Proceedings of Institute of Mathematics of NAS of Ukraine 2002, Vol. 43, Part 2, 486–489

On One Algebra of Temperley–Lieb Type

Nataly POPOVA

Institute of Mathematics of the NAS of Ukraine, 3 Tereshchenkivska Str., 01601 Kyiv-4, Ukraine
E-mail: *popova_n@yahoo.com*

An algebra generated by projections with relations of Temperley–Lieb type is considered. Knowledge of Gröbner basis of the ideal allows to find a linear basis of the algebra. Some questions of representation theory for this algebra were studied in [13]. Obtained in the present paper are the additional relations, which hold in all finite-dimensional irreducible $*$-representations, although they do not hold in the algebra.

1 Introduction

Temperley–Lieb algebras generated by n projections $p_1, \ldots, p_n$ with relations

$$p_i p_j = p_j p_i, \quad |i-j| > 1, \quad p_i p_{i\pm 1} p_i = \tau p_i, \quad \tau \in \mathbb{R},$$

appeared in [1, 2] in the context of ice-type models. On the other hand, they were applied to studying of von Neumann algebras and problems of knots theory by V. Jones (see [3, 4]). Representations of Temperley–Lieb algebras were studied and used by H. Wenzl, F.M. Goodman, P.P. Martin (see, e.g., [5, 6, 7, 8, 9]) and other authors. Values of parameter τ such that the representations exist were found, a description of irreducible representations was given, their dimensions were calculated and other questions were considered.

In [13] we considered the analogous questions of representation theory for modification of Temperley–Lieb algebra: algebra generated by projections $p_1, \ldots, p_n$ with relations

$$p_i p_j = 0, \quad |i-j| > 1, \quad (i,j) \neq (1,n)\,; \quad p_i p_{i\pm 1} p_i = \tau p_i, \quad p_1 p_n p_1 = \tau p_1, \quad p_n p_1 p_n = \tau p_n.$$

In the present paper we find the linear basis of this algebra and consider its properties. Furthermore, some properties of the representations of the algebra are studied by using the results of [13]. New relations in the finite-dimensional irreducible $*$-representations of the algebra allow to prove that the representations obtained by the action of group $\mathbb{Z}_n$ on the operators $P_1, \ldots, P_n$ are equivalent.

The paper is arranged as follows. In Section 2 we give main definitions and designations. A set of values of parameter τ when the finite-dimensional $*$-representations exist and a description of irreducible $*$-representations up to a unitary equivalence are presented (see [13]). In Section 3 we find the linear basis of algebra in question using the Diamond Lemma (see, e.g., [10, 11, 12]) and discovery additional relations in the finite-dimensional irreducible $*$-representations of the algebra.

2 Description of all finite-dimensional irreducible $*$-representations of algebra $TL_{\tau,n,\Gamma}$

We are going to study $*$-algebra generated by n ($n \geq 3$) projections with relations depending on real parameter τ:

$$TL_{\tau,n,\Gamma} = \mathbb{C}\Big\langle e, p_1, \ldots, p_n \mid p_i = p_i^2 = p_i^*, p_i p_j p_i = \gamma_{ij} p_i,$$

$$(\gamma_{ij}) = \Gamma = \left.\begin{pmatrix} 1 & \tau & 0 & \cdots & 0 & \tau \\ \tau & 1 & \tau & 0 & \cdots & 0 \\ 0 & \tau & 1 & \tau & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \tau & 1 & \tau \\ \tau & 0 & \cdots & 0 & \tau & 1 \end{pmatrix}\right\rangle.$$

The theorems giving information about all finite-dimensional $*$-representations of $TL_{\tau,n,\Gamma}$ can be found in [13], but we need some results about these representations here. First of all we give the theorem about the set of the values of parameter τ when the $*$-representations exist and the description of construction of operators of these representations. In the following we consider only nontrivial finite-dimensional irreducible $*$-representations and name them simply 'representations'. If π is a representation of algebra $TL_{\tau,n,\Gamma}$ then P_i will denote $\pi\left(p_i\right)$.

Theorem 1. *Representations of algebra $TL_{\tau,n,\Gamma}$ exist in finite-dimensional space H iff*

$$\tau \in \left[0, \frac{1}{4\cos^2\frac{\pi}{n}}\right] =: \Sigma_n.$$

Then, if $\tau = 0$ all p_i are orthogonal and if $\tau \neq 0$ then a basis of H exists such that operators of the representation are as follows:

$$P_1 = \operatorname{diag}\left(1, 0, \ldots, 0\right),$$

$$P_i = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & \tau_{i-2} & \sqrt{\tau_{i-2} - \tau_{i-2}^2} & 0 & \cdots \\ 0 & \cdots & 0 & \sqrt{\tau_{i-2} - \tau_{i-2}^2} & 1-\tau_{i-2} & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad i = 2, \ldots, n-1,$$

where $\tau_i = \frac{\tau}{1-\tau_{i-1}}$, $i = 1, \ldots, n-3$, $\tau_0 = \tau$ and the number of zeroes on the top of diagonal is equal to $i-2$.

$$P_n = \begin{pmatrix} \tau & l_1 & l_2 & \cdots & l_{n-3} & \lambda & \mu \\ l_1 & \frac{l_1^2}{\tau} & \frac{l_1 l_2}{\tau} & \cdots & \frac{l_1 l_{n-3}}{\tau} & \frac{l_1\lambda}{\tau} & \frac{l_1\mu}{\tau} \\ l_2 & \frac{l_1 l_2}{\tau} & \frac{l_2^2}{\tau} & \cdots & \frac{l_2 l_{n-3}}{\tau} & \frac{l_2\lambda}{\tau} & \frac{l_2\mu}{\tau} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ l_{n-3} & \frac{l_1 l_{n-3}}{\tau} & \frac{l_2 l_{n-3}}{\tau} & \cdots & \frac{l_{n-3}^2}{\tau} & \frac{l_{n-3}\lambda}{\tau} & \frac{l_{n-3}\mu}{\tau} \\ \bar{\lambda} & \frac{l_1\bar{\lambda}}{\tau} & \frac{l_2\bar{\lambda}}{\tau} & \cdots & \frac{l_{n-3}\bar{\lambda}}{\tau} & \frac{|\lambda|^2}{\tau} & \frac{\bar{\lambda}\mu}{\tau} \\ \mu & \frac{l_1\mu}{\tau} & \frac{l_2\mu}{\tau} & \cdots & \frac{l_{n-3}\mu}{\tau} & \frac{\mu\lambda}{\tau} & \frac{\mu^2}{\tau} \end{pmatrix},$$

where $l_i = (-1)^i \tau \prod\limits_{j=0}^{i-1} \frac{\tau_j}{\sqrt{\tau_j - \tau_j^2}}$. λ is such that

$$\left(l_{n-3} + \lambda \frac{\sqrt{\tau_{n-3} - \tau_{n-3}^2}}{\tau_{n-3}}\right)\left(l_{n-3} + \bar{\lambda}\frac{\sqrt{\tau_{n-3} - \tau_{n-3}^2}}{\tau_{n-3}}\right) = \frac{\tau^2}{\tau_{n-3}},$$

and $\mu^2 = \tau - \tau^2 - \sum\limits_{j=1}^{n-3} l_j^2 - |\lambda|^2$, $\mu \geq 0$.

Proof. The proof of this theorem can be found in [13]. ∎

Remark 1. If $\tau \in \Sigma_n \setminus \{0\}$ then dimension of H is equal to n if $\lambda, \mu \neq 0$, to $n-1$ if $\lambda \neq 0, \mu = 0$ and to $n-2$ if $\lambda = \mu = 0$ (i.e. $\tau_{n-3} = 1$).

In the following we assume that $\tau \neq 0$.

Remark 2. Theorem 1 gives explicit construction of operators of representations. One can easily check that different λ's define inequivalent representations. So, we say that each irreducible representation of $*$-algebra $TL_{\tau,n,\Gamma}$ is given by the number λ.

3 Linear basis in the algebra $TL_{\tau,n,\Gamma}$

To found a linear basis in the algebra $TL_{\tau,n,\Gamma}$ we use the Diamond Lemma (see, e.g., [10, 11, 12]).

Let $F_n = \mathbb{C}\langle e, p_1, \ldots, p_n\rangle$ be a free associative algebra and W be a set of words on the alphabet $\{e, p_1, \ldots, p_n\}$ with homogeneous lexicographic order and minimal element e.

Let I be the ideal generated by

$$R = \{p_i^2 - p_i,\, p_i p_{i\pm 1} p_i - \tau p_i,\, p_1 p_n p_1 - \tau p_1,\, p_n p_1 p_n - \tau p_n,\, p_i p_j \mid \\ |i-j| > 1,\, (i,j) \neq (1,n), (n,1)\}.$$

It is not difficult to prove that R is the reduced Gröbner basis of the ideal I. This implies that the next theorem holds:

Theorem 2. *A linear basis of the algebra $TL_{\tau,n,\Gamma}$ is:*

$$e, p_1,\, p_1p_2, \ldots, (p_1p_2\ldots p_n)^k,\, (p_1p_2\ldots p_n)^k p_1\ldots p_j,\; k\in\mathbb{N},\; j = 1,\ldots,n-1;$$
$$p_2,\, p_2p_3, \ldots, (p_2p_3\ldots p_np_1)^k,\, (p_2p_3\ldots p_np_1)^k p_2\ldots p_j,\; k\in\mathbb{N},\; j = 2,\ldots,n;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$p_n,\, p_np_1, \ldots,\, (p_np_1p_2\ldots p_{n-1})^k,\, (p_np_1\ldots p_{n-1})^k p_n\ldots p_j,\; k\in\mathbb{N},\; j = n,1,\ldots,n-2$$

and adjoint elements of these words.

Direct calculations imply that the basis of modification of Temperley–Lieb algebra has the analogous property to the basis of Temperley–Lieb algebra:

Proposition 1. *Product of any two basis elements of algebra $TL_{\tau,n,\Gamma}$ is either zero or a power of τ times another basis element.*

Proposition 2. *For any representation π the following relations hold:*

$$P_1P_2\cdots P_nP_1 = f(\lambda)\,P_1, \quad P_iP_{i+1}\cdots P_nP_1\cdots P_{i-1}P_i = f(\lambda)\,P_i, \quad i = 2,\ldots,n,$$

where

$$f(\lambda) = \left(\tau_{n-3}l_{n-3} + \sqrt{\tau_{n-3} - \tau_{n-3}^2}\bar{\lambda}\right)\prod_{j=0}^{n-4}\sqrt{\tau_j - \tau_j^2}.$$

Note that these relations are not valid in the algebra $TL_{\tau,n,\Gamma}$ because left and right parts of the equations are the elements of the linear basis of the algebra $TL_{\tau,n,\Gamma}$.

Corollary 1. *The algebra $TL_{\tau,n,\Gamma}$ is infinite algebra. But for any finite-dimensional irreducible $*$-representation π the algebra $\pi(TL_{\tau,n,\Gamma})$ is infinite algebra.*

Corollary 2. *(Action on the set $\{P_1, \dots, P_n\}$ of the group $\mathbb{Z}_n$.) Let π, $\tilde{\pi}$ be the representations of the algebra $TL_{\tau,n,\Gamma}$ such that $\pi(p_i) = P_i$, $\tilde{\pi}(p_1) = P_i$, $\tilde{\pi}(p_2) = P_{i+1}$, …, $\tilde{\pi}(p_{n-i+2}) = P_1$, …, $\tilde{\pi}(p_n) = P_{i-1}$ $(i = 1, \dots, n)$. Then π and $\tilde{\pi}$ are equivalent.*

Proof. Theorem 1 implies that $\tilde{\pi}$ is equivalent to the representation $\hat{\pi}$ such that $\hat{\pi}(p_i) = P_i$ (but a parameter $\hat{\lambda}$ which defines this representation is possible different from the parameter λ that defines the representation π), i.e., there exists an unitary operator C that

$$CP_iC^{-1} = P_1, \quad CP_{i+1}C^{-1} = P_2, \quad \dots, \quad CP_1C^{-1} = P_{n-i+2}, \quad \dots, \quad CP_{i-1}C^{-1} = P_n.$$

From Proposition 2 it follows that

$$P_1P_2 \cdots P_nP_1 = f\left(\hat{\lambda}\right) P_1$$

that implies

$$P_iP_{i+1} \cdots P_nP_1 \cdots P_{i-1}P_i = f\left(\hat{\lambda}\right) P_i.$$

But

$$P_iP_{i+1} \cdots P_nP_1 \cdots P_{i-1}P_i = f(\lambda) P_i$$

that implies $f(\lambda) = f\left(\hat{\lambda}\right)$ or $\lambda = \hat{\lambda}$ what proves the statement of Corollary 2. ■

Acknowledgements

The author is truly grateful to Prof. Yu.S. Samoilenko for his advice, fruitful discussions and suggestions.

[1] Temperley H. and Lieb E., Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular plane lattices: some exact results for 'percolation' problem, *J. Proc. Roy. Soc. (London)*, 1971, V.322, 251–280.

[2] Baxter R.J., Exactly solved models in statistical mechanics, London, Acad. Press, 1982.

[3] Jones V., Index for subfactors, *J. Invent. Math.*, 1983, V.72, 1–25.

[4] Jones V., Hecke algebra representations of braid groups and link polynomials, *Annals of Mathematics*, 1987, V.126, 335–388.

[5] Wenzl H., On sequences of projections, *C.R. Math. Rep. Acad. Sci. Canada*, 1987, V.9, N 1, 5–9.

[6] Goodman F.M. and Wenzl H., The Temperley–Lieb algebra at roots of unity, *Pacific Journal of Mathematics*, 1993, V.161, N 2, 307–334.

[7] Martin P.P., All direct sum representations of the Temperley–Lieb algebra, *J. Phys. A: Math. Gen.*, 1987, V.20, L539–L542.

[8] Martin P.P., Temperley–Lieb algebra, group theory and Potts model, *J. Phys. A: Math. Gen.*, 1988, V.21, 577–591.

[9] Connes A., Indice des sous facteurs, algébres de Hecke et théorie des noeuds [D'aprés Vaughan Jones], *Seminaire Bourbaki: 37e année*, 1984–1985, V.647, N Astérisque 133–134, 289–308, 1986.

[10] Bergman G., The Diamond lemma for ring theory, *J. Adv. Math.*, 1978, V.29, N 2, 178–218.

[11] Ufnarovskij V.A., Combinatorial and asymptotic methods in algebra, *Sovr. Prob. Matem. Fundam. Napr.*, 1990, V.57, 5–177 (in Russian).

[12] Rabanovich V.I., Samoilenko Yu.S. and Strelets A.V., On identities in algebras $Q_{n,\lambda}$ generated by idempotents, *Ukr. Math. J.*, 2001, V.53, N 10, 1380–1390 (in Russian).

[13] Popova N.D., On finite-dimensional representations of one algebra of Temperley–Lieb type, *Methods Funct. Anal. Topology*, 2001, V.7, N 3, 80–92.