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Simple Formulae for Higher Order Asymptotic Freedom Corrections

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ABSTRACT

We present simple formulae for the next-to-the-leading order asymptotic freedom corrections to the moments of the non-singlet and the singlet combinations of the deep-inelastic structure functions for electron, muon and neutrino scattering.

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During the past year the higher order calculations, necessary to obtain the next-to-the-leading order asymptotic freedom corrections to the deep-inelastic structure functions, have been completed.[1,2,3] In addition to the calculation of certain renormalization group functions (two-loop β function,[4] two-loop anomalous dimension matrix,[1,3] and one loop gluon and quark Wilson coefficient functions[2,3]), one has to deal with the mixing between quark and gluon operators. The mixing problem is much more difficult in the next-to-the-leading order than in the leading order. The authors of ref. 3 have solved this problem. However their final formulae for the Q^2 dependence of the moments of deep-inelastic structure functions are very complicated.[3,5,6] In particular the expressions of refs. 3 and 5 involve thirteen functions of n, in addition to the familiar exponents d^n_{NS} and $d^n_{\pm}$ of the leading order. Since the functions in question have very complicated analytic expressions, one is led[3,5] to the representation of higher order corrections by a vast array of numbers.*

In this note we present equivalent but simpler expressions for the higher order corrections. In order to simplify the presentation we only state the final results and discuss them in some detail. The derivation of our formulae can be found in a review article by one of us.[7]

We present first the formulae for the moments of $F_2(x, Q^2)$ for an arbitrary deep-inelastic process. We shall comment on other structure functions at the end of our paper. Consider the moments of $F_2(x, Q^2)$

$$\int_0^1 dx\, x^{n-2} F_2(x, Q^2) = M_2^{NS}(n, Q^2) + M_2^{S}(n, Q^2) \tag{1}$$

where "NS" and "S" stand for non-singlet and singlet contributions respectively. The Q^2 dependence of $M_2^{NS}(n, Q^2)$ and $M_2^{S}(n, Q^2)$, up to and including next-to-the-leading order asymptotic freedom corrections, is then given as follows

$$M_2^{NS}(n, Q^2) = \bar{A}_n^{NS}\left[1 + \frac{\bar{R}^{NS}_{2,n}(Q^2)}{\beta_0 \ln \frac{Q^2}{\bar{\Lambda}^2}}\right]\left[\ln \frac{Q^2}{\bar{\Lambda}^2}\right]^{-d^n_{NS}} \tag{2}$$

and

$$M_2^{S}(n, Q^2) = \bar{A}_n^{-}\left[1 + \frac{\bar{R}^{-}_{2,n}(Q^2)}{\beta_0 \ln \frac{Q^2}{\bar{\Lambda}^2}}\right]\left[\ln \frac{Q^2}{\bar{\Lambda}^2}\right]^{-d^n_{-}}$$

$$+ \bar{A}_n^{+}\left[1 + \frac{\bar{R}^{+}_{2,n}(Q^2)}{\beta_0 \ln \frac{Q^2}{\bar{\Lambda}^2}}\right]\left[\ln \frac{Q^2}{\bar{\Lambda}^2}\right]^{-d^n_{+}} . \tag{3}$$

Here

$$\bar{R}^{i}_{2,n}(Q^2) = \bar{R}^{i}_{2,n} - d^n_i \frac{\beta_1}{\beta_0} \ln\ln \frac{Q^2}{\bar{\Lambda}^2} \quad ; \quad i = NS,+,- \tag{4}$$

where β_0 and β_1 are the coefficients of the perturbative expansion of the β function

$$\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{(16\pi^2)^2} + \cdots . \tag{5}$$

We have[4,8] $\beta_0 = 11 - \frac{2}{3} f$ and $\beta_1 = 102 - \frac{38}{3} f$ where f is the number of flavors.

The parameters d_i^n are the familiar exponents of the leading order. On the other hand, the parameters $\bar{R}^i_{2,n}$ are new and we have calculated them on the basis of the results of refs. 1-3. Numerical values for d_i^n and $\bar{R}^i_{2,n}$ for f = 3 and 4 are collected in the Table. Furthermore the constants $\bar{A}_n^{NS}$ and $\bar{A}_n^{\pm}$ are not calculable in QCD perturbation theory. They can be treated as phenomenological parameters to be found by fitting Eqs. (2) and (3) to the data.

As pointed out in ref. 2, the actual numerical values of the parameters $\bar{R}^i_{2,n}$ depend on the definition of the effective coupling constant $\bar{g}^2(Q^2)$ or equivalently on the definition of the scale parameter $\bar{\Lambda}$. The values for $\bar{R}^i_{2,n}$ in the Table and the $\bar{\Lambda}$ which enter Eqs. (1)-(3) correspond to the "minimal scheme bar" ($\overline{MS}$) introduced in ref. 2. Expressions for higher order corrections in different schemes for Λ can be obtained from Eqs. (2) and (3) by changing there

$$\bar{R}^i_{2,n} \rightarrow \bar{R}^i_{2,n} - \beta_o d_i^n \ln \kappa_k^2 \qquad , \qquad i = NS, +, - \tag{6}$$

$$\bar{\Lambda} \rightarrow \Lambda_k \qquad , \tag{7}$$

where

$$\kappa_k = \frac{\Lambda_k}{\bar{\Lambda}} \tag{8}$$

distinguishes between various schemes. Needless to say all schemes are equivalent in the order considered and differ only by higher order corrections $O(1/\ln^2 Q^2/\Lambda^2)$ not included in Eqs. (2)-(3).

For completeness we present the formulae for $\bar{R}^i_{2,n}$ in terms of the renormalization group parameters which have been calculated in refs. 1-4, 8 and 9. We have

$$\bar{R}^{NS}_{2,n} = \bar{B}^{NS}_{2,n} + \frac{\gamma^{(1),n}_{NS}}{2\beta_o} - \frac{\gamma^{(0),n}_{NS}}{2\beta_o^2}\beta_1 \quad , \qquad (9)$$

$$\bar{R}^{-}_{2,n} = \bar{B}^{-}_{2,n} + \frac{\gamma^{(1),n}_{--}}{2\beta_0} - \frac{\lambda^n_-\beta_1}{2\beta_o^2} - \frac{\gamma^{(1),n}_{-+}}{2\beta_o + \lambda^n_- - \lambda^n_+} \quad , \qquad (10)$$

and

$$\bar{R}^{+}_{2,n} = \bar{B}^{+}_{2,n} + \frac{\gamma^{(1),n}_{++}}{2\beta_o} - \frac{\lambda^n_+\beta_1}{2\beta_o^2} - \frac{\gamma^{(1),n}_{+-}}{2\beta_o + \lambda^n_+ - \lambda^n_-} \qquad (11)$$

where

$$\bar{B}^{\mp}_{2,n} = \bar{B}^{\psi}_{2,n} + \frac{(\lambda^n_{\mp} - \gamma^{0,n}_{\psi\psi})}{\gamma^{0,n}_{\psi G}}\bar{B}^{G}_{2,n} \quad . \qquad (12)$$

It should be emphasized that the form of the last terms in Eqs.(10) and (11) is only true for our choice of the matrix $\hat{U}$ used to diagonalize one-loop anomalous dimension matrix (see Eq. 17). $\bar{B}^{NS}_{2,n}$, $\bar{B}^{\psi}_{2,n}$ and $\bar{B}^{G}_{2,n}$ in Eqs. (9) to (12), are the coefficients[2,3] of $g^2/16\pi^2$ in the perturbative expansions of the Wilson coefficient functions of the non-singlet, singlet fermion and gluon operator respectively.

The remaining parameters are defined by the following equations

$$\hat{U}^{-1}\hat{\gamma}^{0,n}\hat{U} = \begin{pmatrix} \lambda^n_- & 0 \\ 0 & \lambda^n_+ \end{pmatrix} \qquad (13)$$

$$\hat{U}^{-1}\hat{\gamma}^{(1),n}\hat{U} = \begin{pmatrix} \gamma^{(1),n}_{--} & \gamma^{(1),n}_{-+} \\ \gamma^{(1),n}_{+-} & \gamma^{(1),n}_{++} \end{pmatrix} \qquad (14)$$

where $\hat{\gamma}^{0,n}$ and $\hat{\gamma}^{(1),n}$ are the one-loop[9] and the two-loop[1,3] anomalous dimension matrices respectively, and $\hat{U}$ is a matrix which diagonalizes $\hat{\gamma}^{0,n}$. Furthermore $\hat{\gamma}_{\psi G}^{0,n}$ is one of the non-diagonal elements of the matrix $\hat{\gamma}^{0,n}$ and $\gamma_{NS}^{0,n}$ and $\gamma_{NS}^{(1),n}$ are the one-loop and the two-loop anomalous dimensions of nonsinglet operators respectively. We also recall that

$$d_i^n = \frac{\lambda_i^n}{2\beta_o} \qquad i = NS, +, - \tag{15}$$

Compilations of analytic formulae and of numerical values necessary for the evaluation of the parameters on the r.h.s. of Eqs. (9)-(12) can be found in refs. 7 and 10 in exactly the same notation as here.

It should be remarked that the matrix $\hat{U}$ is not defined uniquely by Eq. (13). In fact any matrix $\hat{U}'$ which is related to $\hat{U}$ by

$$\hat{U}' = \hat{U}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{16}$$

where a and b are arbitrary finite numbers, satisfies Eq. (13). It can be shown easily however that the numerical values of $\bar{R}^i_{2,n}$ are independent of the choice of the matrix $\hat{U}$ although the form of the last terms in Eqs. (10) and (11) depends on $\hat{U}$. The simple form of Eqs. (10) and (11) is obtained by using $\hat{U}$ for which

$$\hat{U}^{-1} = \begin{bmatrix} 1 & \dfrac{\lambda_-^n - \gamma_{\psi\psi}^{0,n}}{\gamma_{\psi G}^{(1),n}} \\ -1 & \dfrac{\gamma_{\psi\psi}^{0,n} - \lambda_+^n}{\gamma_{G\psi}^{(0),n}} \end{bmatrix} . \tag{17}$$

It should be also added that in the notation of refs. 7 and 10

$$\hat{\gamma}^{0,n} = \begin{pmatrix} \gamma^{0,n}_{\psi\psi} , & \gamma^{0,n}_{G\psi} \\ \gamma^{0,n}_{\psi G} , & \gamma^{0,n}_{GG} \end{pmatrix} \tag{18}$$

and similarly for $\hat{\gamma}^{1,n}$.

The formulae (2) and (3) differ from the corresponding equations in refs. 3, 5 and 6 in three aspects:

i) They are much simpler,

ii) No reference is made to a special value of $Q^2 = Q_o^2$, and

iii) No reference is made to the parton distributions.

Let us briefly discuss points ii) and iii).

Consider Eq. (2). One can choose a particular value of $Q^2 = Q_o^2$ and trade the unknown constants $\bar{A}_n^{NS}$ for the experimentally measured moments $M_2^{NS}(n, Q_o^2)$. One obtains

$$M_2^{NS}(n, Q^2) = M_2^{NS}(n, Q_o^2) \left[1 + \frac{\bar{R}^{NS}_{2,n}(Q^2)}{\beta_o \ln \frac{Q^2}{\bar{\Lambda}^2}} - \frac{\bar{R}^{NS}_{2,n}(Q_o^2)}{\beta_o \ln \frac{Q_o^2}{\bar{\Lambda}^2}} \right] \left[\frac{\ln \frac{Q^2}{\bar{\Lambda}^2}}{\ln \frac{Q_o^2}{\bar{\Lambda}^2}} \right]^{-d^n_{NS}} . \tag{19}$$

It is obvious that Eqs. (2) and (19) are equivalent to each other through the next-to-the-leading order. Equation (19) may appear more convenient than Eq. (2) in phenomenological applications. According to Eq. (19) one is instructed to measure $M_2^{NS}(n, Q^2)$ at an arbitrary value $Q^2 = Q_o^2$. Then $M_2^{NS}(n, Q^2)$ at $Q^2 \neq Q_o^2$ is given in terms of a sole free parameter $\bar{\Lambda}$. It should, however, be kept in mind that the value of Q_o^2 is arbitrary as required by the renormalization group equations and the predictions for the moments $M_2^{NS}(n, Q^2)$ should be independent of Q_o^2. By picking out one particular value of Q_o^2 in order to determine $M_2^{NS}(n, Q_o^2)$ one gives this value specific significance. For consistency one should find $M_2^{NS}(n, Q_o^2)$ from the data by choosing various values of Q_o^2 and check whether expressions (19) with various values of Q_o^2 give results compatible with each other. In order to simplify this procedure and at the same time to impose the independence of the phenomenological fit of Q_o^2, it is convenient to get rid of Q_o^2 and use Eq. (2) instead. The unknown parameters $\bar{A}_n^{NS}$ which appear there are constants which do not depend on Q_o^2. Similar comments apply to Eq. (3) where $\bar{A}_n^{\mp}$ are Q_o^2 independent numbers to be found from experiment. Notice that essentially Eq. (2) can be obtained from Eq. (19) by setting in the latter equation $Q_o^2 = \infty$. We observe therefore that whereas Eq. (19) is an expansion around $\bar{g}^2(Q_o^2) \neq 0$, Eq. (2) is the true asymptotic expansion around the fixed point of the theory which is at $\bar{g}^2(\infty) = 0$.

Regarding point iii) we would like to recall[3,11,12] that the parton distributions cannot be uniquely defined beyond the leading order of asymptotic freedom. Many definitions are possible which differ from each other by next-to-the-leading order corrections. Therefore the study of higher order effects on the Q^2 evolution of quark and gluon distributions does not make much sense because the result of such a study is not a prediction of the theory but depends sensitively on one's definition of parton distributions. Still with a given definition of parton distributions, the parton language is useful in comparing asymptotic freedom predictions in various processes such as deep-inelastic scattering, μ pair production, etc.

Although the formulae for particular structure functions in terms of the effective Q^2 dependent parton distributions may be relatively simple, the expressions for the Q^2 evolution of these parton distributions in terms of their values at a particular Q^2 value, $Q^2 = Q_o^2$, are very complicated. Examples can be found in refs. 3, 5, 6 and 7. Therefore we think that the simplest and most straightforward tests of higher order corrections can be done directly by means of the Eqs. (2) and (3) without any reference to parton distributions.

After having discussed some attractive features of Eqs. (2) and (3) we should mention a possible limitation in the use of Eq. (3). We observe that the last term in the expression for $\bar{R}^-_{2,n}$ in Eq. (10) is singular when $d^n_+ = d^n_- + 1$. While this singularity does not appear for physical values of n and f, it can lead to anomalously large higher order corrections to the "-" contributions and an apparent breakdown of perturbation theory. The singularity in $\bar{R}^-_{2,n}$ is of course spurious and is due to the mixing which can occur when the $\bar{g}^2$ corrections to "-" contributions are of the same order in $\bar{g}^2$ as the leading order "+" contributions. Hence, the singularity found in $\bar{R}^-_{2,n}$ is cancelled by a corresponding singularity in $\bar{A}^+_n$.

Although the procedure is somewhat arbitrary, we can isolate the singularity in $\bar{A}^+_n$ as follows

$$\bar{A}^+_n = A^+_n(Q_o^2) + \frac{\bar{A}^-_n}{2\beta_o^2} \frac{\gamma^{(1),n}_{-+}}{1 + d^n_- - d^n_+} \left[\ln \frac{Q_o^2}{\bar{\Lambda}^2} \right]^{d^n_+ - d^n_- - 1} \tag{20}$$

where $A^+_n(Q_o^2)$ is nonsingular. The freedom associated with choosing the scale Q_o^2 reflects the arbitrariness of the separation of the singular part. Inserting Eq. (20) into Eq. (3), we obtain the following expression for the singlet moments

$$M_2^s(n, Q^2) = \bar{A}_n^- \left[1 + \frac{1}{\beta_o \ln \frac{Q^2}{\bar{\Lambda}^2}} \left(\bar{R}_{2,n}^-(Q^2) + \frac{1}{2\beta_o} \frac{\gamma_{-+}^{(1),n}}{1 + d_-^n - d_+^n} \left(\frac{\ln \frac{Q^2}{\bar{\Lambda}^2}}{\ln \frac{Q_o^2}{\bar{\Lambda}^2}} \right)^{1+d_-^n - d_+^n} \right) \right]$$

$$\times \left[\ln \frac{Q^2}{\bar{\Lambda}^2} \right]^{-d_-^n} + A_n^+(Q_o^2) \left[1 + \frac{\bar{R}_{2,n}^+(Q^2)}{\beta_o \ln \frac{Q^2}{\bar{\Lambda}^2}} \right] \left[\ln \frac{Q^2}{\bar{\Lambda}^2} \right]^{-d_+^n} \quad . \qquad (21)$$

By using Eq. (10), we observe the explicit cancellation of the singularity and for reasonable values of the scale Q_o^2 we do not expect the existence of the singularity to spoil the validity of perturbative nature of the next-to-the-leading corrections.

As we have noted above, the singularity does not occur for physical values of n or f. For integer n the singularities occur for non-integer number of flavors, f. For n = 2, 4, 6, 8 and 10 the position of the singularity is at f = 5.583, 3.788, 1.627, 0.142 and -0.988 respectively. The corresponding residue in $\bar{R}_{2,n}^-$ are equal to −15.43, 1.36, 0.2, 0.007 and -0.02. We observe that the residue of the poles vanishes rapidly for the higher moments as the mixing decreases for large n. As a result only in the case of the second moment is the mixing large enough to require separation of the singular parts as described in Eqs. (20) and (21) and then only when f = 5 or 6. For all other physical values of n and f, the existence of a nearby singularity does not disturb the validity of perturbative nature of the corrections and Eq. (3) can be safely used.

In some cases it is useful to consider interpolation between different numbers of flavors as in an approximate treatment of quark mass effects. In such cases, care must be taken in using Eq. (3).

Another possible difficulty may occur in certain inversion procedures used in reconstructing the structure functions from the moments. Some procedures require a continuation of the moments to non-integer n. The singularities in $\bar{R}^-_{2,n}$ for non-integer n may be expected to provide the leading singularity in this reconstruction. In most situations however the residue is small and will not strongly affect the reconstruction. In other situations, more care must be taken to ensure a proper treatment of the cancellation of this singularity before attempting a proper reconstruction of the structure functions.

So far we have discussed only the structure function F_2. For the longitudinal structure function F_L expressions are much simpler because F_L vanishes in the leading order. The relevant formulae are well known and can be found for instance in refs 7, 11 and 12. For F_3, Eq. (2) or (19) applies with $\bar{R}^{NS}_{2,n}(Q^2)$ replaced by[2,12]

$$\bar{R}_{3,n}(Q^2) = \bar{R}^{NS}_{2,n}(Q^2) - \frac{4}{3}\,\frac{4n+2}{n(n+1)} \tag{22}$$

with n odd for $F_3^{\nu+\bar{\nu}}$ and n even for $F_3^{\nu-\bar{\nu}}$.

Finally we would like to discuss the numerical evaluation of the parameters in Eqs. (2) and (3), and the size of higher order corrections. First we notice that for sufficiently large values of Q^2 and for $n \geq 4$ the next-to-the-leading order corrections to the "-" operator are at least as important as the leading contributions to the "+" operator. This is due to the fact that

$$d^n_+ > d^n_- + 1 \qquad \text{for } n \geq 4 \quad .$$

Therefore for $n \geq 4$ the next-to-the-leading order corrections to $\bar{A}^-$ should be treated on the same footing as the leading order contributions to the $\bar{A}^+$ operator.

Furthermore for n > 8 the former contributions dominate over the latter ones. Similarly the next-to-the-leading order corrections to the "+" operator are for n > 4 and large Q^2 only as important as $1/(\ln^2 Q^2/\Lambda^2)$ corrections to the "-" operator. We further note that for n > 4 $\bar{R}^- \approx \bar{R}^{NS}$ and $d_-^n \approx d_{NS}^n$, which results from the small mixing between quark and gluon operators for large n and the identification of the "-" operator with the singlet quark operator. In addition in the framework of the parton model one expects $\bar{A}_n^-$ to be much larger than $\bar{A}_n^+$ which is confirmed by the data.[13] Thus one expects that for n > 4 the singlet structure function will behave essentially the same as the non-singlet structure function for typical hadronic targets.

In terms of the effective coupling constant

$$\bar{\alpha}(Q^2) \equiv \frac{\bar{g}^2(Q^2)}{4\pi} = \frac{4\pi}{\beta_0 \ln \frac{Q^2}{\bar{\Lambda}^2}} - 4\pi \frac{\beta_1}{\beta_0^3} \frac{\ln\ln \frac{Q^2}{\bar{\Lambda}^2}}{\ln^2 \frac{Q^2}{\bar{\Lambda}^2}} \tag{23}$$

the formulae (2) and (3) can be written as follows

$$M_2^{NS}(n, Q^2) = \bar{A}_n^{NS}[\bar{\alpha}(Q^2)]^{d_{NS}^n} D_{2,n}^{NS}(\bar{\alpha}(Q^2)) \tag{24}$$

$$M_2^S(n, Q^2) = \bar{A}_n^-[\bar{\alpha}(Q^2)]^{d_-^n} D_{2,n}^-(\bar{\alpha}(Q^2)) + \bar{A}_n^+[\bar{\alpha}(Q^2)]^{d_+^n} D_{2,n}^+(\bar{\alpha}(Q^2)) \tag{25}$$

where

$$D_{2,n}^i(\bar{\alpha}) = 1 + \frac{\bar{\alpha}}{4\pi} \bar{R}_{2,n}^i \qquad i = NS, +, - \quad . \tag{26}$$

The quantities $D^{i}_{2,n}(\bar{\alpha})$ are plotted in Fig. 1 as functions of $\bar{\alpha}$. The figure is presented mainly for illustration since the actual size of $D^{i}_{2,n}(\bar{\alpha})$ depends on the definition of Λ or equivalently of $\bar{\alpha}(Q^2)$. The curves in the figure correspond to $\overline{MS}$ scheme[2] for which $0.2 < \bar{\alpha}(Q^2) < 0.5$ for $2 < Q^2 < 100\ \mathrm{GeV}^2$ as extracted from the BEBC data.[2,14]

In summary, the main formulae of this paper are the Eqs. (2), (3) and (9-12) which provide a complete description of leading and next-to-the-leading asymptotic freedom corrections to deep-inelastic structure functions. Phenomenological applications of these expressions are beyond the scope of this paper. We hope that the simple formulae presented here will be useful in the study of asymptotic freedom corrections.

ACKNOWLEDGMENTS

We thank Dennis Duke, Douglas Ross and Chris Sachrajda for discussions. One of us (A.J.B.) would like to thank the SLAC Theory Group for its hospitality where part of this work has been done.

FOOTNOTE

*Unfortunately the Tables 2 and 3 presented in ref. 5 are incorrect due to errors in Eqs. (2.11, 2.12 and 2.19) of Reference 3 which formed the basis of the calculation (D. Duke and D. Ross, private communication). These errors are not related to the calculation of the two loop anomalous dimensions also presented in Reference 3 and used in our calculation of the higher order corrections in this paper.

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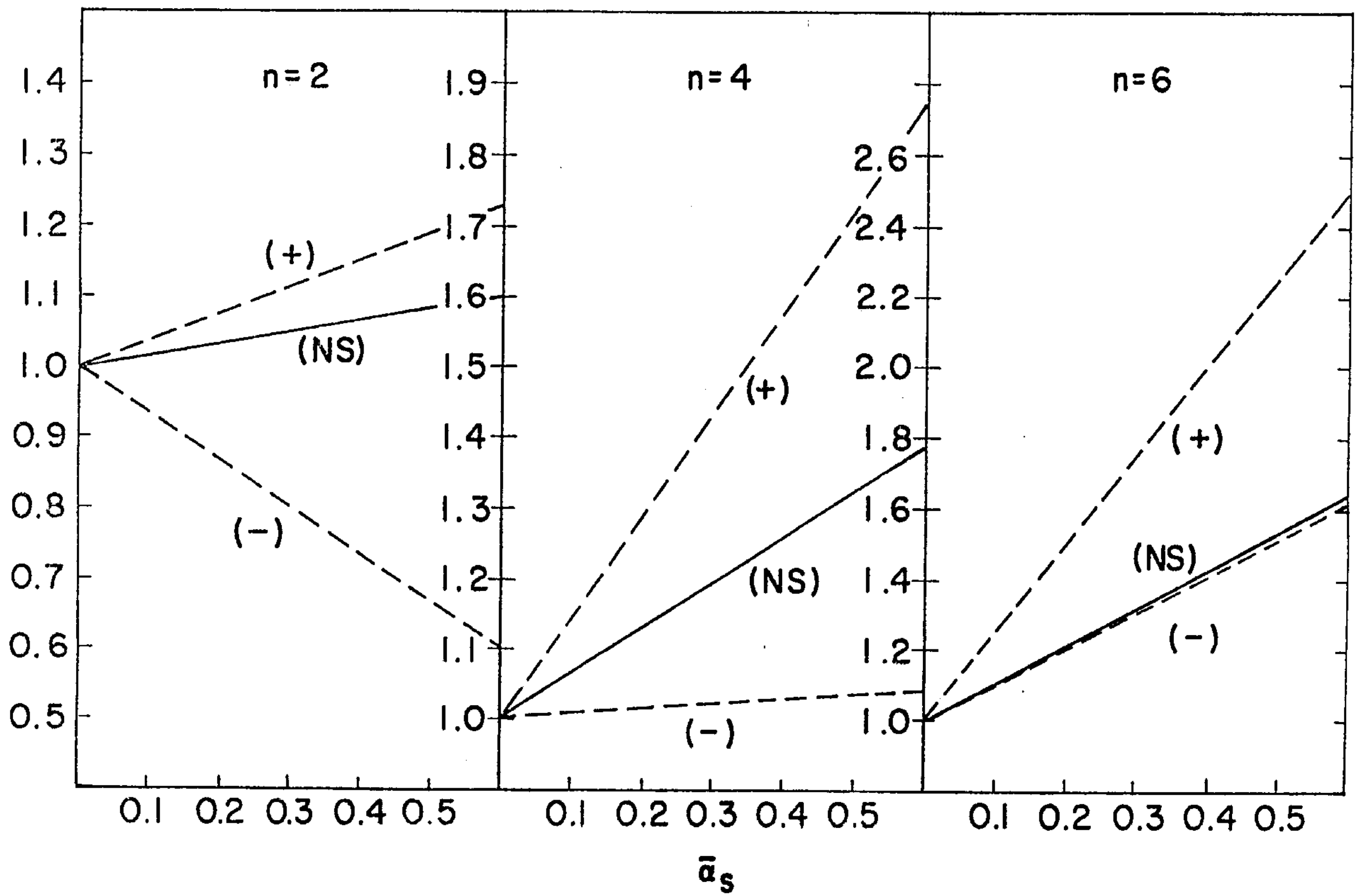
f	n	d^n_{NS}	d^n_-	d^n_+	$\overline{R}^{NS}_{2,n}$	$\overline{R}^-_{2,n}$	$\overline{R}^+_{2,n}$	$\overline{R}_{3,n}$
3	2	0.395	0.000	0.617	1.951	-4.344	3.726	-0.271
	4	0.775	0.760	1.638	7.956	9.078	17.07	6.756
	6	1.000	0.996	2.203	13.19	12.81	30.43	12.36
	8	1.162	1.160	2.587	17.64	17.53	41.72	17.01
	10	1.289	1.287	2.882	21.50	21.44	51.41	20.99
4	2	0.427	0.000	0.747	2.098	-8.117	4.799	-0.124
	4	0.837	0.817	1.852	8.117	0.811	18.17	6.917
	6	1.080	1.074	2.460	13.34	12.99	31.63	12.52
	8	1.255	1.252	2.875	17.78	17.65	43.01	17.15
	10	1.392	1.390	3.192	21.63	21.57	52.78	21.12

Table 1. Numerical values of the parameters d^n_{NS}, $d^n_\pm$, $\overline{R}^{NS}_{2,n}$, $\overline{R}^\pm_{2,n}$ and $\overline{R}^{3,n}$ for f = 3 and f = 4.

FIGURE CAPTION

Fig. 1: Size of the explicit second order corrections $D^i_{2,n}(\overline{\alpha})$ in the $\overline{MS}$ scheme for f = 4.



n=2
n=4
n=6
(+)
(NS)
(−)
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.1 0.2 0.3 0.4 0.5
$\bar{\alpha}_s$

Fig. 1

Erratum

"Simple Formulae for Higher Order Asymptotic Freedom Correction"

W.A. Bardeen and A.J. Buras

Physics Letters 86B, 61, 1979

Eq. 17 contains two misprints. The corrected equation should read as follows:

$$\hat{U}^{-1} = \begin{bmatrix} 1 & \dfrac{\lambda_-^n - \gamma_{\psi\psi}^{0,n}}{\gamma_{\psi G}^{0,n}} \\ -1 & \dfrac{\gamma^{0,n} - \lambda_+^n}{\gamma_{\psi G}^{0,n}} \end{bmatrix} \tag{17}$$

All other equations and results of this paper remain unchanged.

Erratum

"Higher Order Asymptofic-freedom Corrections to Photon-Photon Scattering"

W.A. Bardeen and A.J. Buras

Physical Review D20, 166, 1979

There is a misprint in Eq. (3.29) of this paper, where the second appearance of $\gamma_{G\psi}^{(1),n}$ should read $\gamma_{\psi G}^{(1),n}$. All other equations and results remain unchanged. We thank D.W. Duke and J.F. Owens for informing us about this misprint.