THE MAXIMAL SOLVABLE SUBALGEBRAS OF THE
REAL CLASSICAL LIE ALGEBRAS. II

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ABSTRACT

A list of all conjugacy classes of maximal solvable subalgebras of the Lie algebras $g\ell(n, \mathbb{R})$, $su^*(2r)$, $su(p,q)$, $so^*(2r)$, $sp(2r, \mathbb{R})$ and $usp(2p,2q)$ is proposed and a representative of each class is given in terms of a matrix algebra.

INTRODUCTION

The results presented in this communication have been the subject of 3 publications [1,2,3] and are part of the more general task of determining all the conjugacy classes of the subgroups of some Lie groups of physical interest[1,4,5] undertaken by Patera, Winternitz and Zassenhaus.

Except for the cases of $g\ell(n, \mathbb{R})$, $su^*(2r)$ and $so(p,q)$ where p and g are both odd numbers, there always exists a particular class of maximal solvable subalgebras: the class of the *compact Cartan subalgebras* which are abelian and of dimension equal to the rank of the Lie algebra.

The other conjugacy classes are determined by induction on the dimension of the algebra. This has the advantage of furnishing a explicit procedure for constructing representatives of each class. Three steps are to be considered. We adopt the standard matrix representations for the algebras under consideration. The first step consists of showing that every maximal solvable subalgebra S can be written in one of the two alternative forms

$$S = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & A_5 & A_6 \\ 0 & 0 & A_9 \end{pmatrix} \quad \text{or} \quad S = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}$$

depending on whether the standard representation does or does not admit an invariant bilinear form. One then has to ensure that the blocks A_1 (and A_9) are maximal solvable subalgebras of $g\ell(\alpha, \mathbb{R})$ with $\alpha \le 2$. In the second step it must be shown that the block A_4 and A_5 is a maximal sovable subalgebra of an algebra belonging to the same type as the starting Lie algebra, but of lower dimension. Finally, the third step consists in verifying that a necessary and sufficient condition for two maximal solvable subalgebras S' and S" to be conjugate (by an inner automorphism of the Lie algebra) is that $A_1' \sim A_1''$ and $A_4' \sim A_4''$ or $A_5' \sim A_5''$.

The number of conjugacy classes is often expressible in terms of the Fibonna-

ci numbers

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

(Recall that $F_{n+2} = F_{n+1} + F_n$; $F_0 = 0$, $F_1 = 1$.)

RESULTS

S denotes a representative of a conjugacy class of maximal solvable subalge-
bras; B stands for the matrix associated with an invariant bilinear form (if such
exists) of the fundamental representation of the algebra and N (with some sub-
scripts) denotes the number of conjugacy classes. Also we put

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

1. $\underline{g\ell(n, \mathbb{R})}$

$$S = \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_k \end{pmatrix} \qquad 1 \le k \le n$$

with

$$A_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix} \quad \text{or} \quad A_j = (c_j); \ a_j, b_j, c_j \in \mathbb{R}$$

$$N_n = F_n.$$

Each class is characterized by an ordered set of numbers $(\deg A_1, \ldots, \deg A_k)$,
with $\deg A_j = 1$ or 2 and

$$\sum_{j=1}^{k} \deg A_j = n.$$

The elements in the upper triangle of S $((\nabla))$ are any real numbers. The $s\ell(n, \mathbb{R})$
case is simply obtained by requiring $\text{tr} S = 0$.

2. $\underline{su^*(2r)}$

$$su^*(2r) = \left\{ X \in s\ell(2r, \mathbb{C}); \ T\bar{X} = XT \right\}$$

with

$$T = \text{diag}(\underbrace{J, \ldots, J}_{r})$$

$$(su^*(2) \simeq su(1,1))$$

$$S = \begin{pmatrix} A_{11} & & A_{1r} \\ & \ddots & \\ 0 & & A_{rr} \end{pmatrix}$$

with

$$A_{jj} = \begin{pmatrix} a_{jj} & 0 \\ 0 & a_{jj} \end{pmatrix} \; ; \; A_{j\ell} = \begin{pmatrix} a_{j\ell} & b_{j\ell} \\ -\bar{b}_{j\ell} & \bar{a}_{j\ell} \end{pmatrix} , \quad j < \ell;$$

$$a_{j\ell}, b_{j\ell} \in \mathbb{C} \quad \text{and} \quad \sum_{j=1}^{r} (a_{jj} + \bar{a}_{jj}) = 0$$

$$N_{2r} = 1$$

3. $\underline{su(p,q)}.$ $\quad (p \geq q).$

$$su(p,q) = \left\{ X \in s\ell(p+q,\mathbb{C}), X^{\dagger}B + BX = 0 \right\}.$$

a) Compact cartan subalgebra:

$$S = \text{diag}(ia_1, \ldots, ia_{p+q}); \quad B = \text{diag}(\underbrace{1,\ldots,1}_{p}, \underbrace{-1,\ldots,-1}_{q})$$

with

$$\sum_{j=1}^{p+q} a_j = 0, \qquad a_j \in \mathbb{R}.$$

b) others (possible if q > 0)

$$S = \begin{pmatrix} \lambda_1 & & & & & \star \\ & \ddots & & & & \\ & & \lambda_s & & & \\ & & & C & & \\ & & & & -\bar{\lambda}_s & \\ & & & & & \ddots \\ 0 & & & & & -\bar{\lambda}_1 \end{pmatrix}_{1 \leq s \leq q} \quad , \quad B = \begin{pmatrix} 0 & & & & 1 \\ & & & \ddots & \\ & & \tilde{B} & & \\ & 1 & & & \\ 1 & & & & 0 \end{pmatrix}$$

with $\lambda_j \in \mathbb{C}$ and C is either a compact Cartan subalgebra of su(p-s,q-s) or C = $\emptyset$ (which implies p=q).

$$\tilde{B} = \text{diag}(\underbrace{1,\ldots,1}_{p-s}, \underbrace{-1,\ldots,-1}_{q-s}) \quad \text{or} \quad \tilde{B} = \emptyset$$

$$N_{p,q} = q+1.$$

4. $\underline{so(p,q)}.$ $\quad (p \geq q).$

$$so(p,q) = \left\{ X \in s\ell(p+q, \mathbb{R}); X^{T}B + BX = 0 \right\}.$$

a)   Compact Cartan subalgebra (if p and q are not both odd numbers).

$$S = \text{diag}(A_1,\ldots,A_t); \quad B = \text{diag}(\underbrace{1,\ldots,1}_{p},\underbrace{-1,\ldots,-1}_{q})$$

with

$$A_k = (0), \quad k = \begin{cases} p & \text{if p even} \\ p+1 & \text{if p odd} \end{cases}, \quad t = \frac{k+q}{2}$$

$$A_j = \begin{pmatrix} 0 & -a_j \\ a_j & 0 \end{pmatrix}, \qquad j \neq k; \ a_j \in \mathbb{R}.$$

b)   others (possible if $q \neq 0$).

$$\begin{pmatrix} A_1 & & & & * \\ & \ddots & & & \\ & & A_s & & \\ & & & C & \\ & & & D_s & \\ & & & & \ddots \\ 0 & & & & D_1 \end{pmatrix}_{1 \leq s \leq q} \quad ; \quad B = \begin{pmatrix} 0 & & & & 1 \\ & \ddots & & & \\ & & 1 & \tilde{B} & \\ & & 1 & \ddots & \\ 1 & & & & 0 \end{pmatrix}$$

with

$$A_j = (c_j) \quad \text{or} \quad A_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}, \quad D_j = -A_j;$$

$a_j, b_j, c_j \in \mathbb{R}$.   C is either a compact Cartan subalgebra of so(p-t,q-t),

$$t = \sum_{j=1}^{s} \deg A_j, \text{ (if it is possible) or } C = \emptyset;$$

$$\tilde{B} = \text{diag}(\underbrace{1,\ldots,1}_{p-t},\underbrace{-1,\ldots,-1}_{q-t}) \quad \text{or} \quad \tilde{B} = \emptyset.$$

$$N_{p,q} = \begin{cases} -1+3F_q+2F_{q-1} & \text{if p+q is odd} \\ \frac{1}{2}((-1)^q-1)+2F_q+F_{q-1} & \text{if p+q is even} \end{cases}$$

Each class is characterized by an ordered set of numbers $(\deg A_1,\ldots,\deg A_s)$.

5.   so*(2r)
$$so^*(2r) = \left\{ X \in s\ell(2r,\mathbb{C}); X^T B + BX = 0; T\bar{X} = XT \right\}$$
$$\left(so^*(2) \simeq so(1,1) \right).$$

a)   Compact Cartan subalgebra

$$S = \mathrm{diag}(A_1,\dots,A_r); \quad B = \mathrm{diag}(\underbrace{iK,\dots,iK}_{r}).$$

with

$$A_j = \begin{pmatrix} ia_j & 0 \\ 0 & -ia_j \end{pmatrix}; \quad a_j \in \mathbb{R}.$$

b)   others

with

$$A_j = \begin{pmatrix} \lambda_j & 0 \\ 0 & \bar{\lambda}_j \end{pmatrix}, \quad D_j = -\bar{A}_j \; : \; \lambda_j \in \mathbb{C}.$$

C is either a compact Cartan subalgebra of $so^*(2r-4s)$ or $C = \emptyset$ and

$$\widetilde{B} = \mathrm{diag}(\underbrace{iK,\dots,iK}_{r-2s}) \quad \text{or} \quad \widetilde{B} = \emptyset$$

$$N_r = r+1.$$

6. $\underline{sp(2r,\ \mathbb{R})}$

$$sp(2r,\mathbb{R}) = \{X \in sl(2r,\ \mathbb{R}); \; X^T B + BX = 0 \}$$

$$(sp(2,\ \mathbb{R}) \simeq sl(2,\ \mathbb{R})).$$

a)   Compact Cartan subalgebra

$$S = \mathrm{diag}(A_1,\dots,A_r); \quad B = \mathrm{diag}(\underbrace{J,\dots,J}_{r})$$

with

$$A_j = \begin{pmatrix} 0 & -a_j \\ a_j & 0 \end{pmatrix}; \quad a_j \in \mathbb{R}.$$

b)   others

$$\begin{pmatrix} A_1 & & & & & * \\ & \ddots & & & & \\ & & A_s & & & \\ & & & C_{D_s} & & \\ & & & & \ddots & \\ & & & & & D_1 \\ 0 & & & & & \end{pmatrix}_{1 \le s \le r} \quad ; \quad B = \begin{pmatrix} 0 & & & & 1 \\ & & & & \ddots \\ & & -1 & \widetilde{B}^1 & \ddots \\ & & \ddots & & \\ -1 & \ddots & & & 0 \end{pmatrix}$$

with

$$A_j = (c_j) \quad \text{or} \quad A_j = \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}, \quad D_j = -A_j;$$

a_j, b_j, $c_j \in \mathbb{R}$. C is either a compact Cartan subalgebra of sp $(2r-2t, \mathbb{R})$,

$$t = \sum_{j=1}^{s} \deg A_j, \quad \text{or} \quad C = \emptyset \text{ and}$$

$$\widetilde{B} = \text{diag} \underbrace{(J, \ldots, J)}_{r-t} \text{ or } B = \emptyset$$

$$N_r = F_{r+2}^{-1}.$$

Each class is characterized by an ordered set of numbers (deg $A_1, \ldots$, deg A_s).

7.   <u>usp(2p, 2q)</u>

$$usp(2p, 2q) = \{X \in sl(2p+2q, \mathbb{C}); X^T B + BX = 0; U\bar{X} = XU\}$$

with

$$U = \text{diag}(\underbrace{J, \ldots, J}_{p}, \underbrace{-J, \ldots, -J}_{q})$$

$$(usp(2,0) \simeq su(1,1)).$$

a)   Compact Cartan subalgebra

$$S = \text{diag}(A_1, \ldots, A_r) \quad ; \quad r = p+q; \quad B = \text{diag}(J, \ldots, J)$$

with

$$A_j = \begin{bmatrix} ia_j & 0 \\ 0 & -ia_j \end{bmatrix}; \quad a_j \in \mathbb{R}.$$

b)   others

$$\begin{pmatrix} A_1 & & & & * \\ & \ddots & & & \\ & & A_s & & \\ & & & C_{D_s} & \\ & & & & \ddots \\ O & & & & D_1 \end{pmatrix} \quad ; \quad B = \begin{pmatrix} O & & & & & 1 \\ & & & & \tilde{B}^1 & \ddots \\ & & & -1 & \ddots & \\ & \ddots & -1 & & & \\ -1 & & & & & O \end{pmatrix}$$

$$1 \le s \le q$$

with

$$A_j = \begin{pmatrix} \lambda_j & O \\ O & \bar{\lambda}_j \end{pmatrix}, \quad D_j = -\bar{A}_j; \quad \lambda_j \in \mathbb{C}.$$

C is either a compact Cartan subalgebra of usp (2p-2s, 2q-2s) or C = ∅ and

$$\tilde{B} = \mathrm{diag}(\underbrace{J,\ldots,J}_{r-2s}) \quad \text{or} \quad \tilde{B} = \emptyset$$

$$N_{p,q} = q+1;$$

In these five latter cases, the elements in the upper triangle ((∇)) of S are completely determined by the defining conditions of the algebras (for example $S^T B + BS = 0$ and $U\bar{S} = SU$ for usp (2p,2q)).

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