Aspects of branes in (heterotic) M-theory

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Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. Jürgen Mlynek

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät I:
Prof. Dr. Michael Linscheid

Gutachter:

1. Prof. Dr. Dieter Lüst
2. Prof. Dr. Albrecht Klemm
3. Dr. Klaus Behrndt

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Abstract

First a short introduction to principles and some recent developments in string theory is given.

In a second part the effective action for five- as well as four-dimensional heterotic M-theory in the presence of five-branes is systematically derived from Hořava-Witten theory coupled to N separate M5-brane world-volume theories. The five-dimensional theory is obtained by compactification on a Calabi-Yau three-fold, where we allow for an arbitrary number of Kähler and complex structure moduli. This leads to a five-dimensional $\mathcal{N} = 1$ gauged supergravity theory on the orbifold $S^1/\mathbb{Z}_2$, coupled to four-dimensional $\mathcal{N} = 1$ theories residing on the two orbifold fixed planes and N additional three-branes. We analyze some properties of this action, including the quaternionic structure of the hypermultiplet sector and its relation to the gauged isometry. Further, the multi-domain-wall vacuum solution is given, and the associated four-dimensional effective theory is derived. In particular the Kähler potential and the gauge kinetic functions are determined along with the explicit relations between four-dimensional superfields and five-dimensional component fields.

Next, a truncated form of the previously obtained four-dimensional action and its relation to the five-dimensional domain-wall vacuum state are used to study cosmological rolling-radii solutions. The four-dimensional action is reduced to the minimal geometrically necessary field content including gravity, the dilaton and the T-modulus. To this action we find a one-parameter family of time-dependent solutions and relate them to their approximate five-dimensional counterparts. These are new, generally non-separating solutions corresponding to an evolving pair of domain walls. The five-dimensional solutions are computed to leading non-trivial order in the strong coupling expansion parameter which describes loop corrections to the four-dimensional theory. These loop corrections depend on certain field excitations in the fifth dimension and thus generally vary with time. We point out that the two previously discovered exact five-dimensional separable solutions are precisely the special cases for which the loop corrections are time-independent. At the end, changes induced by the presence of a five-brane are discussed.

In the last part, we study flop-transitions for pure M-theory on Calabi-Yau three-folds, in particular their influence on cosmology in the context of the effective five-dimensional $\mathcal{N} = 1$ supergravity theory. This is a further application of the five-dimensional action obtained earlier, but in the context of pure M-theory without five-branes, and extended to include certain two-brane states. In particular, the two-brane states that correspond to an additional hypermultiplet which becomes massless at the flop-transition is included in the effective action. We find the potential for this hypermultiplet which has quadratic and quartic terms and depends on the Kähler moduli.

By constructing explicit cosmological solutions, it is shown that a flop-transition can dynamically happen, as long as the hypermultiplet is set to zero. Taking into account excitations of the hypermultiplet we find that the transition is generally not completed, that is, the system gets stabilized close to the transition region. Regions of the Kähler moduli space close to flop-transitions can therefore be viewed as dynamically preferred. The generalization of the scenario to heterotic M-theory is discussed.

Keywords:
Heterotic M-theory, M-branes, Flop-transition, Cosmology

Zusammenfassung

Der erste Teil gibt eine kurze Einführung in die Prinzipien und neueren Entwicklungen der String Theorie.

Im zweiten Teil leiten wir systematisch die fünf- und vierdimensionalen Wirkungen der heterotischen M-theorie unter Einbeziehung von Fünfbranen her. Zuerst wird die Theorie von Hořava und Witten an N einzelne Fünfbranen gekoppelt und auf eine dreidimensionale Calabi-Yau Mannigfaltigkeit kompaktifiziert, wobei beliebig viele Moduli der Kähler Klasse und der komplexen Struktur zugelassen werden. Das führt zu einer fünfdimensionalen geeichten $\mathcal{N} = 1$ Supergravitation auf dem Orbifold $S^1/\mathbb{Z}_2$, gekoppelt an vierdimensionale $\mathcal{N} = 1$ supersymmetrische Theorien auf N Dreibranen sowie den zwei Grenzflächen, welche an den Orbifoldfixpunkten fixiert sind. Einige Eigenschaften dieser Wirkung werden betrachtet, insbesondere die quaternionische Struktur des von den Hypermultipletskalaren parametrisierten Moduliraums und die Eichung einer Isometrie dieses Raumes. Danach wird eine BPS-Vakuumlösung bestehend aus parallelen Branen konstruiert und die zugehörige vierdimensionale effektive Wirkung hergeleitet. Dazu werden das Kählerpotential aller skalaren Modulifelder sowie die kinetischen Funktionen für die Eichfelder explizit angegeben. Ausserdem werden die vierdimensionalen Superfelder durch die zugehöhrigen fünfdimensionalen Komponentenfelder ausgedrückt.

Im nächsten Teil wird die vierdimensionale Wirkung der vorangegangenen Arbeit in einer vereinfachten Form angewandt, wobei nur die geometrisch notwendigen Felder betrachtet werden, und das sind die Metrik, das Dilaton und der T-modulus. Eine einparametrige Schar von vierdimensionalen zeitabhängigen Lösungen wird gefunden und in den fünfdimensionalen Zusammenhang gestellt. Die entsprechenden approximativen fünfdimensionalen Lösungen sind eine Entwicklung zu linearer Ordnung im Kopplungsparameter ϵ, welcher Loopkorrekturen zur vierdimensionalen Theorie angibt. Diese Korrekturen sind durch Anregungen gewisser Felder in der fünften Dimension gegeben und können somit zeitabhängig sein. Es zeigt sich, dass gerade jene zwei Lösungen mit zeitunabhängigen Loopkorrekturen den bekannten exakten fünfdimensionalen separierbaren Lösungen entsprechen. Am Ende wird noch disskutiert was sich durch Einbeziehung einer beweglichen Fünfbrane ändert.

Im letzten Teil werden Flopübergänge in der M-theory auf dreidimensionalen Calabi-Yau Mannigfaltigkeiten studiert, inspesondere deren Einfluss auf kosmologische Modelle der fünfdimensionalen $\mathcal{N} = 1$ Supergravitation. Das ist eine weitere Anwendung der Wirkung aus dem zweiten Teil dieser Arbeit, nun jedoch im etwas einfacheren Fall der puren M-theorie ohne Grenzflächen oder Fünfbranen, dafür unter Berücksichtigung spezieller Membranenzustände. Insbesondere wird die Wirkung für jene Membranenzustände hergeleitet, welche einem zusätzlichen Hypermultiplet entsprechen, das beim Flopübergang masselos wird. Das Hypermultiplet hat ein Potential mit quadratischen und kubischen Termen, welche von den Kählermoduli abhängen. Die Konstruktion expliziter zeitabhängiger Lösungen zeigt, dass eine Flopübergang dynamisch realisiert werden kann solange das zusätzliche Hypermultiplet nicht mit einbezogen wird. Anregungen dieses Hypermultiplets ändern die Situation und im allgemeinen findet der Flopübergang nicht mehr vollständig statt, statt dessen wird das System in der Übergangsregion stabilisiert. Somit können solche Regionen um Flopübergange im Kähler Moduliraum als dynamisch bevorzugt betrachtet werden. Die Verallgemeinerung dieses Szenarios auf heterotische M-theorie wird kurz disskutiert.

Schlagwörter:
Heterotische M-theorie, M-branen, Flopübergang, Kosmologie

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Introduction

1.1 The necessity of string theory

Even after the turn of the millennium theoretical physics is still challenged by the two great physical revolutions of the past century, these are the theory of general relativity (GR) and quantum mechanics. The latter has found its most successful manifestation in the quantum field theoretical formulation of the standard model (SM) of particle physics [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], which with astonishing accuracy explains almost all empirical data mainly obtained from particle scattering experiments and decay processes. The SM seems to provide the most accurate description of nature in the microscopic[1] realm down to length scales of $\sim 10^{-16}$cm, or equivalently, up about to the so called electroweak energy scale of $\Lambda_W \sim 200$GeV. On the other hand GR [14, 15, 16, 17, 18] provides the yet unsurpassed theory to describe astrophysical and cosmological phenomena up to length scales of the current Hubble horizon of $\sim 10^{25}$m. Moreover, its characteristic features like the $1/r^2$-behavior of the attractive gravitational force and the equality of inertial mass and gravitational mass as a consequence of the equivalence principle, have been tested down to length scales of ~ 1mm [19]. In spite of the impressive success of both theories they do not provide a coherent physical picture, in fact both theories do not incorporate the essential features of the respective other one, and there is no doubt that both theories need to be extended. It is generally believed that both theories are very good low energy effective approximations to some yet unknown all encompassing theory, for which string theory provides a possible candidate.

To go beyond the SM and GR has proven especially difficult due to the rare data these theories cannot explain which leaves us with hardly any experimental hints as how to proceed. Especially observing signatures of the quantum nature of space-time

[1] One should keep in mind though that QM has reared its funny face also in macroscopic phenomena like for example in superconductivity.

seems out of reach because such effects are not expected to become relevant before the enormous *Planck scale* of $M_P \sim 10^{19}$GeV. Only recently some experimental ideas and hints hopefully involving quantum features of space-time have been addressed, including macroscopic quantum objects [20], high energy cosmic rays [21, 22], elementary particle excitations in gravitational fields [23,24], the possibility of temporal variation of coupling constants [25] and the non-vanishing of the cosmological constant [26,27]. Apart from the last, such issues can at the moment at best give qualitative results like for example providing evidence for the existence of a fundamental length scale and eventual Lorentz violation at corresponding energy scales. Thus the guiding principles that should lead to a unification of the two standard theories are their open questions, aesthetical considerations, mathematical consistency as well as the tight constraints the success of the existing theories puts on any such attempt. The several different approaches taken toward a unified theory of quantum gravity, none of them truly satisfactory yet, are reviewed for example in [28, 29, 30, 31].

Already before the quantization of gravity even seems to become necessary, the high degree of arbitrariness of the SM calls for an explanation. The most prominent ideas to extend the SM of particle physics are *grand unification*, *supersymmetry* and *Kaluza-Klein theories* [32, 33, 34, 35, 36, 37, 38, 39, 40, 41]. In grand unified theories (GUT) the standard model gauge group $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ is embedded in a bigger, simple gauge group G_{GUT} that admits complex representations, such as $SU(5)$, $SO(10)$ or E_6. Above the grand unification scale, roughly estimated to be $\Lambda_{GUT} \sim 10^{16}$GeV, there remains only one gauge coupling. Additional gauge bosons become relevant that mediate new forces between those formerly different SM multiplets that now sit in the same representation of G_{GUT} and as such become indistinguishable above Λ_{GUT}. In general the new interactions violate baryon and lepton number and the proton would no longer be stable but have a lifetime of roughly $\tau_p \sim \Lambda_{GUT}^4/m_p^5$, with m_p the mass of the proton, and this already experimentally excludes the simplest $SU(5)$ model. Other model dependent predictions are the weak mixing angle and sometimes certain mass relations between leptons and quarks. At energies below Λ_{GUT} the SM gauge group is regained after spontaneous symmetry breaking by means of a generalized Higgs mechanism. Though the idea of grand unification seems appealing, it cannot be the end of the story, first it comes with new additional problems in the Higgs sector (problem of doublet-triplet splitting), and moreover it is also not completely consistent with the well known particle spectrum of the SM.

The extrapolation of the running of the gauge couplings in the SM does not show exact unification at some higher scale Λ_{GUT} as needed. One would also expect that as a low energy effective theory the SM should be independent of an upper cut-off scale like Λ_{GUT} at energies much below Λ_{GUT}, which is not the case due to the *hierarchy problem*. It states that the SM Higgs mass m_h gets radiative corrections $\delta m_h \sim \Lambda_{GUT} >> m_h \sim \Lambda_W$ which would render the weak scale unstable and thus the SM strongly dependent one some higher energy scale. As a last point, it seems very improbable that there is a plethora of particles of different masses up to Λ_W, but there should then be a big desert up to Λ_{GUT} in which no new physics appears. All these three problems can be solved by the extension of the SM to the *(minimal) supersymmetric standard model* ((M)SSM), which is nicely reviewed in [42]. To every SM particle a *superpartner sparticle* of equal mass but opposite spin statistics is associated, and in a supersymmetric world the hierarchy problem would vanish due to the cancellation between contributions to loop diagrams of fermionic and bosonic degrees of freedom of superpartners. But since the world obviously is not supersymmetric, this symmetry must be (spontaneously) broken, which leads to a mass split between superpartners of the order of the supersymmetry breaking scale Λ_{SUSY}. Choosing this scale of order of the weak scale $\Lambda_{SUSY} \sim \Lambda_W \sim 200\text{GeV}$ would render the sparticles just heavy enough to be not yet detectable at todays accelerators. Moreover, the hierarchy problem would vanish since the radiative corrections to the Higgs mass would also be of order of the weak scale from contributions below Λ_{SUSY} and only logarithmic in a cut-off of the theory above Λ_{SUSY}, which could be as high as the Planck scale M_P. At the same time the additional light particles above Λ_{SUSY} influence the running of the gauge couplings exactly such that grand unification at Λ_{GUT} is possible. This all sounds nice but it actually involves an enormous amount of arbitrariness because with the enlarged Higgs sector and a supersymmetry breaking mechanism the SSM has around 100 free[2] parameters, i.e. the problem we set out to ameliorate in the SM has become a fivefold worse. The problem of (super)symmetry breaking can be considered far from being solved in high energy particle physics.

An idea to approach the origin of gauge symmetries in the SM is provided by Kaluza-Klein theories [43, 44, 45, 46]. In these theories space-time has more than four dimensions, and for the extra dimensions not to be observable, they are assumed to be curled up in tiny compact *internal* spaces of length scales that cannot be resolved with currently reachable energies. From the Kaluza-Klein perspective a four-dimensional theory is only a low energy effective theory not taking into account the gravitational backreaction of any dynamics onto the extra internal dimensions. For an internal space with an isometry group G_{ISO} the lower dimensional effective theory will have a gauge symmetry group $G \subset G_{ISO}$, even if the higher dimensional theory incorporated gravity only. Thus gauge fields have gravitational origin and the gauge symmetries are remnants of the coordinate invariance of general relativity. Moreover, the gauge coupling depends on the size of the internal space and as such is a geometrical quantity that is apt to become dynamical at high enough energies. There is also a possible explanation to the quantization of mass if all mass is reinterpreted as momentum in internal directions, which is necessarily quantized due to the compactness of the

[2] By "free" we mean "chosen to fit experimental data", or in the case of grand unification "chosen in order to theoretically conform with an appealing idea". The whole situation is even more intricate because the ∼21 free parameters of the SM actually must have very special values to allow for life, the universe and everything!

internal space. This is problematic though because such masses are expected to be of the order of the compactification scale which we previously assumed to be too high to be observable, which is inconsistent with the masses observed. Staying with the given assumption, only the zero momentum modes appearing as massless states are effectively relevant and again a different mechanism for the generation of masses would have to be incorporated. There are various other problems with Kaluza-Klein theories:

- As higher dimensional gravitational theories they are not perturbatively renormalizable and thus do not provide a viable quantum theory.
- There is no dynamical compactification mechanism known that would pick out a special internal space, and thus gauge group and coupling are still arbitrary.
- The size of the internal space need not be stable and thus the coupling "constants" could vary in space-time.
- It has never been possible to exactly extract the SM coupled to gravity by the Kaluza-Klein mechanism, because it cannot generate chiral fermions from a non-chiral theory [47].

So another old and nice idea seems to have failed. There remains one intriguing fact though, a compact internal manifold that can have an isometry group that is big enough to incorporate the SM gauge group needs to be at least seven-dimensional [48]. Together with the four dimensions from external space-time, this leads to eleven dimensions, exactly the highest dimensionality which allows the construction of a consistent supergravity theory [49].

The upshot of the previous paragraphs is that there are very good concepts to go beyond the SM, but all of them bring along lots of problems, most of which are related to the *problem of symmetry breaking*, which is already the most unsatisfactory part in the conventional SM in form of the Higgs mechanism. Moreover, on the one hand all these ideas can consistently be implemented into the SM, but on the other hand this also reflects the fact that there is no hard experimental evidence nor phenomenological need for any of them yet.

All three ideas, grand unification, supersymmetry and Kaluza-Klein compactification reappear in the context of *string theory*. Unfortunately most of the problems associated with these concepts also remain in string theory, though even they are "unified" in the sense that they can all be traced back to one single problem, namely the *problem of background dependence*, as should become clearer along the way. Still, string theory must be taken seriously because so far it provides the only viable perturbative description of quantum gravity, but no longer in the framework of quantum field theory (QFT). At the level of perturbation theory in QFT, the appearing divergencies are related to the absence of a cut-off at short distances due to the point-like nature of the interactions, which in turn is related to the validity of the cherished principle of Lorentz invariance to arbitrarily high energies. This problem itself can be cured in renormalizable theories where the divergencies do not become worse at every order of perturbation theory such that they can be 'absorbed' into parameters. This means that such a parameter (e.g. the gauge coupling constant) is fixed to an empirical value at a certain energy and the only thing that matters thereafter when including higher loop corrections is the *rate of change* of the normalized coupling, described by the so called β-function. Since the coupling constant of gravity, i.e. the Newton constant $G_N = M_P^{-2}$, has dimension of length squared, the true dimensionless expansion parameter of perturbative gravity must be E/M_P, where E is the characteristic energy of the process considered. This shows two unfortunate features, first gravitational interactions become only relevant at enormous energies $E \to M_P$ where perturbation theory breaks down, and second the ratio of an n to a zero graviton exchange amplitude is on dimensional grounds of the order

$$\frac{1}{M_P^{2n}} \left(\int dE_1 E_1 \dots \int dE_{n-1} E_{n-1} \right) ,$$

which in the limit $E_i \to \infty$ diverges the more rapidly the larger n. In position space this limit corresponds to a diagram where all 2n graviton vertices coincide, which is only possible if there is no minimal length scale. Since the divergencies become worse at every order, we would need new renormalizable parameters at every order which is simply neither practicable nor sensible. This is the problem of non-renormalizability of gravity, which in string theory gets solved by the introduction of a powerful symmetry associated to an infinite tower of states as well as a minimal length scale, even without the loss of Lorentz invariance.

String theory starts by replacing point-like particles by one dimensional strings which already implies that string theory cannot be a QFT. But the string length is usually supposed to be close to the Planck length $l_S \simeq L_P = M_P^{-1} \simeq 10^{-33}$cm, and at lower energies the strings look again point-like where QFT yields a low energy effective description of string theory. The extended nature of a string becomes only relevant at the string scale, and it smears out the location of interaction which helps to avoid ultraviolet divergences. The different particle species are interpreted as different vibrational modes of the strings and luckily enough, after quantizing these modes in a Minkowski background space-time, there is also a massless mode which exactly has the properties of a spin-2 graviton with low energy dynamics given by general relativity. So string theory provides the only known perturbatively consistent quantum theory of general relativity. There are actually five different consistent string theories known, and these relativistic quantum theories of strings are highly constrained. It turns out that they are only consistent with

- a ten dimensional space-time,
- a supersymmetric spectrum of excitations,
- all background fields satisfying classical equations of motion, which to lowest order in the expansion parameter $\alpha' = l_S^2$ correspond to the Einstein, Yang-Mills/Maxwell and Klein-Gordon equations,
- possible non-Abelian gauge groups to be $SO(32)$ or $E_8 \times E_8$.

So for reasons of consistency of string theory we see that supersymmetry, the Kaluza-Klein idea, grand unification as well as conventional QFT come back into play!

As explained in more detail later, the five string theories are not really independent but only different special limits (vacua) of an eleven-dimensional theory dubbed *M-theory*. These limits are related by so called *dualities*. Unfortunately, all that is really known of this M-theory is its low energy effective action, namely eleven-dimensional supergravity. As nice as this unifying picture might be, it also raises the question which vacua of M-theory nature has chosen to sit in, if any at all. It is the choice of a background space-time on which to perform the Kaluza-Klein compactification that determines all the features of four-dimensional physics, and there seem to be nearly infinitely many possibilities and so far the SM could not be completely reconstructed in this way, almost though. The main necessary ingredients like chiral fermions, family structure, grand unified gauge groups and Yukawa interactions can be obtained. In short, the high degree of uniqueness in ten or eleven dimensions is completely spoiled when trying to go to lower dimensions. And even if a good vacuum could be found, there would still lack a dynamical mechanism behind this special choice. The true problem behind these difficulties is the present perturbative formulation of string theory that is not defined in a background independent way. It considers only strings in a given constrained background and yields a consistent quantum theory of the fluctuations around this background. The background fields in turn correspond to coherent states of string fluctuations in Minkowski space-time. It is the given background that fixes the causal structure. Moreover, only because there is a consistent quantum theory, it generally is not easily solvable and the spectrum around a given background need not even contain a graviton anymore, and to close the circle, one could again use coherent states of these new fields as background fields in which to quantize a string and then in principle go on like this forever - not very practicable. A solution to this problem is not in sight yet, though one possible attempt is *string field theory*.

In spite of these serious deep problems in M/string theory, it is still interesting to see what can be accomplished by building models derived from the more fundamental M/string theory. Especially such low energy effective models that are possibly valid up to the string scale are of particular interest in early universe cosmology.

1.2 Short introduction to string theory

In this section a brief and sketchy introduction into string theory will be given. The standard references for this subject are ref. [50, 51, 52, 53, 54], nice introductions are also given in ref. [55, 56, 57, 58], and on a non-technical level ref. [59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69] make good reading.

Strings as one-dimensional objects trace out two-dimensional *world-sheets* Σ when moving through a target space-time M, such that the Feynman 'wiring' diagrams get replaced by 'plumbing circuits' as depicted in fig. 1.1. All different Feynman graphs at a given order are replaced by one single string graph of the same order. The string is

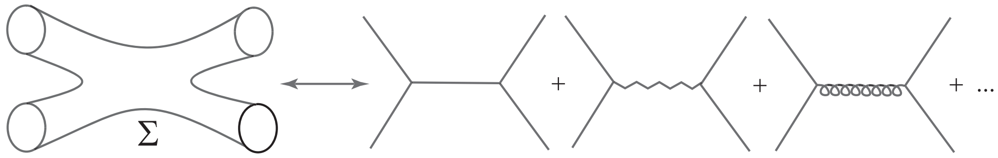

Figure 1.1: Several Feynman diagrams get replaced by one string graph.

described by its embedding into target space, i.e. by the maps $X^I(\sigma^0, \sigma^1) : \Sigma \longrightarrow M$, where X^I, $I = 0, \ldots, D-1$, and σ^α, $\alpha = 0, 1$, are the coordinates of the D-dimensional target space M and the two-dimensional world-sheet Σ, respectively. In general Σ can be an oriented or an unoriented surface and also have boundaries besides those from the external legs. The latter is the case for open strings. For simplicity we consider only closed oriented strings embedded in Minkowski space-time, more specifically, we concentrate on the bosonic string. In analogy to the point particle, the classical Nambu-Goto string action is given by the area of the worldsheet Σ and as such has a direct geometrical interpretation. This latter action is equivalent to the following simpler Polyakov action given by

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^I \partial_\beta X^J \eta_{IJ} \, , \tag{1.1}$$

where α' is the Regge slope with associated string scale $m_s = \alpha'^{-1/2}$, and $T \equiv (2\pi\alpha')^{-1}$ is the string tension giving the energy per length. Further, $h = |\mathrm{det} h_{\alpha\beta}|$ is the determinant of an auxiliary metric $h_{\alpha\beta}$ on Σ that can be eliminated by its algebraic equation of motion to regain the Nambu-Goto action, and η_{IJ} is the flat metric of the Minkowski background space-time. The action (1.1) is a two-dimensional field theory action of a linear sigma model for the D fields X^I living on Σ. Moreover the Polyakov action shows the following three symmetries: (i) invariance under diffeomorphisms of the world-sheet, (ii) Poincaré invariance in D dimensions and (iii) invariance under

Weyl transformations where $h_{\alpha\beta} \to e^{\Lambda(\sigma)} h_{\alpha\beta}$, such that the action (1.1) defines a *conformal* two-dimensional field theory. Upon quantization of this field theory these symmetries must be preserved in order to avoid violation of unitarity, and it is the generically anomalous Weyl symmetry which imposes strong constraints on the theory, it fixes the number of space-time dimensions in which the string action can consistently be quantized. Heuristically speaking, this is because the space-time dimensionality determines the number of degrees of freedom of a string, which in turn affects its symmetry properties.

The quantum string theory is obtained by first quantization, that is the coordinates X^I are promoted to operators in space-time. In terms of the two-dimensional conformal field theory this corresponds to second quantization and the fields X^I are considered as quantum fields on Σ. For quantization there are several methods, these are light cone, old covariant and BRST quantization which differ in the way the symmetry constraints are taken care of. Without going into further details of the quantization process only results are stated here. The independent degrees of freedom are the vibrational excitations of the string transverse to the world-sheet which yield the spectrum of the quantum theory. The spectrum obtained from quantization of the action (1.1) does not contain any space-time fermions but it contains a tachyon, that is a state of negative mass, so we do not get a viable theory. These problems can be fixed by the introduction of fermionic degrees of freedom, and in the so called Ramond-Neveu-Schwarz approach this is done by an extension of the action (1.1) with supersymmetry on the world-sheet Σ. The conformal symmetry of (1.1) gets enhanced to a superconformal symmetry and its preservation under quantization constrains the space-time dimension of the background Minkowski space to be fixed to D=10. Due to the fermionic degrees of freedom the resulting spectrum can strictly speaking no longer be interpreted as vibrational modes of the string in space-time. Moreover, to get rid of the tachyons the spectrum has to be truncated by the so called GSO projection[3]. The surviving spectrum consists of a finite set of massless states together with an infinite tower of massive states. These states are labelled by quantum numbers of the ten-dimensional Lorentz group and thus can be interpreted as space-time 'particles' with the corresponding fields. The massive states have masses quantized in units of $m_s = \alpha'^{-1/2}$ and are usually assumed to be of the order of the Planck mass and thus much too heavy to be of any importance in most situations, but their importance lies in the fact that they are all needed for the quantum theory to be unitary and perturbatively consistent, (i.e. they are necessary for the conformal symmetry to hold quantum mechanically). In the decoupling limit $\alpha' \to 0$, only the massless states survive and this is also called the point particle limit,

[3]The GSO (Gliozzi-Scherk-Olive) projection actually is necessary for the modular invariance of the one-loop vacuum amplitude and it also corresponds to the inclusion of all spin structures of Σ in the corresponding path-integral

moreover from the action (1.1) it can be seen that α' corresponds to the expansion parameter of the two-dimensional quantum field theory on Σ. Therefore, working at tree-level on Σ does not mean that we work at tree-level in space-time, as should become clear next when we consider string interactions and the associated n-point amplitudes (correlation functions).

The interaction of strings is described by world-sheets of different topologies connecting incoming to outgoing string configurations. As shown up to second order for two incoming and two outgoing strings in fig. 1.2, the inclusion of all possible topologies connecting a given configuration of external states is analogous to the loop expansion in field theory (and actually reduces to the latter in the limit $\alpha' \to 0$). The coupling is completely encoded in the topology of the world-sheet interpolating

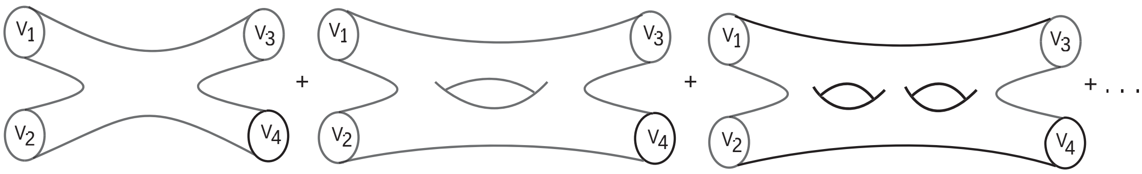

Figure 1.2: Loop expansion of four point amplitude.

between external states, and unlike for point particles there is no longer a choice of coupling constant possible that can depend on the particle species meeting at an interaction vertex. There is only one single coupling constant λ that can be assigned to the three string interaction from which all other interactions can be constructed. Moreover, such world-sheets connecting strings are smooth manifolds without distinguished interaction points in space-time, and this leads to the fact that string interactions are less singular than those of point particles. Correctly, the asymptotic in- and out-going states should be represented as semi-infinite cylinders that are conformally equivalent to punctures in the interpolating world-sheets. Such a puncture can be replaced by the insertion of a so called *vertex operator* $V_i(\sigma_i)$, that is a marginal local operator of the two-dimensional world-sheet field theory. The physical states $|i>$ of the field theory are in one-to-one correspondence with the vertex operators $V_i(\sigma_i)$ which describe the creation or annihilation of the corresponding asymptotic state at the position σ_i on the world-sheet. After this replacement, we are left (in the case of closed oriented strings) with compact closed Riemann surfaces Σ_g that are characterized by their genus[4] g which in turn determines the order of perturbation theory. The n-point amplitude with n external states at order g has $n - \chi$ interaction vertices, such that

[4]The genus g counts the number of holes in a Riemann surface, so for example $g = 0$ for the sphere and $g = 1$ for the torus, also see appendix D for more on Riemann surfaces.

the associated coupling constant is given by $\lambda^{n-\chi}$ where $\chi = 2 - 2g$ is the Euler number of Σ_g. Now the great feature of string theory is that scattering amplitudes of strings in ten-dimensional space-time can be calculated by correlation functions of vertex operators in the two-dimensional superconformal field theory. Schematically an n-point scattering amplitude including closed strings only looks like

$$\begin{aligned} \mathcal{A}(1,\ldots,n) &= \sum_g \lambda^{n-\chi} \langle V_1 \cdots V_n \rangle_g \qquad (1.2) \\ &= \sum_g \lambda^{n-\chi} \int \frac{\mathcal{D}X\,\mathcal{D}h\,\mathcal{D}\psi}{\mathrm{vol(gauge)}} e^{-S[X,h,\psi]} \prod_{i=1}^{n} \int_{\Sigma_g} d^2\sigma_i \sqrt{h(\sigma_i)} V_i(\sigma_i) \end{aligned}$$

where the sum over g corresponds to the loop expansion in terms of the sum over all topologically different Riemann surfaces Σ_g with each contribution weighted with the coupling $\lambda^{n-\chi}$. The vertex operators $V_i(\sigma_i)$, $i = 1,\ldots,n$ are those of the external states and ψ collectively denotes all fermions. The path integral has to be divided by the volume of the world-sheet gauge group in order not to count gauge equivalent configurations separately.

The Polyakov string action (1.1) can be extended to include fermionic degrees of freedom in several ways, moreover one can allow for open and closed strings with oriented or unoriented world-sheets Σ, and furthermore there can be different possibilities to perform the GSO projection. But it turns out that in a Minkowski background there are only five different consistent quantum string theories possible. Demanding the path integral (1.2), or its open string version, to be well defined yields the following constraints. For closed strings there first is the necessity of *modular invariance* of the one-loop partition function. This means that the amplitude (1.2) without any external states and without the sum over g but with Σ_1 the torus must be invariant under *large conformal transformations*, which are reparametrizations of the torus not continuously connected to the unity, or heuristically they correspond to cutting the torus along a non-contractible loop and regluing it after a twist. This symmetry lies at the heart of the soft UV-behavior of closed string theory because it constrains the integration area of the path integral in such a way that it acts like a high energy cut-off. The second constraint comes from possible non-Abelian gauge fields in the string spectrum and the requirement that there are no quantum anomalies in the associated gauge symmetries. For open strings the latter requirement turns out to be equivalent to the requirement of *tadpole cancellation* [70]. To lowest order in the closed string sector there are neither tadpoles nor a cosmological constant due to the invariance under *global conformal transformations*. Another non-trivial observation is that in the case of closed strings both stated requirements are enough for the complete amplitude (1.2) to be well defined.

The five different string theories that fulfill all consistency requirements are listed in table 1.1 where the name, massless bosonic spectrum, amount of supersymmetry, gauge group and types of strings are collected. As can be seen, all theories contain a

name	IIA	IIB	I	SO(32) heterotic	$E_8 \times E_8$ heterotic
string-type	oriented closed	oriented closed	non-oriented open, closed	oriented closed	oriented closed
SUSY	$\mathcal{N}=2$ non-chiral	$\mathcal{N}=2$ chiral	$\mathcal{N}=1$ chiral	$\mathcal{N}=1$ chiral	$\mathcal{N}=1$ chiral
gauge group	U(1)	none	SO(32)	SO(32)	$E_8 \times E_8$
massless bosonic spectrum	g_{IJ}, B_{IJ}, ϕ, A_I, A_{IJK}	g_{IJ}, B_{IJ}, ϕ, A, A_{IJ}, A_{IJKL}	g_{IJ}, ϕ, A_{IJ}, A_I^a	g_{IJ}, B_{IJ}, ϕ, A_I^a	g_{IJ}, B_{IJ}, ϕ, A_I^a

Table 1.1: Five different string theories in ten dimensions.

graviton g_{IJ} and a scalar dilaton ϕ in their massless spectrum, furthermore all theories except type I start from closed oriented strings only and have an antisymmetric two-form tensor field B_{IJ} called Neveu-Schwarz (NS) B-field in their spectrum. These fields make up the universal NS-sector and they are characterized by the fact that these fields couple directly to the closed strings, that is closed strings are charged with respect to the NS-fields. Besides the universal sector each theory individually contains other massless bosonic fields that can consist of non-Abelian gauge fields A_I^a, $a = 1, \ldots,$dimG and/or antisymmetric p-form fields $A_{I_1 \ldots I_p}$ called Ramond-Ramond (RR) p-forms. The strings are not charged with respect to the RR p-forms and thus do not directly couple to them, but in string theory there actually are additional objects called *Dp-branes* with this property. Dp-branes are solitons of string theory that extend in p spatial dimensions and thus have a (p+1)-dimensional world-volume [71, 72, 73, 74, 75, 76]. In appendix B as well as the next chapter there will be more said about the M-theory equivalents of Dp-branes, i.e. M-branes. (After all, the classical treatment of M-branes, certain aspects of their inclusion in (heterotic) M-theory and some cosmological consequences are the main topics of this thesis.) Besides the bosonic field content there are also fermions in the massless spectrum which turns out to be supersymmetric in space-time such that the massless spectra fit into multiplets of ten-dimensional supergravities.

So far only strings in a flat ten-dimensional Minkowski background have been considered, but using the massless string spectra given in table 1.1, one can also consider strings in curved backgrounds with non-trivial field configurations. For simplicity we will only look at the basics of closed strings in non-trivial backgrounds of fields in the universal NS-sector to which closed strings couple directly. In such backgrounds the bosonic part of the Polyakov action (1.1) generalizes to

$$S = S_g + S_B + S_\phi \tag{1.3}$$

with

$$S_g = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{h}\, h^{\alpha\beta}\partial_\alpha X^I \partial_\beta X^J\, g_{IJ}(X)\,, \tag{1.4}$$

$$S_B = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{h}\, \epsilon^{\alpha\beta}\partial_\alpha X^I \partial_\beta X^J\, B_{IJ}(X)\,, \tag{1.5}$$

$$S_\phi = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{h}\, R(h)\,\phi(X)\,, \tag{1.6}$$

where $R(h)$ is the scalar curvature of Σ and $\epsilon^{\alpha\beta}$ is the antisymmetric world-sheet tensor density. This coupling structure is dictated by requiring the symmetries of the simpler action (1.1) to still hold in (1.3). Of course this action would again have to be extended by fermionic terms.

Let us now turn to the question what kind of background configurations yield a consistent quantum theory. The action (1.3) has the form of a non-linear sigma model, and the functions $g_{IJ}(X)$, $B_{IJ}(X)$, $\phi(X)$ can be considered as coupling 'constants' between the fields X^I in the two-dimensional field theory. Weyl invariance (scale invariance) of this field theory does not allow running couplings, therefore the corresponding β-functions must vanish and the equations $\beta^g_{IJ} = \beta^B_{IJ} = \beta^\phi = 0$ turn out to be equations of motion for the background fields. In the two-dimensional field theory these β-functions have loop expansions and can be calculated perturbatively order by order in α'. To first order in α', and for simplicity in a background where $B_{IJ} = const.$ and $\phi = const.$, they read $\beta^g_{IJ} = \alpha' R + O(\alpha'^2)$, $\beta^B_{IJ} = O(\alpha'^2)$ and $\beta_\phi \sim (D-10) + O(\alpha'^2)$. Note that S_ϕ is of higher order in α' and $\beta_\phi = 0$ fixes the space-time dimensionality already at zeroth order. At first order the Einstein equations are reproduced and for general backgrounds the NS-field and the dilaton would also get equations of motion and provide sources of energy-momentum. At this order in α' all equations are of second order in space-time derivatives. Higher derivatives are negligible in this so called regime of *low energy effective theory* only if $\sqrt{\alpha'}R_c^{-1} \ll 1$, where R_c is the characteristic curvature radius of the target space-time, see fig 1.3. The important point is that only solutions to these equations of motion provide a suitable background configuration for the action (1.3) to define a consistent quantum theory, which does not mean that we necessarily know how to solve it and find the explicit string spectrum in such a background. Besides this problem, the space of background solutions is highly degenerate which is the *vacuum degeneracy problem* in string theory. This is characteristic for supersymmetric theories and certain degeneracies are expected to be lifted once supersymmetry is broken. On the other hand, without supersymmetry we loose control of quantum corrections and thus might run into problems of stability of the whole setup considered.

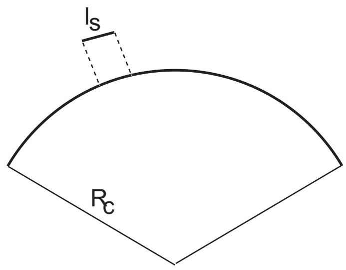

Figure 1.3: In the limit $R_c \gg l_s = \sqrt{\alpha'}$ a background with curvature radius R_c looks flat on scales of order $\mathrm{O}(l_s)$.

Above we have found the equations of motion of the fields of the universal sector only, but there is an alternative way to derive these equations for all massless space-time fields. Using the supersymmetric extension of action (1.3), their n-point functions can be calculated which, to order α', have to be reproduced by scattering amplitudes calculated from the respective effective field actions. The effective actions found in this way are the ten-dimensional supergravities, these are the type I for the heterotic and the type I string and type IIA/B for the corresponding string theories. So the known ten-dimensional supergravities are the low energy approximations to the corresponding string theories.

Next consider using a constant dilaton background $\phi = \langle\phi\rangle$ in (1.6) which yields $S_\phi = \langle\phi\rangle\,\chi$ with $\chi = 2 - 2g$ the Euler number of the Riemann surface Σ_g. Together with (1.2) this determines the physical string coupling as $\lambda_{phys} = \lambda e^{\langle\phi\rangle}$ where an arbitrary constant shift in the dilaton can be used to set $\lambda = \alpha'^2$ such that $\lambda_{phys} = \alpha'^2 g_s$ with the dimensionless string coupling 'constant' defined by $g_s = e^{\langle\phi\rangle}$ [56]. This shows that there is indeed only one single fundamental dimensionful parameter in string theory that can be taken to be α' or λ_{phys}, both being related as just shown. Since g_s is the dimensionless closed string coupling constant, string loop corrections[5] to the effective equations of motion can only be neglected if $e^{\langle\phi\rangle}$ is small enough, which had to be tacitly assumed in the foregoing discussion. Unfortunately it is not yet known what dynamically determines the expectation value $\langle\phi\rangle$, this is part of the major problem of vacuum degeneracy in string theory.

As a last remark, we mention that a closer inspection of (1.3) in the amplitude (1.2) reveals that it has the same effect as inserting the exponentiated (integrated) vertex operators of the background fields into the expectation values taken with respect to the flat background. These exponentiated vertex operators create *coherent states* that correspond to classical background field configurations. For example a curved space-time background in string theory is thus described by a coherent state of gravitons in Minkowski space. In spite of the fact that we only have a background dependent

[5]Not to be confused with the α'-corrections.

formulation of string theory, this hints at some sort of background independence, namely different space-times correspond to different states in one theory, though these states can only be formulated in a Minkowski background.

To summarize, one of the main reasons why string theory has attracted so much attention is that it so far provides the only known perturbatively consistent theory of quantum gravity that reduces to Einstein gravity in a well defined low energy limit, and furthermore it shows a high degree of uniqueness.

1.3 Compactifications and effective actions

As mentioned in the foregoing section, the equations of motion of the massless fields in the string spectrum can be obtained from classical field actions. Such effective actions are a very convenient way of describing massless string states in terms of classical field theory. Since these actions are obtained in an expansion in α', their validity is limited to energy scales below the string scale, but because the latter is assumed to be very high (as high as the Planck scale) the 'low energy' effective theories are extremely useful for applications in particle as well as gravitational physics based on string theory. It must simply be remembered that the validity of effective field theory is limited to background configurations for which the characteristic curvature radius satisfies $R_c \gg \sqrt{\alpha'}$ (see fig 1.3), that is, the background space-time must not be strongly curved and thus, by the Einstein equations, fields cannot behave too violently. Throughout the rest of the thesis we will exclusively work on the level of low energy effective actions.

The existence of a critical dimension D=10 in string theory is in contrast to the observed dimensionality of space-time d=4, and it is therefore necessary to hide the extra six dimensions. Theories in backgrounds with D<10 can be constructed either by specifying an appropriate conformal field theory on the world-sheet, or more easily by a geometrical Kaluza-Klein compactification. Here we consider the latter option only, and since such a compactification will explicitly be performed in this thesis, it is only briefly discussed here. In general, one starts by using as a *background* manifold $M_D = M_4 \times M_{int}$, where, by Poincaré invariance of the vacuum, M_4 must be four-dimensional Minkowski space and M_{int} is an internal space, which is not arbitrary [77]. Along goes a corresponding split of the associated metric and the rest of the spectrum such that all equations of motion are satisfied[6]. Besides this restriction, the internal manifold must be small enough to have escaped its detection at present-day accelerators, moreover it must be Ricci-flat for reasons of modular invariance of (1.3). To define a sensible string theory and regarding the phenomenological importance of

[6]Due to the Pauli principle fermion fields are always set to zero. Strictly speaking there is no such thing as a classical fermion field.

supersymmetry as well as demanding a chiral theory in four dimensions restricts the possibilities for the external theory to minimal $\mathcal{N} = 1$ supergravity. In the simplest cases [78] these constraints together with the desire to get non-Abelian gauge fields in the lower dimensional theory, restricts us to *Calabi-Yau* (CY) compactifications of type I or the heterotic string theories[7], where M_{int} must be a Calabi-Yau three-fold [77], this is a complex Kähler manifold of SU(3) holonomy and of three complex dimensions [80,81,82,83]. Besides this restriction on M_{int} stemming from the gravitino supersymmetry variation, there are further restrictions on the internal gauge fields coming from the gaugino variation.

Whereas string theories are almost unique in the critical dimension, this desirable feature is lost after compactification, which is due to the vacuum degeneracy of string theory. Still, in view of the lower dimensional effective action we are interested in the massless fluctuation spectrum around a given background, and the exact spectrum depends on the choice of M_{int}. Now we have to distinguish between internal and external modes. The external modes are simply the purely external components of the higher-dimensional fields generally depending on the external coordinates only. The internal massless modes are defined by flat directions in the space of background solutions, that is they parametrize the space of deformations of the internal background configuration that leave it a solution to the background field equations of motion. Differently speaking, the background solution actually corresponds to a parametrized family of solutions, and these parameters (integration constants) are then promoted to four-dimensional space-time fields called *moduli* fields. Deformations of purely internal field components lead to scalar fields and mixed components can lead to vector and tensor fields in four dimensions. Purely internal deformations of the metric are usually called geometrical moduli. For CY-compactifications the variation of the Kähler class as well as the complex structure lead to corresponding moduli, which turn out to be determined by the homology of M_{int}. Massless modes of other bosonic fields usually arise as zero-modes of the internal Laplace operator and thus are determined by the harmonic forms on M_{int}, that is again by the homology of M_{int}. Similar considerations also work out for fermions, but usually only the bosonic part has to be considered explicitly and the fermions are then given by supersymmetry.

Explicitly inserting the moduli fields into the solution and then integrating the ten-dimensional action over the internal space M_{int} yields the four-dimensional low-energy effective (moduli space) action. This action describes the dynamics of adiabatic motion through the moduli space of M_{int}, that is the gravitational backreaction of the complete four-dimensional dynamics on the internal geometry of M_{int} is neglected. This can consistently be done as long as the compactification scale is high enough, which is usually assumed to be of order of the Planck scale. This also justifies ne-

[7] In view of the dualities discussed in the next section other compactifications are also interesting. For type I strings the *intersecting brane-worlds* [79] provide very interesting models.

glecting an infinite tower of massive Kaluza-Klein excitations (non-zero modes of the internal Laplacian) since their masses are of order of the compactification scale as well.

The main point here is that a consistent truncation of a ten-dimensional string theory to lower dimensions puts constraints on the possible compactification backgrounds, though they are by no means restrictive enough to uniquely fix such a vacuum. The lower dimensional theory depends highly on the choice made and it describes the physics of the massless perturbative excitations around the chosen background.

1.4 Dualities and M-theory

As discussed in the previous section, there are many ways in reducing the five ten-dimensional string theories to lower dimensions, and there is no satisfactory explanation yet how nature could have taken any particular choice to yield the observed world. The only requirement is that the backgrounds must fulfill the field equations of motion, but this treats all possible solutions with different internal spaces and associated spectrum mathematically on equal footing. This allows to think of all possible backgrounds to a given string theory as one huge moduli space which includes the background geometry as well as expectation values of the various fields in the spectrum. In one area of this moduli space the Calabi-Yau compactifications of the last section are located, and continuously deforming the background by changing the values of the moduli allows to move around in this moduli space. This could for example correspond to changing the volume or the Kähler class of the Calabi-Yau manifold. From classical geometry one would expect two topologically distinct Calabi-Yau spaces not to be continuously deformable into each other, and there would be many disconnected components of the complete moduli space. But it actually turns out that there are certain topology changing processes for which the associated low energy physics behaves completely smooth [84, 85, 86, 87]. Heuristically speaking, a classically singular geometry does not need to look singular when probed with an extended object like a string. At such transition points in moduli space additional massless states appear in the low energy space-time spectrum. An explicit example of a topological change and the influence of such extra states is treated in chapter 4 in a cosmological context. The upshot of this paragraph is that it might be true that in this sense the whole moduli space to one string theory is connected.

There is another manifestation of stringy geometry, namely the duality of perturbatively distinct string theories [54, 55, 85, 88, 89, 90, 91, 92]. One distinguishes perturbative from non-perturbative dualities. The former can already be seen at weak string coupling and do not involve the dilaton. The prime example is *T-duality* that relates different torodial compactifications, most prominently, the type IIA theory on a circle of radius R is equivalent to type IIB on a circle of radius α'/R. Another important manifestation of the phenomenon that different theories on different backgrounds can lead to the same physics is given by mirror symmetry [93]. On the other hand, the non-perturbative dualities identify regions of moduli space which do not simultaneously yield weakly coupled theories. Then the strong coupling limit of one theory is equivalent to another weakly coupled theory, and such theories are called *S-dual*. The non-perturbative dualities cannot yet be proven rigorously but non-trivial checks can be performed by the identification of quantities that, due to supersymmetry, do not receive quantum corrections that would become non-negligible at strong coupling. In particular this is true for the spectrum of solitonic BPS-states [94] (Dp-branes) which become light at strong coupling and then can be identified with the elementary string excitations of the S-dual theory and vice versa. This shows that the string degrees of freedom should not be considered in any way as more fundamental than those of Dp-branes. There is convincing evidence that the type I and heterotic SO(32) theories are S-dual and that the type IIB theory is S-dual to itself.

Now it seems reasonable that all perturbatively distinct string theories are interrelated by an intriguing web of dualities. The missing link is provided by the strong coupling limits of the type IIA and the heterotic $E_8 \times E_8$ theories. On the level of effective actions, the IIA supergravity can be obtained from eleven-dimensional supergravity compactified on a circle. Moreover, in IIA theory a bound state of D0-branes has a mass $\sim n/g$, where $n \in \mathbb{Z}$ and g the coupling constant, and this corresponds exactly to a Kaluza-Klein mass spectrum from a circular compactification. This is interpreted as the fact that the strong coupling limit of type IIA string theory is an eleven-dimensional theory, elusively named *M-theory*, which has the corresponding supergravity as its low energy effective action. Due to its dimensionality, M-theory can no longer be a theory of strings. The fundamental object in M-theory is a membrane, which compactified on a circle gives the type IIA string, as illustrated to the left in fig. 1.4. Similarly, the heterotic $E_8 \times E_8$ string can be obtained from wrapping the

Figure 1.4: Strings from a membrane by compactification.

membrane on an interval, as illustrated to the right in fig. 1.4. The effective action of the strong coupling limit of the heterotic $E_8 \times E_8$ string theory is then given by eleven-dimensional supergravity on an interval, or equivalently on the orbifold $S^1/\mathbb{Z}_2$. Furthermore, for reasons of anomaly cancellation, on each of the two ten-dimensional fixed planes at the end of the interval an $\mathcal{N} = 1$ super Yang-Mills theory with gauge

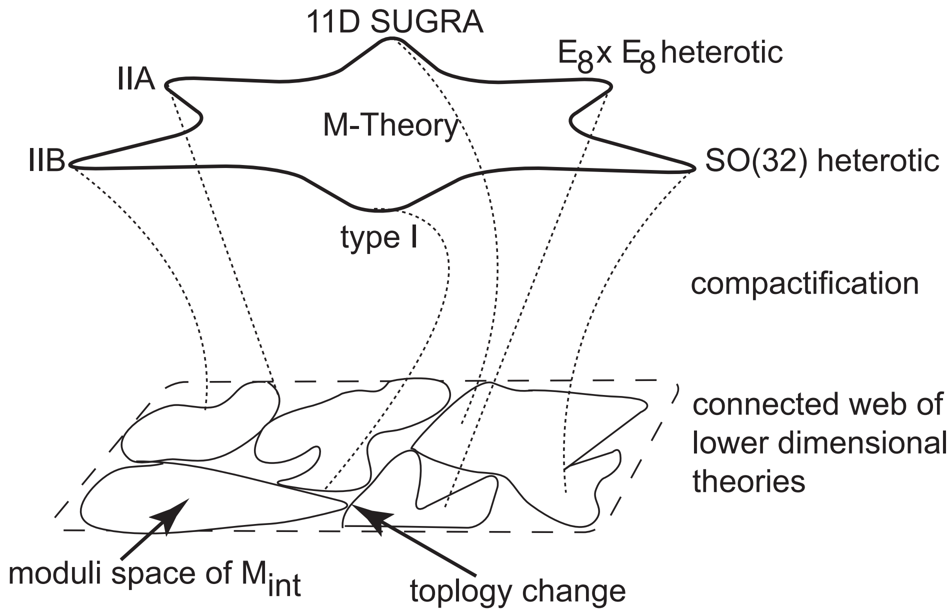

Figure 1.5: Moduli space of M-theory.

group E_8 has to be introduced [95, 96, 97, 98, 99, 100, 101, 102]. This construction is called Hořava-Witten theory or heterotic M-theory, and a special kind of compactification thereof including M5-branes will be one of the major topics of this thesis, see chapter 2.

The upshot of this section is that the moduli spaces of the perturbatively different string theories seem actually to be part of one enormous moduli space to a single underlying theory, named M-theory. The formerly distinct string theories on a specific background then correspond to local 'coordinate patches' on a single moduli space, and they are perturbative descriptions of M-theory around different points in moduli space. This is similar to the description of a topologically non-trivial manifold which cannot be covered by one global set of coordinates. Furthermore, there is no longer an absolute notion of strong or weak coupling nor of solitonic or elementary objects, these all depend on the description used. The huge moduli space of M-theory is illustrated in fig. 1.5.

1.5 Topics and organization of the thesis

If string/M-theory provides the correct fundamental theory of nature, then branes are most likely to play an important part in possible applications, like for example in early universe cosmology. In order to study branes and their classical dynamics it is necessary to have an appropriate low energy effective action at hand. Of course such an action strongly depends on the background and the branes considered, and it must consistently couple world-volume fields to bulk fields. A phenomenologically interesting background is provided by the strong coupling limit of the heterotic $E_8 \times E_8$ string theory. As already noted, on the level of effective actions this limit is called Hořava-Witten theory or more generally heterotic M-theory, and it has been identified as eleven-dimensional supergravity[8] theory compactified on an orbifold $S^1/\mathbb{Z}_2$ with a set of E_8 gauge fields at each ten-dimensional orbifold fixed plane [95, 96]. Eleven-dimensional supergravity has two brane solutions, these are the fundamental electric M2-brane (membrane) and the magnetic M5-brane [103], which are interpreted as the basic 'matter particles' of M-theory.

The main topic of this thesis is the derivation of the effective actions of Calabi-Yau compactifications of (heterotic) M-theory including special brane configurations. Certain time dependent solutions thereof are also considered. All chapters are self-contained and can thus be read independently. Relations between the chapters are indicated whenever useful. Let us next give the plan of the rest of this thesis.

In chapter 2 we derive the $\mathcal{N} = 1$ supersymmetric five- as well as four-dimensional effective supergravity actions of Calabi-Yau compactifications of heterotic M-theory including N separate M5-branes wrapping holomorphic two-cycles within the internal Calabi-Yau manifold. The whole chapter is structured as follows.

After a short introduction in the first section, in the second section the consistent coupling of M5-branes to pure eleven-dimensional M-theory is reviewed along the lines of the nice works of refs. [104, 105]. In the subsequent section this work is extended to heterotic M-theory and the obtained action (2.39) is the first non-trivial new result. It turns out that all relative factors of the different terms in this action are uniquely fixed to values which cannot be obtained from a simple rescaling of fields used in ref. [105]. Next, in section 2.4, we consider an eleven-dimensional (warped) background solution to the given action [106, 107, 108, 109], consisting of a space-time $M = M_4 \times S^1/\mathbb{Z}_2 \times X$ with parallel M5-branes spanning four-dimensional Minkowski space M_4, wrapping holomorphic two-cycles within the internal Calabi-Yau space X and positioned along the orbifold $S^1/\mathbb{Z}_2$. For the first time it is shown that this well known vacuum is a solution to the complete action explicitly including the M5-brane world-volume theories. Then we include the moduli, where the whole homology sectors $H^{1,1}(X)$ and $H^{2,1}(X)$ of the Calabi-Yau space X are taken into account. Moreover, the moduli from the M5-brane world-volume theories as well as gauge matter from the boundary theories are included. In section 2.5 the corresponding gauged $\mathcal{N} = 1$ supersymmetric five-dimensional moduli space supergravity action (2.103) is given. From this five-dimensional viewpoint the M5-branes appear as three-branes. The new results here come from the explicit inclusion of the complex structure

[8] In the following we do not make a difference between M-theory and eleven-dimensional supergravity.

moduli and the world-volume moduli fields into this heterotic M-theory background. Furthermore, following refs. [110, 108], the quaternionic structure of the complete hypermultiplet sector including the complex structure moduli is demonstrated, and it is shown to be in accord with the gauging (of the shift symmetry of the axion) in the universal hypermultiplet and the associated potential. A new dual formulation of the five-dimensional action with electrical coupling of the three-branes concludes this section. Section 2.6 treats the five-dimensional BPS multi-domain wall solution to action (2.103), and it is a generalization of the solution given in refs. [108, 109, 111] to include the three-branes in addition to an arbitrary number of Kähler moduli. Based on this solution, the four-dimensional $\mathcal{N} = 1$ supergravity theory is given in section 2.7, first in component form and then in the corresponding superfield formulation that generalize previous works [112, 113, 109]. The correct parametrization of all superfields in terms of component fields, together with the gauge kinetic functions written in terms of superfields, are the new result of this section.

Some of the results of this chapter have been published in ref. [114], where only a single M5-brane was included in the universal background ($h^{1,1} = 1$), and neither gauge matter nor the complex structure moduli have been taken into account.

In chapter 3 we consider time-dependent solutions to the simplest form of the four-dimensional action (2.135) derived in the previous chapter. That means we consider 'cosmological'[9] solutions of four-dimensional gravity coupled to two scalar fields, geometrically corresponding to the Calabi-Yau volume modulus and the orbifold radius modulus. The influence of the inclusion of a single M5-brane, as was analyzed in [115], is briefly reviewed. This chapter has the following structure.

After an introduction, the second section briefly reviews the correspondence between five- and four-dimensional heterotic M-theory in the simple setting considered. The relation of the considered four-dimensional heterotic M-theory action to the usual starting point of pre-Big-Bang cosmology [116, 117, 118, 119] is also given. In section 3.3 a one-parameter family of new time-dependent solutions is found and, by using the correspondence between the four- and the corresponding five-dimensional theory, these solutions are 'lifted up' to approximate five-dimensional solutions. In the following section 3.4 the role of the fifth (orbifold) dimension is elucidated and its relation to the strong coupling parameter is given. In the last section the results [115] of the generalized treatment that also includes a single M5-brane are stated.

Apart from the last section, this chapter closely follows ref. [120].

In chapter 4 we consider a compactification of pure M-theory including an M2-brane wrapping a holomorphic two-cycle within the internal Calabi-Yau space. The effective action of the associated particle states is derived and for an explicit example

[9] With cosmological solutions we mean time–dependent solutions of Friedmann-Robertson-Walker type.

their influence on the dynamics of a flop transitions is numerically studied. The plan of this chapter is as follows.

An introduction is given in section 4.1. In the subsequent section we first set the stage by quickly reviewing the general structure of eleven- and five-dimensional $\mathcal{N} = 1$ supergravity and their relation by Calabi-Yau compactifications of M-theory. Using the rigid structure of five-dimensional gauged $\mathcal{N} = 1$ supergravity, we then derive the effective action for the hypermultiplet states associated to the M2-brane wrapping a holomorphic two-cycle within the Calabi-Yau manifold. The mass of these states is proportional to the volume of the two-cycle wrapped, and hence these 'transition states' become massless when such a cycle has shrunk to vanishing volume, that is at a so called flop transition. By means of an approximate analytical solution, the temporal dynamics of such a flop transition without the inclusion of the transition states is studied in section 4.3. In section 4.4 the influence of the transition states on the dynamics of the flop transition is numerically studied for an explicit example. Conclusions are made in section 4.5.

The new results of this chapter consist of the effective action for the transition states, the analytical solution of section 4.3 and the numerical results of section 4.4. The complete chapter corresponds to ref. [121].

There are four appendices, the first appendix collects conventions and the notation used in this thesis. In appendix B an introduction to the classical treatment of brane currents is given. Appendix C collects some useful relations from special Kähler geometry, and appendix D reviews a few facts about Riemann surfaces.

Chapter 2

M5-branes in heterotic brane-worlds

2.1 Introduction

Many interesting five-dimensional brane-world models can be constructed by the reduction of Hořava-Witten theory [95, 96, 106] on Calabi-Yau threefolds. As was explicitly shown in refs. [108, 111, 107, 122], this leads to gauged five-dimensional $\mathcal{N} = 1$ supergravity on the orbifold $S^1/\mathbb{Z}_2$ coupled to $\mathcal{N} = 1$ gauge and gauge matter multiplets located on the two four-dimensional orbifold fixed planes. As has already been realized in [106], M5-branes transverse to the orbifold direction, wrapping holomorphic two-cycles within the Calabi-Yau space and stretching across the four uncompactified dimensions, can consistently be incorporated into this picture. The explicit form of the corresponding eleven-dimensional vacuum solution was first given in ref. [109]. Moreover it was shown in refs. [123, 124, 125, 126, 127, 128, 129, 130, 131, 132] that the inclusion of M5-branes helps to realize phenomenologically interesting particle spectra on the orbifold planes by appropriate compactifications, and thus five-dimensional heterotic M-theory with branes in the bulk provides a phenomenologically viable starting point for further applications [133, 134, 135].

Recently such generalized brane-world models including extra branes have attracted some attention [136, 137, 138, 139, 140, 141, 142], especially in the context of cosmology. The purpose of this work is to present a systematic derivation of the five-dimensional effective action of such models in order to provide a firm basis for further investigations [143, 115, 144, 145, 146, 147, 148].

2.2 M5-branes coupled to M-theory

In this section we want to derive the action that consistently couples an M5-brane to eleven-dimensional supergravity. In order to do so, we proceed along the lines of the nice work by Bandos, Berkovits and Sorokin [105] (for recent refinements see [149]). The purpose of this section is twofold. First we want to review the formalism of Pasti, Sorokin and Tonin (PST) which allows a consistent inclusion of (self-)dual fields in so called duality symmetric actions from which the duality constraints follow as equations of motion and need not be imposed by hand. This is needed to formulate a covariant action for the M5-brane world-volume theory, but also to learn how to consistently couple this world-volume action to the background supergravity action. And the second reason is that, in the next section, we want to extend the work of ref. [105] to HW-theory, and this involves the PST-symmetries presented in this section.

This section is structured as follows, first we review the duality symmetric formulation of eleven-dimensional supergravity as well as the covariant action of the M5-brane [104] independently. Before these two actions get coupled, we examine their behavior under rescalings of the fields. Then we couple the actions with the most general relative factors that are allowed by such rescalings and determine a relation between these factors imposed by the PST-symmetries.

2.2.1 Duality symmetric action for D=11 supergravity

The supergravity action

The bosonic part of the action of the standard formulation of eleven dimensional supergravity [49] is given by

$$\begin{aligned}
S_{CJS} &= S_R + S_G \\
&= -\frac{1}{2\kappa_{11}^2}\int_{M_{11}}\left\{d^{11}x\sqrt{-g}\left(\frac{1}{2}R+\frac{1}{4!}(G^{(4)})^2\right)+\frac{2}{3}C^{(3)}\wedge G^{(4)}\wedge G^{(4)}\right\} \\
&= -\frac{1}{2\kappa_{11}^2}\int_{M_{11}}\left\{d^{11}x\sqrt{-g}\frac{1}{2}R - G^{(4)}\wedge *G^{(4)}+\frac{2}{3}C^{(3)}\wedge G^{(4)}\wedge G^{(4)}\right\} \\
&= S_R - \frac{1}{2\kappa_{11}^2}\left\{-\langle G^{(4)},G^{(4)}\rangle - \frac{2}{3}\langle C^{(3)}, *(G^{(4)}\wedge G^{(4)})\rangle\right\}
\end{aligned} \tag{2.1}$$

where the scalar product as defined in (A.6) has been introduced, the field strength for the three-form gauge potential is $G^{(4)} = dC^{(3)}$, and κ_{11} denotes the eleven-dimensional Newton constant.

This action does not contain the six-form potential $C^{(6)}$, and it can also not simply be added as a dynamical field because it does not provide independent degrees of freedom, since its field strength is the Hodge dual of the three-form field strength

$G^{(4)}$. Moreover, due to the Chern-Simons term, there is no dual formulation purely in terms of $C^{(6)}$ instead of $C^{(3)}$ possible. This makes it impossible to minimally couple the M5-brane, which is electrically charged under the six-form potential, simply by adding a minimal coupling term like $\sim T_5 \int_{M_6} C^{(6)}$ (where T_5 is the brane-tension and M_6 is the M5-brane world-volume) to the supergravity action. Thus it is either necessary to explicitly include the six-form $C^{(6)}$ as a dynamical field in the supergravity action, or one must seek a way to magnetically couple the M5-brane to the three-form $C^{(3)}$. A further complication arises since the M5-brane actually is a dyonic object, thus the presence of M5-branes must lead to source contributions in the equation of motion as well as the Bianchi identity of $C^{(3)}$ or $C^{(6)}$.

In the following we present the bosonic[1] part of the duality symmetric formualtion of D=11 supergravity [105] which contains the three-form $C^{(3)}$ as well as the six-form $C^{(6)}$ as dynamical fields. Their equations of motion following from the duality symmetric formulation turn out to be gauge equivalent to the duality constraint

$$G^{(7)} = *G^{(4)}, \tag{2.2}$$

and on-shell, their Bianchi indentities will take the place of the usual equation of motion and Bianchi identity of $C^{(3)}$ which otherwise would follow from the standard formulation (2.1).

To correctly introduce the six-form $C^{(6)}$ and its relation to the dual field strength $G^{(7)}$, the equation of motion of the three-form $C^{(3)}$ following from action (2.1) must be taken into account because it must become the Bianchi identity of the dual field. It is given by

$$d * G^{(4)} = G^{(4)} \wedge G^{(4)}, \tag{2.3}$$

and thus the dual field must be defined by

$$G^{(7)} = dC^{(6)} + C^{(3)} \wedge G^{(4)}, \tag{2.4}$$

such that indeed

$$dG^{(7)} = G^{(4)} \wedge G^{(4)}. \tag{2.5}$$

The explicit appearance of the three-form in the field strength $G^{(7)}$ as in (2.4) is the reason why $C^{(3)}$ can never be completely eliminated from the action (2.1) and be replaced by the six-form $C^{(6)}$ alone. And it is this topological obstruction that makes simple minimal coupling of the M5-brane impossible.

Next it is useful to define "generalized field strengths" by

$$\mathcal{F}^{(4)} \equiv G^{(4)} + *G^{(7)}, \;\; \mathcal{F}^{(7)} \equiv G^{(7)} - *G^{(4)}$$

[1] It is not at all trivial that a at the same time supersymmetric and duality symmetric formulation is possible at all, but actually it is, and for the original work including fermions in a supersymmetric way see [105].

which satisfy the duality relations

$$*\mathcal{F}^{(4)} = -\mathcal{F}^{(7)}, \quad *\mathcal{F}^{(7)} = \mathcal{F}^{(4)}. \tag{2.6}$$

Note that these generalized fields vanish exactly when the duality relation (2.2) is satisfied. It is also necessary to introduce an auxiliary scalar field a and an associated unit, spacelike[2] vector v given by

$$v \equiv \frac{da}{\sqrt{(\partial a)^2}} \quad \Rightarrow \quad v^2 = 1.$$

It is necessary to introduce such a field to be able to write the duality symmetric action in a manifestly covariant form and to enhance its symmetries in such a way, that these extra symmetries will reduce the number of on-shell degrees of freedom exactly to the actual physical value, namely the duality relation (2.2) must be true and thus $C^{(3)}$ and $C^{(6)}$ are not independent. Moreover, the field a turns out to be purely auxiliary and thus has no dynamical degrees of freedom of its own. These symmetries will explicitly be considered shortly.

Now it is possible to construct actions including both gauge fields $G^{(4)}$ and $G^{(7)}$ in a manifestly duality symmetric way, which have equations of motion from which the duality relation (2.2) follows. Furthermore (2.5) is the Bianchi identity for $G^{(7)}$, from which on-shell exactly (2.3) follows.

Three equivalent formulations of such a duality symmetric supergravity action in eleven dimensions are given by [105]

$$S_G^I = \frac{-1}{2\kappa_{11}^2}\left[\langle i_v * G^{(7)}, i_v\mathcal{F}^{(4)}\rangle - \langle i_v * G^{(4)}, i_v\mathcal{F}^{(7)}\rangle + \frac{1}{3}\langle G^{(7)}, *G^{(4)}\rangle\right] \tag{2.7}$$

$$\begin{aligned} S_G^{II} = \frac{-1}{4\kappa_{11}^2}\Big[&- \langle G^{(4)}, G^{(4)}\rangle + \langle i_v\mathcal{F}^{(4)}, i_v\mathcal{F}^{(4)}\rangle - \langle G^{(7)}, G^{(7)}\rangle \\ &+ \langle i_v\mathcal{F}^{(7)}, i_v\mathcal{F}^{(7)}\rangle + \frac{2}{3}\langle G^{(7)}, *G^{(4)}\rangle\Big] \end{aligned} \tag{2.8}$$

$$S_G^{III} = \frac{-1}{2\kappa_{11}^2}\left[- \langle G^{(4)}, G^{(4)}\rangle + \langle i_v\mathcal{F}^{(4)}, i_v\mathcal{F}^{(4)}\rangle - \frac{2}{3}\langle C^{(3)}, *(G^{(4)} \wedge G^{(4)})\rangle\right] \tag{2.9}$$

where the interior product (A.14) has been introduced. These three actions can be obtained from each other by reshuffling terms, discarding total derivatives and using the identity (A.17) and

$$G^{(7)} \wedge G^{(4)} = C^{(3)} \wedge G^{(4)} \wedge G^{(4)} + d(C^{(6)} \wedge G^{(4)})\,. \tag{2.10}$$

[2] One could as well take a timelike unit vector, but this would change certain signs in the following treatment.

Equivalence of these actions means that they all yield the same equations of motion. The first two actions explicitly show the duality symmetric treatment of the three- and six-form, whereas the third formulation is closest to the standard formulation (2.1).

Symmetries and equations of motion

To explicitly see all the symmetries of the actions (2.7-2.9) and to derive the equations of motion for the gauge fields $C^{(3)}$, $C^{(6)}$ and the scalar field a, it is most convenient to calculate the general variation of such an action with respect to these fields. In order to do so, the variations of the corresponding field strengths are needed, and they are given by

$$\begin{aligned}
\delta G^{(4)} &= d\delta C^{(3)}, && (2.11)\\
\delta G^{(7)} &= d\delta C^{(6)} + \delta C^{(3)} \wedge G^{(4)} + C^{(3)} \wedge d\delta C^{(3)} \\
&= d\delta C^{(6)} + d(\delta C^{(3)} \wedge C^{(3)}) + 2\delta C^{(3)} \wedge G^{(4)}, && (2.12)\\
\delta v &= \frac{-1}{\sqrt{(\partial a)^2}} * [v \wedge i_v * d(\delta a)].
\end{aligned}$$

The general variation of the action then is

$$\begin{aligned}
\kappa_{11}^2 \delta S_G &= \int_{M_{11}} \Big\{ \delta G^{(7)} \wedge (v \wedge i_v \mathcal{F}^{(4)}) - \delta G^{(4)} \wedge (v \wedge i_v \mathcal{F}^{(7)}) \\
&\qquad - \frac{d(\delta a)}{\sqrt{(\partial a)^2}} \wedge (v \wedge i_v \mathcal{F}^{(7)} \wedge i_v \mathcal{F}^{(4)}) \Big\} && (2.13)\\
&= -\int_{M_{11}} \Big\{ \Big(\delta C^{(6)} + \delta C^{(3)} \wedge C^{(3)} - \frac{\delta a}{\sqrt{(\partial a)^2}} i_v \mathcal{F}^{(7)} \Big) \wedge d(v \wedge i_v \mathcal{F}^{(4)}) \\
&\qquad + \Big(\delta C^{(3)} - \frac{\delta a}{\sqrt{(\partial a)^2}} i_v \mathcal{F}^{(4)} \Big) && (2.14)\\
&\qquad \wedge \Big(d(v \wedge i_v \mathcal{F}^{(7)}) - 2\, G^{(4)} \wedge v \wedge i_v \mathcal{F}^{(4)} \Big) \Big\}.
\end{aligned}$$

To get from the first to the second line, eqs. (2.11, 2.12) must be used, followed by a partial integration and noting that

$$v \wedge d\Big[\frac{i_v \alpha_p}{\sqrt{(\partial a)^2}} \Big] = -\frac{1}{\sqrt{(\partial a)^2}} d(v \wedge i_v \alpha_p)$$

where α_p stands for any p-form.

From this general variation δS_G several bosonic local symmetries can be deduced [105].
• First, the action is invariant under the general gauge transformations

$$\begin{aligned}\delta_0 C^{(3)} &= d\Lambda^{(2)},\\ \delta_0 C^{(6)} &= d\Lambda^{(5)} + \delta_0 C^{(3)} \wedge C^{(3)},\end{aligned} \tag{2.15}$$

which leave the field strengths invariant. Here $\Lambda^{(p)}(x)$ for p=2,5 are arbitrary p-forms as usual.
• Second, due to the fact that $da \wedge v \sim v \wedge v = 0$, it is also invariant under transformations where the variations of the field strengths are of the form $\delta G^{(i)} = da \wedge \ldots$ for i=4,7. This is the PST1-symmetry that will allow to fix the equations of motion to the desired duality condition (2.2). For the potentials corresponding transformations are given by

$$\delta_1 C^{(3)} = da \wedge \phi^{(2)} \qquad \Rightarrow \qquad \delta_1 G^{(4)} = -da \wedge d\phi^{(2)}, \tag{2.16}$$

$$\delta_1 C^{(6)} = da \wedge \phi^{(5)} - \delta_1 C^{(3)} \wedge C^{(3)} \qquad \Rightarrow \qquad \delta_1 G^{(7)} = -da \wedge [d\phi^{(5)} - 2\phi^{(2)} \wedge G^{(4)}].$$

Here $\phi^{(p)}(x)$ for p=2,5 are arbitrary p-forms too. The second term in $\delta_1 C^{(6)}$ could be absorbed in the first term by taking $\tilde{\phi}^{(5)} \equiv \phi^{(5)} - \phi^{(2)} \wedge C^{(3)}$. But note that as soon as $\delta_1 C^{(3)} \neq 0$ there is also $\delta_1 G^{(7)} \neq 0$, and this induced variation of the dual field by the variation of the three-form cannot be completely absorbed in a variation of the six-form, because the last term in (2.12) is not a total derivative.
• Third, there is the PST2-symmetry that allows to set the field a to any desired value and thus shows that a truly is an auxiliary field. It is given by

$$\begin{aligned}\delta_2 C^{(3)} &= \frac{\varphi}{\sqrt{(\partial a)^2}} i_v \mathcal{F}^{(4)},\\ \delta_2 C^{(6)} &= \frac{\varphi}{\sqrt{(\partial a)^2}} i_v \mathcal{F}^{(7)} - \delta_2 C^{(3)} \wedge C^{(3)},\\ \delta_2 a &= \varphi,\end{aligned} \tag{2.17}$$

with $\varphi(x)$ an arbitrary scalar field.

Of course also the equations of motion can be read off from (2.14) and they are

$$C^{(6)}: \quad d(v \wedge i_v \mathcal{F}^{(4)}) = 0 \tag{2.18}$$

$$C^{(3)}: \quad C^{(3)} \wedge d(v \wedge i_v \mathcal{F}^{(4)}) + d(v \wedge i_v \mathcal{F}^{(7)}) - 2G^{(4)} \wedge v \wedge i_v \mathcal{F}^{(4)} = 0$$

$$\stackrel{(2.18)}{\Longrightarrow} \qquad d[v \wedge i_v \mathcal{F}^{(7)} - 2\, C^{(3)} \wedge v \wedge i_v \mathcal{F}^{(4)}] = 0 \tag{2.19}$$

Due to the duality (2.6) between the “generalized field strengths” this set of equations is already solved if only one of the equations is fulfilled. Explicitly, the equation $d(v \wedge i_v \mathcal{F}^{(4)}) = 0$ has the most general solution

$$v \wedge i_v \mathcal{F}^{(4)} = da \wedge d\Phi^{(2)}, \tag{2.20}$$

with an arbitrary two-form $\Phi^{(2)}$. But this solution turns out to be pure gauge. By using the PST1-symmetry (2.16) to fix the gauge of $G^{(4)}$ by choosing $\delta_1 C^{(3)} = -da \wedge \Phi^{(2)}$, we obtain

$$\begin{aligned}
\delta_1(v \wedge i_v \mathcal{F}^{(4)}) &= v \wedge i_v(\delta_1 G^{(4)} + *\delta_1 G^{(7)}) \\
&= v \wedge i_v(da \wedge d\Phi^{(2)}) + v \wedge i_v * (-2\, da \wedge \Phi^{(2)} \wedge G^{(4)}) \\
&= da \wedge d\Phi^{(2)}
\end{aligned}$$

where eqs. (A.14-A.17) were used. Thus eq. (2.20) is gauge equivalent to

$$v \wedge i_v \mathcal{F}^{(4)} = 0\,,$$

which implies $\mathcal{F}^{(4)} = 0$, and this vanishing of the generalized field strength in turn implies the desired duality relation $*G^{(4)} = G^{(7)}$. Thus we explicitly see how the equation of motion is equivalent to the duality condition. If the duality $G^{(7)} = *G^{(4)}$ holds, then it follows from eq.(2.6) that $\mathcal{F}^{(7)} = -*\mathcal{F}^{(4)} = 0$, and thus both equations of motion (2.18,2.19) are solved. Moreover, and importantly, the solution is independent of a, showing that a really is an auxiliary field without any dynamical degrees of freedom as it is encoded in the PST2-symmetry (2.17).

This completes the treatment of the duality symmetric formulation of pure D=11 supergravity. Before we turn to the coupling of this action to M5-branes, the covariant action for the M5-brane itself is reviewed.

2.2.2 Covariant action for the M5-brane

The M5-brane action

The bosonic part of the covariant action for the M5-brane without its coupling to the eleven-dimensional supergravity background is given by [104]

$$S_5 = -T_5 \int_{M_6} d^6\sigma \left[\sqrt{-\det\left(\gamma_{mn} + i\tilde{H}^*_{mn}\right)} + \frac{\sqrt{-\gamma}}{4} \tilde{H}^{*mn} \tilde{H}_{mn} \right], \tag{2.21}$$

where M_6 is the world-volume of the M5-brane. We use indices $I, J, \ldots = 0, \ldots, 9, 11$ for eleven-dimensional space-time with coordinates x^I and indices $m, n, \ldots = 0, \ldots, 5$ for the six-dimensional five-brane world-volume with coordinates σ^m. Then the various quantities appearing in this action are defined by

$$
\begin{aligned}
H = dB \quad &\longleftrightarrow \quad H_{lmn} = 3\partial_{[l}B_{mn]} \\
(*H)_{lmn} \quad &= \quad \frac{1}{3!}\epsilon_{lmnpqr}H^{pqr} \\
v_l = \frac{\partial_l a}{\sqrt{(\partial a)^2}} \quad &\Rightarrow \quad v^2 = 1 \\
\tilde{H} = i_v H \quad &\longleftrightarrow \quad \tilde{H}_{mn} = v^l H_{lmn} \\
\tilde{H}^* = i_v(*H) \quad &\longleftrightarrow \quad \tilde{H}^*_{mn} = v^l(*H)_{lmn} \\
\gamma = \det(\gamma_{mn}), \quad & \quad \gamma_{mn} = \partial_m X^I \partial_n X^J g_{IJ}
\end{aligned}
$$

with $B(\sigma)$ a two-form gauge potential with selfdual field strength $H = *H$, $a(\sigma)$ an auxiliary PST-scalar and $\gamma_{mn}(\sigma)$ the pullback of the background space-time metric $g_{IJ}(x)$ onto the M5-brane world-volume M_6. The two-form B corresponds to the Goldstone tensor modes (or gauge zero-modes) [150, 151] of the M5-brane solution of D=11 supergravity [103], and boundaries of M2-branes ending on the M5-brane couple to it [152], see fig. 2.1. The other world-volume fields are given by the embedding coordinates $X^I(\sigma)$ of the M5-brane into target space and correspond to Goldstone scalar modes (or scalar zero-modes) from the broken transversal translational invariance of the M5-brane solution. The X^I are hidden in γ_{mn} through the pullback. Together these fields give 3+5=8 bosonic physical degrees of freedom. These bosonic modes have supersymmetric partners, which are the fermionic Goldstone modes from the eight broken supersymmetries of the M5-brane solution. Thus the world-volume fields constitute a $\mathcal{N} = 1$ supersymmetric theory on the M5-brane. For the inclusion of fermions see [153, 154].

The relative factors between the two terms in (2.21) allow the linearization in $\tilde{H}^*$ of the term of Dirac-Born-Infeld type to yield the linearized action

$$
\begin{aligned}
S_{5lin} &= -T_5 \int_{M_6} d^6\sigma\sqrt{-\gamma}\left[1 - \frac{1}{4}\tilde{H}^{*mn}\left(\tilde{H}^*_{mn} - \tilde{H}_{mn}\right)\right] \\
&= -T_5 \left[\ \mathrm{Vol}(M_6) + \frac{1}{2}\langle i_v(*H), i_v(*H - H)\rangle\ \right] \\
&= -T_5 \left[\ \mathrm{Vol}(M_6) + \frac{1}{4}\big(\,\langle H, H\rangle + \langle i_v\mathcal{H}, i_v\mathcal{H}\rangle\,\big)\ \right]
\end{aligned}
\tag{2.22}
$$

with the definition of a "generalized field strength" by $\mathcal{H} \equiv *H - H$. In the linearization the expansion

$$
\det(1 + A) \simeq 1 - \frac{1}{2}trA^2 = 1 + \frac{1}{2}A_{mn}A^{mn}
$$

for any two-form A has been used, and to derive the last equality in (2.22) relation (A.17) is needed.

For simplicity, we from now on use only the linearized action (2.22), though all conclusions of this section would in an accordingly modified form[3] also be true in the more general non-linear case.

Symmetries and equations of motion

To see the symmetries and to obtain the equations of motion it is again convenient to derive the general variation of the action with respect to arbitrary variations of the fields B and a. Using the variation of the unit vector v and the field $\tilde{H}$, which are given by

$$\begin{aligned} \delta v &= \frac{1}{\sqrt{(\partial a)^2}} * [v \wedge i_v * d(\delta a)], & (2.23) \\ \delta \tilde{H} &= i_{\delta v} H + i_v d\delta B = *(\delta v \wedge * H) + i_v d\delta B, & (2.24) \end{aligned}$$

the general variation yields

$$\frac{1}{T_5}\delta S_{5lin} = \int_{M_6} \left(\delta B + \frac{\delta a}{\sqrt{(\partial a)^2}} i_v \mathcal{H}\right) \wedge d(v \wedge i_v \mathcal{H}). \tag{2.25}$$

Now the world-volume versions of the PST1- and PST2-symmetries and a gauge symmetry can be spotted. Up to a total derivative the action (2.22) is invariant under the variations [104]

$$\begin{aligned} \delta B &= d\Lambda^{(1)} + da \wedge \phi^{(1)} - \frac{\varphi}{\sqrt{(\partial a)^2}} i_v \mathcal{H} & (2.26) \\ \delta a &= \varphi \end{aligned}$$

where $\Lambda^{(1)}(\sigma)$, $\phi^{(1)}(\sigma)$ are arbitrary one-forms and $\varphi(\sigma)$ is an arbitrary scalar function. Note that all these symmetries are independent of the relative factor between the volume term an the rest in the action (2.22), which of course reflects the possibility of rescaling the field B independently of the metric.

The equation of motion can be read off to be

$$d(v \wedge i_v \mathcal{H}) = 0, \tag{2.27}$$

with the general solution $v \wedge i_v \mathcal{H} = da \wedge d\phi^{(1)}$. But this is pure gauge and can be gauged away using the PST1-symmetry (2.26). Choosing $\delta B = da \wedge \phi^{(1)}$ and noting that $\delta(v \wedge i_v \mathcal{H}) = -\delta H = da \wedge d\phi^{(1)}$, it follows that in this chosen gauge $v \wedge i_v \mathcal{H} = 0$. Due to the antiselfduality of $\mathcal{H}$ this is only true for $\mathcal{H} = 0$, which implies exactly the desired selfduality condition $*H = H$.

[3] Some of the formulas of course change to a non-linear form. Basically this only changes the "generalized field strength", which in the non-linear case would be given by $i_v \mathcal{H} = \frac{2}{\sqrt{-\gamma}} \frac{\delta \mathcal{L}_{DBI}}{\delta \tilde{H}^*_{mn}} - i_v H$, $\mathcal{L}_{DBI} \equiv \sqrt{-det(\gamma_{mn} + i\tilde{H}^*_{mn})}$, which reduces exactly to $i_v \mathcal{H} = i_v(*H - H)$ in the linear case. For explicit calculations in the non-linear case see for example [155].

Coupling to D=11 supergravity background fields

When the M5-brane is coupled to the eleven-dimensional supergravity background fields the world-volume field H gets modified [152] as

$$H \longrightarrow H = dB - \hat{C}^{(3)}, \tag{2.28}$$

with $\hat{C}^{(3)}$ the pullback of the three-form potential $C^{(3)}$ of D=11 supergravity. This is a gauge invariant combination if for the gauge transformation $\delta_0 C^{(3)} = d\Lambda^{(2)}$ the B-field transforms like $\delta_0 B = \hat{\Lambda}^{(2)}$, such that indeed $\delta_0 H = 0$. This modification changes the Bianchi identity of B to $dH = -d\hat{C}^{(3)}$ and means that there is an electric current present on the world-volume, which can be interpreted as coming from the induced electric field of M2-branes. The situation is illustrated in fig 2.1.

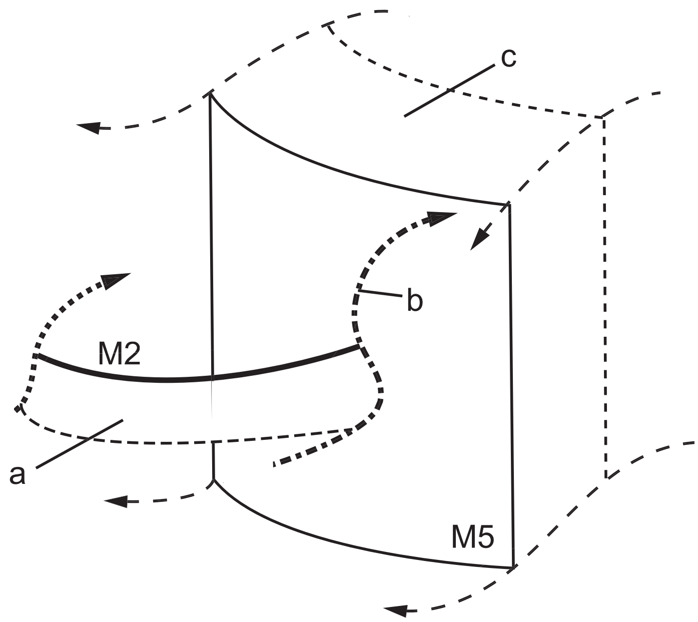

Figure 2.1: M2-brane ending on M5-brane as electric source currents for the fields (a) $C^{(3)}$, (b) B and (c) $C^{(6)}$.

Then as usual there is the minimal coupling term of the form

$$S_{mc} = -T_5 \int_{M_6} -\frac{1}{2}\,\hat{C}^{(6)}, \tag{2.29}$$

with the six-form indirectly given by (2.4). As we have seen, eq. (2.4) is the necessary definition of $C^{(6)}$ such that the Bianchi identity of $G^{(7)}$ is equal to the equation of motion for $G^{(4)}$, as in (2.5). For $G^{(7)}$ to be gauge invariant, $C^{(6)}$ must transform as

$\delta_0 C^{(6)} = d\Lambda^{(5)} - \Lambda^{(2)} \wedge G^{(4)}$. Because this variation is not a total derivative the minimal coupling term (2.29) is not gauge invariant under this variation, thus this cannot be all to couple the M5-brane action to the supergravity background. Furthermore, the PST-symmetries (2.26) would also be broken. To see this, consider the PST1-variation $\delta B = da \wedge \phi^{(1)}$ from which follows that $\delta \tilde{H}^* = 0$, $\delta \tilde{H} = -da \wedge d\phi^{(1)}$, and thus the variation of the action (2.22), in terms of the modified field strength (2.28), turns out to be

$$\begin{aligned} \delta S_{5lin} &= T_5 \, \frac{1}{2} \langle i_v(*H), i_v \delta H \rangle \\ &= T_5 \, \frac{1}{2} \langle H, *\delta H \rangle \\ &= T_5 \frac{1}{2} \int_{M_6} (dB - \hat{C}^{(3)}) \wedge \delta H \\ &= -T_5 \frac{1}{2} \int_{M_6} \left[\hat{C}^{(3)} \wedge \delta H + (\text{total derivative}) \right] . \end{aligned}$$

The term which is not a total derivative must be cancelled and this can be achieved by adding a Wess-Zumino term

$$S_{WZ} = -T_5 \int_{M_6} \frac{1}{2} dB \wedge \hat{C}^{(3)}. \tag{2.30}$$

Because $\hat{C}^{(3)} \wedge \hat{C}^{(3)} = 0$ it does not matter wether dB or H is written in the WZ-term. With the given factor in front of the minimal coupling term (2.29), the WZ-term also cancels the ordinary gauge variation of this term. Furthermore it turns out that the PST2-symmetry holds as well, thus all symmetries are restored by the introduction of this WZ-term.

The complete M5-brane action coupled to the D=11 supergravity backgound fields is then given by [104]

$$S_{M5} = -T_5 \int_{M_6} \left\{ d^6\sigma \sqrt{-\gamma} + \frac{1}{2} \big[i_v(*H) \wedge *i_v \mathcal{H} + H \wedge \hat{C}^{(3)} - \hat{C}^{(6)} \big] \right\}. \tag{2.31}$$

Fair enough, the general variation of this action with respect to B and a is exactly as in (2.25) but in terms of the modified field strength (2.28), and thus all the symmetries on the world-volume are preserved by the coupling to the supergravity background fields. Summarized, all symmetries of (2.31) are determined by the variations

$$\begin{aligned} \delta B &= d\Lambda^{(1)} + da \wedge \phi^{(1)} - \frac{\varphi}{\sqrt{(\partial a)^2}} i_v \mathcal{H} + \hat{\Lambda}^{(2)} \\ \delta a &= \varphi \\ \delta C^{(3)} &= d\Lambda^{(2)} \\ \delta C^{(6)} &= d\Lambda^{(5)} - \Lambda^{(2)} \wedge G^{(4)}. \end{aligned} \tag{2.32}$$

Note on rescalings

Before one of the supergravity actions (2.7-2.9) is coupled to the M5-brane action (2.31), a short comment on different scale conventions is in order. In the action (2.31) for the M5-brane all fields can be rescaled, but it must be noted that not all terms necessarily scale equally because S_{WZ} and S_{mc} are independent of the metric. Any rescaling of the auxiliary scalar a does not affect the action since it contains only the unit vector v.

There are basically two different kinds of rescalings, those that change the action (2.31) only by an overall factor (q_1) and those that imply a new relative factor (q_2) between the volume term and the rest. If we require that the forms of (2.4) and (2.28) should not be altered by any factors between different contributions, then we can write such a rescaling of the fields like

$$\begin{aligned} H \to \sqrt{2q_1q_2}\,H \quad &\Rightarrow \quad B \to \sqrt{2q_1q_2}\,B,\; C^{(3)} \to \sqrt{2q_1q_2}\,C^{(3)},\; C^{(6)} \to 2q_1q_2\,C^{(6)} \\ g_{IJ} \to q_1^{1/3}\,g_{IJ} \quad &\Rightarrow \quad v \to q_1^{1/6}\,v \qquad \Rightarrow \qquad \tilde{H}^{(*)}_{mn} \to q_1^{1/3}\sqrt{q_2}\,\tilde{H}^{(*)}_{mn}, \end{aligned}$$

from which the rescaled action follows to be

$$S_{M5} = -q_1\,T_5 \int_{M_6} \left\{ d^6\sigma\sqrt{-\gamma} + q_2\big[i_v(*H) \wedge *i_v\mathcal{H} + H \wedge \hat{C}^{(3)} - \hat{C}^{(6)}\big] \right\}. \tag{2.33}$$

So only for the special value $q_2 = 1/2$ the M5-brane action (2.31) rescales simply by an overall factor like $S_{M5} \to q_1 S_{M5}$. For any rescaling the factor q_1 simply rescales the brane tension like $T_5 \to q_1\,T_5$, thus q_1 can always be absorbed in the tension. The exact value for T_5 for a given convention in the bulk must be taken from some other source anyway, in our case from ref. [98]. But there remains an ambiguity when we want to combine the supergravity action and the M5-brane action. Since different supergravity conventions are related by arbitrary q_1, q_2 it seems far from clear why exactly the form (2.31) of the M5-brane action, which keeps its form only for the special choice $q_2 = 1/2$, should be added to some supergravity convention that we just have decided on out of the blue. Thus in the following we will combine the M5-brane action (2.33) with general q_2 and correct T_5 with the D=11 supergravity action. Then, for all symmetries of bulk and brane to still hold in the coupled action, it will turn out that q_2 is uniquely related to the coupling strength of the brane current to the field strength in the bulk. Since this coupling strenght is fixed in HW-theory, the factor q_2 will also get fixed.

2.2.3 Combined D=11 supergravity and M5-brane actions

Duality symmetric action coupled to M5-brane

In the duality symmetric formulations (2.7-2.9) of D=11 supergravity the six-form potential $C^{(6)}$ has been introduced as a dynamical field, thus it is now possible to combine the M5-brane action with the supergravity action. But this combination should not alter the duality constraints on the fields and thus it still must produce the duality relations through equations of motion. This is only possible if all the symmetries (2.15,2.16,2.17,2.32) of bulk and brane action are preserved.

The Bianchi identities of the supergravity fields must give the gauge field equations with M5-brane source. Thus the fields get redefined by adding Dirac brane currents (see appendix B) like

$$\begin{aligned} \tilde{G}^{(4)} &\equiv G^{(4)} + q_0\, \Theta(M_6) \\ \tilde{G}^{(7)} &\equiv G^{(7)} - q_0\, H \wedge \Theta(M_6), \end{aligned} \tag{2.34}$$

and on-shell ($\tilde{G}^{(7)} = *\tilde{G}^{(4)}$) this leads to the desired gauge field equations with sources

$$\begin{aligned} d\tilde{G}^{(4)} &= q_0\, \delta(M_6) \\ d * \tilde{G}^{(4)} &= G^{(4)} \wedge \tilde{G}^{(4)} + q_0\, H \wedge \delta(M_6). \end{aligned} \tag{2.35}$$

The factor q_0 is the coupling strength of the magnetic brane current to the electric field strength which will be related to the factor q_2.

Replacing the redefined fields (2.34) in the action S_G^I or S_G^{II} everywhere except in the Chern-Simons term finally allows consistent coupling of the M5-brane to the supergravity action. Using $\tilde{S}_G^I$ for concreteness the complete action is given by

$$\begin{aligned} S &= S_R + \tilde{S}_G^I + S_{M5} + S_c \\ &= \frac{-1}{2\kappa_{11}^2}\Big[\int_{M_{11}} d^{11}x\sqrt{-g}\frac{1}{2}R + \langle i_v * \tilde{G}^{(7)}, i_v\tilde{\mathcal{F}}^{(4)}\rangle \\ &\qquad\qquad - \langle i_v * \tilde{G}^{(4)}, i_v\tilde{\mathcal{F}}^{(7)}\rangle + \frac{1}{3}\langle G^{(7)}, *G^{(4)}\rangle\Big] \\ &\quad - T_5\int_{M_6}\Big[d^6\sigma\sqrt{-\gamma} + q_2\Big(i_v(*H)\wedge *i_v\mathcal{H} + H\wedge\hat{C}^{(3)} - \hat{C}^{(6)}\Big)\Big] \\ &\quad + T_5\, q_2\int_{M_{11}} H\wedge\tilde{G}^{(4)}\wedge\Theta(M_6) \end{aligned} \tag{2.36}$$

with the following important relation for the factors

$$q_0 = -2\kappa_{11}^2\, T_5\, q_2. \tag{2.37}$$

Only when the last term S_c in (2.36) is added and relation (2.37) is satisfied, does the whole action have all desired symmetries. Due to relation (2.37) the factor q_2 will get fixed once T_5 and q_0 are fixed, which will be the case in HW-theory as shown in the next section. Also due to this relation the action (2.36) shows mysteriously nice cancellations between different contributions and the total variation is

$$
\begin{aligned}
\delta S &= \frac{-1}{\kappa_{11}^2}\int_{M_{11}}\Big\{\big(\delta C^{(6)}+\delta C^{(3)}\wedge C^{(3)}-q_0\,\delta B\wedge\Theta(M_6)-\frac{\delta a}{\sqrt{(\partial a)^2}}i_v\tilde{\mathcal{F}}^{(7)}\big)\\
&\quad\wedge d(v\wedge i_v\tilde{\mathcal{F}}^{(4)})+\big(\delta C^{(3)}-\frac{\delta a}{\sqrt{(\partial a)^2}}i_v\tilde{\mathcal{F}}^{(4)}\big)\wedge\big[d(v\wedge i_v\tilde{\mathcal{F}}^{(7)})\\
&\quad -v\wedge i_v\tilde{\mathcal{F}}^{(4)}\wedge(2G^{(4)}+q_0\,\Theta(M_6))+q_0\,v\wedge i_v\mathcal{H}\wedge\delta(M_6)\big]\\
&\quad +q_0\,\big(\delta B+\frac{\delta a}{\sqrt{(\partial a)^2}}i_v\mathcal{H}\big)\wedge\big[d(v\wedge i_v\mathcal{H})-v\wedge i_v\tilde{\mathcal{F}}^{(4)}\big]\wedge\delta(M_6)\Big\}.
\end{aligned}
$$

The symmetries of the action (2.36) can easily be derived from this variation. But for these symmetries to be possible at all, the world-volume auxiliary field a had to be identified with the auxiliary scalar field in the formulation of the duality symmetric D=11 supergravity. This fact and the just mentioned marvelous cancellations show that there is some intriguing relationship between bulk and brane. Explicitly, the symmetries are given by the ordinary gauge symmetries

$$
\begin{aligned}
\delta_{gauge}B &= d\Lambda^{(1)}+\hat{\Lambda}^{(2)}\\
\delta_{gauge}C^{(3)} &= d\Lambda^{(2)}\\
\delta_{gauge}C^{(6)} &= d\Lambda^{(5)}+d\Lambda^{(2)}\wedge C^{(3)}
\end{aligned}
$$

and the PST-symmetries

$$
\begin{aligned}
\delta_{PST}B &= da\wedge\phi^{(1)}-\frac{\varphi}{\sqrt{(\partial a)^2}}i_v\mathcal{H}\\
\delta_{PST}C^{(3)} &= da\wedge\phi^{(2)}+\frac{\varphi}{\sqrt{(\partial a)^2}}i_v\tilde{\mathcal{F}}^{(4)}\\
\delta_{PST}C^{(6)} &= da\wedge\phi^{(5)}+\frac{\varphi}{\sqrt{(\partial a)^2}}i_v\tilde{\mathcal{F}}^{(7)}-\delta_{PST}C^{(3)}\wedge C^{(3)}+q_0\,\delta B\wedge\Theta(M_6)\\
\delta_{PST}a &= \varphi.
\end{aligned}
$$

These gauge and PST-symmetries are parametrized by the p-forms $\Lambda^{(1)}$, $\Lambda^{(2)}$, $\Lambda^{(5)}$, then $\phi^{(1)}$, $\phi^{(2)}$, $\phi^{(5)}$ and φ respectively.

Standard action coupled to M5-brane

It would also be nice to have a formulation of the coupled system where the D=11 supergravity part shows up in its traditional form (2.1) without $C^{(6)}$. This can be achieved by noting that after the replacement of the redefined field strengths (2.34) in the duality symmetric actions $S_G^{I,II,III}$ (everywhere except in the CS-term), the actions $\tilde{S}_G^{I,II}$ are no longer equivalent to $\tilde{S}_G^{III}$. It turns out that this difference is exactly what is needed to couple brane and bulk without the explicit inclusion of $C^{(6)}$. Coupling of the M5-brane to $\tilde{S}_G^{III}$ that leads to the correct equations of motion can be achieved by adding a Dirac term S_{Dirac} instead of a minimal coupling term S_{mc}. Moreover, the term $(i_v\tilde{\mathcal{F}}^{(4)})^2$ can be gauge fixed to zero by the PST1-symmetry and we are left with the following action [105]

$$\begin{aligned} S_{11/M5} &= \tilde{S}_{CJS} + S_{Dirac} + S_{M5}|_{C^{(6)}=0} \\ &= \frac{-1}{2\kappa_{11}^2}\int_{M_{11}}\left\{d^{11}x\sqrt{-g}\frac{1}{2}R - \tilde{G}^{(4)}\wedge *\tilde{G}^{(4)} + \frac{2}{3}C^{(3)}\wedge G^{(4)}\wedge G^{(4)}\right\} \\ &\quad +T_5\, q_2\int_{M_{11}} C^{(3)}\wedge G^{(4)}\wedge\Theta(M_6) \qquad\qquad (2.38) \\ &\quad -T_5\int_{M_6}\left\{d^6\sigma\sqrt{-\gamma} + q_2\left[i_v(*H)\wedge *i_v\mathcal{H} + H\wedge\hat{C}^{(3)}\right]\right\}, \end{aligned}$$

which shows non-minimal coupling of the M5-brane to the standard formulation (2.1) of D=11 supergravity. The total variation with respect to $C^{(3)}$, B and a becomes

$$\begin{aligned} \delta S_{11/M5} &= \frac{1}{\kappa_{11}^2}\int_{M_{11}}\Bigg\{\delta C^{(3)}\wedge\Big[d*\tilde{G}^{(4)} - G^{(4)}\wedge\tilde{G}^{(4)} - q_0\, H\wedge\delta(M_6) \\ &\quad -q_0\, v\wedge i_v\mathcal{H}\Big] + q_0\left(\delta B + \frac{\delta a}{\sqrt{(\partial a)^2}}i_v\mathcal{H}\right)\wedge d(v\wedge i_v\mathcal{H})\wedge\delta(M_6)\Bigg\}, \end{aligned}$$

where again relation (2.37) must be true. After the usual gauge fixing of the equation of motion of the B-field, such that $\mathcal{H}=0$, the $C^{(3)}$-equation of motion exactly becomes the second equation of (2.35), thus what is a Bianchi identity in the duality symmetric formulation (2.36) is now an equation of motion. And the Bianchi identity here is the first equation in (2.35). As required this action (2.38) thus produces the same dynamics as the duality symmetric formulation (2.36), but without the inclusion of the six-form $C^{(6)}$.

It is this last action (2.38) that will be extended to Hořava-Witten theory in the next section.

2.3 M5-branes coupled to Hořava-Witten theory

In this section we finally want to give the standard formulation of Hořava-Witten theory coupled to M5-branes, which is obtained by a generalization of action (2.38) including N separate M5-branes. Furthermore we will see how the factor q_2 gets fixed. That is, it turns out that all relative factors between different terms of the action which couples M5-branes to HW-theory are uniquely determined. This is due to the fact that in HW-theory the coupling between the Yang-Mills theories living on the orbifold fixed planes and the supergravity in the bulk is uniquely determined by considerations of anomaly cancellation. Moreover, the relative strength of the coupling of the M5-brane currents and Yang-Mills currents to the supergravity three-form is also uniquely fixed. Together with the necessary PST-symmetries of the action and the given value for the brane tension this determines all relative factors uniquely. Remarkably, only these unique factors are consistent with the five-brane background solution in eleven dimensions as well as five- and four-dimensional supersymmetry after compactification. Moreover, because away from the orbifold planes HW-theory should locally be indistinguishable from pure supergravity, we expect that in the action that couples pure supergravity to M5-Branes all relative factors must also have the same unique values. Notably though, these factors cannot be obtained by a simple rescaling of the action given in [105], because there the coupling of the M5-brane currents to the bulk three-form seems to have the wrong factor for the conventions chosen in the bulk.

2.3.1 The eleven-dimensional action

In order to obtain eleven-dimensional HW-theory coupled to M5-branes, the action (2.38) must be considered on a background space-time $M = M_{10} \times S^1/\mathbb{Z}_2$, where M_{10} is ten-dimensional Minkowski space, and $S^1/\mathbb{Z}_2$ is an orbifold as depicted in fig. 2.2. The bosonic part of the action then reads [4]

$$
\begin{aligned}
S \;=\; & -\frac{1}{2\kappa^2}\int_M \left\{ d^{11}x\,\sqrt{-g}\,\left(\frac{1}{2}R + \frac{1}{4!}G_{IJKL}G^{IJKL}\right) + \frac{2}{3}C\wedge\mathcal{G}\wedge\mathcal{G}\right\} \\
& -\frac{1}{4\lambda^2}\sum_{j=1}^{2}\int_{M_{10}^j} d^{10}x\sqrt{-g_{10}}\left\{\mathrm{tr}\,F_j^2 - \frac{1}{2}\mathrm{tr}\,R^2\right\} \\
& -\frac{1}{2}T_5\sum_{\hat{\imath}=1}^{N}\int_{M_6^{\hat{\imath}}\cup\tilde{M}_6^{\hat{\imath}}}\Big\{ d^6\sigma_{\hat{\imath}}\sqrt{-\gamma_{\hat{\imath}}}\big[\,1 \\
& \qquad\qquad + v_{\hat{\imath}\,l}(*H_{\hat{\imath}})^{lmn}(*H_{\hat{\imath}} - H_{\hat{\imath}})_{mnp}v_{\hat{\imath}}^p\big] + 2\,dB_{\hat{\imath}}\wedge\hat{C}_{\hat{\imath}}\Big\} \\
& + T_5\sum_{\hat{\imath}=1}^{N}\int_M C\wedge dC\wedge\left[\Theta(M_6^{\hat{\imath}}) + \Theta(\tilde{M}_6^{\hat{\imath}})\right]\;.
\end{aligned}
\tag{2.39}
$$

As mentioned, the structure of eleven-dimensional space-time is $M = M_{10}\times S^1/\mathbb{Z}_2$, where M_{10} is ten-dimensional space-time and we work in the upstairs picture, that is

[4] For the bulk fields we adopt the normalization of ref. [105]. The normalization chosen by Hořava and Witten [96] is obtained by the rescaling $g_{\mathrm{HW}} = 2^{-2/9}g$, $C_{\mathrm{HW}} = \frac{2^{1/6}}{6}C$ and $G_{\mathrm{HW}} = 2^{1/6}G$.

the orbifold is taken as a circle and the $\mathbb{Z}_2$ symmetry is imposed on the fields. As usual,

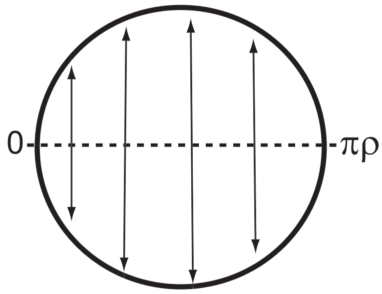

Figure 2.2: Orbifold $S^1/\mathbb{Z}_2$ obtained from a circle S^1 by identifying two opposite sides, with fixed points at 0 and $\pi\rho$.

we define the orbifold coordinate $y = x^{11}$ to be in the range $y \in [-\pi\rho, \pi\rho]$ and let the $\mathbb{Z}_2$ orbifold symmetry act as $y \longrightarrow -y$. This leads to the two fixed ten-dimensional hyperplanes M_{10}^1 and M_{10}^2 located at $y = y_1 = 0$ and $y = y_2 = \pi\rho$, respectively. Further, we have N five-branes with world-volumes $M_6^{\hat{\imath}}$, $\hat{\imath} = 1, \ldots, N$ plus their $\mathbb{Z}_2$ mirrors with world-volumes $\tilde{M}_6^{\hat{\imath}}$ which originate from $M_6^{\hat{\imath}}$ by applying the orbifold map $y \longrightarrow -y$. The latter mirror five-branes are required by consistency in order to keep the theory $\mathbb{Z}_2$ symmetric in the upstairs picture. Physically a five-brane and its mirror correspond to a single brane. Further, to avoid the appearance of additional states [156], we demand that the five-branes world-volumes do not intersect either of the two orbifold fixed planes nor each other. Also due to the upstairs picture, the whole action had to be rescaled by a factor 1/2 with respect to (2.38), and we defined the eleven-dimensional Newton constant like $\kappa^2 = 2\kappa_{11}^2$ [96, 98]. We still use indices $I, J, K, \ldots = 0, \ldots, 9, 11$ for eleven-dimensional space-time with coordinates x^I and indices $m, n, p, \ldots = 0, \ldots, 5$ for the six-dimensional five-brane world-volumes with coordinates $\sigma_{\hat{\imath}}^m$.

Let us next discuss the various sectors of the given action. As usual, the bulk fields consist of the fields in the gravity multiplet of eleven-dimensional supergravity, that is the $\mathbb{Z}_2$–even[5] eleven-dimensional metric g_{IJ}, the $\mathbb{Z}_2$–odd three-index antisymmetric tensor field C_{IJK} and the gravitino Ψ_I, subject to the usual $\mathbb{Z}_2$ truncation [96]. The standard relation $G = dC$ between the three-form potential C and its field strength G is modified due to the presence of source terms as will be explicitly presented shortly. Anomaly cancellation requires the two orbifold fixed planes M_{10}^k to each carry a ten-dimensional $\mathcal{N} = 1$ E_8 gauge multiplet [96], that is an E_8 gauge field A_k with field strength F_k and gauginos χ_k, where $k = 1, 2$. The Yang-Mills coupling λ is fixed in

[5] We call a tensor field $\mathbb{Z}_2$–even if its components orthogonal to the orbifold are even, otherwise we call it $\mathbb{Z}_2$–odd.

terms of the eleven-dimensional Newton constant κ by [96, 98]

$$\lambda^2 = 4\pi(4\pi\kappa^2)^{\frac{2}{3}} \; . \tag{2.40}$$

For an illustration of the Hořava-Witten setup without M5-branes see fig. 2.3.

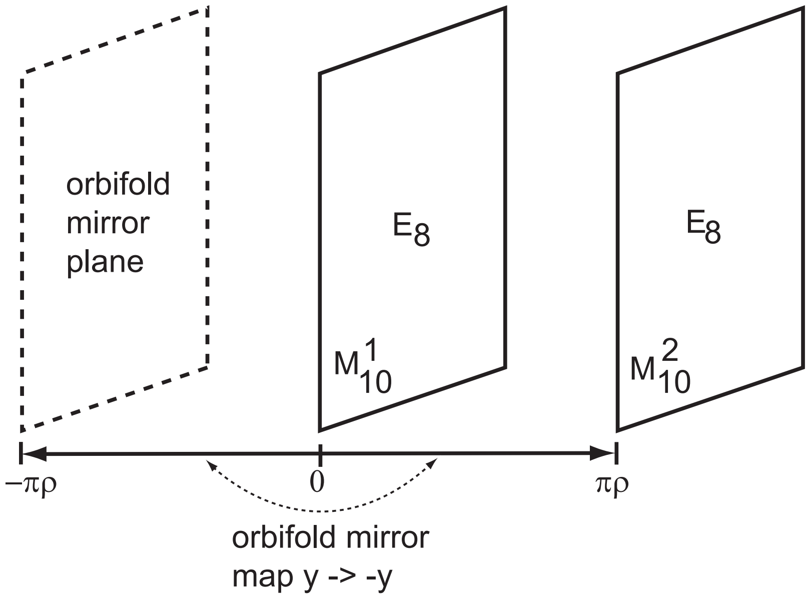

Figure 2.3: The HW-setup consisting of the Minkowski planes M^i_{10}, $i = 1, 2$ sitting at the orbifold fixed points $y = 0$ and $y_{=}\pi\rho$ and carring E_8 gauge fields.

The five-brane world-volume fields consist of the embedding coordinates $X^I_{\hat{\imath}} = X^I_{\hat{\imath}}(\sigma^m_{\hat{\imath}})$ together with the fermions $\theta_{\hat{\imath}}$ and the two-index antisymmetric tensor field $B_{\hat{\imath}mn}$. The five-brane part of the above action is written in the form due to Pasti, Sorokin and Tonin (PST) [104] as was thoroughly reviewed in the previous section. It requires the introduction of an auxiliary scalar field $a_{\hat{\imath}}$ and an associated unit vector field $v_{\hat{\imath}m}$ defined by

$$v_{\hat{\imath}m} = \frac{\partial_m a_{\hat{\imath}}}{\sqrt{\gamma^{np}_{\hat{\imath}}\partial_n a_{\hat{\imath}}\partial_p a_{\hat{\imath}}}} \; . \tag{2.41}$$

The presence of this field enhances the symmetries of the action such that $a_{\hat{\imath}}$ is truly auxiliary and that fixing one of the symmetries turns the equation of motion for $B_{\hat{\imath}}$ into the self-duality condition $*H_{\hat{\imath}} = H_{\hat{\imath}}$. For simplicity, we have chosen to present a linearized form of the PST action as is appropriate for our subsequent discussion. As usual, the metric $\gamma_{\hat{\imath}mn}$ is the pull-back

$$\gamma_{\hat{\imath}mn} = \partial_m X^I_{\hat{\imath}} \partial_n X^J_{\hat{\imath}} g_{IJ} \tag{2.42}$$

of the space-time metric g_{IJ} onto the five-brane world-volume $M_6^{\hat\imath}$. Further, the field strength $H_{\hat\imath}$ of $B_{\hat\imath}$ is defined by

$$H_{\hat\imath} = dB_{\hat\imath} - \hat{C}_{\hat\imath} \tag{2.43}$$

where $\hat{C}_{\hat\imath}$ denotes the pull-back of the bulk field C, that is,

$$\hat{C}_{\hat\imath\, mnp} = \partial_m X^I_{\hat\imath} \partial_n X^J_{\hat\imath} \partial_p X^K_{\hat\imath} C_{IJK} \; . \tag{2.44}$$

The five-brane tension T_5 can be expressed in terms of the 11-dimensional Newton constant as

$$T_5 = \left(\frac{\pi}{2\kappa^4}\right)^{\frac{1}{3}} \; . \tag{2.45}$$

Having introduced all fields we should now specify the source terms in the definition of the bulk antisymmetric tensor field strength. To this end, we introduce

$$G \;=\; dC - \omega_{\rm YM} - \omega_{\rm M5} \, , \tag{2.46}$$

$$\mathcal{G} \;=\; dC - \omega_{\rm YM} \; . \tag{2.47}$$

The field strength $\mathcal{G}$ is defined as in pure HW theory without five-branes, that is, it only contains the "Yang-Mills" sources $\omega_{\rm YM}$ which originate from the orbifold fixed planes and are given by [6]

$$\omega_{\rm YM} = 2k \left[\omega_{(0)} \wedge \delta(y) + \omega_{(N+1)} \wedge \delta(y - \pi\rho)\right] \tag{2.48}$$

with the "Chern-Simons" forms $\omega_{(0)}$ and $\omega_{(N+1)}$ satisfying

$$J^{(0)} \equiv d\omega_{(0)} = \frac{1}{16\pi^2}\left[{\rm tr}\, F_1 \wedge F_1 - \frac{1}{2}{\rm tr}\, R \wedge R\right]_{y=y_0} \, , \tag{2.49}$$

$$J^{(N+1)} \equiv d\omega_{(N+1)} = \frac{1}{16\pi^2}\left[{\rm tr}\, F_2 \wedge F_2 - \frac{1}{2}{\rm tr}\, R \wedge R\right]_{y=\pi\rho} \; .$$

The field strength G, on the other hand, contains both orbifold and five-brane sources where the latter are defined by

$$\omega_{\rm M5} = k \sum_{\hat\imath=1}^{N} \left[\Theta(M_6^{\hat\imath}) + \Theta(\tilde{M}_6^{\hat\imath})\right] \; . \tag{2.50}$$

Here $\Theta(M_6^{\hat\imath})$ is the θ–function associated with the five-brane world-volume $M_6^{\hat\imath}$. In analogy with the ordinary one-dimensional θ–function it satisfies the relation

$$d\Theta(M_6^{\hat\imath}) = \delta(M_6^{\hat\imath}) \, , \tag{2.51}$$

[6] By $\delta(y)$ we denote a δ–function one-form defined by $\hat\delta(y)dy$, where $\hat\delta(y)$ is the ordinary δ–function.

where $\delta(M_6^{\hat{\imath}})$ is the δ–function supported on $M_6^{\hat{\imath}}$ (Analogous expression hold for $\tilde{M}_6^{\hat{\imath}}$.). For later calculations it will be useful to explicitly express these functions in terms of the embedding coordinates $X_{\hat{\imath}}^I$ by writing

$$\begin{aligned}\Theta(M_6^{\hat{\imath}}) &= \frac{1}{4!7!\sqrt{-g}}\, dx^{I_1}\wedge\ldots\wedge dx^{I_4}\epsilon_{I_1\ldots I_{11}} \times \\ &\qquad \int_{M_7^{\hat{\imath}}} dX_{\hat{\imath}}^{I_5}\wedge\ldots\wedge dX_{\hat{\imath}}^{I_{11}}\,\hat{\delta}^{11}(x-X_{\hat{\imath}}(\sigma_{\hat{\imath}}))\,, \\ \delta(M_6^{\hat{\imath}}) &= \frac{-1}{5!6!\sqrt{-g}}\, dx^{I_1}\wedge\ldots\wedge dx^{I_5}\epsilon_{I_1\ldots I_{11}} \times \\ &\qquad \int_{M_6^{\hat{\imath}}=\partial M_7^{\hat{\imath}}} dX_{\hat{\imath}}^{I_6}\wedge\ldots\wedge dX_{\hat{\imath}}^{I_{11}}\,\hat{\delta}^{11}(x-X_{\hat{\imath}}(\sigma_{\hat{\imath}}))\,.\end{aligned} \tag{2.52}$$

We see that the definition of $\Theta(M_6^{\hat{\imath}})$ and, hence, our action (2.39) involves a seven-manifold $M_7^{\hat{\imath}}$ which is bounded by the five-brane world-volume $M_6^{\hat{\imath}}$, that is, $\partial M_7^{\hat{\imath}} = M_6^{\hat{\imath}}$. This seven-manifold is the analogue of a Dirac-string for a monopole in Maxwell theory and is also referred to as Dirac-brane [149,157]. For a summary on brane currents and Dirac-branes see appendix B. There may be a problem in that the action depends on the particular choice of the Dirac-brane. A prescription to resolve this ambiguity has been proposed in ref. [152]. Since our subsequent considerations do not depend on how precisely the Dirac-brane is defined, we will not consider this point in any further detail. The constant k in the above definitions for the field strengths is again fixed in terms of the eleven-dimensional Newton constant and is given by

$$k = \left(\frac{\pi}{2}\right)^{\frac{1}{3}} \kappa^{2/3} = \kappa^2 T_5 = 8\pi^2\,\kappa^2/\lambda^2\,. \tag{2.53}$$

From the definition (4.3) we can now write the Bianchi identity

$$dG = -2k\left[J^{(0)}\wedge\delta(y) + J^{(N+1)}\wedge\delta(y-\pi\rho) + \frac{1}{2}\sum_{\hat{\imath}=1}^{N}\Big(\delta(M_6^{\hat{\imath}}) + \delta(\tilde{M}_6^{\hat{\imath}})\Big)\right], \tag{2.54}$$

for G which will be important later on. The relative factor $1/2$ between the orbifold and five-brane sources accounts for the fact that a five-brane and its mirror really represent the same physical object and should, therefore, not be counted independently. Having the magnetic coupling strength of the five-branes as used in (2.54), this finally fixes the factor q_2, since by comparison with (2.35) and using relation (2.37) we find

$$q_0 = -2k = -2\kappa^2 T_5 = -4\kappa_{11}^2 T_5 = -2q_2\kappa_{11}^2 T_5 \quad\Rightarrow\quad q_2 = 2\,,$$

as has already been used in action (2.39). Because in the bulk away from the boundaries, HW-theory should look the same as pure eleven-dimensional supergravity, we infer that the same factor $q_2 = 2$ must be used in the actions (2.36,2.38).

2.3.2 Symmetries

Let us discuss the symmetries of the action (2.39) some of which will become relevant later on.

In the following we would like to check the BPS property of certain solutions, hence we will need the (bosonic part) of the supersymmetry transformations which we explicitly present for completeness. For the gravitino Ψ_I, the E_8 gauginos χ_k on the two orbifold fixed planes and the five-brane world-volume fermions $\theta_{\hat{i}}$ they are, respectively, given by

$$\begin{aligned} \delta\Psi_I &= D_I\eta + \frac{1}{3!}\left[\frac{1}{4!}\Gamma_{IJ_1J_2J_3J_4} - \frac{2}{3!}g_{IJ_1}\Gamma_{J_2J_3J_4}\right]G^{J_1J_2J_3J_4}\eta && (2.55)\\ \delta\chi_k &= -\frac{1}{4}\Gamma^{\bar{I}\bar{J}}F_{k\bar{I}\bar{J}}\eta && (2.56)\\ \delta\theta_{\hat{i}} &= \eta_{\hat{i}} + P_{\hat{i}+}\kappa_{\hat{i}} && (2.57)\end{aligned}$$

where the projection operators $P_{\hat{i}\pm}$ satisfying $P_{\hat{i}+} + P_{\hat{i}-} = 1$ are defined by

$$P_{\hat{i}\pm} = \frac{1}{2}\left(1 \pm \epsilon^{m_1\ldots m_6}\partial_{m_1}X_{\hat{i}}^{I_1}\ldots\partial_{m_6}X_{\hat{i}}^{I_6}\Gamma_{I_1\ldots I_6}\right) , \qquad (2.58)$$

and $\eta_{\hat{i}}$ is the pullback of η to $M_6^{\hat{i}}$. For simplicity, we have stated these projection operators for the later relevant case $H_{\hat{i}} = 0$. The general expressions can be found in ref. [154]. In the above equations the spinor η parametrizes supersymmetry transformations. Each five-brane world-volume theory is also invariant under an additional fermionic symmetry, namely local κ–symmetry. It is parametrized by the spinor $\kappa_{\hat{i}}$ and appears via the second term in eq. (2.57). Further, D_I is the covariant derivative and $\Gamma_{I_1\ldots I_p}$ denotes the antisymmetrized products of p gamma-matrices Γ_I which satisfy the usual Clifford algebra $\{\Gamma_I, \Gamma_J\} = 2g_{IJ}$.

Besides supersymmetry the action is also, up to total derivatives, invariant under the following gauge variations

$$\delta C = d\Lambda^{(2)}, \qquad \delta B_{\hat{i}} = d\Lambda_{\hat{i}}^{(1)} + \hat{\Lambda}_{\hat{i}}^{(2)}, \qquad (2.59)$$

where $\Lambda_{\hat{i}}^{(1)}$ are arbitrary one-forms. The two-form $\Lambda^{(2)}$ has to be $\mathbb{Z}_2$–odd in order to ensure that the $\mathbb{Z}_2$ properties of C are preserved under the above transformation.

There are two more symmetries on the world-volumes of the M5-branes, namely the "PST-symmetries" given by

$$\delta B_{\hat{i}mn} = (da_{\hat{i}} \wedge \phi_{\hat{i}}^{(1)})_{mn} - \frac{\varphi_{\hat{i}}}{\sqrt{(\partial a_{\hat{i}})^2}}\, v_{\hat{i}}^l(*H_{\hat{i}} - H_{\hat{i}})_{lmn}, \qquad \delta a_{\hat{i}} = \varphi_{\hat{i}} \qquad (2.60)$$

where $\phi_{\hat{i}}^{(1)}$ and $\varphi_{\hat{i}}$ are an arbitrary one-form and a scalar, respectively. As previously mentioned, these symmetries ensure that the self-duality of $H_{\hat{i}}$ follows from the equations of motion and that $a_{\hat{i}}$ is an auxiliary field, as was discussed in great detail in the last section.

2.4 Calabi-Yau background in eleven dimensions

2.4.1 The background solution

Background solutions of heterotic M-theory based on Calabi-Yau three-folds which respect four-dimensional Poincaré invariance and $\mathcal{N} = 1$ supersymmetry were first presented in ref. [106]. This paper also demonstrated how to include five-branes in those backgrounds while preserving the four-dimensional symmetries. The explicit form of these solutions was subsequently given in ref. [109]. All these result were based on the original action derived by Hořava and Witten [96] which does not explicitly include any five-brane world-volume theories. The effect of five-branes on the supergravity background was incorporated by modifying the Bianchi-identity of G to include the five-brane sources as in eq. (2.54). The main purpose of this section is to prove that the solutions obtained in this way can indeed be extended to solutions of the full action (2.39) which does include the five-brane world-volume theories. Practically, this amounts to showing that these solutions correctly match the five-brane source terms in the Einstein equations and that the five-brane world-volume equations of motion are satisfied.

Let us review the solution explicitly given in ref. [109]. Following refs. [106, 109, 158], the solutions are constructed as an expansion in powers of $\kappa^{2/3}$. For considerations on all order solutions see ref. [142]. As can be seen from the action (2.39) and the Bianchi identity (2.54), at lowest order the five-branes and the orbifold fixed planes do neither contribute to the Einstein field equations nor to the equations of motion for the three-form gauge field C. It follows that to zeroth-order the equations of motion in the bulk and on the world-volumes can be solved independently. We consider the space-time structure $M = S^1/\mathbb{Z}_2 \times X \times M_4$, where X is a Calabi-Yau three-fold and M_4 four-dimensional Minkowski space. Coordinates in M_4 are labelled by indices $\mu, \nu, \rho, \ldots = 0, \ldots, 3$. The lowest order Ricci-flat metric on the Calabi-Yau space is denoted by Ω_{AB} with six-dimensional indices $A, B, C, \ldots = 4, \ldots, 9$. The associated Kähler-form ω is defined by $\omega_{a\bar{b}} = i\Omega_{a\bar{b}}$ where $a, b, c, \ldots$ and $\bar{a}, \bar{b}, \bar{c}, \ldots$ are holomorphic and anti-holomorphic indices on the Calabi-Yau space, respectively. The four-form field strength G vanishes at lowest order. This configuration constitutes a solution to the Killing spinor equation $\delta\Psi_I = 0$ and the Bianchi identity since the source terms in eq. (2.54) are proportional to $\kappa^{2/3}$ and, hence, do not contribute at lowest order. At the next order, however, these source terms have to be taken into account and, as a consequence, the field strength G becomes non-vanishing. This induces corrections to the metric which can be computed requiring that $\mathcal{N} = 1$ supersymmetry is preserved and, hence, that the gravitino variation (2.55) vanishes. The size of these corrections is measured by the strong-coupling expansion parameter ϵ_S defined by

$$\epsilon_S \equiv \pi \left(\frac{\kappa}{4\pi}\right)^{2/3} \frac{2\pi\rho}{v^{2/3}} = \frac{\pi\rho}{v^{2/3}}\, T_5 \kappa^2 = \frac{\pi\rho}{v^{2/3}}\, k \tag{2.61}$$

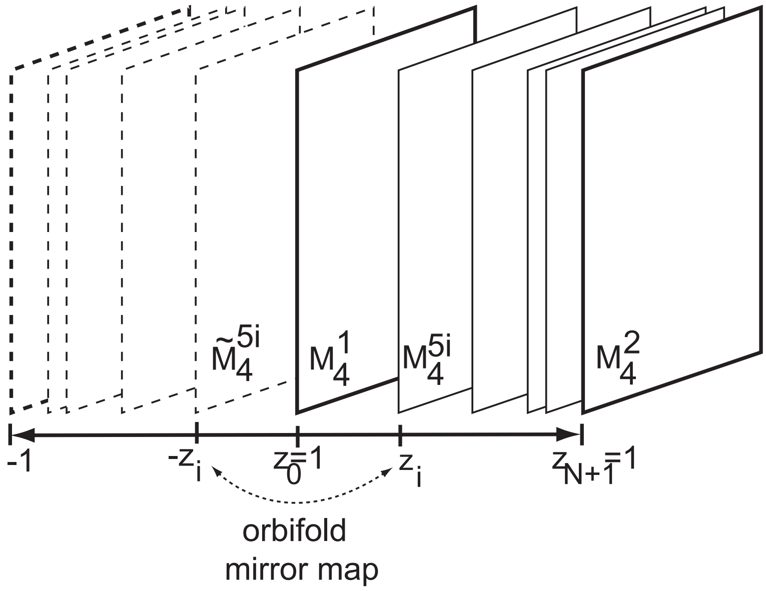

Figure 2.4: N parallel M-branes placed across the orbifold at positions $z_{\hat{\imath}} = Y_{\hat{\imath}}/\pi\rho$ between the boundaries at $z_0 = 0$ and $z_{N+1} = 1$.

where

$$v = \int_X \sqrt{\Omega} \tag{2.62}$$

is the Calabi-Yau volume to lowest order, which we later also use to set the units of six-dimensional volume.

We should now specify the full solutions (to order ϵ_S) and we start with the gauge fields on the orbifold planes. In general, we have non-trivial holomorphic vector bundles on the Calabi-Yau space. These bundles correspond to gauge field backgrounds $\bar{A}_k, k = 1, 2$ in the Calabi-Yau directions which preserve supersymmetry and are, hence, constrained by a vanishing gaugino variation (2.56). This implies that their associated field strengths $\bar{F}_k$ are $(1, 1)$ forms on the Calabi-Yau space. Then, the orbifold sources $J^{(0)}$, $J^{(N+1)}$ in the Bianchi-identity are $(2, 2)$ forms given by

$$\begin{aligned} J^{(0)} &\equiv d\omega_{(0)} = \frac{1}{16\pi^2}\left[\mathrm{tr}\,\bar{F}_1 \wedge \bar{F}_1 - \frac{1}{2}\mathrm{tr}\,R^{(\Omega)} \wedge R^{(\Omega)}\right]_{y=0} , \\ J^{(N+1)} &\equiv d\omega_{(N+1)} = \frac{1}{16\pi^2}\left[\mathrm{tr}\,\bar{F}_2 \wedge \bar{F}_2 - \frac{1}{2}\mathrm{tr}\,R^{(\Omega)} \wedge R^{(\Omega)}\right]_{y=\pi\rho} , \end{aligned} \tag{2.63}$$

where $R^{(\Omega)}$ is the Calabi-Yau curvature tensor associated with the metric Ω.

Next, we should consider the five-brane world-volume fields. Guided by the structure of our action (2.39), we focus on N distinct five-branes (and their $\mathbb{Z}_2$ mirrors) which are taken to be static and parallel to the orbifold fixed planes, as illustrated in fig. 2.4. Furthermore, two spatial dimensions of the world-volumes $M_6^{\hat{\imath}}$ wrap around holomorphic two-cycles $\mathcal{C}_2^{\hat{\imath}}$ of the internal Calabi-Yau space X and the remaining four dimensions stretch across the external Minkowski space-time M_4. Accordingly, we split the five-brane coordinates $\sigma_{\hat{\imath}}^m$ into external and internal coordinates, that is, $\sigma_{\hat{\imath}}^m = (\sigma_{\hat{\imath}}^\mu, \sigma_{\hat{\imath}}^s)$ where $\mu, \nu, \rho, \ldots = 0, \ldots, 3$ and $s, t, \ldots = 4, 5$. Further, we define holomorphic and anti-holomorphic coordinates $\sigma_{\hat{\imath}} = \sigma_{\hat{\imath}}^4 + i\sigma_{\hat{\imath}}^5$ and $\bar{\sigma}_{\hat{\imath}} = \sigma_{\hat{\imath}}^4 - i\sigma_{\hat{\imath}}^5$. With these definitions, the five-brane embedding is specified by

$$X_{\hat{\imath}}^\mu = \sigma_{\hat{\imath}}^\mu\,, \qquad X_{\hat{\imath}}^a = X_{\hat{\imath}}^a(\sigma_{\hat{\imath}})\,, \qquad X_{\hat{\imath}}^{11} = \pm Y_{\hat{\imath}}\,, \tag{2.64}$$

where the position of the $\hat{\imath}$-th five-brane along the orbifold $Y_{\hat{\imath}} \in [0, \pi\rho]$ is a constant, $X_{\hat{\imath}}^a(\sigma_{\hat{\imath}})$ parametrizes the holomorphic curve $\mathcal{C}_2^{\hat{\imath}}$, the index $\hat{\imath} = 1, \ldots, N$ enumerates the five-branes and the two signs in the last equation account for the five-brane $M_6^{\hat{\imath}}$ and its mirror $\tilde{M}_6^{\hat{\imath}}$. The world-volume two-forms $B_{\hat{\imath}}$ are taken to vanish in the background. It can be explicitly shown [109] that this configuration preserves supersymmetry on the five-branes by choosing $\kappa_{\hat{\imath}} = -\eta_{\hat{\imath}}$ in the variation (2.57) and verifying that $P_{\hat{\imath}-}\eta_{\hat{\imath}} = 0$. The five-brane sources in the Bianchi identity then take the specific form

$$\begin{aligned} J_{M5} &\equiv d\omega_{\mathrm{M5}} = k \sum_{\hat{\imath}=1}^{N} J^{\hat{\imath}} \wedge \left[\delta(y - Y_{\hat{\imath}}) + \delta(y + Y_{\hat{\imath}})\right] , \qquad (2.65)\\ J^{\hat{\imath}} &= \delta(\mathcal{C}_2^{\hat{\imath}}) = \frac{1}{2 \cdot 4!\sqrt{\Omega}} dx^{A_1} \wedge \ldots \wedge dx^{A_4} \epsilon_{A_1 \ldots A_4 BC} \ \times \\ &\qquad \frac{1}{v} \int_{\mathcal{C}_2^{\hat{\imath}}} dX_{\hat{\imath}}^B \wedge dX_{\hat{\imath}}^C \hat{\delta}^6(x - X_{\hat{\imath}}(\sigma_{\hat{\imath}}))\,. \end{aligned}$$

The embedding (2.64) implies that $J^{\hat{\imath}}$ are $(2,2)$ forms on the Calabi-Yau space as well [159]. Note that different M5-branes can wrap different holomorphic two-cycles $\mathcal{C}_2^{\hat{\imath}} \in H_{1,1}(X)$, $\hat{\imath} = 1, \ldots, N$.

We have explicitly presented all source terms in the Bianchi identity (2.54), which now can be written as

$$dG = -2k\left[J^{(0)} \wedge \delta(y) + J^{(N+1)} \wedge \delta(y - \pi\rho) + \frac{1}{2}\sum_{\hat{\imath}=1}^{N} J^{\hat{\imath}} \wedge (\delta(y - Y_{\hat{\imath}}) + \delta(y + Y_{\hat{\imath}}))\right]. \tag{2.66}$$

In addition to this Bianchi identity the three-form gauge field C must also satisfy its equation of motion, which, for vanishing world-volume tensor fields $H_{(i)} = 0$ and with the requirement $G_{\mu\nu\sigma\rho} = 0$ from Poincaré invariance in Minkowsi space, simplifies to

$$d * G = 0. \tag{2.67}$$

As shown in refs. [109,158] this set of differential equations (2.66,2.67) can be solved in terms of a two-form $\mathcal{B}$. Moreover, also the metric deformations needed to satisfy the

Einstein equations at linear order in ϵ_S will be given in terms of this same two-form $\mathcal{B}$, as quickly reviewed next.

First, define the Hodge dual of the four-form field strength in terms of a six-form like $-d\mathcal{B}_{(6)} = *G$, such that eq. (2.67) is identically solved. Using harmonic gauge where $d * \mathcal{B}_{(6)} = 0$ and $\Delta = d^\dagger d + dd^\dagger$ together with eq. (A.13), allows to write the Bianchi identity (2.54) as the Poisson equation $\Delta\mathcal{B}_{(6)} = *dG = \ldots$. Because the currents in eq. (2.66) are non-trivial $(2,2)$ forms, i.e. $J^{\hat\imath} \in H^{2,2}(X)$, $\hat\imath = 0, \ldots, N+1$, the only non-vanishing components of $\mathcal{B}_{(6)}$ are $\mathcal{B}_{(6)\,\mu_1\ldots\mu_4 a\bar b} = \epsilon_{\mu_1\ldots\mu_4}\mathcal{B}_{a\bar b}$, which defines the two-form $\mathcal{B}$. This reduces the eleven-dimensional Poisson equation to the equivalent of (2.66), explicitly

$$\begin{aligned}(\Delta_X + \partial_{11}^2)\mathcal{B} \;\; = \;\; -2k *_X \Bigg[& J^{(0)}\hat\delta(y) + J^{(N+1)}\hat\delta(y-\pi\rho) \\ & + \frac{1}{2}\sum_{\hat\imath=1}^{N} J^{\hat\imath}\left(\hat\delta(y - Y_{\hat\imath}) + \hat\delta(y + Y_{\hat\imath})\right)\Bigg] ,\end{aligned} \tag{2.68}$$

where Δ_X denotes the Laplacian and $*_X$ the Hodge star operator restricted to the Calabi-Yau manifold X. Because we are only interested in zero modes we now can expand the two-form $\mathcal{B}$ in terms of harmonic $(1,1)$-forms on X. To this end introduce the set of harmonic $(1,1)-$forms $\{\omega_k,\, k = 1, \ldots, h^{1,1}\}$ forming a basis of $H^{1,1}(X)$ and the moduli space metric defined by

$$G_{jk} = \frac{1}{v}\int_X \omega_j \wedge (*_X\omega_k)\,, \tag{2.69}$$

with inverse G^{jk} which are used to raise and lower $h^{1,1}-$type indices, like e.g. $\omega^j = G^{jk}\omega_k$. Note that whereas $\{\omega_k\}$ are independent of the Calabi-Yau metric, this is no longer true for $\{\omega^k\}$ because of the dependence of G_{jk} on the Calabi-Yau metric through $*_X$ in eq. (2.69). To keep control of the metric dependencies of the various quantities is important when including all Calabi-Yau moduli as is done in section 2.4.2. Now we expand the two-form $\mathcal{B}$ like

$$\mathcal{B} = \sum_{k=1}^{h^{1,1}} b_k\,\omega^k\,. \tag{2.70}$$

Similarly we expand the given currents in terms of harmonic forms like

$$*_X J^{\hat\imath} = \frac{1}{v^{2/3}}\sum_{k=1}^{h^{1,1}} \beta_k^{\hat\imath}\omega^k\,, \qquad \hat\imath = 0, \ldots, N+1, \tag{2.71}$$

and following from eq. (2.69) the dimensionless charges must be defined by

$$\beta_k^{\hat\imath} = \frac{1}{v^{1/3}} \int_X \omega_k \wedge J^{\hat\imath}\,, \qquad k = 1, \ldots, h^{1,1}. \tag{2.72}$$

Note that these charges are independent of the Calabi-Yau metric such that the complete metric dependence of the currents as in the l.h.s of eq. (2.71) is absorbed in the harmonic forms ω^k in the r.h.s.. Furthermore, these charges are topological invariants. For the orbifold sources (2.63) they give the instanton numbers of the Calabi-Yau three-fold, that is the second Chern numbers of the gauge bundle and the first Pontrjagin number of the tangent bundle [80, 81]. For the M5-brane sources the charges are given by the intersection numbers of the $\hat\imath$-th brane on the two-cycle $\mathcal{C}_2^{\hat\imath}$ with the four-cycle $\mathcal{C}_4^k$ Poincaré dual to the two form ω_k. Inserting the expansions eqs. (2.70) and (2.71) into the Poisson equation (2.69) and using (2.61) yields

$$\partial_{11}^2 b_k = -2\frac{\epsilon_S}{\pi\rho}\left[\beta_k^{(0)}\hat\delta(y) + \beta_k^{(N+1)}\hat\delta(y-\pi\rho) + \frac{1}{2}\sum_{\hat\imath=1}^{N}\beta_k^{\hat\imath}\left(\hat\delta(y-Y_{\hat\imath}) + \hat\delta(y+Y_{\hat\imath})\right)\right]$$

which has the solution [158, 109]

$$b_k(y) = -\left[\sum_{\hat\imath=0}^{\hat\jmath}\alpha_k^{\hat\imath}(|y|-Y_{\hat\imath}) + \tilde c\right]\,, \qquad \alpha_k = \frac{\epsilon_S}{\pi\rho}\beta_k \tag{2.73}$$

for $k = 1, \ldots, h^{1,1}$ in the interval

$$Y_{\hat\jmath} \leq |y| \leq Y_{\hat\jmath+1}\,, \tag{2.74}$$

where $Y_0 = 0$, $Y_{N+1} = \pi\rho$ are the locations of the orbifold fixed planes, $Y_{\hat\imath}$, $\hat\imath = 1, \ldots, N$ are the five-brane positions along the orbifold interval, and $\tilde c$ is an integration constant. For consistency the charges must satisfy the condition

$$\sum_{\hat\imath=0}^{N+1}\alpha_k^{\hat\imath} = 0\,, \qquad \forall k = 1, \ldots, h^{1,1}\,, \tag{2.75}$$

which by eq. (2.72) corresponds to the cohomology condition on the currents, which reads

$$\left[\sum_{\hat\imath=0}^{N+1} J^{\hat\imath}\right] = 0$$

as an element of $H^{2,2}(X)$, and by eq. (2.66) this ensures that dG is exact as trivially required. Physically speaking this means that there can be no overall charge in a compact space, because there is nowhere for the flux to go. Finally the solution for the four-form field strength in terms of $\mathcal{B}$ then is

$$G_{ABCD} = \frac{1}{2}\epsilon_{ABCDEF}\partial_{11}\mathcal{B}^{EF} = \sum_{k=1}^{h^{1,1}} \partial_{11}b_k\,(*_X\omega^k)_{ABCD}$$

which by eq. (2.73) then explicitly reads

$$G = -\epsilon(y)\sum_{k=1}^{h^{1,1}}\left(\sum_{\hat{\imath}=0}^{\hat{\jmath}}\alpha_k^{\hat{\imath}}\right)(*_X\omega^k)\,, \quad \text{for} \quad Y_{\hat{\jmath}} \leq |y| \leq Y_{\hat{\jmath}+1}\,, \tag{2.76}$$

where $\epsilon(y) = \text{sign}(y)$, such that G is $\mathbb{Z}_2$-odd as required.

Having now given the explicit solution for the equation of motion and Bianchi identity of the three-form potential, we still need the solution for the metric that solves the Einstein equations (to order ϵ_S). The metric solution turns out to be given by

$$ds^2 = (1-b)\eta_{\mu\nu}dx^\mu dx^\nu + (\Omega_{AB}+h_{AB})dx^A dx^B + (1+2b)dy^2 \tag{2.77}$$

where

$$b = \frac{1}{3}\omega^{AB}\mathcal{B}_{AB} \tag{2.78}$$

$$h_{a\bar{b}} = 2i\,(\mathcal{B}_{a\bar{b}} - \omega_{a\bar{b}}b) \tag{2.79}$$

with ω_{AB} the lowest order Kähler form with $\omega_{a\bar{b}} = i\Omega_{a\bar{b}}$. We would now like to demonstrate that by this configuration the five-brane sources in the Einstein equation are properly matched and the five-brane world-volume equations of motion are satisfied. To verify the former we should consider the singular terms in the Einstein tensor which turn out to be

$$\begin{aligned}(G_{\mu\nu})_{\text{singular}} &= -\frac{3}{2}\partial_y^2 b\,\eta_{\mu\nu} \qquad (2.80)\\ &= \frac{1}{2}\omega^{AB}\sum_{\hat{\imath}=1}^{N}\left[\sum_{k=1}^{h^{1,1}}\alpha_k^{\hat{\imath}}\omega_{AB}^k\right]\left(\hat{\delta}(y-Y_{\hat{\imath}})+\hat{\delta}(y+Y_{\hat{\imath}})\right)\eta_{\mu\nu}\,,\\ (G_{a\bar{b}})_{\text{singular}} &= i\partial_y^2\mathcal{B}_{a\bar{b}} \qquad (2.81)\\ &= -i\sum_{\hat{\imath}=1}^{N}\left[\sum_{k=1}^{h^{1,1}}\alpha_k^{\hat{\imath}}\omega_{a\bar{b}}^k\right]\left(\hat{\delta}(y-Y_{\hat{\imath}})+\hat{\delta}(y+Y_{\hat{\imath}})\right)\,.\end{aligned}$$

These terms have to be compared with the five-brane stress energy tensor which for vanishing $H_{\hat{\imath}}$ is in general given by

$$T_{5\hat{\imath}\,IJ} = T_5\kappa^2\frac{1}{\sqrt{-g}}\int_{M_6^{\hat{\imath}}\cup\tilde{M}_6^{\hat{\imath}}} d^6\sigma_{\hat{\imath}}\hat{\delta}^{11}(x-X_{\hat{\imath}}(\sigma_{\hat{\imath}}))\sqrt{-\gamma_{\hat{\imath}}}\gamma_{\hat{\imath}}^{mn}\partial_m X_{\hat{\imath}\,I}\partial_n X_{\hat{\imath}\,J}\,.$$

Evaluating this expression for the embedding (2.64) and using the expansion (2.71) on the currents (2.65) as well as the relation (2.61), leads to

$$\begin{aligned} T_{5\hat{\imath}\,\mu\nu} &= \frac{1}{2}\eta_{\mu\nu}\left(\hat{\delta}(y-Y_{\hat{\imath}})+\hat{\delta}(y+Y_{\hat{\imath}})\right)\omega^{AB}\sum_{k=1}^{h^{1,1}}\alpha_k^{\hat{\imath}}\omega_{AB}^k\,, \\ T_{5\hat{\imath}\,a\bar{b}} &= -i\left(\hat{\delta}(y-Y_{\hat{\imath}})+\hat{\delta}(y+Y_{\hat{\imath}})\right)\sum_{k=1}^{h^{1,1}}\alpha_k^{\hat{\imath}}\omega_{a\bar{b}}^k \end{aligned} \tag{2.82}$$

with the other components vanishing. Thus each five-brane contribution exactly matches the appropriate delta-function terms in eqs. (2.80, 2.81). We have therefore verified that the five-brane sources in the Einstein equation are properly matched by the solutions. The matching of the boundary sources has been shown in ref. [158].

The only relevant equation of motion on the five-brane world-volumes is the one for the embedding coordinates $X_{\hat{\imath}}^I$. For the case of vanishing $B_{\hat{\imath}}$ it reads

$$\Box X_{\hat{\imath}}^I+\Gamma^I_{JK}\gamma_{\hat{\imath}}^{mn}\partial_m X_{\hat{\imath}}^J\partial_n X_{\hat{\imath}}^K+\frac{2}{6!}\epsilon^{m_1\dots m_6}\partial_{m_1}X_{\hat{\imath}}^{I_1}\dots\partial_{m_6}X_{\hat{\imath}}^{I_6}(*G)^I{}_{I_1\dots I_6}=0\,.$$

The μ and A components of this equation turn out to be trivially satisfied for our solution and it remains to check the eleven component. Using the expressions

$$\Gamma^{11}_{\mu\nu}=\frac{1}{2}\partial_y b\eta_{\mu\nu}\,,\qquad \Gamma^{11}_{a\bar{b}}=-\frac{1}{2}i\partial_y h_{a\bar{b}}$$

for the connection along with the embedding (2.64) and the background (2.76) for G this can be done.

In summary, we have, therefore, explicitly verified that the above background configurations are indeed solutions of the action (2.39).

2.4.2 Inclusion of moduli

In view of the reduction to five dimensions to be carried out in section 2.5, we will now identify the (bosonic) moduli fields of the above background solutions. We include the complete $(1,1)-$ as well as $(2,1)-$(co)homology sectors of the Calabi-Yau moduli space. The indices $\alpha,\beta,\gamma=0,\dots,3,11$ are used to label five-dimensional coordinates.

Geometric moduli from bulk metric

In this subsection we will consider the moduli which arise from the bulk metric. These are the metric deformations in the form of the Kähler and complex structure moduli.

Let us first consider the metric which can be written as

$$ds^2_{11}=V^{-2/3}g_{\alpha\beta}dx^\alpha dx^\beta+g_{AB}dx^A dx^B\,, \tag{2.83}$$

where the Calabi-Yau volume modulus V, the five-dimensional metric $g_{\alpha\beta}$ and the six-dimensional metric g_{AB} on the Calabi-Yau space are functions of the five-dimensional coordinates x^α. The factor $V^{-2/3}$ in the metric is chosen such as to obtain an action in the Einstein frame after reduction. As explained in ref. [108], we have absorbed the metric corrections of eq. (2.79) into the Kähler moduli of the metric g_{AB}. This can only be done because the deformed internal metric of (2.77) turns out to still be a Calabi-Yau metric.

First consider the Kähler moduli and recall that they correspond to the hermitean deformations of the Calabi-Yau metric. The Kähler form associated to the metric g_{AB} is defined by $\omega_{a\bar{b}} = ig_{a\bar{b}}$, which can be expanded in terms of the harmonic $(1,1)$-forms $\{\omega_k,\, k = 1, \ldots, h^{1,1}\}$ as

$$\omega_{AB} = a^k \omega_{k\,AB}\,, \tag{2.84}$$

where the coefficients $a^k = a^k(x^\alpha)$ then are the $(1,1)-$moduli of the Calabi-Yau space. These real Calabi-Yau moduli label the Kähler class. The dimensionless Calabi-Yau volume modulus $V(x^\alpha)$ depends on these moduli like

$$V = \frac{1}{v}\int_X \sqrt{g_6}\, d^6x = \frac{1}{6} d_{ijk} a^i a^j a^k\,, \tag{2.85}$$

where d_{ijk} are the Calabi-Yau intersection numbers defined by

$$d_{ijk} = \frac{1}{v}\int_X \omega_i \wedge \omega_j \wedge \omega_k\,, \qquad i,j,k = 1, \ldots, h^{1,1}\,.$$

Here and in the following the parameter v sets the units of six dimensional volume and so has dimension *length*^{6}, thus, for example, the true Calabi-Yau volume now is vV. We might as well take v to be the lowest order Calabi-Yau volume (2.62) for consistency with the notation of the previous section. Because we want the volume breathing mode V and the remaining $(1,1)$-moduli to be independent, we redefine the shape moduli to[7]

$$b^k = \frac{1}{V^{1/3}} a^k\,, \tag{2.86}$$

such that the volume modulus is scaled out. Then the moduli $\{b^k,\, k = 1, \ldots, h^{1,1}\}$ represent only $h^{1,1} - 1$ independent scalar degrees of freedom because eq. (2.85) becomes the constraint

$$d_{ijk} b^i b^j b^k = 6\,. \tag{2.87}$$

Now it is convenient to introduce a metric $G_{jk}(b)$ on the the Kähler moduli space such that $G_{jk}(b) b^j b^k$ does not scale with V, and this can be achieved by defining

$$G_{jk}(b) = \frac{1}{vV^{1/3}}\int_X \omega_j \wedge (*_X \omega_k) = \frac{1}{2vV^{1/3}}\int_X d^6x \sqrt{g_6}\, \omega_{j\,AB}\omega_k^{AB} \tag{2.88}$$

[7]Though we use the same notation, these shape moduli should not be confused with the coefficients in eq. (2.73).

with the inverse $G^{jk}(b)$. This is a Kähler metric with Kähler potential $K(b)$ defined by

$$K(b) \equiv -\ln\left(d_{ijk}b^i b^j b^k\right)$$

such that

$$G_{jk}(b) = \frac{\partial^2}{\partial b^j\,\partial b^k}\,K(b)\,.$$

It is this metric which in the following will be used to raise and lower the $h^{1,1}$-type indices. Since $G_{jk}(b)$ does not scale with V this implies that neither ω_k nor $\omega^k = G^{kj}(b)\omega_j$ scale with V, though ω^k depend on the moduli b^k simply because the metric does. The scaling behavior of the various quantities is important to correctly include V as a modulus in the solution of the previous section. Note that this moduli metric differs from the one defined in eq. (2.69), since, besides the factor $1/V$, the Hodge star operator depends on the deformed metric g_{AB} which is not the same as the metric Ω_{AB} at lowest order. A more detailed account of Calabi-Yau geometry can be found in [160] or in the appendix of ref. [108], where it is shown that the following useful identity $G(b)_{lm}b^l b^m = 3$ holds.

Next we consider the complex structure moduli of the Calabi-Yau manifold and since they paramatrize $H^{2,1}(X)$ we first introduce a basis $\{\Pi_p,\, p = 1,\ldots,h^{2,1}\}$ of harmonic $(2,1)-$forms. Then the metric deformations associated to the complex structure deformations can be defined by [160, 161]

$$\delta g_{ab} = \bar{\mathfrak{z}}^{\bar p}\,\bar{\mathfrak{b}}_{\bar p\,ab}\,, \qquad\qquad \delta g_{\bar a\bar b} = \mathfrak{z}^p\,\mathfrak{b}_{p\,\bar a\bar b}\,, \tag{2.89}$$

where the $\mathfrak{b}_{p\,\bar a\bar b}$ are given by

$$\mathfrak{b}_{p\,\bar a\bar b} = \frac{-i}{\|\Omega\|^2}\,\bar\Omega_{\bar a}^{\;cd}\Pi_{p\,cd\bar b}\,, \tag{2.90}$$

and Ω_{abc} is the covariantly constant, harmonic $(3,0)-$form on the Calabi-Yau space X with $\|\Omega\|^2 = 1/3!\Omega_{abc}\bar\Omega^{abc}$ constant on X. As is well known, the variations (2.89) can be made to vanish by a non-holomorphic coordinate transformation which would render the metric hermitean again. Actually this is exactly the hermitean metric chosen in eq. (2.83) which must be used to raise and lower Calabi-Yau indices. But the space-time derivatives of the variations (2.89) cannot be made to vanish[8] and thus must be taken care of explicitly in the reduction of the scalar curvature and the supersymmetry transformations. Now since the metric scales like $g^{a\bar b} \sim V^{-1/3}$ and $\Omega_{abc} \sim 1$ we find that $\|\Omega\|^2 \sim V^{-1}$ and so we use the following scale convention

$$\|\Omega\|^2 \equiv V^{-1}.$$

[8]This is analogous to the Christoffel symbols which vanish in a local inertial frame, which does not mean that the curvature vanishes.

This implies that by eq. (2.90) the deformations (2.89) depend also on the volume modulus, which then must be taken care of when taking derivatives. On the complex structure moduli space a metric can be defined by

$$\mathcal{K}_{p\bar{q}}(\mathfrak{z}) \equiv -\frac{\int \Pi_p \wedge \bar{\Pi}_{\bar{q}}}{\int \Omega \wedge \bar{\Omega}} = \frac{-1}{vV\|\Omega\|^2}\int \Pi_p \wedge *\bar{\Pi}_{\bar{q}} = \frac{1}{vV}\int \sqrt{g_6}\, i d^6 z\, \mathfrak{b}_{p\,\bar{a}\bar{b}}\bar{\mathfrak{b}}_{\bar{q}}^{\bar{a}\bar{b}}$$

which is Kähler and can be obtained from the following Kähler potential

$$\mathcal{K}(\mathfrak{z}) \equiv -\ln\left(\frac{i}{v}\int_X \Omega \wedge \bar{\Omega}\right) \tag{2.91}$$

such that

$$\mathcal{K}_{p\bar{q}} = \frac{\partial^2}{\partial \mathfrak{z}^p \partial \bar{\mathfrak{z}}^{\bar{q}}}\,\mathcal{K}(\mathfrak{z}).$$

Zero-modes from bulk tensor field

Let us next turn to the bulk antisymmetric tensor field C. The massless Kaluza-Klein modes of the three-form arise from different suitable cohomology sectors of the Calabi-Yau space which are $H^{1,1}(X)$ as well as $H^{3,0}(X)$, $H^{2,1}(X)$ and their complex conjugates. It turns out to be more convenient not to discuss all three sectors independently but to instead consider $H^{1,1}$ and H^3. Because it is the choice of a specific complex structure that determines the Hodge decomposition $H^3 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$, it is more natural and common to discuss the entire H^3-sector at once. In order to do so let the three-cycles (A^P, B_Q), $P, Q = 0, \ldots, h^{2,1}$ be a canonical homology basis of $H_3(X)$ and the three-forms (α_P, β^Q) the dual cohomology basis of $H^3(X)$, normalized such that

$$\begin{aligned}
\int_X \alpha_Q \wedge \beta^P &= \int_{A^P} \alpha_Q = \delta_Q^P\,, \qquad & \int_X \alpha_P \wedge \alpha_Q &= 0\,,\\
\int_X \beta^P \wedge \alpha_Q &= \int_{B_Q} \beta^P = -\delta_Q^P\,, \qquad & \int_X \beta^P \wedge \beta^Q &= 0\,.
\end{aligned}$$

These relations are invariant under symplectic transformations of the basis three-cycles and this is the geometrical source of the symplectic structure of the manifold parametrized by the hypermultiplet scalars in five-dimensional $\mathcal{N} = 1$ supergravity [160, 162]. Now the three-form potential and its field strength can be expanded like[9]

$$\begin{aligned}
\mathsf{C} &= C + \mathcal{A}^k \wedge \omega_k + (\xi^Q\,\alpha_Q - \tilde{\xi}_Q\,\beta^Q)\,, \\
\mathsf{G} &= G + \mathcal{F}^k \wedge \omega_k + (\mathcal{X}^Q \wedge \alpha_Q - \tilde{\mathcal{X}}_Q \wedge \beta^Q)\,,
\end{aligned} \tag{2.92}$$

[9]The minus sign in the following expansion is pure convention and we use it only for sake of obvious symplectic notation.

and as the low-energy fields we then get, in five dimensions, a three form $C_{\alpha\beta\gamma}$ with field strength $G_{\alpha\beta\gamma\delta}$, $h^{1,1}$ vector fields $\mathcal{A}^k_\alpha$, $k = 1, \ldots, h^{1,1}$ with field strengths $\partial_{[\alpha}\mathcal{A}^k_{\beta]} = \mathcal{F}^k_{\alpha\beta}$, and $2(h^{2,1}+1)$ real scalar fields ξ^Q, $\tilde{\xi}_Q$ with field strengths $\partial_\alpha \xi^Q = \mathcal{X}^Q_\alpha$, $\partial_\alpha \tilde{\xi}_Q = \tilde{\mathcal{X}}_{Q\,\alpha}$ where $Q = 1 \ldots h^{2,1}+1$. In component form these fields are defined by

$$\begin{array}{ll} C_{\alpha\beta\gamma} & G_{\alpha\beta\gamma\delta} = 4\partial_{[\alpha}C_{\beta\gamma\delta]} \\ C_{\alpha a\bar{b}} = \mathcal{A}^k_\alpha\, \omega_{k\,a\bar{b}} & G_{\alpha\beta a\bar{b}} = \mathcal{F}^k_{\alpha\beta}\, \omega_{k\,a\bar{b}} \\ C_{ABC} = \xi^Q\, \alpha_{Q\,ABC} - \tilde{\xi}_Q\, \beta^Q_{ABC} & G_{\alpha ABC} = \mathcal{X}^Q\, \alpha_{Q\,ABC} - \tilde{\mathcal{X}}_Q\, \beta^Q_{ABC}\,. \end{array}$$

In order to understand the multiplet structure of the five-dimensional supergravity obtained in the following, we now want to briefly state how the fields $(\xi^Q, \tilde{\xi}_Q)$ are related to modes associated to the $H^{3,0}(X)$ and $H^{2,1}(X)$ sectors. An alternate expansion of the three-form potential in terms of bases of $H^{3,0}$ and $H^{2,1}$ reads

$$C|_{H^3} = \xi\,\Omega + \bar{\xi}\,\bar{\Omega} + \eta^p\,\Pi_p + \bar{\eta}^{\bar{q}}\,\bar{\Pi}_{\bar{q}}\,, \tag{2.93}$$

where a bar denotes complex conjugation. It is the fields (ξ, η^p) that will turn out to be scalar components of the conventionally identified hypermultiplets, which are the universal hypermultiplet from the $H^{3,0}$-sector and the additional hypermultiplets from the $H^{2,1}$-sector, respectively. The relation between the fields (ξ, η^p) and $(\xi^P, \tilde{\xi}_Q)$ is given by

$$\begin{pmatrix} \xi^P \\ \tilde{\xi}_Q \end{pmatrix} = \begin{pmatrix} \mathcal{Z}^P & f^P_p \\ \mathcal{G}_Q & h_{p\,Q} \end{pmatrix} \begin{pmatrix} \xi \\ \eta^p \end{pmatrix} + c.c.\,,$$

where the periods $(\mathcal{Z}^P, \mathcal{G}_Q)$ and $(f^P_p, h_{p\,Q})$ are defined in (C.4) and (C.11). Since these periods depend on the external coordinates this relation would be different for the expansion coefficients of the field strength G with respect to the two different bases. This is also the reason why it is much more convenient to work with (2.92) than with (2.93), where in the latter case the basis forms depend on the external coordinates through their dependence on the complex structure moduli.

Moduli from boundary gauge fields

Since it has been shown in refs. [123,124,125,126,127,128,129,130,131,132] that a phenomenologically interesting particle spectrum on the orbifold planes can be obtained by appropriate compactifications, we are also interested in the general structure of the moduli from the boundary theories. On both boundaries, over the Calabi-Yau space, we have internal stable holomorphic gauge bundles $V_{R_{\mathcal{I}_i}}$ in the representations $R_{\mathcal{I}_i}$ of the internal gauge groups $\mathcal{I}_i$, $i = 1, 2$. The external low energy gauge groups $\mathcal{H}_i$ are given by the commutants of the internal groups $\mathcal{I}_i$ within E_8. We are interested in the massless modes arising in the decompositions $\mathbf{248}_{E_8} = \sum (R_{\mathcal{I}_i}, R_{\mathcal{H}_i})$ of the adjoint of E_8 under $\mathcal{I}_i \times \mathcal{H}_i$. The number of such modes in a representation $R_{\mathcal{H}_i}$, i.e. the number of families, is given by the dimension of $H^1(X, V_{R_{\mathcal{I}_i}})$, that is by the number of independent harmonic, antiholomorphic one-forms with values in $V_{R_{\mathcal{I}_i}}$. Consider for example the phenomenologically interesting case where on the first boundary we take $\mathcal{I}_1 = SU(4)$ with $\mathcal{H}_1 = SO(10)$ such that [163]

$$\mathbf{248} = (\mathbf{15}, \mathbf{1}) + (\mathbf{1}, \mathbf{45}) + (\mathbf{6}, \mathbf{10}) + (\mathbf{4}, \mathbf{16}) + (\bar{\mathbf{4}}, \bar{\mathbf{16}}) \,,$$

then there are the **45** for the external gauge field, the **16** and $\bar{\mathbf{16}}$ for the gauge matter families and anti-families, and the **10** for the Higgs field, respectively. The case of standard embedding where one internal bundle is taken to be the spin connection $\mathcal{I}_1 = SU(3)$ such that $\mathcal{H}_1 = E_6$ and $H^1(X, \Lambda^1) = H^{1,1}(X)$ has been given in [108].

So in general we denote the external parts of the gauge fields by $A_{1\mu}$, $A_{2\mu}$ with field strengths $F_{1\mu\nu}$, $F_{2\mu\nu}$, which then are gauge fields of vector multiplets of the $\mathcal{N} = 1$ supersymmetric theories on the now four-dimensional orbifold fixed planes M_4^1, M_4^2. In general there are also chiral multiplets. To treat this gauge matter we need to introduce bases

$$\{u_L^x = u_{L\,\bar{a}}^x(R_{\mathcal{I}_i})\, dz^{\bar{a}} \,|\, L = 1, \ldots, \dim(H^1(X, V_{R_{\mathcal{I}_i}}))\,,\, x = 1, \ldots, \dim(R_{\mathcal{I}_i})\}$$

for the cohomology groups $H^1(X, V_{R_{\mathcal{I}_i}})$. In analogy to (2.88) there is also a metric defined by

$$G_{LM}^{(m)}(R_{\mathcal{I}_i}) = \frac{1}{vV^{2/3}} \int_X u_L^{\,x} \wedge * \bar{u}_{Mx} \,, \tag{2.94}$$

where the complex conjugate of the basis one-forms is denoted with lower gauge index. This metric generally depends on the Kähler moduli as well as on the complex structure moduli, whereas the basis one-forms do not depend on the Kähler moduli. Using this fact one can derive the following useful relation

$$u_{L\bar{a}}^x \bar{u}_{M\,b\,x} = -i\Gamma_{LM}^k \omega_{k\,\bar{a}b}, \qquad \Gamma_{LM}^k = G^{kl}(b) \frac{\partial}{\partial b^l} G_{LM}^{(m)} \,, \tag{2.95}$$

that is used in the derivation of the Bianchi identities in five dimensions. We also need the Yukawa couplings which are given by

$$\lambda_{LMN} = \frac{1}{\|\Omega\|^2} \int_X \Omega \wedge u_L^x(R1_{\mathcal{I}_i}) \wedge u_M^y(R2_{\mathcal{I}_i}) \wedge u_N^z(R3_{\mathcal{I}_i})\, f_{xyz}^{(123)} \tag{2.96}$$

where $f_{xyz}^{(123)}$ is completely symmetric and projects onto a possible singlet in the representation $R1_{\mathcal{I}_i} \times R2_{\mathcal{I}_i} \times R3_{\mathcal{I}_i}$. Next we need the generators of $(R_{\mathcal{I}_i}, R_{\mathcal{H}_i})$ and denote and normalize them as follows

$$\begin{gathered} \{T_{xp} = T_{xp}(R_{\mathcal{I}_i}) \,|\, p = 1, \ldots, \dim(R_{\mathcal{H}_i})\} \,, \\ \operatorname{tr} \bar{T}^{xp}(R_{\mathcal{I}_i})\, T_{yq}(R_{\mathcal{I}_i}) = \delta_y^x \delta_q^p \,, \end{gathered} \tag{2.97}$$

such that x, p are the gauge indices of the groups $\mathcal{I}_i$,$\mathcal{H}_i$ in the representation $R_{\mathcal{I}_i}$, $R_{\mathcal{H}_i}$, respectively. These generators satisfy the commutation relation

$$[T_{xp}(R1_{\mathcal{I}_i}), T_{yq}(R2_{\mathcal{I}_i})] = f^{(123)}_{xyz} g_{pqr} \bar{T}^{zr}(R3_{\mathcal{I}_i})$$

which together with eq. (2.97) gives the relation

$$\mathrm{tr}\big([T_{xp}(R1_{\mathcal{I}_i}), T_{yq}(R2_{\mathcal{I}_i})]\, T_{zr}(R3_{\mathcal{I}_i})\big) = f^{(123)}_{xyz} g_{pqr}.$$

Finally we can write the massless modes of the gauge potential like

$$A_{i\,\bar{a}} = \sqrt{2\lambda^2/v} \sum_{R_{\mathcal{I}_i}} u^x_{L\,\bar{a}}(R_{\mathcal{I}_i})\, T_{px}(R_{\mathcal{I}_i})\, C^{Lp}(R_{\mathcal{I}_i})\,,$$

and its field strength defined by $F_{\bar{I}\bar{J}} = \partial_{\bar{I}} A_{\bar{J}} - \partial_{\bar{J}} A_{\bar{I}} + \sqrt{v/2\lambda^2}[A_{\bar{I}}, A_{\bar{J}}]$ has the components

$$\begin{aligned}
F_{i\,\mu\bar{a}} &= \sqrt{2\lambda^2/v} \sum_{R_{\mathcal{I}_i}} u^x_{L\,\bar{a}}\, T_{xp}\,(D_\mu C^{Lp})\,,\\
F_{i\,\bar{a}\bar{b}} &= \sqrt{2\lambda^2/v} \sum_{R_{\mathcal{I}_i}} u^x_{L\,\bar{a}} u^y_{M\,\bar{b}}\,[T_{xp}, T_{yq}]\, C^{Lp} C^{Mq}\,,\\
F_{i\,a\bar{b}} &= \sqrt{2\lambda^2/v} \sum_{R_{\mathcal{I}_i}} \bar{u}_{L\,a\,x} u^y_{M\,\bar{b}}\,[\bar{T}^{xp}, T_{yq}]\, \bar{C}^L_p C^{Mq}\,,
\end{aligned}$$

where $C^{Lp}(R_{\mathcal{I}_i})$ correspond to the scalar gauge matter fields in four dimensions transforming in the representation $R_{\mathcal{H}_i}$ of the gauge group $\mathcal{H}_i$.

Moduli from five-brane world-volume fields

Next we discuss the zero modes on the five-brane world-volumes. The five-branes are allowed to fluctuate in five external dimensions, while internally they can move within the Calabi-Yau space. This leads to the following set of embedding coordinates

$$X^\mu_{\hat{\imath}} = X^\mu_{\hat{\imath}}(\sigma^\nu_{\hat{\imath}})\,, \qquad X^{11}_{\hat{\imath}} = Y_{\hat{\imath}}(\sigma^\nu_{\hat{\imath}})\,, \qquad X^a_{\hat{\imath}} = X^a_{\hat{\imath}}(\sigma_{\hat{\imath}}, M_{\hat{\imath}})$$

where $M_{\hat{\imath}}$ is a set of moduli which parametrizes the moduli space of holomorphic curves with a given homology class $[\mathcal{C}^{\hat{\imath}}_2]$ for the Calabi-Yau space under consideration [126,127,128]. In our low-energy effective action, we will not explicitly take these moduli into account and thus the internal embedding coordinates $X^a_{\hat{\imath}}$ will not show up as moduli fields in the five-dimensional theory. The three-brane surface in five-dimensional space specified by the above embedding coordinates $X^\alpha_{\hat{\imath}}$ is denoted by

$M_4^{5\hat{\imath}}$, $\hat{\imath} = 1, \ldots, N$. Using the above embedding and the bulk metric (2.83), we find the following non-vanishing components of the induced world-volume metrics

$$\begin{aligned} \gamma_{\hat{\imath}\,\mu\nu} &= \partial_\mu X^\alpha_{\hat{\imath}} \partial_\nu X^\beta_{\hat{\imath}}\, g_{\alpha\beta}\,, \\ \gamma_{\hat{\imath}\,st} &= \partial_s X^A_{\hat{\imath}} \partial_t X^B_{\hat{\imath}}\, g_{AB} = (-2i) a^k \partial_\sigma X^a_{\hat{\imath}} \partial_{\bar\sigma} X^{\bar b}_{\hat{\imath}} \omega_{k\,a\bar b}\, \delta_{st}\,. \end{aligned}$$

There are also a number of moduli arising from the world-volume two-form $B_{\hat{\imath}}$ which can be determined from the cohomology of the two-cycle $\mathcal{C}_2^{\hat{\imath}}$. First we introduce a basis of two-cycles $\{\mathcal{W}_2^k\}$ of the second homology group $H_2(X)$ dual to the basis $\{\omega_k\}$ of $H^2(X)$ in the sense that

$$\frac{1}{v^{1/3}} \int_X \omega_k \wedge \delta(\mathcal{W}_2^j) = \frac{1}{v^{1/3}} \int_{\mathcal{W}_2^j} \hat{\omega}_k = \delta_k^j\,,$$

where a hat denotes a pullback, in this case onto the appropriate basis two-cycle $\mathcal{W}_2^j$. The complex curves $\mathcal{C}_2^{\hat{\imath}}$ can then be decomposed with respect to this basis like $\mathcal{C}_2^{\hat{\imath}} = \beta_k^{\hat{\imath}} \mathcal{W}_2^k$, where the coefficients $\beta_k^{\hat{\imath}}$ are defined in (2.72). For arbitrary coefficients not every linear combination of the basis cycles $\{\mathcal{W}_2^k\}$ would be in the homology class of a holomorphic curve, i.e. it need not be an effective curve. Generally the holomorphic cycles sit in a cone in $H^{1,1}(X)$ whereof the Riemann surfaces, i.e. the simply connected holomorphic curves, are again a subclass, and it is such Riemann surfaces on which our five-branes are wrapped. The simple connectedness of the cycles $\mathcal{C}_2^{\hat{\imath}}$ also implies $h^2(\mathcal{C}_2^{\hat{\imath}}) = 1$, thus there exists one independent basis two-form on each such cycle which can be written as an arbitrary linear combination[10]

$$\omega^{\hat{\imath}} \equiv \sum_{k=1}^{h^{1,1}} n_{\hat{\imath}}^k \hat{\omega}_k^{\hat{\imath}}\,, \tag{2.98}$$

where $\hat{\omega}_k^{\hat{\imath}}$ denote the pullbacks of the basis (1,1)-forms onto the cycles $\mathcal{C}_2^{\hat{\imath}}$. One convenient choice of coefficients as taken in [164] is $n_{\hat{\imath}}^k = \beta_k^{\hat{\imath}} / (\sum_{l=1}^{h^{1,1}} \beta_l^{\hat{\imath}2})$, yielding unit volume of the associated cycle. Another natural choice though would be [113] $n_{\hat{\imath}}^k = a^k$ such that $\omega^{\hat{\imath}}$ in (2.98) is the pullback of the Calabi-Yau Kähler form onto the cycle $\mathcal{C}_2^{\hat{\imath}}$. For generality, we simply use the coefficients $n_{\hat{\imath}}^k$, $k = 1, \ldots, h^{1,1}$ in the following. Because $h^2(\mathcal{C}_2^{\hat{\imath}}) = 1$ all harmonic two-forms on $\mathcal{C}_2^{\hat{\imath}}$ must be proportional to the basis two-form $\omega^{\hat{\imath}}$, and for the pullbacks of the basis (1,1)-forms $\omega_k^{\hat{\imath}}$ we find

$$\hat{\omega}_k^{\hat{\imath}} = \frac{\beta_k^{\hat{\imath}}}{(n_{\hat{\imath}}^l \beta_l^{\hat{\imath}})}\, \omega^{\hat{\imath}}$$

[10] Here and in the following no summation over the index $\hat{\imath}$ is implied if not explicitly stated with a sum sign $\sum_{\hat{\imath}}$.

which will be used later. Next we also introduce a basis $\{\lambda_{\hat{\imath}U}\}$ of $H^1(\mathcal{C}_2^{\hat{\imath}})$ where $U, V, W, \ldots = 1, \ldots, 2g_{\hat{\imath}}$ and $g_{\hat{\imath}}$ is the genus of $\mathcal{C}_2^{\hat{\imath}}$. As the low-energy fields on the three-brane $M_4^{5\hat{\imath}}$ we then find the two-form $B_{\hat{\imath}\,\mu\nu}$ with field strength $H_{\hat{\imath}\,\mu\nu\rho}$, $2g_{\hat{\imath}}$ Abelian [11] vector fields $D^U_{\hat{\imath}\,\mu}$ with field strengths $E^U_{\hat{\imath}\,\mu\nu}$ and one independent scalar $s_{\hat{\imath}}$ which shows up in the field strength $j_{\hat{\imath}\,\mu}$. These fields are defined by

$$\begin{array}{ll} B_{\hat{\imath}\,\mu\nu} & H_{\hat{\imath}\,\mu\nu\rho} = (dB - \hat{C})_{\hat{\imath}\,\mu\nu\rho} \\ B_{\hat{\imath}\,\mu s} = D^U_{\hat{\imath}\,\mu}\,\lambda_{\hat{\imath}\,U\,s} & H_{\hat{\imath}\,\mu\nu s} = E^U_{\hat{\imath}\,\mu\nu}\,\lambda_{\hat{\imath}\,U\,s} = (dD^U_{\hat{\imath}})_{\mu\nu}\,\lambda_{\hat{\imath}\,U\,s} \\ B_{\hat{\imath}\,st} = s_{\hat{\imath}}\,\omega^{\hat{\imath}}_{st} & H_{\hat{\imath}\,\mu st} = j_{\hat{\imath}\,\mu}\,\omega^{\hat{\imath}}_{st} = \frac{\beta^{\hat{\imath}}_k}{n^l_{\hat{\imath}}\beta^{\hat{\imath}}_l}(d(n^k_{\hat{\imath}}s_{\hat{\imath}}) - \hat{\mathcal{A}}^k_{\hat{\imath}})_\mu\,\omega^{\hat{\imath}}_{st}\,, \end{array}$$

where as usual $\hat{\imath} = 1, \ldots, N$ enumerates the five-branes. Due to the self-duality condition $*H_{\hat{\imath}} = H_{\hat{\imath}}$ these four-dimensional fields are not all independent though. In order to work out the relations between them we split the $2g_{\hat{\imath}}$ vector fields into two sets, that is, we write $(E^U_{\hat{\imath}}) = (E^u_{\hat{\imath}}, \tilde{E}_{\hat{\imath}\,u})$ where $u, v, w, \ldots = 1, \ldots, g_{\hat{\imath}}$. Then, we find that the self-duality condition reduces to

$$H_{\hat{\imath}} = \frac{(n^k_{\hat{\imath}}\beta^{\hat{\imath}}_k)}{V\,(\beta^{\hat{\imath}}_l b^l)} * j^l_{\hat{\imath}} \tag{2.99}$$

$$\tilde{E}_{\hat{\imath}\,v} = [\mathrm{Im}(\Pi)]_{\hat{\imath}\,vw} * E^w_{\hat{\imath}} + [\mathrm{Re}(\Pi)]_{\hat{\imath}\,vw}E^w_{\hat{\imath}} \tag{2.100}$$

where the star is the four-dimensional Hodge-star operator and $\Pi_{\hat{\imath}\,vw}$ is the period matrix of the complex curve $\mathcal{C}_2^{\hat{\imath}}$. To define this matrix we denote by $(a_{\hat{\imath}\,w}, b_{\hat{\imath}\,w})$ a standard basis of $H_1(\mathcal{C}_2^{\hat{\imath}})$ consisting of α and β cycles and introduce a set of one-forms $(\alpha_{\hat{\imath}\,w})$ satisfying $\int_{a_{\hat{\imath}\,u}} \alpha_{\hat{\imath}\,w} = \delta_{uw}$. Then the period matrix for the $\hat{\imath}$-th two-cycle $\mathcal{C}_2^{\hat{\imath}}$ is given by

$$\Pi_{\hat{\imath}\,uw} \equiv \int_{b_{\hat{\imath}\,u}} \alpha_{\hat{\imath}\,w}.$$

For the case of a torus, $g = 1$, the period matrix is simply a complex number which can be identified with the complex structure τ of the torus. Shortly, we will use the relations (2.99) and (2.100) to eliminate half of the vector fields as well as $B_{\hat{\imath}\,\mu\nu}$ in favour of $s_{\hat{\imath}}$ from our low-energy effective action to arrive at a description in terms of independent fields. A more detailed account of the geometry of Riemann surfaces can be found in appendix D.

The remaining bosonic world-volume fields we should consider are the auxiliary scalar fields $a_{\hat{\imath}}$. If we want the normal vectors $v_{\hat{\imath}}$ to be globally well defined, we cannot allow them to point into the internal directions of the two-cycles only. This is because generally there need not exist a nowhere vanishing vector field on a Riemann surface, as the simple example of a sphere S^2 already demonstrates. Hence, we will take $a_{\hat{\imath}}$

[11] This can be enhanced to non-abelian symmetries if five-branes are "stacked", as discussed in ref. [109]. We do not attempt to incorporate this effect explicitly.

to be independent of the internal coordinates and require them to be functions of the external coordinates only, that is, $a_{\hat{\imath}} = a_{\hat{\imath}}(\sigma^{\mu}_{\hat{\imath}})$. It turns out that these fields will drop out of the five-dimensional effective action after eliminating half of the degrees of freedom [12], using eqs. (2.99) and (2.100).

The non-zero mode

The last ingredient we need to discuss is the non-zero mode (2.76). It consists of the purely internal part of the four-form gauge field strength, but since we now allow the five-branes to fluctuate it must be slightly generalized. Moreover, we have to be careful to correctly include the Calabi-Yau volume modulus. To this end we first define the function

$$\alpha_k(x^\alpha) = \left[\alpha_k^{(0)}\theta(y) + \alpha_k^{(N+1)}\theta(y-\pi\rho) + \sum_{\hat{\imath}=1}^{N}\alpha_k^{\hat{\imath}}\left(\Theta(M_4^{5\hat{\imath}}) + \Theta(\tilde{M}_4^{5\hat{\imath}})\right)\right] \tag{2.101}$$

where $d\Theta(M_4^{5\hat{\imath}}) = \delta(M_4^{5\hat{\imath}})$ and the $\theta-$ and δ–function are defined in analogy with eqs. (2.53). Then from the identity $\delta_k^l = G_{km}G^{ml} = \frac{1}{vV^{1/3}}\int_X \omega_k \wedge *_X\omega^l$ we see that in the combination $\frac{1}{V^{1/3}}G^{kl}(b) *_X \omega_k$ all moduli dependencies cancel when the Hodge star $*_X$ is defined with respect to the deformed CY-metric g_{AB}. This allows to write the non-zero mode like

$$G = -\frac{1}{V^{1/3}}\sum_{k=1}^{h^{1,1}}\alpha_k(x^\alpha)(*_X\omega^k)\,, \tag{2.102}$$

which then is moduli independent as required by the Bianchi identity (2.54), which is independent of the metric. Note that for the static brane configuration (2.64) which implies $\Theta(M_4^{5\hat{\imath}}) = \theta(y - Y_{\hat{\imath}})$ and $\Theta(\tilde{M}_4^{5\hat{\imath}}) = \theta(y + Y_{\hat{\imath}})$, together with $V = 1$ and $g_{AB} \to \Omega_{AB}$, the above expression reduces to the background configuration (2.76) as it should. For this case the function $\alpha_k = \alpha_k(y)$, for a fixed value of k and with two bulk M-branes, is illustrated in fig 2.5.

2.5 The five-dimensional theory

Based on the above background solutions, we would now like to derive the five-dimensional effective action and discuss its properties.

[12] We would like to thank Dmitri Sorokin for a helpful discussion on this point.

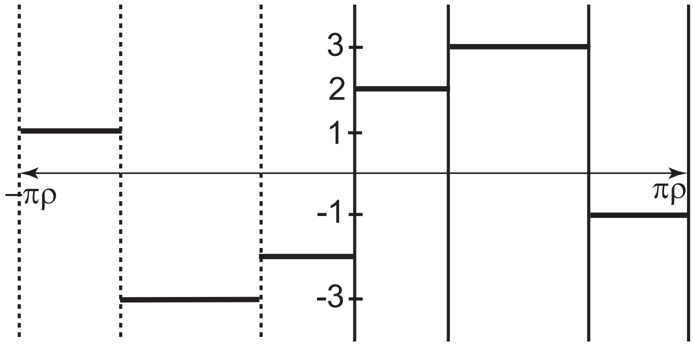

Figure 2.5: Function $\alpha_k(y)$ for charges $\alpha_k^{(0)} = 2$, $\alpha_k^{(1)} = 1$, $\alpha_k^{(2)} = -4$ and $\alpha_k^{(4)} = 1$.

2.5.1 The action

Let us first explain how the previously identified moduli fields fit into supermultiplets. In the bulk, we have[13] $D = 5$, $\mathcal{N} = 1$ supergravity with a gravity multiplet, which, besides the graviton and the gravitino, consists of a vector field, the gravi-photon. So there remain $h^{1,1} - 1$ vectors of which each one must be in a vector multiplet. From the case $h^{1,1} = 1$ [107, 114] we know that there is the universal hypermultiplet with bosonic content $(V, \sigma, \xi, \bar{\xi})$, where σ is the dual of the three-form $C_{\alpha\beta\gamma}$, thus the remaining scalars b^k must be part of the vector multiplets, remembering that by the constraint (2.87) the shape moduli b^i only represent $h^{1,1} - 1$ scalar degrees of freedom. To better understand this structure consider the following alternative description, also illustrated in fig 2.6 [111]. The moduli a^k parametrize a $h^{1,1}$-dimensional Kähler manifold $\mathcal{M}_\mathcal{K}$ and the relation (2.85), or equivalently (2.87), defines the hypermanifolds $\mathcal{M}_V \subset \mathcal{M}_\mathcal{K}$ of constant volume V. Such a $(h^{1,1} - 1)$-dimensional hypersurface can be parametrized by $h^{1,1} - 1$ independent real scalar fields ϕ^x, $x = 1, \ldots, h^{1,1} - 1$ with $b^k = b^k(\phi^x)$, and now a tangent space to $\mathcal{M}_\mathcal{K}$ can naturally be split into vectors tangent and one vector perpendicular to $\mathcal{M}_V$, the former belong to the vector multiplet and are given by $\mathcal{A}^x = (\partial_{\phi^x} b_k)\mathcal{A}^k$, and the latter is the gravi-photon given by $\mathcal{A} = b_k \mathcal{A}^k$. Thus the content of the $h^{1,1} - 1$ vector multiplets is $(\phi^x, \mathcal{A}^x_\alpha, \lambda^{x\,j})$, where $\lambda^{x\,j}$ are the fermions. The complete gravity multiplet then is $(g_{\alpha\beta}, \mathcal{A}_\alpha, \Psi^j_\alpha)$ with Ψ^j_α the gravitino. From the remaining fields of the $(2,1)-$sector we get an additional $h^{2,1}$ hypermultiplets with bosonic content $(\mathfrak{z}^p, \eta^p)$ with complex scalars $\mathfrak{z}^p$, η^p. As will be shown shortly, the hypermultiplet scalars $q^u = (V, \sigma, \mathfrak{z}^p, \xi^Q, \tilde{\xi}_Q)$ parametrize a symplectic, quaternionic manifold $\mathcal{M}_H$, also see [110] and the appendix C. The fermions of the complete hypermultiplet sector we call $\zeta^{a\,i}$. All the five-dimensional fermions are described by symplectic Majorana spinors carrying $SU(2)$ R-symmetry indices $j = 1, 2$. We note that, from their eleven-dimensional origin, the metric, V, $\mathfrak{z}^p$

[13] In four-dimensional counting we have $N = 2$ supersymmetry in five-dimensions which corresponds to eight supercharges.

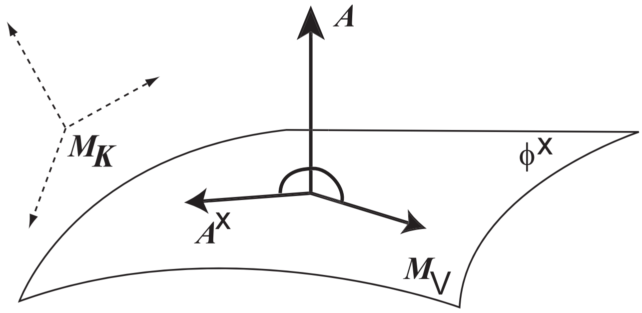

Figure 2.6: Hypermanifold $\mathcal{M}_V \subset \mathcal{M}_\mathcal{K}$ with tangent space split into graviphoton $\mathcal{A}$ and vector-multiplet gauge fields $\mathcal{A}^x$.

and ϕ^x are $\mathbb{Z}_2$–even fields, while $C_{\alpha\beta\gamma}$, $\mathcal{A}^x_\alpha$, $\mathcal{A}_\alpha$, ξ^Q and $\tilde{x}_Q$ are $\mathbb{Z}_2$–odd. To summarize we have found the following supergravity multiplets of bulk fields

gravity multiplet:	$(g_{\alpha\beta}, \mathcal{A}_\alpha, \Psi^j_\alpha)$
universal hypermultiplet:	(V, σ, ξ, ζ)
$h^{2,1}$ hypermultiplets:	$(\mathfrak{z}^p, \eta^p, \zeta^p)$
$h^{1,1} - 1$ vector multiplets:	$(\phi^x, \mathcal{A}^x_\alpha, \lambda^{x\,j})$.

On the four-dimensional fixed planes M^j_4, where $j = 1, 2$, we have $\mathcal{N} = 1$ gauge multiplets, that is gauge fields $A_{j\alpha}$ with field strengths $F_{j\alpha\beta} = (dA_j)_{\alpha\beta}$ and the corresponding gauginos. In addition there are gauge matter fields in $\mathcal{N} = 1$ chiral multiplets with scalar components C^{Jp} transforming in $R_{\mathcal{H}_i}$, together with the corresponding fermions.

On the three-brane world-volume $M^{5\hat{\imath}}_4$, the embedding coordinates $X^\alpha_{\hat{\imath}}$ give rise to a single physical degree of freedom $Y_{\hat{\imath}} = X^{11}_{\hat{\imath}}$, as can be seen from the static gauge choice. This field is part of the $\mathcal{N} = 1$ chiral multiplet with bosonic content $(Y_{\hat{\imath}}, s_{\hat{\imath}})$. We denote the corresponding fermions by $\theta^j_{\hat{\imath}}$. In addition, we have $\mathcal{N} = 1$ gauge multiplets containing Abelian gauge fields $D^u_{\hat{\imath}\alpha}$ with field strenghts $E^u_{\hat{\imath}\,\alpha\beta}$, where $u, v, w, \ldots = 1, \ldots, g_{\hat{\imath}}$ and $g_{\hat{\imath}}$ is the genus of the curve $\mathcal{C}^{\hat{\imath}}_2$ wrapped by the $\hat{\imath}$-th five-brane. In general, there will be additional chiral multiplets parametrizing the moduli space of the five-brane curves $\mathcal{C}^{\hat{\imath}}_2$ but they will not be explicitly taken into account here.

The reduction to five dimensions is not completely straightforward, particularly when dealing with the Chern-Simons and Dirac-term in the eleven-dimensional action. We have, therefore, performed the reduction on the level of the equations of motion. This gives rise to the following effective five-dimensional (moduli space) action

$$S_5 = S_{\text{grav}} + S_{\text{hyper}} + S_{\text{bound}} + S_{\text{matter}} + \sum_{\hat{\imath}=1}^{N} S^{\hat{\imath}}_{3-\text{brane}} \tag{2.103}$$

where the bulk and boundary parts read

$$
\begin{aligned}
S_{\rm grav} &= -\frac{1}{2\kappa_5^2}\int_{M_5}\left\{d^5x\sqrt{-g}\left(\frac{1}{2}R+\frac{1}{4}G_{kl}(b)\partial_\alpha b^k\partial^\alpha b^l+\frac{1}{2}G_{kl}(b)\mathcal{F}^k_{\alpha\beta}\mathcal{F}^{l\,\alpha\beta}\right)\right.\\
&\qquad\qquad\left.+\frac{2}{3}d_{klm}\,\mathcal{A}^k\wedge\mathcal{F}^l\wedge\mathcal{F}^m\right\} \qquad (2.104)\\
S_{\rm hyper} &= -\frac{1}{2\kappa_5^2}\int_{M_5}\left\{d^5x\sqrt{-g}\left(\frac{1}{4}V^{-2}\partial_\alpha V\partial^\alpha V+\frac{1}{4}\mathcal{K}(\mathfrak{z})_{p\bar{q}}\,\partial_\alpha\mathfrak{z}^p\partial^\alpha\bar{\mathfrak{z}}^{\bar{q}}\right.\right.\\
&\quad -V^{-1}(\partial_\alpha\tilde{\xi}_P-\bar{M}_{PQ}(\mathfrak{z})\partial_\alpha\xi^Q)([{\rm Im}M(\mathfrak{z})]^{-1})^{PR}(\partial^\alpha\tilde{\xi}_R-M_{RS}(\mathfrak{z})\partial^\alpha\xi^S)\\
&\qquad\left.+\frac{1}{4!}V^2G_{\alpha\beta\gamma\delta}G^{\alpha\beta\gamma\delta}+V^{-2}G^{kl}(b)\alpha_k\alpha_l\right)\\
&\qquad\left.+2G\wedge\left[(\xi^Pd\tilde{\xi}_P-\tilde{\xi}_Pd\xi^P)-2\alpha_k\mathcal{A}^k\right]\right\} \qquad (2.105)\\
S_{\rm bound} &= -\frac{1}{2\kappa_5^2}\left\{2\int_{M_4^1}d^4x\sqrt{-g_4}\,V^{-1}b^k\alpha_k^{(0)}\right. \qquad (2.106)\\
&\qquad\left.+2\int_{M_4^2}d^4x\sqrt{-g_4}\,V^{-1}b^k\alpha_k^{(N+1)}\right\}\\
S_{\rm matter} &= -\frac{1}{16\pi\alpha_{\rm GUT}}\sum_{j=1}^{2}\int_{M_4^j}d^4x\sqrt{-g_4}\,V\,{\rm tr}(F_{j\mu\nu}F_j^{\mu\nu}) \qquad (2.107)\\
&\quad+\sum_{j=1}^{2}\int_{M_4^j}d^4x\sqrt{-g_4}\sum_{R_{\mathcal{I}_j}}G^{(m)}_{LM}(R_{\mathcal{I}_j})D_\mu C^{Lp}(R_{\mathcal{I}_j})D^\mu\bar{C}^M_p(R_{\mathcal{I}_j})
\end{aligned}
$$

and the three-brane actions are

$$
\begin{aligned}
S^{\hat{\imath}}_{3-{\rm brane}} &= -\frac{1}{2\kappa_5^2}\left\{\int_{M_4^{5\hat{\imath}}\cup\tilde{M}_4^{5\hat{\imath}}}\left[d^4\sigma_{\hat{\imath}}\sqrt{-\gamma_{\hat{\imath}}}\left(V^{-1}\alpha^{\hat{\imath}}_kb^k+2\,\frac{(n^k_{\hat{\imath}}\alpha^{\hat{\imath}}_k)^2}{V(\alpha^{\hat{\imath}}_lb^l)}\,j_{\hat{\imath}\mu}j^\mu_{\hat{\imath}}\right.\right.\right. \qquad (2.108)\\
&\qquad\left.+[{\rm Im}(\Pi)]_{\hat{\imath}uw}E^u_{\hat{\imath}\mu\nu}E^{w\mu\nu}_{\hat{\imath}}\right)\\
&\qquad\left.\left.-4\hat{C}_{\hat{\imath}}\wedge\alpha^{\hat{\imath}}_kd(n^ks_{\hat{\imath}})-2[{\rm Re}(\Pi)]_{\hat{\imath}uw}E^u_{\hat{\imath}}\wedge E^w_{\hat{\imath}}\right]\right\}.
\end{aligned}
$$

The five-dimensional Newton constant κ_5, and the gauge coupling constant $\alpha_{\rm GUT}$ are given by

$$\kappa_5^2=\frac{\kappa^2}{v}\,,\qquad \alpha_{\rm GUT}=\frac{\lambda^2}{4\pi v}\,.$$

In this action all topological terms are written in differential form whereas all other contributions are given in component form. The matrix $M_{PQ}=M_{PQ}(\mathfrak{z})$ is defined in eq.(C.13) in appendix C. The hat denotes the pull-back of a bulk antisymmetric tensor field to the three-brane world-volume. The induced metric $\gamma_{\hat{\imath}\,\mu\nu}$ on the three-brane world-volume is, as usual, defined by

$$\gamma_{\hat{\imath}\,\mu\nu} = \partial_\mu X^\alpha_{\hat{\imath}} \partial_\nu X^\beta_{\hat{\imath}} g_{\alpha\beta} \, .$$

The matrix Π specifying the gauge-kinetic function on the three-branes is the period matrix defined in eq. (2.4.2). We recall that the step-function α_k in the above action has been defined in eq. (2.101) and that the charges $\alpha_k^{\hat{\imath}}$ satisfy the cohomology conditions

$$\sum_{\hat{\imath}=0}^{N+1} \alpha_k^{\hat{\imath}} = 0 \qquad \forall\, k = 1, \ldots, h^{1,1} \, .$$

The fields originating from the three-form in eleven dimensions must still satisfy nontrivial Bianchi identities. The reduction of eq. (2.54) yields

$$\begin{aligned}
(dG)_{5\mu\nu\sigma\rho} &= -4\kappa_5^2 \left(J^{(1)}_{\mu\nu\sigma\rho}\,\delta(y) + J^{(2)}_{\mu\nu\sigma\rho}\,\delta(y-\pi\rho) \right) , \\
(d\mathcal{F}^k)_{5\mu\nu} &= -4\kappa_5^2 \left(J^{(1)k}_{\mu\nu}\,\delta(y) + J^{(2)k}_{\mu\nu}\,\delta(y-\pi\rho) \right) , \\
(d\mathcal{X}^P \mathcal{G}_P - d\tilde{\mathcal{X}}_Q \mathcal{Z}^Q)_{5\mu} &= -4\kappa_5^2 \left(J^{(1)}_{\mu}\,\delta(y) + J^{(2)}_{\mu}\,\delta(y-\pi\rho) \right) ,
\end{aligned}$$

with the various currents given by

$$\begin{aligned}
J^{(i)}_{\mu\nu\sigma\rho} &= \frac{1}{16\pi\alpha_{\mathrm{GUT}}} \left(\mathrm{tr}(F_i \wedge F_i) \right)_{\mu\nu\sigma\rho} , \\
J^{(i)k}_{\mu\nu} &= -i \sum_{R_{\mathcal{I}_i}} \Gamma^k_{LM} (D_\mu C^{Lp} D_\nu \bar{C}^M_p - D_\mu \bar{C}^M_q D_\nu C^{Lq}) \, , \\
J^{(i)}_{\mu} &= \frac{e^{-\mathcal{K}(\mathfrak{z})}}{2V} \sum_{R_{\mathcal{I}_i}} \lambda_{LMN}\, g_{pqr} C^{Lp} C^{Mq} D_\mu C^{Nr} .
\end{aligned}$$

Unlike its eleven-dimensional counterpart (2.54), this Bianchi has only contributions from the orbifold planes since, in five dimensions, the bulk three-brane delta function is a one-form and thus cannot provide a magnetic source to the given fields. Remember that the gauge matter fields C^{Lp} and the quantities Γ^k_{LM}, λ_{LMN}, g_{pqr}, as defined in section (2.4.2), explicitly depend on the representation $R_{\mathcal{I}_i}$, $i = 1, 2$ of the internal holomorphic gauge bundles $V_{R_{\mathcal{I}_i}}$ on the orbifold fixed boundaries. Finally, only terms at most quadratic in space-time derivatives are included in the above action and Bianchi identities.

By construction, the action (2.103) must represent the bosonic part of a five-dimensional $\mathcal{N} = 1$ supergravity theory on the orbifold $S^1/\mathbb{Z}_2$ coupled to two four-dimensional $\mathcal{N} = 1$ theories on the orbifold fixed planes and N additional $\mathcal{N} = 1$

supersymmetric three-branes. Also let us note that for $h^{2,1} = 0$ and once the three-branes are taken away this action reduces, up to rescalings[14], exactly to the action of ref. [108], as it should. Next we want to discuss a few particular properties of the above action.

2.5.2 $\mathcal{N} = 1$ supersymmetric structure

Following the discussions in refs. [110, 107, 108] we will now demonstrate the quaternionic structure of the manifold $\mathcal{M}_H$ parametrized by the hypermultiplets scalars q^u as required by five-dimensional $\mathcal{N} = 1$ supergravity [165]. Furthermore, we find that the gauged isometry and its relation to the potential term as found in [108] are not affected by the inclusion of the complete hypermultiplet sector, that is for $h^{2,1} > 0$. The general structure of five-dimensional $\mathcal{N} = 1$ supergravity can be found in refs. [108, 165].

Let us start by dualizing the three-form $C_{\alpha\beta\gamma}$ to a scalar σ, and in the bulk we have

$$G = \frac{1}{V^2} * [d\sigma - (\tilde{\xi}_Q d\xi_Q - \xi_Q d\tilde{\xi}^Q) + 2\alpha_k(y)\mathcal{A}^k]. \tag{2.109}$$

Using this, the kinetic terms in $S_{\rm hyper}$ can be written in terms of the scalar fields $q^u = (V, \sigma, \mathfrak{z}^p, \xi^Q, \tilde{\xi}_Q)$ like

$$h_{uv} D_\alpha q^u D^\alpha q^v \tag{2.110}$$

where the covariant derivative is generally defined by $D_\alpha q^u = \partial_\alpha q^u + g k_l^u \mathcal{A}_\alpha^l$. Here in our case the Killing vector is $g k_l^\sigma = 2\alpha_l(y)$ such that only σ is charged and thus we have

$$D_\alpha \sigma = \partial_\alpha \sigma + 2\alpha_l \mathcal{A}_\alpha^l. \tag{2.111}$$

The metric h_{uv} is given by

$$\begin{aligned} h_{uv} dq^u \otimes dq^v \;=\; & \frac{1}{8}\mathcal{K}(\mathfrak{z})_{p\bar{q}} d\mathfrak{z}^p \otimes d\bar{\mathfrak{z}}^{\bar{q}} + \frac{1}{8V^2} dV \otimes dV \\ & + \frac{1}{8V^2}[d\sigma - (\tilde{\xi}_Q d\xi^Q - \xi^Q d\tilde{\xi}_Q)]^{\otimes 2} \\ & - \frac{1}{2V}(\mathrm{Im} M)^{-1\,PQ}(d\tilde{\xi}_P - M_{PR} d\xi^R) \otimes (d\tilde{\xi}_Q - \bar{M}_{QR} d\xi^R), \end{aligned} \tag{2.112}$$

where dq^u are one-forms on $\mathcal{M}_H$ and the matrix M is again the one defined in (C.13). The symplectic structure of $\mathcal{M}_H$ refers to the fact that its holonomy group is $Sp\,(2) \times Sp\,(2(h^{2,1}+1))$, and if this is indeed the case it must be possible to write the above metric (2.112) like

$$h_{uv} = V_u^{ia} V_v^{jb} \epsilon_{ij} \Omega_{ab}$$

[14] The rescalings are $C' = \frac{1}{6} 2^{1/6} C$, $G' = 2^{1/6} G$, $\xi' = 2^{1/6}\xi$, $\mathcal{A}' = 2^{1/6}\mathcal{A}$, $V' = 2^{-2/3} V$ and $g'_{5\alpha\beta} = 2^{-2/3} g_{5\alpha\beta}$ where the prime denotes the fields as in ref. [107].

where V_u^{ia} are vielbeins, ϵ_{ij} is the antisymmetric symbol with $\epsilon_{12} = 1$, and Ω_{ab} is the $2(h^{2,1}+1)\times 2(h^{2,1}+1)$ blockdiagonal matrix with blocks $-\epsilon_{ij}$. Therefore, the flat space metric on $\mathcal{M}_H$ is $\epsilon_{ij}\Omega_{ab}$ where $i,j = 1,2$ are $Sp(2) = SU(2)$ and $a,b = 1,\ldots,2(h^{2,1}+1)$ are $Sp(2(h^{2,1}+1))$ indices. Define the vielbeins by

$$V^{ia} = V_u^{ia} dq^u = \frac{1}{\sqrt{2}} \begin{pmatrix} u & \bar{v} & e^a & \bar{E}^a \\ v & -\bar{u} & E^a & -\bar{e}^a \end{pmatrix}^{Xa} \tag{2.113}$$

with one-forms defined by

$$\begin{aligned} e^a &= \frac{1}{2\sqrt{2}} e^a_p d\mathfrak{z}^p\,, \qquad e^a_p \bar{e}_{\bar{q}\,a} = \mathcal{K}_{p\bar{q}}(\mathfrak{z}), \\ E^a &= \frac{1}{2\sqrt{V}}\, e^{-\mathcal{K}/2} P^a_Q (\mathrm{Im}\mathcal{G})^{-1\,QR}(d\tilde{\xi}_R - M_{RS} d\xi^S)\,, \qquad \mathcal{K}_{PQ} = P^a_P \bar{P}_{Q\,a} \\ u &= \frac{1}{\sqrt{V}}\, e^{\mathcal{K}/2} \mathcal{Z}^P (d\tilde{\xi}_P - M_{PQ} d\xi^Q) \\ v &= \frac{1}{2\sqrt{2}V}(dV + i[d\sigma + (\xi^Q d\tilde{\xi}_Q - \tilde{\xi}_Q d\xi^Q)]) \end{aligned}$$

where $\mathcal{K}$ is the Kähler potential (2.91) with associated metric $\mathcal{K}_{p\bar{q}}$ and $\mathcal{Z}^P$ is defined in eq.(C.4) such that $\partial_p \mathcal{Z}^Q P^a_Q = e^a_p$ in accordance with eq.(C.12). Moreover, as defined in the appendix C, $\mathcal{G}(\mathcal{Z})$ is the holomorphic prepotential and $\mathcal{G}_{PQ} = \partial_P\partial_Q\mathcal{G}$. Then by using relation (C.22) it can be shown that

$$h_{uv} dq^u \otimes dq^v = V^{ia} \otimes V^{jb} \epsilon_{ij} \Omega_{ab} = u \otimes \bar{u} + v \otimes \bar{v} + e^a \otimes \bar{e}_a + E^a \otimes \bar{E}_a$$

is indeed exactly given by (2.112).

In order to next demonstrate the quaternionic structure of $\mathcal{M}_H$ and the consistency of the gauging (2.111) with $\mathcal{N} = 1$ supersymmetry, we need the $SU(2)$ connection ω^i_j from which we then can get the triplet of Kähler forms $K^i_j = K^x(\tau_x)^i_j = (K_{uv})^i_j dq^u \wedge dq^v$ by using $-K = d\omega + \omega\wedge\omega$. The Kähler forms then must satisfy the quaternionic algebra

$$(iK^x)(iK^y) = -\delta^{xy} + \epsilon^{xyz}(iK^z). \tag{2.114}$$

The $SU(2)$ connection turns out to be given by [110]

$$\omega^i{}_j = \begin{pmatrix} \frac{1}{2}(v - \bar{v}) + \frac{1}{2}w & -u \\ \bar{u} & -\frac{1}{2}(v-\bar{v}) - \frac{1}{2}w \end{pmatrix}^i{}_j$$

with the one-form w defined by

$$w = e^{\mathcal{K}}(\bar{\mathcal{Z}}_P (\mathrm{Im}\mathcal{G})_{PQ} d\mathcal{Z}^Q - \mathcal{Z}^P (\mathrm{Im}\mathcal{G})_{PQ} d\bar{\mathcal{Z}}^Q). \tag{2.115}$$

Then for the Kähler forms we find

$$K_i^j = \begin{pmatrix} \frac{1}{2}(u\wedge u - v\wedge\bar{v} + e^a\wedge\bar{e}_a - E^a\wedge\bar{E}_a) & u\wedge\bar{v} + e^a\wedge\bar{E}_a \\ v\wedge\bar{u} + E^a\wedge\bar{e}_a & -\frac{1}{2}(u\wedge u - v\wedge\bar{v} + e^a\wedge\bar{e}_a - E^a\wedge\bar{E}_a) \end{pmatrix}_j^{\ i}.$$

which indeed satisfies (2.114). Having the Kähler forms we can finally obtain the prepotentials $\mathcal{P}_l$ from the relation

$$k_l^u K_{uv} = \partial_v \mathcal{P}_l + [\omega_v, \mathcal{P}_l],$$

that relates the Killing vectors to the prepotentials such as to respect the quaternionic structure. We find that the prepotentials are given by

$$g\mathcal{P}_{k\,j}^{i} = \begin{pmatrix} i\frac{\alpha_k}{4V} & 0 \\ 0 & -i\frac{\alpha_k}{4V} \end{pmatrix}_j^{\ i}.$$

These prepotentials we now use to find the potential term of S_{hyper} that must be present for the gauging (2.111) to be consistent with $\mathcal{N}=1$ supersymmetry, and it is correctly given by

$$\begin{aligned} g^2V &= -4g^2G^{kl}(b)\mathrm{tr}\mathcal{P}_k\mathcal{P}_l + 4g^2b^kb^l\mathrm{tr}\mathcal{P}_k\mathcal{P}_l + \frac{1}{2}g^2b^lb^kh_{uv}k_k^uk_l^v \\ &= \frac{1}{2}V^{-2}G^{kl}(b)\alpha_k\alpha_l. \end{aligned} \qquad (2.116)$$

So by supersymmetry this potential term is induced by the gauging of the shift symmetry $\sigma \to \sigma + const.$ of the dilatonic axion. This symmetry is actually an isometry of $\mathcal{M}_H$ that is gauged with the vector field $\alpha_k\mathcal{A}^k$ as the corresponding gauge boson. Unlike in the case without five-branes, the gauge charges change across the bulk as anticipated in ref. [109]. From the definition of α_k, eq. (2.101), these charges are proportional to $\alpha_k^{(0)}$ between the first fixed plane and the first three-brane and proportional to $\alpha_k^{(0)} + \alpha_k^{(1)}$ between the first three-brane and the second three brane etc., see fig 2.5. Note that, similarly to the gauge charge, the potential (2.116) jumps across the three-branes. Further, it is worth pointing out that, while all tension terms in the action (2.103) are proportional to $V^{-1}\alpha_k^{\hat{\imath}}b^k$, where $\hat{\imath} = 0,\ldots,N+1$, the terms on the fixed planes contain an additional factor of two relative to the three-brane term. This factor reflects the nature of the "boundary branes" as being located on $\mathbb{Z}_2$ orbifold fixed planes.

To summarize, we now have shown that the hypermultiplet sector as obtained by direct compactification of Hořava-Witten theory on a Calabi-Yau three-fold nicely fits into five-dimensional $\mathcal{N}=1$ gauged supergravity as required.

2.5.3 Symmetries

Let us start with the supersymmetry transformations of the fermions. Their bosonic part can be obtained either by a reduction from eleven dimensions or, most easily, by using the general supersymmetry transformations of five-dimensional gauged $\mathcal{N} = 1$ supergravity as can be found in [108, 165]. The result is

$$\begin{aligned}
\delta\psi^i_\alpha &= D_\alpha\epsilon^i - \frac{i}{12}\left(\gamma_\alpha{}^{\beta\gamma} - 4\delta^\beta_\alpha\gamma^\gamma\right) b_k\mathcal{F}^k_{\beta\gamma}\epsilon^i + \frac{1}{6}i\gamma_\alpha V^{-1}b^l\alpha_l(y)(\tau_3)^i_j\epsilon^j \\
\delta\zeta^a &= -i\gamma^\alpha\partial_\alpha q^u V^{ja}_u\epsilon_j + 2b^k\alpha_k(y)V^{ja}_\sigma\epsilon_j \\
\delta\lambda^{x\,i} &= -\frac{1}{4}b^x_k\left[\left(i\gamma^\alpha\partial_\alpha b^k + \gamma^{\alpha\beta}\mathcal{F}^k_{\alpha\beta}\right)\epsilon^i + 2iV^{-1}\alpha^k(y)(\tau_3)^i{}_j\epsilon^j\right]
\end{aligned} \tag{2.117}$$

where γ_α are the five-dimensional gamma matrices, τ_3 is a Pauli matrix, $b^x_k = \partial_{\phi^x}b_k$ is the projection onto the ϕ^x–subspace and the vielbeins V^{ia}_u are given in (2.113). The variation of the three-brane world-volume spinors θ^i (assuming the world-volume fields s and D^u vanish) can be obtained by reducing the variation (2.57) which results in

$$\delta\theta^i_{\hat{\imath}} = \epsilon^i_{\hat{\imath}} + (p_{\hat{\imath}+})^i{}_j\kappa^j_{\hat{\imath}} \tag{2.118}$$

where the projection operators $p_{\hat{\imath}\pm}$ are now given by

$$p_{\hat{\imath}\pm} = \frac{1}{2}\left(1 \pm \frac{i}{4!}\epsilon^{\mu_1\ldots\mu_4}\partial_{m_1}X^{\alpha_1}_{\hat{\imath}}\ldots\partial_{m_4}X^{\alpha_4}_{\hat{\imath}}\gamma_{\alpha_1\ldots\alpha_4}\tau_3\right) .$$

Up to total derivatives the action (2.103) is also invariant under the following gauge variations

$$\delta C = d\lambda^{(2)}, \quad \delta\mathcal{A}^k = d\lambda^{k\,(0)}, \quad \delta(n^k s_{\hat{\imath}}) = \hat{\lambda}^{k(0)}_{\hat{\imath}}, \quad \delta\xi = \text{const.}, \quad \delta E^u_{\hat{\imath}} = d\lambda^{(1)u}_{\hat{\imath}}$$

with $\lambda^{(p)}$ being p-form gauge parameters and $\lambda^{k\,(0)}$, $\lambda^{(2)}$ being $\mathbb{Z}_2$–odd. To check this result one must note that the variation of the gauge term $\sim 4G \wedge \alpha_k(x)\mathcal{A}^k$ and the brane terms $\sim 4\alpha^{\hat{\imath}}_k\hat{C}_{\hat{\imath}} \wedge d(n^k_{\hat{\imath}}s)$ cancel each other after partial integration of the former. The above gauge variations of course also follow from the reduction of the gauge symmetries (2.59) in eleven dimensions. Note, however, that there are no remnants of the PST-symmetries (2.60) in our action. This is not surprising since these symmetries have been implicitly gauge-fixed when the self-duality relations (2.99) and (2.100) were used to eliminate half of the degrees of freedom on the three-brane.

2.5.4 The dual form of the action

In our five-dimensional action S_5, eq. (2.103), the three-branes are coupled to the gauge charges $\alpha^{\hat{\imath}}_k$, defined by eqs. (2.72) and (2.73), through the step-functions α_k

defined by eq. (2.101). We can promote the functions α_k to zero-form field strengths which, as follows from eq. (2.101), satisfy the Bianchi identity

$$d\alpha_k = 2\alpha_k^{(0)}\delta(M_4^1) + 2\alpha_k^{(N+1)}\delta(M_4^2) + \sum_{\hat{\imath}=1}^{N} \alpha_k^{\hat{\imath}} \left[\delta(M_4^{5\hat{\imath}}) + \delta(\tilde{M}_4^{5\hat{\imath}})\right] . \tag{2.119}$$

In particular this means that the three-branes couple magnetically to these zero-forms α_k, because the brane sources appear in the Bianchi identity and not in the equation of motion. In analogy with massive IIA supergravity [166], there should now be a dual formulation of the action S_5 where the α_k are replaced by four-forms N^k with five-form field strengths $M^k = dN^k$ to which the three-branes couple electrically. If a dual version of eleven-dimensional supergravity involving only a six-form field existed, we could have derived this dual five-dimensional action directly from eleven dimensions. As explained in section 2.2, such a dual version of eleven-dimensional supergravity is not available and, hence, the reduction necessarily leads to the five-dimensional action S_5 written in terms of α_k. However, there is no obstruction to performing the dualization in five dimensions. This can be done by adding to the action S_5 (with the α_k interpreted as a zero-form field strengths) the terms

$$S_\alpha = \frac{1}{2\kappa_5^2}\left\{ \int_{M_5} N^k \wedge d\alpha_k - 2\alpha_k^{(0)} \int_{M_4^1} \hat{N}_{(0)}^k - 2\alpha_k^{(N+1)} \int_{M_4^2} \hat{N}_{(N+1)}^k - \sum_{\hat{\imath}=1}^{N} \alpha_k^{\hat{\imath}} \int_{M_4^{5\hat{\imath}} \cup \tilde{M}_4^{5\hat{\imath}}} \hat{N}_{\hat{\imath}}^k \right\}$$

where $\hat{N}_{\hat{\imath}}^k$ is the pullback of N^k onto the $\hat{\imath}$-th three-brane. The term S_α is chosen exactly such that the equation of motion for N^k yields the Bianchi identity (2.119) for α_k, by virtue of which the additional term S_α vanishes again, that is on-shell we have added zero. As it should, this leads us back to the original action S_5 with α_k being defined by eq. (2.101). On the other hand, the equation of motion for α_k computed from the action $S_5 + S_\alpha$ is given by

$$\alpha^k = G^{kl}\alpha_l = +\frac{1}{2}V^2 * (4G \wedge \mathcal{A}^k - M^k) . \tag{2.120}$$

Using this relation to replace α^k in favour of M^k, we arrive at the dual version of our five-dimensional brane-world action (2.103). It is given by

$$S_{5,\text{dual}} = S_{\text{grav,d}} + S_{\text{hyper,d}} + S_{\text{bound,d}} + \sum_{\hat{\imath}=1}^{N} S_{3-\text{brane,d}}^{\hat{\imath}} \tag{2.121}$$

where

$$
\begin{aligned}
S_{\text{grav,d}} &= S_{\text{grav}} \\
S_{\text{hyper,d}} &= -\frac{1}{2\kappa_5^2}\int_{M_5}\Big\{d^5x\sqrt{-g}\Big(\frac{1}{4}V^{-2}\partial_\alpha V\partial^\alpha V + \frac{1}{4!}V^2 G_{\alpha\beta\gamma\delta}G^{\alpha\beta\gamma\delta} \\
&\quad - V^{-1}(\partial_\alpha\tilde{\xi}_P - \bar{M}_{PQ}(\mathfrak{z})\partial_\alpha\xi^Q)([\mathrm{Im}M(\mathfrak{z})]^{-1})^{PR}(\partial^\alpha\tilde{\xi}_R - M_{RS}(\mathfrak{z})\partial^\alpha\xi^S) \\
&\qquad + \frac{1}{4}\mathcal{K}(\mathfrak{z})_{p\bar{q}}\,\partial_\alpha\mathfrak{z}^p\partial^\alpha\bar{\mathfrak{z}}^{\bar{q}} + \frac{1}{4\cdot 5!}V^2 G_{kl}(b)M^k_{\alpha_1\ldots\alpha_5}M^{l\,\alpha_1\ldots\alpha_5}\Big) \\
&\quad + 2G\wedge\Big((\xi^P d\tilde{\xi}_P - \tilde{\xi}_P d\xi^P) + V^2 G_{kl}(b)\mathcal{A}^k\wedge *(M^l - 2G\wedge\mathcal{A}^l)\Big)\Big\} \\
S_{\text{bound,d}} &= S_{\text{bound}} - \frac{1}{2\kappa_5^2}\Big\{2\alpha_k^{(0)}\int_{M_4^1}\hat{N}^k + 2\alpha_k^{(N+1)}\int_{M_4^2}\hat{N}^k\Big\} \\
S^{\hat{\imath}}_{3-\text{brane,d}} &= S^{\hat{\imath}}_{3-\text{brane}} - \frac{1}{2\kappa_5^2}\,\alpha_k^{\hat{\imath}}\int_{M_4^{\hat{\imath}}\cup\tilde{M}_4^{\hat{\imath}}}\hat{N}^k_{\hat{\imath}}
\end{aligned}
$$

with S_{grav}, S_{bound} and $S^{\hat{\imath}}_{3-\text{brane}}$ as in eqs. (2.104), (2.107) and (2.108). In this dual form of the action the electric coupling of the boundaries as well as of the bulk three-branes to the electric potential N^k can explicitly be seen. The coupling strengths now are given by the gauge charges $\alpha_k^{\hat{\imath}}$, which thus can be interpreted as electric charges with respect to the potential N^k. As expected there is also a kinetic term for the electric field strength as well as an interaction term.

2.6 The vacuum solution

In this section, we will give the BPS multi-domain-wall solution of the five-dimensional theory defined by the action (2.103). For the case without additional bulk three-branes and including only the universal hypermultiplet, the supersymmetric domain-wall solution of five-dimensional heterotic M-theory has been found in ref. [107], which then was extended to include the whole $(1,1)$-sector in ref. [108]. We now wish to verify that these results can be extended to include the effect of the bulk three-branes and thereby providing a solution of our action (2.103). For the universal case where $h^{1,1} = 1$ this has already been explicitly done in ref. [114].

This solution is needed in section 2.7 where we will explicitly determine the bosonic part of the effective $\mathcal{N} = 1$ four-dimensional theory associated to the general, linearized multi-domain-wall background vacuum solution.

2.6.1 The supersymmetric multi-domain-wall vacuum state

To simplify the subsequent treatement let us note that the action (2.103) can consistently be truncated to have only the five-dimensional metric $g_{\alpha\beta}$, the (1,1)-moduli V, b^k and the three-brane embedding coordinates $X^\alpha_{\hat{\imath}}$ as its field content. The corresponding action is then given by

$$\begin{aligned}
S_5 \;=\; -\frac{1}{2\kappa_5^2}\Bigg\{ & \int_{M_5} d^5x\sqrt{-g}\left(\frac{1}{2}R + \frac{1}{4}V^{-2}\partial_\alpha V\partial^\alpha V + \frac{1}{4}G_{kl}(b)\partial_\alpha b^k\partial^\alpha b^l\right. \\
& \left. + V^{-2}G^{kl}(b)\alpha_k\alpha_l + \lambda(\mathcal{K}-6)\right) \\
& + 2\alpha_k^{(0)}\int_{M_4^1} d^4x\sqrt{-g_4}\,V^{-1}b^k \;+\; 2\alpha_k^{(N+1)}\int_{M_4^2} d^4x\sqrt{-g_4}\,V^{-1}b^k \\
& + \sum_{\hat{\imath}=1}^{N}\alpha_k^{\hat{\imath}}\int_{M_4^{5\hat{\imath}}\cup\tilde{M}_4^{5\hat{\imath}}} d^4\sigma_{\hat{\imath}}\sqrt{\gamma_{\hat{\imath}}}\,V^{-1}b^k\Bigg\},
\end{aligned}$$

where a Lagrange multiplier term has been added to ensure the constraint (2.87). We start with the following Ansatz for the metric, the volume modulus and the shape moduli

$$\begin{aligned}
ds_5^2 &= a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + b^2(y)dy^2 \\
V &= V(y) \\
b^k &= b^k(y) \qquad k = 1\ldots h^{1,1}
\end{aligned} \tag{2.122}$$

with all other bulk fields vanishing. In addition we need to specify the Ansatz for the world-volume fields of the three-branes. We choose a configuration of static three-branes that are parallel to each other as well as to the orbifold fixed planes, as illustrated in fig. 2.4. The ansatz is then given by

$$X^\mu_{\hat{\imath}} = \sigma^\mu_{\hat{\imath}}, \qquad Y_{\hat{\imath}} = \text{const},$$

with all other world-volume gauge fields vanishing. The gauge fields on the boundaries are turned off as well.

This Ansatz leads to a set of equations of motion which are still difficult to solve. Following ref. [108] we can give an implicit solution in terms of certain functions $f^k = f^k(y)$. It turns out that if these functions satisfy the following relations

$$d_{klm}f^lf^m = h_k, \qquad k,l,m = 1,\ldots,h^{1,1} \tag{2.123}$$

with the function $h_k(y)$ given by

$$h_k(y) = -4c\sum_{\hat{\imath}=0}^{\hat{\jmath}}\alpha_k^{\hat{\imath}}\left(|y| - Y_{\hat{\imath}}\right) + c_k \tag{2.124}$$

in the interval

$$Y_{\hat{\jmath}} \leq |y| \leq Y_{\hat{\jmath}+1}, \quad \hat{\jmath} = 0, \ldots, N\,,$$

then the solution can be written as

$$\begin{aligned} V &= (\frac{1}{6}d_{klm}f^k f^l f^m)^2 \\ a &= \tilde{c}\,V^{1/6} \\ b &= c\,V^{2/3} \\ b^k &= V^{-1/6}f^k\,, \end{aligned} \tag{2.125}$$

where c, $\tilde{c}$ and c_k are arbitrary integration constants. One can check that the new features arising from the presence of the N three-branes are properly taken into account. Specifically, the world-volume equations of motion for all N three-branes are independently satisfied and the N three-brane sources in the Einstein equations are properly matched. This implicit solution thus represents a N-domain-wall. This can be explicitly seen from the functions $h_k(y)$ which are harmonic up to the various three-brane and orbifold plane positions and continuous, such that they satisfy

$$\partial_y^2 h_k = -8c\left[\alpha_k^{(0)}\delta(y) + \alpha^{(N+1)}\delta(y-\pi\rho) + \frac{1}{2}\sum_{\hat{\imath}=1}^{N}\alpha_k^{\hat{\imath}}\left(\delta(y-Y_{\hat{\imath}}) + \delta(y+Y_{\hat{\imath}})\right)\right]\,.$$

The δ–functions indicate the positions of the various orbifold planes and three-branes at $y = 0, \pi\rho, Y_{\hat{\imath}}, -Y_{\hat{\imath}}$.

This solution is also a BPS state of the theory since it preserves four of the eight supersymmetries. For the bulk fermions this can be verified by using eqs. (2.117) which for the above Ansatz (2.122), together with $\epsilon^i = \epsilon^i(y)$, are equivalent to

$$\begin{aligned} \frac{a'}{a}\gamma_{\underline{11}}\epsilon^i + \frac{1}{3}\frac{b}{V}(b^k\alpha_k)\,(\tau_3)^i{}_j\epsilon^j &= 0 \\ \epsilon'^i + \frac{1}{6}\frac{b}{V}(b^k\alpha_k)\gamma_{\underline{11}}(\tau_3)^i{}_j\epsilon^j &= 0 \\ \frac{V'}{V}\gamma_{\underline{11}}\epsilon^i + 2\frac{b}{V}(b^k\alpha_k)(\tau_3)^i{}_j\epsilon^j &= 0 \\ b'_k\gamma_{\underline{11}}\epsilon^i + 2\frac{b}{V}\left[\alpha_k - \frac{1}{3}b_k(b^l\alpha_l)\right](\tau_3)^i{}_j\epsilon^j &= 0\,. \end{aligned}$$

These equations are solved by the above solution (2.123)-(2.125) for the Killing spinor given by

$$\epsilon^i = \sqrt{a}\epsilon_0^i\,, \qquad \gamma_{\underline{11}}\epsilon_0^i = (\tau_3)^i{}_j\epsilon_0^j\,,$$

where ϵ_0^i is a constant spinor and the underlined index of the gamma matrix is a tangent space index. Using the spinor $\kappa_{\hat{\imath}}^i = -\epsilon^i$ also solves the supersymmetry variation in eq. (2.118) for the three-brane world-volume fermions.

To summarize, we now have a general BPS vacuum solution given implicitly in terms of the functions f^k which need to satisfy eqs. (2.123) and (2.124). It is these equations that now need to be solved to get an explicit solution on which a further reduction to four dimensions can be performed. In the next subsection such a solution to linear order in the charges $\alpha_k^{\hat{\imath}}$ is found, which in the following section will be used to derive the four-dimensional $\mathcal{N}=1$ effective supergravity theory associated to it.

2.6.2 The linearized solution

A look at the action (2.103) shows that the existence of the domain-walls, i.e the orbifold planes and the three-branes, is an effect which appears at first order in the charges $\alpha_k^{\hat{\imath}}$. Thus it seems natural to look for a solution linear in these charges. Note though, that the general implicit solution (2.123)-(2.125) is of higher order, and thus one could in principle also try to find explicit solutions including higher order terms, but their actual meaning is not quite clear since in a strict sense the five-dimensional effective action (2.103) is already only valid to linear order in the charges $\alpha_k^{\hat{\imath}}$. Nevertheless, it would be interesting to include higher order terms and investigate their influence on four-dimensional physics.

Before we embark on solving eq. (2.123), for later convenience let us first change to normalized orbifold coordinates defined by

$$z = \frac{y}{\pi\rho}\,, \qquad z_{\hat{\imath}} = \frac{Y_{\hat{\imath}}}{\pi\rho}\,, \qquad \hat{\imath} = 0,\ldots,N+1$$

and also shift the integration constants c_k in the definition (2.124) of $h_k(z = y/\pi\rho)$ like

$$h_k(z) \;=\; -4c\, h_k^{\hat{\jmath}}(z) + c_k\,, \qquad \text{for} \qquad |z| \in [z_{\hat{\jmath}}, z_{\hat{\jmath}+1}] \tag{2.126}$$

$$h_k^{\hat{\jmath}}(z) \;=\; \pi\rho\left(\sum_{\hat{\imath}=0}^{\hat{\jmath}} \alpha_k^{\hat{\imath}}(|z| - z_{\hat{\imath}}) - \frac{1}{2}\sum_{\hat{\imath}=0}^{N+1} \alpha_k^{\hat{\imath}}(z_{\hat{\imath}}^2 - 2z_{\hat{\imath}})\right) \tag{2.127}$$

where the shift, i.e the second term in the last line, has been chosen such that

$$\langle h_k - c_k\rangle_{11} = \frac{1}{2}\int_{-1}^{1} dz(h_k(z) - c_k) = 0\,. \tag{2.128}$$

This choice will simplify and clarify the further reduction to four dimensions considerably, as it did not happen in ref. [137].

Now in order to solve equation (2.123) to first non-trivial order we use the Ansatz

$$f^k(z) = A^k(z) + B^k\,, \tag{2.129}$$

with the function $A^k(z)$ given by

$$A^k(z) = \sum_{\hat{\imath}=0}^{\hat{\jmath}} A^k_{\hat{\imath}} \left(|z| - z_{\hat{\imath}}\right) - \frac{1}{2}\sum_{\hat{\imath}=0}^{N+1} A^k_{\hat{\imath}} \left(z^2_{\hat{\imath}} - 2z_{\hat{\imath}}\right) , \quad \text{for} \quad |z| \in [z_{\hat{\jmath}}, z_{\hat{\jmath}+1}] ,$$

where $A^k_{\hat{\imath}}$ and B^k are constants to be determined. Inserting this Ansatz in eqs. (2.123), (2.126) and (2.127) defines these constants in terms of the charges and the integration constants like

$$c_i = d_{ijk} B^j B^k , \qquad -2c\pi\rho\, \alpha^{\hat{\imath}}_i = d_{ijk} A^j_{\hat{\imath}} B^k . \tag{2.130}$$

Now it is more natural to change to a different set of integration constants that will become geometrically easily identifiable low energy fields in the four-dimensional effective theory. Instead of c, c_k, $\tilde{c}$ we change to the constants V_0, R_0 and b^k_0 which are defined such that for vanishing charges $\alpha^{\hat{\imath}}_k \to 0$ they emerge from the five-dimensional fields like

$$V \to V_0, \qquad a \to R_0^{-1/2}, \qquad b \to R_0, \qquad b^k \to b^k_0. \tag{2.131}$$

Using the Ansatz (2.129) and the relations (2.130) in the solution (2.125), the relations between the two sets of integration constants can easily be found. Defining $B = \frac{1}{6} d_{ijk} B^i B^j B^k$ they are given by

$$B^2 = V_0 , \quad B^i = V_0^{1/6} b^i_0 , \quad \tilde{c} = V_0^{-1/6} R_0^{-1/2} , \quad c = \frac{R_0}{V_0^{2/3}} , \quad c_k = 2V_0^{1/3} b_{0k} . \tag{2.132}$$

These relations together with the fact that $2b_i = d_{ijk} b^j b^k$ allow us to explicitly solve the second equation in (2.130) for the constants $A^k_{\hat{\imath}}$ and thus to obtain $f^k(z)$, that is

$$\begin{aligned} A^k_{\hat{\imath}} &= 2\frac{R_0}{V_0^{5/6}} \pi\rho \left[\alpha^{\hat{\imath}\,k} - \frac{1}{2} b^k_0 (\alpha^{\hat{\imath}}_l b^l_0)\right] \\ \Rightarrow \quad f^k(z) &= V_0^{1/6} b^k_0 + 2\frac{R_0}{V_0^{5/6}} \left[h^{\hat{\imath},k}(z) - \frac{1}{2} b^k_0 \left(b^l_0 h^{\hat{\imath}}_l(z)\right)\right] . \end{aligned} \tag{2.133}$$

So finally we are in the position to give the explicit solution by inserting eqs. (2.132) and (2.133) into the implicit solution (2.125) and expand to first order in the charges. In the interval $z_{\hat{\jmath}} \le |z| \le z_{\hat{\jmath}+1}$ the solution then explicitly reads

$$\begin{aligned} V(z) &= V_0 \left[1 - 2\frac{R_0}{V_0}\, b^k_0\, h^{\hat{\jmath}}_k(z)\right] \\ a(z) &= R_0^{-1/2} \left[1 - \frac{R_0}{3V_0}\, b^k_0\, h^{\hat{\jmath}}_k(z)\right] \\ b(z) &= R_0 \left[1 - \frac{4R_0}{3V_0}\, b^k_0\, h^{\hat{\jmath}}_k(z)\right] \\ b^k(z) &= b^k_0 + 2\frac{R_0}{V_0} \left[h^{\hat{\jmath},k}(z) - \frac{1}{3} b^k_0 \left(b^l_0 h^{\hat{\jmath}}_l(z)\right)\right] , \end{aligned}$$

where the function $h_k^{\hat{\jmath}}(z)$ is defined in (2.127). The first thing to note is that this solution actually is an expansion to linear order in the strong coupling expansion parameter

$$\epsilon = \epsilon_S \frac{R_0}{V_0}.$$

Furthermore, these solution show the desired behavior (2.131) for vanishing charges. Finally, from eq. (2.128) it follows that

$$\langle V\rangle_{11} = V_0\,, \quad \langle a\rangle_{11} = R_0^{-1/2}\,, \quad \langle b\rangle_{11} = R_0\,, \quad \langle b^k\rangle_{11} = b_0^k\,,$$

which means that the orbifold averages of the above ϵ-corrections to the solution vanish. In the following section these averaged quantities will be identified with the four-dimensional metric moduli fields, thus in turns out that the bulk fields do not get any corrections to linear order in ϵ from the warping of the fifth dimension.

2.7 The four-dimensional effective theory

The above multi-domain-wall vacuum state is associated to an $\mathcal{N}=1$ effective four-dimensional theory describing fluctuations around this state. We would now like to compute some aspects of this four-dimensional theory.

2.7.1 The action in component form

The bosonic moduli fields in four dimensions from the bulk can only arise from five-dimensional fields that are even across the orbifold because for odd fields no constant ansatz can be made as is necessary for a consistent truncation. So the moduli fields from the bulk are the four-dimensional metric $g_{\mu\nu}$, the Calabi-Yau volume V_0, the orbifold size R_0, the axion $\chi = \mathcal{A}_5$, the complex structure moduli $\mathfrak{z}^p$ and the two-form $B_{\mu\nu} = C_{11\mu\nu}$. The latter can, in four dimensions, be dualized to a scalar σ, and by the usual procedure we find

$$dB = -V_0^{-2}\left[2\sum_{\hat{\imath}=1}^{N} q_k^{\hat{\imath}} z_{\hat{\imath}} * d(n_{\hat{\imath}}^k s) + \frac{1}{2} * d\sigma\right]. \tag{2.134}$$

The $z_{\hat{\imath}}$-dependent terms stem from the WZ-contributions in the brane actions, and thus are not present in the treatement without five-branes [108]. From the three-brane world-volumes, we have the N scalars $z_{\hat{\imath}} = z_{\hat{\imath}}(x^\mu) \in [0,1]$ specifying the position of the three branes and the axions $\nu_{\hat{\imath}} = s_{\hat{\imath}}/\pi\rho$ together with the $\sum_{\hat{\imath}=0}^{N} g_{\hat{\imath}}$ Abelian gauge fields $D_{\hat{\imath}}^u$ with field strengths $E_{\hat{\imath}}^u$, where $u, v, w, \ldots = 1, \ldots, g_{\hat{\imath}}$. Recall that $g_{\hat{\imath}}$ is the genus of the curve $\mathcal{C}_2^{\hat{\imath}}$ within the Calabi-Yau space which is wrapped by the five-brane $M_6^{\hat{\imath}}$. From the boundary fields we get the two gauge fields A_k with field strengths F_k, where $k = 1, 2$, that fall into $\mathcal{N} = 1$ gauge vector multiplets. Moreover there are the chiral gauge matter multiplets with the scalar fields C^{Ip}.

A straightforward but tedious reduction of the action (2.103) leads to

$$S_4 = S_{4,\,\text{gravity}} + S_{4,\,\text{scalar}} + S_{4,\,\text{gauge}} + S_{4,\,\text{matter}} \tag{2.135}$$

where the gravity and scalar parts read

$$\begin{aligned}
S_{4,\,\text{gravity}} &= \frac{-1}{2\kappa_P^2}\int_{M_4} d^4x\sqrt{-g}\,\frac{1}{2}R \\
S_{4,\,\text{scalar}} &= \frac{-1}{2\kappa_P^2}\int_{M_4} d^4x\sqrt{-g}\Bigg\{\frac{3}{4}\frac{(\partial R_0)^2}{R_0^2} + \frac{1}{4}\frac{(\partial V_0)^2}{V_0^2} + \frac{1}{4}\frac{(\partial\sigma)^2}{V_0^2} + \frac{1}{4}G_{kl}\,\partial_\mu b_0^k\,\partial^\mu b_0^l \\
&\qquad + \frac{1}{4}\mathcal{K}_{p\bar{q}}\partial_\alpha \mathfrak{z}^p \partial^\alpha \bar{\mathfrak{z}}^{\bar{q}} + G_{kl}\frac{\partial_\mu\chi^k\,\partial^\mu\chi^l}{R_0^2} + \frac{1}{2}\sum_{\hat{\imath}=1}^{N} q_k^{\hat{\imath}} b_0^k\,\frac{R_0}{V_0}(\partial z_{\hat{\imath}})^2 \\
&\qquad + 2\sum_{\hat{\imath}=1}^{N}\frac{q_k^{\hat{\imath}}}{V_0^2}\,z_{\hat{\imath}}\,\partial_\mu(n_{\hat{\imath}}^k\nu_{\hat{\imath}})\partial^\mu\sigma + 4\sum_{\hat{\imath},\hat{\jmath}=1}^{N}\frac{q_k^{\hat{\imath}}q_l^{\hat{\jmath}}}{V_0^2}\,z_{\hat{\imath}}z_{\hat{\jmath}}\,\partial_\mu(n_{\hat{\imath}}^k\nu_{\hat{\imath}})\partial^\mu(n_{\hat{\jmath}}^l\nu_{\hat{\jmath}}) \\
&\qquad + 2\sum_{\hat{\imath}=1}^{N}\frac{q_k^{\hat{\imath}}q_l^{\hat{\imath}}}{(q_m^{\hat{\imath}}b_0^m)\,V_0R_0}\Big(\partial_\mu(n_{\hat{\imath}}^k\nu_{\hat{\imath}})\partial^\mu(n_{\hat{\imath}}^l\nu_{\hat{\imath}}) + \chi^k\chi^l(\partial z_{\hat{\imath}})^2 \\
&\qquad\qquad - 2\chi^k\partial_\mu(n_{\hat{\imath}}^l\nu_{\hat{\imath}})\partial^\mu z_{\hat{\imath}}\Big)\Big]\Bigg\}
\end{aligned}$$

and the Lagrangians for the gauge fields from boundaries or branes together with the chiral matter are given by

$$\begin{aligned}
S_{4,\,\text{gauge}} &= \frac{-1}{4g_{GUT}^2}\int_{M_4} d^4x\sqrt{-g}\Bigg\{\Big(V_0 + R_0 b_0^k\sum_{\hat{\imath}=0}^{N+1} q_k^{\hat{\imath}}(z_{\hat{\imath}}^2 - 2z_{\hat{\imath}})\Big)\mathrm{tr}F_{1\,\mu\nu}F_1^{\mu\nu} \\
&\qquad + \Big(V_0 + R_0 b_0^k\sum_{\hat{\imath}=0}^{N+1} q_k^{\hat{\imath}} z_{\hat{\imath}}^2\Big)\mathrm{tr}F_{2\,\mu\nu}F_2^{\mu\nu}\Bigg\} \\
&\quad - \int_{M_4} d^4x\sqrt{-g}\Bigg\{\sum_{\hat{\imath}=1}^{N}\frac{1}{4g_{\hat{\imath}}^2}\Big([\mathrm{Im}(\Pi)]_{\hat{\imath}uw}E^u_{\hat{\imath}\mu\nu}E^{w\mu\nu}_{\hat{\imath}} \\
&\qquad - \frac{1}{2}\epsilon_{\hat{\imath}}^{\mu\nu\rho\sigma}[\mathrm{Re}(\Pi)]_{\hat{\imath}uv}E^u_{\hat{\imath}\mu\nu}E^v_{\hat{\imath}\rho\sigma}\Big)\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
S_{4,\,\mathrm{matter}} \;=\; & -\sum_{R_{\mathcal{I}_1}} \int_{M_4} d^4x\sqrt{-g}\frac{1}{R_0}\Bigg\{ G^{(m)}_{LM} \\
& + \frac{R_0}{V_0}\left[\frac{1}{3}G^{(m)}_{LM}b_0^k - \frac{\partial}{\partial b_0^l}G^{(m)}_{LM}(G^{lk}(b_0) - \frac{1}{3}b_0^l b_0^k)\right]\sum_{\hat\imath=0}^{N} q_k^{\hat\imath}(1-z_{\hat\imath})^2\Bigg\} \\
& \qquad\qquad\qquad \times \left(D_\mu C^{Lp}(R_{\mathcal{I}_1})D^\mu \bar{C}^M_p(R_{\mathcal{I}_1})\right) \\
& -\sum_{R_{\mathcal{I}_2}} \int_{M_4} d^4x\sqrt{-g}\frac{1}{R_0}\Bigg\{ G^{(m)}_{LM} \\
& + \frac{R_0}{V_0}\left[\frac{1}{3}G^{(m)}_{LM}b_0^k - \frac{\partial}{\partial b_0^l}G^{(m)}_{LM}(G^{lk}(b_0) - \frac{1}{3}b_0^l b_0^k)\right]\sum_{\hat\imath=0}^{N+1} q_k^{\hat\imath} z_{\hat\imath}^2\Bigg\} \\
& \qquad\qquad\qquad \times \left(D_\mu C^{Lp}(R_{\mathcal{I}_2})D^\mu \bar{C}^M_p(R_{\mathcal{I}_2})\right)
\end{aligned}
$$

with

$$
\kappa_P^2 = \frac{\kappa_5^2}{2\pi\rho}\,, \qquad q_k^{\hat\imath} = \pi\rho\,\alpha_k^{\hat\imath}
$$

$$
4g_{GUT}^2 = 16\pi\alpha_{GUT}\,, \quad 4g_{\hat\imath}^2 = \frac{2\kappa_5^2}{\alpha_k^{\hat\imath} n_{\hat\imath}^k}\,.
$$

As shown next, this lengthy component action can indeed be brought into the nice form of four-dimensional $\mathcal{N} = 1$ supergravity.

2.7.2 Superfield formulation

A four-dimensional $\mathcal{N} = 1$ supergravity action with chiral superfields Φ_i is completely specified by three functions of the superfields. These functions are the holomorphic superpotential $W(\Phi)$, the holomorphic gauge kinetic function (matrix) $f(\Phi)$ and the Kähler potential $K(\Phi, \bar\Phi)$. The Kähler potential and the gauge kinetic functions determine the kinetic terms of the scalar and the gauge fields, respectively. From the superpotential the scalar potential follows.

The scalar fields in the action S_4 fit into chiral multiplets, first there are the complex scalars from the complex structure moduli $\mathfrak{z}^p$ and the chiral gauge matter C^I, both directly corresponding to the bosonic part of the expansion of the associated superfields. The other real scalars have to be paired into components of superfields. These superfields are the dilaton S, the T^i-moduli and the orbifold position moduli $Z_{\hat\imath}$ of the five-branes. The bosonic parts of their expansions in terms of component fields is given by

$$
\begin{aligned}
S &= V_0 + R_0 \sum_{\hat{\imath}=1}^{N} (q_k^{\hat{\imath}} b_0^k) z_{\hat{\imath}}^2 + i(\sigma + 2\sum_{\hat{\imath}=1}^{N} q_k^{\hat{\imath}} \chi^k z_{\hat{\imath}}^2) = V + i\sigma + \sum_{\hat{\imath}=1}^{N} z_{\hat{\imath}}^2 q_k^{\hat{\imath}} T^k \\
T^k &= R_0 b_0^k + 2i\chi^k \qquad (2.136)\\
Z_{\hat{\imath}} &= q_k^{\hat{\imath}} b_0^k R_0 z_{\hat{\imath}} + 2i q_k^{\hat{\imath}} (-n_{\hat{\imath}}^k \nu + \chi^k z_{\hat{\imath}}) = z_{\hat{\imath}} q_k^{\hat{\imath}} T^k - 2i\, q_k^{\hat{\imath}} n_{\hat{\imath}}^k \nu \,.
\end{aligned}
$$

The Kähler potential leading to the action $S_{4,\text{scalar}}$ is given by

$$\kappa_P^2\, K_{\text{scalar}} = K_D(S, T^k, Z_{\hat{\imath}}) + K_T(T^k) + \mathcal{K}(\mathfrak{z}) \qquad (2.137)$$

where

$$
\begin{aligned}
K_D &= -\ln\left[S + \bar{S} - \sum_{\hat{\imath}=1}^{N} \frac{(Z_{\hat{\imath}} + \bar{Z}_{\hat{\imath}})^2}{q_k^{\hat{\imath}}(T^k + \bar{T}^k)}\right], \qquad (2.138)\\
K_T &= -\ln\left[\frac{1}{48} d_{klm}(T^k + \bar{T}^k)(T^l + \bar{T}^l)(T^m + \bar{T}^m)\right],\\
\mathcal{K}(\mathfrak{z}) &= -\ln\left[2i(\mathcal{G} - \bar{\mathcal{G}}) - i(\mathfrak{z}^p - \bar{\mathfrak{z}}^p)\left(\frac{\partial \mathcal{G}}{\partial \mathfrak{z}^p} + \frac{\partial \bar{\mathcal{G}}}{\partial \bar{\mathfrak{z}}^p}\right)\right].
\end{aligned}
$$

The Kählerpotential K_D is a generalization of the result found in ref. [112, 114], to include several five-branes and an arbitrary number of Kähler moduli. K_T and $\mathcal{K}(\mathfrak{z})$ are unchanged to previous results [110, 108], and we note that $\mathcal{K}$ has already been defined in (2.91).

Let us next turn to the fields from the boundary theories. The action $S_{4,\text{matter}}$ can be obtained from the following Kähler potential

$$
\begin{aligned}
K_{\text{matter}} &= e^{K_T/3} \sum_{i=1}^{2} \sum_{R_{\mathcal{I}_i}} \left(G_{LM}^{(m)} + \tilde{\Gamma}_{LM}^k \mathcal{S}_k^i\right) C^{Lp} \bar{C}_p^M\\
\tilde{\Gamma}_{LM}^k &= \frac{2}{S + \bar{S}} \left(\frac{1}{6} G_{LM}^{(m)} (T^k + \bar{T}^k) - 2\Gamma_{LM}^k + \frac{1}{6} \Gamma_{l\,LM} (T^l + \bar{T}^l)(T^k + \bar{T}^k)\right)\\
\mathcal{S}_k^1 &= \sum_{\hat{\imath}=0}^{N} q_k^{\hat{\imath}} \left(1 - \frac{Z_{\hat{\imath}} + \bar{Z}_{\hat{\imath}}}{q_l^{\hat{\imath}}(T^l + \bar{T}^l)}\right)^2\\
\mathcal{S}_k^2 &= \sum_{\hat{\imath}=0}^{N+1} q_k^{\hat{\imath}} \left(\frac{Z_{\hat{\imath}} + \bar{Z}_{\hat{\imath}}}{q_l^{\hat{\imath}}(T^l + \bar{T}^l)}\right)^2
\end{aligned}
$$

This corrects the superfield formulation of previous results [135]. The matter metric $G_{LM}^{(m)}$ was defined in eq. (2.94) and Γ_{LM}^k in eq. (2.95). There we also defined the

Yukawa couplings in eq. (2.96), and they are used in the superpotential that is given by the usual expression [51]

$$W = g_{GUT}\lambda_{LMN} f^{(123)}_{pqr} C^{Lp} C^{Mq} C^{Nr} \, .$$

Turning next to the gauge fields we must give the gauge kinetic functions. The gauge kinetic functions for the gauge fields A_1 and A_2 from the orbifold fixed planes turn out to be

$$\begin{aligned} f_1 &= S - q_k^{N+1} T^k - 2\sum_{\hat{\imath}=1}^{N} Z^{\hat{\imath}} \\ f_2 &= S + q_k^{N+1} T^k \, , \end{aligned} \tag{2.139}$$

such that their real parts in component form are

$$\mathrm{Re} f_1 = V_0 + R_0 b_0^k \sum_{\hat{\imath}=0}^{N} q_k^{\hat{\imath}} (1 - z_{\hat{\imath}})^2 \tag{2.140}$$

$$\mathrm{Re} f_2 = V_0 + R_0 b_0^k \sum_{\hat{\imath}=1}^{N+1} q_k^{\hat{\imath}} z_{\hat{\imath}}^2 \, , \tag{2.141}$$

which by some trivial reshuffling of terms and eq. (2.75) are equivalent to what can be read off from the gauge terms in the action $S_{4,\mathrm{gauge}}$. These component forms are in agreement with the results found in [109], and the new result is again the correct form of the gauge kinetic functions completely in terms of the superfields, as given in eq. (2.139). We note that the threshold terms proportional to T and Z arise as a direct consequence of the domain-wall structure. In fact, the correction is entirely due to the non-trivial orbifold dependence of the dilaton in eq. (2.122). As a consequence, no such threshold correction arises for the three-brane gauge fields, since their kinetic term in eq. (2.108) does not depend on the dilaton. After an appropriate rescaling of the fields $D^u_{\hat{\imath}}$ their gauge-kinetic function is simply proportional to the period matrix (2.4.2) of the holomorphic curve $\mathcal{C}^{\hat{\imath}}_2$, that is

$$f_{\hat{\imath}\, uv} = i\Pi_{\hat{\imath}\, uv} \, .$$

We note that the component form (2.136) of the superfields allows a direct interpretation of this Kähler potential and the resulting moduli field dynamics in five-dimensional terms. From the non-trivial structure of the domain-wall, it is possible to compute loop-corrections of order ϵ to the kinetic terms of the moduli. However, we did not succeed in finding the complex structure and the associated corrected Kähler potential when those corrections were included. It is conceivable that this computation is beyond the range of validity of the five-dimensional theory. This is supported by the observation that the Z–dependent part in the above Kähler potential (2.138) is already suppressed by $\epsilon \sim R/V$ relative to the S–dependent part. This suggests that corrections are already of order ϵ^2 and, therefore, beyond the linear level up to which the five-dimensional theory can generally be trusted.

2.8 Conclusions

To summarize and conclude we can say that we have explicitly found the four- as well as five-dimensional effective $\mathcal{N} = 1$ supergravity actions of heterotic M-theory explicitly including M5-brane world-volume theories. Especially the four-dimensional chiral superfields are given in terms of their component fields which have a direct five-dimensional origin. In four dimensions the fields from the higher dimensional bulk theory do not get corrections to linear order in the strong coupling expansion parameter ϵ from the non-trivial warped structure of the multi-domain-wall background. On the other hand, due to this warping of the fifth dimension, the chiral matter and boundary gauge field Lagrangians get corrections, which directly depend on the five-brane positions and thus on the associated modulus. The corresponding Kähler potential and gauge kinetic functions are given in terms of the superfields.

Chapter 3

Cosmology of heterotic M-theory in four and five dimensions

3.1 Introduction and conclusion

Among the simplest cosmological solutions of string theory are the so called rolling radii solutions [167]. They are characterized by a kinetic–energy driven evolution of the universe. These fundamental solutions of string cosmology provide superinflating cosmological backgrounds as well as expanding backgrounds of Friedmann–Robertson–Walker type [118, 168].

In this chapter, we will analyze and discuss rolling radii solutions from the perspective of the string theory/M–theory relation. In particular, we will discuss how a superinflating phase in string cosmology is embedded into a M–theory context. This will be done within the framework of the four–dimensional $\mathcal{N} = 1$ effective action of the $E_8 \times E_8$ heterotic string theory and the underlying five–dimensional effective action of heterotic M–theory [107, 108, 122], as were obtained in the last chapter from eleven–dimensional Hořava–Witten theory [95, 96] by reduction on a Calabi–Yau three–fold with non–vanishing G–flux. Though here, we will work with the simplest cosmologically reasonable truncations of actions (2.103) and (2.135), i.e. we do not take additional M5-branes in the bulk into account and consistently restrict the field content to the metric, the volume modulus and the orbifold radius. In this context, we will analyze rolling radii solutions and the role the fifth dimension plays in the cosmological evolution they describe. In particular, we will present a new class of non–separating five–dimensional solutions.

Cosmological rolling radii solutions of M–theory related to branes have first been obtained in refs. [169, 170, 171, 172, 173]. The first cosmological solutions of five–dimensional heterotic M–theory have been found in ref. [174, 175]. These latter solutions are generalized rolling radii solutions with an inhomogeneous fifth dimension.

Subsequently, further examples of cosmological solutions to five–dimensional heterotic M–theory have been presented [176, 177, 178, 179, 180]. One purpose of this treatement is to clarify the role of the solutions given in ref. [175] in the present context. Potential–driven inflation and its relation to five–dimensional heterotic M–theory has been first analyzed in ref. [181, 182, 183] . The presentation here is somewhat complementary to this work in that it addresses similar questions, however for the case of kinetic–energy driven inflation. There is also considerable activity, see for example [184, 185, 186, 187], exploring other cosmological aspects of five–dimensional brane–world theories. M–theory rolling radii cosmology based on vacua with a large number of supersymmetries has been investigated in ref. [188]. In the present chapter, we consider a related situation but focus on vacua with the "phenomenological" value of $\mathcal{N} = 1$ supersymmetry in four dimensions. While this situation is of course physically favorable, we have much less control over quantum effects than in the cases analyzed in ref. [188]. Here, we focus on the effect of string loop corrections in four dimensions while we work at lowest order in α'. Also, we will not attempt to include non–perturbative effects.

Let us outline this chapter and summarize its main results. To set the stage, we first briefly review the most important points on the relation between the four– and five–dimensional effective actions of heterotic M–theory, as was presented in the last chapter for much more general settings. In particular, we show how the relevant four–dimensional fields, that is, the four–dimensional metric g_4, the dilaton S_R and the T–modulus T_R arise as moduli of the five–dimensional three–brane vacuum solution. We also review the correspondence between excitations of the bulk fields in the fifth dimension and string loop corrections to the four–dimensional effective action. Correspondingly, the strong coupling expansion parameter $\epsilon \sim T_R/S_R$ can be interpreted as measuring the strength of those bulk excitations as well as the size of the loop corrections. Then, we start with the standard class of four–dimensional rolling radii solutions where we allow the scale factor of the three–dimensional universe, the dilaton and the T–modulus to vary in time. Discarding trivial integration constants, those solutions form a one–parameter set. We then show, using the correspondence between the four– and five–dimensional effective theories, how this complete set can be "lifted up" to approximate solutions of the five–dimensional effective action. Due to the potentials present in the five–dimensional theory, these solutions depend on the fifth coordinate as well as on time and are generically non–separating. They constitute new, non–trivial solution of the five–dimensional effective action of heterotic M–theory that generalize the familiar four–dimensional rolling radii solutions. More specifically, they correspond to a pair of domain wall three–branes with rolling radii. In addition to the overall scaling that is familiar from four–dimensional rolling radii solutions, there is another non–trivial feature of those solutions not visible from a four–dimensional viewpoint. The size of the domain wall bulk excitations (and hence the parameter ϵ) is generically varying in time. It is this time variation of the internal domain wall structure that makes the solutions non–separating and, hence, non–trivial.

We can classify the solutions according to the time–behavior of ϵ. It turns out that there are exactly two solutions (out of the one–parameter set) for which $\epsilon =$ const. In those two cases, one can find exact separable solutions which are precisely the ones that have been given in ref. [175]. For all other cases, ϵ varies in time and the corresponding exact solution must be non–separating. As a result, the separable solutions are exactly the ones for which loop corrections (or equivalently five–dimensional bulk excitations) are independent of time and are, hence, under control at all stages of the evolution. The remaining solutions with non–constant ϵ split into two (one–parameter) subsets, one with increasing ϵ and the other with decreasing ϵ in the negative–time branch. Particularly, the former case of increasing ϵ is interesting. In this case, an effectively four–dimensional solution is subject to increasing loop corrections that can be described by bulk excitations in the five–dimensional theory. When ϵ is of order one, the approximate five–dimensional solution is no longer valid and the subsequent evolution is described by a more complicated non–separating background. In particular, then, the time evolution and the dependence of the fields on the additional dimension are entangled in a complicated way. An interesting question is whether this might help to avoid the curvature singularity at the end of the negative–time branch. Unfortunately, no exact analytic solution is known to us in this non–separating case. However, we present an argument, based on the evolution equations for the induced fields on the boundaries, that a branch change does not occur, even at large values of ϵ.

3.2 Heterotic M-theory in four and five dimensions

A popular starting point for string cosmology is the (lowest order) four–dimensional effective action [118]

$$S_4 = -\frac{1}{16\pi G_N}\int_{M_4}\sqrt{-g_4}e^{-\phi_4}\left[\frac{1}{2}R_4 - \frac{1}{2}\partial_\mu\phi_4\partial^\mu\phi_4 + \frac{3}{4}\partial_\mu\beta_4\partial^\mu\beta_4\right] \tag{3.1}$$

written in the string frame. This action can be viewed as a universal effective action for $\mathcal{N}=1$ compactifications (on Calabi–Yau three–folds) of weakly coupled $E_8\times E_8$ heterotic string theory. In fact, the field content has been truncated to the fields essential for a discussion of string cosmology, that is, gravity and the two universal moduli ϕ_4 and β_4. In terms of the dilaton S and the conventional T–modulus T, we can express those fields as

$$S_R = e^{\phi_4}\ , \qquad T_R = e^{\beta_4}\ . \tag{3.2}$$

Here S_R and T_R denote the real parts of the bosonic component in the respective $\mathcal{N}=1$ superfields.

Given the origin of the above action, it should be possible to relate it to the strong coupling limit of the $E_8 \times E_8$ string theory [95, 96] in its effective formulation via five–dimensional heterotic M–theory [107, 108]. Let us, therefore, discuss the simplest version of this five–dimensional theory briefly. For more details see the previous chapter and ref. [107, 108, 181].

This theory is obtained from its eleven–dimensional counterpart by a reduction on a Calabi–Yau three–fold with a non–vanishing G–flux. Then the five–dimensional space–time has the structure $M_5 = S^1/Z_2 \times M_4$ where M_4 is a smooth 3+1 dimensional space–time. We will use coordinates x^α with indices $\alpha, \beta, \gamma, \dots = 0, 1, 2, 3, 5$ for the full five–dimensional space–time and coordinates x^μ with $\mu, \nu, \rho, \dots = 0, 1, 2, 3$ for M_4. Furthermore, the S^1 coordinate $y \equiv x^5$ is restricted to the range $y \in [-\pi\rho, \pi\rho]$ where ρ is the radius of the orbicircle. In these coordinates, the action of the Z_2 symmetry on S^1 is defined as $y \to -y$. This leads to two four–dimensional fixed planes M_4^1 and M_4^2 at $y = 0$ and $y = \pi\rho$, respectively. The theory on this space–time constitutes a five–dimensional $\mathcal{N}=1$ gauged supergravity theory in the bulk coupled to two four–dimensional $\mathcal{N}=1$ theories on M_4^1 and M_4^2. A simple version [181] of this theory is given by

$$
\begin{aligned}
S_5 = -\frac{1}{2\kappa_5^2}\Big\{ \int_{M_5} \sqrt{-g}\left[\frac{1}{2}R + \frac{1}{4}\partial_\alpha\phi\partial^\alpha\phi + \frac{1}{3}\alpha_b^2 e^{-2\phi}\right] \\
+ \int_{M_4^1}\sqrt{-g}\left(2\alpha_b e^{-\phi}\right) + \int_{M_4^2}\sqrt{-g}\left(-2\alpha_b e^{-\phi}\right)\Big\}\,. \qquad (3.3)
\end{aligned}
$$

This action corresponds to a truncated version of action (2.103) in the universal case ($h^{1,1} = 1$, $h^{2,1} = 0$) without additional M5-branes. In the same spirit as for the four–dimensional action (3.1) above, we have confined ourselves to the field content that is essential for our cosmological discussion. Note that the two boundary actions have opposite signs due to the condition (2.75) for the charge α_b. The field ϕ is the five–dimensional dilaton. Its geometrical interpretation is to measure the size of the internal Calabi–Yau and $V = e^\phi$. The above action constitutes an explicit realization of a five–dimensional brane–world in the context of M–theory, as was first realized in ref. [107].

How precisely are the effective actions (3.1) and (3.3) related? In this given setting this has first been worked out in ref. [107, 108]. As noted, it corresponds to a simplification of the case treated in the last chapter, thus we would like to briefly review the simplified form of those results that will become important in the following. First note that for $\alpha_b \neq 0$, the action (3.3) does not admit flat five–dimensional space–time as a solution. Instead, its "vacuum" is a pair of domain walls or three–branes specified by the solution [107]

$$ds_5^2 = a_0^2 H dx^\mu dx^\nu \eta_{\mu\nu} + b_0^2 H^4 dy^2 \ , \qquad e^\phi = b_0 H^3 \tag{3.4}$$

with harmonic function $H = H(y)$ given by

$$H = c_0 - \frac{2}{3}\epsilon_S h(y) \ , \qquad h(y) = \frac{|y|}{\pi\rho} - \frac{1}{2}, \tag{3.5}$$

where $\epsilon_S = \pi\rho\,\alpha_b$ and a_0, b_0 and c_0 are arbitrary constants[1]. This solution preserves $3+1$–dimensional Poincaré invariance and represents a BPS solution of the five–dimensional supergravity theory described by action (3.3). Hence, a reduction of the five–dimensional theory on this three–brane solution to four dimensions leads to a generally covariant $\mathcal{N} = 1$ supersymmetric theory. This theory is, of course, the four–dimensional effective action of the $E_8 \times E_8$ string whose universal part has been given in eq. (3.1) above, which thus corresponds to the appropriate truncation of action (2.135), written in the string frame. This four–dimensional theory provides an effective description for the moduli of the domain wall solution. To make this explicit, define constants β_4 and ϕ_4 by

$$b_0 = e^{3\beta_4 - 2\phi_4} \ , \qquad c_0 = e^{\phi_4 - \beta_4} \tag{3.6}$$

as well as a four–dimensional metric $g_{4\mu\nu}$ by

$$g_{4\mu\nu} = a_0^2 e^{2\phi_4}\eta_{\mu\nu} \ . \tag{3.7}$$

Note that we can perform a general linear transformation on the coordinates x^μ in the solution (3.4). This converts $\eta_{\mu\nu}$ and, hence, $g_{4\mu\nu}$ into an arbitrary four–dimensional metric. It follows that, in accord with section 2.6.2, to leading order in ϵ_S the solution takes the form

$$\begin{aligned} ds_5^2 &= \left(1 - \frac{2}{3}\epsilon h\right) e^{-\beta_4 - \phi_4} dx^\mu dx^\nu g_{4\mu\nu} + \left(1 - \frac{8}{3}\epsilon h\right) e^{2\beta_4} dy^2 \,, \\ \phi &= \phi_4 - 2\epsilon h \end{aligned} \tag{3.8}$$

where

$$\epsilon = \epsilon_S e^{\beta_4 - \phi_4} \ . \tag{3.9}$$

and $h = h(y)$ is as defined above. The metric $g_{4\mu\nu}$ can be interpreted as the four–dimensional string–frame metric. The moduli ϕ_4 and β_4 measure the internal Calabi–Yau volume $V_0 = e^{\phi_4}$ and the orbifold size $R_0 = e^{\beta_4}$, both averaged over the orbifold

[1] A suitable matching of these integrations constants shows that this solution is given as a special case of the solution given in section 2.6.2.

coordinate [2]. The metric and these moduli can now be promoted to four–dimensional fields depending on x^μ. As discussed in the last chapter and ref. [107, 108], the low–energy dynamics of these fields can be obtained by reducing the five–dimensional action (3.3) using the ansatz (3.9). The resulting dynamics is precisely described by action (3.1). Note that this action depends on all three moduli a_0, b_0 and c_0 of the exact three–brane solution. Modulus a_0 is related to the scale factor in the four–dimensional metric $g_{4\mu\nu}$, and using (3.2) and (3.6) we see that b_0 and c_0 enter the effective four–dimensional action through

$$S_R = b_0 c_0^3 = V_0 \ , \qquad T_R = b_0 c_0^2 = R_0 \ , \tag{3.10}$$

which is indeed a special case of (2.136). This establishes a direct relationship between five–dimensional solutions based on the three–brane (3.9) and solutions of the four–dimensional effective action (3.1) for the moduli. Hence, via eq. (3.9), any cosmological solution of the four–dimensional theory immediately implies an (approximate) cosmological solution in five–dimensions, and vice versa.

As is apparent from the four–dimensional action (3.1), we are working to lowest order in α'. Correspondingly, we have neglected higher derivative terms in the five–dimensional action (3.3) as well. For example, to lowest order we expect R^2 and R^4 terms in the bulk originating from the R^4 term in M–theory [189] as well as R^2 terms on the boundary [158]. Although it would be interesting to include those corrections, particularly from a five–dimensional viewpoint, we will focus on situations where higher–derivative corrections are still small.

There is another requirement for the four–dimensional effective description to be valid. We have used a linearized approximation in

$$\epsilon = \epsilon_0 e^{\beta_4 - \phi_4} \sim \frac{R_0}{V_0} \sim \frac{T_R}{S_R} \tag{3.11}$$

and, hence, ϵ should be smaller than one for the action (3.1) to be sensible. What is the meaning of this last condition? Eq. (3.11) leads us to three different interpretations of the so–called strong coupling expansion parameter ϵ. First, ϵ measures the excitation of bulk gravity in the domain wall solution (3.4) due to the bulk and boundary potentials in the five–dimensional action. That is, ϵ measures the variation of the metric (and the dilaton) as one moves across the orbifold. Second, it measures the (averaged) relative size R_0/V_0 of the orbifold and the internal Calabi–Yau space. And third, it measures the relative size of string–loop corrections to the four–dimensional effective action which is indeed proportional to T_R/S_R. The relation between loop corrections and bulk gravity is not accidental. One can verify that the one loop corrections to the four–dimensional effective action are in fact generated by the non–trivial structure

[2]Note in this context that the function $h(y)$ in eq. (3.5) has been defined so that its orbifold average vanishes.

of the domain wall solution [108]. Hence, the linear approximation in ϵ implies that we are considering a four–dimensional one–loop effective action or, equivalently, five–dimensional bulk excitations that are well approximated by linearized gravity. In the following, we will use the term "bulk excitations" to mean this non–trivial orbifold dependence induced by the potentials in the five–dimensional theory and related to loop corrections. Note that, due to the R^2 corrections on the boundaries mentioned above, higher–derivative corrections will also induce a non–trivial orbifold dependence whenever those corrections become relevant. It would be interesting to include those higher derivative terms, specifically the boundary R^2 terms, in the analysis. A related four–dimensional analysis with R^2 terms has been performed in ref. [190]. Higher curvature terms in five dimensions have been considered in ref. [191], however the boundary R^2 terms were not included in the analysis here. As stated above, here we confine ourselves to the lowest order in α'.

Does the five–dimensional action (3.3) and, correspondingly, its exact domain–wall solution (3.4) encode higher loop–effects as well? Certainly it contains information beyond the one–loop level, since the bulk potential in eq. (3.3) which is uniquely fixed by five–dimensional supersymmetry is of the order $\alpha_b^2 \sim \epsilon^2$. However, higher–order corrections to the five–dimensional action cannot be excluded. Therefore, while one expects higher loop corrections to be described by bulk gravity effects and an action of the type above, there most probably are modifications of the concrete form (3.3) at higher order.

3.3 Cosmological solutions

Based on the above correspondence between the four– and five–dimensional theories we would now like to discuss the simplest type of cosmological solutions, namely rolling radii solutions. These solutions are characterized by an evolution of the universe driven by kinetic energy and they provide superinflating as well as subluminally expanding cosmological string backgrounds [118]. First, we would like to review those solutions in the four–dimensional context. Then we present new five–dimensional solutions that constitute the generalization of rolling radii solutions to heterotic M–theory. Furthermore, we discuss their relation to the known four–dimensional solutions.

Let us first recall the conventional picture that arises in four dimensions. We choose a four–dimensional metric of Friedmann–Robertson–Walker type with flat spatial sections and scale factor $\alpha_4 = \alpha_4(t_4)$, that is,

$$ds_4^2 = g_{4\mu\nu} dx^\mu dx^\nu = -dt_4^2 + e^{2\alpha_4} d\mathbf{x}^2 \ . \tag{3.12}$$

Accordingly, the other two fields are taken to be functions of time only, that is, $\beta_4 = \beta_4(t_4)$ and $\phi_4 = \phi_4(t_4)$. Then the general solution of the four–dimensional action (3.1) is of the form

$$\alpha_4 = p_{4\alpha} \ln |t_4| + \bar{\alpha}_4 \ , \quad \beta_4 = p_{4\beta} \ln |t_4| + \bar{\beta}_4 \ , \quad \phi_4 = p_{4\phi} \ln |t_4| + \bar{\phi}_4 \ , \tag{3.13}$$

where $\bar{\alpha}_4$, $\bar{\phi}_4$ and $\bar{\beta}_4$ are arbitrary integration constants. The expansion powers $\mathbf{p}_4 \equiv (p_{4\alpha}, p_{4\beta}, p_{4\phi})$ are subject to the two constraints

$$3p_{4\alpha} - p_{4\phi} = 1 \ , \qquad 9p_{4\beta}^2 + 4p_{4\phi} + 2p_{4\phi}^2 = 4 \ . \tag{3.14}$$

Apart from trivial integration constants such as $\bar{\alpha}_4$, $\bar{\beta}_4$ and $\bar{\phi}_4$, we therefore have a one–parameter family of solutions specified by the solutions to eq. (3.14). Generically, the scale factor of the universe as well as both moduli fields evolve in time. As usual, for each set of allowed expansion coefficients, we have a solution in the negative–time branch, that is, for $t_4 < 0$ and a solution in the positive–time branch, that is for $t_4 > 0$. As stands, the former evolves into a future curvature singularity while the latter arises from a past curvature singularity. Frequently, for the discussion of superinflating cosmology, specific solutions are chosen from the set specified by (3.14). These specific solutions are characterized by a constant T–modulus and, hence, by the expansion coefficients

$$\mathbf{p}_4^{(T)} = \left(\pm \frac{1}{\sqrt{3}}, 0, \pm\sqrt{3} - 1 \right) \ . \tag{3.15}$$

As discussed in the previous section, the quantity ϵ, defined in eq. (3.11), is of particular importance in our context as it measures the size of the four–dimensional loop corrections as well as the five–dimensional gravitational bulk excitations. Going back to the general class of solutions, we have from eq. (3.11) and (3.13) that

$$\epsilon \sim |t_4|^{p_{4\beta} - p_{4\phi}} \ . \tag{3.16}$$

Generally, therefore, ϵ will be time–dependent. However, we can ask if there are special solutions in the above set for which $p_{4\beta} = p_{4\phi}$ and, hence, ϵ is constant. Such solutions indeed exist and are characterized by the expansion powers

$$\mathbf{p}_4^{(\epsilon)} = \left(\frac{3}{11} \left(1 \pm \frac{4}{3\sqrt{3}} \right), \frac{2}{11} \left(-1 \pm 2\sqrt{3} \right), \frac{2}{11} \left(-1 \pm 2\sqrt{3} \right) \right) \ . \tag{3.17}$$

While, in the following, we will work with the general set of solutions, we will comment on these special cases where appropriate. After this review of the four–dimensional solutions, let us now move on to the five–dimensional case.

Our goal is to specify the five–dimensional origin of the above rolling radii solutions. That is, we would like to find the solutions of the five–dimensional theory (3.3) that, in the small–momentum limit, reduce to the four–dimensional rolling radii solutions. From the action (3.3), it is clear that those solutions, in addition to time, must depend on the orbifold coordinate y, as long as the constant α_b is non–zero. In fact, while models with $\alpha_b = 0$ exist [192], generically α_b is non–vanishing. As a consequence, exact cosmological solutions of the action (3.3) are not easy to find. The first example has been given in ref. [175] using separation of variables and we will come back to this example later on. Some generalizations, also based on separation of variables, including those with curved three–dimensional spatial section have been presented subsequently in ref. [176]. Exact, non–separating solutions have been found for a related action set up to describe the somewhat different physical situation of potential–driven inflation within M–theory [181]. However, exact non–separating solutions for the action (3.3) are hard to find and not a single example is known to us. We will, therefore, content ourselves with giving approximate non–separable solutions. Such solutions can be obtained by "lifting up" the four–dimensional rolling radii solutions to five dimensions using the correspondence (3.9) between the four– and five–dimensional theories. Concretely, by inserting (3.12) and (3.13) into eq. (3.9), we obtain as the approximate solution of the five–dimensional action (3.3)

$$\begin{aligned} ds_5^2 &= (1-2\epsilon h/3)\left(-dt_5^2 + e^{2\alpha_5} d\mathbf{x}^2\right) + (1-8\epsilon h/3)e^{2\beta_5} dy^2 \ , \\ e^{\phi} &= e^{\phi_5}(1-2\epsilon h) \end{aligned} \tag{3.18}$$

where we have introduced the five–dimensional "comoving" time t_5 by

$$dt_5^2 = e^{-\beta_5 - \phi_5} dt_4^2 \ . \tag{3.19}$$

The five–dimensional scale factors α_5, β_5 and ϕ_5 show a power–law behavior

$$\alpha_5 = p_{5\alpha} \ln|t_5| + \bar{\alpha}_5 \ , \quad \beta_5 = p_{5\beta} \ln|t_5| + \bar{\beta}_5 \ , \quad \phi_5 = p_{5\phi} \ln|t_5| + \bar{\phi}_5 \ , \tag{3.20}$$

similar to the one in four dimensions. The expansion coefficients $\mathbf{p}_5 = (p_{5\alpha}, p_{5\beta}, p_{5\phi})$ are subject to the constraints

$$3p_{5\alpha} + p_{5\beta} = 1 \ , \qquad 8p_{5\beta}^2 - 4p_{5\beta} + 3p_{5\phi}^2 = 4 \tag{3.21}$$

and can be obtained from their four–dimensional counterparts using the relations

$$\begin{aligned} p_{5\alpha} &= \frac{2p_{4\alpha} - p_{4\beta} - p_{4\phi}}{2 - p_{4\beta} - p_{4\phi}} \\ p_{5\beta} &= \frac{2p_{4\beta}}{2 - p_{4\beta} - p_{4\phi}} \\ p_{5\phi} &= \frac{2p_{4\phi}}{2 - p_{4\beta} - p_{4\phi}} \ . \end{aligned} \tag{3.22}$$

We recall that the function $h = h(y)$ is defined by

$$h(y) = \frac{|y|}{\pi\rho} - \frac{1}{2} \, . \tag{3.23}$$

The all–important strong coupling expansion parameter ϵ, defined in eq. (3.11), is expressed in terms of five–dimensional quantities as

$$\epsilon = \epsilon_0 e^{\beta_5 - \phi_5} \sim |t_5|^{p_{5\beta} - p_{5\phi}} \, . \tag{3.24}$$

We have now found new approximate solutions of the five–dimensional theory that, via the relations (3.22) between the expansion coefficients, are in one–to–one correspondence with the four–dimensional rolling radii solutions given in (3.12), (3.13). Hence, as is the case for their four–dimensional counterparts, these five–dimensional solutions constitute a one–parameter set specified by the solutions to the constraints (3.21). While the lifting procedure from four dimensions makes it rather easy to obtain those solutions, they are quite non–trivial from a five–dimensional viewpoint. In particular, they are generically non–separating, that is, the time– and orbifold–dependence do not generically factorize. This can, for example, be seen from the function $e^{2\alpha_5}(1 - 2\epsilon h/3)$ that multiplies the three–dimensional spatial part of the metric (3.19). Here, the time–dependence resides in α_5 and ϵ while the orbifold–dependence is encoded in h. Hence, as long as ϵ does depend on time (which it generically does), the variables do not separate. From the discussion of the previous section, the approximation that led us to those solutions is valid as long as higher–derivative terms are negligible and, hence, the momenta $\dot{\alpha}_5$, $\dot{\beta}_5$ and $\dot{\phi}_5$ have to be sufficiently small. Furthermore, the expansion parameter ϵ has to be less than one. The above solutions are direct generalizations of the rolling radii solutions to five dimensions. Apart from α_5, β_5 and ϕ_5 that describe the overall scaling of the domain–wall configuration, there is also a less trivial dependence on time through the expansion parameter ϵ. This dependence implies that the size of transverse gravity excitations (the linear slope in y) varies with time as well.

Can the above approximate five–dimensional solutions be promoted to exact solutions of the action (3.3)? The simplest approach to finding such exact solutions is clearly separation of variables. In fact, in ref. [175] it was shown that the only separable solutions (assuming a flat three–dimensional spatial universe) for the action (3.3) are precisely of the form

$$\begin{aligned} ds_5^2 &= \left(1 - \frac{2}{3}\epsilon h\right)\left(-dt_5^2 + e^{2\alpha_5} d\mathbf{x}^2\right) + \left(1 - \frac{2}{3}\epsilon h\right)^4 e^{2\beta_5} dy^2 \, , \\ e^{\phi} &= \left(1 - \frac{2}{3}\epsilon h\right)^3 e^{\phi_5} \end{aligned} \tag{3.25}$$

where the scale factors α_5, β_5 and ϕ_5 evolve according to the general power law (3.20). However, for the above to be an exact solution the particular values

$$\mathbf{p}_5^{(\epsilon)} = \left(\frac{3}{11}\left(1 \mp \frac{4}{3\sqrt{3}}\right), \frac{2}{11}(1 \pm 2\sqrt{3}), \frac{2}{11}(1 \pm 2\sqrt{3}) \right) . \tag{3.26}$$

for the expansion coefficients must be chosen. These particular coefficients satisfy the constraints (3.21). Therefore, upon linearizing the exact solutions (3.26) in ϵ, we recover particular cases of our approximate solution (3.19). This implies that, from our one–parameter set of approximate solutions, exactly two can be promoted to exact separating solutions while all other exact solutions have to be non–separating. There is another way to characterize the two separating solutions. It can be verified, using the map (3.22) and (3.17), (3.26), that the separating solutions correspond to those four–dimensional solutions with constant strong–coupling expansion parameter. This can also be directly seen in five dimension using eq. (3.24) and the fact that $p_{5\beta} = p_{5\phi}$ for the coefficients (3.26). Hence, we have found that the exact separable solutions to our five–dimensional action are precisely those for which the strong–coupling expansion parameter is constant in time, that is

$$\epsilon = \text{const} . \tag{3.27}$$

Recalling the interpretation of ϵ from the previous section, those exact separable solutions can, therefore, be characterized as precisely the ones for which the ratio of Calabi–Yau and orbifold volumes is constant. Equivalently, they are precisely the ones for which string loop corrections or excitations of bulk gravity are constant in time. If, on the other hand, these quantities vary in time, the corresponding exact solution is non–separable. For this case, no exact explicit solution has been found yet and it may well be that this can only be achieved using numerical methods.

3.4 Role of the fifth dimension

We would now like to discuss the results obtained so far, particularly in view of a kinetic–energy driven phase of inflation and the role of the fifth dimension in such a context.

A solution of the usual problems of standard cosmology requires the scale factor $a = e^{\alpha}$ of the three–dimensional space to accelerate for some period in the early universe. Such a superluminal evolution is realized precisely if [118]

$$\text{sign}(\ddot{a}) = \text{sign}(\dot{a}) . \tag{3.28}$$

where the dot denotes the derivative with respect to comoving time. The condition (3.28) is frame–independent, as it should be. In particular, it can be used either in the four–dimensional string frame or the five–dimensional Einstein frame. Consequently, in writing a we have omitted the subscripts specifying the frame. In general, one expects inflating ($\dot{a} > 0$) as well as deflating ($\dot{a} < 0$) solutions of eq. (3.28), both of which are suited to solve the problems of standard cosmology [118]. In fact, the sign of $\dot{a}$, and hence the notion of expansion and contraction, is not frame–independent. If eq. (3.28) is not satisfied the evolution is decelerated or subluminal. As before there are two cases, namely decelerated expansion ($\dot{a} > 0$) and decelerated contraction ($\dot{a} < 0$).

For the solutions given in the previous section, it is easy to verify that the condition (3.28) is satisfied as long as one chooses the time to be negative. In other words, the complete one–parameter set of solutions leads to accelerated evolution in the negative–time branch. In the positive time branch, on the other hand, the condition (3.28) is never satisfied. The evolution is, therefore, always decelerated. Let us assume in the following discussion that $t_4 < 0$. A convenient way to represent the set of solutions is to plot their expansion powers. This has been done in fig. 3.1 using the coefficients $p_{4\phi}$ and $p_{4\beta}$ for the dilaton and the T–modulus in the four–dimensional string frame, subject to the second condition in (3.14). Let us now discuss the time

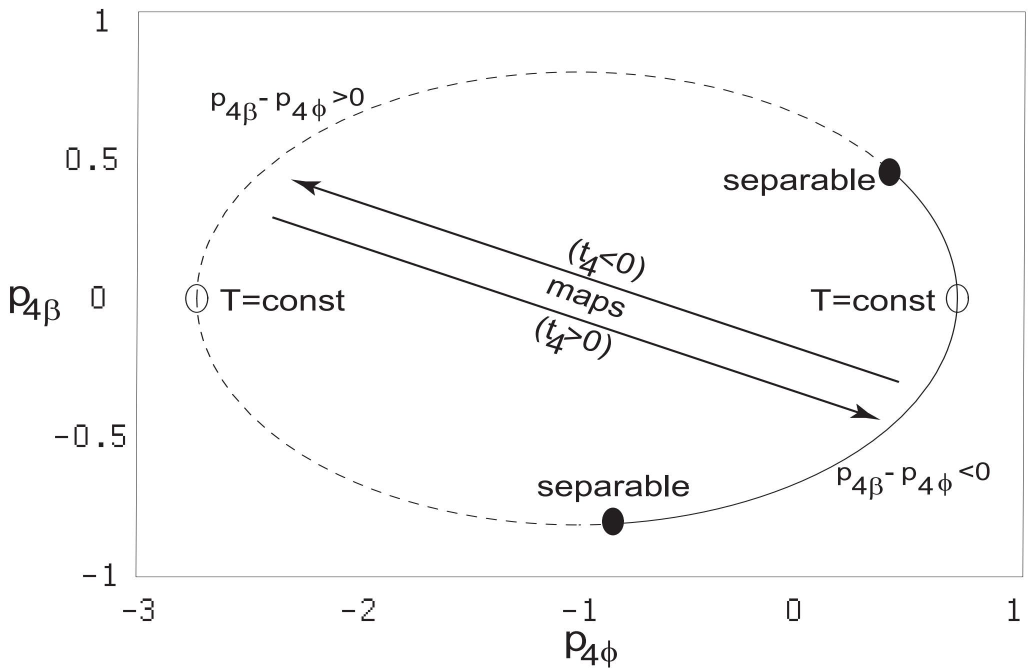

Figure 3.1: The allowed expansion coefficients $p_{4\beta}$, $p_{4\phi}$ w.r.t. the four–dimensional string frame for the rolling radii solutions. The solid dots correspond to the separable solutions, the circles to the solutions with constant T–modulus. The map is induced by the presence of a five–brane.

evolution for the solutions represented in fig. 3.1 in the negative–time branch. We start at $t_4 \to -\infty$, assuming an effective four–dimensional description at this time. All solutions will, of course, eventually develop large higher–derivative (α') corrections as $t_4 \to 0$. For example, the product of the "momenta" $\dot{\alpha}_4$, $\dot{\beta}_4$ and $\dot{\phi}_4$ times the orbifold size is proportional to $|t_4|^{p_{4\beta}-1}$. This increases as $t_4 \to 0$ since $|p_{4\beta}| < 1$ always. The precise time when the lowest order α' approximation is invalidated depends, of course, on initial conditions.

In section 2 we have discussed another sense in which the fifth dimension may become relevant. Namely, the parameter ϵ and, hence, the excitation of fields in the fifth dimension may become large. At the same time, this implies large loop corrections. As we have seen, $\epsilon \sim |t_4|^{p_{4\beta}-p_{4\phi}}$ and, therefore, its qualitative behavior depends on the sign of $p_{4\beta} - p_{4\phi}$. Consequently, unlike the higher–derivative (α') corrections discussed above, ϵ does not always increase in time. Instead, we should distinguish the three cases (for the negative–time branch)

- $p_{4\beta} - p_{4\phi} > 0$: Then ϵ decreases in time, indicating decreasing bulk excitations/loop corrections. The Calabi–Yau space expands faster than the orbifold. Solutions with this property are represented by the dashed line in fig. 3.1.

- $p_{4\beta} - p_{4\phi} = 0$: Then $\epsilon = \text{const}$, corresponding to constant bulk excitations/loop corrections. The Calabi–Yau space expands at the same rate as the orbifold. As discussed this case corresponds precisely to the two exact separable solutions that can be found. These solutions are indicated by the dots in fig. 3.1.

- $p_{4\beta} - p_{4\phi} < 0$: Then ϵ increases in time indicating increasing bulk excitations/loop corrections. The orbifold expands faster than the Calabi–Yau space. The corresponding solutions are represented by the solid line in fig. 3.1.

We see that bulk excitations in the fifth dimension are irrelevant in the first two cases, even as we approach the singularity at $t_4 \to 0$. Of course, the system will still run into a large curvature regime close to the singularity. We note that the "standard" solution with a constant T–modulus and inflation in the $D = 4$ string frame corresponds to the left circle in fig. 3.1. Hence, this solution falls into this category. The right circle, on the other hand, corresponds to a deflating solution in the $D = 4$ string frame and it falls into the third category.

In general, in this third case, bulk excitations become relevant close to the singularity. Whether that happens before or after the systems enters the large curvature regime depends on initial condition. Let us assume that we first enter a large ϵ regime while higher derivative corrections are still small. Then, while ϵ grows, our approximate five–dimensional solution (3.19) quickly becomes invalid. We know that the exact solutions that govern the further evolution have to be non–separating. Consequently, the time evolution and the excitation of bulk modes will be entangled in a complicated way. As we have discussed, we expect this to be described by a five–dimensional action of the type (3.3) possibly with additional higher order corrections. It would, therefore, be interesting to study exact non–separating cosmological solutions of the action (3.3) in the region of large ϵ. Unfortunately, analytic expressions for those solutions are not available and numerical methods might be required.

However, we may try to extract some information about the behavior at large ϵ by looking at the four–dimensional metrics that, for a given five–dimensional cosmological solution, are induced on the two boundaries. For example, it would be of interest to know whether or not a solution which is accelerated for small ϵ can, when ϵ is large, smoothly become decelerated. The answer, unfortunately, is negative, as we now demonstrate. Following ref. [181], let us write a five–dimensional solution in the general form

$$ds_5^2 = -e^{2\nu}dt_5^2 + e^{2\alpha}d\mathbf{x}^2 + e^{2\beta}dy^2 \ , \tag{3.29}$$

were ν, α and β are functions of t_5 and y. Furthermore, we take the dilaton ϕ to be a function of t_5 and y. The equations of motion for such an ansatz, following from the action (3.3), have been presented in ref. [181]. Particularly useful for the present purpose is the 55 component of the Einstein equation which reads explicitly

$$3e^{-2\nu}(\ddot{\alpha} - \dot{\nu}\dot{\alpha}^2) - 3e^{-2\beta}(\alpha'^2 + \nu'\alpha') = -\frac{1}{4}e^{-2\nu}\dot{\phi}^2 - \frac{1}{4}e^{-2\beta}\phi'^2 + \frac{1}{6}\alpha_b^2 e^{-2\phi} \ . \tag{3.30}$$

Here the dot (prime) denotes the derivative with respect to t_5 (y). Furthermore, working in the boundary picture, the functions in the above ansatz have to satisfy the following conditions [181]

$$e^{\phi-\beta}\nu' \mid_{y=y_i} = e^{\phi-\beta}\alpha' \mid_{y=y_i} = \frac{1}{3}\alpha_b \ , \qquad e^{\phi-\beta}\nu' \mid_{y=y_i} = 2\alpha_b \ , \tag{3.31}$$

at the first (second) boundary at $y_1 = 0$ ($y_2 = \pi\rho$). These conditions arise as a consequence of the Z_2 orbifolding and the boundary potentials in the five–dimensional action (3.3). Restricting eq. (3.30) to either one of the boundaries, and using the conditions (3.31), it can be shown that

$$\ddot{\alpha}_i - \dot{\nu}_i\dot{\alpha}_i + 2\dot{\alpha}_i^2 = -\frac{1}{12}\dot{\phi}_i^2 \ . \tag{3.32}$$

Here the subscript i denotes the value of the respective field at the boundary i, that is, for example $\alpha_i(t_5) = \alpha(t_5, y_i)$. We note that the various potential terms occurring in the Einstein equation and the boundary conditions (3.31) cancel in this relation. As a consequence, we have no unusual, linear relationship between the Hubble parameter and the boundary stress energy in eq. (3.31). The possibility of such unconventional relations has been first observed in ref. [181]. As a check, we can now verify that the relation (3.32) is satisfied by our approximate five–dimensional solutions. Putting eq. (3.19) in the form (3.29) and restricting to the boundaries, we can read off the following expressions

$$\alpha_i = p_{5\alpha} \ln |t_5| \mp \frac{1}{12}\epsilon \, , \qquad \phi_i = p_{5\phi} \ln |t_5| \mp \frac{1}{2}\epsilon \, , \qquad \nu_i = \mp \frac{1}{12}\epsilon \tag{3.33}$$

where the upper (lower) sign refers to the boundary $i = 1$ ($i = 2$). Here the expansion coefficient $p_{5\alpha}$, $p_{5\beta}$ and $p_{5\phi}$ satisfy the relations (3.21). Inserting these expressions and using (3.21), we can indeed verify that eq. (3.32) is satisfied to linear order in ϵ, as it should be. We can now go further and use the relations (3.32) to deduce properties of the solutions at arbitrary ϵ. In doing so we have to be careful, of course, since presumably not every set of fields $(\alpha_i, \nu_i, \phi_i)$ satisfying (3.32) can be extended to a full five–dimensional solution. However, conversely, every five–dimensional solution gives rise to induced fields on the boundaries that do satisfy eq. (3.32). It is this latter connection that we are going to use. We introduce the boundary Hubble parameters $H_i = \dot{\alpha}_i$ and choose the five–dimensional time coordinate t_5 such that it becomes comoving time upon restriction to the boundaries. This implies $\nu_i = 0$ and, hence, eq. (3.32) can be written in the form

$$\dot{H}_i = -\left(2H_i^2 + \frac{1}{12}\dot{\phi}_i^{\,2}\right) \, . \tag{3.34}$$

We conclude that $\dot{H}_i$ is always negative. Furthermore, the criterion (3.28) for accelerated evolution can be brought into the form $\mathrm{sign}(\dot{H}_i + H_i^2) = \mathrm{sign}(H_i)$. From eq. (3.34) we conclude that $\dot{H}_i + H_i^2 < 0$, always. Therefore, the evolution is accelerated exactly if $H_i < 0$. In this case, the boundaries deflate. On the other hand, for expanding boundaries, $H_i > 0$, the evolution must be decelerated. Hence, a five–dimensional solution which changes from acceleration to deceleration implies a transition from $H_i < 0$ to $H_i > 0$ for the boundary Hubble rates. This, however, cannot happen in a continuous manner since $\dot{H}_i < 0$. We conclude that a transition from acceleration to deceleration does not take place, even for large values of ϵ. We note, however, that the physically less interesting transition from deceleration to acceleration is not excluded from the above argument. In conclusion, we have shown that the solutions of our five–dimensional theory do not evolve from acceleration to deceleration. This results holds for arbitrarily large ϵ corrections but only to lowest order in α'. It is quite conceivable that the inclusion of higher order α' corrections can change this situation similarly to what happens in four dimensions [190, 193]. Some of those α' correction arise on the boundaries of the five–dimensional theory and, hence, lead to further bulk inhomogeneities. It would be interesting to generalize the present work by including those corrections.

3.5 Inclusion of five-branes

Cosmological scenarios with moving branes have recently received some attention [137, 138]. In the framework of heterotic M-theory the actions (2.103) and (2.135) provide the correct starting point to analyze such scenarios in five and in four dimensions, respectively. Four-dimensional solutions with moving branes have been obtained in ref. [115] and they provide generalizations to the solutions found in this chapter so far. Therefore, and to illustrated the utility of action (2.135), we next briefly state some results of ref. [115].

As can be seen from action (2.135), the generalization of action (3.1) to include a single brane, written in the Einstein frame, is given by

$$S = \frac{-1}{2\kappa_5^2}\int d^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{3}\partial_\mu\phi_4\partial^\mu\phi_4 + \frac{3}{4}\partial_\mu\beta_4\partial^\mu\beta_4 + \frac{q_5}{2}e^{\beta-\phi}\partial_\mu z\partial^\mu z\right], \tag{3.35}$$

where z is the modulus of the brane position along the orbifold. From this action an important implication can already be seen, due to the structure of the kinetic term for z, the fields ϕ and β cannot stay exactly constant once the five-brane moves. Since the dynamics of all three fields is linked, the evolution of z cannot be studied independently (as done in [137]). This situation is not changed by the presence of a non-perturbative potential.

Using the same ansatz as in section 3.3 together with $z = z(t_4)$ yields the following solutions in the Einstein frame [115]

$$\begin{aligned}
\alpha_4 &= \frac{1}{3}\ln\left|\frac{t_4}{T}\right| + \alpha_0 \\
\beta_4 &= p_{4\beta,i}\ln\left|\frac{t_4}{T}\right| + (p_{4\beta,f} - p_{4\beta,i})\ln\left(\left|\frac{t_4}{T}\right|^{\delta} + 1\right)^{-1/\delta} + \beta_0 \\
\phi_4 &= p_{4\phi,i}\ln\left|\frac{t_4}{T}\right| + (p_{4\phi,f} - p_{4\phi,i})\ln\left(\left|\frac{t_4}{T}\right|^{\delta} + 1\right)^{-1/\delta} + \phi_0 \\
z &= d\left(1 + \left|\frac{T}{t_4}\right|^{-\delta}\right)^{-1} + z_0
\end{aligned}$$

where d, α_0, β_0, z_0 and T are arbitrary integration constants. The parameter δ is defined by $\delta = p_{4\beta,i} - p_{4\phi,i}$, and it distinguishes the solutions with $\delta > 0$ in the negative-time branch ($t_4 < 0$) from those with $d < 0$ in the positive-time branch ($t_4 > 0$). Furthermore, the expansion powers are subject to the constraint $3p_{4\beta,n}^2 + p_{4\phi,n}^2 = \frac{4}{3}$ with $n = i, f$, and together with $p_{4\alpha} = 1/3$, these constraints exactly correspond to (3.14) but written with respect to the Einstein frame. From this we can see that asymptotically ($t_4 \to -\infty$, $t_4 \to 0$, $t_4 \to +\infty$) the given solutions exactly show the freely rolling radii behavior found in section 3.3, therefore the five-brane does not move asymptotically. The brane starts moving at times $|t_4| \sim |T|$ and interpolates between the early and late time asymptotic behavior characterized by the expansion coefficients $p_{4\beta,i}, \pi_{4\phi,i}$ and $p_{4\beta,f}, \pi_{4\phi,f}$, respectively. Therefore the presence of a five-brane leads to a transition between two different rolling radii solutions, as indicated with the map arrows in fig. 3.1. This map is actually given by [115]

$$p_{4\beta,f} = \frac{1}{2}(p_{4\beta,i} + p_{4\phi,i}), \qquad p_{4\phi,f} = \frac{3}{2}p_{4\beta,i} - \frac{1}{2}p_{4\phi,i},$$

which has the two solid dots in fig. 3.1, representing the separable solutions, as fixed points. Furthermore, the distinction between the solutions in the negative ($\delta > 0$) and the positive ($\delta < 0$) time branch implies that not all rolling radii solutions provide possible initial or final configurations. In the negative time branch, only the solid part of the ellipse is available as the early time behavior, whereas only the dashed part provides possible late time configurations. For the positive time branch it is exactly the other way round, such that the possible late time behavior of the negative time branch corresponds to the available initial configurations of the positive time branch.

Another interesting observation is that the strong coupling expansion parameter ϵ always grows asymptotically, which, as shown in section 3.4, need not be true for the pure rolling radii solutions. Since the effective action (3.35) is only valid when loop corrections and higher derivative terms are sufficiently small, this limits the validity of the asymptotical solutions. Another situation that goes beyond the validity of the given action is the collision of the five-brane with one of the boundaries, and this would lead to an instanton transition. If this happens or not depends on the initial conditions.

To summarize, we have seen that the presence of a five-brane changes the evolution of the system considerably. It generates a transition between asymptotical rolling radii solutions, but since the system also always evolves towards strong coupling asymptotically where the validity of the effective action (3.35) breaks down, the asymptotic behavior might not be trusted. Brane-collisions can happen but cannot consistently be described in the given framework.

Chapter 4

Flop transition in M-theory cosmology

4.1 Introduction

Space-like curvature singularities arising in cosmological solutions to low-energy string effective actions and their potential resolution constitute a challenging problem in string and M-theory. On the other hand, the string resolution of certain time-like singularities, such as those arising from collapsed cycles in the internal manifold, is, at least in principle, understood. In the course of a string/M-theory phase transition, triggered by cosmological evolution of moduli fields, these singularities may, in fact, arise at a particular instance in time. For example, a flop-transition [194, 84, 85] corresponds to a collapsing two-cycle in the internal Calabi-Yau space while a conifold transition [195, 196, 85] corresponds to a collapsing three-cycle. Clearly, such transitions are of interest for string and M-theory early universe cosmology. For example, one would like to know whether the topological transition can actually be realized dynamically, that is, whether the topology of the internal manifold could have indeed changed during the cosmological evolution. Further, one would like to understand, in this cosmological context, the role of the states which become light at the transition and how precisely the transition effects the evolution of the fields.

In this work, we will be answering these questions for the mildest form of topology change, namely the flop. A related discussion, but in the context of black-hole solutions, has been carried out in ref. [86, 197]. We will be working in the context of M-theory on Calabi-Yau three-folds leading to an effective description in terms of five-dimensional $\mathcal{N} = 1$ supergravity theories [198]– [165]. Flop-transitions arise from collapsing two-cycles within the Calabi-Yau manifold and are, therefore, controlled by the Kähler moduli which, together with $U(1)$ gauge fields, are contained in five-dimensional vector multiplets. Membranes wrapping Calabi-Yau two-cycles lead to hypermultiplet states in five dimensions with a mass proportional to the volume of the cycle. When the cycle collapses at the transition the hypermultiplet becomes massless and can no longer be ignored in the effective theory. In our analysis, we will include these hypermultiplet states explicitly into the five-dimensional effective action. In the following, we will also refer to these states as "transition states". In the case of M-theory on Calabi-Yau three-folds, there are no non-geometrical phases [84], that is, transitions are sharp. This implies that, going through the transition by first collapsing the cycle and then blowing it up in a topologically different way, leads to a another, topologically distinct Calabi-Yau space. While the Hodge numbers of the original and the "flopped" Calabi-Yau space are the same other topological quantities, such as the intersection numbers, change across the transition.

In terms of the five-dimensional effective supergravity theory, the transition can be described in a non-singular way once the additional hypermultiplet is included. For example, the jump in the intersection numbers which appear in the five-dimensional Chern-Simons term is accounted for by loop-corrections involving the hypermultiplet states [84] while the Kähler moduli space metric is continuous across the transition [86]. It turns out that the additional hypermultiplet is charged with respect to a particular linear combination of the vector multiplet gauge fields. Supersymmetry then implies the existence of a potential which depends on the transition states and the vector multiplet scalars. It is this potential which will play an important role in our cosmological analysis. Practically, we will, therefore, study time-evolution in Kähler moduli space close to the flop region including the effect of the transition states and their potential.

The plan of the following chapters is as follows. In the next section, we will review $\mathcal{N}=1$ supergravity in eleven and five dimensions and the five-dimensional effective action for M-theory on Calabi-Yau three-folds. With this machinery at hand, we then go on to derive the effective five-dimensional action of the transition states. Section 4.3 analyzes the cosmology of the five-dimensional theory for arbitrary Calabi-Yau spaces, first with vanishing and then with non-vanishing transition states. In section 4.4, we focus on a specific example of two Calabi-Yau spaces related by a flop and study the cosmological evolution numerically. We conclude in section 4.5.

4.2 The five-dimensional action of M-theory

To set the notation, we will first review $\mathcal{N}=1$ supergravity in eleven and five dimensions and the structure of the five-dimensional effective action for M-theory on Calabi-Yau three-folds. Subsequently, we will show how to couple to this action the hypermultiplet which contains the transition states. The five-dimensional effective action including this hypermultiplet will be the basis for the subsequent cosmological analysis.

4.2.1 Supergravity in eleven and five dimensions

The bosonic part of eleven-dimensional supergravity is given by [49]

$$S_{11} = \frac{-1}{2\kappa^2}\int_{M_{11}} \left\{ d^{11}x\,\sqrt{-g}\,\big(\frac{1}{2}R + \frac{1}{4!}G_{IJKL}G^{IJKL}\big) + \frac{2}{3}C \wedge G \wedge G \right\}, \tag{4.1}$$

where $G = dC$ is the field strength of the three-form potential C and κ is the 11-dimensional Newton constant, as usual. Indices $I, J, K, \cdots = 0, \ldots, 10$ label the 11-dimensional coordinates x^I. Later, we will also need the bosonic part of the membrane action [199, 200]

$$S_{M_3} = -T_2 \int_{M_3} \left\{ d^3\sigma\sqrt{-\gamma} + 2\hat{C} \right\} \tag{4.2}$$

which couples to C. The membrane world-volume is parametrized by coordinates σ^n, where $n, m, p, \cdots = 0, 1, 2$, and its embedding into 11-dimensional space-time is specified by $X^I = X^I(\sigma)$. The pull-backs γ_{nm} and $\hat{C}$ of the space-time metric and the three-form are defined by

$$\begin{aligned} \gamma_{nm} &= \partial_n X^I \partial_m X^J g_{IJ} \,, \\ \hat{C}_{nmp} &= \partial_n X^I \partial_m X^J \partial_p X^K C_{IJK} \,, \end{aligned}$$

as usual. In terms of the 11-dimensional Newton constant, the membrane tension T_2 is given by

$$T_2 = \left(\frac{8\pi}{\kappa^2}\right)^{\frac{1}{3}} .$$

Let us now move on to five-dimensional $\mathcal{N} = 1$ supergravity focusing on the aspects relevant to this work. For a more complete account we refer to the literature [198]–[165].

We denote five-dimensional space-time indices by $\alpha, \beta, \gamma, \cdots = 0, 1, 2, 3, 4$. In addition to the supergravity multiplet, consisting of the vielbein, an Abelian vector field and the gravitini, there are two types of matter multiplets, namely vector- and hyper-multiplets. In general, one can have any number, $n_{\rm V}$, of vector multiplets each containing a real scalar field and an Abelian vector field plus fermionic partners and any number, $n_{\rm H}$, of hypermultiplets each containing four real scalars plus fermions. It is useful to treat the Abelian gauge fields in the vector multiplets and the supergravity multiplet on the same footing and collectively denote them by A^i_α where $i, j, k, \cdots = 0, \ldots, n_{\rm V}$. The real scalars contained in the vector multiplets are described by $n_{\rm V} + 1$ fields b^i. These define a manifold of very special geometry [201] with metric

$$G_{ij} = -\partial_i \partial_j \ln K \;, \tag{4.3}$$

which is given in terms of the degree three homogeneous polynomial

$$K = d_{ijk} b^i b^j b^k \,. \tag{4.4}$$

Here d_{ijk} are constant coefficients. The b^i are subject to the constraint

$$K = 6 \tag{4.5}$$

which reduces the number of independent fields to $n_{\rm V}$, as required.

Further, we denote by Q^u, where $u, v, w, \dots = 1, \dots, 4n_{\rm H}$, the hypermultiplet scalars. They parametrize a quaternionic manifold, that is, a manifold with holonomy $SU(2) \times Sp(2n_{\rm H})$. The metric on this manifold, h_{uv}, is hermitian with respect to the three complex structures

$$J_u{}^v = J_u^{av} \tau_a \tag{4.6}$$

satisfying the quaternionic algebra

$$J^a_u{}^v J^b_v{}^w = -\delta^{ab} \delta_u{}^w + \epsilon^{abc} J^c_u{}^w \,, \tag{4.7}$$

where $a, b, c, \dots = 1, 2, 3$. Here τ_a are the hermitian Pauli matrices so that the complex structures fill out the adjoint of $SU(2)$. The associated triplet of Kähler forms is given by

$$\mathcal{K}_{uv} = J_u{}^w h_{wv} \,.$$

We also need to introduce the $SU(2)$ part $\omega_u = \omega_u^a \tau_a$ of the spin connection.

Let us assume that the metric h_{uv} admits $n_{\rm V} + 1$ Killing vectors k_i^u. These Killing vectors should respect the quaternionic structure which means they originate from prepotentials $\mathcal{P}_i = \mathcal{P}_i^a \tau_a$ via the relation

$$k_i^u \mathcal{K}_{uv} = \partial_v \mathcal{P}_i + [\omega_v, \mathcal{P}_i] \,. \tag{4.8}$$

With these conventions the bosonic part of the supergravity and vector multiplet action reads

$$\begin{aligned} S_{\rm V} \;=\; & \frac{-1}{2\kappa_5^2} \int_{M_5} \left\{ d^5x \sqrt{-g} \left(\frac{1}{2} R + \frac{1}{4} G_{kl} \partial_\alpha b^k \partial^\alpha b^l + \frac{1}{2} G_{kl} F^k_{\alpha\beta} F^{l\,\alpha\beta} \right) \right. \\ & \left. + \frac{2}{3} d_{klm} \, A^k \wedge F^l \wedge F^m \right\} , \end{aligned} \tag{4.9}$$

where κ_5 is the five-dimensional Newton constant. The bosonic part of the hypermultiplet action takes the form

$$S_{\rm H} = -\frac{1}{2\kappa_5^2} \int_{M_5} d^5x \sqrt{-g} \left\{ h_{uv} D_\alpha Q^u D^\alpha Q^v + V \right\} \,, \tag{4.10}$$

with the potential V given by

$$V = \frac{1}{2}g^2 \left[4(G^{ij} - b^i b^j)\mathrm{tr}(\mathcal{P}_i \mathcal{P}_j) + \frac{1}{2} b^i b^j h_{uv} k_i^u k_j^v \right] . \tag{4.11}$$

The trace in this expression is performed over the Pauli matrices. The covariant derivative D_α includes the gauging of the hypermultiplets with respect to the vector fields A^i and is defined by

$$D_\alpha Q^u = \partial_\alpha Q^u + g\, A^i k_i^u , \tag{4.12}$$

where g is the gauge coupling. Note that the appearance and the structure of the potential (4.11) is directly linked to this gauging of the hypermultiplets.

4.2.2 M-theory on Calabi-Yau three-folds

Let us now briefly review the reduction [202] of the action (4.1) for 11-dimensional supergravity on a Calabi-Yau three-fold X with Hodge numbers $h^{1,1}$ and $h^{2,1}$. This leads to a five-dimensional $\mathcal{N} = 1$ supergravity theory of the type described in the previous subsection with $n_{\mathrm{V}} = h^{1,1} - 1$ vector multiplets, $n_{\mathrm{H}} = h^{2,1} + 1$ hypermultiplets. In pure M-theory there is no gauging of the hypermultiplets, which is in contrast to the case of heterotic M-theory as treated in chapter 2.

We now need to specify the geometrical origin of some of these five-dimensional fields. The 11-dimensional metric on the direct product space $M_{11} = M_5 \times X$ can be written as

$$ds_{11}^2 = \mathcal{V}^{-2/3} g_{\alpha\beta} dx^\alpha dx^\beta + g_{AB} dx^A dx^B ,$$

where $A, B, C, \dots = 5, \dots, 10$ label the coordinates on the Calabi-Yau space, g_{AB} is the Ricci-flat metric on X and $g_{\alpha\beta}$ is the five-dimensional metric. The Calabi-Yau volume modulus $\mathcal{V}$ is defined by

$$\mathcal{V} = \frac{1}{v} \int_X d^6x \sqrt{g_6} ,$$

where v is an arbitrary six-dimensional reference volume. We note that the five-dimensional Newton constant κ_5 is related to its 11-dimensional counterpart κ by

$$\kappa_5^2 = \frac{\kappa^2}{v} .$$

The Kähler form ω_{AB} of X can be expanded as

$$\omega_{AB} = a^i \omega_{iAB} \tag{4.13}$$

into a basis $\{\omega_{iAB}\}$ of $(1,1)$ forms. We take this basis of $(1,1)$ forms to be dual to an (effective) basis $\{\mathcal{W}^i\}$ of the second homology, that is,

$$v^{-1/3}\int_{\mathcal{W}^i}\omega_j = \delta^i_j \; .$$

The $h^{1,1}$ expansion coefficients a^i are the Kähler moduli of the Calabi-Yau space. As stands, they are, of course, not independent from the volume modulus $\mathcal{V}$. However, we can define volume-independent moduli b^i by

$$b^i = \mathcal{V}^{-1/3}a^i \; .$$

These constitute $n_{\rm V} = h^{1,1} - 1$ independent fields as they can be shown to satisfy the constraint (4.5). They should be interpreted as the scalar fields in the vector multiplets. The coefficients d_{ijk} which appear in eq. (4.4) should then be identified as the intersection numbers of the Calabi-Yau space.

The three-form C can be expanded as

$$C = \bar{C} + A^i\wedge\omega_i + \xi\wedge\Omega + \bar{\xi}\wedge\bar{\Omega}\,, \tag{4.14}$$

where Ω is the holomorphic $(3,0)$ form on X. The $h^{1,1}$ five-dimensional vector fields A^i_α account for the gauge fields in the vector multiplets and in the gravity multiplet. The five-dimensional three-form $\bar{C}$ can be dualized to a scalar and forms, together with the complex scalar ξ and the volume modulus $\mathcal{V}$, the universal hypermultiplet. There are $h^{2,1}$ additional hypermultiplets which originate from the complex structure moduli and the $(2,1)$ part of C which we have omitted in eq. (4.14). These standard hypermultiplets will not be of particular importance, in the following.

Let us now review some relevant features of M-theory flop transitions following refs. [84, 86]. A flop constitutes a transition from the Calabi-Yau space X to a topologically different space $\tilde{X}$ due to a complex curve $\mathcal{C}$ in X shrinking to zero size and subsequently being blown up in a topologically distinct way. Concretely, let us expand the class $\mathcal{W}$ of the curve $\mathcal{C}$ in our homology basis as

$$\mathcal{W} = \beta_i\mathcal{W}^i$$

with constant coefficients β^i. The volume of $\mathcal{C}$ can then be written as

$$\mathrm{Vol}(\mathcal{C}) = \int_{\mathcal{C}}\omega = (v\mathcal{V})^{1/3}b$$

where we have introduced the particular linear combination

$$b = \beta_i b^i \tag{4.15}$$

of vector multiplet moduli. Within the moduli space of X we have $b^i > 0$, for all i, as well as $b > 0$ and the limit $b \to 0$ corresponds to approaching the flop. Continuing further to negative values of b leads into the moduli space of the birationally equivalent Calabi-Yau space $\tilde{X}$. This new Calabi-Yau space $\tilde{X}$ has the same Hodge numbers and, hence, the same five-dimensional low-energy spectrum as the original space X. However, the intersection numbers have changed across the transition [84]. More specifically, setting $(\beta_i) = (1, 0, \ldots, 0)$ for simplicity the new intersection numbers $\tilde{d}_{ijk}$ (expressed in terms of the field basis b^i) are given by

$$\tilde{d}_{111} = d_{111} - \frac{1}{6} \tag{4.16}$$

with all other components unchanged. Sometimes a new basis of fields $\tilde{b}^i$, defined by

$$\tilde{b}^1 = -b^1 \ , \qquad \tilde{b}^i = b^i - b^1 \ , \tag{4.17}$$

for all $i \neq 1$, is introduced [86] to cover the moduli space of $\tilde{X}$. These new fields have the advantage of being positive throughout the moduli space of $\tilde{X}$ which is not the case for the original fields b^i. For our applications we will find it usually more practical to use a single set of fields to cover the moduli spaces for both X and $\tilde{X}$.

How does the five-dimensional effective theory change across the transition? Inspection of the action (4.9), (4.10) without gauging and potential shows that only the vector multiplet part (4.9) is affected through the change (4.16) in the intersection numbers. From eq. (4.3), the metric G_{ij} takes the specific form

$$G_{ij} = -d_{ijk}b^k + \frac{1}{4}d_{ikl}d_{jmn}b^k b^l b^m b^n \ ,$$

where we have used that $K = 6$. This form shows that, despite the jump (4.16) in the intersection number the metric remains continuous across the flop since d_{111} is always multiplied by b^1 which vanishes at the transition. We remark that the associated connection

$$\Gamma^k_{ij} = \frac{1}{2}G^{kl}\frac{\partial G_{ij}}{\partial b^l} \tag{4.18}$$

which appears in the five-dimensional equations of motion contains a term proportional to d_{ijk} (without additional fields b^i) and, hence, jumps across the flop. Given the continuity of the metric G_{ij} the only discontinuous term in the action is the Chern-Simons term in eq. (4.9) which is proportional to the intersection numbers. It has been shown [84], that its jump can be accounted for by loop corrections which involve the transition states. Let us now discuss these additional states in more detail.

4.2.3 The transition states

The five-dimensions particles which become massless at the flop originate from a membrane which wraps the collapsing complex curve $\mathcal{C}$ with homology class $\mathcal{W}$, as

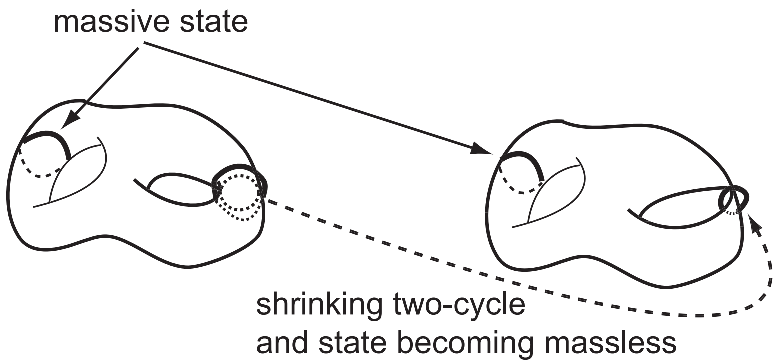

Figure 4.1: Membrane wrapping a collapsing cycle.

illustrated in fig 4.1. We can find the world-line action for these transition states by starting with the membrane action (4.2). Introducing a complex world-volume coordinate $\sigma = \sigma^1 + i\sigma^2$ and world-time $\tau = \sigma^0$ we consider an embedding of the membrane into 11-dimensional space of the form

$$X^\alpha = X^\alpha(\tau)\ , \qquad X^A = X^A(\sigma)\ , \qquad X^{\bar{A}} = X^{\bar{A}}(\bar{\sigma})\ ,$$

where here A and $\bar{A}$ are holomorphic and anti-holomorphic indices on the Calabi-Yau space, respectively, and $X^A = X^A(\sigma)$ parametrizes the complex curve $\mathcal{C}$. The reduction of the membrane action on this curve leads to the following world-line action

$$S_p = -(v^{1/3}T_2)\int_{\mathbb{R}}\left\{dt\,(\beta_i b^i)\sqrt{-\partial_\tau X^\alpha \partial_\tau X^\beta g_{\alpha\beta}} + 2\,\beta_i \hat{A}^i\right\}\ . \tag{4.19}$$

This particle has four transverse (scalar) degrees of freedom and must, hence, form a hypermultiplet in five dimensions. We denote the scalars in this hypermultiplet by q^u, where $u, v, w, \cdots = 1, 2, 3, 4$. It is charged with respect to the particular linear combination

$$\mathcal{A} \equiv \beta_i A^i\ , \tag{4.20}$$

of vector fields with associated gauge coupling

$$g = 2\,v^{1/3}T_2 = 2\left(\frac{8\pi}{\kappa_5^2}\right)^{1/3}\ , \tag{4.21}$$

as can be seen from the last term in (4.19). From the first term in the world-line action we can read off the mass which is given by

$$m = \frac{1}{2}gb = T_2\mathcal{V}^{-1/3}\mathrm{Vol}(\mathcal{C})\ . \tag{4.22}$$

What does this information tell us about the five-dimensional effective action of these transition states? Clearly, these states being hypermultiplets, their effective action must be of the general form (4.10). We assume that the associated hypermultiplet moduli space metric is flat, so that

$$h_{uv} = \delta_{uv} \,. \tag{4.23}$$

As we will see shortly, this assumption is consistent with the above properties of the transition states and the constraints enforced by five-dimensional supergravity. To work this out explicitly, let us first recall the quaternionic structure on the four-dimensional flat moduli space. Introducing the t'Hooft η-symbols [203]

$$\begin{aligned} \eta^a_{bc} = \ & \bar{\eta}^a_{bc} &= \epsilon^a_{\ bc} \,, \\ \eta^a_{b0} = \ & \bar{\eta}^a_{0b} &= \delta^a_b \,, \end{aligned}$$

which satisfy the properties

$$\begin{array}{rcl} [\eta^i, \bar{\eta}^j] & = & 0 \,, \\ (\eta^i)^T = (\eta^i)^{-1} \,, & & (\bar{\eta}^i)^T = (\bar{\eta}^i)^{-1} \,, \end{array} \tag{4.24}$$

the triplet of complex structures can be written as

$$J^{a\ v}_{\ u} \equiv -\bar{\eta}^a_{uw} \delta^{wv} \,,$$

which satisfy the quaternionic algebra (4.7), as required. The associated triplet of Kähler forms is given by

$$\mathcal{K}^a_{uv} = -\bar{\eta}^a_{uv}. \tag{4.25}$$

We know that the transition states are charged under the particular combination of gauge fields (4.20). Hence the vectors k^u_i must be proportional to β_i and, at the same time, be Killing vectors on flat four-dimensional space. We know that this gauging must lead to a potential of the form (4.11). As $q^u \to 0$ this potential must vanish so that the moduli b^i indeed parametrize flat directions in this limit. This implies that the Killing vectors k^u_i should not correspond to translations but rather to rotations and, hence, be of the form

$$k^u_i = \beta_i t^u{}_v q^v \,, \tag{4.26}$$

where t is an arbitrary anti-symmetric matrix. In addition, these Killing vectors must originate from a prepotential, that is, they must satisfy eq. (4.8). This is the case precisely if $[t, \bar{\eta}^a] = 0$ for $a = 1, 2, 3$, or, equivalently, if the matrix t is of the form

$$t_{uv} = n_a \eta^a_{uv} \,, \tag{4.27}$$

where n_a are real coefficients. This matrix represents the generator of $SO(2)$ in the representation $\mathbf{2} \oplus \mathbf{2}$. We require the standard normalisation $\mathrm{tr}(t^2) = -4$ or, equivalently, $n_a n^a = 1$. The associated prepotential then reads

$$\mathcal{P}_i^a = \frac{1}{2}\beta_i q^v (\bar{\eta}^a_{vw}\, n_b \eta^{b\,w}{}_u) q^u + \xi_i^a \,, \tag{4.28}$$

where ξ_i^a are arbitrary integration constant. They represent the generalisation of Fayet-Illiopoulos terms to five-dimensional $\mathcal{N} = 1$ supergravity. As they lead to terms in the potential which do not vanish for vanishing q^u we will set them to zero in the following.

Inserting (4.23), (4.26), (4.27) and (4.28) into the general hypermultiplet action (4.10) we obtain

$$S_q = -\frac{1}{2\kappa_5^2}\int_{M_5} d^5x\sqrt{-g}\,\{D_\alpha q^u D^\alpha q_u + V)\} \,, \tag{4.29}$$

with the potential

$$V = \frac{1}{4}g^2\left[b^2 q_u q^u + 4(G^{kl}\beta_k\beta_l - b^2)(q_u q^u)^2\right] \tag{4.30}$$

and the covariant derivative

$$D_\alpha q^u = \partial_\alpha q^u + g\mathcal{A} t_{uv} q^v \,. \tag{4.31}$$

Consequently, the hypermultiplet current j_α which couples to the gauge field $\mathcal{A}_\alpha$ is given by

$$j_\alpha = g q^u t_{uv}\partial_\alpha q^v \,. \tag{4.32}$$

We recall that $b = \beta_i b^i$, defined in eq. (4.15), is proportional to the volume of the collapsing cycle and the generator t has been given in eq. (4.27). So far, we have only used that the transition states are charged under the gauge field $\mathcal{A}$. Clearly, the gauge coupling g which appears in the covariant derivative (4.31) has to be identified with the value (4.21) obtained from the reduction of the membrane action. Then, the above hypermultiplet action (4.29) is completely fixed. From the first term in the potential (4.30) we can now read off the mass of the hypermultiplet which is given by $gb/2$. This value indeed coincides with the one obtained from the membrane reduction, eq. (4.22), as it should for consistency.

In addition, we have found a potential term quartic in the transition states which was not anticipated from the membrane reduction but was imposed on us by five-dimensional supergravity. This quartic term plays an important role in lifting "unwanted" flat directions. While the potential should be flat for vanishing transition states, $q^u = 0$, and arbitrary b^i, a flat direction along the flop at $b = 0$ and arbitrary

q^u would be a surprise [1]. Fortunately, this potential flat direction is lifted by the second term in eq. (4.30).

4.3 Cosmology

4.3.1 Cosmological ansatz and equations of motion

Let us briefly summarize the discussion so far. We have seen that M-theory on a Calabi-Yau three-fold X with Hodge numbers $h^{1,1}$ and $h^{2,1}$ is effectively described by the five-dimensional supergravity action (4.9), (4.10) with $n_{\rm V} = h^{1,1} - 1$ vector multiplets and $n_{\rm H} = h^{2,1} + 1$ hypermultiplets and no gauging. When a flop-transition to a topologically distinct Calabi-Yau space $\tilde{X}$ occurs the Hodge numbers and hence the number of massless particles remains the same while the structure of the five-dimensional action changes in accordance with the change (4.16) in the intersection numbers. In addition, at and near the flop-transition region another light hypermultiplet appears whose action (4.29) has to be added to the previous one for an accurate description across the transition. It is important not to confuse these transition hypermultiplet states which arises at the flop with the standard hypermultiplets associated with the complex structure moduli space of the Calabi-Yau space.

Which parts of this five-dimensional effective action are we actually interested in for our cosmological applications? Since we would like to study flop-transitions which arise by moving in the Calabi-Yau Kähler moduli space we should certainly consider the associated moduli fields, that is the vector multiplet scalars b^i. Clearly, we should also keep the transition states q^u which become light at the flop. However, these states are charged and generically source the vector fields. Hence, it seems we have to allow for non-trivial vector field backgrounds for consistency. Fortunately, we can avoid such a considerable complication by setting all scalars q^u equal to each other, that is, $q \equiv 2q^u$ for all $u = 1, 2, 3, 4$ and a single scalar q. This configuration is consistent with the q^u equations of motion, as can be seen from (4.30), and leads to a vanishing current (4.32). Consequently, the vector fields can be consistently set to zero in this case. The standard hypermultiplets, in fact, completely decouple from the other fields and are, hence, not essential for our purpose. From these standard hypermultiplet scalars we will only keep the dilaton $\mathcal{V} = e^{\phi}$ since it represents the overall volume of the internal Calabi-Yau space and is, therefore, of particular physical relevance.

In summary, the spectrum of our five-dimensional effective action can be consistently truncated to the five-dimensional metric, the $h^{1,1} - 1$ Kähler moduli space

[1] Such a flat direction with non-vanishing transition states would correspond to a Higgs branch where the gauge symmetry corresponding to the vector field $\mathcal{A}$ is broken. The resulting change in the number of light vector multiplets would be inconsistent with the fact that Hodge numbers are unchanged across the flop.

scalars b^i, the universal transition scalar q and the dilaton ϕ. From eq. (4.9) and eq. (4.29), the accordingly truncated effective action then reads

$$\begin{aligned} S_5 &= -\frac{1}{2\kappa_5^2}\int_{M_5} d^5x\sqrt{-g}\Big\{\frac{1}{2}R+\frac{1}{4}\partial_\alpha\phi\partial^\alpha\phi+\frac{1}{4}G_{kl}\partial_\alpha b^k\partial^\alpha b^l+\lambda(K-6) \\ &\qquad +\partial_\alpha q\partial^\alpha q+V\Big\}, \qquad (4.33) \\ V &= \frac{1}{4}g^2\left[(\beta_l b^l)^2q^2+(G^{kl}\beta_k\beta_l-(\beta_l b^l)^2)q^4\right] . \qquad (4.34) \end{aligned}$$

A Lagrange multiplier term has been added to enforce the constraint $K=6$, eq. (4.5), on the moduli b^i. The value of the gauge coupling g has been given in eq. (4.21). We also recall that the "Kähler potential" K and the metric G_{ij} have been defined in eq. (4.4) and eq. (2.46), respectively.

We are now ready to consider the cosmological evolution of our system. We focus on backgrounds depending on time τ only and a metric with a three-dimensional maximally symmetric subspace which we take to be flat, for simplicity. Accordingly, we consider the following Ansatz

$$\begin{aligned} ds^2 &= -e^{2\nu(\tau)}d\tau^2+e^{2\alpha(\tau)}d\mathbf{x}^2+e^{2\beta(\tau)}dy^2 \\ \phi &= \phi(\tau) \\ b^i &= b^i(\tau) \\ q &= q(\tau)\ , \end{aligned}$$

where $\mathbf{x}=(x^1,x^2,x^3)$ and $y=x^4$. Note that α and β are the scale factors of the three-dimensional universe and the additional spatial dimension, respectively. For later convenience, we have also included a lapse function ν. The equations of motion for this Ansatz, derived from the action (4.33), are given by

- Einstein equations:

$$\begin{aligned} 3(\dot\alpha^2+\dot\alpha\dot\beta) &= +\tfrac{1}{4}\left(\dot\phi^2+G_{ij}\dot b^i\dot b^j+4\dot q^2\right)+e^{2\nu}V \\ 3(\ddot\alpha-\dot\nu\dot\alpha+2\dot\alpha^2) &= -\tfrac{1}{4}\left(\dot\phi^2+G_{ij}\dot b^i\dot b^j+4\dot q^2\right)+e^{2\nu}V \qquad (4.35) \\ 2\ddot\alpha+\ddot\beta+3\dot\alpha^2+\dot\beta^2+2\dot\alpha\dot\beta-2\dot\nu\dot\alpha-\dot\nu\dot\beta &= -\tfrac{1}{4}\left(\dot\phi^2+G_{ij}\dot b^i\dot b^j+4\dot q^2\right)+e^{2\nu}V \end{aligned}$$

• Field equations of motion:

$$\ddot{\phi} + (3\dot{\alpha} + \dot{\beta} - \dot{\nu})\dot{\phi} = 0 \tag{4.36}$$

$$\ddot{b}^k + (3\dot{\alpha} + \dot{\beta} - \dot{\nu})\dot{b}^k + \Gamma^k_{ij}\dot{b}^i\dot{b}^j + 2e^{2\nu}\left(G^{kj}\frac{\partial V}{\partial b^j} - \frac{2}{3}b^i V\right) = 0 \tag{4.37}$$

$$\ddot{q} + (3\dot{\alpha} + \dot{\beta} - \dot{\nu})\dot{q} + \frac{1}{2}e^{2\nu}\frac{\partial V}{\partial q} = 0$$

$$K = 6\,.$$

In these equations, we have already used the result $\lambda = -V/9$ for the Lagrange multiplier λ which follows by contracting the b^i equations of motion [2] with $b_i = G_{ij}b^j$. The connection Γ^k_{ij} associated to the moduli space metric G_{ij} has been defined in eq. (4.18).

The above action and evolution equations have been written for a definite topology of the internal space with intersection numbers d_{ijk}. When an evolution leads to a flop transition, the Kähler potential K, the metric G_{ij} and the connection Γ^k_{ij} have to be changed "by hand"in accordance with the change (4.16) in the intersection numbers to obtain the equations of motion for the new topology. As discussed earlier, this implies continuity of K and the metric while the connection jumps across the transition. We also note that, from eq. (4.34), the potential V is continuous while its derivatives with respect to b^i contain the connection and, hence, jump. From these properties and the equations of motion we conclude that all fields and their first time derivatives and, hence, the stress energy for all fields is continuous across the flop.

4.3.2 An approximate solution for vanishing transition states

It is clear from the eqs. (4.35) and (4.37) that the transition state q can be set to zero consistently and we will now analyse the cosmological evolution in this case. From the structure of the equations of motion, the configuration $q = 0$ seems rather non-generic and having a non-vanishing, perhaps small initial value for q appears to be more plausible. We will study this generic case further below. However, setting $q = 0$ corresponds to the conventional picture of a flop transition as being induced by slow free rolling in moduli space. It is, therefore, useful to consider this case in some detail, if only as a point of reference.

Setting the transition state q and, hence, the potential V, to zero simplifies the equations of motion considerably. Another simplification arises if we choose the gauge $\nu = 3\alpha + \beta$ for the lapse function. In this gauge, we will denote time by τ in the following. The second term in the b^i equations vanishes for this choice and multiplying

[2] Some relations for very special geometry which are useful in this context have been collected in ref. [108].

the remainder by $\dot{b}_i$ we find by integration that

$$k \equiv G_{ij} \frac{\partial b^i}{\partial \tau} \frac{\partial b^j}{\partial \tau} = \text{const.}\,.$$

Hence, the kinetic energy k of the Kähler moduli is constant on hyper-surfaces of constant time τ. Note that this is no longer true for proper cosmological time related to τ by $dt^2 = e^{6\alpha+2\beta} d\tau^2$. It is straightforward to integrate the Einstein equations (4.35) and the ϕ equation of motion for time τ. The result can be easily rewritten in terms of proper time t where it takes the form

$$\alpha = p_\alpha \ln|t|\,, \qquad \beta = p_\beta \ln|t|\,, \qquad \phi = p_\phi \ln|t|\,. \tag{4.38}$$

Here, we have dropped trivial additive integration constants for all three fields and the origin of time, for simplicity. The expansion powers p_α, p_β and p_ϕ must satisfy the constraints

$$\begin{aligned} 3p_\alpha + p_\beta &= 1\,, & (4.39)\\ p_\alpha^2 + p_\alpha p_\beta &= \frac{1}{12}\left(p_\phi^2 + k\right)\,, & (4.40) \end{aligned}$$

and the relation between proper time t and τ is simply

$$\tau = \ln|t|\,.$$

We still need to find the explicit form of b^i for a complete solution. To do this, we have to solve the following system of equations

$$\begin{aligned} \ddot{b}^i + \Gamma^i_{jk}\dot{b}^j\dot{b}^k &= 0 & (4.41)\\ d_{ijk}b^i b^j b^k &= 6 & (4.42)\\ G_{ij}\dot{b}^i\dot{b}^j &= k\,. & (4.43) \end{aligned}$$

Here the dot denotes the derivative with respect to τ. Hence, in this time coordinate, the fields b^i move along geodesics in moduli space subject to the constraint (4.42) from special geometry and the kinetic energy constraint (4.43). Unfortunately, the equation (4.41) is hard to solve in general due to the second non-linear term. However, for a sufficiently small time interval and slow motion this term can be neglected. In other words, the geodesics are well approximated by straight lines

$$b^i = c^i + p^i\tau + O(\tau^2) \tag{4.44}$$

in moduli space, where c^i and p^i are constants, as long as

$$|\tau| \ll \left| \frac{2p^i}{\Gamma^i_{jk}(c)p^i p^j} \right|\,. \tag{4.45}$$

This approximation also implies that we neglect the kinetic energy of the fields b^i compared to the other fields, that is, from eq. (4.43), we consider a solution with $k \simeq 0$. Accordingly, this value for k has to be inserted into the relation (4.40). Within our approximation, the special geometry constraint (4.42) turns into two conditions, namely

$$d_{ijk}c^i c^j c^k = 6 \,, \qquad d_{ijk}c^i c^j p^k = 0 \,. \tag{4.46}$$

These algebraic equations can be easily solved for given intersection numbers. This completes our approximate solution.

Let us now apply this result to a flop transition. We assume, for simplicity, that the flop occurs in the b^1 direction and at $b^1 = 0$. By setting $c^1 = 0$ in our solution (4.44) we can, in fact, arrange the flop to take place at time $\tau = 0$. Now we consider two solutions of the above type with intersection numbers d_{ijk} and $\tilde{d}_{ijk}$. We recall that d_{111} is the only intersection number which changes (as given in eq. (4.16)) across the transition. Since we have set $c^1 = 0$ this particular intersection number drops out of the constraints (4.46) which need to be satisfied for a valid solution. We are, therefore, free to choose the same constants c^i, p^i, solving the constraints (4.44), on both sides of the flop. This leads to two solutions, for either topology, which can be continuously matched together at the flop transition for $\tau = 0$. Further, our approximation is valid for a certain period of time before and after the flop as quantified by the condition (4.45). The scale factors α and β and the dilaton ϕ are unaffected by the transition in that they evolve according to (4.38) with the same expansion powers p_α, p_β and p_ϕ on both sides of the transition. These results suggest that the system indeed evolves through the transition into the moduli space of the topologically distinct Calabi-Yau space and, hence, that the topology change is dynamically realized. This picture will be confirmed by our numerical integration further below.

4.3.3 Evolution for non-vanishing transition states

If the transition states no longer vanish, that is $q \neq 0$, the potential becomes operative and the conclusions of the previous subsection do not apply any more. Clearly, we should not expect to find analytic solutions in this case any more. However, some qualitative features of the evolution can be inferred from the structure of the potential (4.34).

Let us consider small values of the transition state such that the potential (4.34) is dominated by the first term. It is then approximately given by $V \sim b^2 q^2$. Note, that this potential, for fixed non-zero q has a minimum at $b = 0$, that is, precisely at the flop point. From this observation and the general shape of the potential it is intuitively clear that a generic evolution will lead to oscillations around $b = 0$ and will finally settle down to this point. In other words, there is a clear preference for the system to settle down at the transition point rather than complete the transition.

The complete potential (4.34) must still have a minimum at small b for fixed non-zero q at least as long as q is sufficiently small. This suggests that the above argument generalizes to this case and this will indeed be confirmed by our numerical integration. In conclusion, this suggest that the system behaves quite differently if we allow non-zero values of the transition states. Previously, for vanishing transition states, we have found that the topology does change dynamically. If $q \neq 0$, on the other hand, the potential becomes important and favours the region in moduli space close to the flop transition. In this case, the system tends to settle down in the transition region so that the topology change is not completed.

4.4 An explicit model

In this section, we would like to substantiate our previous claims by numerically studying the cosmological evolution of our system. To do this we need to consider a particular example, that is, a particular pair of Calabi-Yau manifolds related by a flop transition which provides us with a concrete set of intersection numbers. We will use the Calabi-Yau spaces described in refs. [204, 205, 206] and applied to black hole physics in refs. [86, 197].

Concretely, we consider two elliptically fibred Calabi-Yau spaces X and $\tilde{X}$, both with a Hirzebruch F_1 base space and with $h^{1,1} = 3$. These spaces share a boundary of the Kähler moduli space which corresponds to a flop transition. Following ref. [86], both moduli spaces can be covered by a single set of coordinates (W, U, T) with the flop transition along $W = U$. The Kähler moduli space of X corresponds to the coordinate range

$$U > W > 0 \,, \qquad T > \frac{3}{2}U \,, \tag{4.47}$$

and the associated Kähler potential is given by

$$K = \frac{9}{4}U^3 + 3T^2U - W^3 \,. \tag{4.48}$$

We can use the constraint $K = 6$ to solve for T in terms of the other two moduli resulting in

$$T = \frac{1}{6}\left(-\frac{27U^3 - 12W^3 - 72}{U}\right)^{1/2} \,.$$

The fields b^i which we have previously used are related to (W, U, T) by

$$b^1 = U - W \,, \qquad b^2 = W \,, \qquad b^3 = T - \frac{3}{2}U \,.$$

This definition implies, from eq. (4.47), that the fields b^i are indeed positive throughout the moduli space of X. The flop transition is approached as $b^1 \to 0$. Hence, the coefficients (β^i) which enter the potential (4.34) are given by $(\beta^1, \beta^2, \beta^3) = (1, 0, 0)$.

For the second Calabi-Yau space $\tilde{X}$ the moduli are in the range

$$W > U > 0, \qquad T > W + \frac{1}{2}U\,, \tag{4.49}$$

and the Kähler potential is given by

$$\tilde{K} = \frac{5}{4}U^3 + 3U^2W - 3UW^2 + 3T^2U\ . \tag{4.50}$$

Solving $\tilde{K} = 6$ for T, as before, leads to

$$\tilde{T} = \frac{1}{6}\left(-\frac{15U^3 + 36U^2W - 36UW^2 - 72}{U}\right)^{1/2} \ .$$

From eq. (4.17), fields $\tilde{b}^i$ which are positive in the moduli space of $\tilde{X}$ can be defined by

$$\tilde{b}^1 = -b^1 = W - U\ , \qquad \tilde{b}^2 = b^2 - b^1 = 2W - U\ , \qquad \tilde{b}^3 = b^3 - b^1 = T - \frac{5}{2}U + W\ .$$

Note, that the two moduli spaces (4.47) and (4.49) indeed have a common boundary at $b^1 = U - W = 0$ which is where the flop transition occurs. Also, using the basis b^i as defined above, it is easy to see that the Kähler potentials (4.48) and (4.50) are related by the shift (4.16) in the intersection numbers, as is required for a flop transition.

We will now study the above example using W and U or, equivalently, b^1 and b^2 as the independent variables. It is useful to plot the potential (4.34) as a function of these variables for a fixed value of q. This has been done in fig. 4.2 for a value of $q = 1/3$. Obviously, this potential has a minimum in both directions which happens to be at

$$U = W = \left(\frac{3}{10}\right)^{1/3} \tag{4.51}$$

independent of the value of q. The associated potential value at the minimum (in units where $g = 1$) is

$$V_{\rm min} = \frac{300^{2/3}}{16}q^4\ . \tag{4.52}$$

From our previous general argument, we did anticipate a minimum in the direction $b^1 = U - W$. However, a minimum for both fields, located precisely at the flop and, hence, at field values (4.51) independent of q, comes as a surprise. We do not

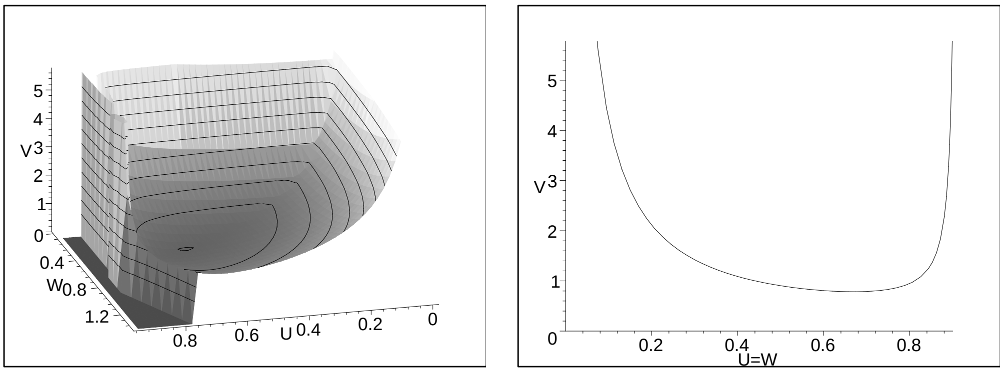

Figure 4.2: Potential V in terms of W,U and along the flop transition line at W=U, both for fixed q=1/3.

know whether this is a general feature of the potential (4.34) near flop transitions or particular to this example. Incidentally, we note that, having fixed all moduli b^i, the shape of the potential (4.52) in the remaining q-direction is well-suited for inflation. Unfortunately, to be in the slow-roll regime we need that $q \gg 1$ which, in turn, implies that $V \gg 1$ in units of the fundamental Planck scale. Such potential values clearly go beyond the region of validity of our five-dimensional effective action and it is, therefore, difficult to conclude anything definite about this tantalizing possibility. In any case, we will restrict field values such that V does not become too large in the following.

Before studying the flop-transition numerically, let us briefly discuss the other boundaries of the moduli space as described by the conditions (4.47) and (4.49) in relation to the potential. We note that the potential steeply increases towards the boundary directions $W \to 0$, $W \to \infty$ and $U \to 0$. Hence, as long as the potential is operative (that is, q is non-zero) it prevents evolution towards these boundaries. The same is true for the boundary prescribed by $U^3 \to (W^3 + 6)/9$ as long as one stays in the X part of the moduli space, $b^1 = U - W > 0$, and W is sufficiently small. The potential barrier rapidly vanishes in this direction for increasing W in the $\tilde{X}$ part of the moduli space. In our numerical evolution, we will simply avoid this direction of moduli space by choosing suitable initial conditions.

We have numerically integrated the system of equations (4.35), (4.37) for the above example, that is for the Kähler potentials (4.48) and (4.50). Here, we present the results for three characteristic sets of initial conditions which lead to an evolution towards the flop transition region. The precise initial values of all fields are specified in table 4.1. Fig. 4.3 and fig. 4.4 show the corresponding evolution of the fields as a function of proper time t. The first set of initial conditions in table 4.1 leads to

	U	$\dot{U}$	W	$\dot{W}$	q	$\dot{q}$	ϕ	$\dot{\phi}$	α	$\dot{\alpha}$	β
1^{st}	4/5	-1/5	1/8	1/9	0	0	1/2	-1/10	3/10	1/2	1/10
2^{nd}	1/2	-1/5	1/5	1/10	1/5	1/8	2/3	1/10	1/3	2/5	1/10
3^{rd}	4/5	-1/3	1/2	1/10	3/4	1/9	2/3	1/5	3/10	1/3	1/10

Table 4.1: Table of initial conditions in order of increasing initial value of $q = q(0)$.

a vanishing transitions state, that is, we have chosen $q(0) = 0$ and $\dot{q}(0) = 0$. This is precisely the case we have discussed in subsection (4.3.2). The resulting evolution is shown fig. 4.3. It can be seen that the system, starting off in the moduli space

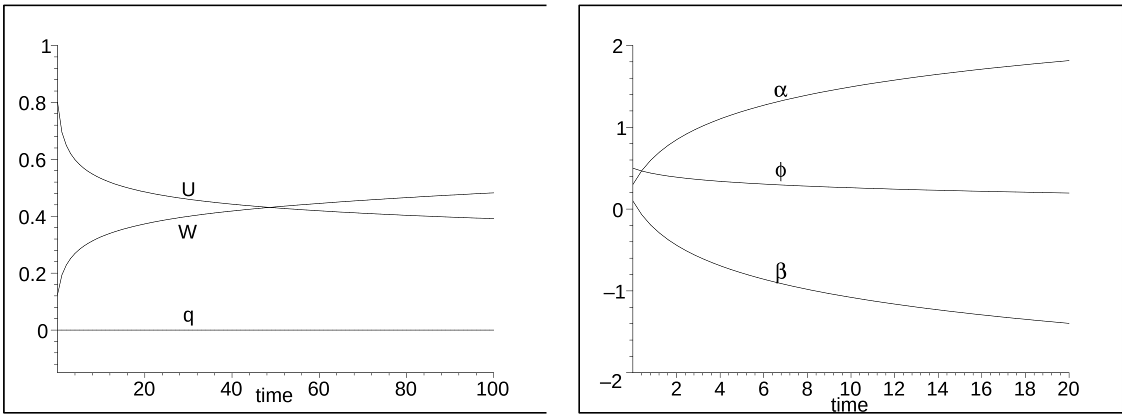

Figure 4.3: Evolution of the fields W(t), U(t) and $\alpha(t)$, $\beta(t)$, $\phi(t)$, respectively, for vanishing transition states and for the first set of initial conditions given in table 4.1.

of X at $b^1 > 0$, evolves towards the flop transition $b^1 = U - W \to 0$ and then moves on to negative values of b^1, corresponding to the moduli space of $\tilde{X}$. Hence, the topological transition is indeed dynamically realized as suggested by the previous analytic solution. The picture changes considerably once we allow for a non-vanishing transition state. The second set of initial conditions in table 4.1 corresponds to small, non-vanishing values of $q(0)$ and $\dot{q}(0)$. Again, the system starts out in the X moduli space and evolves towards the flop. The associated plots in the first row of fig. 4.4 show that after a few large initial oscillations around $b^1 = 0$ the system stabilizes at $b^1 = U - W \simeq 0$ and, hence, at the flop transition. A similar behaviour can be observed for larger initial values $q(0)$, as in the third set in table 4.1 with associated plots in the second row of fig. 4.4.

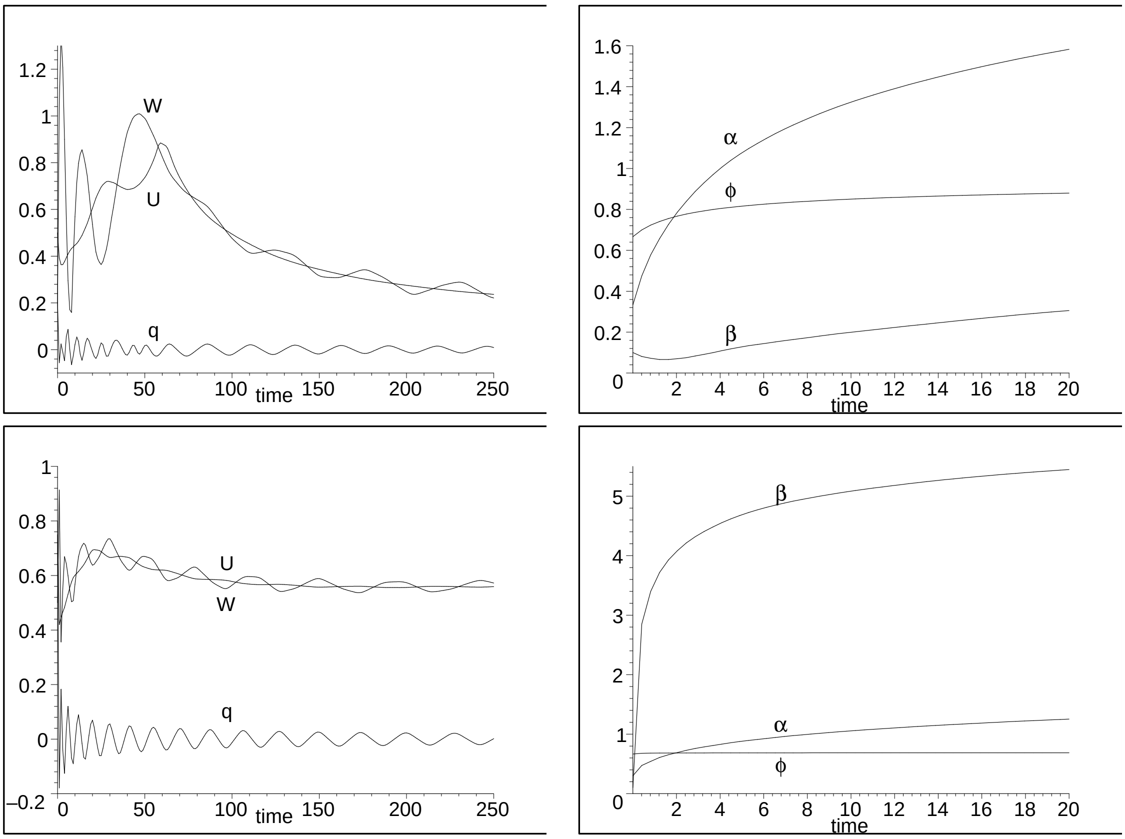

Figure 4.4: Evolution of the fields W(t), U(t), q(t) and $\alpha(t)$, $\beta(t)$, $\phi(t)$, respectively, for the second and third set of initial conditions given in table 4.1.

4.5 Conclusions

We have shown in this work that the dynamics of M-theory flop transitions strongly depends on whether or not the transition states which become light at the flop are taken into account. If these modes are exactly set to zero the moduli space evolution proceeds freely and the topology change can indeed be dynamically realized, that is, the system moves between two topologically different Calabi-Yau spaces related by a flop for appropriate but generic initial conditions. For non-vanishing values of the transition modes, however, a potential becomes operative which generically stabilizes moduli fields at the flop. Hence, the system does not really evolve into the moduli space of the flopped Calabi-Yau manifold and the transition remains incomplete. One may argue that this latter case is likely since non-vanishing values of the transition states represent a more generic set of field configurations in the early universe. If this is indeed the case, the region in moduli space close to a flop is preferred by the dynamics of the system.

It is likely that our results can be transferred to heterotic M-theory which provides a more realistic setting for low-energy physics from M-theory. To do this, we have to compactify the fifth dimension on a line interval and couple the five-dimensional $\mathcal{N} = 1$ bulk supergravity used in this chapter to $\mathcal{N} = 1$ theories on the two boundaries [107, 108]. The vacuum state of this theory is a static BPS domain wall which corresponds to a certain path in the Calabi-Yau Kähler moduli space as one moves between the boundaries. Using this property of the vacuum state, one can, in fact, construct static vacua of heterotic M-theory with an inherent flop transition [207], that is, vacua with the flop occurring at a particular point in the interval. Assuming the results of this work indeed transfer to heterotic M-theory, it is these inherently flopped vacua which would be preferred by the dynamical evolution of the system.

Bibliography

[1] K. Hagiwara et al. *Review of particle physics.* Phys. Rev. D66, 010001 (2002).

[2] F Jegerlehner. *Electroweak Theory and LEP Physics.* Lectures given at the Ecole Polytechnique, Lausanne.

[3] F Jegerlehner. *Renormalizing the Standard Model.* TASI Lectures 1990, Boulder Colorado.

[4] M. E. Peskin and Schroeder D. V. *An Introduction to Quantum Field Theory.* (1995). Addison Wesley.

[5] Zoltan Kunszt. *Bread and butter standard model.* (1999), `hep-ph/0004103`.

[6] S. F. Novaes. *Standard model: An introduction.* (1999), `hep-ph/0001283`.

[7] H. Spiesberger, M. Spira, and P. M. Zerwas. *The standard model: Physical basis and scattering experiments.* (2000), `hep-ph/0011255`.

[8] Guido Altarelli. *The standard electroweak theory and beyond.* (2000), `hep-ph/0011078`.

[9] A. Pich. *The Standard model of electroweak interactions.* (1994), `hep-ph/9412274`.

[10] Chris Quigg. *The electroweak theory.* (2002), `hep-ph/0204104`.

[11] Chris Quigg. *Electroweak symmetry breaking and the Higgs sector.* Acta Phys. Polon. B30, 2145–2192 (1999), `hep-ph/9905369`.

[12] Chris Quigg. *Visions: The coming revolutions in particle physics.* Eur. Phys. J. direct C4S1, 40 (2002), `hep-ph/0204075`.

[13] Jonathan L. Rosner. *The standard model in 2001.* (2001), `hep-ph/0108195`.

[14] Albert Einstein. *The Field Equations of Gravitation.* Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 1915, 844–847 (1915).

[15] Albert Einstein. *The Foundation of the General Theory of Relativity*. Annalen Phys. 49, 769–822 (1916).

[16] Thorne K. S. Misner, C. W. and J. A. Wheeler. *Gravitation*. (1973). Freeman, New York.

[17] H. Weyl. *Raum-Zeit-Materie*. (1923). Springer-Verlag, Berlin.

[18] John C. Baez. *The Meaning of Einstein's Equation*. (2001), `gr-qc/0103044`.

[19] Clifford M. Will. *The confrontation between general relativity and experiment*. Living Rev. Rel. 4, 4 (2001), `gr-qc/0103036`.

[20] Raymond Y. Chiao. *Conceptual tensions between quantum mechanics and general relativity: Are there experimental consequences?* (2002), `gr-qc/0211078`.

[21] Giovanni Amelino-Camelia. *Quantum-gravity phenomenology: Status and prospects*. Mod. Phys. Lett. A17, 899–922 (2002), `gr-qc/0204051`.

[22] Giovanni Amelino-Camelia. *Quantum-gravity phenomenology with gamma rays and UHE cosmic rays*. (2002), `astro-ph/0201047`.

[23] R. Colella, A. W. Overhauser, and S. A. Werner. *Observation of gravitationally induced quantum interference*. Phys. Rev. Lett. 34, 1472–1474 (1975).

[24] Hartmut Abele, Stefan Baessler, and Alexander Westphal. *Quantum states of neutrons in the gravitational field and limits for non-Newtonian interaction in the range between 1-mu-m and 10-mu-m*. (2003), `hep-ph/0301145`.

[25] Jean-Philippe Uzan. *The fundamental constants and their variation: Observational status and theoretical motivations*. (2002), `hep-ph/0205340`.

[26] S. Perlmutter et al. *Measurements of Omega and Lambda from 42 High-Redshift Supernovae*. Astrophys. J. 517, 565–586 (1999), `astro-ph/9812133`.

[27] A. Goobar et al. *The acceleration of the universe: Measurements of cosmological parameters from type Ia supernovae*. Phys. Scripta T85, 47–58 (2000).

[28] Luis Alvarez-Gaume and M. A. Vazquez-Mozo. *Topics in string theory and quantum gravity*. (1992), `hep-th/9212006`.

[29] S. Carlip. *Quantum gravity: A progress report*. Rept. Prog. Phys. 64, 885 (2001), `gr-qc/0108040`.

[30] Carlo Rovelli. *Strings, loops and others: A critical survey of the present approaches to quantum gravity*. (1997), `gr-qc/9803024`.

[31] C. Rovelli. *Quantum space-time: What do we know?* In *Callender, C. (ed.): Physics meets philosophy at the Planck scale* 101-122.

[32] Michael E. Peskin. *Beyond the standard model.* (1997), `hep-ph/9705479`.

[33] A.N. Schellekens. *Beyond the standard model.* (1995). Lectures given at NATO school of theoretical physics, Dalfsen.

[34] John R. Ellis. *Beyond the standard model for hillwalkers.* (1998), `hep-ph/9812235`.

[35] Keith A. Olive. *Introduction to supersymmetry: Astrophysical and phenomenological constraints.* (1999), `hep-ph/9911307`.

[36] Edward Witten. *Quest for unification.* (2002), `hep-ph/0207124`.

[37] M. F. Sohnius. *Introducing Supersymmetry.* Phys. Rept. 128, 39–204 (1985).

[38] Jose Miguel Figueroa-O'Farrill. *BUSSTEPP lectures on supersymmetry.* (2001), `hep-th/0109172`.

[39] Adel Bilal. *Introduction to supersymmetry.* (2001), `hep-th/0101055`.

[40] Joseph D. Lykken. *Introduction to supersymmetry.* (1996), `hep-th/9612114`.

[41] J. Wess and J. Bagger. *Supersymmetry and supergravity.* Princeton, USA: Univ. Pr. (1992) 259 p.

[42] Jan Louis, Ilka Brunner, and Stephan J. Huber. *The supersymmetric standard model.* (1998), `hep-ph/9811341`.

[43] M. J. Duff. *Kaluza-Klein theory in perspective.* (1994), `hep-th/9410046`.

[44] M. J. Duff, B. E. W. Nilsson, and C. N. Pope. *Kaluza-Klein Supergravity.* Phys. Rept. 130, 1–142 (1986).

[45] T. Kaluza. *On the problem of unity in physics.* Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) K1, 966–972 (1921).

[46] O. Klein. *Quantum theory and five-dimensional theory of relativity.* Z. Phys. 37, 895–906 (1926).

[47] Edward Witten. *Fermion Quantum Numbers in Kaluza-Klein Theory.* In *Appelquist, T. (Ed.) et Al.: Modern Kaluza-Klein theories*, 438-511. (In *Shelter Island 1983, Proceedings, Quantum Field Theory And The Fundamental Problems Of Physics*, 227-277).

[48] Edward Witten. *Search for a realistic Kaluza-Klein theory*. Nucl. Phys. B186, 412 (1981).

[49] E. Cremmer, B. Julia, and J. Scherk. *Supergravity theory in 11 dimensions*. Phys. Lett. B76, 409–412 (1978).

[50] Michael B. Green, J. H. Schwarz, and Edward Witten. *Superstring Theory. Vol. 1: Introduction*. Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics).

[51] Michael B. Green, J. H. Schwarz, and Edward Witten. *Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies and Phenomenology*. Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics).

[52] D. Lust and S. Theisen. *Lectures on string theory*. Lect. Notes Phys. 346, 1–346 (1989).

[53] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge, UK: Univ. Pr. (1998) 402 p.

[54] J. Polchinski. *String theory. Vol. 2: Superstring theory and beyond*. Cambridge, UK: Univ. Pr. (1998) 531 p.

[55] Michael Haack, Boris Kors, and Dieter Lust. *Recent developments in string theory: From perturbative dualities to M-theory*. (1998), `hep-th/9904033`.

[56] Thomas Mohaupt. *Introduction to string theory*. (2002), `hep-th/0207249`.

[57] John H. Schwarz. *Introduction to superstring theory*. (2000), `hep-ex/0008017`.

[58] Richard J. Szabo. *BUSSTEPP lectures on string theory: An introduction to string theory and D-brane dynamics*. (2002), `hep-th/0207142`.

[59] Edward Witten. *Magic, mystery, and matrix*. Not. Amer. Math. Soc. 45, 1124–1129 (1998).

[60] Joseph Polchinski. *String duality: A colloquium*. Rev. Mod. Phys. 68, 1245–1258 (1996), `hep-th/9607050`.

[61] Joseph Polchinski. *Quantum gravity at the Planck length*. Int. J. Mod. Phys. A14, 2633–2658 (1999), `hep-th/9812104`.

[62] Ashoke Sen. *Recent developments in superstring theory*. Nucl. Phys. Proc. Suppl. 94, 35–48 (2001), `hep-lat/0011073`.

[63] Ashoke Sen. *Developments in superstring theory*. (1998), `hep-ph/9810356`.

[64] W. Lerche. *Recent developments in string theory.* (1997), `hep-th/9710246`.

[65] C. P. Bachas. *String/M theory.* (1999), `hep-ph/0003259`.

[66] Augusto Sagnotti and Alexander Sevrin. *Strings, gravity and particle physics.* (2002), `hep-ex/0209011`.

[67] Sunil Mukhi. *Understanding fields using strings: A review for particle physicists.* Pramana 54, 543–559 (2000), `hep-ph/0002005`.

[68] Sunil Mukhi. *Recent developments in string theory: A brief review for particle physicists.* (1997), `hep-ph/9710470`.

[69] Jan de Boer. *String theory: An update.* (2002), `hep-th/0210224`.

[70] John H. Schwarz. *Anomaly cancellation: A retrospective from a modern perspective.* Int. J. Mod. Phys. A17S1, 157–166 (2002), `hep-th/0107059`.

[71] Joseph Polchinski. *Lectures on D-branes.* (1996), `hep-th/9611050`.

[72] Joseph Polchinski, Shyamoli Chaudhuri, and Clifford V. Johnson. *Notes on D-Branes.* (1996), `hep-th/9602052`.

[73] Constantin P. Bachas. *Lectures on D-branes.* (1998), `hep-th/9806199`.

[74] Donald Marolf. *String/M-branes for relativists.* (1998), `gr-qc/9908045`.

[75] K. S. Stelle. *BPS branes in supergravity.* (1998), `hep-th/9803116`.

[76] Carlo Angelantonj and Augusto Sagnotti. *Open strings.* Phys. Rept. 371, 1–150 (2002), `hep-th/0204089`.

[77] P. Candelas, Gary T. Horowitz, Andrew Strominger, and Edward Witten. *Vacuum Configurations For Superstrings.* Nucl. Phys. B258, 46–74 (1985).

[78] G. L. Cardoso et al. *Non-Kaehler string backgrounds and their five torsion classes.* (2002), `hep-th/0211118`.

[79] Ralph Blumenhagen, Boris Kors, Dieter Lust, and Tassilo Ott. *Intersecting brane worlds on tori and orbifolds.* Fortsch. Phys. 50, 843–850 (2002), `hep-th/0112015`.

[80] P. Candelas. *Lectures on complex manifolds.* In *Trieste 1987, Proceedings, Superstrings '87* 1-88.

[81] M. Nakahara. *Geometry, topology and physics.* Bristol, UK: Hilger (1990) 505 p. (Graduate student series in physics).

[82] S. I. Goldberg. *Curvature and Homology.* (1962). Dover Publications, New York.

[83] M. Haack. *Calabi-Yau fourfold compactifications in string theory.* Fortsch. Phys. 50, 3–99 (2002).

[84] Edward Witten. *Phase Transitions In M-Theory And F-Theory.* Nucl. Phys. B471, 195–216 (1996), `hep-th/9603150`.

[85] Brian R. Greene. *String theory on Calabi-Yau manifolds.* (1996), `hep-th/9702155`.

[86] Aaron Chou et al. *Critical points and phase transitions in 5d compactifications of M-theory.* Nucl. Phys. B508, 147–180 (1997), `hep-th/9704142`.

[87] Klaus Behrndt, Dieter Lust, and Wafic A. Sabra. *Moving moduli, Calabi-Yau phase transitions and massless BPS configurations in type II superstrings.* Phys. Lett. B418, 303–311 (1998), `hep-th/9708065`.

[88] Ashoke Sen. *An introduction to non-perturbative string theory.* (1998), `hep-th/9802051`.

[89] Atish Dabholkar. *Lectures on orientifolds and duality.* (1997), `hep-th/9804208`.

[90] P. K. Townsend. *Four lectures on M-theory.* (1996), `hep-th/9612121`.

[91] P. K. Townsend. *M-theory from its superalgebra.* (1997), `hep-th/9712004`.

[92] Gabriel Lopes Cardoso, Dieter Lust, and Thomas Mohaupt. *Duality symmetries and supersymmetry breaking in string compactifications.* (1994), `hep-th/9409095`.

[93] S. Hosono, A. Klemm, and S. Theisen. *Lectures on mirror symmetry.* (1994), `hep-th/9403096`.

[94] A. Bilal. *Duality, superstrings and M-theory.* (1998), `hep-ph/9805469`.

[95] Petr Horava and Edward Witten. *Heterotic and type I string dynamics from eleven dimensions.* Nucl. Phys. B460, 506–524 (1996), `hep-th/9510209`.

[96] Petr Horava and Edward Witten. *Eleven-Dimensional Supergravity on a Manifold with Boundary.* Nucl. Phys. B475, 94–114 (1996), `hep-th/9603142`.

[97] S. P. de Alwis. *Anomaly cancellation in M-theory.* Phys. Lett. B392, 332–334 (1997), `hep-th/9609211`.

[98] Jan O. Conrad. *Brane tensions and coupling constants from within M- theory.* Phys. Lett. B421, 119–124 (1998), hep-th/9708031.

[99] Troels Harmark. *Coupling constants and brane tensions from anomaly cancellation in M-theory.* Phys. Lett. B431, 295–302 (1998), hep-th/9802190.

[100] Adel Bilal, Jean-Pierre Derendinger, and Roger Sauser. *M-theory on S(1)/Z(2): Facts and fakes.* Nucl. Phys. B576, 347–374 (2000), hep-th/9912150.

[101] Michael Faux, Dieter Lust, and Burt A. Ovrut. *Intersecting orbifold planes and local anomaly cancellation in M-theory.* Nucl. Phys. B554, 437–483 (1999), hep-th/9903028.

[102] Krzysztof A. Meissner and Marek Olechowski. *Anomaly cancellation in M-theory on orbifolds.* Nucl. Phys. B590, 161–172 (2000), hep-th/0003233.

[103] R. Gueven. *Black p-brane solutions of $D = 11$ supergravity theory.* Phys. Lett. B276, 49–55 (1992).

[104] Paolo Pasti, Dmitri P. Sorokin, and Mario Tonin. *Covariant action for a $D = 11$ five-brane with the chiral field.* Phys. Lett. B398, 41–46 (1997), hep-th/9701037.

[105] Igor Bandos, Nathan Berkovits, and Dmitri P. Sorokin. *Duality-symmetric eleven-dimensional supergravity and its coupling to M-branes.* Nucl. Phys. B522, 214–233 (1998), hep-th/9711055.

[106] Edward Witten. *Strong Coupling Expansion Of Calabi-Yau Compactification.* Nucl. Phys. B471, 135–158 (1996), hep-th/9602070.

[107] Andre Lukas, Burt A. Ovrut, K. S. Stelle, and Daniel Waldram. *The universe as a domain wall.* Phys. Rev. D59, 086001 (1999), hep-th/9803235.

[108] Andre Lukas, Burt A. Ovrut, K. S. Stelle, and Daniel Waldram. *Heterotic M-theory in five dimensions.* Nucl. Phys. B552, 246–290 (1999), hep-th/9806051.

[109] Andre Lukas, Burt A. Ovrut, and Daniel Waldram. *Non-standard embedding and five-branes in heterotic M- theory.* Phys. Rev. D59, 106005 (1999), hep-th/9808101.

[110] S. Ferrara and S. Sabharwal. *Quaternionic Manifolds For Type II Superstring Vacua Of Calabi-Yau Spaces.* Nucl. Phys. B332, 317 (1990).

[111] Klaus Behrndt and Sergei Gukov. *Domain walls and superpotentials from M theory on Calabi- Yau three-folds.* Nucl. Phys. B580, 225–242 (2000), hep-th/0001082.

[112] Jean-Pierre Derendinger and Roger Sauser. *A five-brane modulus in the effective N = 1 supergravity of M-theory.* Nucl. Phys. B598, 87–114 (2001), hep-th/0009054.

[113] Gregory W. Moore, Grigor Peradze, and Natalia Saulina. *Instabilities in heterotic M-theory induced by open membrane instantons.* Nucl. Phys. B607, 117–154 (2001), hep-th/0012104.

[114] Matthias Brandle and Andre Lukas. *Five-branes in heterotic brane-world theories.* Phys. Rev. D65, 064024 (2002), hep-th/0109173.

[115] Edmund J. Copeland, James Gray, and Andre Lukas. *Moving five-branes in low-energy heterotic M-theory.* Phys. Rev. D64, 126003 (2001), hep-th/0106285.

[116] M. Gasperini and G. Veneziano. *Inflation, deflation, and frame independence in string cosmology.* Mod. Phys. Lett. A8, 3701–3714 (1993), hep-th/9309023.

[117] M. Gasperini and G. Veneziano. *Pre - big bang in string cosmology.* Astropart. Phys. 1, 317–339 (1993), hep-th/9211021.

[118] G. Veneziano. *String cosmology: The pre-big bang scenario.* (2000), hep-th/0002094.

[119] M. Gasperini and G. Veneziano. *The pre-big bang scenario in string cosmology.* (2002), hep-th/0207130.

[120] Matthias Brandle, Andre Lukas, and Burt A. Ovrut. *Heterotic M-theory cosmology in four and five dimensions.* Phys. Rev. D63, 026003 (2001), hep-th/0003256.

[121] Matthias Brandle and Andre Lukas. *Flop transitions in M-theory cosmology.* (2002), hep-th/0212263.

[122] John R. Ellis, Zygmunt Lalak, Stefan Pokorski, and Witold Pokorski. *Five-dimensional aspects of M-theory dynamics and supersymmetry breaking.* Nucl. Phys. B540, 149–186 (1999), hep-ph/9805377.

[123] Bjorn Andreas. *On vector bundles and chiral matter in N = 1 heterotic compactifications.* JHEP 01, 011 (1999), hep-th/9802202.

[124] Gottfried Curio. *Chiral matter and transitions in heterotic string models.* Phys. Lett. B435, 39–48 (1998), hep-th/9803224.

[125] Bjorn Andreas, Gottfried Curio, and Albrecht Klemm. *Towards the standard model spectrum from elliptic Calabi- Yau.* (1999), hep-th/9903052.

[126] Ron Donagi, Andre Lukas, Burt A. Ovrut, and Daniel Waldram. *Non-perturbative vacua and particle physics in M-theory.* JHEP 05, 018 (1999), hep-th/9811168.

[127] Ron Donagi, Andre Lukas, Burt A. Ovrut, and Daniel Waldram. *Holomorphic vector bundles and non-perturbative vacua in M- theory.* JHEP 06, 034 (1999), hep-th/9901009.

[128] Ron Donagi, Burt A. Ovrut, and Daniel Waldram. *Moduli spaces of fivebranes on elliptic Calabi-Yau threefolds.* JHEP 11, 030 (1999), hep-th/9904054.

[129] Ron Donagi, Burt A. Ovrut, Tony Pantev, and Daniel Waldram. *Standard model vacua in heterotic M-theory.* (1999), hep-th/0001101.

[130] Ron Donagi, Burt A. Ovrut, Tony Pantev, and Daniel Waldram. *Standard models from heterotic M-theory.* Adv. Theor. Math. Phys. 5, 93–137 (2002), hep-th/9912208.

[131] R. Donagi, Burt A. Ovrut, T. Pantev, and D. Waldram. *Non-perturbative vacua in heterotic M-theory.* Class. Quant. Grav. 17, 1049–1056 (2000).

[132] Ron Donagi, Burt A. Ovrut, Tony Pantev, and Dan Waldram. *Standard-model bundles.* Adv. Theor. Math. Phys. 5, 563–615 (2002), math.ag/0008010.

[133] Petr Horava. *Gluino condensation in strongly coupled heterotic string theory.* Phys. Rev. D54, 7561–7569 (1996), hep-th/9608019.

[134] Andre Lukas, Burt A. Ovrut, and Daniel Waldram. *Gaugino condensation in M-theory on S**1/Z(2).* Phys. Rev. D57, 7529–7538 (1998), hep-th/9711197.

[135] Andre Lukas, Burt A. Ovrut, and Daniel Waldram. *Five-branes and supersymmetry breaking in M-theory.* JHEP 04, 009 (1999), hep-th/9901017.

[136] Andre Lukas, Burt A. Ovrut, and Daniel Waldram. *Heterotic M-theory vacua with five-branes.* Fortsch. Phys. 48, 167–170 (2000), hep-th/9903144.

[137] Justin Khoury, Burt A. Ovrut, Paul J. Steinhardt, and Neil Turok. *The ekpyrotic universe: Colliding branes and the origin of the hot big bang.* Phys. Rev. D64, 123522 (2001), hep-th/0103239.

[138] Justin Khoury, Burt A. Ovrut, Paul J. Steinhardt, and Neil Turok. *A brief comment on 'The pyrotechnic universe'.* (2001), hep-th/0105212.

[139] Renata Kallosh, Lev Kofman, and Andrei D. Linde. *Pyrotechnic universe.* Phys. Rev. D64, 123523 (2001), hep-th/0104073.

[140] R. Arnowitt and Bhaskar Dutta. *Yukawa textures in Horava-Witten M-theory.* Nucl. Phys. B592, 143–163 (2001), `hep-th/0006172`.

[141] Gottfried Curio and Axel Krause. *G-fluxes and non-perturbative stabilisation of heterotic M- theory.* Nucl. Phys. B643, 131–156 (2002), `hep-th/0108220`.

[142] Gottfried Curio and Axel Krause. *Four-flux and warped heterotic M-theory compactifications.* Nucl. Phys. B602, 172–200 (2001), `hep-th/0012152`.

[143] Andre Lukas. *Moving five-branes and cosmology.* (2002), `hep-th/0210026`.

[144] Nuno D. Antunes, Edmund J. Copeland, Mark Hindmarsh, and Andre Lukas. *Kinky brane worlds.* (2002), `hep-th/0208219`.

[145] Edmund J. Copeland, James Gray, Andre Lukas, and David Skinner. *Five-dimensional moving brane solutions with four- dimensional limiting behaviour.* Phys. Rev. D66, 124007 (2002), `hep-th/0207281`.

[146] Mar Bastero-Gil, Edmund J. Copeland, James Gray, Andre Lukas, and Michael Plumacher. *Baryogenesis by brane-collision.* Phys. Rev. D66, 066005 (2002), `hep-th/0201040`.

[147] A. Notari and A. Riotto. *Isocurvature perturbations in the ekpyrotic universe.* Nucl. Phys. B644, 371–382 (2002), `hep-th/0205019`.

[148] D. G. Cerdeno and C. Munoz. *Phenomenology of heterotic M-theory with five-branes.* Phys. Rev. D66, 115007 (2002), `hep-ph/0206299`.

[149] K. Lechner, P. A. Marchetti, and M. Tonin. *Anomaly free effective action for the elementary M5-brane.* Phys. Lett. B524, 199–207 (2002), `hep-th/0107061`.

[150] David M. Kaplan and Jeremy Michelson. *Zero Modes for the D=11 Membrane and Five-Brane.* Phys. Rev. D53, 3474–3476 (1996), `hep-th/9510053`.

[151] Tom Adawi, Martin Cederwall, Ulf Gran, Bengt E. W. Nilsson, and Behrooz Razaznejad. *Goldstone tensor modes.* JHEP 02, 001 (1999), `hep-th/9811145`.

[152] S. P. de Alwis. *Coupling of branes and normalization of effective actions in string/M-theory.* Phys. Rev. D56, 7963–7977 (1997), `hep-th/9705139`.

[153] Igor Bandos et al. *Covariant action for the super-five-brane of M-theory.* Phys. Rev. Lett. 78, 4332–4334 (1997), `hep-th/9701149`.

[154] Piet Claus, Renata Kallosh, and Antoine Van Proeyen. *M 5-brane and superconformal (0,2) tensor multiplet in 6 dimensions.* Nucl. Phys. B518, 117–150 (1998), `hep-th/9711161`.

[155] Igor Bandos, Alexei Nurmagambetov, and Dmitri P. Sorokin. *The type IIA NS5-brane*. Nucl. Phys. B586, 315–330 (2000), `hep-th/0003169`.

[156] Ori J. Ganor and Amihay Hanany. *Small E_8 Instantons and Tensionless Non-critical Strings*. Nucl. Phys. B474, 122–140 (1996), `hep-th/9602120`.

[157] Kurt Lechner and Pieralberto Marchetti. *Dirac branes, characteristic currents and anomaly cancellations in 5-branes*. Nucl. Phys. Proc. Suppl. 102, 94–99 (2001), `hep-th/0103161`.

[158] Andre Lukas, Burt A. Ovrut, and Daniel Waldram. *On the four-dimensional effective action of strongly coupled heterotic string theory*. Nucl. Phys. B532, 43–82 (1998), `hep-th/9710208`.

[159] Katrin Becker, Melanie Becker, and Andrew Strominger. *Five-branes, membranes and nonperturbative string theory*. Nucl. Phys. B456, 130–152 (1995), `hep-th/9507158`.

[160] Philip Candelas and Xenia de la Ossa. *Moduli space of Calabi-Yau manifolds*. Nucl. Phys. B355, 455–481 (1991).

[161] M. Bodner, A. C. Cadavid, and S. Ferrara. *(2,2) vacuum configurations for type IIA superstrings: N=2 supergravity Lagrangians and algebraic geometry*. Class. Quant. Grav. 8, 789–808 (1991).

[162] A. Ceresole, R. D'Auria, and S. Ferrara. *The Symplectic Structure of N=2 Supergravity and its Central Extension*. Nucl. Phys. Proc. Suppl. 46, 67–74 (1996), `hep-th/9509160`.

[163] R. Slansky. *Group theory for unified model building*. Phys. Rept. 79, 1–128 (1981).

[164] Eduardo Lima, Burt A. Ovrut, and Jaemo Park. *Five-brane superpotentials in heterotic M-theory*. Nucl. Phys. B626, 113–164 (2002), `hep-th/0102046`.

[165] Anna Ceresole and Gianguido Dall'Agata. *General matter coupled $N = 2$, $D = 5$ gauged supergravity*. Nucl. Phys. B585, 143–170 (2000), `hep-th/0004111`.

[166] E. Bergshoeff, M. de Roo, Michael B. Green, G. Papadopoulos, and P. K. Townsend. *Duality of Type II 7-branes and 8-branes*. Nucl. Phys. B470, 113–135 (1996), `hep-th/9601150`.

[167] Mark Mueller. *Rolling radii and a time dependent dilaton*. Nucl. Phys. B337, 37 (1990).

[168] James E. Lidsey, David Wands, and Edmund J. Copeland. *Superstring cosmology.* Phys. Rept. 337, 343–492 (2000), hep-th/9909061.

[169] Andre Lukas, Burt A. Ovrut, and Daniel Waldram. *Cosmological solutions of type II string theory.* Phys. Lett. B393, 65–71 (1997), hep-th/9608195.

[170] Andre Lukas, Burt A. Ovrut, and Daniel Waldram. *String and M-theory cosmological solutions with Ramond forms.* Nucl. Phys. B495, 365–399 (1997), hep-th/9610238.

[171] Andre Lukas, Burt A. Ovrut, and Daniel Waldram. *The cosmology of M-theory and type II superstrings.* (1998), hep-th/9802041.

[172] Nemanja Kaloper, Ian I. Kogan, and Keith A. Olive. *Cos(M)ological solutions in M and string theory.* Phys. Rev. D57, 7340–7353 (1998), hep-th/9711027.

[173] Hong Lu, S. Mukherji, C. N. Pope, and K. W. Xu. *Cosmological solutions in string theories.* Phys. Rev. D55, 7926–7935 (1997), hep-th/9610107.

[174] Andre Lukas, Burt A. Ovrut, and Daniel Waldram. *Cosmology and heterotic M-theory in five-dimensions.* (1998), hep-th/9812052.

[175] Andre Lukas, Burt A. Ovrut, and Daniel Waldram. *Cosmological solutions of Horava-Witten theory.* Phys. Rev. D60, 086001 (1999), hep-th/9806022.

[176] Harvey S. Reall. *Open and closed cosmological solutions of Horava-Witten theory.* Phys. Rev. D59, 103506 (1999), hep-th/9809195.

[177] H. A. Chamblin and H. S. Reall. *Dynamic dilatonic domain walls.* Nucl. Phys. B562, 133–157 (1999), hep-th/9903225.

[178] James E. Lidsey. *Separable brane cosmologies in heterotic M-theory.* Class. Quant. Grav. 17, L39–L45 (2000), gr-qc/9911066.

[179] Mariusz P. Dabrowski. *Mixmaster Horava-Witten cosmology.* (2000), hep-th/0102065.

[180] Ulrich Ellwanger. *Cosmological evolution in compactified Horava-Witten theory induced by matter on the branes.* Eur. Phys. J. C25, 157–164 (2002), hep-th/0001126.

[181] Andre Lukas, Burt A. Ovrut, and Daniel Waldram. *Boundary inflation.* Phys. Rev. D61, 023506 (2000), hep-th/9902071.

[182] Justin Khoury, Paul J. Steinhardt, and Daniel Waldram. *Inflationary solutions in the brane world and their geometrical interpretation*. Phys. Rev. D63, 103505 (2001), `hep-th/0006069`.

[183] Andre Lukas and David Skinner. *Brane-world inflation and the transition to standard cosmology*. JHEP 09, 020 (2001), `hep-th/0106190`.

[184] Pierre Binetruy, Cedric Deffayet, and David Langlois. *Non-conventional cosmology from a brane-universe*. Nucl. Phys. B565, 269–287 (2000), `hep-th/9905012`.

[185] Takeshi Nihei. *Inflation in the five-dimensional universe with an orbifold extra dimension*. Phys. Lett. B465, 81–85 (1999), `hep-ph/9905487`.

[186] Panagiota Kanti, Ian I. Kogan, Keith A. Olive, and Maxim Pospelov. *Cosmological 3-brane solutions*. Phys. Lett. B468, 31–39 (1999), `hep-ph/9909481`.

[187] Panagiota Kanti, Ian I. Kogan, Keith A. Olive, and Maxim Pospelov. *Single-brane cosmological solutions with a stable compact extra dimension*. Phys. Rev. D61, 106004 (2000), `hep-ph/9912266`.

[188] Tom Banks, Willy Fischler, and Lubos Motl. *Dualities versus singularities*. JHEP 01, 019 (1999), `hep-th/9811194`.

[189] Michael B. Green, Michael Gutperle, and Pierre Vanhove. *One loop in eleven dimensions*. Phys. Lett. B409, 177–184 (1997), `hep-th/9706175`.

[190] Ignatios Antoniadis, J. Rizos, and K. Tamvakis. *Singularity - free cosmological solutions of the superstring effective action*. Nucl. Phys. B415, 497–514 (1994), `hep-th/9305025`.

[191] John R. Ellis, Nemanja Kaloper, Keith A. Olive, and Jun'ichi Yokoyama. *Topological R**4 inflation*. Phys. Rev. D59, 103503 (1999), `hep-ph/9807482`.

[192] Andre Lukas and Burt A. Ovrut. *Symmetric vacua in heterotic M-theory*. (1999), `hep-th/9908100`.

[193] C. Cartier, Edmund J. Copeland, and R. Madden. *The graceful exit in string cosmology*. JHEP 01, 035 (2000), `hep-th/9910169`.

[194] Paul S. Aspinwall, Brian R. Greene, and David R. Morrison. *Calabi-Yau moduli space, mirror manifolds and spacetime topology change in string theory*. Nucl. Phys. B416, 414–480 (1994), `hep-th/9309097`.

[195] Andrew Strominger. *Massless black holes and conifolds in string theory*. Nucl. Phys. B451, 96–108 (1995), `hep-th/9504090`.

[196] Brian R. Greene, David R. Morrison, and Andrew Strominger. *Black hole condensation and the unification of string vacua.* Nucl. Phys. B451, 109–120 (1995), `hep-th/9504145`.

[197] Ingo Gaida, Swapna Mahapatra, Thomas Mohaupt, and Wafic A. Sabra. *Black holes and flop transitions in M-theory on Calabi-Yau threefolds.* Class. Quant. Grav. 16, 419–433 (1999), `hep-th/9807014`.

[198] Ali H. Chamseddine and H. Nicolai. *Coupling the SO(2) supergravity through dimensional reduction.* Phys. Lett. B96, 89 (1980).

[199] Paul S. Howe and R. W. Tucker. *A locally supersymmetric and reparametrization invariant action for a spinning membrane.* J. Phys. A10, L155–L158 (1977).

[200] E. Bergshoeff, E. Sezgin, and P. K. Townsend. *Supermembranes and eleven-dimensional supergravity.* Phys. Lett. B189, 75–78 (1987).

[201] M. Gunaydin, G. Sierra, and P. K. Townsend. *The geometry of N=2 Maxwell-Einstein supergravity and Jordan algebrasS.* Nucl. Phys. B242, 244 (1984).

[202] A. C. Cadavid, A. Ceresole, R. D'Auria, and S. Ferrara. *Eleven-dimensional supergravity compactified on Calabi-Yau threefolds.* Phys. Lett. B357, 76–80 (1995), `hep-th/9506144`.

[203] Gerard 't Hooft. *Computation of the quantum effects due to a four- dimensional pseudoparticle.* Phys. Rev. D14, 3432–3450 (1976).

[204] J. Louis, J. Sonnenschein, S. Theisen, and S. Yankielowicz. *Non-perturbative properties of heterotic string vacua compactified on K3 x T**2.* Nucl. Phys. B480, 185–212 (1996), `hep-th/9606049`.

[205] David R. Morrison and Cumrun Vafa. *Compactifications of F-Theory on Calabi–Yau Threefolds – II.* Nucl. Phys. B476, 437–469 (1996), `hep-th/9603161`.

[206] Philip Candelas, Anamaria Font, Sheldon Katz, and David R. Morrison. *Mirror symmetry for two parameter models. 2.* Nucl. Phys. B429, 626–674 (1994), `hep-th/9403187`.

[207] Brian R. Greene, Koenraad Schalm, and Gary Shiu. *Dynamical topology change in M theory.* J. Math. Phys. 42, 3171–3187 (2001), `hep-th/0010207`.

[208] Claudio Teitelboim. *Monopoles of higher rank.* Phys. Lett. B167, 69 (1986).

[209] Rafael I. Nepomechie. *Magnetic monopoles from antisymmetric tensor gauge fields.* Phys. Rev. D31, 1921 (1985).

[210] K. Lechner and P. A. Marchetti. *Interacting branes, dual branes, and dyonic branes: A unifying lagrangian approach in D dimensions.* JHEP 01, 003 (2001), `hep-th/0007076`.

[211] Georges de Rham. *Differentiable manifolds: Forms, Currents, Harmonic Forms.* (1984). Springer-Verlag Berlin.

[212] K. Lechner and P. A. Marchetti. *Duality-invariant quantum field theories of charges and monopoles.* Nucl. Phys. B569, 529–576 (2000), `hep-th/9906079`.

[213] Sergio Ferrara and Stefan Theisen. *Modulis Spaces, Effective Actions And Duality Symmetry In String Compactifications.* Based on lectures given at 3rd Hellenic Summer School, Corfu, Greece, Sep 13-23, 1989.

[214] Jan Louis and Andrei Micu. *Type II theories compactified on Calabi-Yau three-folds in the presence of background fluxes.* Nucl. Phys. B635, 395–431 (2002), `hep-th/0202168`.

[215] H. Farkas and I. Kra. *Riemann Surfaces.* (1991). Springer-Verlag.

Appendix A

Conventions

In this appendix the notation and conventions are summarized. This includes indices as well as the conventions used for the Levi-Civita tensor, p-forms, Hodge-star operator, scalar product, exterior derivative and interior product. The mostly plus metric $(-,+,+,\ldots+)$ which has negative signature is used, and here n denotes the dimensionality of space-time. In the definitions below the general indices $l_i, m_i, n_i \ldots$ are used, which can stand for any of the real space-time indices of the following list.

• **indices:**

$I, J, K \ldots$	$= 0, \ldots 9, 11$	: 11-dim. space-time M
$\bar{I}, \bar{J}, \bar{K} \ldots$	$= 0, \ldots 9$	: 10-dim space-time M_{10} perpendicular to orbifold $S^1/\mathbb{Z}_2$
$A, B, C \ldots$	$= 4, \ldots 9$	: real indices on 6-dim. Calabi-Yau manifold X
$a, \bar{a}, b, \bar{b} \ldots$		: (anti-)holomorphic indices on X
$\alpha, \beta, \gamma \ldots$	$= 0, \ldots 3, 11$	: 5-dim. space-time M_5
$\mu, \nu, \rho \ldots$	$= 0, \ldots 3$	: 4-dim. space-time M_4 and M5-brane world-volume parallel to M_4 in static gauge
$l, m, n \ldots$	$= 0, \ldots 5$	: 6-dim. M5-brane world-volume M_6
s, t	$= 4, 5$	: 2-dim. M5-brane world-volume M_6 on two-cycle $\mathcal{C}_2$ within X
$\sigma, \bar{\sigma}$		: (anti-)holomorphic M_6 on $\mathcal{C}_2$
i, j	$= 1, 2$	: No. of boundary M_{10}^i or $SU(2)$ indices
$\hat{\imath}, \hat{\jmath}$	$= 0, \ldots N+1$	: No. of Five-brane
$L, M, \ldots$	$= 1, \ldots$	: No. of basis one-forms of $H^1(X, V_{R_{\mathcal{I}_i}})$
x, y, z	$= 1, \ldots \dim(R_{\mathcal{I}_i})$	: gauge indices of $R_{\mathcal{I}_i}$
$p, q, \ldots$	$= 1, \ldots \dim(R_{\mathcal{H}_i})$	: gauge indices of $R_{\mathcal{H}_i}$
$p, \bar{p}, q, \bar{q} \ldots$	$= 1, \ldots h^{2,1}$	: No. of complex structure moduli
$k, l, \ldots$	$= 1, \ldots h^{1,1}$	: No. of Kähler moduli

• Levi-Civita tensor:
By $\hat{\epsilon}_{m_1\ldots m_n}$ we denote the pure number antisymmetric symbol, and the associated Levi-Civita tensor is defined by

$$\epsilon_{m_1\ldots m_n} \equiv sgn(g)\sqrt{|g|}\hat{\epsilon}_{m_1\ldots m_n} \quad \Rightarrow \quad \epsilon^{m_1\ldots m_n} = \frac{1}{\sqrt{|g|}}\hat{\epsilon}^{m_1\ldots m_n} \tag{A.1}$$

$$\epsilon_{m_1..m_p n_1..n_{n-p}}\epsilon^{m_1..m_p l_1..l_{n-p}} = sgn(g)p!(n-p)!\delta^{[l_1}_{n_1}\ldots\delta^{l_{n-p}]}_{n_{n-p}} \tag{A.2}$$

• p-forms:

$$\alpha_p \equiv \frac{1}{p!}\alpha_{p\ n_1\ldots n_p}dx^{n_1}\wedge\ldots\wedge dx^{n_p} \tag{A.3}$$

•Hodge-star operator:

$$(*\alpha_p)_{l_1\ldots l_{n-p}} \equiv \frac{1}{p!}\epsilon_{l_1\ldots l_{n-p}m_1\ldots m_p}\alpha_p{}^{m_1\ldots m_p} \tag{A.4}$$

$$**\alpha_p = sgn(g)(-1)^{p(n-p)}\alpha_p \tag{A.5}$$

•scalar product:

$$\langle\alpha_p,\,\beta_p\rangle \equiv \int \alpha_p\wedge *\bar{\beta}_p \tag{A.6}$$

$$= sgn(g)(-1)^{p(n-p)}\int d^n x\sqrt{|g|}\frac{1}{p!}\alpha_{p\,l_1\ldots l_p}\bar{\beta}_p^{l_1\ldots l_p} \tag{A.7}$$

•exterior derivative:

$$d\alpha_p \equiv \frac{1}{p!}\partial_m\alpha_{p\,l_1\ldots l_p}dx^m\wedge dx^{l_1}\wedge\ldots\wedge dx^{l_p} \tag{A.8}$$

$$d(\alpha_p\wedge\beta_q) = d\alpha_p\wedge\beta_q + (-1)^p\alpha_p\wedge d\beta_q \tag{A.9}$$

$$\tag{A.10}$$

$$\int_M d\alpha_p\wedge\beta_q = (\alpha_p\wedge\beta_q)|_{\partial M} + (-1)^{p+1}\int_M \alpha_p\wedge d\beta_q \tag{A.11}$$

$$\langle d\alpha_{p-1},\beta_p\rangle = \langle\alpha_{p-1},d^\dagger\beta_p\rangle \tag{A.12}$$

$$d^\dagger = sgn(g)(-1)^{np+n+1}*d* \tag{A.13}$$

•interior product:

$$i_v\alpha_p \equiv \frac{1}{(p-1)!}v^m\alpha_{p\,ml_1\ldots l_{p-1}}dx^{l_1}\wedge\ldots\wedge dx^{l_{p-1}} \tag{A.14}$$

$$i_v\alpha_p = sgn(g)(-1)^{np+1}*(v\wedge *\alpha_p) \tag{A.15}$$

$$\langle i_v\alpha_p,\beta_{p-1}\rangle = (-1)^{n+1}\langle\alpha_p, v\wedge\beta_{p-1}\rangle \tag{A.16}$$

$$\langle i_v\alpha_p, i_v\beta_p\rangle = (-1)^{n+1}v^2\langle\alpha_p,\beta_p\rangle - sgn(g)\langle i_v*\alpha_p, i_v*\beta_p\rangle \tag{A.17}$$

Appendix B

Branes, currents and Dirac branes

This appendix summarizes some general background material needed to understand classical brane currents [157,208,209,210,211] and their coupling to fields. The concept of a Dirac brane is introduced to couple a magnetic brane to an electric field. Moreover a geometrical interpretation of the Dirac quantization condition is given. Nothing will be said on the quantum mechanical description of D-branes and their world-volume gauge fields by means of open string theory. For reviews on D-branes see [71, 72, 73, 74, 75, 76].

When a p-brane, electrically charged under a $(p+1)$-form potential A, moves through d-dimensional space-time M_d ($d \geq p+1$), it sweeps out a $(p+1)$-dimensional world-volume W_{p+1} and thereby produces an electric $(d-p-1)$-*current* J_e which acts as a source for the field strength $F = dA$ like

$$d * F = J_e. \tag{B.1}$$

This equation reflects nothing more than a generalized form of Maxwells inhomogenuous equation for general $(p+1)$-form potentials. It directly follows from the action[1]

$$S_A = S_{kin} + S_{mc} = \int_{M_d} \left\{ \frac{1}{2} F \wedge *F + (-1)^{p+1} A \wedge J_e \right\} \tag{B.2}$$

after variation with respect to A. The second term is called *minimal coupling* term. If the p-brane world-volume is parametrized by the coordinates $\{\sigma^m,\, m = 0 \ldots p\}$ and embedded into space-time by the maps

$$\begin{array}{rlll} X^\mu : & W_{p+1} & \longrightarrow M_d, & \mu = 0 \ldots d-1, \\ & \sigma^m & \longmapsto X^\mu(\sigma^m), & m = 0 \ldots p, \end{array}$$

[1]The annoying factors of (-1) appearing in the following are such as to be consistent with the conventions given in appendix A.

then the minimal coupling term, explicitly showing the natural coupling of a $(p+1)$-form to a $(p+1)$-dimensional surface, is more conventionally written like

$$\begin{aligned} S_{mc} &= T_p \int_{W_{p+1}} \hat{A} \\ &= T_p \int_{W_{p+1}} \frac{1}{(p+1)!} \hat{A}_{m_1 \ldots m_{p+1}} \, d\sigma^{m_1} \wedge \ldots \wedge d\sigma^{m_{p+1}} \\ &= T_p \int_{W_{p+1}} \frac{A_{\mu_1 \ldots \mu_{p+1}}}{(p+1)!} \, \partial_{m_1} X^{\mu_1} \ldots \partial_{m_{p+1}} X^{\mu_{p+1}} \, d\sigma^{m_1} \wedge \ldots \wedge d\sigma^{m_{p+1}}, \end{aligned} \tag{B.3}$$

such that the pullback $\hat{A}$ of the (p+1)-form A from space-time M_d onto the world-volume W_{p+1} is given by

$$\hat{A}_{m_1 \ldots m_{p+1}} = A_{\mu_1 \ldots \mu_{p+1}} \, \partial_{m_1} X^{\mu_1} \ldots \partial_{m_{p+1}} X^{\mu_{p+1}}.$$

The factor T_p is the brane tension and is a mass density [2]. In order to relate the coupling term in (B.2) to (B.3), the integral over the world-volume in (B.3) has to be extended to the whole space-time with the help of a *generalized* δ*−function* defined exactly such that [3]

$$\int_{W_{p+1}} \hat{A} \equiv \int_{M_d} \delta(W_{p+1}) \wedge A. \tag{B.4}$$

It obviously follows now that the electric current must be given by

$$J_e = (-1)^{d-p-1} \, T_p \, \delta(W_{p+1}), \tag{B.5}$$

such that *generalized* δ*-functions and currents can be identified.* An explicit form of such a δ−function (distribution) is given by

$$\begin{aligned} \delta(W_{p+1})_{\mu_1 \ldots \mu_{d-p-1}} &= \frac{sgn(g)}{(p+1)! \sqrt{|g|}} \, \epsilon_{\mu_1 \ldots \mu_{d-p-1} \nu_1 \ldots \nu_{p+1}} \\ &\quad \times \int_{W_{p+1}} dX^{\nu_1} \wedge \ldots \wedge dX^{\nu_{p+1}} \delta^{(d)}(x - X(\sigma)) \, , \end{aligned} \tag{B.6}$$

where $\epsilon_{\ldots}$ is the Levi-Civita tensor (A.1) and $\delta^{(d)}(x-X)$ is the normal δ−function in d−dimensions. Using this explicit form and the fact that $dX^{\nu_1} \wedge \ldots \wedge dX^{\nu_{p+1}} = \partial_{m_1} X^{\nu_1} \ldots \partial_{m_{p+1}} X^{\nu_{p+1}} \, d\sigma^{m_1} \wedge \ldots \wedge d\sigma^{m_{p+1}}$ allows to check the consistency of eq.(B.3) with eq.(B.4).

[2] Strictly speaking T_p should be a charge density, but for BPS-states charge and mass can be identified.

[3] Note that for d and p even, there arises an ambiguity because the potential as well as the δ−function are odd forms so their order in (B.4) matters; we simply choose the convention where the δ−function stands to the left of the $(p+1)$-form, consistent with appendix (D).

An important implication of this identification of currents and δ-functions is that generally currents are distributions and so their properties are only defined in a distributional sense. For example from eq.(B.1) need not directly follow the conservation of current $dJ_e = 0$, though in the case of smooth forms this would immediately follow by $d^2 = 0$. As shown next, this difference manifests itself clearly in the consistency of gauge invariance of the action (B.2) and conservation of current.

Up to now it was not specified if the world-volume W_{p+1} was open or closed, that is, if W_{p+1} has a boundary or not. But consider a gauge transformation $\delta A = d\Lambda$, then, using Stokes theorem, the variation of the minimal coupling term (B.3) is

$$\delta S_{mc} = T_p \int_{W_{p+1}} d\hat{\Lambda} = T_p \int_{\partial W_{p+1}} \hat{\Lambda}$$

which generally vanishes only if $\partial W_{p+1} = 0$, i.e. W_{p+1} must be closed. So action (B.2) is only gauge invariant for closed world-volumes W_{p+1}. This of course must be consistent with eqs. (B.4, B.5) which implies the conservation of current in the distributional sense like

$$\delta S_{mc} = \int_{M_d} d\Lambda \wedge J_e = (-1)^{p+1} \int_{M_d} \Lambda \wedge dJ_e = 0 \quad \Longleftrightarrow \quad dJ_e = 0$$

and thus

$$dJ_e = 0 \quad \Longleftrightarrow \quad d\,\delta(W_{p+1}) = 0 \quad \Longleftrightarrow \quad \partial W_{p+1} = 0.$$

So it turns out that the current associated to a closed surface is also closed, at least in the distributional sense. But on the other hand the current of a open brane need not be conserved, though the equation of motion eq. (B.1) is still true, but again only in a distributional sense.

If the world-volume W_{p+1} is open, the action (B.2) must be modified to be gauge invariant, and in order to electrically couple the current of an open p-brane in a gauge invariant way, a compensating boundary term and an associated boundary p-form field B must be introduced. The minimal coupling term then becomes

$$S_{mc} = T_p \int_{W_{p+1}} \hat{A} - d\hat{B} = T_p \int_{W_{p+1}} \hat{A} - T_p \int_{\partial W_{p+1}} \hat{B}\,,$$

and the boundary field must transform like $\delta B = \Lambda$ under a gauge transformation such that the combination $\hat{A} - d\hat{B}$ is gauge invariant. This modification does not change the equation of motion (B.1) for A and it also does not lead to the conservation of the electric current alone. Now the boundary term looks exactly like a minimal coupling term, but here of the $p-$form potential B to the world-volume of the boundary ∂W_{p+1}, which is trivially closed ($\partial^2 W_{p+1} = 0$). In general the potential B need not be defined on the whole space-time M_d but might only corresponds to a world-volume field of another q-brane ($q \geq p$) on which the p-brane can end. Thus from the q-brane perspective there exist closed $(p-1)$-branes confined to the $q-$brane world-volume, charged with respect to the field B, which from a space-time perspective correspond to the boundaries of p-branes ending on the q-brane. This geometrical picture makes the non-conversation of the electrical current reasonable because the electric charge can flow from the p-brane onto the q-brane on which it ends. In case that this q-brane is a magnetic brane it actually becomes a dyonic object by this inflow of electric charge from electric objects ending on it. This is for example the case for membranes ending on M5-branes in M-theory, as illustrated in fig 2.1.

There is another important observation to be made here. For open world-volumes consistency of the definition (B.4) of the δ-function with Stokes law implies

$$\begin{aligned}
\int_{W_{p+1}} d\hat{\Lambda} &= \int_{\partial W_{p+1}} \hat{\Lambda} = \int_{M_p} \delta(\partial W_{p+1}) \wedge \Lambda \\
&= \int_{M_d} \delta(W_{p+1}) \wedge d\Lambda = \int_{M_d} (-1)^{d-p} d\delta(W_{p+1}) \wedge \Lambda \\
\Longrightarrow &\quad \delta(\partial W_{p+1}) = (-1)^{d-p} d\delta(W_{p+1})
\end{aligned}$$

such that a δ-function for the open surface W_{p+1} can be viewed as the Θ-function for its boundary ∂W_{p+1}, i.e.

$$\Theta(\partial W_{p+1}) \equiv (-1)^{d-p} \delta(W_{p+1}) \quad \Longrightarrow \quad d\Theta(\partial W_{p+1}) = \delta(\partial W_{p+1}). \tag{B.7}$$

This shows that to every boundary there exists a generalized $\Theta-$function and thus the current associated to a boundary is exact and therefore trivially closed. Due to eq. (B.7) an explicit form of such a $\Theta-$function is given in analogy to eq. (B.6). On topologically trivial spaces such distributions are *globally* well defined, but care has to be taken on compact spaces that allow cycles, i.e. non-trivially closed surfaces that are not boundaries of any other surfaces. Also note that the Θ-function to a given boundary ∂W_{p+1} is not unique, it is only defined up to an exact piece. Geometrically this is shown in figure B.1 and means that two surfaces with the same boundary differ at most by a boundary, because let $\tilde{W}_{p+1}$ and W_{p+1} be different surfaces sharing the same boundary $\partial W_{p+1} = \partial \tilde{W}_{p+1}$, then there exists an interpolating surface I_{p+2} such that $\partial I_{p+2} = \tilde{W}_{p+1} - W_{p+1}$. Now associate the exact current $d\Theta(\partial I_{p+2}) = \delta(\partial I_{p+2})$ to this boundary and find the following correspondence between surfaces and currents

$$\begin{array}{rcl}
\tilde{W}_{p+1} = W_{p+1} + \partial I_{p+2} & \stackrel{\partial^2=0}{\Longrightarrow} & \partial \tilde{W}_{p+1} = \partial W_{p+1} \\
& \Longleftrightarrow & \\
\Theta(\tilde{W}_{p+1}) = \Theta(W_{p+1}) + \delta(\partial I_{p+2}) & \stackrel{d^2=0}{\Longrightarrow} & d\Theta(\tilde{W}_{p+1}) = d\Theta(W_{p+1}) = \delta(\partial W_{p+1}).
\end{array} \tag{B.8}$$

So a trivially closed current and the associated $\Theta-$function have a nice geometrical interpretation as a trivially closed world-volume that is the boundary of another open world-volume, which is not unique though.

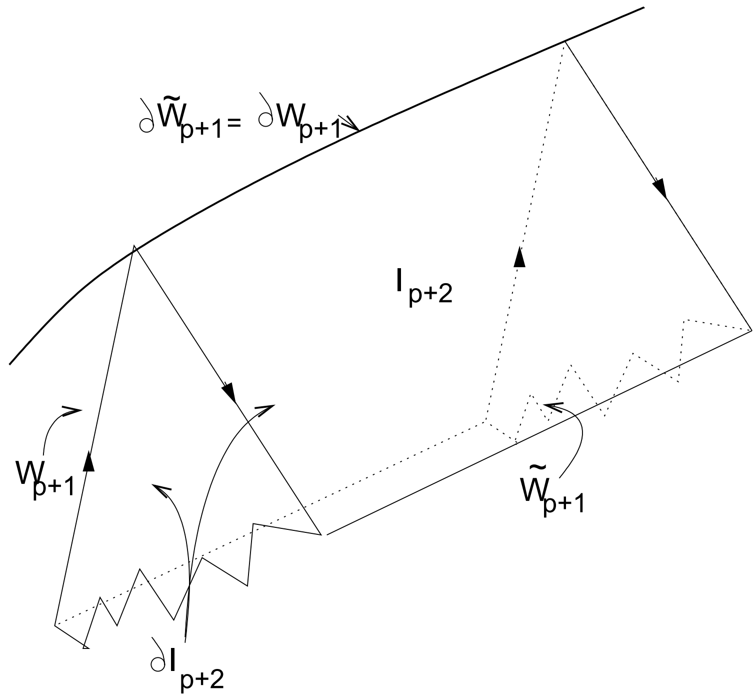

Figure B.1: Geometry of current ∂W_{p+1} and associated Θ-functions W_{p+1} and $\tilde{W}_{p+1}$.

Up to now W_{p+1} was considered as the physical world-volume of an actual p-brane and ∂W_{p+1} as its boundary, but of course one could as well consider $W_p \equiv \partial W_{p+1}$ as the world-volume of a trivially closed $(p-1)$-brane, then W_{p+1} is not physical and can be interpreted as a *Dirac brane* attached to the $(p-1)$-brane, in analogy to a *Dirac string* attached to a magnetic monopole in Maxwell theory. This analogy reaches further and Dirac branes are actually needed to couple magnetic branes to electric fields as follows.

To every electric $p-$brane there exists a dual q-brane with $q = d - p - 4$, magnetically charged with respect to the $(p+1)$-form A. The problem now arises because the magnetic q-brane with world-volume W_{q+1} produces a magnetic $(d-q-1 = p+3)$-current J_m which couples to the electric field by a non-trivial Bianchi identity like

$$dF = J_m = (-1)^{d-q-1}\, T_q\, \delta(W_{q+1})$$

which is in direct contradiction with $dF = d^2 A = 0$ to be true everywhere, thus the field strength cannot everywhere be derived from a globally well defined potential only, i.e. $F \neq dA$ globally on M_d. Thus a redefined field strength is introduced by adding a Dirac brane current like

$$F \equiv dA + (-1)^{d-q-1}\, T_q\, \Theta(W_{q+1}) \tag{B.9}$$

such that the Bianchi identity is given by

$$dF = (-1)^{d-q-1}\, T_q\, d\Theta(W_{q+1}) = (-1)^{d-q-1}\, T_q\, \delta(W_{q+1}) = J_m, \tag{B.10}$$

as required. At the classical level only the field strength (B.9) must be independent of the choice of the Dirac brane, thus under such a change as in eq.(B.8) the potential must also transform. Then the field strength F is invariant under a transformation given by

$$\begin{aligned} \Theta(W_{q+1}) &\longrightarrow \Theta(W_{q+1}) + \delta(\partial I_{q+3}) \\ A &\longrightarrow A - (-1)^{d-q-1} T_q\, \Theta(\partial I_{q+3}) + d\Lambda, \end{aligned} \tag{B.11}$$

where a normal gauge transformation has been included. This is the classical statement of Dirac brane independence, but on a quantum mechanical level the condition of the non observability of a Dirac brane leads to Dirac quantization conditions for the brane tensions that can be derived in the given formulation. Consider the situation illustrated in fig. B.2 where an electrical p-brane probes the field of a dual magnetic $q-$brane by moving through the field of the latter along a closed world-volume E_{p+1} not intersecting the Dirac brane. Then the integration of the potential A over E_{p+1}

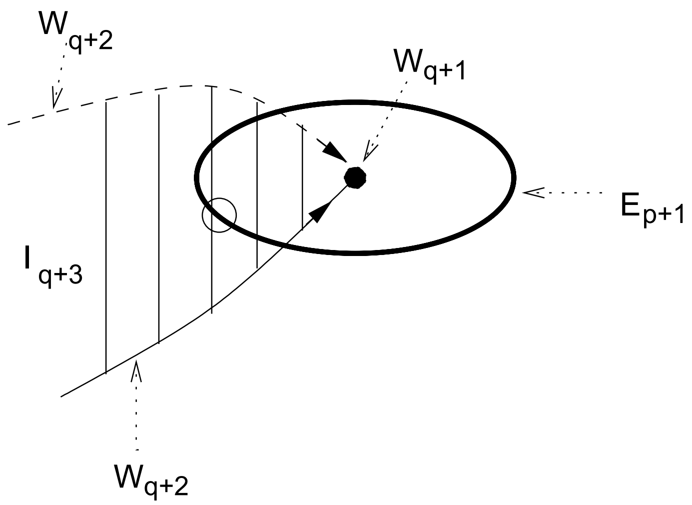

Figure B.2: Electric p-brane probing the field of its dual magnetic $q-$brane.

can be considered as a higher dimensional analogue of a Wilson loop, or equivalently as the phase picked up by the $p-$brane “wavefunction” after the motion along E_{p+1}. For the position of the Dirac brane to be unobservable this phase can at most change by an integral multiple of 2π under a change of Dirac brane, which implies the Dirac quantization condition. Formally the condition reads

$$\exp\big(-i\, T_p \int_{E_{p+1}} A\big) \overset{!}{=} \exp\big(-i\, T_p \int_{E_{p+1}} A - (-1)^{d-q-1} T_q \Theta(\partial I_{q+3})\big)$$

which, by eqs. (B.4,B.7), is exactly true if

$$T_p\, T_q \int_{M_d} \delta(E_{p+1}) \wedge \delta(I_{q+3}) = 2\pi\, k, \quad k \in \mathbb{Z}. \tag{B.12}$$

This last integral gives the intersection number of the two surfaces E_{p+1} and I_{q+3} and is thus an integer [211], possibly equal to one, and therefore we find the Dirac quantization condition

$$T_p\, T_q = 2\pi\, n, \quad n \in \mathbb{Z}\,.$$

Note that by a deformation of the world-volume E_{p+1} by a closed surface D_{p+1}, the integral (B.12) can always be made well defined, as illustrated in fig. B.3. This allows for a nice interpretation of normal gauge transformations $d\Lambda_p = \delta(D_{p+1})$, as the inclusion of a trivially closed (p+1)-dimensional Dirac-brane D_{p+1} somewhere in space-time, which does not affect the field strength in any way.

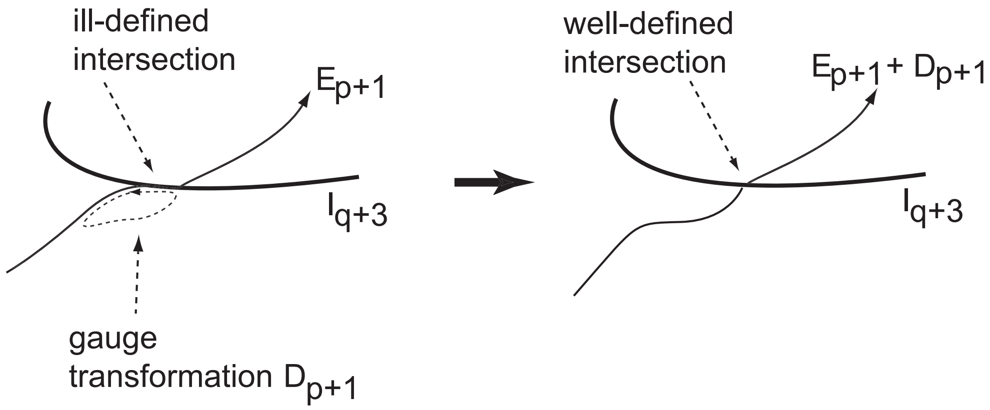

Figure B.3: Getting a well-defined intersection point by a gauge transformation.

Generally the need to introduce Dirac branes can be understood quite easily, because formally a magnetic monopole is treated as one pole of a dipole with the other pole sitting at infinity, both poles being connected by the Dirac brane. This construction assures that the magnetic field lines close, since the magnetic flux emitted from the monopole to infinity flows back along the Dirac brane. Moving the second pole around at infinity should certainly not affect the field of the monopole, and this corresponds to the requirement of Dirac brane independence.

At first sight it looks as if this whole coupling procedure is only applicable to magnetic $q-$branes with trivially closed world-volumes, i.e. $W_{q+1} = \partial W_{q+2}$, because only for such world-volumes globally well defined $\Theta-$functions exist, but eqs. (B.9, B.10) need not be globally defined, but must only be true on local coordinate patches, where in overlapping regions the Θ-functions and the potentials must be related by a transformation (B.11).

As a last remark, this way of coupling electric and magnetic branes is very asymmetric since the two corresponding currents are treated in quite a different way. For a different, duality symmetric way which treats magnetic and electric branes on equal footings see [212, 210], but this alternative way still depends on the introduction of Dirac branes though.

To summarize and generalize, to every p-chain $W_p = \sum_{i=1}^n a_i N_p^i,\ a_i \in \mathbb{K}$ of p-dimensional submanifolds N_p^i of a d-dimensional embedding space M_d, a $(d-p)$-current $\delta(W_p)$ can be associated, that is, a linear functional on the space $\Lambda^{(p)}(M_d)$ of smooth p-forms on M_d, like

$$\begin{aligned} \delta : W_p \longmapsto \delta(W_p) :\ \ \Lambda^{(p)}(M_d) &\longrightarrow \mathbb{C} \\ A_p &\longmapsto \int_{M_d} \delta(W_p) \wedge A_p = \sum_{i=1}^n a_i \int_{N_p^i} \hat{A}_p. \end{aligned}$$

Thus currents are generalized distributions, or loosely speaking sums of forms whose coefficients are distributions. Moreover, the current corresponding to a closed or exact p-chain is in the sense of distributions itself closed or exact, respectively. Restricting this correspondence to cycles on compact spaces we have the Poincaré duality, which maps the homology class of a p-cycle to the cohomology class of $(d-p)$-form, thus the notion of currents associated to chains can be viewed as an extension of Poincaré duality [157, 211].

Appendix C

Special Kähler geometry

In this appendix we briefly summarize the general structure of a special Kähler manifold. Here all the necessary relations of special geometry which we needed in the main text are given.

As is well known $\mathcal{N} = 1$ supersymmetry requires the moduli space parametrized by the scalar component fields of the vector- and hyper-multiplets to be a product space $\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H$ with no couplings between the two types of fields. In the context of Calabi-Yau compactifications the space $\mathcal{M}_V$ is a Kähler manifold parametrized by the Kähler moduli (ϕ^x, $x = 1, \ldots, h^{1,1}$) associated to the deformations of the Kähler class defined by the $H^{1,1}$ sector of the Calabi-Yau manifold. In the heterotic case the space $\mathcal{M}_H$ is a $4(h^{2,1} + 1) = 4n_H$-dimensional quaternionic manifold that factorizes like $\mathcal{M}_H = \mathcal{M}_{3,0} \times \mathcal{M}_{2,1}$, where the factors arise from the $H^{3,0}$ and $H^{2,1}$ sectors, respectively. The $H^{3,0}$ sector gives rise to the complex scalar ξ that pairs with the dilaton (volume modulus V) and the space-time axion σ to the scalar components of the universal hypermultiplet and they parametrize $\mathcal{M}_{3,0} = SU(1,1)/U(1)$. The manifold $\mathcal{M}_{2,1}$ is parametrized by the complex scalars $\mathfrak{z}^p$ and η^p, $p = 1, \ldots, h^{2,1}$ arising from the complex structure deformations of the metric and the three-form potential, respectively. For $\eta^p = 0$ this manifold reduces to a submanifold like $\mathcal{M}_H \stackrel{\eta^p=0}{\longrightarrow} \mathcal{M}_{SK}$ to a special Kähler manifold $\mathcal{M}_{SK}$ parametrized by the complex structure moduli $\mathfrak{z}^p$ only. It is this special Kähler geometry we are interested in. It turns out to be more convenient to treat the whole $H^3 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$ sector at once and then let the different sectors emerge through the choice of a complex structure specified by the $\mathfrak{z}^p$.

Let us first introduce a canonical homology basis of H_3 consisting of three-cycles (A^P, B_Q), $P, Q = 0, \ldots, h^{2,1}$ together with the dual cohomology basis of H^3 given by the three-forms (α_P, β^Q), normalized such that

$$\int_X \alpha_Q \wedge \beta^P = \int_{A^P} \alpha_Q = \delta^P_Q\,, \qquad \int_X \alpha_P \wedge \alpha_Q = 0\,, \tag{C.1}$$

$$\int_X \beta^P \wedge \alpha_Q = \int_{B_Q} \beta^P = -\delta^P_Q\,, \qquad \int_X \beta^P \wedge \beta^Q = 0\,. \tag{C.2}$$

These relations are invariant under symplectic transformations [160] like $(\vec{\alpha}, \vec{\beta}) \to T(\vec{\alpha}, \vec{\beta})$ with $T \in \mathrm{Sp}(2b^{2,1} + 2; \mathbb{Z})$, which is the reason for the symplectic structure of the manifold $\mathcal{M}_H$. Now we can expand the holomorphic (3,0)-form Ω with respect to the basis (α_P, β^Q) in a symplectically invariant way like

$$\Omega = \mathcal{Z}^Q \alpha_Q - \mathcal{G}_Q \beta^Q\,, \tag{C.3}$$

with the periods $(\mathcal{Z}^Q, \mathcal{G}_Q)$ defined by

$$\mathcal{Z}^Q = \int_X \Omega \wedge \beta^Q = \int_{A^Q} \Omega\,, \qquad \mathcal{G}_P = \int_X \Omega \wedge \alpha_P = \int_{B_P} \Omega\,. \tag{C.4}$$

It is known that the complex structure of the Calabi-Yau space is entirely determined by the $\mathcal{Z}^Q$, and turning the argument around, the choice of complex structure determines $\Omega = \Omega(\mathcal{Z})$ and thus also $\mathcal{G}_Q = \mathcal{G}_Q(\mathcal{Z})$. It is a non-trivial result [160] that the periods $\mathcal{G}_Q$ actually derive from a holomorphic prepotential $\mathcal{G} = \mathcal{G}(\mathcal{Z})$ like $\mathcal{G}_Q = \frac{\partial}{\partial \mathcal{Z}^Q}\mathcal{G}$. Then from the first relation in (C.4) it follows that $\Omega(\lambda\mathcal{Z}) = \lambda\Omega(\mathcal{Z})$ and thus $\mathcal{G}_Q(\lambda\mathcal{Z}) = \lambda\mathcal{G}_Q(\mathcal{Z})$, $\mathcal{G}(\lambda\mathcal{Z}) = \lambda^2\mathcal{G}(\mathcal{Z})$ under a rescaling $\mathcal{Z} \to \lambda\mathcal{Z}$. For the prepotential this yields the important relations

$$\mathcal{G} = \frac{1}{2}\mathcal{G}_Q\mathcal{Z}^Q, \qquad \mathcal{G}_Q = \mathcal{G}_{QR}\mathcal{Z}^R\,, \tag{C.5}$$

where $\mathcal{G}_{QR} = \partial_Q\partial_R\mathcal{G}$. Moreover the scaling behavior just given shows that the periods $\mathcal{Z}^Q$ are projective coordinates that can be related to the affine special coordinates for example by $\mathfrak{z}^p = \mathcal{Z}^p/\mathcal{Z}^0$ wherever $\mathcal{Z}^0 \neq 0$.

Now the Kähler potential can be defined by

$$\mathcal{K}(\mathcal{Z}) = -\ln\left(i\int_X \Omega \wedge \bar{\Omega}\right) = -\ln\left(\mathcal{G}_Q\bar{\mathcal{Z}}^Q - \bar{\mathcal{G}}_Q\mathcal{Z}^Q\right) = -\ln\left(2i\,\mathrm{Im}\mathcal{G}_{PQ}\bar{\mathcal{Z}}^P\mathcal{Z}^Q\right)\,,$$

which demonstrates its invariance under symplectic transformations, that cannot so easily be seen in terms of special coordinates where it is given by

$$\mathcal{K}(\mathfrak{z}) = -\ln\left[2i(\mathcal{G} - \bar{\mathcal{G}}) - i(\mathfrak{z}^p - \bar{\mathfrak{z}}^p)\left(\frac{\partial\mathcal{G}}{\partial\mathfrak{z}^p} + \frac{\partial\bar{\mathcal{G}}}{\partial\bar{\mathfrak{z}}^p}\right)\right]\,. \tag{C.6}$$

This shows that the Kähler potential is completely determined by a holomorphic prepotential $\mathcal{G}$ and the coordinates $\mathfrak{z}^p$ or $\mathcal{Z}^P$ respectively, and it can be shown that such Kähler potentials satisfy the "special geometry"constraint [213]

$$R_{p\bar{q}r\bar{s}} = \mathcal{K}_{p\bar{q}}\mathcal{K}_{r\bar{s}} + \mathcal{K}_{p\bar{s}}\mathcal{K}_{r\bar{q}} - e^{2\mathcal{K}} C_{prt}\bar{C}_{\bar{q}\bar{s}\bar{u}}\mathcal{K}^{t\bar{u}}\,, \tag{C.7}$$

where

$$\mathcal{K}_{p\bar{q}}(\mathfrak{z}) \;=\; \frac{\partial}{\partial \mathfrak{z}^p}\frac{\partial}{\partial \bar{\mathfrak{z}}^{\bar{q}}}\mathcal{K}(\mathfrak{z})\,, \qquad C_{pqr} = \frac{\partial}{\partial \mathfrak{z}^p}\frac{\partial}{\partial \mathfrak{z}^q}\frac{\partial}{\partial \mathfrak{z}^r}\mathcal{G}(\mathfrak{z})\,, \tag{C.8}$$

and $R_{p\bar{q}r\bar{s}}$ is the curvature. That is why one is talking of a *special* Kähler manifold in this case.

Let us next consider the $H^{2,1}$ sector as a part of H^3. To do this we need a result of Kodaira that states [160]

$$\partial_p\Omega = -\partial_p\mathcal{K}(\mathfrak{z}) + \Pi_p\,, \tag{C.9}$$

with $\{\Pi_p\}$ a basis of $H^{2,1}$. Using this together with (C.3) and the explicit form of the Kähler potential we can expand the basis forms Π_p with respect to the basis (α_P, β^Q) like

$$\Pi_p = f_p{}^Q\alpha_Q - h_{p\,Q}\beta^Q\,, \tag{C.10}$$

with the periods $(f_p{}^Q, h_{p\,Q})$ given by

$$\begin{aligned} f_p^Q &= \partial_p\mathcal{Z}^Q + \partial_p\mathcal{K}(\mathfrak{z})\,\mathcal{Z}^Q = \partial_p\mathcal{Z}^P\tilde{f}_P^Q\,, \qquad \tilde{f}_P^Q = \delta_P^Q + \partial_P\mathcal{K}\mathcal{Z}^Q\,,\\ h_{p\,Q} &= \partial_p\mathcal{G}_Q + \partial_p\mathcal{K}(\mathfrak{z})\,\mathcal{G}_Q = \mathcal{G}_{QR}f_p^R\,, \end{aligned} \tag{C.11}$$

where by using $\mathcal{Z}^P\partial_P\mathcal{K} = -1$ it can easily be shown that $\tilde{f}_p^Q$ is a projector. The Kähler metric can now also be written like

$$\mathcal{K}_{p\bar{q}} = f_p^P\mathcal{K}_{PQ}\bar{f}_{\bar{q}}^Q = -ie^{\mathcal{K}}(h_{p\,P}\bar{f}_{\bar{q}}^P - f_p^P\bar{h}_{\bar{q}\,P}) = -\frac{\int_X \Pi_p\wedge\bar{\Pi}_{\bar{q}}}{\int_X \Omega\wedge\bar{\Omega}}\,. \tag{C.12}$$

Next we introduce a matrix M defined by [162]

$$M_{PQ} = \bar{\mathcal{G}}_{PQ} + T_{PQ}\,, \qquad T_{PQ} = 2i\frac{\mathrm{Im}\mathcal{G}_{PR}\mathcal{Z}^R\,\mathrm{Im}\mathcal{G}_{QS}\mathcal{Z}^S}{\mathcal{Z}^R\mathrm{Im}\mathcal{G}_{RS}\mathcal{Z}^S}\,, \tag{C.13}$$

and this matrix has the properties

$$\mathcal{G}_P = M_{PQ}\mathcal{Z}^Q, \qquad h_{p\,P} = \bar{M}_{PQ}f_p^Q, \qquad \mathrm{Im}M_{PQ}\mathcal{Z}^P\bar{\mathcal{Z}}^Q = -\frac{1}{2}e^{-\mathcal{K}}, \tag{C.14}$$

or equivalently

$$T_{PQ}\mathcal{Z}^Q = 2i\,\mathrm{Im}\mathcal{G}_{PQ}\mathcal{Z}^Q, \qquad \bar{T}_{PQ}f_p^Q = 0, \qquad \mathrm{Im}T_{PQ}\mathcal{Z}^P\bar{\mathcal{Z}}^Q = -e^{-\mathcal{K}}. \tag{C.15}$$

This matrix M shows up in the kinetic terms of the hypermultiplet scalars in $\mathcal{N}=1$ supergravity[1]. This is due to the following relations [214]

$$\int_X \alpha_P \wedge *\beta^Q = -\mathrm{Re}M_{PR}\,(\mathrm{Im}M)^{-1\,RQ}\,, \tag{C.16}$$

$$\int_X \alpha_P \wedge *\alpha_Q = -\left[\mathrm{Im}M_{PQ} + \mathrm{Re}M_{PR}(\mathrm{Im}M)^{-1\,RS}\mathrm{Re}M_{SQ}\right]\,, \tag{C.17}$$

$$\int_X \beta^P \wedge *\beta^Q = -(\mathrm{Im}M)^{-1\,PQ}\,, \tag{C.18}$$

which are needed in the process of compactification. The inverse of ImM is explicitly given by

$$(\mathrm{Im}M)^{-1\,PQ} = -(\mathrm{Im}\mathcal{G})^{-1\,PQ} - 2e^{\mathcal{K}}\left[\mathcal{Z}^P\bar{\mathcal{Z}}^Q + \bar{\mathcal{Z}}^P\mathcal{Z}^Q\right]\,. \tag{C.19}$$

Other useful relations are [162]

$$U^{PQ} = f_p^P\mathcal{K}^{p\bar{q}}\bar{f}_{\bar{q}}^Q = -\frac{1}{2}e^{-\mathcal{K}}(\mathrm{Im}M)^{-1\,PQ} - \bar{\mathcal{Z}}^P\mathcal{Z}^Q. \tag{C.20}$$

Using the second relation in (C.15) we see that

$$U^{PQ}M_{QR} = U^{PQ}\bar{\mathcal{G}}_{QR}, \qquad \mathcal{G}_{PQ}U^{QR} = \bar{M}_{PQ}U^{QR}\,. \tag{C.21}$$

The last relation we give is useful in the demonstration of the symplectic quaternionic structure of the hyper-multiplet sector as done in section (2.5.2), and it is

$$(\mathrm{Im}M)^{-1\,PQ} = -\frac{1}{2}e^{-\mathcal{K}}\left[\mathrm{Im}\mathcal{G}^{-1\,PR}\mathcal{K}_{RS}\mathrm{Im}\mathcal{G}^{-1\,SQ}\right] - 2\,e^{\mathcal{K}}\mathcal{Z}^P\bar{\mathcal{Z}}^Q. \tag{C.22}$$

[1]In the case of suitable compactifications of Type II theories it is also related to kinetic and topological terms F^2 and $F\tilde{F}$ of vector fields [214], though not in the considered heterotic case.

Appendix D

Riemann surfaces

This appendix is intended to give a very short review of some basic facts on the structure of the holonomy of a Riemann surface, which are needed to understand the origin and structure of the worldvolume gauge fields on the three-brane arising as zero modes from the M5-brane worldvolume theory wrapped on such a Riemann surface. We closely follow the treatment in [215], to which we refer for proofs and details.

A *Riemann surface* is a connected, complex analytic manifold M of one complex (two real) dimension(s). Thus there is a set of charts $\{U_\alpha, z_\alpha\}_{\alpha\in I}$ consisting of an open cover $\{U_\alpha\}_{\alpha\in I}$ of M such that $M = \bigcup_{\alpha\in I} U_\alpha$, and coordinate functions $z_\alpha : U_\alpha \to \mathbb{C}$ which are homeomorphisms onto open sets of the complex plane, such that the transition functions $t_{\alpha\beta} = z_\alpha \circ z_\beta : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta)$ are holomorphic for $U_\alpha \cap U_\beta \neq \emptyset$.

To each Riemann surface of genus g a $4g-$sided polygon can be associated, which can be represented by its *symbol* $\Pi_{u=1}^{g}\, b_u\, a_u\, b_u^{-1}\, a_u^{-1}$. As shown in figure D.1, each side of the polygon corresponds to a cycle (homology class) of the Riemann surface, and by glueing pairs of sides a_u, a_u^{-1} and b_u, b_u^{-1} respectively, one can obtain the surface from the polygon. Because there are $2g$ independent cycles a_u, b_u for $i = 1 \ldots g$ on a compact Riemann surface M of genus g, the homology group $H_1(M)$ of cycles is $2g$ dimensional.

With every simple (not self-intersecting) closed curve c on M a real, closed one-form current $\delta(c)$ can be associated such that[1]

$$\langle \alpha, *\delta(c)\rangle = -\int_M \alpha \wedge \delta(c) = \int_c \alpha \tag{D.1}$$

for any closed one-form α and where the scalar product and Hodge star are as defined in appendix A. Because every cycle c on M, that is, every closed curve that itself is

[1]The last integral is the integration along the curve c of the pullback of α to this curve, for which we do not introduce extra notation here.

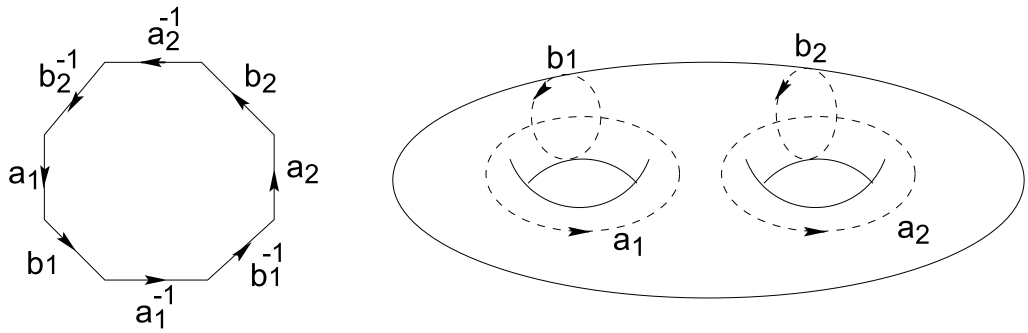

Figure D.1: Riemann surface of genus 2 with cycles a_u, b_u as a standard basis of first homology group $H_1(M)$.

not a boundary, is a finite sum of simple cycles, there exists such an associated real current $\delta(c)$ with the above property. For two given cycles c_1 and c_2 on the Riemann surface M this allows to define the *intersection number* by

$$c_1 \cdot c_2 = \int_M \delta(c_1) \wedge \delta(c_2) = \langle \delta(c_1), - * \delta(c_2) \rangle, \in \mathbb{Z}. \tag{D.2}$$

The intersection properties of the standard basis $\{a_1, b_1, \ldots, a_g, b_g\}$ are

$$a_u \cdot b_v = \delta_{uv}, \qquad a_u \cdot a_v = 0 = b_u \cdot b_v. \tag{D.3}$$

Relabeling the cycles by putting all $a-$cycles in front of the $b-$cycles like

$$h_U = a_U, \quad U = 1 \ldots g, \qquad h_U = b_{U-g}, \quad U = g+1 \ldots 2g, \tag{D.4}$$

leads to a basis $\{h_1 \ldots h_{2g}\}$ of $H_1(M)$ with the *intersection matrix* given by

$$h_U \cdot h_U = \mathbf{J}_{UV}, \qquad \mathbf{J} = \begin{bmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{bmatrix}, \tag{D.5}$$

where $\mathbf{1}_g$ is the $g \times g$ identity matrix. Such a basis with intersection matrix $\mathbf{J}$ is called *canonical homology basis*. To a given canonical homology basis define the currents

$$\begin{aligned} \lambda_U &= \delta(h_{U+g}) = \delta(b_U), \quad U = 1 \ldots g, \\ \lambda_U &= -\delta(h_{U-g}) = -\delta(a_{U-g}), \quad U = g+1 \ldots 2g, \end{aligned}$$

then it follows from Eqs.(D.1-D.5) that

$$\int_{h_U} \lambda_V = \delta_{UV}, \tag{D.6}$$

moreover note that $-\langle\lambda_U, *\lambda_V\rangle = \mathbf{J}_{UV}$. The set $\{\lambda_1 \dots \lambda_{2g}\}$ is the unique real, dual basis of the cohomology group $H^1(M)$, that is, the $2g-$dimensional vector space of harmonic one-forms on M, with the property (D.6). This allows to uniquely expand every harmonic one-form like $\lambda = \sum_{U=1}^{2g} p_U \lambda_U$ with *periods* $p_U = \int_{h_U} \lambda$. Because the Hodge star operator preserves the space of real harmonic forms, the Hodge duals of the basis one-forms can then be expanded in terms of the basis itself like

$$*\lambda_U = \mathbf{T}_U{}^V \lambda_V, \tag{D.7}$$

or in different words, the Hodge star operator is represented by a $2g \times 2g$ matrix $\mathbf{T}$ in the space of harmonic one-forms. It turns out that the matrix $\mathbf{\Gamma} = \mathbf{T}(\mathbf{J}^T)$ is given by

$$\mathbf{\Gamma}_{UV} = \mathbf{T}_{UW}(\mathbf{J}^T)^W{}_V = \langle\lambda_U, \lambda_V\rangle \tag{D.8}$$

and it is symmetric and positive definite. Writing the matrix $\mathbf{T}$ in block form like

$$\mathbf{T} = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{bmatrix}, \tag{D.9}$$

the symmetry and positive definiteness of $\mathbf{\Gamma}$ imply

$$\Lambda_4 = -\Lambda_1^T, \qquad \Lambda_2 = \Lambda_2^T, \qquad \Lambda_3 = \Lambda_3^T, \qquad \Lambda_2 > 0, \qquad \Lambda_3 < 0, \tag{D.10}$$

and so Λ_2 and Λ_3 are invertible, moreover from $\mathbf{T}^2 = -\mathbf{1}_{2g}$ $(\Leftrightarrow\ *^2 = -1)$ follows

$$\Lambda_1^2 + \Lambda_2\Lambda_3 + \mathbf{1}_g = 0, \qquad \Lambda_1\Lambda_2 = \Lambda_2(\Lambda_1^T), \qquad \Lambda_3\Lambda_1 = (\Lambda_1^T)\Lambda_3. \tag{D.11}$$

So far only real-valued forms have been considered, but on a complex surface we of course can introduce (anti-) holomorphic forms. A one-form ω is called *holomorphic* when locally $w = df$ with f a holomorphic function, and this is true if and only if $\omega = \beta + i * \beta$ for some harmonic one-form β. Thus introduce holomorphic one-forms by

$$\omega_U = \lambda_U + i * \lambda_U, \qquad U = 1 \dots 2g, \tag{D.12}$$

then the space of complex harmonic one-forms decomposes as

$$H^1(M) = H^{(1,0)}(M) \oplus H^{(0,1)}(M) \tag{D.13}$$

where the set $\{\omega_1 \dots \omega_g\}$ $(\{\bar{\omega}_1 \dots \bar{\omega}_g\})$ is a basis of the vector space of holomorphic (antiholomorphic) one-forms $H^{(1,0)}(M)$ $(H^{(0,1)}(M))$, which thus is $g-$dimensional. To summarize, on a compact Riemann surface M of genus g the vector space $H^{(1,0)}(M)$ of holomorphic one-forms has dimension g, and $\{\omega_1 \dots \omega_g\}$ forms a basis thereof.

The following matrix

$$\begin{aligned}\frac{1}{2}\langle\omega_U,\omega_V\rangle = & \quad \langle\lambda_U,\lambda_V\rangle - i\langle\lambda_U,*\lambda_V\rangle & = (\mathbf{\Gamma}+i\mathbf{J})_{UV} \\ = & \qquad\qquad \langle\lambda_U,\omega_V\rangle & = \begin{cases} -i\int_{b_V}\omega_U, & V=1\dots g \\ +i\int_{a_{V-g}}\omega_U, & V=g+1\dots 2g \end{cases}\end{aligned}$$

can be interpreted as the *period matrix* of the holomorphic forms $\{\omega_U,\ U=1\dots 2g\}$ with respect to the standard cycles $\{a_u,\ b_u,\ u=1\dots g\}$ as seen in the second line, moreover the first line allows to directly relate this period matrix to the matrix $\mathbf{T}$ through Eq. (D.8). By a clever change of basis we can now engineer a $g\times g$ block of this period matrix to any desired form, especially consider the following basis

$$\alpha_u = (-i\ \Lambda_3^{-1})_u{}^v\ \omega_{v+g}, \tag{D.14}$$

then $\{\alpha_u,\ u=1\dots g\}$ is the unique basis of the space of holomorphic one-forms $H^{(1,0)}(M)$ with the property

$$\int_{a_u}\alpha_v = \delta_{uv}, \tag{D.15}$$

and moreover the period matrix defined by

$$\mathbf{\Pi}_{uv} \equiv \int_{b_u}\alpha_v \tag{D.16}$$

is given by

$$\mathbf{\Pi} = (-\Lambda_3)^{-1}\ \Lambda_1^T + i\ (-\Lambda_3)^{-1}, \tag{D.17}$$

showing that by Eq. (D.10) it is symmetric and has positive definite imaginary part.

To summarize, we now have explicitly constructed a basis of holomorphic one-forms $\{\alpha_u,\ u=1\dots g\}$ of $H^{(1,0)}(M)$ associated to the standard basis of cycles $\{a_u,\ b_u,\ u=1\dots g\}$ of the first homology group $H_1(M)$ satisfying the standard normalization (D.15). Moreover Eqs. (D.9, D.17) gives the relation between the period matrix $\mathbf{\Pi}$ and the matrix representation $\mathbf{T}$ of the Hodge star operator.

Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfasst zu haben und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Matthias Brändle
19. Februar 2003