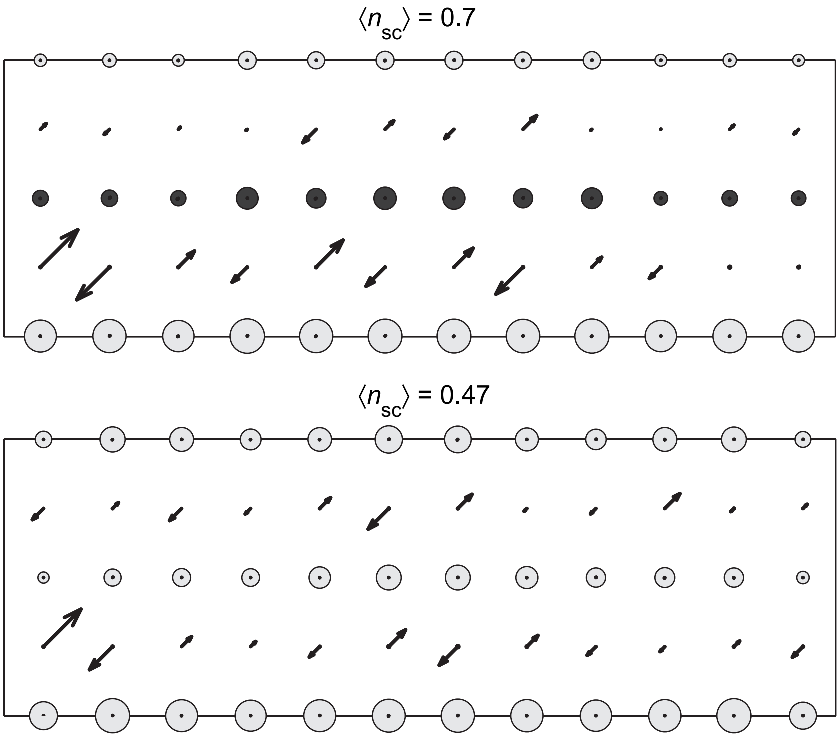

Figure 3. The average pair field $\phi = \langle c^\dagger_\uparrow c^\dagger_\downarrow \rangle$ and spin $\langle S^z \rangle$ measured in DMRG calculations for the 5×12 systems with alternating U, as described in the text. The two calculations differ in the average density of electrons in the superconducting chains, $\langle n_{\rm sc} \rangle$ (upper panel: $\langle n_{\rm sc} \rangle = 0.7$, lower panel: $\langle n_{\rm sc} \rangle = 0.47$). The size of the circles indicate the magnitude of ϕ, and their color indicate its sign (bright-positive sign, dark-negative sign). The arrows indicate the magnitude and sign of $\langle S^z \rangle$. In each calculation, a positive pair field $\Delta = 0.1$ was applied to the lower chain, and a Zeeman field $h = 0.1$ was applied to the leftmost site of the second row from the bottom.

$x = 1/8$ doping) both orientations of stripes are present, and an ARPES experiment would see the average of the picture in figure 4 and its rotation by $\pi/2$.

6. Order parameter theory of the PDW state

In this section, we explore the aspects of the theory of a PDW that can be analyzed without reference to microscopic mechanisms. We focus on the properties of ordered states at $T = 0$, far from the point of any quantum phase transition, where for the most part fluctuation effects can be neglected. (The one exception to the general rule is that, where we discuss effects of disorder, we will encounter various spin-glass related phases where fluctuation effects, even at $T = 0$, can qualitatively alter the phases.) For simplicity, most of our discussion is couched in terms of a Landau theory, in which the effective free energy is expanded in powers of the order parameters; this is formally *not* justified deep in an ordered phase, but it is a convenient way to exhibit the consequences of the order parameter symmetries.

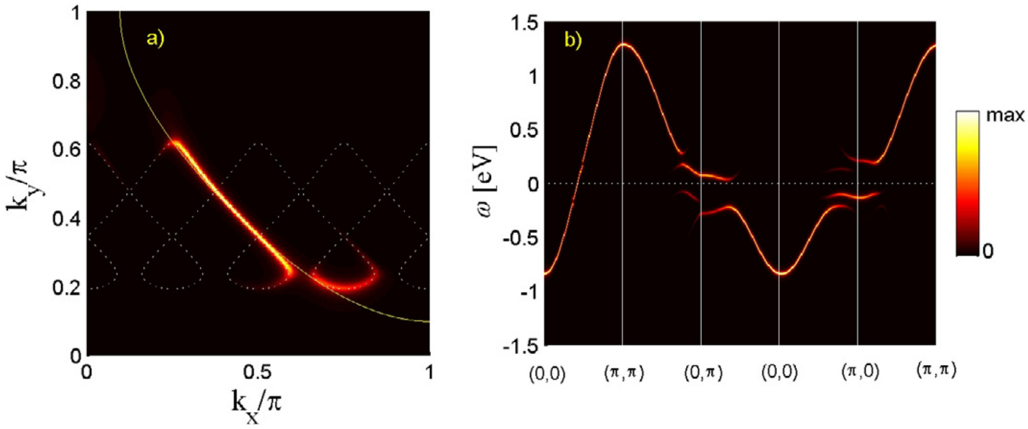

Figure 4. (a) The spectral function $A(\mathbf{k}, \omega = 0)$ for a striped superconductor. The band parameters used in the calculation were fitted to the ARPES spectrum of LSCO [45]: $t = 0.25$, $t' = -0.031863$, $t'' = 0.016487$ and $t''' = 0.0076112$, where t, t', ... are nearest neighbor hopping, second-nearest neighbor hopping and so on, chemical potential $\mu = -0.16235$. (All the parameters above are measured in eV.) The striped superconducting order parameter has a wavevector of $\mathbf{Q} = (2\pi/8, 0)$, and its magnitude is $\Delta_\mathbf{Q} = 60\,\text{meV}$. The order parameter is of 'd-wave character', in the sense that it is of opposite sign on x- and y-oriented bonds. The thin solid line shows the underlying bare Fermi surface, and the dotted line shows the Fermi surface in the presence of the PDW. (b) $A(\mathbf{k}, \omega)$ for the same model parameters along a cut in k-space.

6.1. Order parameters and symmetries

We will now define the various order parameters introduced in this section and discuss their symmetry properties. The striped superconducting order parameter $\Delta_\mathbf{Q}$ is a charge $2e$ complex scalar field, carrying momentum $\mathbf{Q}$. To define it microscopically, we write the superconducting order parameter as

$$\begin{aligned}\phi(\mathbf{r}, \mathbf{r}') &\equiv \langle \psi_\uparrow^\dagger(\mathbf{r}) \psi_\downarrow^\dagger(\mathbf{r}') \rangle \\ &= F\left(\mathbf{r} - \mathbf{r}'\right)\left[\Delta_0 + \Delta_\mathbf{Q} \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot\mathbf{R}} + \Delta_{-\mathbf{Q}} \mathrm{e}^{-\mathrm{i}\mathbf{Q}\cdot\mathbf{R}}\right],\end{aligned} \tag{18}$$

where $\mathbf{R} = (\mathbf{r} + \mathbf{r}')/2$, $F(\mathbf{r} - \mathbf{r}')$ is some short-range function (for a 'd-wave-like' striped superconductor, $F(\mathbf{r})$ changes sign under 90° rotation) and Δ_0 is the uniform $\mathbf{Q} = \mathbf{0}$ component of the order parameter[10]. In the rest of this subsection, we set $\Delta_0 = 0$. The effect of Δ_0 is discussed in section 6.3. To be concrete, we assume that the host crystal is tetragonal, and that there are therefore two potential symmetry related ordering wavevectors, $\mathbf{Q}$ and $\bar{\mathbf{Q}}$, which are mutually orthogonal, so $\Delta_{\bar{\mathbf{Q}}}$ must be treated on an equal footing with $\Delta_\mathbf{Q}$. (The discussion

[10] A state in which both components of the SC order parameter *coexist*, $\Delta_0 \neq 0$ *and* $\Delta_\mathbf{Q} \neq 0$ is certainly not 'uniform'. Even a weak $\Delta_\mathbf{Q} \neq 0$ implies the existence of a modulation of the local amplitude of the SC order parameter, and a SC state is 'truly uniform' only if $\Delta_\mathbf{Q} = 0$. Nevertheless, as we will see in section 6.3, the properties of a SC state in which both order parameters coexist are largely dominated by the 'uniform' component Δ_0, and the striking features of the PDW state are not directly observable. In this sense, the uniform-PDW coexisting SC state is effectively 'uniform'.

is easily generalized to crystals with other point-group symmetries.) Similarly, for simplicity, spin–orbit coupling is assumed to be negligible.

The order parameters that may couple to $\Delta_{\mathbf{Q}}$ and their symmetry properties are as follows: the nematic order parameter N is a real pseudo-scalar field; the CDW $\rho_{\mathbf{K}}$ with $\mathbf{K}=2\mathbf{Q}$ is a scalar field; $\vec{S}_{\mathbf{Q}}$ is a neutral spin-vector field. All these order parameters are electrically neutral. Under spatial rotation by $\pi/2$, $N\to -N$, $\rho_{\mathbf{K}}\to\rho_{\bar{\mathbf{K}}}$, $\vec{S}_{\mathbf{Q}}\to\vec{S}_{\bar{\mathbf{Q}}}$ and $\Delta_{\mathbf{Q}}\to\pm\Delta_{\bar{\mathbf{Q}}}$, where $\pm$ refers to a d-wave or s-wave version of the striped superconductor. Under spatial translation by $\mathbf{r}$, $N\to N$, $\rho_{\mathbf{K}}\to \mathrm{e}^{\mathrm{i}\mathbf{K}\cdot\mathbf{r}}\rho_{\mathbf{K}}$, $\vec{S}_{\mathbf{Q}}\to \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot\mathbf{r}}\vec{S}_{\mathbf{Q}}$ and $\Delta_{\mathbf{Q}}\to \mathrm{e}^{\mathrm{i}\mathbf{Q}\cdot\mathbf{r}}\Delta_{\mathbf{Q}}$. Note that since the SDW and CDW orders are real, $\vec{S}^{\star}_{\mathbf{Q}}=\vec{S}_{-\mathbf{Q}}$ and $\rho^{\star}_{\mathbf{K}}=\rho_{-\mathbf{K}}$. Generally, $\Delta_{\mathbf{Q}}$ and $\Delta^{\star}_{\mathbf{Q}}$ are independent.

6.2. Landau theory

Specifically, the emphasis in this section is on the interrelation between striped superconducting order and other orders. There is a necessary relation between this order and CDW and nematic (or orthorhombic) order, since the striped superconductor breaks both translational and rotational symmetries of the crystal. From the microscopic considerations, above, and from the phenomenology of the cuprates, we also are interested in the relation of superconducting and SDW order. The Landau effective free energy density can then be expanded in powers of these fields:

$$\mathcal{F}=\mathcal{F}_2+\mathcal{F}_3+\mathcal{F}_4+\cdots \tag{19}$$

where $\mathcal{F}_2$, the quadratic term, is simply a sum of decoupled terms for each order parameter,

$$\begin{aligned}\mathcal{F}_3=\gamma_s\left[\rho_{-\mathbf{K}}\vec{S}_{\mathbf{Q}}\cdot\vec{S}_{\mathbf{Q}}+\rho_{-\bar{\mathbf{K}}}\vec{S}_{\bar{\mathbf{Q}}}\cdot\vec{S}_{\bar{\mathbf{Q}}}+\mathrm{c.c.}\right]+\gamma_\Delta\left[\rho_{-\mathbf{K}}\Delta^{\star}_{-\mathbf{Q}}\Delta_{\mathbf{Q}}+\rho_{-\bar{\mathbf{K}}}\Delta^{\star}_{-\bar{\mathbf{Q}}}\Delta_{\bar{\mathbf{Q}}}+\mathrm{c.c.}\right]\\ +g_\Delta N\left[\Delta^{\star}_{\mathbf{Q}}\Delta_{\mathbf{Q}}+\Delta^{\star}_{-\mathbf{Q}}\Delta_{-\mathbf{Q}}-\Delta^{\star}_{\bar{\mathbf{Q}}}\Delta_{\bar{\mathbf{Q}}}-\Delta^{\star}_{-\bar{\mathbf{Q}}}\Delta_{-\bar{\mathbf{Q}}}\right]\\ +g_sN\left[\vec{S}_{-\mathbf{Q}}\cdot\vec{S}_{\mathbf{Q}}-\vec{S}_{-\bar{\mathbf{Q}}}\cdot\vec{S}_{\bar{\mathbf{Q}}}\right]+g_cN\left[\rho_{-\mathbf{K}}\rho_{\mathbf{K}}-\rho_{-\bar{\mathbf{K}}}\rho_{\bar{\mathbf{K}}}\right]\end{aligned} \tag{20}$$

and the fourth order term, which is more or less standard, is shown explicitly below.

The effect of the cubic term proportional to γ_s on the interplay between the spin and charge components of stripe order has been analyzed in depth in [84]. Similar analysis can be applied to the other terms. In particular, the γ_Δ and g_Δ terms imply[11] that the existence of superconducting stripe order ($\Delta_{\mathbf{Q}}\neq 0$ and $\Delta_{\bar{\mathbf{Q}}}=0$), implies the existence of nematic order ($N\neq 0$) and charge stripe order with half the period ($\rho_{2\mathbf{Q}}\neq 0$). However, the converse statement is not true: while CDW order with ordering wavevector $2\mathbf{Q}$ or nematic order tends to promote PDW order, depending on the magnitude of the quadratic term in $\mathcal{F}_2$, PDW order may or may not occur.

One new feature of the coupling between the PDW and CDW order is that it produces a sensitivity to disorder which is not normally a feature of the superconducting state. In the

[11] Note that the γ_Δ term is odd under a particle–hole transformation, which takes $\rho_{\mathbf{K}}\to-\rho_{\mathbf{K}}$. Therefore, if the system has exact particle–hole symmetry, this term vanishes, and there is no necessary relation between $\Delta_{\mathbf{Q}}$ and $\rho_{\mathbf{K}}$. Microscopic systems are generically not symmetric under charge conjugation. However, some real systems (e.g. the cuprates) are not too far from being particle–hole symmetric, and therefore in these systems γ_Δ is expected to be relatively small.

presence of quenched disorder, there is always some amount of spatial variation of the charge density, $\rho(\mathbf{r})$, of which the important portion for our purposes can be thought of as being a pinned CDW, that is, a CDW with a phase which is a pinned, slowly varying function of position, $\rho(\mathbf{r}) = |\rho_{\mathbf{K}}| \cos[\mathbf{K}\cdot\mathbf{r} + \phi(\mathbf{r})]$. Below the nominal striped superconducting ordering temperature, we can similarly express the PDW order in terms of a slowly varying superconducting phase, $\Delta(\mathbf{r}) = |\Delta_{\mathbf{Q}}| \exp[\mathrm{i}\mathbf{Q}\cdot r + \mathrm{i}\theta_{\mathbf{Q}}(\mathbf{r})] + |\Delta_{-\mathbf{Q}}| \exp[-\mathrm{i}\mathbf{Q}\cdot\mathbf{r} + \mathrm{i}\theta_{-\mathbf{Q}}(\mathbf{r})]$. The resulting contribution to $\mathcal{F}_3$ is

$$\mathcal{F}_{3,\gamma} = 2\gamma_{\Delta} |\rho_{\mathbf{K}} \Delta_{\mathbf{Q}} \Delta_{-\mathbf{Q}}| \cos[2\theta_{-}(\mathbf{r}) - \phi(\mathbf{r})], \tag{21}$$

where

$$\begin{aligned} \theta_{\pm}(\mathbf{r}) &\equiv [\theta_{\mathbf{Q}}(\mathbf{r}) \pm \theta_{-\mathbf{Q}}(\mathbf{r})]/2, \\ \theta_{\pm\mathbf{Q}}(\mathbf{r}) &= [\theta_{+}(\mathbf{r}) \pm \theta_{-}(\mathbf{r})]. \end{aligned} \tag{22}$$

The aspect of this equation that is notable is that the disorder couples directly to a piece of the superconducting phase, θ_{-}. No such coupling occurs in usual 0 momentum superconductors.

It is important to note that the condition that $\Delta(\mathbf{r})$ be single valued implies that $\theta_{\mathbf{Q}}(\mathbf{r})$ and $\theta_{-\mathbf{Q}}(\mathbf{r})$ are defined modulo 2π. Correspondingly, $\theta_{\pm}$ are defined modulo π, subject to the constraint that if $\theta_{\pm} \to \theta_{\pm} + \pi m_{\pm}$ then $m_{+} + m_{-}$ must be an even integer. Since ϕ and θ_{-} are locked to each other at long distances, the possible topological excitations of the coupled PDW–CDW system are thus point defects in 2D and line defects in 3D classified by the circulation of θ_{+} and ϕ on any enclosing contour. The elementary topological defects thus are: (a) an ordinary superconducting vortex, about which $\Delta\theta_{+} = 2\pi$ and $\Delta\phi = 0$. (b) A bound state of a half vortex and a dislocation[12], about which $\Delta\theta_{+} = \pi$ and $\Delta\phi = 2\pi$. (c) A double dislocation (or dislocation bound state) about which $\Delta\theta_{+} = 0$ and $\Delta\phi = 4\pi$. All these defects have a logarithmically divergent energy in 2D, or energy per unit length in 3D; the prefactor of the logarithm is determined by the superfluid stiffness for the vortex, the elastic modulus of the CDW for the double vortex, and an appropriate sum of these two stiffnesses for the half vortex. Consequences of this rich variety of topological defects are discussed in [6, 25, 86].

An important consequence of the coupling between the superconducting and CDW phase is that the effect of quenched disorder, as in the case of the CDW itself, destroys long-range superconducting stripe order. (This statement is true [87], even for weak disorder, in dimensions $d < 4$.) Naturally, the way in which this plays out depends on the way in which the CDW state is disordered.

In the most straightforward case, the CDW order is punctuated by random, pinned dislocations, i.e. 2π vortices of the ϕ field. The existence of the coupling in equation (21) implies that there must be an accompanying π vortex in θ_{-}. The condition of single-valuedness implies that there must also be an associated half-vortex or anti-vortex in θ_{+}. If these latter vortices are fluctuating, they destroy the superconducting state entirely, leading to a resistive state with short-ranged striped superconducting correlations. If they are frozen, the resulting state is analogous to the ordered phase of an XY spin-glass: such a state has a non-vanishing Edwards–Anderson order parameter, spontaneously breaks time-reversal symmetry, and, presumably, has vanishing resistance but no Meissner effect and a vanishing critical current. In 2D, according to conventional wisdom, a spin-glass phase can only occur at $T = 0$, but in 3D there can be a finite temperature glass transition [88].

[12] The possibility of half vortices in a striped superconductor and their effect on the phase diagram in the clean case was discussed by Agterberg and Tsunetsugu [85].

In 3D there is also the exotic possibility that, for weak enough quenched disorder, the CDW forms a Bragg-glass phase, in which long-range order is destroyed, but no free dislocations occur [89]–[91]. In this case, ϕ can be treated as a random, but single-valued function—correspondingly, so is θ_-. The result is a superconducting Bragg-glass phase which preserves time reversal symmetry and, presumably, acts more or less the same as a usual superconducting phase. It is believed that a Bragg-glass phase is not possible in 2D [90].

Another perspective on the nature of the superconducting state can be obtained by considering a composite order parameter which is proportional to $\Delta_Q\Delta_{-Q}$. There is a cubic term which couples a uniform, charge $4e$ superconducting order parameter, Δ_4, to the PDW order:

$$\mathcal{F}_3' = g_4\{\Delta_4^\star[\Delta_{\mathbf{Q}}\Delta_{-\mathbf{Q}} + \Delta_{\bar{\mathbf{Q}}}\Delta_{-\bar{\mathbf{Q}}}] + \text{c.c.}\} \tag{23}$$

This term implies that whenever there is PDW order, there is also necessarily charge $4e$ uniform superconducting order. However, since this term is independent of θ_-, it would be totally unaffected by Bragg-glass formation of the CDW. The half-vortices in θ_+ discussed above can simply be viewed as the fundamental $(hc/4e)$ vortices of a charge $4e$ superconductor.

Some additional physical insight can be gained by examining the quartic terms ($\mathcal{F}_4$ in equation (19)). Let us write all the possible fourth order terms consistent with symmetry:

$$\begin{aligned}
\mathcal{F}_4 = {} & u\left(\vec{S}_{\mathbf{Q}}\cdot\vec{S}_{\mathbf{Q}}\Delta_{\mathbf{Q}}^\star\Delta_{-\mathbf{Q}} + \vec{S}_{\bar{\mathbf{Q}}}\cdot\vec{S}_{\bar{\mathbf{Q}}}\Delta_{\bar{\mathbf{Q}}}^\star\Delta_{-\bar{\mathbf{Q}}} + \text{c.c.}\right)\\
& + \left(v_+\left[\vec{S}_{-\mathbf{Q}}\cdot\vec{S}_{\mathbf{Q}} + \vec{S}_{-\bar{\mathbf{Q}}}\cdot\vec{S}_{\bar{\mathbf{Q}}}\right] + \tilde{v}_+[|\rho_{\mathbf{K}}|^2 + |\rho_{\bar{\mathbf{K}}}|^2]\right)\\
& \times\left(|\Delta_{\mathbf{Q}}|^2 + |\Delta_{-\mathbf{Q}}|^2 + |\Delta_{\bar{\mathbf{Q}}}|^2 + |\Delta_{-\bar{\mathbf{Q}}}|^2\right)\\
& + \left(v_-\left[\vec{S}_{-\mathbf{Q}}\cdot\vec{S}_{\mathbf{Q}} - \vec{S}_{-\bar{\mathbf{Q}}}\cdot\vec{S}_{\bar{\mathbf{Q}}}\right] + \tilde{v}_-[|\rho_{\mathbf{K}}|^2 - |\rho_{\bar{\mathbf{K}}}|^2]\right)\\
& \times\left(|\Delta_{\mathbf{Q}}|^2 + |\Delta_{-\mathbf{Q}}|^2 - |\Delta_{\bar{\mathbf{Q}}}|^2 - |\Delta_{-\bar{\mathbf{Q}}}|^2\right)\\
& + vN^2\left\{\left(|\Delta_{\mathbf{Q}}|^2 + |\Delta_{-\mathbf{Q}}|^2\right) + \left(|\Delta_{\bar{\mathbf{Q}}}|^2 + |\Delta_{-\bar{\mathbf{Q}}}|^2\right)\right\}\\
& + \lambda_+\left\{\left(|\Delta_{\mathbf{Q}}|^2 + |\Delta_{-\mathbf{Q}}|^2\right)^2 + \left(|\Delta_{\bar{\mathbf{Q}}}|^2 + |\Delta_{-\bar{\mathbf{Q}}}|^2\right)^2\right\}\\
& + \lambda_-\left\{\left(|\Delta_{\mathbf{Q}}|^2 - |\Delta_{-\mathbf{Q}}|^2\right)^2 + \left(|\Delta_{\bar{\mathbf{Q}}}|^2 - |\Delta_{-\bar{\mathbf{Q}}}|^2\right)^2\right\}\\
& + \lambda(|\Delta_{\mathbf{Q}}|^2 + |\Delta_{-\mathbf{Q}}|^2)(|\Delta_{\bar{\mathbf{Q}}}|^2 + |\Delta_{-\bar{\mathbf{Q}}}|^2)\\
& + \cdots
\end{aligned} \tag{24}$$

where we have explicitly shown all the terms involving $\Delta_{\mathbf{Q}}$, while the terms $\cdots$ represent the remaining quartic terms all of which, with the exception of those involving N, are exhibited explicitly in [84].

There are a number of features of the ordered phases which depend qualitatively on the sign of various couplings. Again, this is very similar to what happens in the case of CDW order—see, for example [92, 93]. For instance, depending on the sign of λ, either unidirectional (superconducting stripe) or bidirectional (superconducting checkerboard) order is favored.

On physical grounds, we have some information concerning the sign of various terms in $\mathcal{F}_4$. The term proportional to u determines the relative phase of the spin and superconducting stripe order—we believe $u > 0$ which thus favors a $\pi/2$ phase shift between the SDW and the striped superconducting order, i.e. the peak of the superconducting order occurs where the spin order passes through zero. The other interesting thing about this term is that it implies an effective cooperativity between spin and striped superconducting order. The net effect, i.e. whether spin and striped superconducting order cooperate or fight, is determined by the sign of $|u| - v_+ - v_-$, such that they cooperate if $|u| - v_+ - v_- > 0$ and oppose each other if $|u| - v_+ - v_- < 0$. It is an interesting possibility that spin order and superconducting stripe order can actually favor each other even with all 'repulsive' interactions. The term proportional to λ_- determines whether the superconducting stripe order tends to be real ($\lambda_- > 0$), with a superconducting order that simply changes sign as a function of position, or a complex spiral, which supports ground-state currents ($\lambda_- < 0$).

6.3. *Coexisting uniform and striped order parameters*

Finally, we comment on the case of coexisting uniform and striped superconducting order parameters. Such a state is not thermodynamically distinct from a regular (uniform) superconductor coexisting with a CDW, even if the uniform superconducting component is in fact weaker than the striped component. Therefore, we expect many of the special features of the striped superconductor (such as its sensitivity to potential disorder) to be lost. Here, we extend the Landau free energy to include a uniform superconducting component, and show that this is indeed the case.

We will now analyze the coupling of a striped superconducting order parameter $\Delta_{\mathbf{Q}}$ to a uniform order parameter, Δ_0. In this case, we have to consider in addition to the order parameters introduced in section 6 a CDW order parameter with wavevector $\mathbf{Q}$, denoted by $\rho_{\mathbf{Q}}$. The additional cubic terms in the Ginzburg–Landau free energy are

$$\mathcal{F}_{3,u} = \gamma_{\mathbf{Q}}\Delta_0^\star\left[\rho_{\mathbf{Q}}\Delta_{-\mathbf{Q}} + \rho_{-\mathbf{Q}}\Delta_{\mathbf{Q}} + \rho_{\bar{\mathbf{Q}}}\Delta_{-\bar{\mathbf{Q}}} + \rho_{-\bar{\mathbf{Q}}}\Delta_{\bar{\mathbf{Q}}}\right] + \text{c.c.} + g_\rho\left[\rho_{-2\mathbf{Q}}\rho_{\mathbf{Q}}^2 + \rho_{-2\bar{\mathbf{Q}}}\rho_{\bar{\mathbf{Q}}}^2 + \text{c.c.}\right]. \tag{25}$$

Equation (25) shows that if both Δ_0 and $\Delta_{\mathbf{Q}}$ are nonzero, there is necessarily a coexisting nonzero $\rho_{\mathbf{Q}}$, through the $\gamma_{\mathbf{Q}}$ term. The additional quartic terms involving Δ_0 are

$$\begin{aligned}\mathcal{F}_{4,u} = {} & u_\Delta\left(\Delta_0^{\star 2}\Delta_{\mathbf{Q}}\Delta_{-\mathbf{Q}} + \Delta_0^{\star 2}\Delta_{\bar{\mathbf{Q}}}\Delta_{-\bar{\mathbf{Q}}} + \text{c.c.}\right) + \delta|\Delta_0|^2[|\Delta_{\mathbf{Q}}|^2 + |\Delta_{\bar{\mathbf{Q}}}|^2] \\ & + |\Delta_0|^2[u_\rho(|\rho_{\mathbf{Q}}|^2 + |\rho_{\bar{\mathbf{Q}}}|^2) + u'_\rho(|\rho_{2\mathbf{Q}}|^2 + |\rho_{2\bar{\mathbf{Q}}}|^2)] \\ & + v'|\Delta_0|^2[\vec{S}_{-\mathbf{Q}}\cdot\vec{S}_{\mathbf{Q}} + \vec{S}_{-\bar{\mathbf{Q}}}\cdot\vec{S}_{\bar{\mathbf{Q}}}].\end{aligned} \tag{26}$$

Let us now consider the effect of quenched disorder. Following the discussion preceding equation (22), we write the order parameters in real space as

$$\Delta(\mathbf{r}) = |\Delta_0|e^{i\theta_0} + |\Delta_{\mathbf{Q}}|e^{i(\theta_{\mathbf{Q}}+\mathbf{Q}\cdot\mathbf{r})} + \left|\Delta_{-\mathbf{Q}}\right| e^{i(\theta_{-\mathbf{Q}}-\mathbf{Q}\cdot\mathbf{r})} \tag{27}$$

and

$$\rho(r) = |\rho_{\mathbf{Q}}|\cos\left(\mathbf{Q}\cdot\mathbf{r} + \phi_{\mathbf{Q}}\right) + |\rho_{2\mathbf{Q}}|\cos\left(2\mathbf{Q}\cdot\mathbf{r} + \phi\right). \tag{28}$$

Let us assume that the disorder nucleates a point defect in the CDW, which in this case corresponds to a 2π vortex in the phase $\phi_{\mathbf{Q}}$. By the g_ρ term in equation (25), this induces a 4π vortex in ϕ. (Note that in the presence of $\rho_{\mathbf{Q}}$, a 2π vortex in ϕ is not possible.) The γ_Δ term in equation (21) then dictates a 2π vortex in the phase $\theta_- = (\theta_{\mathbf{Q}} - \theta_{-\mathbf{Q}})/2$. However, unlike before, this vortex does not couple to the global superconducting phase $\theta_+ = (\theta_{\mathbf{Q}} + \theta_{-\mathbf{Q}})/2$. Therefore, an arbitrarily small uniform superconducting component is sufficient to remove the sensitivity of a striped superconductor to disorder, and the system is expected to behave more or less like a regular (uniform) superconductor, albeit with a modulated amplitude of the order parameter.

Since the usual (uniform) superconducting order and the PDW break distinct symmetries, nothing can be said, in general, about the conditions in which they will coexist. However, microscopic considerations can, in some cases, yield generic statements, too. For example, in a striped SC, a uniform component of the order parameter can be generated by dimerizing the stripe order, such that the positive and negative strips of superconducting order are made alternately broader and narrower. In any structure (such as the LTT structure of LBCO), in which there is zero Josephson coupling between neighboring layers, a coupling is generated, thus lowering the energy of the system, in proportion to the square of the dimerization. Presumably, so long as the PDW period is incommensurate with the underling lattice, there is also a quadratic energy cost to dimerization, which is related to an appropriate generalized elastic constant of the PDW. However, if the PDW has a long period, this elastic constant will be vanishingly small. Thus, any long period, incommensurate PDW may generically be expected to be unstable toward the generation of a small amount of uniform SC order.

7. Non-collinear order and time reversal symmetry breaking

In a layered system, PDW order in the planes can lead to frustration of the inter-plane Josephson coupling, which naturally explains the layer decoupling seen in $1/8$ doped LBCO[13]. In analogy with frustrated magnetic systems (in which the superconducting order is thought of as an XY pseudo-spin), this frustration can also lead to various forms of non-collinear order. In the PDW case, such non-collinear orders break time-reversal symmetry and are accompanied by spontaneous equilibrium currents.

In this section, we give detailed predictions for the patterns of bulk time-reversal symmetry breaking and spontaneous currents in various lattice geometries. We will discuss this problem at zero temperature and at a classical level. It is worth noting that the non-collinear order, where it occurs, results in a partial lifting of the frustration. In the case of a PDW in the LTT structure (relevant to $La_{2-x}Ba_xCuO_4$), we shall show that it results in a non-vanishing effective Josephson coupling between planes, and hence, in a sense, spoils the strict layer decoupling we have touted. However, this effective Josephson coupling is equivalent to a higher order coupling [6] (due to coherent tunneling of two Cooper pairs), both in terms of its small magnitude, and its dependence on the cosine of twice the difference of the superconducting phases on neighboring planes. (See equations (30) and (32)). Note also that defects (such as point defects, domain walls or twin boundaries) can lead to additional intra-plane time reversal symmetry breaking, that can drive the system into a glassy superconducting state (as discussed in section 6)[14].

[13] The problem of the 3D phase transition in a system with an effective layer decoupling is largely unsolved. See, however, the recent work of Raman *et al* [94].

[14] An in-plane magnetic field can also change the inter-layer frustration, leading to small violations of the layer decoupling effect. If large enough such effects can be used to detect a PDW state. A similar effect can also take place in junctions between an FFLO state and a uniform superconductor [95].

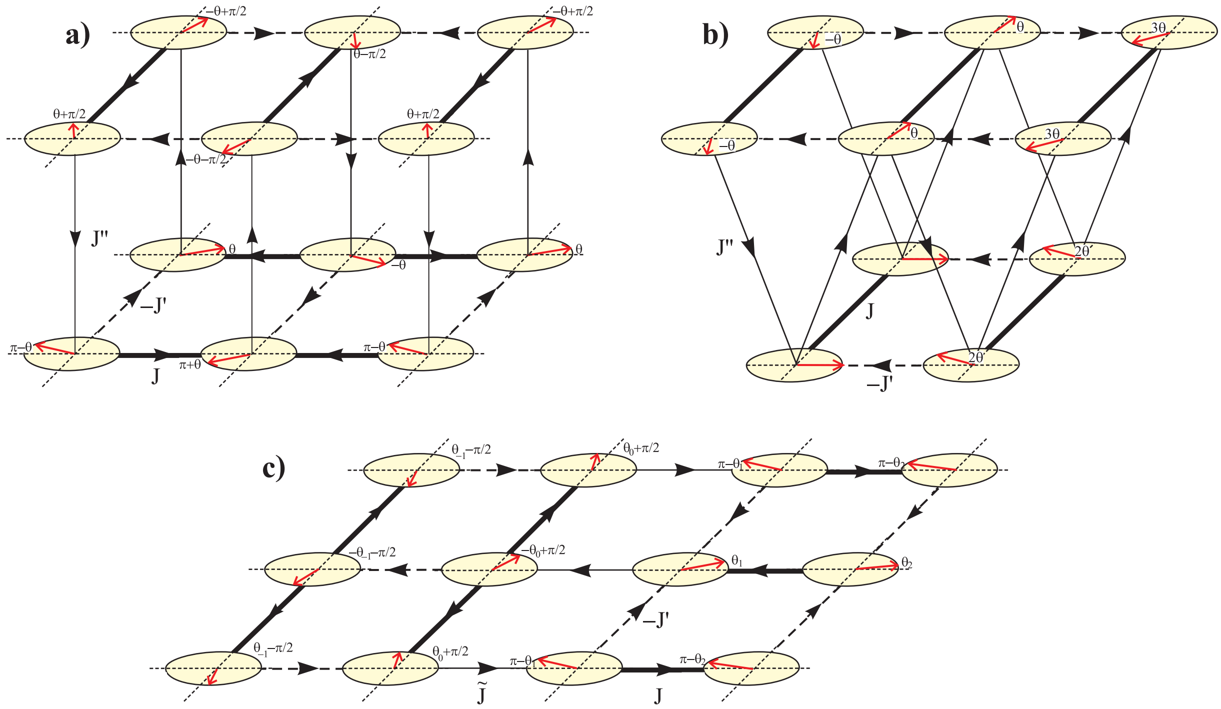

Figure 5. (a) Model for a striped superconductor with an LTT structure. Solid (dashed) lines represent positive (negative) Josephson couplings. The arrow on the center of each link indicates the direction of the equilibrium current across that link. The red arrows on the vertices represent the superconducting phases. (b) Same as (a) for an orthorhombic striped superconductor, where the charge stripes are shifted by half a period from one layer to the next. (c) An in-plane domain wall.

Let us start with the case of the LBCO LTT structure, in which the stripe direction rotates by 90° between adjacent planes. We model the system by a 3D discrete lattice of Josephson junctions, shown in figure 5(a)[15]. The lattice spacing in the plane is the inter-stripe distance λ, and c is the inter-plane distance. Each lattice point has a single degree of freedom $\theta_{\mathbf{r}}$, which is the local value of the superconducting phase at that point. $J, -J', J''$ are the intra-stripe, the inter-stripe and the inter-plane Josephson couplings, respectively. We assume that $J > J' \gg J'' > 0$, corresponding to a unidirectional striped superconductor in the planes. For any collinear configuration, the Josephson coupling between the planes vanishes. However, if the staggered order parameter in each plane is rotated by 90° relative to its neighbors, then the energy can be lowered by distorting the phases periodically with respect to the collinear

[15] Note that we are actually considering a simplified version of the LBCO LTT structure. The structure in figure 5(a) has two planes per unit cell, while the LBCO LTT structure has four. The difference is that in LBCO, the charge stripes in second neighboring planes (which are parallel to each other) are shifted by half a period relative to one another, while in figure 5(a) they are not. However, the considerations we discuss here are the same for two structures, and the resulting non-collinear ground states are similar.

configuration in each plane. We use a variational *ansatz* for the phases $\theta_{\mathbf{r}}$ of the form

$$\theta_{\mathbf{r}} = \frac{1+(-1)^z}{2} y\pi + \frac{1-(-1)^z}{2}\left(x+\frac{1}{2}\right)\pi + (-1)^{x+y+z}\theta, \tag{29}$$

where $\mathbf{r} = (x, y, z)$ is the integer valued position vector (x and y are measured in units of λ, and z is measured in units of c), and the distortion angle θ is a variational parameter. The Josephson energy per site as a function of θ is

$$E_{\mathrm{LTT}}(\theta) = -\left(J + J'\right)\cos 2\theta - J''\sin 2\theta. \tag{30}$$

The inter-plane coupling energy gain is linear in θ, whereas the cost in intra-plane coupling energy is quadratic in θ. Thus the distortion occurs for any nonzero value of the inter-plane coupling J''. Minimizing equation (30), we obtain

$$\tan 2\theta = \frac{J''}{J+J'}. \tag{31}$$

The equilibrium currents across the three types of links are $\mathcal{J} = J\sin 2\theta$, $\mathcal{J}' = J'\sin 2\theta$ and $\mathcal{J}'' = J''\cos 2\theta = \mathcal{J} + \mathcal{J}'$, where equation (31) was used in the last relation. The directions of the currents are as indicated in figure 5(a). Associated with these currents is a magnetic field with nonzero components in all three directions. The wavevector associated with this pattern is $\mathbf{Q} = (\frac{\pi}{\lambda}, \frac{\pi}{\lambda}, \frac{\pi}{c})$, where λ is the inter-stripe distance (for LBCO at $x = 1/8$, $\lambda \approx 4a$ where a is the Cu–Cu distance) and c is the inter-plane distance.

The non-collinear distortion in the above pattern induces an effective nonzero inter-plane coupling. However, the effective inter-layer coupling is (taking the limit $J'' \ll J, J'$)

$$J_{\mathrm{eff}} \simeq \frac{(J'')^2}{4(J+J')} \tag{32}$$

and is therefore much smaller than the bare inter-plane coupling J''. Note, moreover, that the induced Josephson coupling between two neighboring planes with PDW superconducting phases θ_i and θ_j has the form $J_{\mathrm{eff}}\cos[2(\theta_i - \theta_j)]$, i.e. its period in the relative phase is π.

Next, we consider the case of an orthorhombic structure (such as the LTO phase of LBCO). In this case, rotational symmetry in the plane is broken in the same way in every plane, and the stripes are all in the same direction. However, we assume that due to Coulomb interactions, the charge stripes are shifted by half a period between adjacent planes. (Such a shift is indeed observed between second neighbor planes in the LTT phase of LBCO, in which the stripe direction is parallel.) Therefore, the inter-plane coupling is frustrated due to the resulting 'zigzag' geometry. We shall show below that the ground state has spiral order, which partially relieves this frustration. Introducing a spiral twist angle θ, such that $\theta_{\mathbf{r}} = 2x\theta$ (as shown in figure 5(b)), costs an energy $E_{\mathrm{ORT}}(\theta)$ per stripe, given by

$$\frac{E_{\mathrm{ORT}}(\theta)}{L} = J'\cos 2\theta - 2J''\cos\theta, \tag{33}$$

where L is the length of each stripe. The minimum is for $\cos\theta = \frac{J''}{2J'}$. Therefore, a spiral distortion occurs for any $J'' < 2J'$. The currents along this links are $\mathcal{J}'' = -\mathcal{J}' = J''\sin\theta$, and their directions are indicated in figure 5(b). Each plane carries a uniform current that flows perpendicular to the stripes, and an equal and opposite current flows between the planes. The magnetic field associated with these currents is pointing parallel to the stripe direction, and its lowest Fourier component is at wavevector $\mathbf{Q} = (0, 0, \frac{2\pi}{c})$.

Finally, we turn to the case of a domain wall in the PDW order, depicted in figure 5(c). (Such a defect is very costly energetically, but it is favored by a twin boundary in the crystal structure.) The Josephson coupling across the domain wall vanishes for any collinear configuration. The energy can be lowered by distorting the phases in the pattern shown in figure 5(c), which is closely analogous to the minimum energy configuration in the LTT case (figure 5(a)). The superconducting phases $\theta_{\mathbf{r}}$ are given by

$$\theta_{\mathbf{r}} = \begin{cases} (x+\frac{1}{2})\pi - (-1)^y \theta_x, & (x < 1), \\ y\pi + (-1)^y \theta_x, & (x \geqslant 1), \end{cases} \tag{34}$$

where the distortion angle θ_x depends on the distance from the domain wall x. (In our notation, $x = 0$ and 1 are the two columns on either side of the domain wall.) The energy is

$$\begin{aligned} \frac{E_{\mathrm{DIS}}(\{\theta_x\})}{L} = &- J \sum_{x=1}^{\infty} \cos(\theta_{x+1} - \theta_x) - J' \sum_{x=1}^{\infty} \cos(2\theta_x) \\ &- J' \sum_{x=-\infty}^{0} \cos(\theta_{x-1} - \theta_x) - J \sum_{x=-\infty}^{0} \cos(2\theta_x) \\ &- \tilde{J} \sin(\theta_{x=1} + \theta_{x=0}). \end{aligned} \tag{35}$$

Here, $\tilde{J}$ is the Josephson coupling across the domain wall, and L is the number of sites along the domain wall. For simplicity, we assume that $\tilde{J} \ll J, J'$, in which case $\theta_x \ll 1$ and we may expand equation (35) to second order in θ_x. Minimizing $E_{\mathrm{DIS}}(\{\theta_x\})$, we obtain the following solution:

$$\theta_x = \begin{cases} \theta_< \mathrm{e}^{\alpha x}, & (x < 1), \\ \theta_> \mathrm{e}^{-\beta x}, & (x \geqslant 1), \end{cases} \tag{36}$$

where $\alpha = 2\sinh^{-1}\left(\sqrt{\frac{J'}{J}}\right)$, $\beta = 2\sinh^{-1}\left(\sqrt{\frac{J}{J'}}\right)$, $\theta_< = \frac{\tilde{J}}{J'(1-\mathrm{e}^{-\alpha_<})+4J}$ and $\theta_> = \frac{\tilde{J}}{J(1-\mathrm{e}^{-\alpha_>})+4J'}$. Associated with the distortion of the superconducting phases is a periodic pattern of spontaneous currents, shown in figure 5(c), with periodicity of two inter-stripe distances.

Similar considerations apply to an in-plane Josephson junction between a striped superconductor and a uniform superconductor, if the boundary is perpendicular to the stripe direction. Therefore, in such a junction time reversal symmetry is also broken. The critical current is of order $\frac{\tilde{J}^2}{\min\{J,J'\}}$. (This follows from the same considerations as the effective inter-plane coupling in the LTT case, equation (32).) It is thus suppressed relative to the critical current of a Josephson junction between uniform superconductors, which is of order $\tilde{J}$, as a result of the frustration of the Josephson coupling across the junction. Similarly to inter-plane coupling in the LTT case, the period of the coupling between a uniform and a striped superconductor in the relative phase is π, i.e. *half* of the period of the coupling between two uniform superconductors.

8. Connections and history

The notion of a superconducting state with spontaneously generated oscillations in the sign of the order parameter has cropped up, under various guises, a number of times in the past. It is worthwhile to recount some of these circumstances, not only in the interest of scholarship, but also to broaden the range of phenomena which can be addressed within the same conceptual framework.

8.1. Josephson π junctions

Since the superconducting order parameter is a charge $2e$ scalar field, it is often assumed that it is possible to think of the superconducting state as a Bose condensed state of charge $2e$ bosons. In contrast, most classic treatments of the subject [96] emphasize that many features of BCS theory, especially those associated with quasi-particle coherence factors, cannot be understood in this way. At the very least, a bosonic theory is inadequate to capture basic features of the ground state of any superconductor which has gapless quasi-particles, either because of the order parameter symmetry (e.g. d-wave) or because of scattering from magnetic impurities (gapless superconductor).

Even ignoring the possibility of gapless quasi-particles, there are qualitative possibilities in a fermionic system that cannot occur in a bosonic system. A feature of a time reversal invariant bosonic system is that the ground-state can be chosen to be real and nodeless. Thus, the order parameter in a Bose-condensed system must have a phase that is independent of position. The π junctions, which we have been discussing, are possible only because of the composite character of the superconducting order parameter [71].

There have been several previous theoretical studies that have found circumstances under which π junctions might occur [71, 72, 97]. More recently, the existence of such π junctions in the predicted circumstances have been confirmed by experiment. The first such experiments [74, 75] were significant as the 'phase sensitive' measurements that definitively established the d-wave symmetry of the superconducting order in the cuprates. More recently, however, mesoscopic π junctions between two s-wave superconductors have been constructed and characterized [73]. In our opinion, these latter experiments are also landmarks in the study of superconductivity. They establish that π junctions, the essential ingredient for the existence of striped superconductors, are physically possible.

8.2. FFLO states

In a superconductor with negligible spin–orbit coupling, it is possible to generate an imbalance in the population of up and down spin quasi-particles, either by applying a magnetic field in a geometry in which it predominantly couples to the electron spins, or by injecting a non-equilibrium population of quasi-particles from a neighboring ferromagnet [98]. In the related systems of cold fermionic atomic gases, it is possible to vary the population of up and down spin atoms independently, and to study the effect of this population imbalance on the superfluid state [25, 99, 100]. While a first-order quenching of the superconducting state is possible under these circumstances, there has also been considerable discussion of the possibility of spatially modulated superconducting states, so-called FFLO states [11, 12]. Two distinct states of this sort have been considered: (i) The FF state [11], in which the order parameter has constant magnitude but a phase that twists as a function of position according to $\theta = \Delta\mathbf{k}_{\mathrm{F}} \cdot \mathbf{r}$, where $\Delta\mathbf{k}_{\mathrm{F}}$ is the difference between the up spin and down spin Fermi momentum. (ii) The LO state [12], in which the order parameter remains real, but oscillates in sign with a period $L = 2\pi/|\Delta\mathbf{k}_{\mathrm{F}}|$.

The LO state is similar in structure to the striped superconductor considered here. In the order parameter theory presented in section 6.2, it corresponds to $\lambda_{-} > 0$ in equation (24). The parallel with the FF state (which is realized in the order parameter theory for $\lambda_{-} < 0$) is less crisp, but when superconducting striped spirals, which spontaneously break time reversal symmetry arise due to the appropriate type of geometric frustration of the Josephson couplings (as discussed in section 7), states that are in many ways analogous to the FF state also occur in striped superconductors. Thus, many of the physical phenomena we have discussed in this paper are pertinent to the FFLO phases in more weakly correlated systems, with the added richness [25] in the case of cold atomic gases that there are conserved quantities associated with the continuous rotational invariance of the underlying Hamiltonian.

However, the FFLO states arise from the explicit breaking of time reversal symmetry. In the absence of a magnetic field, Kramer's theorem implies perfect nesting between time-reversed pairs of states on opposite sides of the Fermi surface, so BCS pairing always occurs preferentially at $\mathbf{k} = \mathbf{0}$. This constraint is removed when time reversal symmetry is explicitly broken. One can think of the FFLO states as taking advantage of the 'best' remaining approximate nesting vector, $\Delta \mathbf{k}_{\mathrm{F}}$, in the two-particle channel. Alternatively, one can think of the LO state as consisting of a set of discommensurations [25, 98] such that the excess spin-up quasi-particles are incorporated in mid-gap states localized near the core of the discommensuration.

The energetic considerations that lead to the FFLO states are thus very different than the strong-coupling physics that gives rise to the striped superconductor[16]. The fact that the FFLO states explicitly break time reversal symmetry implies that they are macroscopically distinct (as phases of matter) from the striped superconductors that preserve this symmetry. Even in comparison with striped states that spontaneously break time reversal symmetry, the distinction remains that the FFLO states have a net magnetization, while the striped superconductor does not. Conversely, the FFLO states generally have no particular relation to other flavors of electronic ordering, while striped superconductors, as is characteristic of all electronic liquid crystals, embody a subtle interplay between multiple ordering tendencies. Specifically, since the striped superconductor seems to be generally associated with the strong coupling physics of doped antiferromagnets, there is a natural sense in which antiferromagnetism, CDW formation and striped superconductivity are intertwined.

8.3. Intertwined orders and emergent symmetries

One explicit way in which the relation between several order parameters can be more intimate than in a generic theory of 'competing orders' is if there is an emergent symmetry at low energies which unifies them. In particular, the order parameter structure of the PDW state, involving several order parameters coupled to each other, evokes the $SO(5)$ approach of a unified description of antiferromagnetism and uniform d-wave superconductivity [13, 14]. Indeed, by tuning the parameters of the effective Landau–Ginzburg theory that we presented in other sections it is possible to achieve an effective enlarged symmetry that makes it possible to 'rotate' the striped superconducting order and charge stripe order parameters into each other. Even if the enlarged symmetry is not exact, a rotation of the order parameters is possible but with a finite energy cost (similar to a 'spin flop'). It is also worth noting that a symmetry that allows a similar form of unification of d-wave superconductivity, electron nematicity and d-density wave (dDW) order [61] has recently been found to exist under special circumstances by Kee *et al* [103]. It is therefore possible that there could exist additional forms of striped superconducting states, which interleave these orders.

Thus, it is possible to view the PDW state as a 'liquid crystalline' analogue of the $SO(5)$ scenario. Indeed, the possibility of an $SO(5)$ 'spiral' was discussed previously by Zhang [104].

[16] FFLO states in the absence of magnetic fields have been shown to exist for special band structures in 1D [101] and 2D [102].

However, it should be noted that in the context of any conventional Landau–Ginzburg treatment of a system of competing orders, a general theorem [105] precludes a sign change of any component of the order parameter, and hence precludes the existence of spirals. In order to get a PDW state from an interplay between d-wave superconductivity and antiferromagnetism, unconventional gradient-dependent interactions between the different order parameters, such as those discussed in [105], must play a significant role in the physics.

In other words, in addition to the standard couplings allowed by a theory with several order parameters, the existence of a stripe order (for instance) in the charge order parameter must be able to induce a texture in the superconducting order as well. A useful analogy to keep in mind is the McMillan–deGennes theory of the nematic–smectic transition in classical liquid crystals in which the nematic order parameter acts as a component of a gauge field thus coupling to the *phase* of the smectic order, or in blue phases of liquid crystals. (For a detailed discussion of these topics in liquid crystals see, e.g. [106, 107].) In fact, Radzihovsky and Vishwanath [25] present a theory of FFLO states in ultra-cold atoms with gauge-like couplings (i.e. covariant derivative couplings) that relate the stripe (and spiral) order to the superconducting order.

In addition to the conceptual advantages, noted above, the liquid-crystal picture of the PDW state offers a direct way to classify the phase transitions (both quantum and thermal) out of this state. Thus, in addition to a direct transition to a normal state, intermediate phases characterized with composite order parameters, are also possible leading to an interesting phase diagram. We will explore these issues in a separate publication [86].

8.4. PDW states in Hubbard and t–J models

In the context of the cuprates, there have been several studies looking for a striped superconducting state in the t–J or Hubbard models. On the one hand, extensive, but not exhaustive DMRG calculations by White and Scalapino [77, 108] have consistently failed to find evidence in support of any sort of spontaneously occurring π junctions. On the other hand, a number of variational Monte Carlo and renormalized mean field calculations have concluded that the striped superconductor is either the ground state of such a model [109], under appropriate circumstances, or at least close in energy to the true ground state [80, 110, 111]. These latter calculations are certainly encouraging, in the sense that they suggest that there is no obvious energetic reason to *rule out* the existence of spontaneously occurring PDW order in strongly correlated electronic systems. However, the fact remains that no spontaneous π-junction formation has yet been observed in DMRG or other 'unbiased' studies of the t–J or the purely repulsive Hubbard models, indicating that there remain basic unsettled issues concerning the microscopic origins of π junctions.

9. Final thoughts

In this paper, we have introduced the PDW phase and studied its properties theoretically. In terms of symmetry, the PDW is distinct from the standard uniform superconductor. While some of its properties are similar to those of a uniform superconductor (e.g. zero resistance), others are markedly different: most importantly, the existence of a Fermi surface (and hence a finite density of states) in the ordered phase [25, 79], the possibility of frustration of the inter-layer coupling (depending on the lattice geometry), and the strong sensitivity to (non-magnetic) disorder. Generically, the PDW state in the presence of weak disorder is expected to give way to a 'superconducting glass' phase, in which the configuration average of the *local* superconducting order parameter vanishes, but the Edwards–Anderson order parameter is nonzero (and hence gauge symmetry is broken).

Even though the ordered PDW state itself is time reversal invariant, time reversal symmetry breaking is a very natural consequence of PDW order, either in the superconducting glass phase, or as a way of relieving the frustration of the Josephson couplings in some crystal structures. Specifically, frustration can lead to non-collinear ground state configurations of the superconducting pseudo-spins (representing the local phase of the superconducting order), which are analogous to the non-collinear ground states that are often found in frustrated spin systems. An even more exotic state that can naturally emerge from a 'parent' PDW state is a superconductor with a charge $4e$ order parameter [6, 7, 25], which can result when the CDW part of the PDW order is melted by either quantum or thermal fluctuations.

The occurrence of PDW states in microscopic models is an intrinsically strong coupling effect, since PDW order (much like CDW or SDW) is not an instability of a generic Fermi surface. In this paper, we have provided a 'proof of principle' of a not-too-contrived, strongly correlated, microscopic model with a PDW ground state. This model mimics some features of the striped state found in the cuprates (e.g. it has charge stripes separated by π-phase-shifted spin stripes). Whether a PDW state can be found in more realistic models, which include features such as uniformly repulsive interactions and a d-wave-like order parameter, remains to be settled.

Doped Mott insulators are strongly correlated systems whose ground states have a strong tendency to form liquid-crystalline-like [8] inhomogeneous phases [112]–[116]. In this regard, the PDW state is an electronic liquid crystal phase in which the superconducting and charge/spin orders do not compete with each other but rather are intertwined. As some of us have noted earlier [117, 118], the observation of a high pairing scale in such an electronically inhomogeneous state is suggestive of the existence of an optimal degree of inhomogeneity for superconductivity. Indeed, recent ARPES data suggest that the stripe order that develops in $La_{2-x}Ba_xCuO_4$ does not suppress the pairing scale[17]. The fact that the pairing scale is large in this material suggests that the development of charge stripe order suppresses the development of superconducting coherence but not pairing. In fact, it gives credence to the argument that there is a connection between the emergence of charge order and the mechanism of superconducting pairing [117, 118].

However, at present, it is unclear to what extent PDW order should be expected to be common where stripe order occurs. On the purely theoretical side, PDW order has proven elusive in DMRG studies of models [77, 108] with entirely repulsive interactions. Indeed, in a previous publication [7], we showed that in any weakly interacting superconductor, π junctions can only occur under exceedingly fine-tuned circumstances. It is clear from variational calculations [80], [109]–[111] that for strong interactions, the differences in energy between the PDW and uniform sign superconducting states in striped systems is relatively small; what particular features of the microscopic physics tip the balance one way or another is still not clear. Accordingly, it is not clear, in the absence of unambiguous experimental evidence, whether in the context of the cuprates, we should expect the PDW state to be a rare occurrence, perhaps stabilized by some particular detail of the electronic structure of $La_{2-x}Ba_xCuO_4$, or if instead

[17] ARPES data in $La_{2-x}Ba_xCuO_4$ show a substantial and weakly doping-dependent anti-nodal gap across $x = 1/8$ [44, 45], where the signatures of the PDW state are strongest.

we should infer that some degree of local PDW order exists in any cuprate in which evidence of local stripe correlations can be adduced.

To close this section, we turn to discuss the evidence for PDW states in the cuprate high temperature superconductors. The analysis of the PDW state was motivated by the experimental observations on $La_{2-x}Ba_xCuO_4$. Having studied the nature of this phase, we will now discuss to what extent the signatures of the PDW state are consistent with experiment. Finally, we speculate on the possible relevance of these ideas to other members of the cuprate family.

9.1. Striped SC phases in $La_{2-x}Ba_xCuO_4$ and 214 cuprates

As already discussed in section 3, the onset of clearly identifiable 2D superconducting correlations in $La_{2-x}Ba_xCuO_4$ with $x = \frac{1}{8}$ occurs at $\sim$40 K, together with the onset of static spin-stripe order. It would be natural to associate this behavior with the simultaneous onset of local PDW order; however, an attempt to reach a consistent interpretation of a broad range of results leads to a more nuanced story.

The original motivation for applying the PDW concept to $La_{1.875}Ba_{0.125}CuO_4$ was to explain the dynamical layer decoupling through the frustration of the interlayer Josephson coupling in the LTT phase [6, 109], as discussed in section 7. It provides a compelling account[18] for the induced dynamical layer decoupling produced in underdoped $La_{2-x}Sr_xCuO_4$ by a modest c-axis magnetic field [2]. Moreover, the sensitivity of the PDW to disorder, which limits the growth of the superconducting correlation length within the planes, provides a natural explanation for the existence of an enormously enhanced 'superconducting fluctuation' regime, characterized by enhanced contributions of local superconductivity to the electrical conductivity and to (strongly anisotropic) diamagnetism, but with no global phase coherence. Thus, it naturally accounts for the most dramatic aspects of the experimental data [1] below the spin ordering temperature T_{SDW}. We consider this strong evidence that the basic ingredients of the theory are applicable to the stripe ordered state of $La_{2-x}Ba_xCuO_4$ and closely related materials. In addition, the observed transition at temperature $T_{3\mathrm{D}}$ into a state with zero resistance in all directions has a natural interpretation in terms of an assumed PDW state as the superconducting glass transition [7]. Besides having zero resistance, the glass phase presumably shows no Meissner effect and zero critical current. If this latter identification is correct, it leads to the further prediction that this phase should be characterized by various phenomena associated with slow dynamics, characteristic of spin glasses, as well as with breaking of time reversal symmetry. The experimental detection of such phenomena below $T_{3\mathrm{D}}$ supercurrents) would serve as further confirmation of the existence of a PDW in this material. (For example, the glass phase would likely exhibit a metastable zero-field Kerr effect [58].)

One can also look for evidence for the PDW in single-particle properties. One of the key features of the PDW stripes is the gapping of single-particle excitations in the antinodal region, as illustrated in figure 4; in contrast, the nodal states remain ungapped. From the underlying band-structure, one sees that the largest contribution to the density of states with energies near E_{F} comes from the antinodal regions (where the dispersion is relatively flat); thus, the onset

[18] Since $La_{2-x}Sr_xCuO_4$ retains the LTO structure to low temperatures, and the spin correlations in the c-direction measured at zero field are extremely short-ranged [119], it is unclear whether the charge stripes in neighboring planes tend to be perpendicular to each other, as in the LTT materials, or parallel but offset by half a period from each other, as in the $YBa_2Cu_3O_{6+x}$ bilayers. In either case, the interlayer Josephson coupling for a PDW would be highly frustrated.

of local PDW order should have a major impact on properties sensitive to the total density of states. Conversely, properties that are largely determined by near nodal quasi-particle dynamics, which presumably includes the quasi-particle contribution to the in-plane conductivity, may be less strongly affected.

Observed striking changes in various transport properties of several stripe order cuprates can be interpreted in this light as being suggestive of the appearance of local PDW order at the onset of *charge*-stripe order at T_{CO} (which is generally somewhat higher than T_{SDW}). In $La_{2-x}Ba_xCuO_4$ and Nd- and Eu-doped $La_{2-x}Sr_xCuO_4$, it is observed that the in-plane thermopower drops dramatically below T_{CO} [1], [120]–[122] as does the Hall resistivity [123]–[125]. Furthermore, the opening of a superconducting-like gap as the temperature drops below T_{CO} results in an observed [126] suppression of the in-plane optical conductivity at frequencies below 40 meV. In contrast, the in-plane dc-resistivity changes relatively little [1, 127] upon cooling through T_{CO}.

Putting aside the issue of the onset-temperature, the notion that stripe-ordered cuprates exhibit local PDW order is also supported by ARPES studies. For example, measurements on stripe-ordered $La_{1.48}Nd_{0.4}Sr_{0.12}CuO_4$ at $T = 15\,K$ ($>2T_c$) reveal a gapless nodal arc of states covering roughly a third of the nominal Fermi surface, as well as a gap reaching 30 meV in the antinodal region [128]. Temperature-dependent ARPES measurements on $La_{2-x}Ba_xCuO_4$ with $x = \frac{1}{8}$ indicate that, for temperatures above the spin-ordering transition, there is a gapless nodal arc of states, together with a substantial antinodal gap [45].

However, there are several aspects of this story, which require further analysis. Firstly, there is the issue that different aspects of the crossovers we would like to identify with the onset of local PDW order appear to onset at different temperatures. This is not necessarily inconsistent, as a crossover (as opposed to a phase transition) can appear to occur at somewhat different temperatures depending on what quantity is measured and how the data are analyzed. Nonetheless, the drop in the thermopower and Hall number appears to have a very sharp onset at T_{CO}, while the superconducting-like drop in the in-plane resistivity at T_{SDW} is also very sharp, at least in $\frac{1}{8}$-doped $La_{2-x}Ba_xCuO_4$. (In this sense, it is reminiscent of the situation [129] in O-doped La_2CuO_4, where the sharply defined spin ordering and superconducting ordering transitions occur at the same temperature (in zero field) with very small uncertainty.)

A still more perplexing issue arises in correlating the onset of the signatures of 2D superconductivity in $La_{1.875}Ba_{0.125}CuO_4$ with the thermal evolution of the ARPES [44, 45] spectrum. Below T_{SDW}, there is clear evidence of the appearance of a d-wave-like gap in the nodal region, with the scale of this second gap being smaller than the pre-existing antinodal gap [45]. This behavior suggests that uniform d-wave superconductivity develops and coexists with the PDW superconductor below T_{SDW}. However, this is somewhat problematic, as the proposed explanation of the dynamical interlayer decoupling and the bounded growth of superconducting correlations that occurs below T_{SDW} rests on the assumed (near) absence of a uniform component of the order parameter in each plane. Reconciling the uniform d-wave component of the order parameter inferred spectroscopically from ARPES studies of $La_{2-x}Ba_xCuO_4$ with the apparently almost complete absence of such a component inferred from bulk transport measurements on the same material is a challenge for future work. It may be significant, however, that ARPES studies of $La_{1.6-x}Nd_{0.4}Sr_xCuO_4$ [128] and $La_{1.8-x}Eu_{0.2}Sr_xCuO_4$ [130] appear consistent with pure PDW order (i.e. there is no d-wave gap in the nodal region), although the PDW appears to set in at around T_{CO}, which can be substantially greater than T_{SDW} in these materials.

9.2. Dynamical layer decoupling and quasi-2D behavior in the cuprates

The cuprate superconductors are layered materials with varying degrees of quasi-2D behavior. Evidence for quasi-2D behavior (and for dimensional crossover) in the cuprates has existed for a long time and it is well documented. It is thus useful to compare and contrast this well-known behavior with the unexpected layer decoupling effect observed in $La_{2-x}Ba_xCuO_4$.

In a quasi-2D system, as a continuous thermodynamic superconducting phase transition is approached, the in-plane correlation length grows very rapidly. While at first the fluctuations have a markedly 2D character, very close to the phase transition they rapidly cross over to their ultimate 3D behavior. Dimensional crossover is observed, for instance, in dynamical probes of some cuprates. High frequency ($\sim$100 GHz) conductivity measurements in $Bi_2Sr_2CaCu_2O_{8+\delta}$ (the most quasi-2D material among the cuprates) by Corson *et al* [131] showed that (at those frequencies) the fluctuation conductivity is 2D-like and exhibits Kosterlitz–Thouless behavior, as if the CuO_2 planes were effectively decoupled. Similarly, quasi-2D behavior in the dynamic conductivity (with frequencies in the range 1–10 GHz) has been observed in underdoped $La_{2-x}Sr_xCuO_4$ near T_c by Kitano *et al* [132]. By probing the system at finite frequency, these experiments explore the correlations at a frequency dependent mesoscopic length scale, where sufficiently weak 3D couplings have negligible effect on the physics. By contrast, the resistive transition both in $Bi_2Sr_2CaCu_2O_{8+\delta}$ and in $La_{2-x}Sr_xCuO_4$, measured at zero frequency in macroscopic samples, is not of the 2D XY (Kosterlitz–Thouless) type, but rather reflects the 3D nature of these materials. Even in a case ($Bi_2Sr_2CaCu_2O_{8+\delta}$, $T_c \approx 85$ K [133]) where nonlinear current–voltage behavior appears to indicate a Kosterlitz–Thouless transition for in-plane transport, *c*-axis measurements indicate that an interlayer Josephson current sets in at a temperature 2 K higher.

In contrast, the unusual layer decoupling effect observed in *stripe-ordered* $La_{2-x}Ba_xCuO_4$ takes place in a temperature range where the CuO_2 planes appear to become superconducting (well above the 3D critical temperature) [1, 28]. The layer decoupling effect is observed in the resistive transition, and is thus not a dimensional crossover. As we noted above, in this regime $La_{2-x}Ba_xCuO_4$ behaves as if for some reason the effective inter-layer Josephson coupling is either turned off (which is unphysical) or is somehow *frustrated*.

Support for this idea is provided by recent Josephson resonance experiments in $La_{2-x}Sr_xCuO_4$ by Schafgans *et al* [2], which essentially measure the *c*-axis superfluid stiffness, ρ_c. In the absence of an external magnetic field, ρ_c has the expected [134]–[136] magnitude, i.e. ρ_c is proportional to the normal state conductivity at T_c. However, for underdoped materials, ρ_c becomes unmeasurably small in the presence of moderate magnetic fields ($B \leqslant 8T$). Magnetic fields are known to induce static spin-stripe order (as detected by neutron scattering experiments [41]) in precisely the same range of field strengths and hole concentration. These experiments thus suggest [2] that the ‘fluctuating stripe order’ [15] seen in $La_{2-x}Sr_xCuO_4$ at zero field may actually be of the PDW type and that dynamical layer decoupling occurs as static stripe order is stabilized in a magnetic field[19]. Indeed, in materials, including $La_{2-x}Ba_xCuO_4$ and $La_{1.6-x}Nd_{0.4}Sr_xCuO_4$, which exhibit stripe order in zero field, ρ_c is found [137] to be orders of magnitude smaller than its ‘expected’ value on the basis of the normal state conductivity.

[19] While it is tempting to reinterpret in hindsight the results of Corson *et al* [131] as being indicative of ‘fluctuating PDW order’ in $Bi_2Sr_2CaCu_2O_{8+\delta}$, we should note that the STM data on this material [5] show a glassy pattern of short-range stripe order at high bias. However, as we explained elsewhere in this paper, a glassy version of the PDW state would not exhibit a sharp layer decoupling effect.

9.3. Possible relevance to other cuprates

Although there are still open issues, the PDW state (or its glassy version) seems to offer a rather compelling explanation for what is otherwise an extremely surprising set of phenomena observed in stripe ordered cuprates. Could these ideas also be relevant to a broader range of phenomena in the cuprates? The direct empirical information available [15] concerning the structure of any sort of static or fluctuating stripe order present in cuprates outside the 214 family is much less clear[20]. Consequently, any attempt to achieve a theoretical understanding based on the assumed existence of a PDW state is necessarily speculative. We therefore present the discussion of this final section in the spirit of provocative conjectures, which we believe are deserving of further investigation.

ARPES studies of $Bi_2Sr_2CaCu_2O_{8+\delta}$ [82] and $La_{2-x}Sr_xCuO_4$ [138, 139] have revealed 'Fermi arcs' of gapless states between antinodal pseudogaps. There has been a great deal of controversy over the nature of the antinodal pseudogap [140, 141]. Two recent studies of $Bi_2Sr_2CaCu_2O_{8+\delta}$ have reported signatures of Bogoliubov quasi-particles in the antinodal gap region [63, 142], which was interpreted as being suggestive that the pseudogap is, at least in part, produced by superconducting fluctuations. On cooling through T_c, a d-wave gap appears along the nodal arc [138, 139, 143]. In near optimally doped samples, as $T \to 0$, this d-wave gap and the pseudo-gap merge to form a single gap with a simple $[\cos(k_x) - \cos(k_y)]$ form. However, in underdoped samples, even as $T \to 0$, the nodal gap appears to have a different energy scale than the antinodal gap (i.e. they do not merge to form a simple d-wave gap) [139, 143]. Thus, in some ways it is clear that there are two distinct gaps—an antinodal pseudo-gap that might be associated with some sort of 'competing' order, and a nodal gap, which is clearly superconducting in the sense that it onsets quite sharply at T_c. However, in other ways it seems that all the gaps have some unifying superconducting character.

We propose that this puzzle may be resolved by postulating that there are two distinct gaps, both with superconducting character in the sense that one is associated with uniform the other with modulated superconducting order. Indeed, the measured quasi-particle spectral function in the pseudogap looks somewhat like that of the PDW state (see figure 4)[21]. Moreover, just such a combination of modulated and uniform superconducting orders has been previously proposed on phenomenological grounds to explain [64, 92] STM spectra [3, 65], [144]–[146] in $Bi_2Sr_2CaCu_2O_{8+\delta}$ and other cuprates [5, 147].

Seemingly more direct evidence of superconducting fluctuations in the normal state of $La_{2-x}Sr_xCuO_4$, $Bi_2Sr_2CaCu_2O_{8+\delta}$ and $Bi_2Sr_{2-y}La_yCuO_6$ has been reported by Ong and co-workers [148, 149] based on measurements of the Nernst effect and diamagnetism. Nernst

[20] The results of recent neutron scattering studies of underdoped $YBa_2Cu_3O_{6+x}$ by Hinkov *et al* [55] have confirmed [52] the existence of a nematic phase, derived from the weak melting of a stripe ordered state, onsetting below a temperature comparable to the pseudogap onset-temperature, $T^\star$. Still more recently [56], the same authors have demonstrated that modest magnetic fields stabilize static spin-stripe order where primarily fluctuating (nematic) order existed at zero field.

[21] Technically, the electron-hole-mixed quasi-particles in the antinodal region of a PDW state are not perfectly symmetric with respect to E_F (see figure 4), in contrast to Bogoliubov quasi-particles; however, to detect this distinction, one would need to measure a sample containing a single-domain PDW state. Measurements on a nematic PDW state in $Bi_2Sr_2CaCu_2O_{8+\delta}$ would average over stripe orientations; furthermore, the experimental 'quasi-particle' peaks are quite broad in the pseudogap state [63, 142], so that any fine details are hidden by damping. The overdamping also fills in the spectral weight at E_F, in contrast to the true gap that is found for $T < T_c$ [63, 138].

measurements on $YBa_2Cu_3O_{6+x}$ [150] and STM studies of $Bi_2Sr_2CaCu_2O_{8+\delta}$ [151] suggest that disorder may be important to the existence of the fluctuation effects over a substantial temperature range. It is intriguing that the onset temperatures of the enhanced Nernst response in $La_{2-x}Sr_xCuO_4$ has a maximum at $x \sim 0.1$ [148], close to the optimum doping for stripe order. Moreover, Taillefer and co-workers [152] have found close correlations between an enhanced Nernst signal and stripe order. Neither the observed sensitivity to disorder nor the association with stripe order, by themselves, necessarily negate the interpretation of these effects in terms of superconducting fluctuations; however, both would be unusual in the case of a simple, homogeneous d-wave superconductor. While we are far from having an explicit theory, it seems to us that these general trends are consistent with the existence of a disordered PDW state over at least a portion of the pseudogap phase. Specifically, Ong and co-workers [153] have reported the observation of a sublinear dependence of the magnetization on magnetic field ($M \sim -B^{\alpha}$ with $\alpha < 1$) in a relatively narrow but non-vanishing range of temperatures above T_c in crystals of $Bi_2Sr_2CaCu_2O_{8+\delta}$. This behavior, if it truly persists in the limit $B \to 0$, must signify the existence of a distinct phase of matter in this range of temperatures, which we very tentatively propose could be a superconducting glass formed from a disordered PDW[22].

One of the most intriguing recent discoveries in the cuprates involve several distinct observations of a rather subtle, and not fully understood, form of time reversal symmetry breaking in the pseudogap phase of $YBa_2Cu_3O_{6+x}$ [57, 58] and $HgBa_2CuO_{4+\delta}$ [59]. As we have seen, various forms of subtle time-reversal symmetry breaking can occur when frustration is added into the PDW mix. It is our hope that, with further work, a relation can be established between these two rather vague statements.

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