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No nonminimally coupled massless scalar hair for spherically symmetric neutral black holes

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ABSTRACT

We provide a remarkably compact proof that spherically symmetric neutral black holes cannot support static nonminimally coupled massless scalar fields. The theorem is based on causality restrictions imposed on the energy-momentum tensor of the fields near the regular black-hole horizon.

1. Introduction

The non-linearly coupled Einstein-scalar field equations have attracted the attention of physicists and mathematicians for more than five decades. Interestingly, the composed Einstein-scalar system is characterized by very powerful and elegant no-hair theorems [1–3] which rule out the existence of asymptotically flat black-hole solutions with regular event horizons that support various types of scalar (spin-0) matter configurations.

The early no-hair theorems of Chase [4] and Bekenstein [5] have ruled out the existence of regular black holes supporting static minimally coupled massless scalar field configurations. The no-hair theorems of Bekenstein [5] and Teitelboim [6] have excluded the existence of black-hole hair made of minimally coupled massive scalar fields [7–9]. Later no-hair theorems of Heusler [10] and Bekenstein [11] have ruled out the existence of neutral black-hole spacetimes supporting static matter configurations made of minimally coupled scalar fields with positive semidefinite self-interaction potentials.

The physically interesting regime of scalar fields nonminimally coupled to gravity has been investigated by several authors. In a very interesting paper, Mayo and Bekenstein [12] have proved that spherically symmetric charged black holes cannot support matter configurations made of charged scalar fields nonminimally coupled to gravity with generic values of the dimensionless coupling parameter ξ [the physical parameter ξ quantifies the nonminimal coupling of the field to gravity, see Eq. (4) below]. Intriguingly, the rigorous derivation of a no-hair theorem for *neutral* scalar fields nonminimally coupled to gravity seems to be a mathematically more challenging task. In particular, the important no-hair theorems of [12,13] can be used to rule out the existence of spherically symmetric scalar hairy configurations in the restricted physical regimes $\xi < 0$ and $\xi \geq 1/2$ [14].

The main goal of the present paper is to present a (remarkably compact) unified no-hair theorem for neutral massless scalar fields nonminimally coupled to gravity which is valid for *generic* values of the dimensionless coupling parameter ξ (in particular, below we shall extend the interesting no-scalar hair theorems of [12] and [13] to the physical regime of nonminimally coupled neutral scalar fields with $0 < \xi < 1/2$). Our theorem, to be proved below, is based on simple physical restrictions imposed by causality on the energy-momentum tensor of the fields near the regular horizon of the black-hole spacetime.

2. Description of the system

We consider a non-linear physical system composed of a neutral black hole of horizon radius r_H and a massless scalar field ψ with nonminimal coupling to gravity. The composed black-hole-scalar-field system is assumed to be static and spherically symmetric, in which case the spacetime outside the black-hole horizon is characterized by the curved line element [12] (we shall use natural units in which $G = c = 1$)

$$ds^2 = -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ , \qquad (1)$$

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$ [here (t, r, θ, ϕ) are the Schwarzschild coordinates]. As explicitly proved in [12], regardless of the matter content of the curved spacetime, a non-extremal regular black hole is characterized by the near-horizon relations [15]

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$$e^{-\lambda} = L \cdot x + O(x^2) \quad \text{where} \quad x \equiv \frac{r - r_{\rm H}}{r_{\rm H}} \quad ; \quad L > 0 \tag{2}$$

and

$$\lambda' r_{\rm H} = -\frac{1}{x} + O(1) \quad ; \quad \nu' r_{\rm H} = \frac{1}{x} + O(1) \, . \tag{3}$$

The curved black-hole spacetime is non-linearly and non-minimally coupled to a real massless scalar field ψ whose action is given by [12]

$$S = S_{EH} - \frac{1}{2}\int \left(\partial_\alpha \psi \partial^\alpha \psi + \xi R \psi^2\right)\sqrt{-g} d^4 x \, , \tag{4}$$

where the dimensionless physical parameter ξ quantifies the strength of the nonminimal coupling of the field to gravity, $R(r)$ is the scalar curvature of the spacetime, and $S_{\rm EH}$ is the Einstein–Hilbert action. As explicitly shown in [12], in the near-horizon $x \ll 1$ region, R is given by the simple expression

$$R = \frac{4L-2}{r_{\rm H}^2} \cdot [1 + O(x)] \, . \tag{5}$$

From the action (4) one finds the characteristic differential equation [12]

$$\partial_\alpha \partial^\alpha \psi - \xi R \psi = 0 \tag{6}$$

for the nonminimally coupled scalar field. Using the metric components (1) of the curved black-hole spacetime, one can express the scalar radial equation in the form

$$\psi'' + \frac{1}{2}\Big(\frac{4}{r} + \nu' - \lambda'\Big)\psi' - \xi R e^{\lambda} \psi = 0 \, . \tag{7}$$

(Here a prime $'$ denotes a spatial derivative with respect to the radial coordinate r.)

The action (4) also yields the compact expressions [12]

$$T^t_t = e^{-\lambda} \frac{\xi(4/r - \lambda')\psi\psi' + (2\xi - 1/2)(\psi')^2 + 2\xi \psi \psi''}{1 - 8\pi\xi\psi^2} \tag{8}$$

and

$$T^t_t - T^\phi_\phi = e^{-\lambda} \frac{\xi(2/r - \nu')\psi\psi'}{1 - 8\pi\xi\psi^2} \tag{9}$$

for the components of the energy-momentum tensor. As explicitly proved in [12], regardless of the matter content of the theory, a regular hairy black-hole spacetime must be characterized by finite mixed components of the energy-momentum tensor:

$$\{|T^t_t|, |T^r_r|, |T^\theta_\theta|, |T^\phi_\phi|\} < \infty \, . \tag{10}$$

In addition, it was proved in [12] that causality requirements enforce the characteristic inequalities[1]

$$|T^\theta_\theta| = |T^\phi_\phi| \leq |T^t_t| \geq |T^r_r| \tag{11}$$

on the components of the energy-momentum tensors of physically acceptable systems. Note that the relations [12]

$$\mathrm{sgn}(T^t_t) = \mathrm{sgn}(T^t_t - T^r_r) = \mathrm{sgn}(T^t_t - T^\phi_\phi) \tag{12}$$

provide necessary conditions for the validity of the characteristic energy conditions (11).

3. The no-hair theorem for static nonminimally coupled massless scalar fields

In the present section we shall explicitly prove that a spherically symmetric non-extremal neutral black hole *cannot* support non-linear hair made of static nonminimally coupled massless scalar fields.

We start our proof with the scalar radial equation (7) which, in the near-horizon $x \ll 1$ region, can be written in the form [see Eqs. (2), (3), and (5)]

$$\frac{d^2\psi}{dx^2} + \frac{1}{x}\frac{d\psi}{dx} + \frac{\beta}{x}\psi = 0 \quad ; \quad \beta \equiv \xi(2 - 4L)/L \, . \tag{13}$$

The general mathematical solution of the ordinary differential equation (13) can be expressed in terms of the familiar Bessel functions (see Eq. 9.1.53 of [16])

$$\psi(x) = A \cdot J_0(2\beta^{1/2}x^{1/2}) + B \cdot Y_0(2\beta^{1/2}x^{1/2}) \quad \text{for} \quad x \ll 1 \, , \tag{14}$$

where $\{A, B\}$ are constants. Using Eqs. 9.1.8 and 9.1.12 of [16], one finds the asymptotic near-horizon behavior

$$\psi(x \to 0) = A \cdot [1 - \beta x + O(x^2)] + B \cdot [\pi^{-1}\ln(\beta x) + O(1)] \tag{15}$$

of the radial scalar function. Substituting Eqs. (2), (3), and (15) into Eq. (8) and taking cognizance of the energy condition (10) [12], one immediately realizes that the coefficient of the singular term in the asymptotic near-horizon solution (15) should vanish [17]:

$$B = 0 \, . \tag{16}$$

We therefore find that the nonminimally coupled scalar field is characterized by the near-horizon behavior

$$\psi(x \ll 1) = A \cdot J_0(2\beta^{1/2}x^{1/2}) \, . \tag{17}$$

Substituting (17) into (8) and (9) and using the near-horizon relations (2) and (3), one obtains the simple expressions

$$T^t_t = \xi \cdot \frac{L\psi\psi'}{r_{\rm H}(1 - 8\pi\xi\psi^2)} \cdot [1 + O(x)] \tag{18}$$

and

$$T^t_t - T^\phi_\phi = -\xi \cdot \frac{L\psi\psi'}{r_{\rm H}(1 - 8\pi\xi\psi^2)} \cdot [1 + O(x)] \tag{19}$$

for the components of the energy-momentum tensor in the near-horizon $x \ll 1$ region. We immediately find from (18) and (19) the near-horizon relation

$$T^t_t = -(T^t_t - T^\phi_\phi) \, , \tag{20}$$

in *contradiction* with the characteristic relation (12) imposed by causality on the energy-momentum components of physically acceptable systems.

[1] As explicitly shown by Bekenstein and Mayo [12], for spherically symmetric spacetimes one can write $\epsilon = -T^t_t - \sum_{i=1}^3 c_i^2(T^t_t - T^i_i)$ and $j^\mu j_\mu = -(T^t_t)^2 - \sum_{i=1}^3 c_i^2[(T^t_t)^2 - (T^i_i)^2]$, where $\epsilon \equiv T_{\mu\nu}u^\mu u^\nu$ and $j^\mu \equiv -T^\mu_\nu u^\nu$ are respectively the energy density and the Poynting vector according to a physical observer with a 4-velocity u^ν, and the coefficients $\{c_i\}_{i=0}^3$ are characterized by the normalization condition $-c_0^2 + \sum_{i=1}^3 c_i^2 = -1$ (this relation guarantees that $u^\mu u_\mu = -1$ [12]). For physically acceptable systems in which the transfer of energy is not superluminal, the energy density should be of the same sign as $-T^t_t$ and the Poynting vector should be non-spacelike ($j^\mu j_\mu \leq 0$) for all observers [12] (that is, for all choices of the coefficients $\{c_i\}_{i=0}^3$). These physical requirements yield the characteristic energy conditions (11) [12].

4. Summary

In this compact analysis, we have proved that *if* a spherically symmetric neutral black hole can support non-linear configurations made of nonminimally coupled massless scalar fields, then in the near-horizon $(r-r_{\text{H}})/r_{\text{H}} \ll 1$ region the energy momentum components of the fields must be characterized by the relation $T^t_t = -(T^t_t - T^\phi_\phi)$ [see Eq. (20)]. However, one realizes that this near-horizon behavior is in *contradiction* with the characteristic relation $\text{sgn}(T^t_t) = \text{sgn}(T^t_t - T^\phi_\phi)$ [see Eq. (12)] which, as explicitly proved in [12], is imposed by causality on the energy-momentum components of generic physically acceptable systems. Thus, there are no physically acceptable solutions for the eigenfunction of the external nonminimally coupled massless scalar fields except the trivial one, $\psi \equiv 0$.

We therefore conclude that spherically symmetric neutral black holes cannot support static configurations made of nonminimally coupled massless scalar fields with *generic* values of the dimensionless physical parameter ξ.

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