CONSTRAINTS AND SPONTANEOUS SYMMETRY BREAKING IN QUANTUM FIELD THEORY

An application of Rieffel induction and other methods

Dissertation
submitted for the degree of Doctor of Philosophy
at the University of Cambridge, U.K.

by
Urs Achim Wiedemann
from
Weil am Rhein bei Basel

Wolfson College
Cambridge
1994

Declaration

The material presented herein, with the exception of the introductory exposition and otherwise where explicitly indicated, constitutes an original contribution to the study of constraints and spontaneous symmetry breaking in quantum field theory. Nothing herein has been submitted for any degree, diploma or other qualification at any other University. The main results of Chapter II and III have been published in a collaborative work with N.P. Landsman [1]. The calculations presented in these chapters have been done by the author, yet they are part of a work done in collaboration and would not have found their present form without that. No part of Chapters IV, V, VI and VII is the outcome of work done in collaboration. The major part of Chapter VI has been published essentially without modifications [2]. Chapter V is in preparation for a publication.

Summary

The first part of this thesis applies the Rieffel induction procedure, recently advocated by Landsman for the quantization of systems with constraints, to certain linear quantum field theories. After a preparatory chapter in which Rieffel induction is used to implement constraints on the Heisenberg algebra $[q_\mu, p_\nu] = -ig_{\mu\nu}$, Landsman's proposal is applied to free quantum electrodynamics (QED). Starting from the Fock representation of the unconstrained field algebra, a new representation of the field algebra on the Rieffel-induced Hilbert space $\mathcal{H}_{phys}$ is constructed, which carries a trivial action of the gauge group. This leads to a new type of gauge fixing, lying conceptually between the Coulomb gauge and the Gupta-Bleuler gauges. The characteristic features of this formulation of free QED are presented in detail (Hamiltonian, propagator, n-point correlation functions, (semi)-positivity of the metric, implementation of the Poincaré group, action of gauge transformations, etc.). Also, some steps are undertaken to apply the method to functional representations and for a 3-dimensional vector potential with $A_0 = 0$.

Subsequently, the Rieffel induction procedure is applied to a simple model showing spontaneous symmetry breaking, viz. the Stückelberg-Kibble model. Here, a physical state space $\mathcal{H}_{phys}$ is constructed which carries a massive representation of the Poincaré group. Its longitudinal one-particle component arises from a particular Bogoliubov-transformation of the five (unphysical) degrees of freedom one has started with.

The second, smaller part of this thesis contains two more chapters on spontaneous symmetry breaking. In the first one, formal properties of the effective potential of a scalar field theory are investigated. One finds that the effective potential is exactly one-fold differentiable at the value of the vacuum expectation value, and this turns out to be crucial for the validity of the perturbative loop expansion. In a second chapter, the algebraic characterization of the vacuum expectation value of a scalar field as an element in the center of the weak closure of certain representations of the field algebra is used in an attempt to simplify a particular type of gauge-invariant interaction terms.

Acknowledgements

First of all, I would like to thank my supervisor, Dr. N.P. Landsman, for providing the encouragement and the freedom without which this thesis would not have been realised. Also I am grateful to him for suggesting this interesting line of research to me and for the unwavering help and support I have received from him throughout the last three years.

I have benefitted from discussions with many others who have often clarified or corrected the ideas I tried to follow up. Special thanks are due to: A. Amann, G. Bacciagaluppi, T. Breuer, D. Buchholz, A.C. Davis, M. Earnshaw, M. Gaberdiel, H. Grundling, M. Hindmarsh, R. Horgan, C. Jäkel, M. Köhler, G. Morchio, M. Müger, A. Petkos, M. Redhead, R. Rivers, F. Strocchi, R. Verch.

At different stages of this work, financial support has been received from: the Science and Engeneering Research Council (SERC), the Konrad-Adenauer Foundation, the Ribbands Scholarship and, most importantly, the European Communities Research Training Fellowship under grant No. ERBCHBICT920018 which supported the last two years of this work. Furthermore, I would like to acknowledge financial help towards Conference Fees received from Wolfson College Cambridge and the Cambridge Philosophical Society, as well as the help received during visits towards travel and living expenses from the Istituto Nazionale di Fisica Nucleare, Pisa, Italy, the Schrödinger Institut Wien, Austria, the Banff Thermal Field Workshop Local Organizing Committee, Alberta, Canada and the Rencontres de Moriond, Les Arcs, France.

Also, thanks are due to K. Fredenhagen who allowed me to spend the academic year 93/94 at the II. Institut für Theoretische Physik, Hamburg, FRG as well as to DESY Hamburg and DAMTP Cambridge for providing a working place and the basic equipment.

Last but not least, I have to thank my parents and Anita.

Adler, dulde mein kreisen
Neben der sonne und dir.
Ist deine beute auch mir?
Traue mir, niemals entreißen
Andere unser revier.[3]

List of Contents

Introductory Part

Rieffel Induction

Spontaneous Symmetry Breaking

CHAPTER I:
INTRODUCTION

Two strategies offer themselves for imposing constraints on a quantum field theory (QFT). Either one starts from quantizing a classical system with constraints imposed, or the unconstrained quantized theory is taken as starting point. The first part of this Introduction provides a brief survey of different methods for dealing with constraints in QFTs according to either of these strategies. some problems arising in QFTs showing spontaneous symmetry breaking (SSB). In the case of scalar QFTs without gauge fields, we focus on the effective potential, which monitors spontaneous symmetry breakdown. In the case of gauge QFTs, we draw attention to the fact that SSB is a gauge-dependent concept. While most technical prerequisites are developed in the subsequent chapters, this Introduction aims at relating these chapters to each other and to the literature.

1. Theories of Constraints

We give a brief outline of Dirac's treatment of constraints, following [4] and [5]. After recalling the Gupta-Bleuler indefinite metric method, and its relation to Dirac's treatment, we discuss two more recent proposals for imposing constraints: the T-procedure, advocated by Grundling and Hurst [6, 7, 8], which takes the quantized theory as starting point, and the Rieffel induction procedure proposed by Landsman [9]. Both proposals are formulated in an operator-algebraic setting which, besides conceptual clarity, offers the advantage of avoiding certain representation-dependent difficulties (e.g. the spectral problem in Dirac's theory of constraints, mentioned below). Throughout our work, this may be regarded as the main motivation for adopting an algebraic approach whenever applicable.

Dirac's treatment of constraints

Historically, the first treatment of constraints in quantum theories has been proposed by Dirac [10]. It starts from a classical theory formulated on TQ, the tangent space of a manifold Q, on which the dynamics is specified by a Lagrangian $L(q,\dot{q})$, with $(q,\dot{q}) \in TQ$, and the corresponding equations of motion

$$\frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial \dot{q}^i \partial q^j}\dot{q}_j =: f_i(q,\dot{q}). \tag{1.1.1}$$

These equations have a solution only if the vector $f_i(q,\dot{q})$ lies in the range of $W_{ij} = \frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j}$. Generally, W may be singular, in which case $\det W = 0$ and solutions exist only for $(q,\dot{q}) \in (TQ)^{(1)} \subset TQ$, a certain submanifold of TQ. Changing by Legendre transformation, $p_i = \frac{\partial L(q,\dot{q})}{\partial \dot{q}^i}$, to the cotangent manifold T^*Q, a set of s *primary constraints*

$$\phi^s(q,p) = 0, \qquad (q,p) \in T^*Q, \tag{1.1.2}$$

specifies $(T^*Q)^{(1)} \subset T^*Q$, the image of the Legendre transform. These primary constraints ensure that on $(T^*Q)^{(1)}$, at least the initial value problem is well-defined. Furthermore, to ensure that the constraints are consistent with the dynamics and with each other, one requires that the Poisson brackets of ϕ^s with the Hamiltonian and with themselves have to be constraints. In general, this is not the case and the obtained Poisson brackets are used as *secondary constraints* for a further reduction of the phase space. Finally, iteration shows either that the theory is inconsistent or it leads to a set of constraints which are consistent with the dynamics and with each other. Then, the resulting final constraint manifold C is a submanifold of $(T^*Q)^{(1)}$.

Dirac classified constraints by a compatibility requirement: a constraint is called a *first class constraint* if its Poisson bracket with arbitrary constraints is a constraint. Otherwise, it is called a *second class constraint*.

First class constraints $\phi^\alpha(q,p)$ generate transformations within C for which C decays into equivalence classes under 'gauge transformations',

$$(q_i, p_i) \to (q_i + \epsilon\{q_i, \phi^\alpha\}, p_i + \epsilon\{p_i, \phi^\alpha\}) \tag{1.1.3}$$

According to Dirac's ideas, second class constraints can be eliminated explicitly. Dirac dealt with them by replacing Poisson brackets $\{.,.\}$ with Dirac brackets $\{.,.\}_D$ [1] which have the same algebraic properties as Poisson brackets but satisfy

$$\{f, \chi_s\}_D = 0, \tag{1.1.4}$$

where $\{\chi_s\}$ denotes the set of all second class constraints and f is an arbitrary constraint. The essential point is that one may go to a new set of canonical variables (Q_i, P_i), $i = 1, ..., N$ so that the

[1] For a definition and discussion of Dirac brackets, cf. [5, 10].

S different second class constraints are functions of the $i = N - S + 1, ..., N$ components only. Then, the dynamics can be discussed on the $2(N - S)$-dimensional manifold on which one is left with first class constraints only. Since second class constraints are not the subject of what follows, we refer to the literature for a deeper discussion, cf. [5, 10].

For quantizing classical systems, Dirac postulated his correspondence principle between Poisson brackets and commutators, $\{f, g\} \rightarrow \frac{1}{i\hbar}[\hat{f}, \hat{g}]$, associating with each function $f(q, p)$ over phase space an operator $\hat{f}$ given in a representation on some Hilbert space $\mathcal{H}$. The proposal is then to impose the quantized first class constraints as *supplementary conditions* on $|\,\psi\rangle \in \mathcal{H}$,

$$\hat{\phi}^{\alpha}(q, p)|\,\psi\rangle = 0, \tag{1.1.5}$$

thereby singling out a subspace $\mathcal{H}_{phys} \subset \mathcal{H}$.

Remarks:

- If $\hat{\phi}^{\alpha}(q, p)$ were a second class constraint, condition (1.1.5) would be inconsistent. E.g., $\phi^1 = q$, $\phi^2 = p$, $\hat{q}|\,\psi\rangle = \hat{p}|\,\psi\rangle = 0$ but $[\hat{q}, \hat{p}]|\,\psi\rangle = i\hbar|\,\psi\rangle \neq 0$. This problem is circumvented by introducing Dirac brackets.

- If 0 does not lie in the point spectrum of $\hat{\phi}^{\alpha}(q, p)$, then there are no normalizable vectors $|\,\psi\rangle \in \mathcal{H}$ for which (1.1.5) holds. E.g., consider free QED with the gauge transformation $A_i \rightarrow A_i + \nabla_i g$, generated by $\phi(x) = \nabla_i E_i(x)$. From Maxwell's equations, we obtain the Gauss law $\phi(x) = 0$ whereas the commutation relations $[A_i, \phi(y)] = i\hbar\nabla_{y,i}\delta(x - y)$ forbid us to interpret $\phi = 0$ as an operator equation. One may try to adopt Dirac's treatment, taking ϕ as supplementary condition in the sense of (1.1.5), but there is still the difficulty that ϕ has a purely continuous spectrum [11, 12].[2]

The Gupta-Bleuler method

One way to circumvent the problem of finding normalizable vectors satisfying condition (1.1.5) is to introduce an indefinite metric η in $\mathcal{H}$, such that $|\langle\psi\mid\eta\mid\psi\rangle| < \infty$. The Gupta-Bleuler method [13, 14], starts from this idea and gives a procedure of how to regain a physical interpretation in such an indefinite metric formalism:

[2] In fact, in the finite dimensional case, according to v. Neumann's uniqueness theorem, the Weyl form of A_i, ϕ has essentially only one irreducible representation and in this representation, the operators have a purely continuous spectrum, i.e., there is no $|\,\psi\rangle$ with $\langle\psi, \psi\rangle < \infty$, satisfying (1.1.5). As pointed out by Narnhofer and Thirring, this problem reminds one of the improper eigenfunctions of the position operator x, which are something like δ-functions but not normalizable, $\int\delta^2 = \infty$, [12].

Given a Hilbert space $\mathcal{H}$ with positive definite scalar product $\langle .,. \rangle$, the physical expectation values are computed in terms of an indefinite sesquilinear form $(.,.) = \langle ., \eta . \rangle$ with metric operator η. This product is positive semidefinite on a proper and maximal subspace $\mathcal{H}' \subset \mathcal{H}$. The physical Hilbert space $\mathcal{H}_{phys}$ is then defined as the quotient

$$\mathcal{H}_{phys} = \mathcal{H}'/\mathcal{H}'', \tag{1.1.6}$$

where $\mathcal{H}''$ contains all elements of $\mathcal{H}'$ with vanishing $(.,.)$-norm.

The non-physical aspect of an indefinite metric in the Gupta-Bleuler formalism has been found repellent by many physicists. But there is a no-go-theorem of Strocchi [15, 16] saying that only a vector potential A_μ which is both non-covariant and non-local can satisfy Maxwell's equations as operator equations on a Hilbert space. The latter indeed happens in Coulomb gauge QED. In the light of Strocchi's theorem, the indefinite metric formalism appears to be if not the unique then at least a very economical way of obtaining a local and covariant formulation of gauge theories. Also, with the advent of the Faddeev-Popov path integral method for gauge fixing [17], the Gupta-Bleuler method has turned out to be very suitable for perturbative calculations, while on the other hand, a rigorous mathematical frame has been given to it (in the sense of a modified Wightman field theory) by the work of Strocchi and Wightman for the abelian case of free QED, cf. [18].

The T-procedure

Strocchi's theorem indicates that one cannot get rid of an indefinite metric formalism without paying a certain prize. The T-procedure, advocated by Grundling and Hurst [6, 7, 8], and going back to earlier proposals by Carey, Gaffney and Hurst [19, 20, 21, 22], does not use an indefinite metric formalism. Let us look for the mathematical features of this approach.

The T-procedure which we review only for first class constraints, starts from the quantized theory without recourse to a classical formulation. The theory is assumed to be given in the framework of algebraic quantum theory, first advocated by Segal and later effectively used in local quantum field theory by Haag and Kastler.[3] As starting point, Grundling and Hurst use an unconstrained C^*-field

[3]Though we do not aim at reviewing this approach to QFT, the following should be mentioned: the fundamental mathematical object of a Haag-Kastler quantum field theory is a map $\mathcal{O} \longrightarrow \mathcal{A}(\mathcal{O})$, mapping each bounded open region $\mathcal{O}$ in Minkowski space into a C^*-algebra $\mathcal{A}(\mathcal{O})$. Isotony of this net of algebras $[\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)]$ allows one to define the C^*-algebra of quasilocal observables $\mathcal{A}$ as inductive limit, $\mathcal{A} := \overline{\cup_{\mathcal{O}} \mathcal{A}(\mathcal{O})}$. $\mathcal{A}$ satisfies what is known as Haag-Kastler axioms, a set of physical requirements: isotony, Einstein causality, transformation properties under the Poincaré group, etc., cf. [16, 23]. States are defined as normalized positive linear functionals on $\mathcal{A}$.
According to the philosophy of algebraic QFT, the field algebra should be constructed from the algebra of observables by investigating the superselection structure of the theory. This program, first advocated by Borchers and by Doplicher, Haag and Roberts, has been completed for special cases. The T-procedure (as well as the Rieffel induction procedure discussed below) deviate from a Haag-Kastler setting in so far, that $\mathcal{A}$ is taken to be the field algebra and not the algebra of observables.

algebra, a set $\mathcal{U}$ of n unitary operators $U_i(\lambda)$ generated by first class constraints and the set of Dirac states

$$\mathcal{S}_D = \{\omega \in \mathcal{S}_\mathcal{A} \mid \omega(U_i(\lambda)) = 1; i = 1, ..., n\}, \tag{1.1.7}$$

where $\mathcal{S}_\mathcal{A}$ denotes the set of all states of the field algebra $\mathcal{A}$. Then, the constraint-free algebra of physical observables $\mathcal{R}$ is given by

$$\mathcal{R} = \mathcal{C}/\mathcal{D}, \tag{1.1.8}$$

where, (cf. the footnote for a more precise definition), $\mathcal{C}$ denotes the subset of elements in $\mathcal{A}$, commuting with the constraints and $\mathcal{D}$ denotes a two-sided ideal in $\mathcal{A}$ built from the constraints.[4] This structure will be discussed further in the following chapters on the Heisenberg CCR algebra and free QED. Here, we mention only what is known about the physical states on $\mathcal{A}$ and $\mathcal{R}$:

1. $\mathcal{S}_\mathcal{D}$ is not empty iff $1 \notin \mathcal{F}(L)$.[5]

2. There exists a one-to-one correspondence between the set of Dirac states $\mathcal{S}_\mathcal{D}(\mathcal{C})$ and the set of all states on $\mathcal{R}$, $\mathcal{S}(\mathcal{R})$. This one-to-one correspondence exists between the subsets of all pure Dirac states $\mathcal{S}^p_\mathcal{D}$ and all pure states on $\mathcal{R}$, $\mathcal{S}^p(\mathcal{R})$, too.

3. At least for the important case of a CCR algebra with hermitian supplementary conditions (1.1.5) and corresponding unitaries $U_\alpha(\lambda) := e^{i\lambda\hat{\phi}_\alpha}$, $\lambda \in \mathbb{R}$, no Dirac state $\omega \in \mathcal{S}_D$ is regular [11].

One may look for alternatives to the T-procedure on pragmatic or conceptual grounds. We mention the following two points:

- The input data do not arise from a classical theory via a quantization prescription. Where does the set of unitaries come from? From an algebraic point of view, this information should be encoded in the underlying C^*-algebraic net, obviating the need for specification as input data.

- For CCR-algebras, the GNS-representation of $\pi_\omega(\mathcal{A})$ on $\mathcal{H}_\omega$, corresponding to Dirac states ω, is non-regular on non-physical quantities. This implies, e.g. for free QED, that the vector potential

[4] To be more precise:

$$\mathcal{F}(L) := C^*(\mathcal{U} - 1),$$
$$\mathcal{D} := [\mathcal{A}\mathcal{F}(L)] \cap [\mathcal{F}(L)\mathcal{A}],$$
$$\mathcal{C} := \{A \in \mathcal{A} \mid [A, H] \in \mathcal{D}; H \in \mathcal{D}\}.$$

Here, $[X]$ denotes the linear space for an arbitrary set $X \subset \mathcal{A}$, closed in the C^*-norm and $C^*(X)$ is the C^*-algebra generated by X. Furthermore, Grundling and Hurst have shown that $\mathcal{D}$ is an ideal of $\mathcal{C}$ as is required for making sense of (1.1.8), cf. [6].

[5] Grundling and Hurst have shown that this condition is equivalent to having only first class constraints in the theory, cf. [6].

does not exist as an operator on $\mathcal{H}_\omega$. This will certainly complicate perturbative calculations in this setting [12].

Rieffel induction procedure

In contrast to the T-procedure, the Rieffel induction procedure does make contact with an underlying classical theory. The starting point of this method, recently proposed by Landsman [9], is the observation that the phase space of many constrained classical systems can be written as a Marsden-Weinstein quotient [24], combined with the proposal that a Rieffel induction procedure provides the appropriate quantum analogue for this classical reduction procedure. Here, we specify the two most important ingredients of Landsman's proposal, the Marsden-Weinstein reduction and the Rieffel induction procedure:

1. Marsden-Weinstein reduction procedure: the starting point is a classical theory, given by a symplectic space (M, B) on which a Lie group G (the gauge group) is acting. Provided certain technical assumptions are satisfied, one can define a so-called moment map $J : M \to \mathrm{g}^*$ by $J_X(m) = \langle J(m), X \rangle$, where g^* is the topological dual of the Lie algebra g of G. Then, the Marsden-Weinstein reduced space M^0 is defined by the Marsden-Weinstein quotient

$$M^0 = J^{-1}(0)/G. \tag{1.1.9}$$

This space is equipped with a reduced symplectic form $\tilde{B}$. Since Marsden-Weinstein reduction is not employed actively in our later work, we restrict ourselves here to a footnote [6] briefly illustrating the notions involved for the example of classical electrodynamics.

2. Rieffel induction procedure: this originates from the idea to find a quantum analogue of the Marsden-Weinstein procedure. Schematically, given a quantization prescription $Q_\hbar$ which relates the symplectic space (M, B) to some field algebra $\tilde{\mathcal{A}}$ (e.g., the Weyl CCR algebra over M), G to some algebra $\mathcal{B}$ generated by G (namely the group algebra when G is finite dimensional) and $(M^0, \tilde{B})$ to some (a priori unknown) algebra of observables

$$\begin{array}{ccc} (M,B);G & \xrightarrow{Q_\hbar} & \tilde{\mathcal{A}};\mathcal{B} \\ {\scriptstyle \text{Marsden-Weinstein Reduction}}\big\downarrow & & \big\downarrow{\scriptstyle \text{Rieffel Induction}} \\ (M^0,\tilde{B}) & \xrightarrow{Q_\hbar} & \mathcal{A}_{obs} = ? \end{array} \tag{1.1.10}$$

[6] E.g., for classical electrodynamics, choose the space of solutions of the wave equation $M = \{A_\mu \mid \Box A_\mu = 0; A_\mu \in L^2(R^3) \otimes C^4\}$, with symplectic form $B(A, C) = \mathrm{Im} \int \frac{d^3\mathrm{p}}{(2\pi)^3 2p_0} A_\mu(\mathrm{p}) \overline{C}^\mu(\mathrm{p}) =: \mathrm{Im}(A, C)_M$ and $G = \{g \in \mathcal{S}'(R^4) \mid \Box g = 0; dg \in M\}$. G acts on M as a gauge group via $A_\mu \to A_\mu + \partial_\mu g$. The moment map may be computed as $J_g(A) = \mathrm{Im}(dg, A)_M$ and $J^{-1}(0) = \{A_\mu \in M \mid \partial_\mu A^\mu = 0\}$. Now, the quotient $J^{-1}(0)/G$ removes the gauge degeneracy of the symplectic form B with respect to the action of G on $J^{-1}(0)$, i.e., loosely speaking, M^0 contains the gauge-independent part of M, cf. [1].

the question is: which method replaces the Marsden-Weinstein procedure on the quantum side of this scheme such that the diagram is commutative?
Landsman's proposal starts from a C^*-field algebra $\tilde{\mathcal{A}}$, quantizing the unconstrained system, an algebra of constraints $\mathcal{B}$ and a subspace $L \subset \mathcal{H}$ of a Hilbert space $\mathcal{H}$ on which the algebra of weak observables $\mathcal{A} = \mathcal{B}' \cap \tilde{\mathcal{A}}$ ($\mathcal{B}'$ being the commutant of $\mathcal{B}$) and the constraint algebra $\mathcal{B}$ are acting from the left and from the right, respectively. The aim of the Rieffel induction procedure is to obtain a representation $\pi^\chi(\mathcal{A})$ on a Hilbert space $\mathcal{H}^\chi$ by induction from a representation $\pi_\chi(\mathcal{B})$ on a Hilbert space $\mathcal{H}_\chi$.[7] Then, $\pi^\chi(\mathcal{A})$ can be identified with the representation-independent observable algebra $\mathcal{A}_{obs}$. The essential tool of the Rieffel induction procedure is the rigging map $\langle .,. \rangle_\mathcal{B}$, defined on $L \times L$ and taking values in $\mathcal{B}$,

$$\langle .,. \rangle_\mathcal{B} : L \times L \longrightarrow \mathcal{B}, \tag{1.1.11}$$

$L \subset \mathcal{H}$ not necessarily dense. By definition, this rigging map satisfies the following conditions for all $\psi, \varphi \in L$:

1. $\langle \lambda\psi, \mu\varphi \rangle_\mathcal{B} = \overline{\lambda}\mu \langle \psi, \varphi \rangle_\mathcal{B}$ for all $\lambda, \mu \in \mathbb{C}$;

2. $\langle \psi, \varphi \rangle_\mathcal{B}^* = \langle \varphi, \psi \rangle_\mathcal{B}$;

3. $\langle \psi, \varphi B \rangle_\mathcal{B} = \langle \psi, \varphi \rangle_\mathcal{B} B$ for all $B \in \mathcal{B}$,

4. $\langle A\psi, \varphi \rangle_\mathcal{B} = \langle \psi, A^*\varphi \rangle_\mathcal{B}$ for all $A \in \mathcal{A}$.

5. $\langle A\psi, A\psi \rangle \leq \parallel A \parallel^2 \langle \psi, \psi \rangle$ for all $\psi \in L$, $A \in \mathcal{A}$.

It is possible to construct π^χ only in case that π_χ is L-positive, i.e., for all $\psi \in L$, $\pi_\chi(\langle \psi, \psi \rangle_\mathcal{B}) \geq 0$ as an operator on $\mathcal{H}_\chi$. Then, one forms the tensor product space $L \otimes \mathcal{H}_\chi$, endowed with a bilinear form $(.,.)_0$,

$$(\psi \otimes v, \varphi \otimes w)_0 = (\pi_\chi(\langle \varphi, \psi \rangle_\mathcal{B})v, w)_\chi, \tag{1.1.12}$$

where $(.,.)_\chi$ is the inner product in $\mathcal{H}_\chi$. $\mathcal{H}^\chi$ is the completion of the quotient of $L \otimes \mathcal{H}_\chi$ by the subspace $\mathcal{H}_N \subset L \otimes \mathcal{H}_\chi$ of vectors with vanishing $(\ldots)_0$-norm,

$$\mathcal{H}^\chi = ((L \otimes \mathcal{H}_\chi)/\mathcal{H}_N)^-, \tag{1.1.13}$$

[7] To phyicists, induction methods are known from Wigner's classification of all irreducible unitary representations of the Poincaré group $\mathcal{P}$, cf. [25, 26]. More generally, the method of induced representations for groups allows to construct via a general induction procedure a representation of the complete group, once a representation of a subgroup is given. Similar induction methods have been developed for group algebras and (which is relevant for our work) to C^*-algebras by Rieffel, cf. [27]. Landsman's proposal specifies the input data in such a way that Rieffel's method for the construction of representations of C^*-algebras applies.

where $(.)^-$ denotes the completion of the pre-Hilbert space. For all $\psi\tilde{\otimes}v \in \mathcal{H}^\chi$, the action of $A \in \mathcal{A}$ is given by $\pi^\chi(A)\psi\tilde{\otimes}v = (\pi(A)\psi)\tilde{\otimes}v$.

The difficult part in applying this scheme is to specify a rigging map (1.1.11). To indicate the ideas employed, we give the following

Example:

Take $\mathcal{B} = C^*(G)$, the C^*-group algebra of a locally compact group G. Then, one may try to define a rigging map (1.1.11) [8] on $L \times L$ as a function on G given by

$$\langle\psi,\varphi\rangle_{\mathcal{B}}(x) = (U(x)\varphi,\psi), \tag{1.1.19}$$

[8]To see that (1.1.19) defines a rigging map, one has to check the five properties given below (1.1.11): In the case of a convolution algebra $C_c(G)$ of functions on a locally compact group G, one finds

1. $\langle\lambda\psi,\mu\varphi\rangle_{\mathcal{B}}(x) = \overline{\lambda}\mu(U(x)\varphi,\psi)$.
2. Using $f^*(x) = \overline{f(x^{-1})}$ for the unimodular group G, we find
$$\begin{aligned}\langle\psi,\varphi\rangle^*_{\mathcal{B}}(x) &= (U(x)\varphi,\psi)^* = (\psi,U(x^{-1})\varphi) = (U(x)\psi,\varphi)\\ &= \langle\varphi,\psi\rangle_{\mathcal{B}}(x).\end{aligned} \tag{1.1.14}$$
3. Using the multiplication of elements in a convolution algebra, $A * B(x) := \int d\mu(y)A(xy^{-1})B(y)$, we obtain
$$\begin{aligned}\langle\psi,\varphi B\rangle_{\mathcal{B}}(x) &:= \langle\psi,\pi(B)\varphi\rangle_{\mathcal{B}}(x) = \int d\mu(g)\langle\psi,\pi(g^{-1})B(g)\varphi\rangle_{\mathcal{B}}(x)\\ &= \int d\mu(g)(\pi(x)\pi(g^{-1})\varphi,\psi)B(g) = \int d\mu(g)\langle\psi,\varphi\rangle_{\mathcal{B}}(xg^{-1})B(g)\\ &= (\langle\psi,\varphi\rangle_{\mathcal{B}} * B)(x).\end{aligned} \tag{1.1.15}$$
4. $\pi(A) \in \pi(\mathcal{A}) \subset \pi(\mathcal{B})'$ and hence, $\pi(\mathcal{A}) \subset \pi(G)'$. This allows us to write
$$\begin{aligned}\langle A\psi,\varphi\rangle_{\mathcal{B}}(x) &= (U(x)\varphi,A\psi) = (A^*U(x)\varphi,\psi) = (U(x)A^*\varphi,\psi)\\ &= \langle\psi,A^*\varphi\rangle_{\mathcal{B}}(x).\end{aligned} \tag{1.1.16}$$
5. For all states of a GNS-triplet $(\pi_\omega,\Omega_\omega,\mathcal{H}_\omega)$, we may write in the case of a compact group G:
$$\begin{aligned}\omega(\langle A\psi,A\psi\rangle_{\mathcal{B}}) &= (\int_G d\mu(g)\langle A\psi,A\psi\rangle_{\mathcal{B}}(g)\pi_\omega(g)\Omega_\omega,\Omega_\omega)_{\mathcal{H}_\omega} = \int_G d\mu(g)(\pi(g)A\psi,A\psi)_{\mathcal{H}}(\pi_\omega(g)\Omega_\omega,\Omega_\omega)_{\mathcal{H}_\omega}\\ &= \int_G d\mu(g)((\pi\otimes\pi_\omega)(g)(A\otimes 1)\psi\otimes\Omega_\omega,(A\otimes 1)\psi\otimes\Omega_\omega)_{\mathcal{H}\otimes\mathcal{H}_\omega} \le \| A \|^2\omega(\langle\psi,\psi\rangle_{\mathcal{B}})\end{aligned} \tag{1.1.17}$$
where we have used in the last line that $\int_G d\mu(g)(\pi\otimes\pi_\omega)(g)$ is a projection operator which is positive and commutes with all $A \in \mathcal{A}$,
$$\begin{aligned}\omega(\langle\psi,\psi\rangle_{\mathcal{B}}) &= \int_G d\mu(g)((\pi\otimes\pi_\omega)(g)\psi\otimes\Omega_\omega,\psi\otimes\Omega_\omega)_{\mathcal{H}\otimes\mathcal{H}_\omega}\\ &= ((\pi\otimes\pi)_\omega(1_G)\psi\otimes\Omega_\omega,\psi\otimes\Omega_\omega)_{\mathcal{H}\otimes\mathcal{H}_\omega} \ge 0\end{aligned} \tag{1.1.18}$$
For a non-compact, locally compact group G, this argument has to be modified, cf. [9], Prop. 3.

where U is a continuous unitary representation of G on $\mathcal{H}$, $x \in G$.[9] Provided certain technical assumptions hold, [10] (1.1.19) satisfies all conditions for a rigging map specified above, if the actions of $\mathcal{A}$ and U commute. If we induce from the trivial representation $\mathcal{H}_\chi = \mathbb{C}$, we may write on the dense subspace $L \subset \mathcal{H}$

$$(\psi, \varphi)_0 = \int_G dx (U(x)\psi, \varphi) \tag{1.1.20}$$

Finally, let us compare Landsman's proposal with the T-procedure and the Gupta-Bleuler method:

- $(.,.)_0$ is positive semidefinite on the complete 'unphysical' space $L \otimes \mathcal{H}_\chi$ (in contrast to the Gupta-Bleuler method), though the construction of $\mathcal{H}^\chi$ is close in spirit to the Gupta-Bleuler quotient $\mathcal{H}'/\mathcal{H}''$.

- In contrast to the T-procedure, the Rieffel induction procedure makes contact with an underlying classical theory.

The first part of this thesis (Chapters II - IV) consists in an application of Landsman's proposal to free QED.

[9] For a unimodular group G, for which the right and left Haar measure $d\mu$ are equal, consider the completion of the vector space $C_c(G)$ of continuous functions $f : G \to \mathbb{C}$ with compact support, equipped with a $*$-operation $(f)^*(x) = \overline{f(x^{-1})}$ (implied by the unimodularity) and with one of the following two multiplications:

1. (untwisted) $(f * g)(x) = \int_G d\mu(y) f(xy^{-1}) g(y)$
2. (twisted) $(f *_c g)(x) = \int_G d\mu(y) f(xy^{-1}) g(y) c(xy^{-1}, y)$, where $c : G \times G \to U(1)$ is a 2-cocycle.

Vector spaces with this structure define C^*-algebras. They are called 'twisted (or untwisted) C^*-group algebras' and are denoted $C^*(G, c)$, (or $C^*(G)$) respectively.
These group algebras can be endowed either with a Euclidean topology where the Haar measure dx is the Lebesgue measure $d^n x$, or with the discrete topology where the Haar measure $d\mu(x)$ is the Dirac measure, i.e., every point has measure 1, all sets are open and all functions continuous.
E.g., the Weyl CCR-algebra over $G = \mathbb{R}^{2n}$ turns our to be isomorphic to the twisted group algebra with discrete topology and 2-cocycle $c((x_1, x_2), (y_1, y_2)) = \exp\left[\frac{i}{2}(x_1 y_2 - x_2 y_1)\right]$, cf. [28]. In the present example, G is endowed with the continuous topology. Only then one finds that (1.1.19) takes values in $\mathcal{B}$.

[10] One has to find a dense subspace $L \subset \mathcal{H}$, such that L is invariant under the right action of $\mathcal{B}$, defined by $\pi^-(f) = \int_G d\mu(x) f(x) U(x^{-1})$ and (1.1.20) is finite. If L is not left invariant by the action of $\mathcal{B}$, one may still change the scheme such that (1.1.20) is defined (cf. the following chapters on free QED).

2. Spontaneous Symmetry Breaking

In practice, an order parameter is used to determine whether a symmetry group G of a field algebra $\mathcal{A}$ is either unitarily implemented or spontaneously broken in a realization of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$.[11] In scalar QFTs without gauge fields, the vacuum expectation value of a certain scalar field is a suitable order parameter, which allows one to decide this question. Its value can be determined by the minimum of the effective potential of the theory. Here, our discussion focuses entirely on the formal properties of this effective potential.

In gauge QFTs, however, already the selection of a suitable order parameter is a highly non-trivial task. As we shall discuss, even the characterization of the Higgs-mechanism via SSB turns out to be gauge-dependent.

SSB for scalar theories

In scalar QFTs, the absolute minimum of the effective potential gives the value of the vacuum expectation value of the scalar field, which is a good order parameter. Hence, the effective potential, originated in the early sixties in proofs of Goldstone's theorem [30, 31], has played a crucial rôle in the discussion of spontaneous symmetry breaking and mechanisms of phase transitions, once a tractable perturbative evaluation was established with the loop expansion [32, 33, 34]. More recently, the effective potential has been used widely in phenomenological applications, e.g. for the discussion of phase transitions in the early universe. Without reviewing this very active field of research, we mention that there, a non-convex form of the effective potential is frequently assumed to have an energy-density interpretation. Faced by such heuristic applications, the following questions arise:

- It is obvious that the loop expansion of the effective potential is not asymptotic to its non-perturbative form [35, 36, 37]: for a non-convex classical potential, the loop expansion leads to

[11] Let us recast this point in an algebraic language: we say that a group G is a global *symmetry group* of the field algebra $\mathcal{A}$ if there exists a group of automorphisms $\{\alpha_g\}_{g\in G}$ on $\mathcal{A}$ satisfying $\alpha_{g_1}\alpha_{g_2}(A) = \alpha_{g_1 g_2}(A)$ for all $g_1, g_2 \in G$, $A \in \mathcal{A}$, commuting with space-time translations α_x. An automorphism α_g of $\mathcal{A}$ is said to be *unitarily implemented* on a Hilbert space $\mathcal{H}$, if $\mathcal{H}$ carries a representation $\pi(\mathcal{A})$ of $\mathcal{A}$ and there exists a unitary operator $U(g)$ on $\mathcal{H}$ for all $g \in G$, such that $\pi(\alpha[A]) = U(g)\pi(A)U^{-1}(g)$ for all $A \in \mathcal{A}$, $g \in G$, and $U(g_1)U(g_2) = U(g_1 g_2)$ for all $g_1, g_2 \in G$. A symmetry α_g of $\mathcal{A}$ is said to be *spontaneously broken* in a representation π of $\mathcal{A}$, if it is not unitarily implemented in the corresponding Hilbert space $\mathcal{H}$ [29].
Now, choose a translationally invariant (vacuum) state $\Omega \in \mathcal{H}$ and assume that we have $(\Omega, \pi(A)\Omega) \neq (\Omega, \pi(\alpha_g(A))\Omega)$ for some element $A \in \mathcal{A}$, $g \in G$. There are two possibilities: either α_g is spontaneously broken in π, or there exists a unitary operator $U(g)$ but $U(g)\Omega = \Omega_g \neq \Omega$. In the latter case, Ω_g is a translationally invariant (vacuum) state, since α_g commutes with α_x. We say that the vacuum is *degenerate*, which implies that the representation π on $\mathcal{H}$ is reducible. Consequently, in the irreducible representations of $\mathcal{A}$ on Hilbert spaces $\mathcal{H}_g$ obtained by GNS construction from one of the degenerate vacua Ω_g, α_g will be spontaneously broken. In this sense, the vacuum expectation value $(\Omega, \pi(A - \alpha_g(A))\Omega)$ is an order parameter and $(\Omega, \pi(A - \alpha_g(A))\Omega) \neq 0$ indicates spontaneous symmetry breaking.

a non-convex expression for the effective potential, despite a formal convexity argument [38]. Hence, we ask: in how far does one obtain non-perturbative physical information about the effective potential from the perturbative loop expansion?

- Given that the exact form of the effective potential is convex [37, 39], in which sense has either the non-convex expression for the effective potential or the convex one an energy density interpretation?

- How far is it possible to justify the heuristic interpretation of the perturbatively calculated effective potential despite the failure of the loop expansion? And are there other perturbative methods which lead to a convex expression?

Chapter VI is concerned with these questions. To be as precise as possible, we restrict our arguments to $P(\phi)_2$-theories whenever necessary. For this class of scalar field theories, we can use a plethora of important results which have been established within the program of constructive field theory. However, it should become clear that our arguments can be expected to hold true in a much more general setting.

SSB for gauge theories

The aim of this last introductory section is to draw attention to some problems arising in the characterization of the Higgs mechanism, thereby motivating the calculations reported in Chapter VII.
In the conventional formulation of continuum gauge field theories with Faddeev-Popov gauge fixing, a non-zero vacuum expectation value is assumed [40]. E.g., this assumption is considered to be necessary for the appearance of massive vector bosons in the electroweak theory. However, as pointed out by Wightman [41], the following apparently inconsistent results and assumptions exist in literature:

- In lattice versions of gauge theories which are believed to show spontaneous symmetry breaking, the vacuum expectation value of the scalar field is generically zero [42]. This is nothing but a simple consequence of the manifest gauge invariance of the measure in the usual lattice formulation without Faddeev-Popov gauge fixing.

- In lattice gauge theories with Faddeev-Popov gauge fixing, it has been shown that the gauge-invariant two-point function of the scalar field decays exponentially in the radiation gauge [43] and in all Gupta-Bleuler gauges [44]. This result holds formally in the continuum limit and shows that there is no SSB in these gauges [43, 44, 45, 46].

- In lattice gauge theories with Faddeev-Popov gauge fixing of the Landau type, the gauge-invariant two point function of the scalar field approaches a non-zero value in the large distance limit and one does obtain a non-zero vacuum expectation value [44, 45].

These results show that SSB is a gauge-dependent concept, not indicative for the Higgs mechanism in general [47].[12]
As shall be explained at the end of Chapter III, our application of the Rieffel induction procedure to free QED leads to a new gauge, the 'Landsman gauge', which is a particular cross-breed of the Coulomb and Gupta-Bleuler gauges. In the light of the results reviewed in this subsection, it is clearly of interest to exploit the features of the Landsman gauge for QFTs showing SSB. This is done in Chapter V, where we apply the Rieffel induction procedure to the Stückelberg-Kibble model.
Our main motivation for the approach in Chapter VII has been the idea that a gauge-independent algebraic characterization of the vacuum expectation value should help to clarify the situation described by Wightman. Though we have not been able to contribute much to the problems mentioned by Wightman, we report here on our first step, which consisted in emphasizing that the vacuum expectation value of a scalar field has an interpretation as a central element in the weak closure of a certain class of representations of the field algebra. Motivated by Haag's treatment of the BCS-model (which we briefly review) [48], we finally try to exploit this algebraic characterization of the vacuum expectation value in a formal analysis of a particular type of interaction terms.

[12] It should be clear that this observation provides excellent motivation for studying methods unrelated to the Faddeev-Popov path integral formulation of QFT, which implement constraints (and gauge-fixing conditions) in QFTs showing SSB. Indeed, this has been the starting point of our work on the Rieffel induction procedure described in Chapters II - V.

CHAPTER II:

CONSTRAINTS IN THE HEISENBERG ALGEBRA

As a preparation, in this chapter we apply the Rieffel induction procedure to the Heisenberg algebra of canonical commutation relations (CCR) of four degrees of freedom,

$$[q_\mu, p_\nu] = -ig_{\mu\nu}, \tag{2.1}$$

$g_{\mu\nu} = (1, -1, -1, -1)$. Therefore, in the first section we specify the necessary input data, cf. (1.1.10), namely an algebra of observables and an algebra of constraints, as well as representations of these algebras.

The Heisenberg algebra defines a toy model,[1] mimicking free QED by taking the electromagnetic vector potential A_μ to be frozen at the value $\mathring{A}_\mu = A_\mu(\mathring{p})$, where $\mathring{p}$ denotes a fixed momentum four vector. Most of the results obtained will survive our discussion of free QED in Chapter III with minor changes, thereby simplifying our presentation considerably.

1. Weyl algebra formulation of the model and its representations

Here, we consider the CCR-algebra $\triangle(\mathbb{C}^4, B)$ over the vector space $\mathbb{C}^4$ with symplectic form $B(.,.)$, defined by the Weyl form of the canonical commutation relations

$$W(f)W(f') = W(f+f')e^{[-\frac{i}{2}B(f,f')]}, \tag{2.1.1}$$

$$B(f, f') = 2\mathrm{Im}(f, f')_M, \tag{2.1.2}$$

for all $f, f' \in \mathbb{C}^4$, where $\mathrm{Im}(.,.)_M$ denotes the imaginary part of

$$(f, f')_M := f_\mu g^{\mu\nu} \overline{f'}_\nu. \tag{2.1.3}$$

[1]This investigation has been motivated by the paper of Carey, Gaffney and Hurst [19] who analyze (2.1) by methods different from ours.

To see how (2.1) and (2.1.1) are connected, one may formally write $W(c,d) = \exp[-i(c_\mu p^\mu + d_\mu q^\mu)]$ for $f = c + id$, which allows one to 'derive' (2.1.1) from (2.1) with the help of the Baker-Campbell-Hausdorff formula. Yet, this connection remains formal since $\triangle(\mathbb{C}^4, B)$ is a C^*-algebra, given independently of any representation while the generators q_μ, p_ν exist in regular representations of $\triangle(\mathbb{C}^4, B)$ only, [49].

The algebras of weak observables and constraints

Here, we specify subalgebras of $\triangle(\mathbb{R}^8, B)$ by their transformation properties under the group $E(2)$. Then, we briefly motivate this choice.

We start from the Weyl algebra $\triangle(\mathcal{K}, B)$ over a linear space $\mathcal{K}$ with symplectic form B. A symplectic transformation of $\mathcal{K}$ is defined by a linear operator Z on $\mathcal{K}$, which leaves B invariant, $B(.,.) = B(Z.,Z.)$. Z defines an automorphism α of $\triangle(\mathcal{K}, B)$ by

$$\alpha(W(f)) := W(Zf). \tag{2.1.4}$$

In our concrete case, we interpret the $f = c + id$ as elements $(c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4)$ in $\mathcal{K} = \mathbb{R}^8$ and we specify a representation Z of $E(2)$ on $\mathbb{R}^8$ by the generators:

$$E_1 = \begin{pmatrix} E'_1 & 0 \\ 0 & E'_1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} E'_2 & 0 \\ 0 & E'_2 \end{pmatrix} \qquad M_3 = \begin{pmatrix} M'_3 & 0 \\ 0 & M'_3 \end{pmatrix}$$

$$E'_1 = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad E'_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad M'_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The automorphisms $\alpha_{(\theta,l_1,l_2)}$ of $\triangle(\mathbb{R}^8, B)$ are then defined by specifying Z, $Z(\theta, x_1, x_2) = \exp[\theta M'_3 + x_1 E'_1 + x_2 E'_2]$ in (2.1.4). For later use, we present this operator, using the short-hand $a = (1 - \cos\theta)$:

$$Z(\theta, x_1, x_2) = \begin{pmatrix} \cos\theta & \sin\theta & -\frac{x_1}{\theta}\sin\theta - \frac{x_2}{\theta}a & \frac{x_1}{\theta}\sin\theta + \frac{x_2}{\theta}a \\ -\sin\theta & \cos\theta & -\frac{x_2}{\theta}\sin\theta + \frac{x_1}{\theta}a & \frac{x_2}{\theta}\sin\theta - \frac{x_1}{\theta}a \\ \frac{x_1}{\theta}\sin\theta - \frac{x_2}{\theta}a & \frac{x_1}{\theta}a + \frac{x_2}{\theta}\sin\theta & 1 - \frac{1}{\theta^2}(x_1^2 + x_2^2)a & \frac{1}{\theta^2}(x_1^2 + x_2^2)a \\ \frac{x_1}{\theta}\sin\theta - \frac{x_2}{\theta}a & \frac{x_1}{\theta}a + \frac{x_2}{\theta}\sin\theta & -\frac{1}{\theta^2}(x_1^2 + x_2^2)a & 1 + \frac{1}{\theta^2}(x_1^2 + x_2^2)a \end{pmatrix}.$$

To make contact with a simpler expression for Z, given by Weinberg [50], we use the substitution

$$\begin{aligned} l_1 &= \frac{x_1}{\theta}\sin\theta - \frac{x_2}{\theta}(1 - \cos\theta) \\ l_2 &= \frac{x_1}{\theta}(1 - \cos\theta) + \frac{x_2}{\theta}\sin\theta, \end{aligned} \tag{2.1.5}$$

which leads to[2]

$$Z(\theta, l_1, l_2) = \begin{pmatrix} \cos\theta & \sin\theta & -l_1\cos\theta - l_2\sin\theta & l_1\cos\theta + l_2\sin\theta \\ -\sin\theta & \cos\theta & l_1\sin\theta - l_2\cos\theta & -l_1\sin\theta + l_2\cos\theta \\ l_1 & l_2 & 1 - \frac{1}{2}(l_1^2 + l_2^2) & \frac{1}{2}(l_1^2 + l_2^2) \\ l_1 & l_2 & \frac{-1}{2}(l_1^2 + l_2^2) & 1 + \frac{1}{2}(l_1^2 + l_2^2) \end{pmatrix},$$

$$Z(\theta, l_1, l_2)^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta & l_1 & -l_1 \\ \sin\theta & \cos\theta & l_2 & -l_2 \\ -l_1\cos\theta - l_2\sin\theta & l_1\sin\theta - l_2\cos\theta & 1 - \frac{1}{2}(l_1^2 + l_2^2) & \frac{1}{2}(l_1^2 + l_2^2) \\ -l_1\cos\theta - l_2\sin\theta & l_1\sin\theta - l_2\cos\theta & \frac{-1}{2}(l_1^2 + l_2^2) & 1 + \frac{1}{2}(l_1^2 + l_2^2) \end{pmatrix}.$$

Following Carey, Gaffney and Hurst [19], we are led to subspaces N, T, S of $\mathbb{R}^8$, the first two of which allow for the construction of $E(2)$-invariant subalgebras of $\triangle(\mathbb{R}^8, B)$,

$$\begin{aligned} N &= \{f \in \mathbb{R}^8 \mid f = (a_1, a_2, a, a, b_1, b_2, b, b)\}, \\ T &= \{f \in \mathbb{R}^8 \mid f = (0, 0, a, a, 0, 0, b, b)\}, \\ S &= \{f \in \mathbb{R}^8 \mid f = (a_1, a_2, 0, 0, b_1, b_2, 0, 0)\}. \end{aligned} \tag{2.1.6}$$

We mention the following properties of these subspaces and their corresponding Weyl algebras:

- T and N are invariant under $E(2)$, whereas $N = S \oplus T$. S is not invariant under the action of $E(2)$, i.e., the action of $E(2)$ on N is reducible but indecomposable.
- Hence, $\triangle(T, B)$ and $\triangle(N, B)$ are invariant under the action of $E(2)$, but $\triangle(S, B)$ is not. Yet, $\triangle(S, B)$ is isomorphic to the quotient $\triangle(T, B)/\triangle(N, B)$ with the obvious equivalence relation understood, [19].

We have specified $E(2)$-invariant subalgebras of $\triangle(\mathbb{C}^4, B)$ to obtain a setup, closely related to the Weyl algebra of the vector potential of free QED. In fact, we shall already encounter the main features of our later application of the Rieffel induction procedure to free QED in the simpler setting of the Heisenberg Weyl algebra. Especially, the rôle of the stability group $E(2)$ of the massless representations of the Poincaré group can be studied already in this toy model.

Representations of the Weyl algebra and their equivalence

As a second preparatory step, we introduce two representations of $\triangle(\mathbb{C}^4, B)$, the Schrödinger representation on $L^2(\mathbb{R}^4, d^4x)$ and the Fock representation on a symmetrized Fock space over the one-

[2] From our substitution, it is obvious that $Z(\theta, x_1, x_2) = Z(-\theta, -x_1, -x_2)^{-1}$, whereas $Z(\theta, l_1, l_2) \neq Z(-\theta, -l_1, -l_2)^{-1}$.

particle Hilbert space $\mathbb{C}^4$. Then we introduce the Bargmann transform which defines a unitary map intertwining both representations. Let us start by specifying these representations:

1. Schrödinger representation π_S on $\mathcal{H} = L^2(\mathbb{R}^4, d^4x)$, endowed with the scalar product

$$(\psi, \varphi) = \int d^4x \psi(x)\overline{\varphi}(x) \tag{2.1.7}$$

for all $\psi, \varphi \in L^2(\mathbb{R}^4, d^4x)$. The representation $\pi_S(\triangle(\mathbb{R}^8, B))$ on $L^2(\mathbb{R}^4, d^4x)$ is defined by

$$\pi_S(W(a,b))\psi(x) = e^{\frac{1}{2}ia_\mu b^\mu} e^{-ib_\mu x^\mu}\psi(x-a). \tag{2.1.8}$$

2. Fermi representation π_F on $\mathcal{H} = S\mathbb{C}^4 := \bigoplus_{n=0}^{\infty}\mathcal{H}^{(n)}$; $\mathcal{H}^{(n)} = \mathbb{C}^4_{(1)}\otimes_s ... \otimes_s \mathbb{C}^4_{(n)}$, where $\otimes_s$ denotes the symmetrized tensor product. We define the representation π_F on the dense subspace $E \subset S\mathbb{C}^4$ which is the span of the total subset of exponential vectors [51]

$$E = \{\sum_{i=1}^{N} \lambda_i e^{z_i} \mid \lambda_i \in \mathbb{C}\setminus\{0\}, z_i \in \mathbb{C}^4; N < \infty\}; \tag{2.1.9}$$

$$e^z = 1 \oplus z \oplus \frac{1}{\sqrt{2}} z \otimes z \oplus \frac{1}{\sqrt{3!}} z \otimes z \otimes z \oplus \ldots \tag{2.1.10}$$

with scalar product

$$(e^x, e^y) = e^{(x,y)_E}, \tag{2.1.11}$$

where $(g,f)_E = g_\nu \delta_{\nu\mu}\overline{f}_\mu$. The representation is defined by

$$\pi_F(W(f))e^x = e^{\frac{-1}{2}(f,f)_E + (x,\tilde{f})_E} e^{(x-\tilde{f})}, \tag{2.1.12}$$

where $\tilde{f} = \begin{pmatrix} -\overline{f}_0 \\ f_i \end{pmatrix}$. This definition can be extended to all of $\mathcal{H}$.

Remarks:

- The heuristic idea behind the Fermi representation is the so-called 'Fermi trick' to interchange the role of the 0-components of the annihilation and creation operators (thereby replacing $-g_{\mu\nu}$ by the Euclidean metric $\delta_{\mu\nu}$, obtaining a positive definite inner product and a Fock representation). In the present case, this is done formally by specifying $\hat{a}_\mu$, $\hat{a}^*_\mu$ via (remark the minus signs):

$$\begin{aligned} q_0 &:= \frac{1}{\sqrt{2}}(\hat{a}_0 + \hat{a}^*_0) \quad ; \quad p_0 := \frac{i}{\sqrt{2}}(\hat{a}_0 - \hat{a}^*_0) \\ q_i &:= \frac{1}{\sqrt{2}}(\hat{a}_i + \hat{a}^*_i) \quad ; \quad p_i := \frac{-i}{\sqrt{2}}(\hat{a}_i - \hat{a}^*_i) \end{aligned} \tag{2.1.13}$$

These definitions for q_ν, p_μ satisfy (2.1) if

$$[\hat{a}_\mu, \hat{a}_\nu^*] = \delta_{\mu\nu}. \tag{2.1.14}$$

The action of these operators on the n-particle Hilbert spaces $\mathcal{H}^{(n)}$ is given by (symmetrization is understood)

$$\hat{a}_\mu(f_\mu) z_1 \otimes \ldots \otimes z_n = \sqrt{n}(z_1, f)_E z_2 \otimes \ldots \otimes z_n, \tag{2.1.15}$$

$$\hat{a}_\mu^*(f_\mu) z_1 \otimes \ldots \otimes z_n = \sqrt{n+1} f \otimes z_1 \otimes \ldots \otimes z_n, \tag{2.1.16}$$

where $\hat{a}_\mu(f_\mu) = \hat{a}_0 \overline{f}_0 + \hat{a}_i \overline{f}_i$, $\hat{a}_\nu(f_\nu)^* = \hat{a}_0^* f_0 + \hat{a}_i^* f_i$, $[\hat{a}_\mu(f_\mu), \hat{a}_\nu^*(g_\nu)] = (g, f)_E$, and consequently

$$e^{\hat{a}_\mu(f_\mu)} e^x = e^{(x,f)_E} e^x; \tag{2.1.17}$$

$$e^{\hat{a}_\mu^*(f_\mu)} e^x = e^{(x+f)}. \tag{2.1.18}$$

Introducing (2.1.13) into the formal expression $W(c, d) = e^{[-i(c_\mu p^\mu + d_\mu q^\mu)]}$ we obtain (2.1.12) by acting with

$$\pi_F(W(f)) = e^{[\hat{a}_\mu(\tilde{f}_\mu) - \hat{a}_\mu(\tilde{f}_\mu)^*]} \tag{2.1.19}$$

on exponential vectors, i.e., the heuristic Fermi trick leads to the Fermi representation.

- The Schrödinger representation and the Fermi representation are unitarily equivalent. To see this, we introduce the Bargmann transform $\check{B}$ with kernel $\check{B}(z, x)$, which defines a unitary map[52] from $L^2(\mathbb{R}^4, d^4x)$ into the space of entire analytic functions $\mathcal{F}$ in $L^2(\mathbb{C}^4, e^{[-\sum_\mu z_\mu \overline{z}_\mu]} d^4 z)$

$$\check{B} : L^2(\mathbb{R}^4, d^4x) \longrightarrow \mathcal{F} := \{F \mid F \text{ entire on } \mathbb{C}; \|F\|_F^2 = \int |F(z)|^2 e^{[-\sum_\mu z_\mu \overline{z}_\mu]} d^4 z < \infty\},$$

$$(\check{B}f)(z) := \frac{1}{\pi} \int f(x) e^{[\sum_\mu (x_\mu z_\mu - \frac{1}{2} x_\mu^2 - \frac{1}{4} z_\mu^2)]} d^4 x = \int \check{B}(z, x) f(x) d^4 x. \tag{2.1.20}$$

The inverse of $\check{B}$ is given by

$$(\check{B}^{-1} F)(x) = \int \check{B}(\overline{z}, x) F(z) e^{[\frac{-1}{2} \sum_\mu z_\mu \overline{z}_\mu]} d^4 z.$$

Using for $\mathcal{F}$ the orthonormal basis $\{\xi_\alpha(z) = \frac{1}{\sqrt{\alpha!}} z^\alpha \mid \alpha \in I_\alpha\}$, one can introduce the unitary map R,

$$R : \mathcal{F} \longrightarrow S\mathbb{C}^4,$$

$$R\left(\frac{1}{\sqrt{\alpha!}} z_1^{\alpha_1} \ldots z_4^{\alpha_4}\right) := e_1^{\alpha_1} \otimes_s \ldots \otimes_s e_4^{\alpha_4}. \tag{2.1.21}$$

Here, α denotes a multiindex, $I_\alpha = \{\alpha \mid \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{N}^4\}$, $\alpha! = \alpha_1!\alpha_2!\alpha_3!\alpha_4!$, $\{e_j\}_1^4$ are the standard basis vectors in $\mathbb{C}^4$, $e_j^{\alpha_j}$ is the α_j-times symmetric tensor product of e_j and $z_i \in \mathbb{C}$. Especially, it follows for the Fock vacuum $\psi_v(x) = \frac{1}{\pi}e^{[\frac{-1}{2}\sum_\mu x_\mu^2]}$ that $(\check{B}\psi)(z) = 1$, $R(\check{B}\psi_v) = e^0$ and

$$R(e^z) = \sum_{\alpha \in I_\alpha} \frac{1}{\sqrt{|\alpha|!}} z_1^{\alpha_1} \dots z_4^{\alpha_4}. \qquad (2.1.22)$$

2. Rieffel Induction for the Schrödinger representation on $\mathcal{H} = L^2(\mathbb{R}^4, d^4x)$

As mentioned in the Introduction, one has to specify three inputs to carry out the Rieffel induction procedure: the right $\mathcal{B}$-module[3] L which carries a left representation of $\mathcal{A}$ and a right representation of $\mathcal{B}$, the Hilbert space $\mathcal{H}_\chi$ which carries the representation of $\mathcal{B}$ one induces from and the rigging map $\langle .,. \rangle_\mathcal{B} : L \times L \to \mathcal{B}$. Here, we specify these inputs for the Schrödinger representation of the Heisenberg CCR algebra where the calculation is particularly short and simple.

Imposing the constraints

We choose the following input data:

1. as constraint algebra: $\mathcal{B} = C^*(\mathbb{R}^2)$, the group algebra over $\mathbb{R}^2 \simeq T$ with continuous Euclidean topology. T is now regarded as additive group and will be understood as a gauge group.

2. as field algebra of weak observables: $\mathcal{A} = \Delta(N, B) \subset \Delta(\mathbb{C}^4, B)$
   $\mathcal{A}$ is contained in the commutant of the abelian algebra $C^*(T) = \Delta(T, B)$ which lies in the center of $\mathcal{A}$.[4]

3. Hilbert space $\mathcal{H}_\chi$ with scalar product $(.,.)_\chi$, used to induce from: $\mathcal{H}_\chi = \mathbb{C}$, carrying the trivial representation π_χ of $\mathcal{B}$; $\pi_\chi(f) = \int_T d^2x f(x)$ for $f \in \mathcal{B}$. We write $\pi_\chi(f) = \hat{f}(0)$, where $\hat{}$ denotes the Fourier transform. Later on, we shall consider Fourier transformations $\hat{f}(x_{ab})$ at arbitrary points $x_{ab} := (0, 0, a, a, 0, 0, b, b) \in T$.

4. Hilbert space $\mathcal{H}$ with scalar product $(.,.)$: $\mathcal{H} = L^2(\mathbb{R}^4, d^4x)$, carrying two representations:

[3] A right $\mathcal{B}$-module L is a complex vector space L carrying a linear anti-representation (a 'right representation') π_R of $\mathcal{B}$, i.e., $\pi_R(AB) = \pi_R(B)\pi_R(A)$.

[4] In what follows, we shall assume $\mathcal{A} = \Delta(N, B) = \Delta(T, B)'$. This has been claimed as a theorem by Grundling and Hurst [7] but according to private communication with Grundling, the proof contains a gap. We will assume this theorem to be true nevertheless. If the gap turns out to be irrepairable, our approach still carries through with the weak observable algebra defined as $\Delta(N, B)$, since in any case $\Delta(N, B) \subseteq \Delta(T, B)'$ is sufficient to perform Rieffel induction.

(a) the left representation of $\mathcal{A}$, defined by the Schrödinger representation (2.1.8), restricted to $\mathcal{A}$.

(b) the right representation of $\mathcal{B}$, defined by $\pi^-(f) = \int_T d^2x f(x)U(x)$. Here, $U(x) := \pi_S(W(x))$ defines a continuous unitary representation, since π_S is a regular representation and hence strongly continuous.[5]

5. rigging map on $L \times L$, where L is taken to be $C_c(\mathbb{R}^4)$: for all $x_{ab} \in T$, we specify [6]

$$\langle \psi, \varphi \rangle_{\mathcal{B}}(x_{ab}) = (U(x_{ab})\varphi, \psi). \tag{2.2.1}$$

With the above input, we obtain the following

Result: The 'physical' representation π^χ is L-positive, $\pi^\chi(\langle \psi, \psi \rangle_{\mathcal{B}}) \geq 0$. The corresponding Hilbert space $\mathcal{H}^\chi$ is isomorphic to $L^2(\mathbb{R}^2, d^2x)$ and carries a unitary implementation of the group of automorphisms $\alpha_{(\theta, l_1, l_2)}$.

Calculation: To establish L-positivity of π_χ, note that

$$(U(x_{ab})\psi, \psi) = \int d^4x e^{[ib(x_3 - x_4)]}\psi(x_1, x_2, x_3 - a, x_4 - a)\overline{\psi}(x_1, x_2, x_3, x_4),$$

and hence

$$\pi_\chi(\langle \psi, \psi \rangle_{\mathcal{B}}) = \int_T dadb(U(x_{ab})\psi, \psi) = \int dx_1 dx_2 \mid \int da\psi(x_1, x_2, a, a)|^2 \geq 0.$$

For the construction of $\mathcal{H}^\chi$, we calculate $(.,.)_0$ on $L \otimes \mathbb{C} = L$ (this allows us to drop $v, w \in \mathbb{C}$ in (1.1.12)).

$$\begin{aligned}(\psi, \varphi)_0 &= \int_{\mathbb{R}^4}\int_T d^4x dadb e^{[ib(x_3 - x_4)]}\psi(x_1, x_2, x_3 - a, x_4 - a)\overline{\psi}(x_1, x_2, x_3, x_4) \\ &= \int dx_1 dx_2 \int da\psi(x_1, x_2, a, a)\int dx_3 \overline{\varphi}(x_1, x_2, x_3, x_3).\end{aligned} \tag{2.2.2}$$

Now, we consider the mapping

$$\begin{aligned}&V : L \subset L^2(\mathbb{R}^4, d^4x) \longrightarrow L^2(\mathbb{R}^2, d^2x), \\ &\psi(x_1, x_2, x_3, x_4) \longrightarrow (V\psi)(x_1, x_2) = \int da\psi(x_1, x_2, a, a),\end{aligned} \tag{2.2.3}$$

[5] Here, we use the one-to-one correspondence between the non-degenerate representations π^- of the group algebra $C^*(T)$ and the continuous unitary representations U of the group T [28].

[6] That this expression satisfies the conditions for a rigging map (1.1.11), follows from our remark in the footnote to (1.1.19).

which satisfies

$$(V\psi, V\varphi) = (\psi, \varphi)_0. \tag{2.2.4}$$

From (2.2.2) we find that the null space $\mathcal{H}_N$ of $(.,.)_0$ is $\mathcal{H}_N = \ker V$. Furthermore, V quotients to a unitary mapping $\tilde{V}$

$$\tilde{V} : ((L \odot \mathcal{H}_\chi)/\mathcal{H}_N)^- \longrightarrow \mathcal{H}^\chi = L^2(\mathbb{R}^2, d^2x). \tag{2.2.5}$$

Finally, let us look for a unitary representation U^χ of $E(2)$ on $\mathcal{H}^\chi = L^2(\mathbb{R}^2, d^2x)$, which implements the automorphism $\alpha_{(\theta,l_1,l_2)}$ on $\mathcal{H}^\chi$ via

$$\pi^\chi(\alpha_{(\theta,l_1,l_2)}[W(f)]) = \pi^\chi(W(Z(\theta,l_1,l_2)f)) = U^\chi(\theta,l_1,l_2)\pi^\chi(W(f))U^\chi(\theta,l_1,l_2)^{-1},$$

for $f \in N$. In the present case, there naturally exists already a unitary representation $U_\mathcal{H}$ of $E(2)$ on $\mathcal{H} = L^2(\mathbb{R}^4, d^4x)$, defined by

$$(U_\mathcal{H}(\theta, l_1, l_2)\psi)(x) = \psi(Z(\theta, l_1, l_2)^{-1}x) \tag{2.2.6}$$

for all $\psi \in L^2(\mathbb{R}^4, d^4x)$. $U_\mathcal{H}$ is unitary on $L^2(\mathbb{R}^4, d^4x)$, since $(\varphi, \psi) = (U_\mathcal{H}\varphi, U_\mathcal{H}\psi)$ due to the $E(2)$-invariance of the measure d^4x. $U_\mathcal{H}(\theta, l_1, l_2)$ implements the automorphism $\alpha_{(\theta,l_1,l_2)}$ on $L^2(\mathbb{R}^4, d^4x)$. This allows us to obtain a representation U^χ of $E(2)$ on $\mathcal{H}^\chi$ as a quotient of $U_\mathcal{H}$ by using the map V:

$$U^\chi(\theta, l_1, l_2)(V\psi)(x_1, x_2) = (VU_\mathcal{H}(\theta, l_1, l_2)\psi)(x_1, x_2) = \int da\psi(e^{-i\theta M'_3}(x_1, x_2, a, a)). \tag{2.2.7}$$

To see that U^χ is well-defined, one checks that $U_\mathcal{H}\mathcal{H}_N \subset \mathcal{H}_N$, by calculating in (2.2.3) that $(VU_\mathcal{H}\psi)(x_1, x_2) = \int da\psi(y_1, y_2, a, a)$, where $y_1 = x_1\cos\theta + x_2\sin\theta$, $y_2 = -x_1\sin\theta + x_2\cos\theta$. One sees that U^χ is unitary since it follows from (2.2.2) that $(\psi, \varphi)_0 = (U^\chi\psi, U^\chi\varphi)_0$.

Remarks:

- From the representation theory of $E(2)$, we know that the generators E'_1 and E'_2 of the abelian subgroup of $E(2)$ have to be represented by zero in all irreducible finite dimensional unitary representation. Hence, U^χ is a non-trivial representation for the generator M'_3 of $E(2)$ only, cf. (2.2.7).

- The 'vacuum' state on $\mathcal{H} = L^4(\mathbb{R}^4, d^4x)$ is

$$\psi_v(x_1, x_2, x_3, x_4) = \frac{1}{\pi}e^{[\frac{-1}{2}\sum x_\mu^2]}, \tag{2.2.8}$$

which is not invariant under the action of $U_\mathcal{H}$. But $(V\psi_v)(x_1, x_2) = \psi_0(x_1, x_2) \in \mathcal{H}^\chi = L^2(\mathbb{R}^2, d^2x)$,

$$\psi_0(x_1, x_2) = \int da\psi_v(x_1, x_2, a, a) = \frac{1}{\sqrt{\pi}}e^{[\frac{-1}{2}(x_1^2+x_2^2)]}. \tag{2.2.9}$$

defines an $E(2)$-invariant cyclic[7] vacuum state.

Comparison with the literature

Finally, let us make contact with the work of Carey, Gaffney and Hurst [19], by considering the most general form

$$\pi_\chi^{(2)}(F) = \hat{F}(\xi_1, \xi_2) \tag{2.2.10}$$

of an irreducible representation to be induced from. This leads to

$$\pi_\chi^{(2)}((\psi, \psi)_B) = (2\pi)^2 \int dx_1 dx_2 \, | \int da e^{\frac{i\xi_1 a}{2}} \psi(x_1, x_2, \frac{1}{2}(a - \xi_2), \frac{1}{2}(a + \xi_2))|^2 \geq 0. \tag{2.2.11}$$

Again, we can construct $\mathcal{H}^\chi$, going through the discussion given above with the mapping $V^{(2)}$,

$$(V^{(2)}\psi)(x_1, x_2) = \int da e^{\frac{i\xi_1 a}{2}} \psi(x_1, x_2, \frac{1}{2}(a - \xi_2), \frac{1}{2}(a + \xi_2)), \tag{2.2.12}$$

where $V^{(2)}\psi \in L^2(\mathbb{R}^2, d^2x)$. Here, we remark that in their method to impose the constraint algebra $\mathcal{B}$ of this model on a physical state space, Carey, Gaffney and Hurst obtain the closely related expression

$$g(x_1, x_2, \lambda_1, \lambda_2) = \frac{1}{\sqrt{\pi}} \int du f(x_1, x_2, \frac{1}{2}(u - \lambda_2), \frac{1}{2}(u + \lambda_2)) e^{\frac{iu\lambda_1}{2}} \tag{2.2.13}$$

in a very different way: they are looking for a direct integral decomposition of $\mathcal{H} = L^2(\mathbb{R}^4, d^4x)$,

$$\mathcal{H} = \int d\lambda'_1 d\lambda'_2 \mathcal{H}_{\lambda'_1 \lambda'_2}, \tag{2.2.14}$$

in which all the elements of $\pi_S(\triangle(N, B))$ are in diagonal form. This decomposition is obtained by specifying the subspaces of $\mathcal{H}$ on which the central elements $\pi_S(\triangle(T, B)) = \mathcal{Z}[\pi_S(\triangle(N, B))]$[8] are represented by a c-number

$$\pi_S(W(h(a, b)))g(x_1, x_2, \lambda_1, \lambda_2) = e^{-i(a\lambda_1 + b\lambda_2)} g(x_1, x_2, \lambda_1, \lambda_2). \tag{2.2.15}$$

The improper restriction $\Pi_{\lambda'_1 \lambda'_2}(\triangle(N, B))$ of $\pi_S(\triangle(N, B))$ to a single improper subspace $\mathcal{H}_{\lambda'_1 \lambda'_2}$ is irreducible and, more importantly, it is a representation of the $E(2)$-invariant factor algebra for $\mathcal{H}_{00}$. The problem with this approach is that the vectors

$$g(\cdot, \cdot, \lambda_1, \lambda_2)\delta(\lambda_1 - \lambda'_1)\delta(\lambda_2 - \lambda'_2) \in \mathcal{H}_{\lambda'_1 \lambda'_2} = L^2(\mathbb{R}^2, d^4x) \tag{2.2.16}$$

which have been specified by the restriction $\Pi_{\lambda'_1 \lambda'_2}$, do not lie in $\mathcal{H}$. This problem is circumvented in the Rieffel induction procedure by constructing the physical state space $\mathcal{H}^\chi = L^2(\mathbb{R}^2, d^2x)$ in which $V^{(2)}\psi$ lives.

[7] A state $\psi \in \mathcal{H}$ is called cyclic if the action of the algebra on ψ spans a dense subspace of $\mathcal{H}$.

[8] $\mathcal{Z}[\pi(\mathcal{A})]$ denotes the center of $\pi(\mathcal{A})$, i.e., the set of all elements $Z \in \pi(\mathcal{A})$ for which $[Z, A] = 0$ for all $A \in \pi(\mathcal{A})$.

3. Rieffel Induction for the Fermi Representation on $\mathcal{H} = S\mathbb{C}^4$

As a second illustration, Rieffel induction is applied again to the Heisenberg CCR algebra, changing the input data of our previous discussion only by replacing $\mathcal{H} = L^2(\mathbb{R}^4, d^4x)$ with $\mathcal{H} = S\mathbb{C}^4$. Now, the calculations are more involved and the main difficulties of our subsequent analysis of the Fock particle representation of free QED are encountered.

Calculation of $(.,.)_0$ and analysis

In close analogy to (2.2.2), we calculate $(.,.)_0$ for the elementary vectors $e^x, e^y \in E$ in (2.1.9). Using $U(x_{ab}) := \pi_F(W[(a+ib)(0,0,1,1)])$, and $L = E$, defined in (2.1.9), we obtain

$$
\begin{aligned}
(e^x, e^y)_0 &= \int_T dadb(U(x_{ab})e^x, e^y) \\
&= \int_T dadb e^{-(a^2+b^2)-x_0(a-ib)+x_3(a+ib)+\overline{y}_0(a+ib)-\overline{y}_3(a-ib)+x_0\overline{y}_0+x_i\overline{y}_i} \\
&= e^{[-x_0x_3-\overline{y}_0\overline{y}_3+(x_1\overline{y}_1+x_2\overline{y}_2)]},
\end{aligned}
\tag{2.3.1}
$$

where the usual summation convention over the index $i = 1, 2, 3$ is understood.
Before analyzing this expression, we introduce the dense subspace $\mathcal{D} \subset \mathcal{H}$:

Definition of $\mathcal{D}$: $\mathcal{D}$ is defined as the set of finite linear combinations of symmetric n-particle states $x^{(1)}\otimes_s x^{(2)}\otimes_s \ldots \otimes_s x^{(n)}$ with arbitrary finite number of particles n.

Remarks:

- For our calculation, we shall use the fact that symmetric n-particle state in $\mathcal{D}$ are obtained by differentiation of elementary exponential vectors

$$
x^{(1)}\otimes_s x^{(2)}\otimes_s \ldots \otimes_s x^{(n)} = \frac{1}{\sqrt{n!}} \frac{d}{dr_1} \ldots \frac{d}{dr_n} e^{[\sum r_i x^{(i)}]}|_{r_i=0}. \tag{2.3.2}
$$

- So far, we have specified $(.,.)_0$ as a positive semi-definite[9] quadratic form on $\mathcal{H}$ with domain $E \cup \mathcal{D}$.[10] One may ask whether $(.,.)_0$ can be extended to $\mathcal{H}$. This is not the case since it has been shown by Landsman [1], that $(.,.)_0$ is not even closable as a quadratic form with domain $E \cup \mathcal{D}$ and hence it can certainly not be extended to $\mathcal{H}$.

[9] That $(.,.)_0$ is positive semidefinite will be shown in the argument following (2.3.15).

[10] With $E \cup \mathcal{D}$, we do not denote the set theoretic union but the set of all vectors in $\mathcal{H}$ which are finite linear combinations of vectors in E and $\mathcal{D}$, e.g., $e^{x_1} + x_2 \otimes_s x_3 \in E \cup \mathcal{D}$.

- $\mathcal{D}$ is not invariant under the action of any dense subalgebra of $\triangle(\mathbb{R}^8, B)$, $\pi_F(\triangle(\mathbb{R}^8, B))\mathcal{D} \not\subset \mathcal{D}$, whereas E is invariant under the action of the dense subalgebra of $\triangle(\mathbb{R}^8, B)$, containing all finite linear combinations of Weyl elements. Nevertheless, we are particularly interested in performing the Rieffel induction procedure on $L = \mathcal{D}$, since this will lead us from (2.3.2) to the 'physical' n-particle states. To know under the action of which algebra $\mathcal{D}$ is left invariant, we observe that $\mathcal{D}$ is obtained by the action of the generators of $\Pi_F(\triangle(M, B))$ on Ω (these generators are essentially the annihilation and creation operators). The set of all finite sums of polynomials of these generators defines an unbounded operator algebra under whose action $\mathcal{D}$ is stable by construction.

For a detailed analysis of $(.,.)_0$ on $\mathcal{D}$, we introduce the following

Combinatorial Notation:

- $I_n = \{1, 2, ..., n\}$ index set with subsets $I_{n;2q} = \{1, 2, ..., 2q\}$, $I_{n,q} = \{2q+1, ..., n\}$, i.e. $I_n = I_{n;2q} \cup I_{n,q}$.

- $\mathcal{P}(I_k)$, the set of permutations of I_k

- $\mathcal{P}_{I_{n,q}}$, the set of different partitions of I_n into index sets $p_q(I_{n;2q}) = I^{p_q}_{n;2q}$ and $p_q(I_{n,q}) = I^{p_q}_{n,q} = \{t_1^{p_q}, ..., t_{n-2q}^{p_q}\}$ of $2q$ and $(n-2q)$ elements, respectively. Consequently, $\mathcal{P}'_{I^{p_q}_{n;2q},q'}$ is the set of different partitions of $I^{p_q}_{n;2q}$ into index sets $p'_{q'}(I^{p_q}_{n;2q;2q'})$ and $p'_{q'}(I^{p_q}_{n;2q,q'})$ of $2q'$ and $(n-2q-2q')$ elements, respectively.

- $\mathcal{S}_q^{p_q} = \{s_q^p\}$, the set of all sets of non-ordered pairs s_q^p of $p_q(I_{n;2q})$,
  $s_q^p = \{(m_i^q, \tilde{m}_i^q) \mid i = 1, ..., q; m_i^q, \tilde{m}_i^q \in p_q(I_{n;2q}), m_i^q \neq m_j^q \neq \tilde{m}_{i'}^q \neq \tilde{m}_{j'}^q, i \neq j, i' \neq j'\}$

With this notation, the positive semi-definite product $(.,.)_0$ on elementary vectors in $\mathcal{D}$ reads:

$$
\begin{aligned}
&(x^{(1)}\otimes_s ... \otimes_s x^{(n)}, y^{(1)}\otimes_s ... \otimes_s y^{(m)})_0 \\
&= \frac{1}{\sqrt{n!m!}} \frac{d}{dr_1} \cdots \frac{d}{dr_n} \frac{d}{ds_1} \cdots \frac{d}{ds_m} e^{[-r_i x_0^{(i)} r_j x_3^{(j)} - s_i \overline{y}_0^{(i)} s_j \overline{y}_3^{(j)} + r_i x_1^{(i)} s_j \overline{y}_1^{(j)} + r_i x_2^{(i)} s_j \overline{y}_2^{(j)}]}\Big|_{r_i = s_j = 0}. \\
&= \frac{1}{\sqrt{n!m!}} \sum_{q,q'=0} \sum_{\mathcal{P}_{I_{n,q}}, \mathcal{P}'_{I_{m,q'}}} \left(\sum_{s_q^p \in \mathcal{S}_q^p} \prod_{i=1}^{q} (-1)^q (x_0^{(m_i^q)} x_3^{(\tilde{m}_i^q)} + x_0^{(\tilde{m}_i^q)} x_3^{(m_i^q)}) \right) \\
&\quad \times \left(\sum_{s_{q'}^{p'} \in \mathcal{S}_{q'}^{p'}} \prod_{i=1}^{q'} (-1)^{q'} (\overline{y}_0^{(m_i^{q'})} \overline{y}_3^{(\tilde{m}_i^{q'})} + \overline{y}_0^{(\tilde{m}_i^q)} \overline{y}_3^{(m_i^q)}) \right) \\
&\quad \times \left(\sum_{p \in \mathcal{P}(I^{p_q}_{m,q'})} \prod_{i=1}^{n-2q} (x_1^{(t_i^{p_q})} \overline{y}_1^{(p(t_i^{p'_{q'}}))} + x_2^{(t_i^{p_q})} \overline{y}_2^{(p(t_i^{p'_{q'}}))}) \right) \delta_{n-2q, m-2q'}. \qquad (2.3.3)
\end{aligned}
$$

Example:

To illustrate this awkward looking formula, we mention as an example:

$$(x^{(1)}\otimes_s x^{(2)}, y^{(1)}\otimes_s y^{(2)})_0 = \frac{1}{\sqrt{2!2!}}(x_0^{(1)}x_3^{(2)} + x_0^{(2)}x_3^{(1)})(\overline{y}_0^{(1)}\overline{y}_3^{(2)} + \overline{y}_0^{(2)}\overline{y}_3^{(1)})$$
$$+\frac{1}{\sqrt{2!2!}}(x_1^{(1)}\overline{y}_1^{(1)} + x_2^{(1)}\overline{y}_2^{(1)})(x_1^{(2)}\overline{y}_1^{(2)} + x_2^{(2)}\overline{y}_2^{(2)})$$
$$+\frac{1}{\sqrt{2!2!}}(x_1^{(1)}\overline{y}_1^{(2)} + x_2^{(1)}\overline{y}_2^{(2)})(x_1^{(2)}\overline{y}_1^{(1)} + x_2^{(2)}\overline{y}_2^{(1)}) \qquad (2.3.4)$$

Remark:

The next step is to determine the null vectors of $\mathcal{D}$ with respect to $(.,.)_0$ and to construct the physical state space $\mathcal{H}^\chi$ by quotienting. Since we know from the work of Carey, Gaffney and Hurst that the observable algebra of the model is isomorphic to $\Delta(S, B)$, we expect to find in $\mathcal{H}^\chi$ the 'transversal' components x_1, x_2 of the single particle state $x \in \mathcal{H}^{(1)}$, only. [11] From (2.3.1), however, this is not obvious, since $(.,.)_0$ involves the components x_3 and x_0, too. That these components finally end up in the null space $\mathcal{H}_N$, can be established via (2.3.3) by the following

Result: Arbitrary n-particle states $x^{(1)}\otimes_s...\otimes_s x^{(n)} \in \mathcal{D}$ can be written in the form

$$x^{(1)}\otimes_s...\otimes_s x^{(n)} = \vec{v}_T + \lambda_x\Omega + \vec{n}, \qquad (2.3.5)$$

where $\vec{v}_T \in \mathcal{D}$ denotes a linear combination of vectors with purely transversal components, Ω is the vacuum, λ_x is a constant generally depending on the vector $x^{(1)}\otimes_s...\otimes_s x^{(n)} \in \mathcal{D}$ and $\vec{n} \in \mathcal{H}_N$.

Example:

Let us illustrate the above result by deriving the decomposition (2.3.5) of a 2-particle state $x^{(1)}\otimes_s x^{(2)}$. For this, we calculate the $(.,.)_0$-product of $x^{(1)}\otimes_s x^{(2)} - \lambda_x 1$ with arbitrary m-particle states. According to (2.3.3),

$$(x^{(1)}\otimes_s x^{(2)} - \lambda_x\,\Omega, y^{(1)}\otimes_s...\otimes_s y^{(m)})_0$$
$$= \frac{-1}{\sqrt{2}}(x_0^{(1)}x_3^{(2)} + x_0^{(2)}x_3^{(1)})(\Omega, y^{(1)}\otimes_s...\otimes_s y^{(m)})_0 - \lambda_x(\Omega, y^{(1)}\otimes_s...\otimes_s y^{(m)})_0$$

[11] For one-particle states, the projection onto the transversal components is given by the operator P_T, $P_T x = P_T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} =: x_T$. Furthermore, we define $x_\perp := P_\perp \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_0 \end{pmatrix}$

$$+\frac{1}{\sqrt{2m!}}\sum_{\mathcal{P}'_{I_{m,q'}}}\left(\sum_{p\in\mathcal{P}_2}(x_1^{(1)}\overline{y}_1^{(p(t_1^{p'_{q'}}))}+x_2^{(2)}\overline{y}_2^{(p(t_1^{p'_{q'}}))})(x_1^{(2)}\overline{y}_1^{(p(t_2^{p'_{q'}}))}+x_2^{(1)}\overline{y}_2^{(p(t_2^{p'_{q'}}))})\right)$$

$$\times\left(\sum_{s^{p'}_{q'}\in\mathcal{S}^{p'}_{q'}}\prod_{i=1}^{q'}(\overline{y}_0^{(m_i^{q'})}\overline{y}_3^{(\tilde{m}_i^{q'})}+\overline{y}_0^{(\tilde{m}_i^{q'})}\overline{y}_3^{(m_i^{q'})})\right)\delta_{2,m-2q'}, \tag{2.3.6}$$

for m even and zero otherwise, where $\mathcal{P}_2$ denotes the permutation group of two elements and $q' = \frac{m}{2}-1$. Here, choosing

$$\lambda_x = (x^{(1)}\otimes_s x^{(2)},\Omega)_0 = \frac{-1}{\sqrt{2}}(x_0^{(1)}x_3^{(2)}+x_0^{(2)}x_3^{(1)}), \tag{2.3.7}$$

all the longitudinal x-components disappear from the $(.,.)_0$-product and we find that

$$x^{(1)}\otimes_s x^{(2)} = P_T x^{(1)}\otimes_s P_T x^{(2)} + \lambda_x\Omega + \vec{n},$$

$\vec{n}\in\mathcal{H}_N$. This means, that in the sense of (2.3.5), $x^{(1)}\otimes_s x^{(2)}$ can be decomposed into a purely transversal part, a multiple of the vacuum vector Ω and a vector in the null space.

Calculation:

Now we derive (2.3.5) for the general case. We denote a symmetrized n-particle state built up of one particle states $x^{(i)}$, $i\in I_*$, arbitrary index set with w elements, by $x^{()}\otimes_s\ldots\otimes_s x^{(w)}|_{I_*}$. This allows us to simplify (2.3.3) by writing

$$(x^{()}\otimes_s\ldots\otimes_s x^{(2q)}|_{p_q(I_{2q})},\Omega)_0 := \frac{1}{\sqrt{(2q)!}}\left(\sum_{s_q^p\in\mathcal{S}_q^p}\prod_{i=1}^{q}(-1)^q(x_0^{(m_i^q)}x_3^{(\tilde{m}_i^q)}+x_0^{(\tilde{m}_i^q)}x_3^{(m_i^q)})\right);$$

$$(\Omega,y^{()}\otimes_s\ldots\otimes_s y^{(2q')}|_{p_{q'}(I_{2q'})})_0 := \frac{1}{\sqrt{(2q')!}}\left(\sum_{s^{p'}_{q'}\in\mathcal{S}^{p'}_{q'}}\prod_{i=1}^{q'}(-1)^{q'}(\overline{y}_0^{(m_i^{q'})}\overline{y}_3^{(\tilde{m}_i^{q'})}+\overline{y}_0^{(\tilde{m}_i^{q})}\overline{y}_3^{(m_i^{q})})\right);$$

$$\begin{aligned}(x^{(1)}\otimes_s\ldots\otimes_s x^{(n)},y^{(1)}\otimes_s\ldots\otimes_s y^{(m)})_0 = &\sum_{q,q'=0}\sum_{\mathcal{P}_{I_{n,q}},\mathcal{P}'_{I_{m,q'}}}\sqrt{\frac{(2q)!(2q')!(n-2q)!(m-2q')!}{n!m!}}\\ &\times(x^{()}\otimes_s\ldots\otimes_s x^{(2q)}|_{p_q(I_{2q})},\Omega)_0(\Omega,y^{()}\otimes_s\ldots\otimes_s y^{(2q')}|_{p'_{q'}(I_{2q'})})_0\\ &\times(P_T x^{()}\otimes_s\ldots\otimes_s P_T x^{(n-2q)}|_{p_q(I_{n,q})},y^{()}\otimes_s\ldots\otimes_s y^{(m-2q')}|_{p'_{q'}(I_{m,q'})})_0\\ &\times\delta_{n-2q,m-2q'}.\end{aligned} \tag{2.3.8}$$

Now, it is clear that

$$\begin{aligned}&(P_T x^{()}\otimes_s\ldots\otimes_s P_T\, x^{(n-2q)},y^{()}\otimes_s\ldots\otimes_s y^{(m)})_0\\ &= \sum_{q'=0}\sum_{\mathcal{P}'_{I_{m,q'}}}\sqrt{\frac{(2q')!(m-2q')!}{m!}}(\Omega,y^{()}\otimes_s\ldots\otimes_s y^{(2q')}|_{p'_{q'}(I_{2q'})})_0\\ &\times(P_T x^{()}\otimes_s\ldots\otimes_s P_T x^{(n-2q)},y^{()}\otimes_s\ldots\otimes_s y^{(m-2q')}|_{p'_{q'}(I_{m,q'})})_0\delta_{n-2q,m-2q'}.\end{aligned} \tag{2.3.9}$$

$$(x^{(1)}\otimes_s...\otimes_s x^{(n)}, y^{(1)}\otimes_s...\otimes_s y^{(m)})_0 = \sum_{q=0}\sum_{\mathcal{P}_{I_{n,q}}} \sqrt{\frac{(2q)!(n-2q)!}{n!}}(x^{()}\otimes_s...\otimes_s x^{(2q)}|_{p_q(I_{2q})}, \Omega)_0$$
$$\times (P_T x^{()}\otimes_s...\otimes_s P_T x^{(n-2q)}, y^{()}\otimes_s...\otimes_s y^{(m)})_0. \tag{2.3.10}$$

(2.3.5) is now a consequence of the fact, that the vector

$$\vec{n}_\lambda = x^{(1)}\otimes_s...\otimes_s x^{(n)} - \sum_{q=0}\sum_{\mathcal{P}_{I_{n,q}}} \lambda_{p_q(I_{n,q})} P_T x^{()}\otimes_s...\otimes_s P_T x^{(n-2q)}|_{p_q(I_{n,q})} \tag{2.3.11}$$

becomes a null vector for the choices

$$\lambda_{p_q(I_{n,q})} = \sqrt{\frac{(2q)!(n-2q)!}{n!}}(x^{()}\otimes_s...\otimes_s x^{(2q)}|_{p_q(I_{2q})}, \Omega)_0. \tag{2.3.12}$$

Remarks:

- $\mathcal{S}_q^p$ contains $\frac{(2q)!}{2^q}$ different sets of numbered pairs, i.e., $\frac{(2q)!}{2^q}\frac{1}{q!}$ different sets of not ordered pairs. $\prod_{i=1}^q (x_0^{(m_i^q)} x_3^{(\tilde{m}_i^q)} + x_0^{(\tilde{m}_i^q)} x_3^{(m_i^q)})$ is an expression with 2^q terms, each term containing as many x_0's as x_3's, i.e., $(x^{()}\otimes_s...\otimes_s x^{(2q)}|_{p_q(I_{2q})}, \Omega)_0$ contains $\frac{(2q)!}{q!}$ different terms. Since all indices are symmetrized, we may equally well use the symmetrization $\frac{1}{(2q)!}\sum_{\mathcal{P}(I_{2q})}$ of one term. Putting all these factors together, we obtain the better looking expression

$$(x^{()}\otimes_s...\otimes_s x^{(2q)}|_{p_q(I_{2q})}, \Omega)_0 = \frac{(-1)^q}{q!\sqrt{(2q)!}} \sum_{\mathcal{P}(I_{2q})} x_0^{(p(1))}...x_3^{(p(2q))},$$

  where the sum is over all permutations $p \in \mathcal{P}(I_{2q})$.

- It follows from the form of (2.3.11) that

$$x_\perp^{(1)}\otimes_s...\otimes_s x_\perp^{(l)}\otimes_s x_T^{(l+1)}\otimes_s...\otimes_s x_T^{(n)} = \sqrt{\frac{l!(n-l)!}{n!}}(x_\perp^{(1)}\otimes_s...\otimes_s x_\perp^{(l)}, \Omega)_0 x_T^{(l+1)}\otimes_s...\otimes_s x_T^{(n)} + \vec{n}. \tag{2.3.13}$$

- All elements of the null space $\mathcal{H}_N \cap \mathcal{D}$ are linear combinations of vectors of the form $\vec{n}_\lambda$, as can be seen from their construction.

Conventions:

- In what follows, we denote by $[x] \in \mathcal{H}^\chi$ the equivalence class of all vectors $x + \vec{n} \in \mathcal{H}, \vec{n} \in \mathcal{H}_N$.

- All Hilbert spaces of the same dimension are isomorphic, but they are not necessarily isomorphic as carriers of representations. Saying that $\mathcal{H}^\times$ can be 'naturally identified' with $S\mathbb{C}^2$, we mean that the pertinent representations of the observable algebra and of $E(2)$ on these spaces are unitarily equivalent.

Now, we are in a position to discuss the physical Hilbert space $\mathcal{H}^\times$, which carries a constraint-free representation of the algebra $\mathcal{A} = \Delta(N, B)$.

Result: The induced 'physical' Hilbert space $\mathcal{H}^\times$ can be naturally identified with $S\mathbb{C}^2$. The $(.,.)_0$-inner product, used for the construction of $\mathcal{H}^\times$, is positive semi-definite.

Calculation:
On E, we define the map $V : E \to S\mathbb{C}^2$ by linear extension of

$$V(e^x) = (e^x, \Omega)_0 e^{[x_T]}. \tag{2.3.14}$$

To see that this linear extension is well-defined (since the basis of exponential vectors $\{e^x\}$ is over-complete, the existence of a linear extension is not guaranteed), we consider the map $\hat{V} : \mathcal{D} \to S\mathbb{C}^2$, defined by (cf. (2.3.11))

$$\hat{V}(x^{(1)}\otimes_s \ldots \otimes_s x^{(n)}) \longrightarrow \sum_{q=0} \sum_{P_{I_{n,q}}} \lambda_{p_q(I_{n,q})} P_T x^{()} \otimes_s \ldots \otimes_s P_T x^{(n-2q)}|_{p_q(I_{n,q})} \tag{2.3.15}$$

and extended to $\mathcal{D}$ by linearity. The essential point is now that $\hat{V}$ can be extended to E and coincides on E with V. This can be seen by using (2.3.13) and

$$x^{\{1\}}\otimes_s \ldots \otimes_s x^{\{n\}} = \sum_l \frac{n!}{l!(n-l)!} x_\perp^{\{1\}} \otimes_s \ldots \otimes_s x_\perp^{\{l\}} \otimes_s x_T^{\{l+1\}} \otimes_s \ldots \otimes_s x_T^{\{n\}},$$

where $x = x_\perp + x_T$. One obtains

$$\begin{aligned}
V(e^x) &= (e^x, \Omega)_0 e^{[x_T]} \\
&= \sum_{l,n>l} \frac{1}{\sqrt{l!}} (x^{\{1\}} \otimes_s \ldots \otimes_s x^{\{l\}}, \Omega)_0 \frac{1}{\sqrt{(n-l)!}} x_T^{\{l+1\}} \otimes_s \ldots \otimes_s x_T^{\{n\}} \\
&= \sum_{l,n>l} \frac{\sqrt{n!}}{l!(n-l)!} x_\perp^{\{1\}} \otimes_s \ldots \otimes_s x_\perp^{\{l\}} \otimes_s x_T^{\{l+1\}} \otimes_s \ldots \otimes_s x_T^{\{n\}} \\
&= \sum_n \frac{1}{\sqrt{n!}} \hat{V} \left(\sum_l \frac{n!}{l!(n-l)!} x_\perp^{\{1\}} \otimes_s \ldots \otimes_s x_\perp^{\{l\}} \otimes_s x_T^{\{l+1\}} \otimes_s \ldots \otimes_s x_T^{\{n\}} \right) \\
&= \sum_n \frac{1}{\sqrt{n!}} \hat{V} \left(x^{\{1\}} \otimes_s \ldots \otimes_s x^{\{n\}} \right).
\end{aligned} \tag{2.3.16}$$

Here, the indices $\{i\}$ should not be mistaken for labels of different vectors $x^{(1)}$, $x^{(j)}$. They count the multiplicity of one and the same vector x in the tensor products.
It follows from our calculation that $\hat{V}$ can be extended to $E \cup \mathcal{D}$. With the expression

$$(e^x, e^y)_0 = (e^x, \Omega)_0(\Omega, e^y)_0 e^{(x_T, y_T)}$$

following from (2.3.3) and with the map (2.3.14), we obtain

$$(\hat{V}X, \hat{V}Y) = (X, Y)_0 \tag{2.3.17}$$

for all $X, Y \in E \cup \mathcal{D}$. From this we conclude that $(.,.)_0$ is positive semi-definite on $E \cup \mathcal{D}$. Furthermore, the $(.,.)_0$-null space $\mathcal{H}_N$ is $\mathcal{H}_N = \ker \hat{V}$. This implies that $\hat{V}$ quotients to a unitary mapping $\check{V}$ which can be extended to the completion of $(\mathcal{D} \cup E)/\mathcal{H}_N$,

$$\check{V} : ((\mathcal{D} \cup E)/\mathcal{H}_N)^{-} \longrightarrow \mathcal{H}^{\chi} \simeq \mathcal{S}\mathbb{C}^2.$$

Result: The 'generating' functional $\omega^{\chi}(f) := (\pi_F(W(f))\Omega, \Omega)_0$ satisfies $\omega^{\chi}(P_T f) = \omega^{\chi}(f)$ for $f \in N$.
Furthermore, $(\pi_F(W(f))\Omega, \Omega)_0 = (\Omega, \pi_F(W(-f))\Omega)_0$ if and only if $f \in N$.

Calculation:
Using (2.1.12), we may calculate

$$\begin{aligned}(\pi_F(W(f))\Omega, \pi_F(W(g))\Omega)_0 &= e^{\frac{-1}{2}(f,f)_E} e^{\frac{-1}{2}(g,g)_E} (e^{-\tilde{f}}, e^{-\tilde{g}})_0 \\ &= e^{\frac{-1}{2}(f,f)_E} e^{\frac{-1}{2}(g,g)_E} e^{\overline{f}_0 f_3 + g_0 \overline{g}_3} e^{f_1 \overline{g}_1 + f_2 \overline{g}_2} \\ &= e^{\frac{-1}{2}(f_1\overline{f}_1 + f_2\overline{f}_2 + g_1\overline{g}_1 + g_2\overline{g}_2)} e^{f_1\overline{g}_1 + f_2\overline{g}_2} \\ &\quad \times e^{\frac{-1}{2}(f_3\overline{f}_3 + f_0\overline{f}_0 - 2\overline{f}_0 f_3)} e^{\frac{-1}{2}(g_3\overline{g}_3 + g_0\overline{g}_0 - 2g_0\overline{g}_3)},\end{aligned} \tag{2.3.18}$$

where $\tilde{f} = \begin{pmatrix} -\overline{f}_0 \\ f_i \end{pmatrix}$. For $g = 0$, $f \in N$, we obtain from (2.3.18) the 'generating' functional

$$\omega^{\chi}(f) = e^{\frac{-1}{2}(f_1\overline{f}_1 + f_2\overline{f}_2)} = e^{\frac{1}{2}(f,f)_M} \tag{2.3.19}$$

and hence $\omega^{\chi}(f) = \omega^{\chi}(P_T f)$. Furthermore, it follows from (2.3.18) that $(\pi_F(W(f))\Omega, \Omega)_0 = (\Omega, \pi_F(W(-f))\Omega)_0 e^{\overline{f}_0 f_3 - f_0 \overline{f}_3}$.

Remarks:

- For a discussion of the reason why $(.,.)_0$ does not preserve the adjoint, we refer the reader to our chapter on QED. There it is argued that the preservation of the adjoint for all $f \in N$ is sufficient at least for some practical purposes.

- For $[\Omega] = \check{V}(\Omega)$ and for all $f \in N$, we have

$$(\pi^\chi(W(f))[\Omega], [\Omega])^\chi = \omega^\chi(f) \tag{2.3.20}$$

i.e., ω^χ determines $\mathcal{H}^\chi$ by the GNS-construction [53].

- From (2.3.19) it is clear that ω^χ is invariant under all Lorentz transformations and hence under the subgroup $E(2)$ of the Lorentz group. This means that there exists a unitary representation U^χ of $E(2)$, implementing the automorphism $\alpha_{(\theta,l_1,l_2)}(W(f)) = W(Z(\theta,l_1,l_2)f)$ on $\mathcal{H}^\chi$. U^χ is defined by

$$U^\chi(\theta, l_1, l_2)\pi^\chi(W(f))[\Omega] := \pi^\chi(W(Z(\theta, l_1, l_2)f))[\Omega]. \tag{2.3.21}$$

To obtain an explicit expression for U^χ, defined in (2.3.21), on elements in $\mathcal{D}$, our strategy is as follows: We show that in the decomposition of vectors (2.3.22), the prefactors of the form (2.3.12) are not affected by $f \to Z(\theta, l_1, l_2)f$. This allows us to show that (2.3.21) is given explicitly on n-particle states by the action on one-particle states, $U^\chi[P_T f] = [P_T Z(\theta, l_1, l_2)f] = [e^{-i\theta M_3'} P_T f]$. Therefore, we establish the following

Result: For $f^{(i)} \in N$, all n-particle states

$$\frac{d}{dr_1}\cdots\frac{d}{dr_n}\pi_F(W(-\sum_i r_i f^{(i)}))\Omega|_{r_i=0} \tag{2.3.22}$$

can be written in terms of the 'transversal' components $P_T f^{(i)}$ only, with prefactors which do not change under the substitution $f \to Z(\theta, l_1, l_2)f$, cf. (2.3.29).

Example:

To illustrate this result, we give two simple examples:

$$\frac{d}{dr_1}\pi_F(W(-\sum_i r_i f^{(i)}))\Omega|_{r_i=0} = \tilde{f}^{(1)} = P_T\tilde{f}^{(1)} + \vec{n}$$

$$\begin{aligned}\frac{d}{dr_1}\frac{d}{dr_2}\pi_F(W(-\sum_i r_i f^{(i)}))\Omega|_{r_i=0} &= \sqrt{2}\tilde{f}^{(1)}\otimes_s\tilde{f}^{(2)} - \frac{1}{2}\Big((f^{(1)}, f^{(2)})_E + (f^{(2)}, f^{(1)})_E\Big)\Omega \\ &= \sqrt{2}P_T\tilde{f}^{(1)}\otimes_s P_T\tilde{f}^{(2)} - \frac{1}{2}\Big((f^{(1)}, f^{(2)})_E + (f^{(2)}, f^{(1)})_E\Big)\Omega, \\ &\quad - \Big(\tilde{f}_0^{(1)}\tilde{f}_3^{(2)} + \tilde{f}_3^{(1)}\tilde{f}_0^{(2)}\Big)\Omega + \vec{n}\end{aligned} \tag{2.3.23}$$

where we have used (2.3.7). Using (cf. (2.1.12)) $\tilde{f}_0 = -\overline{f}_0$, $\tilde{f}_3 = f_3$ and $f_0 = f_3$ since $f \in N$, we obtain

$$\frac{d}{dr_1}\frac{d}{dr_2}\pi_F(W(-\sum_i r_i f^{(i)}))\Omega|_{r_i=0} = -\frac{1}{2}\Big((P_T f^{(1)}, P_T f^{(2)})_E + (P_T f^{(2)}, P_T f^{(1)})_E\Big)\Omega + \vec{n}$$
$$+\sqrt{2}P_T\tilde{f}^{(1)}\otimes_s P_T\tilde{f}^{(2)}, \tag{2.3.24}$$

which is an expression in purely 'transversal' components.

Calculation:

We investigate the n-particle states obtained by differentiation

$$\frac{d}{dr_1}\cdots\frac{d}{dr_n}\pi_F(W(-\sum_i r_i f^{(i)}))\Omega|_{r_i=0} = \frac{d}{dr_1}\cdots\frac{d}{dr_n}e^{\frac{-1}{2}(\sum_i r_i f^{(i)},\sum_i r_i f^{(i)})_E}e^{\sum_i r_i\tilde{f}^{(i)}}|_{r_i=0}$$
$$=\sum_{q=0}\sum_{\mathcal{P}_{I_{n,q}}}\left(\sum_{s_q^p\in\mathcal{S}_q^p}\prod_{i=1}^{q}\frac{(-1)^q}{2^q}[(f^{(m_i^q)}, f^{(\tilde{m}_i^q)})_E + (f^{(\tilde{m}_i^q)}, f^{(m_i^q)})_E]\right)$$
$$\times\sqrt{(n-2q)!}\tilde{f}^{()}\otimes_s\ldots\otimes_s\tilde{f}^{(n-2q)}|_{p_q(I_{n,q})}. \tag{2.3.25}$$

Using

$$\frac{-1}{2}[(f^{(m_i^q)}, f^{(\tilde{m}_i^q)})_E + (f^{(\tilde{m}_i^q)}, f^{(m_i^q)})_E]$$
$$= -(\tilde{f}^{(m_i^q)}\otimes_s\tilde{f}^{(\tilde{m}_i^q)}, \sqrt{2}\Omega)_0 - \frac{1}{2}(P_T f^{(m_i^q)}, P_T f^{(\tilde{m}_i^q)})_E - \frac{1}{2}(P_T f^{(\tilde{m}_i^q)}, P_T f^{(m_i^q)})_E, \tag{2.3.26}$$

and (2.3.11), this allows us to write

$$\frac{d}{dr_1}\cdots\frac{d}{dr_n}\pi_F(W(-\sum_i r_i f^{(i)}))\Omega|_{r_i=0} =$$

$$\sum_{q=0}\sum_{\mathcal{P}_{I_{n,q}}}\left(\sum_{s_q^p\in\mathcal{S}_q^p}\prod_{i=1}^{q}[-\frac{1}{2}(P_T f^{(m_i^q)}, P_T f^{(\tilde{m}_i^q)})_E - \frac{1}{2}(P_T f^{(\tilde{m}_i^q)}, P_T f^{(m_i^q)})_E - (\tilde{f}^{(m_i^q)}\otimes_s\tilde{f}^{(\tilde{m}_i^q)}, \sqrt{2}\Omega)_0]\right)\times$$
$$\sum_{q'=0}\sum_{\mathcal{P}'_{I^{pq}_{n,q,q'}}}\sqrt{(2q')!(n-2q-2q')!}(\tilde{f}^{()}\otimes_s\ldots\otimes_s\tilde{f}^{(2q')}|_{p'_{q'}(I^{pq}_{n,q,q'})}, \Omega)_0 P_T\tilde{f}^{()}\otimes_s\ldots\otimes_s P_T\tilde{f}^{(n-2q-2q')}|_{p'_{q'}(I^{pq}_{n,q})}$$
$$\tag{2.3.27}$$

In a first step, we consider those terms in the last equation, which contain no factor $(P_T f^{(m_i^q)}, P_T f^{(\tilde{m}_i^q)})_E$ and for which $n = 2q + 2q'$. Using

$$\sqrt{(2q')!}(\tilde{f}^{()}\otimes_s\ldots\otimes_s\tilde{f}^{(2q')}|_{p'_{q'}(I^{pq}_{n,q,q'})}, \Omega)_0 = \sum_{s^p_{q'}\in\mathcal{S}^p_{q'}}\prod_{i=0}^{q'}(\tilde{f}^{(m_i^{q'})}\otimes_s\tilde{f}^{(\tilde{m}_i^{q'})}, \sqrt{2}\Omega)_0,$$

we find for these terms

$$\sum_{q=0}(-1)^q \sum_{\mathcal{P}_{I_{n,q}}} \sum_{s_q^p \in \mathcal{S}_q^p} \sum_{s_{q'}^p \in \mathcal{S}_{q'}^p} \prod_{i=0}^{q} \prod_{j=0}^{q'} (\tilde{f}^{(m_i^q)} \otimes_s \tilde{f}^{(\tilde{m}_i^q)}, \sqrt{2}\Omega)_0 (\tilde{f}^{(m_j^{q'})} \otimes_s \tilde{f}^{(\tilde{m}_j^{q'})}, \sqrt{2}\Omega)_0$$

$$= \sum_{q=0}(-1)^q a(q)(\tilde{f}^{(1)} \otimes_s \ldots \otimes_s \tilde{f}^{(n)}, \sqrt{n!}\Omega)_0. \tag{2.3.28}$$

Let us consider this expression for fixed q. Then, $a(q)$ may be determined by a simple combinatorial argument: the number of elements in $\mathcal{P}_{I_{n,q}}$ is $\frac{n!}{(n-2q)!(2q)!}$ and there are $\frac{(2q)!}{2^q}$ different sets of ordered pairs, i.e., $\frac{(2q)!}{2^q}\frac{1}{q!}$ different sets of not ordered pairs in $\mathcal{S}_q^p$. Now, $a(q)$ is given by counting the numbers of sums on the RHS and LHS of the equation given above:

$$\frac{n!}{(n-2q)!(2q)!}\frac{(2q)!}{2^q}\frac{1}{q!}\frac{(n-2q)!}{2^{n-q}}\frac{1}{(\frac{n}{2}-q)!} = a(q)\frac{n!}{2^n}\frac{1}{\frac{n}{2}!}.$$

Hence, $a(q)$ is given as a binomial coefficient

$$a(q) = \frac{\frac{n}{2}!}{q!(\frac{n}{2}-q)!} = \binom{\frac{n}{2}}{q}.$$

This implies that $\sum_q a(q)(-1)^q = 0$ and hence, the $n = 2q + 2q'$ contribution vanishes.

The above argument can be taken over to all other terms with a non-zero number of non-transversal components. The point is to keep fixed the subset of indices of transversal factors.[12]

We conclude that only terms with purely transversal components remain and hence

$$\frac{d}{dr_1}\cdots\frac{d}{dr_n}\pi_F(W(-\sum_i r_i f^{(i)}))\Omega|_{r_i=0}$$

$$= \sum_{q=0}\sum_{\mathcal{P}_{I_{n,q}}}\left(\sum_{s_q^p \in \mathcal{S}_q^p}\prod_{i=1}^{q}\left(\frac{-1}{2}\right)^q [(P_T f^{(m_i^q)}, P_T f^{(\tilde{m}_i^q)})_E + (P_T f^{(\tilde{m}_i^q)}, P_T f^{(m_i^q)})_E]\right)$$

$$\times\sqrt{(n-2q)!}P_T\tilde{f}^{()}\otimes_s\ldots\otimes_s P_T\tilde{f}^{(n-2q)}|_{p_q(I_{n,q})} + \vec{n}. \tag{2.3.29}$$

Remark:

- Putting (2.3.21) and (2.3.29) together, we obtain the explicit form of $U^\times$ on n-particle states in $\mathcal{H}^\times$,

$$U^\times(\theta, l_1, l_2)[P_T\tilde{f}^{(1)}\otimes_s\ldots\otimes_s P_T\tilde{f}^{(n)}] = [e^{-i\theta M'_3}P_T\tilde{f}^{(1)}\otimes_s\ldots\otimes_s e^{-i\theta M'_3}P_T\tilde{f}^{(n)}].$$

[12] E.g. Consider all terms in (2.3.27) with $n = 2q + 2q'$ and one factor $[\frac{-1}{2}(P_T f^{(k)}, P_T f^{(l)})_E - \frac{1}{2}(P_T f^{(l)}, P_T f^{(k)})_E]$. Writing this factor for k, l fixed in front of the sums of (2.3.27), we restrict the sums over the possible permutations and partitions to the subset of $(n-2)$ indices from which k and l are excluded. The remaining expression has the form of (2.3.28) (formulated for the smaller set of $(n-2)$ indices) and the argument given above implies that this contribution vanishes.
Similarly, all terms in (2.3.27) with an arbitrary fixed number of factors $[\frac{-1}{2}(P_T f^{(k)}, P_T f^{(l)})_E - \frac{1}{2}(P_T f^{(l)}, P_T f^{(k)})_E]$ and an arbitrary fixed $(n-2q-2q')$-particle state can be treated by restricting the above argument to the remaining set of indices.

- If $x \in T$, then $\pi_F(W(x))F = F + \vec{n}$ for all $F \in \mathcal{D} \cup E$, $\vec{n} \in \mathcal{H}_N$. From this it follows that $\pi^\chi(\triangle(N, B))$ can be identified with $\triangle(N/T, \check{B})$ which we call the algebra of observables, $\check{B}$ being the non-degenerate symplectic form obtained from B by quotienting [1].

4. T-procedure for the Mini Model

To make the connection between the Rieffel induction procedure and the T-procedure more explicit, we sketch in this section how the observable algebra is obtained in the T-procedure. To apply the T-procedure to the Heisenberg CCR algebra with $\mathcal{A} = \triangle(N, B)$ and the set of unitary constraints $\mathcal{U} = \{U_i(\lambda)\}$, $U_i(\lambda) = W(\lambda v_i)$, $v_i \in T$ basis vectors, we start by specifying the C^*-algebras $\mathcal{F}(L)$, $\mathcal{D}$ and $\mathcal{C}$:

$$\mathcal{F}(L) = C^*(\mathcal{U} - 1), \tag{2.4.1}$$

is the C^*-algebra generated by $(U_i(\lambda) - 1)$, and

$$\mathcal{D} = [\mathcal{A}\mathcal{F}(L)] \cap [\mathcal{F}(L)\mathcal{A}] = [\mathcal{A}\mathcal{F}(L)], \tag{2.4.2}$$

since $\mathcal{A}$ commutes with $\mathcal{F}(L)$. The 'weak commutant' $\mathcal{C}$ of $\mathcal{D}$ reads

$$\mathcal{C} = \mathcal{D}'_w = \{A \in \mathcal{A} \mid [A, H] \in \mathcal{D};\, H \in \mathcal{D}\} = \mathcal{A}, \tag{2.4.3}$$

since $[A_1, A_2 \sum_i c_i(U_i(\lambda) - 1)] = [A_1, A_2] \sum_i c_i(U_i(\lambda) - 1) \in \mathcal{D}$ for all $A_1, A_2 \in \mathcal{A}$, $c_i \in \mathbb{C}$. From this, we obtain (cf. (1.1.8)) the algebra of observables

$$\mathcal{R} = \mathcal{A}/[AC^*(\mathcal{U} - 1)]. \tag{2.4.4}$$

The essential point is now that due to a general theorem of Grundling (cf. [8], Thm. 5.2), this algebra is isomorphic to $\triangle(N/T, \check{B})$, where $\check{B}$ denotes the non-degenerate symplectic form obtained from B by quotienting. Hence,

$$\mathcal{R} \simeq \triangle(N/T, \check{B}) \simeq \triangle(S, B), \tag{2.4.5}$$

i.e., $\mathcal{R}$ is an $E(2)$-invariant quotient algebra, isomorphic to $\triangle(S, B)$.

CHAPTER III: CONSTRAINTS IN FREE QED

In this chapter, the Rieffel induction procedure is applied to the free electromagnetic field, given by the canonical commutation relations of the vector potential

$$[A_\mu(x), A_\nu(y)] = -ig_{\mu\nu} D(x-y), \tag{3.1}$$

where D denotes the commutator function satisfying $\Box D = 0$, with initial conditions $D(\mathrm{x}, 0) = 0$, $\frac{\partial}{\partial t} D(\mathrm{x}, t)|_{t=0} = -\delta(x)$. In close analogy to our discussion of the Heisenberg CCR-algebra, we start again by specifying the Weyl algebra and subalgebras of (3.1) and some of its representations. Then we construct the Rieffel induced Hilbert space which carries a representation of the observable algebra and we discuss its properties. As already in the last chapter, the choice of our starting point has been motivated by the paper of Carey, Gaffney and Hurst [19].

1. Weyl algebra formulation and its representations

In what follows, we consider the Weyl algebra $\triangle(M, B)$ over the vector space $M = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ with symplectic form $B(.,.)$, defined by the Weyl form of the canonical commutation relations

$$W(\phi)W(\phi') = W(\phi + \phi')e^{-\frac{i}{2}B(\phi,\phi')} \tag{3.1.1}$$

$$B(\phi, \phi') = 2\mathrm{Im}(\phi, \phi')_M \tag{3.1.2}$$

for all ϕ, $\phi' \in M$, where

$$(\phi, \phi')_M := \int \frac{d^3\mathrm{k}}{(2\pi)^3 2k_0} \phi_\mu(\mathrm{k}) g^{\mu\nu} \overline{\phi'_\nu(\mathrm{k})}.$$

Remarks:

- Let us comment on the connection between (3.1.1) and (3.1). Smearing the vector potential with test functions in Schwartz space $f \in \mathcal{S}(\mathbb{R}^4) \otimes \mathbb{R}^4$, $A(f) = \int d^4x A_\mu(x) f^\mu(x)$, (3.1) reads

$$[A(f), A(g)] = i\sigma(f, g),$$

$$\sigma(f,g) = -\int d^4x d^4y D(x-y) f^\mu(x) g_\mu(y).$$

This allows us to introduce the operators $U(f) = e^{[iA(f)]}$ which (according to the Baker-Campbell-Haussdorff formula, $e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$ for $[A,[A,B]] = [B,[A,B]] = 0$) satisfy the Weyl form of the CCR

$$U(f)U(g) = U(f+g)e^{[-\frac{1}{2}\sigma(f,g)]}.$$

To obtain a one-to-one correspondence between Weyl operators and test functions, one uses the map $f \to \phi$, $\phi_\mu = D * f_\mu$. Let us define the space M_0 of real solutions of the wave equation $\Box\phi_\mu = 0$,

$$M_0 = \{\phi = D * f \mid f \in \mathcal{S}(\mathbb{R}^4) \otimes \mathbb{R}^4\}. \tag{3.1.3}$$

Then the operators $W(\phi) = U(f)$, $\phi \in M_0$ satisfy (3.1.1) with symplectic form B induced by σ (cf. (3.1.2)):[1]

$$B(\phi,\phi') = -\int d^3\mathrm{x}[\phi^\mu(\mathrm{x})\dot{\phi'}_\mu(\mathrm{x}) - \dot{\phi}^\mu(\mathrm{x})\phi'_\mu(\mathrm{x})].$$

$$B(\phi,\phi') = -i\int \frac{d^3\mathrm{k}}{(2\pi)^3}[\phi^\mu(\mathrm{k})\overline{\phi'_\mu}(\mathrm{k}) - \overline{\phi^\mu}(\mathrm{k})\phi'_\mu(\mathrm{k})]. \tag{3.1.4}$$

- The Weyl algebra is defined by specifying M and B. B is given above. M is taken to be the completion of M_0 in the Fermi inner product[2]

$$\langle\phi,\phi'\rangle_F = \int \frac{d^3\mathrm{k}}{(2\pi)^3 2k_0}[\phi_i(\mathrm{k})\overline{\phi'_i}(\mathrm{k}) + \overline{\phi_0}(\mathrm{k})\phi'_0(\mathrm{k})].$$

Here, $\langle\phi,\phi'\rangle_F$ defines an L^2-norm $\|\phi\|_F^2 = \langle\phi,\phi\rangle_F$ and M_0 is dense in $L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$. Hence, we obtain as completion of M_0, cf. [54]

$$M = L^2(\mathbb{R}^3)\otimes\mathbb{C}^4. \tag{3.1.5}$$

- In what follows, it is useful to introduce formal annihilation and creation operators a_μ, a^*_μ for the free electromagnetic field by

$$A_\mu(x) = \int \frac{d^3\mathrm{k}}{(2\pi)^3 2k_0}[e^{-ikx}a_\mu(\mathrm{k}) + e^{ikx}a^*_\mu(\mathrm{k})]|_{k_0=\mathrm{k}}, \text{ and hence}$$

[1] In what follows, the symbol ϕ is used for three different but closely related mathematical objects: 1. the functions $\phi \in M$, satisfying $\Box\phi_\mu = 0$; 2. the Cauchy data $\{\phi_\mu(\mathrm{x}), \dot{\phi}_\mu(\mathrm{x})\}$ of the equation of motion for fixed time t; and 3. the Fourier transform

$$\phi^\mu(\mathrm{x},t) = \frac{1}{(2\pi)^3}\int \frac{d^3\mathrm{k}}{2k_0}[\phi^\mu(\mathrm{k})e^{-ikx} + \overline{\phi^\mu}(\mathrm{k})e^{ikx}].$$

[2] B is the imaginary part of $\langle.,.\rangle_F$, as may be seen from $\langle\phi,\phi'\rangle_F = B(\phi,J\phi') + iB(\phi,\phi')$. Here, the operator J is defined as $J\phi_\mu(\mathrm{k}) = -(i\phi_0(\mathrm{k}), -i\phi_j(\mathrm{k}))$, $J^2 = -1$.

$$iA(f) = \int \frac{d^3\mathrm{k}}{(2\pi)^3 2k_0}[a_\mu(\mathrm{k})\overline{\phi_\mu}(\mathrm{k}) - a^*_\mu(\mathrm{k})\phi_\mu(\mathrm{k})] = a_\mu(\phi^\mu) - a_\mu(\phi^\mu)^*, \qquad (3.1.6)$$

where $\phi_\mu(k) = D(k)f_\mu(k) = -i\epsilon(k_0)\delta(k^2)f_\mu(k)$. Here, (3.1) leads to

$$[a_\mu(f^\mu), a_\nu(g^\nu)^*] = (g, f)_M,$$

$$W(\phi^\mu) = \exp\,[a_\mu(\phi^\mu) - a_\mu(\phi^\mu)^*]. \qquad (3.1.7)$$

Notation:

In what follows, we use $\tilde{dk} = \frac{d^3\mathrm{k}}{(2\pi)^3 2k_0}$.

Subalgebras of $\triangle(M, B)$

The algebraic automorphisms $\alpha_{\Lambda,a}$ implementing Poincaré transformations on $\triangle(M, B)$ are induced by the symplectic transformations $\gamma_{\Lambda,a}$ on M,

$$\alpha_{\Lambda,a}(W(\phi^\mu)) = W(\gamma_{\Lambda,a}(\phi^\mu)) \qquad \text{with} \qquad (\gamma_{\Lambda,a}(\phi^\mu))(x) = \Lambda^\mu_\nu \phi^\nu(\Lambda^{-1}(x-a)). \qquad (3.1.8)$$

Here, we introduce the subspaces N and T of M which are invariant under $\gamma_{\Lambda,a}$:

$$N = \{\phi_\mu \in M \mid k^\mu\phi_\mu(\mathrm{k}) = 0\} = \{\phi_\mu \in M \mid \partial^\mu\phi_\mu(x) = 0\},$$

$$T = \{\phi_\mu \in M \mid \phi^\mu(\mathrm{k}) = ik^\mu g(\mathrm{k})\} = \{\phi_\mu \in M \mid \phi_\mu(x) = \partial^\mu g(x), \Box g(x) = 0\}. \qquad (3.1.9)$$

The corresponding subalgebras are denoted by $\triangle(N, B)$ and $\triangle(T, B)$, respectively. Furthermore, we introduce the subspace $S \subset M$,

$$S = \{\phi_\mu \in M \mid \phi_0(\mathrm{k}) = 0 = k^i\phi_i(\mathrm{k})\} = \{\phi_\mu \in M \mid \phi_0(x) = 0 = \nabla_i\phi_i(x)\}$$

with corresponding subalgebra $\triangle(S, B)$.

Remarks:

- Here, $N = S \oplus T$, N and T are left invariant by $\gamma_{\Lambda,a}$ while S is not. Correspondingly, $\triangle(N, B)$ and $\triangle(T, B)$ are invariant under the action of $\mathcal{P}$, whereas $\triangle(S, B)$ is not invariant under $\alpha_{\Lambda,a}$.
- Carey, Gaffney and Hurst [19] whose discussion we follow by singling out subalgebras of $\triangle(M, B)$, point out that $\triangle(S, B)$ is isomorphic to the Poincaré-invariant 'algebra of physical photons', $\triangle(N, B)/\triangle(T, B)$, though it is not isomorphic to it as carrier of the action of $\mathcal{P}$.

Some Representations of $\triangle(M, B)$

As a final preparatory step, we introduce the representations of $\triangle(M, B)$ on the symmetric (Fermi) Fock space $\mathcal{H} = \mathcal{S}(L^2(\mathbb{R}^3) \otimes \mathbb{C}^4)$, and a functional representation which may be regarded as the analogue of the Schrödinger representation in our discussion of the Heisenberg CCR algebra. We specify:

1. The Fermi representation Π_F is given on $\mathcal{H} = \mathcal{S}(L^2(\mathbb{R}^3) \otimes \mathbb{C}^4)$, the symmetric Fock space over the one-particle Hilbert space $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$. We define Π_F on the dense subspace $E \subset \mathcal{S}(L^2(\mathbb{R}^3) \otimes \mathbb{C}^4)$, which is the span of the total subset of exponential vectors

$$E = \{\sum_{i=1}^{N} \lambda_i e^{\psi^{(i)}} \mid \lambda_i \in \mathbb{C}, \psi^{(i)} \in \mathcal{H}, N < \infty\}; \tag{3.1.10}$$

$$e^{\psi} = 1 \oplus \psi \oplus \frac{1}{\sqrt{2}}\psi \otimes \psi \oplus \frac{1}{\sqrt{3!}}\psi \otimes \psi \otimes \psi \oplus \ldots \tag{3.1.11}$$

with scalar product

$$(e^{\psi}, e^{\varphi}) = e^{(\psi,\varphi)_E}, \tag{3.1.12}$$

$$(\psi, \varphi)_E = \int \frac{d^3\mathrm{k}}{(2\pi)^3 2k_0}\psi_\mu(\mathrm{k})\delta^{\mu\nu}\overline{\varphi}_\nu(\mathrm{k}). \tag{3.1.13}$$

The representation is defined by[3]

$$\Pi_F(W(\phi))e^{\psi} = e^{\frac{-1}{2}(\phi,\phi)_E + (\psi,\tilde{\phi})_E} e^{(\psi - \tilde{\phi})}, \tag{3.1.14}$$

where $\tilde{\phi}_\mu = \begin{pmatrix} -\overline{\phi}_0 \\ \phi_i \end{pmatrix}$. The definition (3.1.14) extends to all of $\mathcal{H}$.

2. A representation Π_L of $\triangle(M, B)$ on $L^2(L^2_{\mathbb{R}}(\mathbb{R}^3)\otimes_{\mathbb{R}}\mathbb{R}^4, \mu)$ defined by

$$\Pi_L(W(\phi_\mu))\psi(A) = e^{\frac{-1}{2}i(\phi^{(1)},\phi^{(2)})_E} e^{-i(\phi^{(2)},A)_E}\psi(A - \phi^{(1)}),$$

where $\phi_\mu(\mathrm{k}) = \phi^{(1)}_\mu(\mathrm{k}) + i\phi^{(2)}_\mu(\mathrm{k})$, and the scalar product is given by $\mu(A) = [dA]e^{-(A,A)}$,

$$(\psi, \varphi)_{L^2L^2} = \int [dA]e^{-(A,A)}\psi(A)\overline{\varphi(A)}$$

[3] To see that this defines a representation, we check that

$$\Pi_F(W(\phi))\Pi_F(W(\varphi)) = e^{[i\,\mathrm{Im}[(\overline{\phi}_0,\overline{\varphi}_0)_E + (\phi_i,\varphi_i)_E]]}\Pi_F(W(\phi^\mu + \varphi^\mu)),$$

where

$$\mathrm{Im}[(\overline{\phi}_0, \overline{\varphi}_0)_E + (\phi_i, \varphi_i)_E] = \mathrm{Im}[\langle\varphi, \phi\rangle_F] = B(\varphi, \phi).$$

for all $\psi, \varphi \in L^2(L^2(\mathbb{R}^3) \otimes \mathbb{R}^4, \mu)$ [55]. We mention this functional representation for the sake of completeness only. A calculation of a Rieffel-induced inner product $(.,.)_0$ in a functional representation will be given for QED in the temporal gauge, discussed in the next chapter. For the specification of subspaces $L \subset L^2(L^2(\mathbb{R}^3) \otimes \mathbb{R}^4, \mu)$, on which Rieffel induction can be performed, we refer to the definition of (4.4.1).

Remark:

In close analogy to our discussion of the Heisenberg CCR algebra, we comment on the connection between the Fermi trick and the Fermi representation Π_F. The idea is to start from canonical commutation relations for the annihilation and creation operators $\hat{a}_\mu$, $\hat{a}^*_\mu$,

$$[\hat{a}(f), \hat{a}^*(g)] = (g, f)_E, \tag{3.1.15}$$

$$\hat{a}(f) = \hat{a}_\mu(f^\mu) = \int \frac{d^3\mathrm{k}}{(2\pi)^3 2k_0}[\hat{a}_0(\mathrm{k})\overline{f}_0(\mathrm{k}) + \hat{a}_i(\mathrm{k})\overline{f}_i(\mathrm{k})].$$

which allow for a representation of (3.1.7) on $\mathcal{H}$ as

$$\Pi_F(W(\phi^\mu)) = e^{[\hat{a}_0(\overline{\phi}_0)^* - \hat{a}_i(\phi_i)^* + \hat{a}_i(\phi_i) - \hat{a}_0(\overline{\phi}_0)]} = e^{[\hat{a}_\mu(\bar{\phi}_\mu) - \hat{a}_\mu(\bar{\phi}_\mu)^*]}. \tag{3.1.16}$$

The action of $\Pi_F(\triangle(M, B))$ on $\mathcal{H} = \mathcal{S}(L^2(\mathbb{R}^3) \otimes \mathbb{C}^4)$ is now given by $\Pi_F(W(\phi^\mu))$ and the following action of the annihilation and creation operators on the n-particle Hilbert space (symmetrization is understood):

$$\hat{a}_\mu(\phi^\mu)\psi_1 \otimes \ldots \otimes \psi_n = \sqrt{n}(\psi_1, \phi)_E \psi_2 \otimes \ldots \otimes \psi_n;$$

$$\hat{a}_\mu(\phi^\mu)^*\psi_1 \otimes \ldots \otimes \psi_n = \sqrt{n+1}\phi \otimes \psi_1 \otimes \ldots \otimes \psi_n.$$

Especially, this leads to

$$e^{\hat{a}_\mu(\phi^\mu)} e^\psi = e^{(\psi,\phi)_E} e^\psi;$$

$$e^{\hat{a}_\mu(\phi^\mu)^*} e^\psi = e^{(\psi+\phi)}$$

and hence to (3.1.14). Again, we remark that whereas $W(\phi)$ exists independently of the representation chosen, the vector potential A_μ and its creation and annihilation operators exist only as operators on those Hilbert spaces which (like the Fock space $\mathcal{H}$) carry a regular representation of $\triangle(M, B)$.

2. Rieffel induction on $\mathcal{H} = \mathcal{S}(L^2(\mathbb{R}^3) \otimes \mathbb{C}^4)$

We are now in a position to specify the input data from which we want to carry out the Rieffel induction procedure for free QED on $\mathcal{H} = \mathcal{S}(L^2(\mathbb{R}^3) \otimes \mathbb{C}^4)$. Firstly, we present the mathematical setup, secondly we comment on a mathematical difficulty concerning the definition of our path integral. Then we calculate and analyse the inner product $(.,.)_0$, obtaining results similar to those obtained for the Heisenberg CCR-algebra.

Imposing the constraints

Though our choice of algebras and representations for free QED parallels our discussion of the Heisenberg algebra, there is one additional problem: the gauge group $G = T$ (regarded as additive group) is not locally compact and hence it is unclear whether one can make sense of the integration (1.1.20). For the same reason, the group algebra $C^*(T)$ is not defined.[4] On the other hand, our discussion of the Heisenberg algebra has shown that what is needed is neither $C^*(T)$ nor the rigging map, but a continuous representation of the gauge group T and a well-defined inner product $(.,.)_0$. These observations lead us to a slight modification of the Rieffel induction procedure [1], reflected by the following input data:[5]

1. as constraint algebra: we do not define a constraint algebra (as mentioned above, we can not rely on the existence of $\mathcal{B} = C^*(T)$, which would be a tentative choice). All we need is a continuous representation of the gauge group T on the Hilbert space $\mathcal{H}$.

2. as field algebra of weak observables: $\mathcal{A} = \triangle(N, B)$ which is the commutant in $\triangle(M, B)$ of $\triangle(M, B) = C^*(T_d)$, the group algebra over T with discrete topology.

3. as Hilbert space $\mathcal{H}_\chi$ with scalar product $(.,.)_\chi$, used to induce from: $\mathcal{H}_\chi = \mathbb{C}$, carrying the trivial representation π_χ of T.

4. as Hilbert space $\mathcal{H}$ with scalar product $(.,.)$: $\mathcal{H} = \mathcal{S}(L^2(\mathbb{R}^3) \otimes \mathbb{C}^4)$, carrying two representations:

   (a) the left representation of $\mathcal{A}$, defined by the Fermi Fock representation Π_F.

---

[4] Recently, Grundling has defined a group algebra for groups which are obtained as topological inductive limits of locally compact subgroups,cf. [56]. We have not tried to exploit this work for our purposes.

[5] From the definition of the rigging map, $(.,.)_0$ inherits the properties

$$\begin{aligned}
&(\Psi, \Psi)_0 \geq 0, \\
&(A\Psi, \Phi)_0 = (\Psi, A^*\Phi)_0, \\
&(A\Psi, A\Psi)_0 \leq \| A \|^2 (\Psi, \Psi)_0,
\end{aligned} \tag{3.2.1}$$

for all $\Psi, \Phi \in L$, $A \in \mathcal{A}$. These conditions may be required independently of a rigging (which needs not to exist). They are sufficient for obtaining an 'induced' representation $\pi^\chi(\mathcal{A})$ on $\mathcal{H}^\chi$, cf. [1]. Though, strictly speaking, we give up Rieffel induction by starting our discussion rigorously with the specification of the semidirect inner product satisfying (3.2.1), our point is that Rieffel induction provides a systematic mechanism leading to (3.2.1).

(b) the right representation of T, defined by $U(\partial_\mu g) = \Pi_F(W(\partial_\mu g))$ which is continuous, $\partial_\mu g \in T$.

5. For the specification of $(.,.)_0$, we consider the 'rigging map' on $E \times E$: for all $\partial_\mu g \in T$, this map is specified by

$$\langle \psi, \varphi \rangle_{\mathcal{B}}(\partial_\mu g) = (U(\partial_\mu g)\varphi, \psi). \tag{3.2.2}$$

By a slight abuse of notation, $\langle .,. \rangle_{\mathcal{B}}$ will be called 'rigging map', though it does not take values in a C^*-algebra.

Calculation and analysis of $(.,.)_0$

We start with an expression for the rigging map:

Result: On elementary vectors e^ψ, e^φ, the rigging map reads[6]

$$(U(\partial_\mu g)e^\psi, e^\varphi) = e^{-i[(k_0\psi_0,\overline{g})+(k_i\psi_i,g)+(\overline{g},k_0\overline{\varphi}_0)+(g,k_i\overline{\varphi}_i)]}e^{(\psi,\varphi)_E}, \tag{3.2.3}$$

Calculation:
For $\phi^\mu(\mathrm{k}) = ik^\mu g(\mathrm{k}) \in T$, we write $\tilde{\phi}_\mu = \begin{pmatrix} -\overline{\phi}_0 \\ \phi_i \end{pmatrix} = \begin{pmatrix} ik_0\overline{g} \\ ik_i g \end{pmatrix}$,

$$\pi_F(W(\phi^\mu)) = e^{-i[k_0 a_0(\overline{g})^* + k_i a_i(g)^*]}e^{-i[k_0 a_0(\overline{g}) + k_i a_i(g)]}e^{\frac{-1}{2}(k_\mu g(\mathbf{k}), k_\mu g(\mathbf{k}))_E}.$$

Using the action of $\Pi_F(W(\phi))$ on e^ψ, we obtain (3.2.3).

Remark:
In analogy to (2.3.1), we would like to calculate $(.,.)_0$ for the elementary vectors e^ψ, $e^\varphi \in E$, starting from

$$(e^\psi, e^\varphi)_0 = \int_T [\mathcal{D}\phi](\pi_F(W(\phi))e^\psi, e^\varphi), \tag{3.2.4}$$

where $[\mathcal{D}\phi]$ denotes a (path integral) measure over the group T. In contrast to (2.3.1) however, T is not locally compact. The following result specifies in which sense (3.2.4) is well-defined.

[6] Here g_1, g_2 denote the real and imaginary components of $g(\mathbf{k}) = g_1(\mathbf{k}) + ig(\mathbf{k})$ and $(f,g) = \int d\tilde{k} f(\mathbf{k})g(\mathbf{k})$. Furthermore, we use the formal expression $(k_0\psi_0, \overline{g})$ for the scalar product $(k_0\psi_0, \overline{g}) := \int d\tilde{k}\psi_0(\mathbf{k})k_0\overline{g}(\mathbf{k})$.

Result: (Landsman,[1])

Consider an isotonic family $\{T_n\}_n$ of n-dimensional Hilbert subspaces $T_n \subset T_{n+1} \subset T$, such that T is the algebraic inductive limit of this family. Then

$$(\Psi, \Phi)_0 = \lim_{n\to\infty} \int_{T_n} \frac{d^n g}{\pi^{\frac{n}{2}}} (U(\partial_\mu g)\Psi, \Phi) \tag{3.2.5}$$

exists for all $\Psi, \Phi \in E$. On elementary vectors,

$$(e^\psi, e^\varphi)_0 = e^{(\psi,\varphi)_E} \int_T d\mu(g) e^{-i[(k_0\psi_0,\overline{g})+(k_i\psi_i,g)+(\overline{g},k_0\overline{\varphi}_0)+(g,k_i\overline{\varphi}_i)]}, \tag{3.2.6}$$

where μ is a promeasure.[7]

Argumentation:(Landsman,[1])

To obtain (3.2.5) for all elements of E, it is sufficient to establish (3.2.5) for elementary vectors, since the result holds automatically for finite linear combinations of elementary vectors.

The essential point is that the functions $k_0\psi_0$, $k_i\psi_i$, $k_0\overline{\varphi}_0$, $k_i\overline{\varphi}_i$, occuring in (3.2.6), span a finite-dimensional Hilbert space $\mathcal{K}$. Hence,

$$f(g) = e^{-i[(k_0\psi_0,\overline{g})+(k_i\psi_i,g)+(\overline{g},k_0\overline{\varphi}_0)+(g,k_i\overline{\varphi}_i)]}$$

is a tame function (cf. footnote for definitions) and this implies that the right hand side of

$$\int_{T_n} \frac{d^n g}{\pi^{\frac{n}{2}}} (U(\partial_\mu g)\Psi, \Phi) = \int_T d\mu(g) f(P_n g)$$

can be explicitly evaluated, cf. [57]. Since $P_n \to 1$ weakly by construction of the family $\{T_n\}_n$, one sees that the limit (3.2.5) exists.

Notation:

In the subsequent discussion, we use the following projection operators on $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$:

$$\begin{aligned}
(P_T\psi)_\mu(\mathbf{k}) &= \left(0, (\delta_{ij} - \frac{k_i k_j}{k_0^2})\psi_j(\mathbf{k})\right), \\
(P_L\psi)_\mu(\mathbf{k}) &= \left(0, \frac{k_i k_j}{k_0^2}\psi_j(\mathbf{k})\right), \\
(P_0\psi)_\mu(\mathbf{k}) &= (\psi_0(\mathbf{k}), 0, 0, 0), \\
(P_\perp\psi)_\mu(\mathbf{k}) &= ((P_L + P_0)\psi)_\mu(\mathbf{k}).
\end{aligned} \tag{3.2.7}$$

[7] The T_n are regarded as real Hilbert spaces. Given 1. projectors $P_n : T \to T_n$ with $P_i P_j = P_i$ for $i \leq j$ and $T_n \subset T$ a finite-dimensional (Hilbert) space of T, and 2. arbitrary Borel sets $A \in T_n$ with measures $\mu_n(A) = \int_A \frac{d^n g}{\pi^{\frac{n}{2}}} e^{-(k_0^2 g_1, g_1)-(k_0^2 g_2, g_2)}$. Then the promeasure μ is defined on cylinder sets by $\mu(P_n^{-1}(A)) = \mu_n(A)$. Since the covariance of μ is the unit operator (which is not nuclear), it cannot be extended to a Borel measure on T. Yet, 'tame' functions, satisfying $f(x) = f_n(P_n x)$, (f_n a Borel function on $T_n \subset T$) can be integrated with respect to μ, cf. [57].

Result: On elementary vectors in E, we obtain

$$(e^{\psi}, e^{\varphi})_0 = e^{\int d\tilde{k}[-k_\mu(P_0\psi)_\mu(\mathbf{k})k_\nu(P_L\psi)_\nu(\mathbf{k})-k_\mu(P_0\overline{\varphi})_\mu(\mathbf{k})k_\nu(P_L\overline{\varphi})_\nu(\mathbf{k})+(P_T\psi)_\mu(\mathbf{k})(P_T\overline{\varphi})_\mu(\mathbf{k})]}. \quad (3.2.8)$$

Calculation:

Carrying our the Gaussian integrals in

$$\int_T [\mathcal{D}g(\mathbf{k})](\pi_F(W(\phi^\mu(\mathbf{k})))e^{\psi}, e^{\varphi})$$
$$= \int_T [\mathcal{D}g_1(\mathbf{k})][\mathcal{D}g_2(\mathbf{k})]e^{-(k_0^2 g_1, g_1)-(k_0^2 g_2, g_2)}e^{-i[(k_0\psi_0,\overline{g})+(k_i\psi_i,g)+(\overline{g},k_0\overline{\varphi}_0)+(g,k_i\overline{\varphi}_i)]}e^{(\psi,\varphi)_E}, \quad (3.2.9)$$

we obtain

$$(e^{\psi}, e^{\varphi})_0 = e^{\int d\tilde{k}F(\mathbf{k})},$$
$$F(\mathbf{k}) = [-\frac{k_0\psi_0(\mathbf{k})k_i\psi_i(\mathbf{k})}{k_0^2} - \frac{k_0\overline{\varphi}_0(\mathbf{k})k_i\overline{\varphi}_i(\mathbf{k})}{k_0^2}] + [\psi_i(\mathbf{k})(\delta_{ij} - \frac{k_i k_j}{k_0^2})\overline{\varphi}_j(\mathbf{k})]. \quad (3.2.10)$$

With the help of $\frac{k_i}{k_0^2}\psi_i(\mathbf{k}) = k_\mu(P_L\psi)_\mu(\mathbf{k})$; $k_0\psi_0 = k_\mu(P_0\psi)_\mu(\mathbf{k})$, this takes the form of (3.2.8).

Remark:

(3.2.8) may be analysed by exploiting the close relation with (2.3.1). Identifying x_0 with $k_\mu(P_0\psi)_\mu$, x_3 with $k_\mu(P_L\psi)_\mu$, $(x_1, x_2, 0, 0)$ with $(P_T\psi)_\mu$ and y_μ with the corresponding expressions for φ_μ, the combinatorial machinery developed in our discussion of the Heisenberg CCR-algebra can be taken over without alterations. Again, we are interested in the structure of n-particle states which we shall obtain in our calculations as derivatives of elementary exponential vectors,

$$\psi^{(1)}\otimes_s\psi^{(2)}\otimes_s\ldots\otimes_s\psi^{(n)} := \frac{1}{\sqrt{n!}}\frac{d}{dr_1}\ldots\frac{d}{dr_n}e^{\sum r_i\psi^{(i)}}|_{r_i=0}.$$

We start with the

Definition of $\mathcal{D}$: $\mathcal{D}$ is the set of finite linear combinations of n-particle states $\psi^{(1)}\otimes_s\psi^{(2)}\otimes_s\ldots\otimes_s\psi^{(n)}$, $\psi^{(i)} \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$.

The $(.,.)_0$-product on $\mathcal{D}$ reads (cf. the previous chapter)

$$(\psi^{(1)} \otimes_s\ldots\otimes_s\psi^{(n)}, \varphi^{(1)}\otimes_s\ldots\otimes_s\varphi^{(m)})_0 = \frac{1}{\sqrt{n!m!}}\sum_{q,q'=0}\sum_{\mathcal{P}_{I_{n,q}},\mathcal{P'}_{I_{m,q'}}}$$
$$\times(\sum_{s_q^p\in\mathcal{S}_q^p}\prod_{i=1}^{q}\int d\check{k}^{(i)}(k_\mu^{(i)}P_0\psi_\mu^{(m_i^q)}(\mathbf{k}^{(i)}))(k_\mu^{(i)}P_L\psi_\mu^{(\tilde{m}_i^q)}(\mathbf{k}^{(i)}))$$

$$+(k_\mu^{(i)} P_0 \psi_\mu^{(\tilde{m}_i^q)}(\mathrm{k}^{(i)}))(k_\mu^{(i)} P_L \psi_\mu^{(m_i^q)}(\mathrm{k}^{(i)})))$$

$$\times \Big(\sum_{s_{q'}^{p'} \in \mathcal{S}_{q'}^{p'}} \prod_{i=1}^{q'} \int d\tilde{k}^{(i)} (k_\mu^{(i)} P_0 \overline{\varphi}_\mu^{(m_i^{q'})}(\mathrm{k}^{(i)}))(k_\mu^{(i)} P_L \overline{\varphi}_\mu^{(\tilde{m}_i^{q'})}(\mathrm{k}^{(i)}))$$

$$+(k_\mu^{(i)} P_0 \overline{\varphi}_\mu^{(\tilde{m}_i^{q'})}(\mathrm{k}^{(i)}))(k_\mu^{(i)} P_L \overline{\varphi}_\mu^{(m_i^{q'})}(\mathrm{k}^{(i)})))$$

$$\times \left(\sum_{p \in \mathcal{P}(I_{m,q'}^{pq})} \prod_{i=1}^{n-2q} \int d\tilde{k}^{(i)} P_T \psi_\mu^{(t_i^{pq})}(\mathrm{k}^{(i)}) \overline{\varphi}_\mu^{(p(t_i^{p'q'}))}(\mathrm{k}^{(i)}) \right) \delta_{n-2q, m-2q'} (-1)^{q+q'}. \quad (3.2.11)$$

Remark:

In analogy to the case of the Heisenberg algebra, $\mathcal{D}$ is not left invariant by the action of $\triangle(M, B)$, $\Pi_F(\triangle(M,B))\mathcal{D} \not\subset \mathcal{D}$, but remains stable under the action of the unbounded operator algebra generated by the creation and annihilation operators.

We start our analysis of $\mathcal{D}$ by deriving the decomposition properties of vectors into transverse components.

Result: Arbitrary n-particle states $\psi^{(1)} \otimes_s \ldots \otimes_s \psi^{(n)} \in \mathcal{D}$ can be decomposed into transverse components up to a vector $\vec{n}$ in the null space, $\mathcal{H}_N$,

$$\psi^{(1)} \otimes_s \ldots \otimes_s \psi^{(n)} = \sum_{q=0} \sum_{\mathcal{P}_{I_{n,q}}} \lambda_{p_q(I_{n,q})} (P_T \psi^{()}) \otimes_s \ldots \otimes_s (P_T \psi^{(n-2q)})|_{p_q(I_{n,q})} + \vec{n}, \quad (3.2.12)$$

$$\lambda_{p_q(I_{n,q})} = \sqrt{\frac{(2q)!(n-2q)!}{n!}} (\psi^{()} \otimes_s \ldots \otimes_s \psi^{(2q)}|_{p_q(I_{n,q})}, \Omega)_0. \quad (3.2.13)$$

Calculation:

The calculation follows step by step the derivation of (2.3.5) via (2.3.11) and (2.3.12). Especially, we may write again

$$(\psi^{()} \otimes_s \ldots \otimes_s \psi^{(2q)}|_{p_q(I_{2q})}, \Omega)_0 = \frac{(-1)^q}{q! \sqrt{(2q)!}} \sum_{\mathcal{P}(I_{2q})} (P_0 \psi)^{(p(1))} \ldots (P_L \psi)^{(p(2q))},$$

where each term contains as many $(P_0 \psi)$'s as $(P_L \psi)$'s. The counterpart of (2.3.13) now reads

$$(P_\perp \psi)^{(1)} \otimes_s \ldots \otimes_s (P_\perp \psi)^{(l)} \otimes_s (P_T \psi)^{(l+1)} \otimes_s \ldots \otimes_s (P_T \psi)^{(n)}$$
$$= \sqrt{\frac{l!(n-l)!}{n!}} ((P_\perp \psi)^{(1)} \otimes_s \ldots \otimes_s (P_\perp \psi)^{(l)}, \Omega)_0 (P_T \psi)^{(l+1)} \otimes_s \ldots \otimes_s (P_T \psi)^{(n)} + \vec{n}. \quad (3.2.14)$$

Result: $(.,.)_0$ is positive semi-definite. The induced 'physical' Hilbert space $\mathcal{H}^\chi$ can be identified naturally with the symmetric Fock space $\mathcal{S}(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)$.

Calculation:

The argument for the positivity of $(.,.)_0$ parallels the one in our discussion of the Heisenberg algebra. We define a map $V : E \to \mathcal{S}(L^2(\mathbb{R}) \otimes \mathbb{C}^2)$ by the analogue of (2.3.14) and a corresponding map $\hat{V} : \mathcal{D} \to \mathcal{S}(L^2(\mathbb{R}) \otimes \mathbb{C}^2)$ by

$$\hat{V}(\psi^{(1)} \otimes_s \ldots \otimes_s \psi^{(n)}) = \sum_{q=0} \sum_{\mathcal{P}_{I_{n,q}}} \lambda_{p_q(I_{n,q})} P_T \psi^{()} \otimes_s \ldots \otimes_s P_T \psi^{(n-2q)}|_{p_q(I_{n,q})}.$$

Again, $\hat{V}$ can be extended to $E \cup \mathcal{D}$ and we finally obtain that $(\hat{V}\Psi, \hat{V}\Phi) = (\Psi, \Phi)_0$ which implies that $(.,.)_0$ is positive semi-definite on $E \cup \mathcal{D}$ and that the null space $\mathcal{H}_N$ is $\mathcal{H}_N = \ker \hat{V}$. Consequently, $\hat{V}$ quotients to a mapping $\tilde{V}$ which can be extended to the completion of $(E \cup \mathcal{D})/\mathcal{H}_N$,

$$\tilde{V} : ((E \cup \mathcal{D})/\mathcal{H}_N)^- \longrightarrow \mathcal{H}^\times \simeq \mathcal{S}(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2).$$

Remark:

All infinite-dimensional Hilbert spaces of the same cardinality are isomorphic, but they are not isomorphic as carriers of representations of algebras $\Delta(M, B)$, $\Delta(N, B)$, etc. or of the Poincaré group $\mathcal{P}$. In this sense, the quotient $(E \cup \mathcal{D})/\mathcal{H}_N$ specifies a particular Hilbert space, carrying a unitary representation of $\mathcal{P}$ as we shall see in what follows.

Firstly, however, we turn to results about the subspace $N \subset M$ and the subalgebra $\Delta(N, B)$. For the representation of the Weyl algebra $\Delta(N, B)$ with $\psi^{(i)} \in N$, satisfying $k^\mu \psi^{(i)}_\mu(\mathrm{k})$, we obtain (cf. (2.3.29)) the

Result: For $\psi^{(i)} \in N$, satisfying $k^\mu \psi^{(i)}_\mu(\mathrm{k}) = 0$,

$$\begin{aligned}
&\frac{d}{dr_1} \cdots \frac{d}{dr_n} \pi_F(W(-\sum_i r_i \psi^{(i)}))\Omega|_{r_i=0} \\
&= \sum_{q=0} \sum_{\mathcal{P}_{I_{n,q}}} \Big(\sum_{s^p_q \in S^p_q} \prod_{i=1}^{q} \left(\frac{-1}{2}\right)^q [(P_T \psi^{(m^q_i)}(\mathrm{k}^{(i)}), P_T \psi^{(\tilde{m}^q_i)}(\mathrm{k}^{(i)}))_E \\
&+ (P_T \psi^{(\tilde{m}^q_i)}(\mathrm{k}^{(i)}), P_T \psi^{(m^q_i)}(\mathrm{k}^{(i)})_E)]\Big) \\
&\times \sqrt{(n-2q)!}\, P_T \tilde{\psi}^{()} \otimes_s \ldots \otimes_s P_T \tilde{\psi}^{(n-2q)}|_{p_q(I_{n,q})} + \vec{n},
\end{aligned} \tag{3.2.15}$$

Calculation:

The calculation step by step follows the derivation of (2.3.29). The point is that the exponential factor $e^{\frac{-1}{2}(\psi,\psi)}$ in $\Pi_F(W(-\psi))\Omega = e^{\frac{-1}{2}(\psi,\psi)} e^{\tilde{\psi}}$ cancels the longitudinal components stemming from the derivatives of $e^{\tilde{\psi}}$, $\psi = \sum_i r_i \psi^{(i)}$.

Result: The 'generating' functional

$$\omega^\chi(\phi^\mu) = (\pi_F(W(\phi^\mu))\Omega, \Omega)_0$$

takes the form

$$\omega^\chi(\phi^\mu) = e^{\frac{1}{2}(\phi_\mu,\phi_\mu)_M} e^{-(\frac{1}{k_0}\phi_0, k_\mu\phi^\mu)}. \tag{3.2.16}$$

Especially, it is Poincaré invariant on all elements $\phi \in N$.

In the remainder of this section, we discuss in three subsections the transformation properties of $\mathcal{H}^\chi$ under Poincaré transformations, gauge transformations and regular states, and, finally, the Hamiltonian and its spectrum.

Poincaré transformations on $\triangle(N, B)$

Leaving $\triangle(N, B)$ invariant, the Poincaré group $\mathcal{P}$ acts as an automorphism group on $\triangle(N, B)$, cf. (3.1.8). On the other hand, we have, in anology with (2.3.20)

$$\omega^\chi(\phi) = (\pi^\chi(W(\phi))[\Omega], [\Omega])^\chi = e^{\frac{1}{2}(\phi,\phi)_M},$$

for $[\Omega] = \tilde{V}(\Omega)$ and for all $\phi \in N$, i.e., ω^χ determines $\mathcal{H}^\chi$ by the GNS-construction. The Lorentz invariance of ω^χ on the 'algebra of weak observables' $\triangle(N, B)$ implies the

Result: $\mathcal{H}^\chi$ carries a canonical unitary representation U^χ of $\mathcal{P}$. U^χ coincides with the massless photon representation.

Calculation:
For all elements $\pi_F(W(\phi^\mu)) \in \pi_F(\triangle(N, B))$, $k_\mu\phi^\mu = 0$, the generating functional of the induced representation is Poincaré invariant. Especially, we have a Poincaré-invariant vacuum $[\Omega] \in \mathcal{H}^\chi$, and there exists a canonical unitary representation U^χ of $\mathcal{P}$ on $\mathcal{H}^\chi$, defined by

$$U^\chi(\Lambda, a)[\pi_F(W(\psi))\Omega] = [\pi_F(\alpha_{\Lambda,a}(W(\psi)))\Omega], \tag{3.2.17}$$

$\psi \in N$ which implements the automorphism (3.1.8) of $\triangle(N, B)$.
Since the irreducible unitary representations of the Poincaré group are classified completely by the eigenvalue of $P_\mu P^\mu$ and the action of the little group [25, 26], we can identify U^χ with the massless photon representation in the following way: calculating the Hamiltonian in (3.2.21), we see that the spectrum of the Hamiltonian has no mass gap and hence $P^\mu P_\mu = 0$ on one-particle states. The action of the corresponding little group $E(2)$ on $\mathcal{H}^\chi$ is then given in complete analogy with (2.3.29).

We can obtain U^χ explicitly by combining (3.2.17) and (2.3.29). For the one-particle state, this leads to

$$U^\chi(\Lambda, a)[P_T\psi] = \frac{d}{dr}U^\chi(\Lambda, a)[\Pi_F(W(-r\psi))\Omega]|_{r=0} = [P_T\gamma_{\Lambda,a}(\psi)],$$

where $\psi \in N$. From inspection of (2.3.29), we see that this expression can easily be extended to arbitrary n-particle states.[8]

Gauge transformations and regularity of states

In the T-procedure of Grundling and Hurst [6] as well as in the work of Narnhofer and Thirring [12], one starts with gauge-invariant 'Dirac' states ω_D over the unconstrained field algebra, for which

$$\omega_D(W(\partial_\mu g)) = 1 \qquad \text{for all} \qquad \partial_\mu g \in T.$$

As a consequence, one finds that such states must be non-regular, cf. [11].

In contrast to the T-procedure, the Rieffel induction procedure starts with a representation of the unconstrained field algebra on the Fock space $\mathcal{H}$, i.e., the states considered are regular and the vector potentials $\hat{A}_\mu$ exist as operator-valued distributions, $\Pi_F(W(D * f)) = e^{i\hat{A}(f)}$, $f \in \mathcal{S}(\mathbb{R}^4) \otimes \mathbb{R}^4$. For $f_\mu = \partial_\mu g$, we know from (3.2.4) that

$$(\Pi_F(W(\partial_\mu g))e^\psi, e^\varphi)_0 = (e^\psi, e^\varphi)_0, \tag{3.2.18}$$

which according to the above is equivalent to the gauge condition

$$(\partial_\mu \hat{A}^\mu(x)\Psi, \Phi)_0 = 0 \tag{3.2.19}$$

for all $\Psi, \Phi \in E$, and, by extension, for all elements in $\mathcal{D} \cup E$. On the other hand, we have for all $\Psi \in \mathcal{D} \cup E$,

$$\Pi_F(W(\partial_\mu g))\Psi = \Psi + \vec{n}, \tag{3.2.20}$$

$\vec{n} \in \mathcal{H}_N$, which is as good as having Dirac states. Hence, while $(\cdot\Psi, \Psi)$ are not gauge invariant but regular, the physically relevant states $(\cdot\Psi, \Psi)_0$ are gauge-invariant but defined on $\triangle(N, B)$ only.[9] We summarize this discussion by the following

[8] Here, N/T is isomorphic to S, but only N/T carries a representation of $\mathcal{P}$. Hence, $\mathcal{S}(P_T(L^2(\mathbb{R}^4) \otimes \mathbb{C}^4))$ cannot be identified naturally with $\mathcal{H}^\chi$.

[9] Here, the essential point is that for a state ω with $\omega(A) = (\pi_F(A)\Psi, \Psi)_0$, we require positivity, $\omega(A^*A) \geq 0$, which is guaranteed if $(.,.)_0$ preserves the adjoint, $(\pi_F(A^*)\pi_F(A)\Psi, \Psi)_0 = (\pi_F(A)\Psi, \pi_F(A)\Psi)_0$. As we shall see in what follows, the latter only holds on $\triangle(N, B)$.

Result: The vectors $\Psi \in \mathcal{D} \cup E$ define regular gauge-invariant states $(\cdot\Psi, \Psi)_0$ on $\triangle(N, B)$. Yet, no vector Ψ is gauge-invariant. Gauge transformations map vectors Ψ into themselves, plus a vector in the null space, cf. (3.2.20).

Remark:

The Dirac states investigated by Grundling and Hurst are regular on the subalgebra $\triangle(N, B)$. In this sense, Rieffel induction provides a construction of a particular class of Dirac states.

The Hamiltonian on $\Pi_F(\triangle(N, B))$

We establish a Gupta-Bleuler type expression for the Hamiltonian.

Result: On the Hilbert space $\mathcal{H}$, which carries the Fock representation of $\Pi_F(\triangle(N, B))$, the time evolution τ_t is implemented by the Hamiltonian

$$H_F = -\int d\tilde{k}\sqrt{\mathbf{k}^2}\hat{a}^*_\mu(\mathbf{k})g^{\mu\nu}\hat{a}_\nu(\mathbf{k}). \tag{3.2.21}$$

On the quotient space $\mathcal{H}^\chi$, the corresponding Hamiltonian satisfies the positive spectrum condition.

Calculation:

The time evolution on $\triangle(M, B)$ is given by the automorphism group τ_t,

$$\tau_t(W(\phi_\mu)) = W(e^{it\sqrt{D}}\phi_\mu),$$

$$(D\phi)_\mu = (-\triangle\phi_0, -\triangle\phi_1, -\triangle\phi_2, -\triangle\phi_3).$$

The Hamiltonian H_F is a representation-dependent operator, defined up to a constant (since Π_F is irreducible) by

$$e^{itH_F}\Pi_F(W(\phi_\mu))e^{-itH_F} = \Pi_F(W(e^{it\sqrt{D}}\phi_\mu)).$$

Comparing with the explicit form of the representation

$$\Pi_F(W(e^{it\sqrt{D}}\phi_\mu)) = e^{[e^{-it\sqrt{D}}\hat{a}_0(\overline{\phi}_0)^* - e^{it\sqrt{D}}\hat{a}_i(\phi_i)^* + e^{-it\sqrt{D}}\hat{a}_i(\phi_i) - e^{it\sqrt{D}}\hat{a}_0(\overline{\phi}_0)]},$$

we obtain (3.2.21), where we have used $e^{i\alpha a^* a}e^{[z^* a - z a^*]}e^{-i\alpha a^* a} = e^{[e^{-i\alpha}z^* a - e^{i\alpha}z a^*]}$. H_F has positive eigenvalues for transversal states. From (3.2.14), we know that arbitrary n-particle states decompose into tensor products of purely transversal components up to vectors in the null space, and hence

$$(\Psi, H_F\Psi)_0 \geq 0 \tag{3.2.22}$$

for all $\Psi = \psi^{(1)}\otimes_s \dots \otimes_s \psi^{(n)}$. This shows that the positive semi-definiteness of $(.,.)_0$ implies that the Hamiltonian in the induced representation on $\mathcal{H}^\chi$ has positive spectrum.

3. The propagator for free QED on $S(L^2(\mathbb{R}^3)\otimes\mathbb{C}^4)$

In this section, we undertake the first step in testing the suitability of the Rieffel induction procedure for perturbation theory: we calculate the Feynman propagator.

Definition: The Feynman propagator $D_F(x,y)$ on $S(L^2(\mathbb{R}^3)\otimes\mathbb{C}^4)$ is the sum of two amplitudes

$$iD_F(x,y)_{\mu\nu} := \left(\hat{A}_\mu(y)\Omega,\hat{A}_\nu(x)\Omega\right)_0\vartheta(x_0-y_0)+\left(\hat{A}_\nu(x)\Omega,\hat{A}_\mu(y)\Omega\right)_0\vartheta(y_0-x_0), \quad (3.3.1)$$

where $\vartheta(x_0)=1$ for $x_0\geq 0$ and 0 otherwise.

Remark:

Π_F is a regular representation of $\triangle(N,B)$. Hence, the generator $i\hat{A}_\mu(f^\mu)$ obtained as the derivative of $\Pi_F(W(\lambda\phi^\mu))$ at $\lambda=0$ exists as an operator on $\mathcal{H}$ and $\hat{A}_\mu(f)\Omega = i\tilde{\phi}_\mu(\mathrm{k})$, $\phi_\mu\in L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$. Hence, $\hat{A}_\mu(x)$ makes sense as an operator-valued distribution.
In (3.2.11), we are given the $(.,.)_0$-product for objects like $(\hat{A}_\mu(f)\Omega,\hat{A}_\nu(f')\Omega)_0 = (\phi,\phi')_0$. To calculate (3.3.1), we are interested in the formal action of $\hat{A}_\mu(x)$ on Ω. Therefore, we choose formally the 'test function' $f^\mu_x(z)=\vec{e}^\mu\delta^{(4)}(z-x)$, $\phi^{f_x}_\mu(\mathrm{k})=-ie^{ikx}\epsilon(k_0)\delta(k^2)\vec{e}^\mu$, so that $\hat{A}_\nu(x)\Omega = \hat{A}_\mu(f^\mu_x\delta_{\mu\nu})\Omega = i\int d\tilde{k}(-i\widetilde{e^{ikx}\delta_{\mu\nu}})\hat{a}^*_\mu\Omega$. Taking this as a starting point, we obtain the following

Result: The Feynman propagator $D_F(x,y)_{\mu\nu}$ is of Coulomb-gauge type.

Calculation:

$$\begin{aligned}&(\hat{A}_\mu(y)\Omega,\hat{A}_\nu(x)\Omega)_0\theta(x_0-y_0)\\ &=\int\frac{d^3\mathrm{k}}{(2\pi)^3 2k_0}\frac{d^3\mathrm{k}'}{(2\pi)^3 2k'_0}(-i\widetilde{e^{ik'y}\delta_{\mu\mu'}})\overline{(-i\widetilde{e^{ikx}\delta_{\nu'\nu}})}\theta(x_0-y_0)(\hat{a}_{\mu'}(\mathrm{k})^*\Omega,\hat{a}_{\nu'}(\mathrm{k}')^*\Omega)_0. \quad (3.3.2)\end{aligned}$$

Using (3.2.11), we may write

$$\begin{aligned}(\hat{a}_\mu(\mathrm{k})^*\Omega,\hat{a}_\nu(\mathrm{k}')^*\Omega)_0 &= \int\frac{d^3\mathrm{p}}{(2\pi)^3 2p_0}(2\pi)^3 2p_0\delta^{(3)}(\mathrm{k}-\mathrm{p})\vec{e}_\mu{}^t\left(P_T(2\pi)^3 2p_0\delta^{(3)}(\mathrm{k}'-\mathrm{p})\right)\vec{e}_\nu\\ &=(2\pi)^3 2k'_0\delta^{(3)}(\mathrm{k}-\mathrm{k}')\delta_{i\mu}\left(\delta_{ij}-\frac{k_ik_j}{\mathrm{k}^2}\right)\delta_{j\nu}. \quad (3.3.3)\end{aligned}$$

Here, the 0-components do not contribute and with $\tilde{\phi}_i=\phi_i$, we obtain

$$(A_\mu(y)\Omega,A_\nu(x)\Omega)_0\theta(x_0-y_0)=\int\frac{d^3\mathrm{k}}{(2\pi)^3 2k_0}e^{-ik(x-y)}\theta(x_0-y_0)\delta_{i\mu}\left(\delta_{ij}-\frac{k_ik_j}{\mathrm{k}^2}\right)\delta_{j\nu}.$$

$$D_F(x,y)_{\mu\nu} = -i\int \frac{d^3\mathrm{k}}{(2\pi)^3 2k_0}[e^{-ik(x-y)}\theta(x_0-y_0) + e^{ik(x-y)}\theta(y_0-x_0)]\delta_{i\mu}\left(\delta_{ij} - \frac{k_i k_j}{\mathrm{k}^2}\right)\delta_{j\nu}$$
$$= \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2+i\epsilon}\delta_{i\mu}\left(\delta_{ij} - \frac{k_i k_j}{\mathrm{k}^2}\right)\delta_{j\nu}, \qquad (3.3.4)$$

which is a Coulomb-gauge type propagator.

Remarks:

- For a time-like unit vector $\eta_\mu = (1,0,0,0)$ and $\hat{k}^\mu = \frac{k^\mu - (k\eta)\eta^\mu}{\sqrt{(k\eta)^2 - k^2}}$,

$$\delta_{i\mu}\left(\delta_{ij} - \frac{k_i k_j}{\mathrm{k}^2}\right)\delta_{j\nu} = -g_{\nu\mu} + \eta_\nu\eta_\mu - \hat{k}_\nu\hat{k}_\mu.$$

  This is the form of the Feynman propagator D_F in the radiation gauge, cf. [58]. In the traditional setting, η_μ is time-like in the frame in which canonical quantization has been carried out, and the vectors η_μ, $\hat{k}^\mu$ together with the two polarization vectors form an orthonormal basis. The manifest Lorentz covariance of general S-matrix elements is then eventually established by current conservation. In the present setting, we start from the Poincaré-invariant functional $\omega^\backslash$ at the very beginning. A Lorentz frame is singled out by the definition of $(\psi,\varphi)_E$.

- The Feynman propagator D_F is defined as the probability amplitude for the creation of a photon at space-time point x and its reabsorption into the vacuum at y and at a later time $y_0 > x_0$. $(.,.)_0$ is antilinear in the second component and hence we write the created states in the first component and the annihilated states in the second component.

A major difference with respect to the usual setting in perturbation theory is that the $(.,.)_0$-product does not preserve the adjoint.

Result: $(.,.)_0$ on $\mathcal{D}$ preserves the adjoint for test functions in N only. In general,

$$(A_\mu(x)e^0, A_\nu(y)e^0)_0 \neq (e^0, A_\mu(x)A_\nu(y)e^0)_0$$

Calculation:

For arbitrary test functions, one finds

$$-(\Omega, iA(g)iA(f)\Omega)_0 = -(\Omega, \frac{d}{dr_1}\frac{d}{dr_2}\Pi_F(e^{iA(r_1 g)}e^{iA(r_2 f)})\Omega)_0|_{r_i=0} = T(\phi^g,\phi^f), \qquad (3.3.5)$$

where $\phi^g_\mu(\mathrm{k}) = -i\epsilon(k_0)\delta(k^2)g_\mu(\mathrm{k})$ and

$$T(\phi^g,\phi^f) = \frac{1}{2}(\phi^g,\phi^f)_E + \frac{1}{2}(\phi^f,\phi^g)_E + i\mathrm{Im}(\phi^g,\phi^f)_M$$

$$+\int d\tilde{k}\frac{k_0\phi_0^g(\mathrm{k})k_i\overline{\phi_i^f}(\mathrm{k})}{k_0^2}+\frac{k_0\phi_0^f(\mathrm{k})k_i\overline{\phi_i^g}(\mathrm{k})}{k_0^2}$$
$$=(\phi^g,P_T\phi^f)_E+\int d\tilde{k}\left(\frac{k^\mu\phi_\mu^g(\mathrm{k})k^\nu\overline{\phi_\nu^f}(\mathrm{k})}{k_0^2}-2i\mathrm{Im}[\frac{k^\mu\phi_\mu^g(\mathrm{k})k^0\overline{P_0\phi^f}(\mathrm{k})}{k_0^2}]\right). \quad (3.3.6)$$

On the other hand, one obtains

$$(A(g)\Omega,A(f)\Omega)_0=(\phi^g,P_T\phi^f)_E.$$

These expressions coincide for ϕ^g, $\phi^f\in N$, only.

4. n-point-functions for free QED on $S(L^2(\mathbb{R}^3)\otimes\mathbb{C}^4)$

$\Pi_F(W(\phi^\mu))$ is not unitary with respect to $(.,.)_0$ on L, as may be seen from

$$(\Pi_F(W(\phi^\mu))\Pi_F(W(\varphi))\Omega,\Pi_F(W(\psi))\Omega)_0$$
$$=e^{-[(\tilde{\varphi},\tilde{\phi})_E-(\tilde{\phi},\tilde{\varphi})_E]}e^{-[(\tilde{\psi},\tilde{\phi})_E-(\tilde{\phi},\tilde{\psi})_E]}(\Pi_F(W(\varphi))\Omega,\Pi_F(W(\phi^\mu))\Pi_F(W(\psi))\Omega)_0. \quad (3.4.1)$$

Here, we investigate how this non-unitarity manifests itself in the expressions of the n-point functions.

> **Result:** Factorization properties for n-point functions in free QED are derived. Wick's theorem is found to have 'unusual' consequences since the $(.,.)_0$ vacuum expectation values of normal ordered products do not vanish in general.

Calculation:

We start from the expression

$$(i)^nA(\psi^{(n)})...A(\psi^{(1)})\Omega=\frac{d}{dr_1}...\frac{d}{dr_n}\Pi_F(e^{iA(\sum_i r_i\psi^{(i)})})e^{i\sum_{l>m}\mathrm{Im}(\psi^{(l)},\psi^{(m)})_M r_l r_m}\Omega|_{r_i=0}$$
$$=\frac{d}{dr_1}...\frac{d}{dr_n}(e^{-\frac{1}{2}(\sum_i r_i\psi^{(i)},\sum_i r_i\psi^{(i)})_E}$$
$$\times e^{i\sum_{l>m}\mathrm{Im}(\psi^{(l)},\psi^{(m)})_M r_l r_m}e^{-\sum_i r_i\tilde{\psi}^{(i)}})|_{r_i=0}, \quad (3.4.2)$$

where, with a slight abuse of notation, we write $\psi=\phi^g$. Using the general formula for $(e^\psi,e^\varphi)_0$, this leads to

$$(-i)^n(\Omega,A(\psi^{(n)})...A(\psi^{(1)})\Omega)_0$$
$$=\frac{d}{dr_1}...\frac{d}{dr_n}(e^{-\frac{1}{2}(\sum_i r_i\psi^{(i)},\sum_i r_i\psi^{(i)})_E}$$

$$
\begin{aligned}
&\times e^{-i\sum_{l>m}\mathrm{Im}(\psi^{(l)},\psi^{(m)})_M r_l r_m} e^{-\int d\tilde{k}\,\frac{k_0(\sum_i r_i\overline{\tilde{\psi}_0^{(i)}})(\mathbf{k})k_j(\sum_i r_i\overline{\tilde{\psi}_j^{(i)}})(\mathbf{k})}{k_0^2}})\Big|_{r_i=0} \\
= \;& \frac{d}{dr_1}\cdots\frac{d}{dr_n}(e^{-\frac{1}{2}\sum_{l>m} r_l r_m[-\frac{1}{2}(\psi^{(l)},\psi^{(m)})_E-\frac{1}{2}(\psi^{(m)},\psi^{(l)})_E-i\mathrm{Im}(\psi^{(l)},\psi^{(m)})_M]} \\
&\times e^{-\frac{1}{2}\sum_{l>m} r_l r_m[\int d\tilde{k}\,\frac{k_0\psi_0^{(l)}(\mathbf{k})k_i\overline{\psi_i^{(m)}}(\mathbf{k})}{k_0^2}+\frac{k_0\psi_0^{(m)}(\mathbf{k})k_i\overline{\psi_i^{(l)}}(\mathbf{k})}{k_0^2}]})\Big|_{r_i=0} \\
= \;& \frac{d}{dr_1}\cdots\frac{d}{dr_n}\Big(e^{-\sum_{l>m} r_l r_m T(\psi^{(l)},\psi^{(m)})}\Big)\Big|_{r_i=0}.
\end{aligned}
\tag{3.4.3}
$$

More generally, one may consider $2n$-point functions of the form

$$
\begin{aligned}
&(A(\psi^{(m+1)})\ldots A(\psi^{(2n)})\Omega, A(\psi^{(m)})\ldots A(\psi^{(1)})\Omega)_0 \\
&= (-1)^{n-m}\frac{d}{dr_{m+1}}\cdots\frac{d}{dr_{2n}}\frac{d}{ds_1}\cdots\frac{d}{ds_m} e^{-\frac{1}{2}(\sum_i r_i\psi^{(i)},\sum_i r_i\psi^{(i)})_E} e^{i\sum_{l<m}\mathrm{Im}(\psi^{(l)},\psi^{(m)})_M r_l r_m} \\
&\quad\times e^{-\frac{1}{2}(\sum_i s_i\psi^{(i)},\sum_i s_i\psi^{(i)})_E} e^{-i\sum_{l>m}\mathrm{Im}(\psi^{(l)},\psi^{(m)})_M s_l s_m}(e^{-\sum_i r_i\tilde{\psi}^{(i)}}, e^{-\sum_i s_i\tilde{\psi}^{(i)}})_0\Big|_{r_i=s_j=0} \\
&= (-1)^{n-m}\frac{d}{dr_{m+1}}\cdots\frac{d}{dr_{2n}}\frac{d}{ds_1}\cdots\frac{d}{ds_m}(e^{-\sum_{l>m} r_l r_m T(\psi^{(l)},\psi^{(m)})} e^{-\sum_{l>m} s_l s_m T(\psi^{(l)},\psi^{(m)})} \\
&\quad\times e^{-\sum_{l,m} r_l s_m(\psi^{(l)},P_T\psi^{(m)})_E})\Big|_{r_i=s_j=0} \\
&= (-1)^{n-m}\sum_{s\in\mathcal{S}^{(n)}}\prod_{i=1}^{n} T_{n,m}(\psi^{(m_i)},\psi^{(\tilde{m}_i)}),
\end{aligned}
\tag{3.4.4}
$$

where $T_{n,m}(\psi^{(m_i)},\psi^{(\tilde{m}_i)}) = T(\psi^{(m_i)},\psi^{(\tilde{m}_i)})$ for $m_i,\tilde{m}_i > m$ or $m_i,\tilde{m}_i < m$ and $T_{n,m}(\psi^{(m_i)},\psi^{(\tilde{m}_i)}) = (\psi^{(m_i)},P_T\psi^{(\tilde{m}_i)})_E$ otherwise.

Remarks:

- The reason for this unusual factorization property may be seen in the vacuum expectation value $(\Omega,..\Omega)_0$, used in the calculation. While Wick's theorem still holds in our setting, the vacuum expectation value of normal ordered products is not necessarily zero any more, e.g., $(\Omega, a^*a^*\Omega)_0 \neq 0$. This causes the deviation from conventional results.

- For $\psi^{(i)} \in N$, the situation is much better since $T_{n,m}(\psi^{(m_i)},\psi^{(\tilde{m}_i)}) = (\psi^{(m_i)},P_T\psi^{(\tilde{m}_i)})_E$ for arbitrary $m_i,\tilde{m}_i$. Hence we obtain on this restricted test function space the typical factorization property of n-point functions in a non-interacting theory:

$$
\begin{aligned}
(\Omega, A(\psi^{(2n)})\ldots A(\psi^{(1)})\Omega)_0 &= (A(\psi^{(m+1)})\ldots A(\psi^{(2n)})\Omega, A(\psi^{(m)})\ldots A(\psi^{(1)})\Omega)_0 \\
&= \sum_{s\in\mathcal{S}^{(n)}}\prod_{i=1}^{n}(A(\psi^{(m_i)})\Omega, A(\psi^{(\tilde{m}_i)})\Omega)_0.
\end{aligned}
\tag{3.4.5}
$$

Consequently, the structure of time-ordered n-point functions, valid for smearing with test functions in N only, reads

$$\begin{aligned}(\Omega, \mathcal{T} A(x^{(2n)})...A(x^{(1)})\Omega)_0 &= \sum_{s\in\mathcal{S}^{(n)}} \prod_{i=1}^{n} (\Omega, \mathcal{T} A(x^{(m_i)}) A(x^{(\tilde{m}_i)})\Omega)_0 \\ &= (i)^n \sum_{s\in\mathcal{S}^{(n)}} \prod_{i=1}^{n} D_F(x^{(m_i)}, x^{(\tilde{m}_i)}), \end{aligned} \tag{3.4.6}$$

where $\mathcal{T}$ denotes the time-ordering operator.

Consequences for perturbation theory

In this subsection, we make a short remark on the difficulties encountered in setting up a perturbation theory with the Rieffel induction procedure. To be concrete, let us consider scalar QED, given by the lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - ieA^\mu(\partial_\mu \phi^\dagger \phi - \phi^\dagger \partial_\mu \phi) + e^2 A^2 \phi^\dagger \phi. \tag{3.4.7}$$

This theory of a radiation field A_μ interacting with a complex scalar field ϕ is invariant under the gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x) \qquad ; \qquad \phi(x) \to e^{ie\lambda(x)} \phi(x). \tag{3.4.8}$$

Perturbation theory aims at calculating the time-ordered vacuum expectation values

$$\begin{aligned}&G(x_1, ..., x_n; x_{n+1}, ..., x_{2n}; y_1, ..., y_p) \\ &= \langle \Omega \mid \mathcal{T} \phi(x_1)...\phi(x_n)\phi^\dagger(x_{n+1})...\phi^\dagger(x_{2n}) A(y_1)...A(y_p)\Omega\rangle,\end{aligned}$$

by a perturbative expansion of

$$\begin{aligned}&G(x_1, ..., x_n; x_{n+1}, ..., x_{2n}; y_1, ..., y_p) \\ &= \frac{\langle \Omega \mid \mathcal{T} \phi_{IN}(x_1)...\phi^\dagger_{IN}(x_{2n}) A_{IN}(y_1)...A_{IN}(y_p) e^{-i\int d^4x \mathcal{H}_{int}[A^\mu_{IN}(x), \phi_{IN}(x), \phi^\dagger_{IN}(x)]}\Omega\rangle}{\langle \Omega \mid \mathcal{T} e^{-i\int d^4x \mathcal{H}_{int}[A^\mu_{IN}(x), \phi_{IN}(x), \phi^\dagger_{IN}(x)]}\Omega\rangle},\end{aligned}$$

where $\mathcal{H}_{int}$ denotes the interaction hamiltonian.

Here, the free IN-fields A^μ_{IN}, ϕ_{IN} and $\phi^\dagger_{IN}$ satisfy canonical commutation relations and we may specify the corresponding Weyl algebra $\mathcal{A}_{scalar}$. In practice, the calculation of $G(x_1, ..., y_p)$ amounts to a perturbative expansion around $e = 0$. This implies that in the perturbative setting, Rieffel induction uses a gauge group, defined by the transformations $A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda(x)$, $\phi(x) \to \phi(x)$, and obtained from (3.4.8) for $e = 0$.

From this, it is clear that the Rieffel induction procedure reduces essentially to the one carried out for free QED. Especially, one will encounter to zeroth order in e the same unusual factorization properties of n-point functions as for free QED.

Let us now think in terms of Feynman diagrams.

For the external lines of the Feynman diagrams of this theory, we may argue that smearing with test functions $\psi^{(i)} \in N$ is sufficient to calculate physical amplitudes. The point is that the vector potential used in the calculation of scattering amplitudes reads

$$A_{(\lambda)}(x) = \int \frac{d^3\mathrm{k}}{(2\pi)^3 2k_0} \epsilon^{\mu}_{(\lambda)}(\mathrm{k})[e^{-ikx} a_\mu(\mathrm{k}) + e^{ikx} a^*_\mu(\mathrm{k})], \tag{3.4.9}$$

where $\epsilon_{(\lambda)}(\mathrm{k})$ denotes a polarization operator satisfying $k_\mu \epsilon^{\mu}_{(\lambda)}(\mathrm{k}) = 0$ and λ labels the two transversal components corresponding to physical photons. Clearly, every smearing of a polynomial in $A_{(\lambda)}(x)$ with arbitrary test functions in $L^2(\mathbb{R}^3)$ amounts to a smearing of a polynomial in $A_\mu(x)$ with elements f such that their convolutions $D * f$ lie in N. For internal lines, however, we do not have a physical reason why our results should be stable under smearing with a subclass of test-functions only.

These simple considerations should make it clear, that compared with conventional approaches, imposing constraints via a Rieffel induction procedure will make a perturbative treatment more complicated. Hence, we have not tried to develop such a perturbation theory.

Remark:

In general, after having imposed constraints on the Weyl algebra of free QED, one still has the freedom of performing gauge transformations [21]. Here however, the Rieffel induction procedure employed has led to a gauge fixing, as may be seen from the Coulomb-type expression obtained for the propagator (3.3.1). This gauge-fixing does not coincide with any of the other gauges known for free QED [18]: it is a gauge leading to a Coulomb-type propagator, a Gupta-Bleuler type Hamiltonian and an 'unusual' factorization property of n-point functions. In what follows, we shall call this gauge the Landsman gauge.

CHAPTER IV:

RIEFFEL INDUCTION FOR FREE QED ON $S(L^2(\mathbb{R}^3) \otimes \mathbb{C}^3)$

In applications in which a fixed Lorentz frame is singled out, e.g. thermal field theory, one often takes as starting point a three-dimensional, real-valued vector potential $\mathbf{A} = (A_1, A_2, A_3)$ with canonical equal time commutation relations

$$[A_k(\mathrm{x}), E_l(\mathrm{y})]_{et} = i\delta_{kl}\delta^{(3)}(\mathrm{x} - \mathrm{y}). \tag{4.1}$$

Here, E_l denotes the electric field, the conjugate momentum of the vector potential A_l. In this chapter, the Rieffel induction procedure is carried out for the Weyl algebra corresponding to these canonical commutation relations. We briefly comment on Rieffel induction for functional representations and on an application to thermal field theory.

1. Weyl algebra and representations

In close analogy to the Weyl algebra formulation of (3.1), we consider the Weyl algebra $\triangle_{et}(M_{et}, B_{et})$ of equal time commutation relations with $M_{et} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^3$,

$$B_{et}(f, g) = 2\mathrm{Im}(f, g)_3, \tag{4.1.1}$$

$$(f, g)_3 = \int \frac{d^3\mathrm{k}}{(2\pi)^3} f_i(\mathrm{k})\delta_{ij}\overline{g}_j(\mathrm{k}). \tag{4.1.2}$$

For all $f, g \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^3$, the Weyl relations read

$$W(f)W(g) = e^{-i\mathrm{Im}(g,f)_3} W(f + g), \tag{4.1.3}$$

Remarks:

- The connection between (4.1) and (4.1.3) can be seen from the formal expression

$$W(f) = e^{iA(f^{(1)})+iE(f^{(2)})}, \tag{4.1.4}$$

where $f = f^{(1)} + if^{(2)} \in M_{et}$, $f_k^{(1)} \in L^2_{\mathbb{R}}(\mathbb{R}^3)$, the set of real-valued test functions and

$$A(f^{(1)}) = \int d^3\mathrm{x} A_k(\mathrm{x}) f_k^{(1)}(\mathrm{x}),$$
$$E(f^{(2)}) = \int d^3\mathrm{x} E_l(\mathrm{x}) f_l^{(2)}(\mathrm{x}).$$

- Again, it is useful to specify annihilation and creation operators a_k, a_k^* for the free electromagnetic field,

$$A_k(\mathrm{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}[a_k(\mathrm{k})e^{i\mathbf{kx}} + a_k(\mathrm{k})^* e^{-i\mathbf{kx}}],$$
$$E_k(\mathrm{x}) = i\int \frac{d^3\mathbf{k}}{(2\pi)^3} k_0[-a_k(\mathrm{k})e^{i\mathbf{kx}} + a_k(\mathrm{k})^* e^{-i\mathbf{kx}}]. \tag{4.1.5}$$

Using $f^{(i)}(\mathrm{k}) = \int d^3\mathrm{x} e^{-i\mathbf{kx}} f^{(i)}(\mathrm{x})$ with $f^{(i)}(\mathrm{k})^* = f^{(i)}(-\mathrm{k})$, we obtain

$$iA(f^{(1)}) + iE(f^{(2)}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} i[a_k(\mathrm{k})(f_k^{(1)}(\mathrm{k})^* - if_k^{(2)}(\mathrm{k})^*) + a_k(\mathrm{k})^*(f_k^{(1)}(\mathrm{k}) + if_k^{(2)}(\mathrm{k}))]$$
$$= ia(F) + ia(F)^* = a(-iF) + a(iF)^*, \tag{4.1.6}$$

where $F_k(\mathrm{k}) = f_k^{(1)}(\mathrm{k}) + if_k^{(2)}(\mathrm{k})$ and

$$[a(F), a(G)^*] = (G, F)_3. \tag{4.1.7}$$

Constraints

To select a gauge group, we consider the Maxwell equations for $A_0 = 0$,

$$\partial_0 \nabla_j A_j = 0,$$
$$\Box A_i + \partial_i \nabla_j A_j = 0. \tag{4.1.8}$$

These equations determine the wave equation for the transverse components of $\mathbf{A}$ and the Gauss law constraint

$$\nabla_i E_i = 0. \tag{4.1.9}$$

This motivates the choice of $M_{et}^T \subset M_{et}$ as a gauge group,

$$M_{et}^T = \{f_k = f_k^{(1)} + if_k^{(2)} \in M_{et} \mid f_k^{(2)} = \partial_k g; g \in L^2_{\mathbb{R}}(\mathbb{R}^3)\}. \tag{4.1.10}$$

Remark:

- $\triangle_{et}(M_{et}^T, B_{et})$ generates gauge transformations consistent with the classical ones $A_i \to A_i + \nabla_i g$, $E_i \to E_i$:

$$W(i\nabla g)W(f)W(i\nabla g)^* = e^{i(f^{(1)},\nabla g)_{et}}W(f) = e^{i(A+\nabla g)(f^{(1)})+iE(f^{(2)})}. \tag{4.1.11}$$

- The corresponding algebra of weak observables $\mathcal{A} = \triangle_{et}(M_{et}, B_{et}) \cap (\triangle_{et}(M_{et}^T, B_{et}))'$ is $\mathcal{A} = \triangle_{et}(M_{et}^W, B_{et})$,

$$M_{et}^W = \{f_k = f_k^{(1)} + if_k^{(2)} \in M_{et} \mid \partial_k f_k^{(1)} = 0\}. \tag{4.1.12}$$

Representations of $\triangle_{et}(M_{et}^T, B_{et})$

We specify two representations of $\triangle_{et}(M_{et}, B_{et})$:

1. The Fermi Fock representation Π_3 of $\triangle_{et}(M_{et}, B_{et})$ on $S(L^2(\mathbb{R}^3) \otimes \mathbb{C}^3)$ is given on a dense subspace $L_3 \subset S(L^2(\mathbb{R}^3) \otimes \mathbb{C}^3)$ which is the span of the total subset of exponential vectors

$$L_3 = \{\sum_{i=1}^{N} \lambda_i e^{\psi^{(i)}} \mid \lambda_i \in \mathbb{C}, \psi^{(i)} \in \mathcal{H}, N < \infty\};$$

$$e^{\psi} = 1 \oplus \psi \oplus \frac{1}{\sqrt{2}}\psi \otimes \psi \oplus \frac{1}{\sqrt{3!}}\psi \otimes \psi \otimes \psi \oplus \ldots \tag{4.1.13}$$

with scalar product

$$(e^{\psi}, e^{\varphi}) = e^{(\psi,\varphi)_3}. \tag{4.1.14}$$

The representation is defined by

$$\Pi_3(W(F))e^{\psi} = e^{\frac{-1}{2}(F,F)_3 + (\psi,F)_3} e^{(\psi - F)} \tag{4.1.15}$$

and can be extended to all of $\mathcal{H}$.

2. A functional representation Π_{L^2} of the algebra of canonical equal time commutation relations $\triangle_{et}(M_{et}, B_{et})$ on $L^2(L^2_{\mathbb{R}}(\mathbb{R}^3) \otimes \mathbb{R}^3, \mu)$, defined by

$$(\Pi_{L^2}(W(\phi_1 + i\phi_2))\psi)(A) = e^{\frac{-1}{2}(\phi_1,\phi_2)_R + i(A,\phi_2)_R}\psi(A - \phi_1), \tag{4.1.16}$$

$$(\psi, \varphi)_{L^2} = \int [\mathcal{D}A] e^{-(A,A)_R}\psi(A)\overline{\varphi(A)}, \tag{4.1.17}$$

$$(\phi_1, \phi_2)_R = \int \frac{d^3\mathrm{k}}{(2\pi)^3}\phi_1(\mathrm{k})\overline{\phi_2}(\mathrm{k}). \tag{4.1.18}$$

Here $\overline{\phi_i}(\mathrm{k}) = \phi_i(-\mathrm{k})$ which implies $(\phi_1, \phi_2)_R = (\phi_2, \phi_1)_R$ and $\Psi(A) \in L^2(L^2_{\mathbb{R}}(\mathbb{R}^3) \otimes \mathbb{R}^3, \mu)$,where $A_i(\mathrm{k}) = \overline{A}_i(-\mathrm{k}) \in L^2(L^2_{\mathbb{R}}(\mathbb{R}^3) \otimes \mathbb{R}^3, \mu)$.

Remarks:

- (4.1.15) is related to the annihilation and creation operators on $S(L^2(\mathbb{R}^3)\otimes\mathbb{C}^3)$ via

$$\Pi_3(W(F)) = e^{i[\tilde{a}(F)^* + \tilde{a}(F)]}, \tag{4.1.19}$$

  where the action of $\tilde{a}_k$, $\tilde{a}_k^*$ reads (symmetrization is understood):

$$\begin{aligned}\tilde{a}_k(F_k)\psi_1 \otimes \ldots \otimes \psi_n &= \sqrt{n}(\psi_1, F)_3\psi_2 \otimes \ldots \otimes \psi_n;\\ \tilde{a}_k(F_k)^*\psi_1 \otimes \ldots \otimes \psi_n &= \sqrt{n+1}F \otimes \psi_1 \otimes \ldots \otimes \psi_n.\end{aligned} \tag{4.1.20}$$

- In the section on functional representations, we consider a non-dense subspace

$$L_w = \{\psi \in L^2(L^2(\mathbb{R}^3)\otimes\mathbb{R}^3, \mu) \mid \Psi(A) = (\psi, A)_R\}, \tag{4.1.21}$$

  as well as the closely related set of exponentiated functions (4.4.1). Remark that $(\Psi, A)_R \neq (A, \Psi)_R$, since ψ_i takes values in $\mathbb{C}^3$.

2. Rieffel induction on $\mathcal{H} = S(L^2(\mathbb{R}^3)\otimes\mathbb{C}^3)$

Again, we start by specifying input data which result from a slight modification of the Rieffel induction procedure (cf. our discussion in the last chapter):

1. constraint algebra: We do not define a constraint algebra (again, we can not rely on the existence of $\mathcal{B} = C^*(M_{et}^T)$, since M_{et}^T is not locally compact). It is sufficient to have a continuous representation of the gauge group M_{et}^T on the Hilbert space $\mathcal{H}$.
2. field algebra of weak observables: $\mathcal{A} = \triangle_{et}(M_{et}^W, B_{et})$
3. Hilbert space $\mathcal{H}_\chi$ with scalar product $(.,.)_\chi$, used to induce from: $\mathcal{H}_\chi = \mathbb{C}$, carrying the trivial representation π_χ of M_{et}^T.
4. Hilbert space $\mathcal{H}$ with scalar product $(.,.)$: $\mathcal{H} = S(L^2(\mathbb{R}^3)\otimes\mathbb{C}^3)$, carrying two representations:
   (a) the left representation of $\mathcal{A}$, defined by the Fermi Fock representation Π_3.
   (b) the right representation of M_{et}^T, defined by $U(\nabla g) = \Pi_3(W(\nabla g))$.
5. rigging map on $L_3 \times L_3$: for all $f_k = i\partial_k g \in M_{et}^T$, $\psi, \varphi \in L^2(\mathbb{R}^3)\otimes\mathbb{C}^3$, the 'rigging map' reads

$$\langle e^\psi, e^\varphi\rangle_{\mathcal{B}}(f_k) = (\Pi_3(W(f_k))e^\psi, e^\varphi). \tag{4.2.1}$$

Calculation of $(.,.)_0$

We introduce the following

Notation:

On the one-particle space $L^2(\mathbb{R}^3) \otimes \mathbb{C}^3)$, polarization operators $\tilde{P}_T$, $\tilde{P}_L$ are defined as follows:

$$(\tilde{P}_T\psi)_i(\mathrm{k}) := \left(\delta_{ij} - \frac{k_i k_j}{\mathrm{k}^2}\right)\psi_j(\mathrm{k}),$$
$$(\tilde{P}_L\psi)_i(\mathrm{k}) := \left(\frac{k_i k_j}{\mathrm{k}^2}\right)\psi_j(\mathrm{k}). \tag{4.2.2}$$

Result: On elementary vectors in L_3, we obtain

$$(e^\psi, e^\varphi)_0 = e^{\int \frac{d^3\mathbf{k}}{(2\pi)^3}[(\tilde{P}_T\psi)_i(\mathbf{k})(\tilde{P}_T\overline{\varphi})_i(\mathbf{k}) - \frac{1}{2\mathbf{k}^2}k_i(\tilde{P}_L\psi)_i(\mathbf{k})k_j(\tilde{P}_L\psi)_j(-\mathbf{k}) - \frac{1}{2\mathbf{k}^2}k_i(\tilde{P}_L\overline{\varphi})_i(\mathbf{k})k_j(\tilde{P}_L\overline{\varphi})_j(-\mathbf{k})]}. \tag{4.2.3}$$

Calculation:

The $(.,.)_0$-product reads

$$\begin{aligned}
(e^\psi, e^\varphi)_0 &= \int_{M_{et}^T}[\mathcal{D}g](\Pi_3(W(i\partial_k g))e^\psi, e^\varphi) \\
&= \int_{M_{et}^T}[\mathcal{D}g]e^{\frac{-1}{2}(\nabla g,\nabla g)_3}e^{(\psi,i\nabla g)_3}e^{-(i\nabla g,\varphi)_3}e^{(\psi,\varphi)_3} \\
&= \int_{M_{et}^T}[\mathcal{D}g]e^{\int d^3x[\frac{-1}{2}g(x)(-\triangle)g(x)+ig(x)(\nabla_i\psi_i(x)+\nabla_i\overline{\varphi}_i(x))]}e^{(\psi,\varphi)_3} \\
&= e^{\int d^3x[\frac{-1}{\frac{1}{2}4(-\triangle)}(\nabla_i\psi_i(x)+\nabla_i\overline{\varphi}_i(x))^2]}e^{(\psi,\varphi)_3}.
\end{aligned} \tag{4.2.4}$$

Here, the integration over M_{et}^T can be defined in the sense of (3.2.5) as the limit of integrations over a family of finite dimensional locally compact spaces which eventually exhaust M_{et}^T. Substituting the Fourier transforms, $\psi_i(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}e^{ikx}\psi_i(\mathrm{k})$, we obtain

$$(e^\psi, e^\varphi)_0 = e^{\int \frac{d^3\mathbf{k}}{(2\pi)^3}\psi_i(\mathbf{k})(\delta_{ij}-\frac{k_i k_j}{\mathbf{k}^2})\overline{\varphi}_j(\mathbf{k})}e^{-\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\mathbf{k}^2}[k_i\psi_i(\mathbf{k})k_j\psi_j(-\mathbf{k})+k_j\overline{\varphi}_j(\mathbf{k})k_i\overline{\varphi}_i(-\mathbf{k})]}.$$

Remark:

(4.2.3) allows for the same combinatorial treatment as (3.2.8) and (2.3.1). In what follows, indentify (cf. (2.3.1)) x_0 with $\frac{1}{\sqrt{2}}k_i(\tilde{P}_L\psi)_i(\mathrm{k})$, x_3 with $\frac{1}{\sqrt{2}}k_i(\tilde{P}_L\psi)_i(-\mathrm{k})$, $(x_1, x_2, 0, 0)$ with $(\tilde{P}_T\psi)_i$ and y_i

with the corresponding expressions for φ_i. Again, we start from n-particle states, obtained in our calculation as derivatives

$$\psi^{(1)}\otimes_s\psi^{(2)}\otimes_s\ldots\otimes_s\psi^{(n)}=\frac{1}{\sqrt{n!}}\frac{d}{dr_1}\cdots\frac{d}{dr_n}e^{\sum r_i\psi^{(i)}}|_{r_i=0}.$$

Definition of $\mathcal{D}$: $\mathcal{D}$ denotes the set of finite linear combinations of n-particle states $\psi^{(1)}\otimes_s\psi^{(2)}\otimes_s\ldots\otimes_s\psi^{(n)}$, $\psi^{(i)}\in L^2(\mathbb{R}^3)\otimes\mathbb{C}^3$.

The $(.,.)_0$-product on $\mathcal{D}$ reads:

$$(\psi^{(1)}\otimes_s\ldots\otimes_s\psi^{(n)},\varphi^{(1)}\otimes_s\ldots\otimes_s\varphi^{(m)})_0=\frac{1}{\sqrt{n!m!}}\sum_{q,q'=0}\sum_{\mathcal{P}_{I_{n,q}},\mathcal{P}'_{I_{m,q'}}}$$
$$\left(\sum_{s_q^p\in\mathcal{S}_q^p}\prod_{i=1}^{q}\int d\tilde{k}^{(i)}(k_j^{(i)}\tilde{P}_L\psi_j^{(m_i^q)}(\mathbf{k}^{(i)}))(k_j^{(i)}\tilde{P}_L\psi_j^{(\tilde{m}_i^q)}(-\mathbf{k}^{(i)}))\right)$$
$$\times\left(\sum_{s_{q'}^{p'}\in\mathcal{S}_{q'}^{p'}}\prod_{i=1}^{q'}\int d\tilde{k}^{(i)}(k_j^{(i)}\tilde{P}_L\overline{\varphi}_j^{(m_i^{q'})}(\mathbf{k}^{(i)}))(k_j^{(i)}\tilde{P}_L\overline{\varphi}_j^{(\tilde{m}_i^{q'})}(-\mathbf{k}^{(i)}))\right)$$
$$\times\left(\sum_{p\in\mathcal{P}(I^{p_q}_{m,q'})}\prod_{i=1}^{n-2q}\int d\tilde{k}^{(i)}\tilde{P}_T\psi_j^{(t_i^{p_q})}(\mathbf{k}^{(i)})\overline{\varphi}_j^{(p(t_i^{p'_{q'}}))}(\mathbf{k}^{(i)})\right)\delta_{n-2q,m-2q'}. \quad (4.2.5)$$

Remark:

- In the derivation of (4.2.5), we have used that

$$\int d\tilde{k}^{(i)}\frac{1}{2}(k_j^{(i)}\tilde{P}_L\psi_j^{(m_i^q)}(\mathbf{k}^{(i)}))(k_j^{(i)}\tilde{P}_L\psi_j^{(\tilde{m}_i^q)}(-\mathbf{k}^{(i)}))$$
$$+\int d\tilde{k}^{(i)}\frac{1}{2}(k_j^{(i)}\tilde{P}_L\psi_j^{(\tilde{m}_i^q)}(\mathbf{k}^{(i)}))(k_j^{(i)}\tilde{P}_L\psi_j^{(m_i^q)}(-\mathbf{k}^{(i)}))$$
$$=\int d\tilde{k}^{(i)}(k_j^{(i)}\tilde{P}_L\psi_j^{(m_i^q)}(\mathbf{k}^{(i)}))(k_j^{(i)}\tilde{P}_L\psi_j^{(\tilde{m}_i^q)}(-\mathbf{k}^{(i)})). \quad (4.2.6)$$

In close analogy to our discussion of free QED on $S(L^2(\mathbb{R}^3)\otimes\mathbb{C}^4)$, we exploit (4.2.5). Without repeating the calculations, we give the following

Results:

- decomposition of n-particle states into transversal components:

$$\psi^{(1)}\otimes_s\ldots\otimes_s\psi^{(n)}=\sum_{q=0}\sum_{\mathcal{P}_{I_{n,q}}}\lambda_{p_q(I_{n,q})}(P_T\psi^{()})\otimes_s\ldots\otimes_s(P_T\psi^{(n-2q)})|_{p_q(I_{n,q})}+\vec{n},$$
$$\lambda_{p_q(I_{n,q})}=\sqrt{\frac{(2q)!(n-2q)!}{n!}}(\psi^{()}\otimes_s\ldots\otimes_s\psi^{(2q)}|_{p_q(I_{n,q})},\Omega)_0. \quad (4.2.7)$$

- induced physical Hilbert space $\mathcal{H}^\chi = S(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)$:
  This is obtained from the mapping V with $\ker V$ as $(.,.)_0$-null space

$$V : S(L^2(\mathbb{R}) \otimes \mathbb{C}^3) \longrightarrow \mathcal{H}^\chi = \overline{\mathcal{D} \cup L_3}/\mathcal{H}_N \simeq S(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2). \qquad (4.2.8)$$

- gauge transformations:
  The gauge invariance of the $(.,.)_0$-product can be checked immediately:

$$(\Pi_3(W(i\nabla_i g))e^{\psi}, e^{\varphi})_0 = (e^{\psi}, e^{\varphi})_0. \qquad (4.2.9)$$

The kernel of π^χ on $\mathcal{H}^\chi$ is the abelian subalgebra $\Delta_{et}(M_{et}^T, B_{et})$, since

$$\Pi_3(W(f))\Psi = \Psi + \vec{n}.$$

- time evolution automorphism and Hamiltonian:
  this will be discussed in the following application of our formalism to the discussion of Planck's law by Hertle and Honegger.

3. An application: thermal equilibrium states for free QED

In a first part of this section, we sketch a derivation of Planck's law, given by Hertle and Honegger. This derivation has the drawback that the 'unphysical longitudinal photons are thermalized' and hence one has to heuristically adjust the expression for Planck's energy density by a factor $\frac{2}{3}$. The aim in the second part of this section is to show how Rieffel induction allows one to do the same calculation on the physical Hilbert space $\mathcal{H}^\chi$, where this difficulty does not occur.

Planck's law: the result of Hertle and Honegger

Hertle and Honegger start from the quasi-local Weyl algebra $\Delta_{et}(M_{et}, B_{et}) = \overline{\bigcup_\Lambda \mathcal{W}(L^2(\Lambda))}$, where $L^2(\Lambda)$ denotes the square-integrable complex-valued functions with support in the open subset $\Lambda \subset \mathbb{R}^3$, $\mathcal{W}(L^2(\Lambda)) = \Delta_{et}(L^2(\Lambda), B_{et})$. The local time evolution is given by the automorphism group τ_t^Λ,

$$\tau_t^\Lambda(\check{W}(f)) = \check{W}(e^{it\sqrt{D_\Lambda}}f),$$

$$D_\Lambda f = (-\Delta_\Lambda f_1, -\Delta_\Lambda f_2, -\Delta_\Lambda f_3)$$

for $f_i \in L^2(\Lambda)$ in the domain of the Laplacian with Dirichlet boundary conditions $-\Delta_\Lambda$. For an orthonormal base $(e_n^\Lambda)_{n\in\mathbb{N}} \in L^2(\Lambda)$ of eigenvectors of $\sqrt{D_\Lambda}$ with eigenvalues ϵ_n^Λ, $\sqrt{D_\Lambda}e_n^\Lambda = \epsilon_n^\Lambda e_n^\Lambda$, we have the energy operator

$$d\Gamma(\sqrt{D_\Lambda}) = \sum_{n=0}^{\infty} \epsilon_n^\Lambda \tilde{a}^*(e_n^\Lambda)\tilde{a}(e_n^\Lambda)$$

and the local Gibbs equilibrium state for inverse temperature $\beta > 0$ on $\mathcal{W}(L^2(\Lambda))$ is defined by

$$\omega_\Lambda(A) = \frac{\mathrm{Tr}(e^{-\beta d\Gamma(\sqrt{D_\Lambda})}\Pi_3(A))}{\mathrm{Tr}(e^{-\beta d\Gamma(\sqrt{D_\Lambda})})}.$$

In this setting, Hertle and Honegger have shown that for very general regions Λ, the local states ω_Λ, extended to states on $\Delta_{et}(M_{et}, B_{et})$, converge in the weak* sense to a limiting Gibbs state ω. To obtain $\omega_\Lambda(A)$, they start from the generating functional $\omega_{HH}(W(f)) = e^{\frac{-1}{4}(f,f)_3}$ which implies[1] ([49], Prop 5.2.28)

$$\omega_{HH,\Lambda}(W(f)) = e^{\frac{-1}{4}(f,f)_3 - \frac{1}{2}t_\Lambda(f,f)},$$

$$t_\Lambda(f,g) = (f, \frac{e^{-\beta\sqrt{D_\Lambda}}}{1-e^{-\beta\sqrt{D_\Lambda}}}g)_3.$$

Especially, $t_\Lambda(f,g) = \omega_{HH,\Lambda}(\tilde{a}^*(g)\tilde{a}(f))$, and the expectation values $\omega_{HH,\Lambda}(d\Gamma(u(\sqrt{D_\Lambda})))$ for some class of functions u, reads

$$u(\sqrt{D_\Lambda}) = \sum_{n=0}^{\infty}\sum_{i=1}^{3} u(\epsilon_n^\Lambda)|\, e_{n,i}^\Lambda\rangle\langle e_{n,i}^\Lambda\,|,$$

$$\omega_{HH,\Lambda}(d\Gamma(u(\sqrt{D_\Lambda}))) = \sum_{n=0}^{\infty}\sum_{i=1}^{3}(e_{n,i}^\Lambda, u(\sqrt{D_\Lambda})\frac{e^{-\beta\sqrt{D_\Lambda}}}{1-e^{-\beta\sqrt{D_\Lambda}}}e_{n,i}^\Lambda)_3$$

$$= 3\mathrm{Tr}(P_\Lambda u(\sqrt{-\Delta_\Lambda})\frac{e^{-\beta\sqrt{\Delta_\Lambda}}}{1-e^{-\beta\sqrt{\Delta_\Lambda}}}P_\Lambda). \tag{4.3.1}$$

Remark:

In their work, Hertle and Honegger have taken the Tr in ω_Λ to be the trace on $S(L^2(\mathbb{R}^3)\otimes\mathbb{C}^3)$, thereby thermalizing the unphysical photons, too. This results in a factor 3 (instead of 2) in (4.3.1) and one is forced to adjust the result for the energy-density of the black body radiation by a factor $\frac{2}{3}$, stemming from the restriction to physical photons. Our formalism allows to replace this ad hoc procedure by calculating the generating functional ω^χ of the representation on the physical Hilbert space $\mathcal{H}^\chi$ given in (4.2.8).

[1] Hertle and Honegger start from the Weyl relations $W(f)W(g) = e^{-\frac{i}{2}\mathrm{Im}(f,g)}W(f+g)$, cf. [59], whose symplectic form differs by a factor 2 from (4.1.3).

Result: The generating functional ω^χ for the representation of $\Delta_{et}(M_{et}^W, B_{et})$ reads

$$\omega^\chi(f) := (\Pi_3(W(f))\Omega, \Omega)_0 = e^{-\frac{1}{2}(f,\tilde{P}_T f)_3} \tag{4.3.2}$$

for all $f \in M_{et}^W$.

Calculation:

Using (4.2.3) and (4.1.15), we obtain

$$\begin{aligned}
\omega^\chi(f) &= (\Pi_3(W(f))\Omega, \Omega)_0 \\
&= e^{-\frac{1}{2}(f,f)_3}(e^{-f}, \Omega)_0 \\
&= e^{-\frac{1}{2}(f,f)_3} e^{-\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}k_i f_i(k)k_j f_j(-k)} \\
&= e^{-\frac{1}{2}(f,f)_3} e^{-\frac{1}{2}\int \frac{d^3k}{(2\pi)^3} f_i(k)\frac{k_ik_j}{k^2}\overline{f_j(k)}} e^{-\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}k_i f_i(k)k_j[\overline{f_j(k)}+f_j(-k)]}.
\end{aligned} \tag{4.3.3}$$

Now, considering the complex conjugate of $f_j(k) = \int d^3x e^{-ikx}[f_j^{(1)}(x) + i f_j^{(2)}(x)]$, one finds that

$$\overline{f_j(k)} + f_j(-k) = 2\int d^3x e^{ikx} f_j^{(1)}(x).$$

This implies that the last term in (4.3.3) vanishes for $f \in M_{et}^W$, which leads to (4.3.2).

Consequences:

Since D_Λ is rotation invariant, we may specify an orthonormal base $(e_{n,i}^\Lambda)_{n\in\mathbb{N}} \in L^2(\Lambda)$, $i = 1, 2, 3$ of eigenvectors of $\sqrt{D_\Lambda}$ with eigenvalues ϵ_n^Λ, $\sqrt{D_\Lambda}e_{n,i}^\Lambda = \epsilon_n^\Lambda e_{n,i}^\Lambda$, $\tilde{P}_L e_{n,i}^\Lambda = \delta_{3i}e_{n,i}^\Lambda$, $\tilde{P}_T e_{n,i}^\Lambda = (\delta_{1i}+\delta_{2i})e_{n,i}^\Lambda$. (cf. (4.2.2) for the definition of these projection operators.)

The generating (vacuum) functional obtained by Rieffel induction in the setting of Hertle and Honegger reads (for the rescaling of the exponent of (4.3.3) by a factor 2, cf. the foonote given above),

$$\omega_{HH}^\chi(W(f)) = e^{\frac{-1}{4}(f,\tilde{P}_T f)_3}.$$

Following the arguments of Hertle and Honegger with this generating functional as starting point, we are led to

$$\omega_{HH,\Lambda}^\chi(W(f)) = e^{\frac{-1}{4}(f,\tilde{P}_T f)_3 - \frac{1}{2}t_\Lambda(f,\tilde{P}_T f)},$$

$$t_\Lambda(f,f) = (f, \frac{e^{-\beta\sqrt{D_\Lambda}}}{1-e^{-\beta\sqrt{D_\Lambda}}}\tilde{P}_T f)_3,$$

for all $f \in M_{et}^w$. Hence, we obtain

$$\omega_{HH,\Lambda}^\chi(d\Gamma(u(\sqrt{D_\Lambda}))) = \sum_{n=0}^{\infty}\sum_{i=1}^{3}(e_{n,i}^\Lambda, u(\sqrt{D_\Lambda})\frac{e^{-\beta\sqrt{D_\Lambda}}}{1-e^{-\beta\sqrt{D_\Lambda}}}\tilde{P}_T e_{n,i}^\Lambda)_3$$

$$= 2\mathrm{Tr}(P_\Lambda u(\sqrt{-\Delta_\Lambda})\frac{e^{-\beta\sqrt{\Delta_\Lambda}}}{1-e^{-\beta\sqrt{\Delta_\Lambda}}}P_\Lambda),$$

where the sum in the last equation receives no contribution from $i=3$ in contrast to (4.3.1). The methods of Hertle and Honegger can be taken over without alterations to discuss the weak* convergence of $\omega^\lambda_{HH,\Lambda}$ to a limiting Gibbs state. This leads to Planck's law with correct prefactor,

$$\tilde{E}(u) = \frac{1}{\pi^2}\int_0^\infty \frac{u(\kappa)\kappa^2}{e^{\beta\kappa}-1}d\kappa = \lim_{\Lambda\to\infty}\frac{1}{\Lambda}\omega^\chi_{HH,\Lambda}(d\Gamma(u(\sqrt{D_\Lambda}))),$$

where $u(\kappa)=\kappa\chi_I(\kappa)$, $I\subseteq[0,\infty[$ in the case of an expression for the energy density.

4. Rieffel induction for functional representations

Here, we would like to sketch what Rieffel Induction looks like with different input data. We start from the Hilbert space $\mathcal{H}=L^2(L^2(\mathbb{R}^3)\otimes\mathbb{R}^3,\mu)$ which carries a functional left representation of $\mathcal{A}$ and a right representation of M^T_{et}.

Rieffel induction on $L^2(L^2(\mathbb{R}^3)\otimes\mathbb{R}^3,\mu)$

We start from the input data $\mathcal{A}=\Delta_{et}(M_{et},B_{et})$, $\mathcal{H}=L^2(L^2(\mathbb{R}^3)\otimes\mathbb{R}^3,\mu)$, M^T_{et}, $\mathcal{H}_\chi=\mathbb{C}$ and the 'rigging map' $\langle.,.\rangle_{\mathcal{B}}$ on $L^{exp}_w\otimes L^{exp}_w$,

$$L^{exp}_w = \{\sum_{i=1}^N \lambda_i e^{\Psi^{(i)}} \mid \Psi^{(i)}\in L_w, \lambda_i\in\mathbb{C}, N<\infty\}, \tag{4.4.1}$$

defined by

$$\langle\Psi,\Upsilon\rangle_{\mathcal{B}}(i\partial_i g) = (\Pi_{L^2}(W(i\partial_i g))\Psi,\Upsilon)_{L^2} \tag{4.4.2}$$

for all $f_i=i\partial_i g\in M^T_{et}$; $\Psi,\Upsilon\in L^2(L^2(\mathbb{R}^3)\otimes\mathbb{R}^3,\mu)$, $g\in L^2_{\mathbb{R}}(\mathbb{R}^3)$. With the rigging map defined on $L^{exp}_w\otimes L^{exp}_w$, we obtain the following

Result:

$$(e^\Psi,e^\Upsilon)_0 = e^{\frac{1}{4}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\left(\psi_i(\mathbf{k})(\delta_{ij}-\frac{k_ik_j}{\mathbf{k}^2})\psi_j(-\mathbf{k})+\overline{\nu}_i(\mathbf{k})(\delta_{ij}-\frac{k_ik_j}{\mathbf{k}^2})\overline{\nu}_j(-\mathbf{k})+2\psi_i(\mathbf{k})(\delta_{ij}-\frac{k_ik_j}{\mathbf{k}^2})\overline{\nu}_j(\mathbf{k})\right)}. \tag{4.4.3}$$

Calculation:

$$(e^\Psi,e^\Upsilon)_0 = \int_{M^T_{et}}[\mathcal{D}g][\mathcal{D}A]e^{-(A,A)_R}e^{i(A,\nabla g)_R}e^{(\psi,A)_R}e^{(\overline{\Upsilon},A)_R}$$

$$
\begin{aligned}
&= \int_{M_{et}^T} [\mathcal{D}g][\mathcal{D}A] e^{-(A,A)_R} e^{A(\psi+i\nabla g+\overline{\nu})_R} \\
&= \int_{M_{et}^T} [\mathcal{D}g] e^{\frac{1}{4}(\psi+i\nabla g+\overline{\nu})^2} \\
&= \int_{M_{et}^T} [\mathcal{D}g] e^{-\frac{1}{4}(\nabla g,\nabla g)} e^{-\frac{i}{2}g(\nabla_i\psi_i+\nabla_i\overline{\nu}_i)} e^{\frac{1}{4}(\psi+\overline{\nu})^2} \\
&= e^{\int d^3x \frac{-1}{(-\Delta)}\frac{1}{4}(\nabla_i\psi_i+\nabla_i\overline{\nu}_i)^2+\frac{1}{4}(\psi+\overline{\nu})^2} \\
&= e^{\frac{1}{4}\int d^3x\left(\psi_i(x)\psi_i(x)-\frac{\nabla_i\psi_i(x)\nabla_j\psi_j(x)}{(-\Delta)}+\overline{\nu}_i(x)\overline{\nu}_i(x)-\frac{\nabla_i\overline{\nu}_i(x)\nabla_j\overline{\nu}_j(x)}{(-\Delta)}+2\psi_i(x)\overline{\nu}_i(x)-2\frac{\nabla_i\psi_i(x)\nabla_j\overline{\nu}_j(x)}{(-\Delta)}\right)}
\end{aligned}
\tag{4.4.4}
$$

which leads to (4.4.3).

Remark:

Longitudinal, real-valued components do not show up in the $(.,.)_0$ product as long as we restrict ourselves to $\Delta_{et}(M_{et}^W, B_{et})$. This may be seen from the first line of the calculation given above, since $\psi \to \psi + i\nabla\lambda$ can be absorbed by shifting $g \to g - \lambda$.

The one-particle space

To specify the one-particle space, let us calculate $(.,.)_0$ on L_w, defined in (4.1.21). There, the $(.,.)_0$-product reads

$$
\begin{aligned}
(\Psi,\Upsilon)_0 &= \int_{M_{et}^T} [\mathcal{D}g](\Pi_{L^2}(W(i\partial_i g))\Psi,\Upsilon)_{L^2} \\
&= \int_{M_{et}^T} [\mathcal{D}g][\mathcal{D}A] e^{-(A,A)_R} e^{i(A,\nabla g)_R}\Psi(A)\overline{\Upsilon(A)} \\
&= \int_{M_{et}^T} [\mathcal{D}g][\mathcal{D}A] e^{-(A-\frac{i}{2}\nabla g,A-\frac{i}{2}\nabla g)_R} e^{-\frac{1}{4}(\nabla g,\nabla g)_R}\Psi(A)\overline{\Upsilon(A)} \\
&= \int_{M_{et}^T} [\mathcal{D}g][\mathcal{D}A] e^{-(A,A)_R} e^{-\frac{1}{4}(g,-\Delta g)_R}[\Psi(A)+\frac{i}{2}\Psi(\nabla g)][\overline{\Upsilon(A)}+\frac{i}{2}\overline{\Upsilon(\nabla g)}] \\
&= \frac{1}{2}\int d^3x[\psi_i(x)\overline{\upsilon}_i(x) - \frac{\nabla_i\psi_i(x)\nabla_j\overline{\upsilon}_j(x)}{-\Delta}] \\
&= \frac{1}{2}\int \frac{d^3\mathrm{k}}{(2\pi)^3}\psi_i(\mathrm{k})\left(\delta_{ij} - \frac{k_ik_j}{\mathrm{k}^2}\right)\overline{\upsilon}_j(\mathrm{k}).
\end{aligned}
\tag{4.4.5}
$$

Remarks:

- The null space of L_w is $\mathcal{H}_N \cap L_w = \{\Psi \mid \psi = \check{P}_L\upsilon, \Upsilon \in L_w\}$. Accordingly, one can specify a physical space by $\mathcal{H}^\chi = L_w/\mathcal{H}_N \cap L_w \simeq \check{P}_T L_w$. This indicates that our choice of L_w has singled out (part of) the one-particle space.

- In principle, there is no obstruction to complete the Rieffel induction procedure for functional representations. We haven't worked out further details.

CHAPTER V:

SPONTANEOUS SYMMETRY BREAKING AND RIEFFEL INDUCTION

In this chapter, we apply the Rieffel induction procedure to a linear QFT showing SSB, viz. the Stückelberg-Kibble model. This model has often been used as testing ground for the investigation of the structures underlying the Higgs mechanism [47]. The motivation for what follows is two-fold. Firstly, one wants to extend the application of the Rieffel induction procedure to a theory with massive particles, thereby constructing the Rieffel-induced physical Hilbert space as a carrier space of the massive representations of the Poincaré group. Secondly, however, we have seen that the Rieffel induction procedure on the Fermi-Fock Hilbert space leads to a new gauge, the Landsman gauge, which is a particular cross-breed of the Coulomb and Gupta-Bleuler gauges. As mentioned in the Introduction, the Higgs mechanism is characterized differently in different gauges and hence, it is clearly of interest to observe its features in the Landsman gauge. Especially, we find that in the Landsman gauge, the 'rearrangement of would-be Goldstone bosons' can be exhibited in great clarity.

1. The Weyl algebra of the Stückelberg-Kibble model

The Stückelberg-Kibble model is an abelian Higgs model with the modulus η of the scalar field $\phi(x) = \eta(x)e^{\varphi(x)}$ frozen to unity: $\eta(x) = 1$. It is given by the lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(\partial_\mu\varphi + eA_\mu\right)\left(\partial^\mu\varphi + eA^\mu\right) + g.f. \tag{5.1.1}$$

Remark:

The equations of motion may be written as those of a free, massive, divergenceless vector field

$$\left(\Box + e^2\right) j^\mu = 0 \qquad ; \qquad \partial_\mu j^\mu = 0, \tag{5.1.2}$$

where

$$j^\mu = \partial^\mu\varphi + eA^\mu \tag{5.1.3}$$

is a gauge-invariant observable of the theory. In this form, the theory is not suitable for Rieffel induction since we have got rid of the gauge group already, and this gauge group plays an essential role in the construction of the Rieffel induced inner product. To carry out the Rieffel induction procedure, we have to choose equations of motion for the gauge-dependent fields A_μ and φ which are consistent with the dynamics given above but allow us to define field algebras with gauge-dependent elements. This amounts to a particular choice of the constraints.

Equations of Motion and gauge transformations

In what follows, we choose the 't Hooft constraint[1]

$$\partial_\mu A^\mu = e\varphi. \tag{5.1.4}$$

With this constraint, we may start from the following equations of motion, consistent with (5.1.2):

$$\left(\Box + e^2\right) A^\mu = 0 \qquad ; \qquad \left(\Box + e^2\right) \varphi = 0, \tag{5.1.5}$$

These equations of motion are invariant under the gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \lambda \qquad ; \qquad \varphi \to \varphi - e\lambda, \tag{5.1.6}$$

where λ has to satisfy

$$\left(\Box + e^2\right) \lambda = 0. \tag{5.1.7}$$

The Weyl algebra $\triangle_{sk}(M_{sk}, B_{sk})$

Here, we define the Weyl algebras corresponding to the space of classical solutions M_{sk} of (5.1.5). We start from the canonical commutation relations of the fields A_μ and φ:

$$\begin{aligned} [\varphi(x), \varphi(y)] &= i\triangle(x-y), \\ [A_\mu(x), A_\nu(y)] &= -ig_{\mu\nu}\triangle(x-y), \end{aligned} \tag{5.1.8}$$

where $(\Box + e^2)\triangle(x) = 0$ with initial conditions $\triangle(\mathrm{x}, 0) = 0$, $\frac{\partial}{\partial t}\triangle(\mathrm{x}, t)|_{t=0} = -\delta^{(3)}(\mathrm{x})$.
The Weyl algebra $\triangle_{sk}(M_{sk}, B_{sk})$ is defined by[2]

$$M_{sk} = \{(\phi_\mu, \phi) \mid \phi_\mu \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^4, \phi \in L^2(\mathbb{R}^3); \left(\Box + e^2\right)\phi_\mu = \left(\Box + e^2\right)\phi = 0\}, \tag{5.1.9}$$

[1] We call (5.1.4) the 't Hooft constraint since it resembles the gauge fixing condition used by t'Hooft in his proof of the renormalizability of gauge theories showing spontaneous symmetry breaking.

[2] Here, M_{sk} is the completion of the space of Cauchy data of the real solutions of the wave equation, cf. (3.1.3).

and the symplectic form

$$B_{sk}(\phi_\mu, \phi; \phi'_\mu, \phi') = 2\mathrm{Im}(\phi_\mu, \phi_\mu)_M - 2\mathrm{Im}(\phi, \phi), \tag{5.1.10}$$

and the Weyl form of the canonical commutation relations reads

$$W(\phi_\mu, \phi)W(\phi'_\mu, \phi') = W(\phi_\mu + \phi'_\mu, \phi + \phi')e^{-\frac{i}{2}B_{sk}(\phi_\mu,\phi;\phi'_\mu,\phi')} \tag{5.1.11}$$

Remark:

The formal connection between (5.1.8) and (5.1.11) may be seen by smearing the fields A_μ, φ with test functions f_μ, f in the corresponding Schwartz spaces, and considering the Weyl operators

$$W(\phi_\mu, \phi) = e^{iA_\mu(f^\mu)+i\varphi(f)},$$

where $\phi_\mu = \triangle * f_\mu$, $\phi = \triangle * f$.

Subalgebras of $\triangle_{sk}(M_{sk}, B_{sk})$

We are interested in Poincaré-invariant subalgebras of $\triangle_{sk}(M_{sk}, B_{sk})$. The automorphisms $\alpha_{\Lambda,a}$ implementing $\mathcal{P}$ on $\triangle_{sk}(M_{sk}, B_{sk})$ are induced by the symplectic transformations $\gamma_{\Lambda,a}$ on M_{sk},

$$\alpha_{\Lambda,a}(W(\phi_\mu, \phi)) = W(\gamma_{\Lambda,a}(\phi_\mu, \phi)) \qquad \text{with} \qquad (\gamma_{\Lambda,a}(\phi_\mu, \phi))(x) = (\Lambda^\nu_\mu \phi_\nu, \phi)(\Lambda^{-1}(x - a))). \tag{5.1.12}$$

We introduce the following subspaces of M_{sk}, invariant under $\gamma_{\Lambda,a}$:

$$\begin{aligned} N_{sk} &= \{(\phi_\mu, \phi) \mid \partial^\mu \phi_\mu = e\phi\} \subset M_{sk}, \\ T_{sk} &= \{(\phi_\mu, \phi) \mid \phi_\mu = \partial_\mu g, \phi = -eg; g \in L^2(\mathbb{R}^3); \left(\Box + e^2\right) g = 0\} \subset M_{sk}. \end{aligned} \tag{5.1.13}$$

Representations of $\triangle_{sk}(M_{sk}, B_{sk})$

We define a Fermi Fock representation Π_F of $\triangle_{sk}(M_{sk}, B_{sk})$ on a dense subset L of the Fock space

$$\mathcal{H} = S\left(L^2(\mathbb{R}^3) \otimes \mathbb{C}^4\right) \otimes S\left(L^2(\mathbb{R}^3)\right). \tag{5.1.14}$$

Here, L is defined by

$$\begin{aligned} L &= L_1 \otimes L_2 \\ L_1 &= \{\sum_i^N \lambda_i e^{\psi_\mu^{(i)}} \mid \psi_\mu^{(i)} \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^4; \lambda_i \in \mathbb{C}, N < \infty\}, \\ L_2 &= \{\sum_i^N \lambda_i e^{\psi^{(i)}} \mid \psi^{(i)} \in L^2(\mathbb{R}^3); \lambda_i \in \mathbb{C}, N < \infty\}. \end{aligned} \tag{5.1.15}$$

On elementary vectors in L, the scalar product reads

$$(e^{\psi_\mu} \otimes e^{\psi}, e^{\chi_\mu} \otimes e^{\chi}) = e^{(\psi_\mu,\chi_\mu)_E + (\psi,\chi)}$$

and the action of Π_F is defined by (cf. (3.1.14) for definition of $\tilde{\phi}_\mu$)

$$\Pi_F(W(\phi_\mu, \phi))e^{\psi_\mu} \otimes e^{\psi} = e^{\frac{-1}{2}(\phi_\mu,\phi_\mu)_E + (\psi_\mu,\tilde{\phi_\mu})_E} e^{\frac{-1}{2}(\phi,\phi)+(\psi,\phi)} e^{\psi_\mu - \tilde{\phi_\mu}} \otimes e^{\psi-\phi}. \tag{5.1.16}$$

Of course, these definitions can be extended to all of $\mathcal{H}$.

2. Rieffel induction for the Stückelberg-Kibble model

We start by specifying the following input data:

1. as constraint algebra: again, we do not specify a constraint algebra, but a continuous right representation of the gauge group T_{sk} on $\mathcal{H}$.
2. as field algebra of weak observables: $\mathcal{A} = \Delta_{sk}(N_{sk}, B_{sk})$
3. as Hilbert space $\mathcal{H}_\chi$ with scalar product $(.,.)_\chi$, used to induce from: $\mathcal{H}_\chi = \mathbb{C}$, carrying the trivial representation π_χ of T_{sk}.
4. Hilbert space $\mathcal{H}$ with scalar product $(.,.)$: $\mathcal{H} = S(L^2(\mathbb{R}^3) \otimes \mathbb{C}^4) \otimes S(L^2(\mathbb{R}^3))$, carrying two representations:
   (a) the left representation of $\mathcal{A}$, defined by the Fermi Fock representation Π_F.
   (b) the right representation of T_{sk}, defined by $U(t) = \Pi_F(W(t))$ for all $t \in T_{sk}$.
5. as 'rigging map' on $L \times L$: for all $(\partial_\mu g, -eg) \in T_{sk}$, the rigging map $\langle .,. \rangle_B$ on $L \otimes L$ reads

$$\langle e^{\psi_\mu} \otimes e^{\psi}, e^{\chi_\mu} \otimes e^{\chi} \rangle_B(\partial_\mu g, -eg) = (\pi_F(W(\partial_\mu g, -eg))e^{\psi_\mu} \otimes e^{\psi}, e^{\chi_\mu} \otimes e^{\chi}). \tag{5.2.1}$$

Calculation of $(.,.)_0$

In close analogy to the calculations in the previous chapter, we obtain the following

Result: On elementary vectors in L,[3]

$$(e^{\psi_\mu} \otimes e^{\psi}, e^{\chi_\mu} \otimes e^{\chi})_0 = e^{\int \tilde{dk} \frac{-1}{k_0^2}[(k_i\psi_i - ie\psi)k_0\psi_0 + (k_i\overline{\chi}_i + ie\overline{\chi})k_0\overline{\chi}_0]}$$
$$\times e^{\int \tilde{dk}\psi_i\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)\overline{\chi}_j + \left(\frac{e}{k_0}\psi_i + i\frac{k_i}{k_0}\psi\right)\frac{k_i k_j}{\mathbf{k}^2}\overline{\left(\frac{e}{k_0}\chi_j + i\frac{k_j}{k_0}\chi\right)}}. \tag{5.2.2}$$

[3] Here, $k_0 = \sqrt{\mathbf{k}^2 + e^2}$.

Calculation:

The Rieffel inner product reads

$$(e^{\psi_\mu} \otimes e^{\psi}, e^{\chi_\mu} \otimes e^{\chi})_0 = \int_{T_s k} [\mathcal{D}g](\Pi_F(W(\partial_\mu g, -eg))e^{\psi_\mu} \otimes e^{\psi}, e^{\chi_\mu} \otimes e^{\chi}), \qquad (5.2.3)$$

where the integration over T_{sk} can be defined in the sense of (3.2.5). Carrying out the calculation of $(e^{\psi_\mu} \otimes e^{\psi}, e^{\chi_\mu} \otimes e^{\chi})_0$, we obtain (cf. the calculation of (3.2.8))

$$\begin{aligned}
&(e^{\psi_\mu} \otimes e^{\psi}, e^{\chi_\mu} \otimes e^{\chi})_0 \\
&= \int [\mathcal{D}g] e^{\frac{-1}{2}(k_\mu g, k_\mu g)_E} e^{-i[(k_0\psi_0, \overline{g}) + (k_i\psi_i, g) + (\overline{g}, k_0\chi_0) + (g, k_i\chi_i)]} e^{\frac{-e^2}{2}(g,g) - e(\psi, g) + e(g,\chi)} e^{(\psi_\mu, \chi_\mu)_E + (\psi,\chi)} \\
&= \int [\mathcal{D}g] e^{-(k_0^2 g, g)} e^{-ig_1[k_0\psi_0 + k_i\psi_i + k_0\overline{\chi}_0 + k_i\overline{\chi}_i + ie\overline{\chi} - ie\psi]} e^{ig_2[k_0\psi_0 - k_i\psi_i - k_0\overline{\chi}_0 + k_i\overline{\chi}_i + ie\overline{\chi} + ie\psi]} e^{(\psi_\mu, \chi_\mu)_E + (\psi,\chi)} \\
&= e^{\frac{-1}{k_0^2}[k_0\psi_0 k_i\psi_i + k_0\overline{\chi}_0 k_i\overline{\chi}_i + ie\overline{\chi} k_i\overline{\chi}_i - ie\psi k_i\psi_i]} e^{\psi_i\left(\delta_{ij} - \frac{k_i k_j}{k_0^2}\right)\overline{\chi}_j - \frac{e^2}{k_0^2}\psi\overline{\chi} + \psi\overline{\chi} - \frac{1}{k_0^2}(iek_i\psi_i\overline{\chi} - ie\psi k_i\overline{\chi}_i)}.
\end{aligned} \qquad (5.2.4)$$

Decomposing this expression with projection operators,

$$\left(\delta_{ij} - \frac{k_i k_j}{k_0^2}\right) = \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right) + \frac{e^2 k_i k_j}{k_0^2 \mathbf{k}^2},$$

we obtain (5.2.2), which completes our calculation.

To extract information about the n-particle states from (5.2.2), we introduce

Definition of $\mathcal{D}$: $\mathcal{D}$ is the set of finite linear cominations of n-particle states[4]

$$\frac{1}{\sqrt{n!}} \frac{d}{dr_1} \cdots \frac{d}{dr_n} e^{\sum_i r_i \psi_\mu^{(i)}} \otimes e^{\sum_j r_j \psi^{(j)}} \Big|_{r_i = 0},$$

where $\psi_\mu^{(i)} \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ and $\psi^{(i)} \in L^2(\mathbb{R}^3)$.

We are particularly interested in the finite particle space $\mathcal{D}$ in which we expect to find besides the transverse also massive longitudinal components. For this analysis, we introduce the following

Notation:

- Given a (gauge-dependent) one-particle state $\psi_\mu \otimes \Omega + \Omega \otimes \psi$, the *Bogoliubov-transformed components* ψ_L, ψ_N are defined by:

$$\begin{aligned}
\psi_{L,i}(\mathbf{k}) &:= \cos\theta \frac{k_i k_j \psi_j(\mathbf{k})}{\mathbf{k}^2} + i \sin\theta \frac{k_i \psi(\mathbf{k})}{|\mathbf{k}|}, \\
\psi_{N,i}(\mathbf{k}) &:= -\sin\theta \frac{k_i k_j \psi_j(\mathbf{k})}{\mathbf{k}^2} + i \cos\theta \frac{k_i \psi(\mathbf{k})}{|\mathbf{k}|},
\end{aligned} \qquad (5.2.5)$$

[4] Here, it is notationally simpler to specify n-particle states as n-th derivatives of elementary exponential vectors, than to write them out explicitly. A one-particle state reads $\psi_\mu \otimes \Omega' + \Omega'' \otimes \psi$, a two-particle state is

$$\frac{1}{2}\left(\psi_\mu^{(1)} \otimes_s \psi_\mu^{(2)} \otimes \Omega' + \psi_\mu^{(1)} \otimes_s \psi^{(2)} + \psi_\mu^{(2)} \otimes_s \psi^{(1)} + \Omega'' \otimes \psi^{(1)} \otimes_s \psi^{(2)}\right),$$

and explicit expressions for $n > 2$ are cumbersome. Here, as in what follows, we have denoted the vacuum by $\Omega = \Omega'' \otimes \Omega'$

where $\cos\theta = \frac{e}{k_0}$, $\sin\theta = \frac{|\mathrm{k}|}{k_0}$.

- Furthermore, we specify the five-component vector $\psi_*^{(i)}$

$$\psi_*^{(i)}(\mathrm{k}) := \left(P_T\psi_\mu^{(i)}(\mathrm{k}), \psi_L^{(i)}(\mathrm{k}), \psi_N^{(i)}(\mathrm{k}), \psi_0^{(i)}(\mathrm{k})\right) \tag{5.2.6}$$

and the projection operator P_p onto the 'physical' components

$$(P_p\psi_*^{(i)})(\mathrm{k}) = \left(P_T\psi_\mu^{(i)}(\mathrm{k}), \psi_L^{(i)}(\mathrm{k}), 0, 0\right). \tag{5.2.7}$$

Also, for notational convenience, we set

$$\psi_*^{(1)} \times \ldots \times \psi_*^{(n)} := \frac{1}{\sqrt{n!}}\frac{d}{dr_1}\cdots\frac{d}{dr_n} e^{\sum_i r_i\psi_\mu^{(i)}} \otimes e^{\sum_j r_j\psi^{(j)}}\Big|_{r_i=0}. \tag{5.2.8}$$

Our next step is to write (5.2.2) for finite particle states in $\mathcal{D}$:

$$(\psi_*^{(1)} \times \ldots \times \psi_*^{(n)}, \chi_*^{(1)} \times \ldots \times \chi_*^{(m)})_0 = \frac{1}{\sqrt{n!m!}} \sum_{q,q'=0} \sum_{\mathcal{P}_{I_{n,q}},\mathcal{P}'_{I_{m,q'}}}$$

$$\times\Big(\sum_{s_q^p\in\mathcal{S}_q^p}\prod_{i=1}^{q}\int d\tilde{k}^{(i)}\frac{1}{k_0^{(i)2}}(k_i^{(i)}\psi_i^{(m_i^q)}(\mathrm{k}^{(i)}) - ie\psi_i^{(m_i^q)}(\mathrm{k}^{(i)}))(k_0^{(i)}\psi_0^{(\tilde{m}_i^q)}(\mathrm{k}^{(i)}))$$

$$+\frac{1}{k_0^{(i)2}}(k_i^{(i)}\psi_i^{(\tilde{m}_i^q)}(\mathrm{k}^{(i)}) - ie\psi_i^{(\tilde{m}_i^q)}(\mathrm{k}^{(i)}))(k_0^{(i)}\psi_0^{(m_i^q)}(\mathrm{k}^{(i)}))$$

$$\times\Big(\sum_{s_{q'}^{p'}\in\mathcal{S}_{q'}^{p'}}\prod_{i=1}^{q'}\int d\tilde{k}^{(i)}\frac{1}{k_0^{(i)2}}(k_i^{(i)}\overline{\chi}_i^{(m_i^q)}(\mathrm{k}^{(i)}) + ie\overline{\chi}_i^{(m_i^q)}(\mathrm{k}^{(i)}))(k_0^{(i)}\overline{\chi}_0^{(\tilde{m}_i^q)}(\mathrm{k}^{(i)}))$$

$$+\frac{1}{k_0^{(i)2}}(k_i^{(i)}\overline{\chi}_i^{(\tilde{m}_i^q)}(\mathrm{k}^{(i)}) + ie\overline{\chi}_i^{(\tilde{m}_i^q)}(\mathrm{k}^{(i)}))(k_0^{(i)}\overline{\chi}_0^{(m_i^q)}(\mathrm{k}^{(i)}))\delta_{n-2q,m-2q'}(-1)^{q+q'}$$

$$\times\left(\sum_{p\in\mathcal{P}(I_{m,q'}^{pq})}\prod_{i=1}^{n-2q}\int d\tilde{k}^{(i)} P_T\psi_\mu^{(t_i^{pq})}(\mathrm{k}^{(i)})\overline{\chi}_\mu^{(p(t_i^{p'q'}))}(\mathrm{k}^{(i)}) + \psi_L^{(t_i^{pq})}(\mathrm{k}^{(i)})\overline{\chi}_L^{(p(t_i^{p'q'}))}(\mathrm{k}^{(i)})\right). \tag{5.2.9}$$

The decomposition properties of elements in $\mathcal{D}$ are now given in complete analogy to free QED, cf. (2.3.11) and (2.3.12):

> **Result:** Arbitrary n-particle states $\psi_*^{(1)} \times \ldots \times \psi_*^{(n)} \in \mathcal{D}$ can be decomposed into physical components $P_p\psi_*$ up to a vector in the $(.,.)_0$-null space:

$$\psi_*^{(1)} \times \ldots \times \psi_*^{(n)} = \sum_{q=0}\sum_{\mathcal{P}_{I_{n,q}}} \lambda_{p_q(I_{n,q})}(P_p\psi_*^{()})\otimes_s\ldots\otimes_s(P_p\psi_*^{(n-2q)})|_{p_q(I_{n,q})} + \vec{n}, \tag{5.2.10}$$

$$\lambda_{p_q(I_{n,q})} = \sqrt{\frac{(2q)!(n-2q)!}{n!}}(\psi_*^{(1)} \times \ldots \times \psi_*^{(2q)}|_{p_q(I_{n,q})}, \Omega)_0. \tag{5.2.11}$$

Calculation:

The analysis of (5.2.9) parallels our analysis of (3.2.11).

Now, we turn to the calculation of the generating functional.

Result: The generating functional ω^χ is Poincaré-invariant on $\triangle_{sk}(N_{sk}, B_{sk})$,

$$\omega_\chi(\phi^\mu, \phi) := (\Pi_F(W(\phi_\mu, \phi))\Omega, \Omega)_0 = e^{\frac{1}{2}(\phi_\mu,\phi_\mu)_M} e^{-\frac{1}{2}(\phi,\phi)}. \tag{5.2.12}$$

Choosing a Lorentz frame, we may write

$$\omega^\chi(\phi_\mu, \phi) = e^{-\frac{1}{2}(\phi_\mu, P_T\phi_\mu)_E} e^{-\frac{1}{2}(\phi_L,\phi_L)}. \tag{5.2.13}$$

Calculation:

$$\begin{aligned}(\Pi_F(W(\phi_\mu, \phi))\Omega, \Omega)_0 &= e^{-\frac{1}{2}(\phi_\mu,\phi_\mu)_E} e^{-\frac{1}{2}(\phi,\phi)} (e^{-\tilde{\phi}_\mu} \otimes e^{-\phi}, \Omega)_0 \\ &= e^{-\frac{1}{2}(\phi_\mu,\phi_\mu)_E} e^{-\frac{1}{2}(\phi,\phi)} e^{-\frac{1}{k_0^2}(k_0\overline{\phi}_0(-k_i\phi_i+ie\phi))} \\ &= e^{\frac{1}{2}(\phi_\mu,\phi_\mu)_M} e^{-\frac{1}{2}(\phi,\phi)} e^{-\frac{1}{k_0^2}(k_0\overline{\phi}_0(k_\mu\phi_\mu+ie\phi))},\end{aligned} \tag{5.2.14}$$

which leads to (5.2.12) for $(\phi_\mu, \phi) \in N_{sk}$. Now, (5.2.13) can be obtained by appropriate rearrangement of the five components in (5.2.12), using $k_0\phi_0 = k_i\phi_i - ie\phi$,

$$\begin{aligned}e^{\frac{1}{2}(\phi_\mu,\phi_\mu)_M} e^{-\frac{1}{2}(\phi,\phi)} &= e^{-\frac{1}{2}(\phi_\mu,P_T\phi_\mu)_E} e^{-\frac{1}{2}\overline{\phi}_i\frac{k_ik_j}{k^2}\phi_j} e^{\frac{1}{2}\overline{\phi}_0\phi_0} e^{-\frac{1}{2}(\phi,\phi)} \\ &= e^{-\frac{1}{2}(\phi_\mu,P_T\phi_\mu)_E} e^{-\frac{1}{2}\overline{\phi}_i\frac{k_ik_j}{k^2}\phi_j} e^{\frac{1}{2k_0^2}(k_i\overline{\phi}_i+ie\overline{\phi})(k_i\phi_i-ie\phi)} e^{-\frac{1}{2}(\phi,\phi)} \\ &= e^{-\frac{1}{2}(\phi_\mu,P_T\phi_\mu)_E} e^{-\frac{1}{2}\overline{\phi}_i\frac{e^2k_ik_j}{k_0^2k^2}\phi_j} e^{-\frac{1}{2}\overline{\phi}\frac{k_ik_j}{k_0^2}\phi} e^{-\frac{1}{2}(-ie\overline{\phi}\frac{k_i}{k_0^2}\phi_i+ie\frac{k_i}{k_0^2}\overline{\phi}_j\phi)} \\ &= e^{-\frac{1}{2}(\phi_\mu,P_T\phi_\mu)_E} e^{-\frac{1}{2}(\frac{e}{k_0}\phi_i+i\frac{k_i}{k_0}\phi)\frac{k_ik_j}{k^2}(\frac{e}{k_0}\overline{\phi}_j-i\frac{k_i}{k_0}\overline{\phi})},\end{aligned} \tag{5.2.15}$$

from which one obtains (5.2.13).

Result: The Hamiltonian corresponding to (5.1.5) satisfies the positive spectrum condition on $\mathcal{H}^\chi$.

Calculation:

The time evolution on $\triangle(M, B)$ is given by the automorphic group τ_t,

$$\tau_t(W(\phi_\mu, \phi)) = W(e^{it\sqrt{D+e^2}}\phi_\mu, e^{it\sqrt{D+e^2}}\phi), \tag{5.2.16}$$

where $(D\phi)_\mu = (-\triangle\phi_0, -\triangle\phi_1, -\triangle\phi_2, -\triangle\phi_3)$.[5] The Hamiltonian H is a representation-dependent operator, implementing this time evolution in the representation Π_F by

$$e^{itH}\Pi_F(W(\phi_\mu, \phi))e^{-itH} = \Pi_F(W(e^{it\sqrt{D+e^2}}\phi_\mu, e^{it\sqrt{D+e^2}}\phi)).$$

Comparing this with the explicit form of the representation in terms of annihilation and creation operators $\hat{a}^*_\mu$, $\hat{a}_\mu$ for the electromagnetic field and $\hat{b}^*$, $\hat{b}$ for the scalar field, we obtain

$$H = -\int d\tilde{k}\sqrt{\mathrm{k}^2 + e^2}\hat{a}^*_\mu(\mathrm{k})g^{\mu\nu}\hat{a}_\nu(\mathrm{k}) + \int d\tilde{k}\sqrt{\mathrm{k}^2 + e^2}\hat{b}^*(\mathrm{k})\hat{b}(\mathrm{k}). \tag{5.2.17}$$

Now, it is easy to see that

$$(\Psi, H\Psi)_0 \geq 0$$

for all $\Psi \in \mathcal{H}_{\mathrm{phys}}$. The point is that arbitrary (normalized) components of the physical one-particle state space, $\left(\delta_{ij} - \frac{k_i k_j}{\mathrm{k}^2}\right)\psi_j$ and $\frac{k_i}{\mathrm{k}}\psi_i \cos\theta + i\psi \sin\theta$ pick up (the same) positive energy contributions. For multi-particle states, this holds true due to their decomposition into such components. The elements of $\mathcal{H}$ corresponding to a negative energy have ended up in the null space. Finally, we comment on the behaviour of elements in $\mathcal{H}^\chi$ under Poincaré transformations. From the Lorentz-invariance of ω^χ (given in (5.2.12)) on the 'algebra of weak observables' $\triangle(N_{sk}, B_{sk})$, we obtain the following

Result: $\mathcal{H}^\chi$ carries a canonical unitary representation U^χ of $\mathcal{P}$. U^χ is a massive particle representation.

Calculation:

In complete analogy to our discussion of free QED, we conclude from the Poincaré invariance of ω^χ that there exists a Poincaré invariant vacuum $[\Omega] \in \mathcal{H}^\chi$ and a representation U^χ of $\mathcal{P}$ on $\mathcal{H}^\chi$, defined by (cf. (3.2.17))

$$U^\chi(\Lambda, a)[\pi_F(W(\phi_\mu, \phi))\Omega] = [\pi_F(W(\gamma_{\Lambda,a}(\phi_\mu, \phi)))\Omega] \tag{5.2.18}$$

for all $(\phi_\mu, \phi)) \in N_{sk}$.

Again, we may use the spectrum of the Hamiltonian, which shows a mass gap, to argue this time that we are dealing with a massive representation ($m^2 = e^2$) of the Poincaré group. More explicitly, we can see that the three components of the vector $P_p\psi^{(i)}_*$ transform as a massive one-particle state under the action of the little group $SO(3)$. Namely, $P_T\psi_\mu$ transforms as transverse components do, whereas ψ_L defined by (5.2.5), has obviously the transformation properties of a longitudinal component.

[5] τ_t corresponds to the equations of motion (5.1.5).

CHAPTER VI:

THE EFFECTIVE POTENTIAL

This chapter contains a discussion of formal properties of the zero temperature effective potential V of a real scalar $P(\phi)_d$ quantum field theory with Z_2-symmetry. Our main result is that in the spontaneously broken case the effective potential is only one-fold differentiable at points of pure quantum corrected ground states [2, 60]. It turns out to be this non-differentiability which justifies the approximation of the quantum corrected ground states as minima in a naïve loop expansion. Furthermore, we employ a thermodynamic language, based on the notion of Gibbs potentials, to give an argument for the energy density interpretation of V in its affine section.

1. The effective potential in $P(\phi)_2$-theories

In this section, we firstly list some results obtained for $P(\phi)_2$-theories. Then, we point out a difficulty in the definition of the effective potential, and its resolution. On the basis of these results, we obtain a statement about the differentiability of the effective potential.

The model

We consider the theory of one self-interacting Bose field in Euclidean space, described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\lambda \phi)^2 + \frac{1}{2}m^2\phi^2 + P(\phi), \tag{6.1.1}$$

where $P(\phi)$ is a polynomial in the field ϕ and m is the renormalized mass. The generating functional for the Euclidean Green's functions is given by

$$\begin{aligned} Z[J] &= \lim_{\Omega\to\infty} \int [d\phi]_\Omega \exp\left(\frac{-1}{\hbar}\int_\Omega [\mathcal{L}(\phi(x)) - J(x)\phi(x)]dx\right) \\ &= \lim_{\Omega\to\infty} \int \exp\left(-\int [P(\phi(x)) - J(x)\phi(x)]dx\right) d\mu_C^\Omega, \end{aligned} \tag{6.1.2}$$

where $d\mu_C^\Omega$ is a Gaussian measure with a free covariance, converging to $C = (-\partial^2 + m^2)^{-1}$ in the infinite volume limit. $J(x)\phi(x)$ is the usual Schwinger source term, Ω denotes a finite space-time volume, with which we approach finally the infinite-volume limit and Planck's constant is set to one in the second expression.

The generating functional of connected Green's functions $W[J]$ is given by

$$W[J] = \ln Z[J] \tag{6.1.3}$$

and its Legendre transform is the effective action $\Gamma[\phi_c]$

$$\begin{aligned}\Gamma[\phi_c(x)] &= W[J] - \int J(x)\phi_c(x)dx;\\ \phi_c(x) &= \frac{\delta}{\delta J(x)} W[J].\end{aligned} \tag{6.1.4}$$

Definition of the effective potential

Usually, the effective potential $\hat{V}(\hat{\phi}_c)$ is defined by setting ϕ_c in the effective action to be a constant $\hat{\phi}_c$ and dividing through the total space volume Ω:

$$\lim_{\Omega\to\infty} \frac{1}{\Omega} \Gamma_\Omega[\phi_c]_{\phi_c=\hat{\phi}_c} = -\hat{V}(\hat{\phi}_c). \tag{6.1.5}$$

It is however crucial to note that $\hat{V}$ is only well-defined for all field values $\hat{\phi}_c$ if the functional derivative of $W[J]$ in (6.1.4) exists for all J. More importantly: we know nothing a priori about the existence of the limits in (6.1.2) and (6.1.5) and their dependence on boundary conditions.

To obtain an effective potential which is always well-defined, let us start from the density of $W[J]$ for constant source term

$$w(J) = \lim_{\Omega\to\infty} \frac{1}{\Omega} \ln \int_\Omega \exp[-\int_\Omega [P(\phi(x)) - J\phi(x)]dx] d\mu_c^\Omega. \tag{6.1.6}$$

In the case of $P(\phi)_2$-theories, this density can be shown to exist. It is strictly convex[1] and the limit is independent of a large number of classical boundary conditions [61, 62]. Following Jona-Lasinio [63] and Slade [64], we define the effective potential as a Fenchel transform of $w(J)$

$$V(\hat{\phi}_c) := \sup_J [J\hat{\phi}_c - w(J)]. \tag{6.1.7}$$

For strictly convex $w(J)$, this Fenchel transform is well-defined [65].

Remark:

[1] We call a function f of a real variable x strictly convex, if it is a convex function which has nowhere an affine section. In an affine section, f is given by a straight line.

Recalling that the Legendre transformation of a function $w(J)$ plots the intersection of the tangent of w with the w-axis against its slope (Figure 1), it is easy to see that for a differentiable $w(J)$ both definitions of the effective potential are equivalent,

$$\hat{V}(\hat{\phi}_c) = V(\hat{\phi}_c), \tag{6.1.8}$$

whereas for non-differentiable $w(J)$, only (6.1.7) is well-defined everywhere.

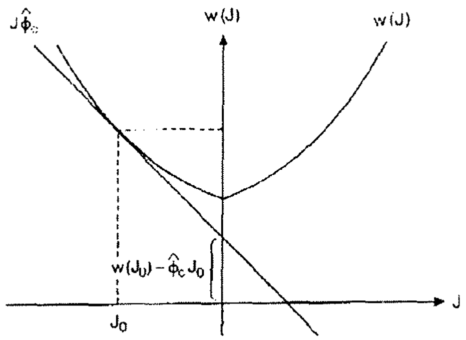

Figure 1: Example of a strictly convex $w(J)$ with non-differentiability at $J = 0$ and geometrical construction of the Legendre transformation.

Non-differentiability of the effective potential

In what follows, we consider (6.1.1) with an even polynomial $P(\phi(x))$ which allows for spontaneous symmetry breaking. Introducing left and right derivatives $\frac{d^-}{dJ}$ and $\frac{d^+}{dJ}$ respectively, we obtain from convexity

$$\frac{d^-}{dJ}w(0) = -\frac{d^+}{dJ}w(0). \tag{6.1.9}$$

However, in the case of a scalar field theory with spontaneous symmetry breaking, the left and right derivatives of $w(J)$ are nothing but the degenerate vacuum expectation values [61]

$$\frac{d^\pm}{dJ}w(0) = \phi_\pm. \tag{6.1.10}$$

Glimm, Jaffe and Spencer have shown that for ϕ_2^4 quantum field theories, there exists a phase transition in the sense that $\phi_\pm \neq 0$ [66]. Hence, for $J = 0$, $w(J)$ is not differentiable, whereas it is differentiable for all non-zero values of J. As can be seen from Figure 2, the effective potential, given by (6.1.7), therefore has a linear section between the two quantum corrected ground states $\phi_\pm$, and is strictly convex elsewhere. This is sufficient to obtain the following

Result: Let $V(\hat{\phi}_c) = \sup_J[J\hat{\phi}_c - w(J)]$ where $\frac{d^\pm}{dJ}w(0) = \phi_\pm$ and $w(J)$ is strictly convex, differentiable for $J \neq 0$ and non-differentiable for $J = 0$. Then $V(\hat{\phi}_c)$ is exactly one-fold differentiable for $\hat{\phi}_c = \phi_\pm$.

Argument:

To discuss the differentiability of V at ϕ_+, it is sufficient to calculate its right derivatives. All left derivatives are zero, since for $\phi_- \le \hat{\phi}_c \le \phi_+$, V is affine (the affinity of V can easily be seen from the geometrical constuction, described in the text, cf. Figures 1 and 2). Hence for $\epsilon_i > 0$ we consider the difference

$$\begin{aligned} V(\phi_+ + \epsilon_1) - V(\phi_+ + \epsilon_2) &= \sup_J[J(\phi_+ + \epsilon_1) - w(J)] - \sup_J[J(\phi_+ + \epsilon_2) - w(J)] \\ &= J_1(\phi_+ + \epsilon_1) - w(J_1) - J_2(\phi_+ + \epsilon_2) + w(J_2) \\ &= \frac{1}{2}\frac{d^{+2}}{dJ^2}w(0)\left(J_1^2 - J_2^2\right) + O\left(J_i^3\right) \\ &= \frac{1}{2}\left(\frac{d^{+2}}{dJ^2}w(0)\right)^{-1}(\epsilon_1 - \epsilon_2)(\epsilon_1 + \epsilon_2) + O\left(J_i^3\right), \end{aligned} \tag{6.1.11}$$

where we have defined J_i by

$$\begin{aligned} \epsilon_i &= \frac{d^{+2}w(0)}{dJ^2}J_i + O\left(J_i^2\right); \\ &\frac{d^+}{dJ}w(J_i) = \phi_+ + \epsilon_i \end{aligned} \tag{6.1.12}$$

and used a Taylor expansion for $w(J_i)$

$$w(J_i) = w(0) + \frac{d^+w(0)}{dJ}J_i + \frac{1}{2}\frac{d^{+2}w(0)}{dJ^2}J_i^2 + \dots.$$

Due to the strict convexity of $w(J)$, the inverse of its second right derivative is strictly positive and we obtain from (6.1.11)

$$\begin{aligned} \frac{d^+V(\phi_+ + \epsilon)}{d\hat{\phi}_c} &= \left(\frac{d^{+2}w(0)}{dJ^2}\right)^{-1}\epsilon + O(\epsilon^2); \\ \frac{d^{+2}V(\phi_+ + \epsilon)}{d\hat{\phi}_c^2} &= \left(\frac{d^{+2}w(0)}{dJ^2}\right)^{-1} + O(\epsilon) > 0 \end{aligned} \tag{6.1.13}$$

for small enough ϵ. The same line of reasoning goes through for ϕ_-. This completes our argument.

Remark:

We conclude that in the spontaneously broken case, the effective potential V is a convex function with a linear section between ϕ_- and ϕ_+. It is infinitely often differentiable for $\hat{\phi}_c \neq \phi_\pm$ and exactly one-fold differentiable at $\hat{\phi}_c = \phi_\pm$.

2. Perturbative evaluation of the effective potential

In the first section, we have established the non-perturbative form of V. Now, we turn to the question, how far perturbative calculations allow us to approximate this form. After recalling the loop expansion

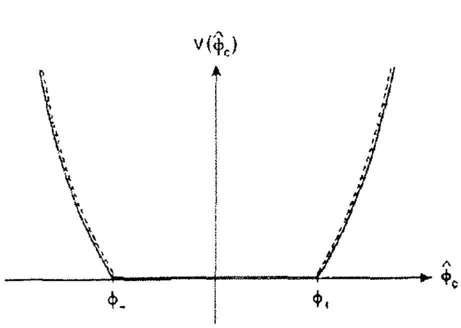

Figure 2: The effective potential $V(\hat{\phi}_c)$ (straight line) and $\hat{V}(\hat{\phi}_c)$ (dotted line) as defined in the text. Remark that $\hat{V}$ is only defined for non-minimal values of V.

of V, we point out that it is exactly the one-fold differentiability of V established in the last section, which allows us to determine ϕ_+ and ϕ_- as minima of V. We close with a short remark on the Wilson recursion formula which provides a different approximation scheme of V.

The loop expansion

The usual perturbative approach to the evaluation of the effective potential is the loop expansion [32, 33, 34]. Interpreting the effective action in (6.1.4) as a generating functional of the one-particle irreducible Green's functions, one makes different expansions of $\Gamma[\phi_c]$ either in powers of ϕ_c or in powers of momentum

$$\begin{aligned} \Gamma[\phi_c] &= \sum_n \frac{1}{n!} \int dx_1 ... dx_n \Gamma^n(x_1, ..., x_n) \phi_c(x_1) ... \phi_c(x_n) \\ &= \int dx[-\hat{V}(\phi_c) + \frac{1}{2}(\partial_\lambda \phi_c)^2 Z(\phi_c) + ...]. \end{aligned} \tag{6.2.1}$$

Then one can easily see that the n-th derivative of $-\hat{V}$ is the sum of all 1PI Feynman graphs with n vanishing external momenta

$$\hat{V}(\hat{\phi}_c) = -\sum_n \frac{1}{n!} \tilde{\Gamma}^n(0, ..., 0) \hat{\phi}_c^n, \tag{6.2.2}$$

where $\tilde{\Gamma}^n$ denotes the Fourier transform of Γ^n. The loop expansion is a tractable resummation of (6.2.2): first summing all 1PI tree diagrams, then the diagrams with one loop, etc., one approximates $-\hat{V}$. As can be seen from the generating functional $Z[J]$ in (6.1.2), each vertex in these diagrams yields a factor $\frac{1}{\hbar}$ while each propagator, being the inverse of the differential operator in the quadratic part of the Lagrangian, yields a factor $\hbar$. Hence the loop expansion is formally an expansion in powers of $\hbar$,

$$\hat{V}(\hat{\phi}_c, \hbar)' =' \sum_{n=0}^{\infty} v_n(\hat{\phi}_c) \hbar^n \tag{6.2.3}$$

For Euclidean scalar $P(\phi)_2$ quantum field theories, Slade has shown [64] that the series $\{V_m(\hat{\phi}_c, \hbar)\}$ with

$$V_m(\hat{\phi}_c, \hbar)' =' \sum_{n=0}^{m} v_n(\hat{\phi}_c)\hbar^n \tag{6.2.4}$$

is asymptotic[2] to $V(\hat{\phi}_c, \hbar)$ in (6.1.7) for all values of $\hat{\phi}_c$, satisfying $\mid \hat{\phi}_c \mid \geq \phi_+$,

$$\lim_{m\to\infty} V_m(\hat{\phi}_c, \hbar) \sim V(\hat{\phi}_c, \hbar) = \sup_J[J\hat{\phi}_c - w(J)].$$

Here, $v_n(\hat{\phi}_c)$ denotes the negative sum of all renormalized 1PI n-loop diagrams with lines corresponding to free covariance of mass $\left(P''(\hat{\phi}_c)\right)^{\frac{1}{2}}$, and the formal dependence of V on $\hbar$ is denoted explicitly. For field values $\mid \hat{\phi}_c \mid < \phi_+$, the series $\{V_m(\hat{\phi}_c, \hbar)\}$ is not asymptotic to V, cf. [64]. This leads us to the question whether we do obtain any information about $\phi_\pm$, defined in (6.1.10), from the loop expansion.

Information about $\phi_\pm$ from the loop expansion

We define the maximum of all k local minima $\hat{\phi}_{m,i}$ of V_m, $(i = 1, 2, ..., k)$:

$$\hat{\phi}_m = \max_i[\hat{\phi}_{m,i}], \tag{6.2.5}$$

where

$$\frac{dV_m(\hat{\phi}_{m,i})}{d\hat{\phi}_c} = 0. \tag{6.2.6}$$

Furthermore, we introduce the Taylor expansion of the first derivative of V_m around $\hat{\phi}_m$,

$$\frac{dV_m(\hat{\phi}_c, \hbar)}{d\hat{\phi}_c} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^{(n+1)}V_m(\hat{\phi}_m, \hbar)}{d\hat{\phi}_c^{(n+1)}} (\hat{\phi}_c - \hat{\phi}_m)^n. \tag{6.2.7}$$

With this notation, we formulate the following

Result: If the Taylor expansion (6.2.7) converges for $\mid \hat{\phi}_c - \phi_+ \mid \leq \mid \hat{\phi}_m - \phi_+ \mid$ and $\frac{d^2V_m(\phi_*,\hbar)}{d\hat{\phi}_c^2} > 0$ for $\phi_* > \min(\phi_m, \phi_+)$, then

$$\lim_{m\to\infty} \hat{\phi}_m \sim \phi_+. \tag{6.2.8}$$

[2]We call a power series $\sum_{n=0}^{N} a_n z^n$ asymptotic to a function $f(z)$, $f(z) \sim \sum_{n=0}^{N} a_n z^n$, if $\lim_{z\to 0} \frac{1}{z^n} \mid f(z) - \sum_{n=0}^{N} a_n z^n \mid = 0$ holds for all integer N.

Argument:

Since the Taylor expansion of $\frac{dV_m(\phi_+,\hbar)}{d\hat{\phi}_c}$ converges for $\mid \hat{\phi}_c - \phi_+ \mid \leq \mid \hat{\phi}_m - \phi_+ \mid$, we can write

$$\frac{dV_m(\phi_+,\hbar)}{d\hat{\phi}_c} = \frac{dV_m(\hat{\phi}_m,\hbar)}{d\hat{\phi}_c} + \frac{d^2V_m(\phi_*,\hbar)}{d\hat{\phi}_c^2}(\phi_+ - \hat{\phi}_m), \tag{6.2.9}$$

where the Taylor remainder term on the r.h.s. is taken for some ϕ_*, satisfying $\mid \phi_+ - \phi_m \mid \geq \mid \phi_+ - \phi_* \mid$. Now, since the loop expansion is asymptotic at ϕ_+, we write

$$\lim_{\hbar\to 0}\frac{1}{\hbar^m}\left| \frac{dV(\phi_+,\hbar)}{d\hat{\phi}_c} - \frac{dV_m(\phi_+,\hbar)}{d\hat{\phi}_c}\right| = 0. \tag{6.2.10}$$

Substituting the Taylor expansion (6.2.9) into (6.2.10), we obtain

$$\lim_{\hbar\to 0}\frac{1}{\hbar^m}\frac{d^2V_m(\phi_*,\hbar)}{d\hat{\phi}_c^2}\mid \phi_+ - \hat{\phi}_m \mid = 0. \tag{6.2.11}$$

Since $\frac{d^2v_0(\phi_*)}{d\hat{\phi}_c^2} \neq 0$, cf. (6.2.3), and since the second derivative of V_m is positive according to our assumption, we conclude from (6.2.10) that

$$\lim_{\hbar\to 0}\frac{1}{\hbar^m}\mid \phi_+ - \hat{\phi}_m \mid = 0. \tag{6.2.12}$$

Conclusion:

- The naïve loop expansion contains all asymptotic information about the form of the effective potential: the loop expansion is asymptotic to V in its strictly convex regions and it approaches asymptotically the extremals of the affine section of V.

Remarks:

- A result analogous to (6.2.8) holds for the $m \to \infty$-limit of the minimum of all local minima of V_m, being asymptotic to ϕ_-. We conclude that the largest (smallest) minima of the m-th order loop expansion are asymptotic to ϕ_+ (ϕ_-) respectively.

- The assumptions leading to the result (6.2.8) are well justified: the second derivative of V_m is asymptotic to a positive number at ϕ_+, since it follows from (6.1.13) that

$$\lim_{m\to\infty}\frac{d^2V_m(\phi_+)}{d\hat{\phi}_c^2} \sim \left(\frac{d^{+2}w(0)}{dJ^2}\right)^{-1} > 0. \tag{6.2.13}$$

  This is a direct consequence of the non-differentiability of V. The assumption about the positivity of the second derivative of V_m is clearly consistent both with (6.2.13) and with the strict convexity of V outside its affine region.

The Wilson Recursion Formula

We have seen that while the loop expansion of V leads to a non-convex expression, it approximates asymptotically $\phi_\pm$, thereby providing all necessary information for the determination of the correct convex form of V. In this subsection, for the sake of completeness, we draw attention to the Wilson recursion formula which provides an approximation scheme leading to a convex expression for V. The effective potential $V(\phi_c)$ is a function of the zero momentum mode ϕ_c of the field ϕ and may be obtained by integrating out the high momentum modes iteratively. According to arguments due to Wilson [67, 39], this amounts to fixing a momentum cut-off Λ and following the subsequent iteration scheme:

$$
\begin{aligned}
U_0(\phi) &= V_{cl}(\phi), \\
U_{l+1}(\phi) &= -\left(\frac{\Lambda}{2^l}\right)^d \ln\left[\frac{f(U_l(\phi))}{f(U_l(0))}\right], \\
f(U_l(\phi)) &= \int_{-\infty}^{\infty} \exp\left[-\left(\frac{\Lambda}{2^l}\right)^{-d+2} y^2 - \left(\frac{\Lambda}{2^l}\right)^{-d} \frac{1}{2}\{U_l(y+\phi) + U_l(-y+\phi)\}\right] dy, \\
V(\phi_c) &= U_\infty(\phi_c).
\end{aligned}
\tag{6.2.14}
$$

We restrict our discussion to the following [68]

Remarks:

- This Wilson recursion fomula is closely related to the usual loop expansion. Indeed, if one expands the l-th iterated potential $U_l(y+\phi)$ up to second order in y, ignoring higher order terms, one obtains an expression for U_{l+1} which does not depend on f. Then, the crude approximation $\frac{d^2U_l(\phi)}{d\phi^2} = \frac{d^2U_\infty(\phi)}{d\phi^2}$ leads to

$$
U_\infty(\phi) = U_0(\phi) + \sum_{l=0}^{\infty} k_l^d \frac{1}{2} \ln\left[\frac{2k_l^2 + U"_\infty(\phi)}{2k_l^2 + U"_\infty(0)}\right],
$$

  which is exactly the formula obtained by an one-loop calculation.

- The iteration scheme (6.2.14) allows to calculate recursion formulae for the higher order derivatives of $U_l(\phi)$. From the behaviour of these derivatives one can argue that in the $l \longrightarrow \infty$-limit, the potentials $U_l(\phi)$ approach a convex expression, as expected from general arguments.

- Numerical iterations of (6.2.14) indicate too that $U_l(\phi)$ converges to a convex expression for $l \rightarrow \infty$.

3. Energy-density interpretation of the effective potential

In this section, we present an argument to the effect that the convex form of the effective potential V has an energy density interpretation. Following heuristic arguments [31, 33, 34], one usually assumes that

$$V(\hat{\phi}_c) = \lim_{\Omega\to\infty} \frac{\int_\Omega Tr\left(\rho_c \hat{H}_\Omega(\phi)\right)}{\Omega}, \tag{6.3.1}$$

where $\hat{H}_\Omega$ is the Hamiltonian in the finite region[3] Ω and ρ_c is a projection operator of unit trace, denoting the state for which

$$\hat{\phi}_c = \lim_{\Omega\to\infty} \frac{\int_\Omega Tr(\rho_c\phi)}{\Omega}. \tag{6.3.2}$$

In the case of $P(\phi)_2$-theories, equation (6.3.1) has been obtained for values of $\hat{\phi}_c$ which do not minimize V [69]. Here, we show that the energy density interpretation of V holds for its linear section, too. We start with the following assumption, underlying a theory of phase transitions:[4]

> **Assumption:** There is a finite set $(Q_1, ..., Q_n)$ of extensive quantities such that, if the corresponding densities $(\hat{q}_1, ..., \hat{q}_n)$ of these quantities are constrained to take values, say $(q_1, ..., q_n)$, then the entropy density functional $\hat{s}(\rho)$ is maximized by precisely one translationally invariant state ρ.

Our argument is based on the following thermodynamic **setting**:
We introduce the finite-volume Helmholtz free energy functional $\hat{F}_\rho$ of an arbitrary state ρ

$$\hat{F}_\Omega(\rho) = Tr\left(\rho \hat{H}_\Omega(\phi) - T\hat{S}_\Omega(\rho)\right), \tag{6.3.3}$$

with the entropy

$$\hat{S}(\rho) = -k\rho\ln\rho, \tag{6.3.4}$$

and the infinite-volume density

$$\hat{f}(\rho) = \lim_{\Omega\to\infty} \frac{\hat{F}_\Omega(\rho)}{\Omega}. \tag{6.3.5}$$

In the zero temperature case, $\hat{f}$ is nothing but the energy density functional. The field ϕ in (6.3.3) is a linear functional of the state ρ

$$\phi(\rho) = \lim_{\Omega\to\infty} \frac{\int_\Omega Tr(\rho\phi)}{\Omega}. \tag{6.3.6}$$

[3] Here, (6.3.1) is time-independent and hence we may choose for Ω either a space or a space-time region.
[4] Sewell [70] states this assumption for extensive conserved quantities Q_i in the case of a lattice system with a finite number of degrees of freedom per lattice site. In our case, the family of conserved quantities which commute with the Hamiltonian is not sufficient to specify the state of the system completely. Hence we choose a more general formulation.

We interpret ϕ as a density functional in the sense of the fundamental assumption and constrain it to take a value $\hat{\phi}_c$. For lattice systems with a finite number of degrees of freedom per lattice site [70], this leads to the constrained Helmholtz free energy density

$$f_c(\hat{\phi}_c, T) = \inf_\rho\{\hat{f}(\rho) \mid \phi(\rho) = \hat{\phi}_c\} \tag{6.3.7}$$

and the Gibbs potential

$$g(J,T) = \inf_\rho\{\hat{f}(\rho) - J\phi(\rho)\}, \tag{6.3.8}$$

where the intensive variable J is the conjugate of ϕ. Following Sewell [70] we assume that the expressions for $f_c(\hat{\phi}_c, T)$ and $g(J,T)$ hold for infinite continuous systems.

Notation: We denote by $\{\tilde{\rho}(J)\}_J$ an arbitrary one-parameter family of equilibrium states $\tilde{\rho}(J)$ which minimize $\hat{f}(\rho) - J\phi(\rho)$. Remark that $f_c(\hat{\phi}_c, T)$ is convex in $\hat{\phi}_c$ and $g(J,T)$ is concave in J [70].

Result: Assume that $\sup_J[g(J,T)] = g(0,T)$, $\mid \phi(\tilde{\rho}(0)) \mid \leq \mid \frac{d^+g(0,T)}{dJ} \mid$, $\mid \phi(\tilde{\rho}(J)) \mid > \mid \frac{d^+g(J,T)}{dJ} \mid$ for $J \neq 0$ and $\phi(\tilde{\rho}(J))$ continuous for $J \neq 0$.
Then

$$f_c(\hat{\phi}_c, T) = \sup_J[J\hat{\phi}_c + g(J,T)] \tag{6.3.9}$$

for $\frac{d^+g(0,T)}{dJ} \leq \hat{\phi}_c \leq \frac{d^-g(0,T)}{dJ}$, and $\hat{\phi}_c$ minimizes f_c.

Argument:
The variable T is omitted.

1. It follows from the concavity of $g(J)$ together with $\mid \frac{d^\pm g(0)}{dJ} \mid \geq \mid \hat{\phi}_c \mid$, that $\mid \frac{d^\pm g(J)}{dJ} \mid > \mid \hat{\phi}_c \mid$, for all $J \neq 0$. The assumptions $\sup_J[g(J)] = g(0)$ and $\frac{d^+g(0)}{dJ} \leq \hat{\phi}_c \leq \frac{d^-g(0)}{dJ}$ then ensure that the slope of $g(J)$ is steeper than the slope of the straight line $J\hat{\phi}_c$ for all J (cf. Figure 3). The supremum on the r.h.s. of (6.3.9) is hence obtained for $J = 0$. Denoting by $\{\tilde{\rho}(J)\}_c$ an arbitrary one-parameter family of states which minimize $\hat{f}(\rho) - J\phi(\rho)$, we write

$$\begin{aligned} \sup_J[J\hat{\phi}_c + g(J)] &= [J\hat{\phi}_c + \inf_\rho\{\hat{f}(\rho) - J\phi(\rho)\}]_{J=0} \\ &= [J\hat{\phi}_c + \hat{f}\,(\tilde{\rho}(J)) - J\phi\,(\tilde{\rho}(J))]_{J=0} \\ &= \inf_\rho \hat{f}(\rho) \\ &= \hat{f}\,(\tilde{\rho}(0))\,. \end{aligned} \tag{6.3.10}$$

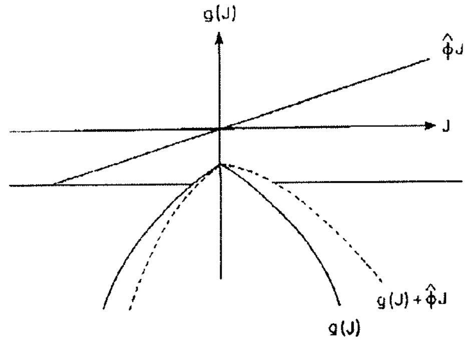

Figure 3: Example of a concave Gibbs energy density $g(J)$ with non-differentiability at $J = 0$. As is easily seen, the supremum of $[\hat{\phi}J + g(J)]$ (dotted line) is obtained for $J = 0$, if $|\frac{d^{\pm}g(0)}{dJ}| \geq \hat{\phi}_c$.

2. Since $\phi(\tilde{\rho}(J))$ is continuous for $J > 0$, there is a one-parameter family $\{\tilde{\rho}_+(J)\}_J$, such that $\phi(\tilde{\rho}_+(J))$ is continuous for $J \geq 0$. Denoting $\phi_\pm = -\frac{d^{\pm}g(0,T)}{dJ}$, we conclude $\phi(\tilde{\rho}_+(0)) = \phi_+$. Using i), we observe that the r.h.s. of (6.3.9) is constant for $\phi_- \leq \phi_c \leq \phi_+$. This allows us to write

$$\begin{aligned}\hat{f}(\tilde{\rho}_+(0)) &= \sup_J[J\hat{\phi}_c + g(J)] \\ &= [J\phi_+ + \hat{f}(\tilde{\rho}_+(J)) - J\phi(\tilde{\rho}_+(J))]_{J=0} \\ &= \inf_{\tilde{\rho}_+(J)}\{\hat{f}(\tilde{\rho}_+(J)) \mid \phi(\tilde{\rho}_+(J)) = \phi_+\} \\ &= f_c(\phi_+). \qquad (6.3.11)\end{aligned}$$

3. The same argument as in 2. holds for negative J: there is a one-parameter family of states $\{\tilde{\rho}_-(J)\}_J$, such that $\phi(\tilde{\rho}_-(J))$ is continuous for $J \leq 0$ and hence

$$\begin{aligned}\phi(\tilde{\rho}_-(0)) &= \phi_-; \\ \hat{f}(\tilde{\rho}_-(0)) &= f_c(\phi_-). \qquad (6.3.12)\end{aligned}$$

4. From 1., 2. and 3., we conclude that

$$\inf_\rho \hat{f}(\rho) = f_c(\phi_+) = f_c(\phi_-). \qquad (6.3.13)$$

Since f_c attains its absolute minimum at ϕ_+ and ϕ_-, it follows from its convexity that f_c has an affine section between ϕ_+ and ϕ_-. As seen in i), the r.h.s. of (6.3.9) is constant for $\phi_- \leq \phi_c \leq \phi_+$. Hence

$$f_c(\hat{\phi}_c) = \sup_J[J\hat{\phi}_c + g(J)]. \qquad (6.3.14)$$

Remarks:

- The connection between thermodynamics and the field theoretic formulation of (6.1.1) is the following: for $T = 0$, it follows from the strict convexity of $w(J)$ in (6.1.6) together with (6.1.10) that

$$\sup_J[-w(J)] = -w(0). \tag{6.3.15}$$

$w(J)$ can be written in terms of a finite volume partition function $Z_\Omega(J)$ [69]

$$w(J) = \lim_{\Omega\to\infty} w_\Omega(J) = \lim_{\Omega\to\infty} \frac{1}{\Omega} \ln \frac{Z_\Omega(J)}{Z_\Omega(0)}. \tag{6.3.16}$$

Hence $w(J)$ is the (negative) expected energy density in the presence of an external field, i.e. the (negative) Gibbs potential. Since the vacuum expectation value of ϕ in presence of a positive external source J is greater than ϕ_+ [66] and since $-w(J)$ satisfies (6.1.9), the result given above applies. Hence the effective potential

$$V(\hat{\phi}_c) = \sup_J[J\hat{\phi}_c - w(J)] \tag{6.3.17}$$

has an energy density interpretation for all $\hat{\phi}_c$ which minimize V.

- Our results can be embedded in a much wider context. Every non-differentiable Gibbs potential $g(J)$ of the type considered will lead to a Fenchel transform f_c for which analogs of our results exist. The non-differentiability of f_c will then justify the assumption that every infinitely often differentiable function which is asymptotic to f_c in its strictly convex regions, asymptotically approaches the extremals of the affine section of f_c with its minima. In particular, this is true for every polynomial ansatz for f_c in its strictly convex regions. This gives an interesting justification for the heuristic Landau-Ginzburg approach to phase transitions.
More precisely, in heuristic applications, the non-convex expression obtained from the loop expansion is often interpreted as a free energy density. This may be understood as a Landau-type argument: choosing a polynomial ansatz (6.2.2) for $\hat{V}$, one speaks of the 'roll-down of the vacuum expectation value' in a non-convex potential. Though it is difficult to give a mathematically precise meaning to this picture[5], the success of the Landau-Ginzburg ansatz in the description of phase transitions in solid state physics may illustrate the heuristic value of this approach.

[5] In fact, our result contradicts one of the main features of this picture, the non-convexity.

Non-relativistic example of a non-differentiable energy density

As an illustration of our remarks on the energy density of systems showing spontaneous symmetry breaking, we mention the non-relativistic spin-boson model, defined by the Hamiltonian

$$H = \epsilon\sigma_1 \otimes I + I \otimes \sum_{n=1}^{F} \omega_n a_n^* a_n + \sigma_3 \otimes \sum_{n=1}^{F} \lambda_n (a_n + a_n^*) \tag{6.3.18}$$

on the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{S}(L^2(\mathbb{N}))$.[6] Here, σ_1 and σ_3 denote Pauli matrices, and the coupling constants ω_n and λ_n are required to satisfy $\Lambda := \sum_{n=1}^{\infty} \frac{\lambda_n^2}{\omega_n} < \infty$ (which guarantees that H is bounded from below) and $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$ [71, 72]. H may be thought to model the interaction of a 2-level molecule with a radiation field, described by creation and annihilation operators a_n^*, a_n for the discrete modes $n = 1, ..., F$, where F is eventually taken to infinity. ϵ denotes the splitting between the two energy levels of the molecule. As in the case of (6.1.1), the model shows a 'chiral' $\mathbb{Z}_2$-symmetry J,[7] implemented by

$$P = \sigma_1 \otimes \prod_{n=1}^{F} (-1)^{a_n^* a_n}. \tag{6.3.19}$$

This model has been used to discuss the chirality of molecules [71]. The simplest method to get information about the energy density of this model is to make an ansatz for the eigenvectors of H in terms of product states $\psi = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \phi$, $\phi \in \mathcal{S}(L^2(\mathbb{N}))$, $|\psi| = 1$, ψ in the domain of H. Using a variational principle [71], one obtains the minimal energy states

1. for $|\epsilon| \leq 2\Lambda$:[8]

$$\psi_\pm = \left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{1 \pm \left(1 - \frac{\epsilon^2}{4\Lambda^2}\right)^{\frac{1}{2}}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathrm{sgn}(\epsilon)\sqrt{1 \pm (-1)\left(1 - \frac{\epsilon^2}{4\Lambda^2}\right)^{\frac{1}{2}}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$
$$\otimes\left(\otimes_{n=1}^{F} W\Big(\pm\frac{-\lambda_n}{\omega_n}\left(1 - \frac{\epsilon^2}{4\Lambda^2}\right)^{\frac{1}{2}}\Big) \mid 0\rangle\right) \tag{6.3.20}$$

2. for $|\epsilon| \geq 2\Lambda$:

$$\psi = \left(\frac{1}{\sqrt{2}}\right)\begin{pmatrix} 1 \\ -\mathrm{sgn}(\epsilon) \end{pmatrix} \otimes \mid 0\rangle.$$

with corresponding energies

1. for $|\epsilon| \leq 2\Lambda$: $E = \langle\psi_\pm \mid H \mid \psi_\pm\rangle = -\left(1 + \frac{\epsilon^2}{4\Lambda^2}\right)\Lambda$;

---

[6] Here, $\mathcal{S}(L^2(\mathbb{N}))$ denotes the symmetric Fock space over the one-particle space $L^2(\mathbb{N})$.
[7] $J(\sigma_1) := \sigma_1$; $J(\sigma_3) := -\sigma_3$; $J(a_n) := -a_n$; $J(a_n^*) := -a_n^*$
[8] We use the Weyl operators $W(f) := e^{\sum_{n=1}^{F} a_n^* f_n - f_n^* a_n}$, $f \in L^2(\mathbb{N})$.

2. for $|\epsilon| \geq 2\Lambda$: $E = \langle\psi \mid H \mid \psi\rangle = -|\epsilon|$.

One easily obtains a suitable order parameter:

1. for $|\epsilon| \leq 2\Lambda$: $m_{\epsilon,\pm} = \langle\psi_\pm \mid \sigma_3 \mid \psi_\pm\rangle = \pm\frac{1}{\Lambda}(4\Lambda^2 - \epsilon^2)$
2. for $|\epsilon| \geq 2\Lambda$: $m_\epsilon = \langle\psi \mid \sigma_3 \mid \psi\rangle = 0$

Remarks:

- For $|\epsilon| \leq 2\Lambda$, the ground state is not an eigenvector of P, i.e., the symmetry i is broken.
- Writing $E(\delta)$ as a function of $\delta = \epsilon - 2\Lambda$, one finds that E is exactly one-fold differentiable at the point of phase transition, $\epsilon = 2\Lambda$.
- For $|\epsilon| \leq 2\Lambda$, ϵ, Λ fixed, the energy as a function of the order parameter is a constant in the interval $m \in [m_{\epsilon,-}, m_{\epsilon,+}]$,
$$E(m) = \left(1 + \frac{\epsilon^2}{4\Lambda^2}\right)\Lambda, \tag{6.3.21}$$
since mixed states $\gamma\psi_+ + (1-\gamma)\psi_-$, $\gamma \in [0,1]$ exist for arbitrary order parameter m in the interval $[m_{\epsilon,-}, m_{\epsilon,+}]$. Hence, E has an affine section with endpoints corresponding to the pure states ψ_+, ψ_-.[9]
- We do not have an expression for the lowest lying energy states ψ_* with $m_* > m_{\epsilon,+}$, but from the results given above it follows that $E(m) < E(m_*)$ and that to some order in m, a non-differentiability in $E(m)$ at $m = m_{\epsilon,+}$ (or $m_{\epsilon,-}$) has to be expected.

We regard this model as a nice example for a non-differentiable energy density which indicates a non-trivial phase structure at zero temperature. To shorten our illustration, however, we have based our remarks on the results of Pfeiffer [71]. This remains unsatisfactory since product states are not the most general eigenstates of H. For a more satisfactory though technically much more involved treatment of (6.3.18), we refer to Spohn's work [73] where ground states have been constructed as zero-temperature limits of KMS-states. An energy density $E(m)$ based on Spohn's result can be found in [72]. Our remarks apply in this case, too.

[9] These pure states lie in different superselection sectors if one regards all operators which act non-trivially on a finite number of modes of the photon field as local observables. I.e., chirality is a superselection rule of this model. In fact, this has been the main motivation for Pfeifer [71] to investigate this model.

CHAPTER VII:
THE VACUUM EXPECTATION VALUE

In the first part of this chapter, we show that the vacuum expectation value of a scalar field can be understood as the expectation value of an element in the center of the weak closure of a class of representations of the field algebra.[1] The possibility of extending this argument to gauge field theories is discussed shortly.

In the second part of this chapter, we try to exploit this algebraic information about the vacuum expectation value in an unconventional attempt of model building. Our discussion tries to parallel Haag's treatment of the Bardeen-Cooper-Schrieffer model [48] in the case of a gauge field theory. We caution the reader that this last part is speculative.

1. The vacuum expectation value for scalar field theories

In this section, we show that in a certain class of representations, the vacuum expectation value corresponds to an element of the center of the field algebra $\mathcal{A}$. The methods we employ are essentially algebraic. We start from the smeared Wightman fields $\phi(f)$. We then imagine that a procedure exists[2] by which we can pass from the unbounded operators $\phi(f)$ to bounded ones which for simplicity we denote by the same symbol. This step is necessary for in what follows we will use some mathematical techniques which are rigorously valid for bounded operators only. For all functions f with compact support in the space-time region $\mathcal{O}$, we consider the $\phi(f)$ to be elements of a local algebra $\mathcal{A}(\mathcal{O})$ and the algebra $\mathcal{A}$ is then defined as in the footnote above (1.1.7).

Our result concerns the quantity

$$\hat{\phi}(f) = \lim_{\mathcal{O}\to\infty} \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} d^4x \alpha_x(\phi(f)), \tag{7.1.1}$$

[1] In what follows, we shall speak of the 'center of the field algebra' instead of 'the center of the weak closure of a class of representations of the field algebra', as long as no confusion can arise.

[2] For a discussion about the problems in passing from Wightman fields to Haag-Kastler nets of bounded operators, cf. [23, 16].

where α_x denotes the space-time translation automorphism and $|\mathcal{O}|$ is the volume of the finite space-time region $\mathcal{O}$. This object will be constructed from

$$\tilde{\phi}_V(f) = \frac{1}{V}\int_V d^3\bar{x}\alpha_{\bar{x}}(\phi(f)), \tag{7.1.2}$$

where $\alpha_{\bar{x}}$ is the space translation automorphism and V is a finite space volume. To see that the integral (7.1.2) is well-defined, we mention that the operator norm $\| \alpha_{\bar{x}}(\phi(f)) \|$ is independent of $\bar{x}$ and that this norm is subadditive. Hence the integrand can be bounded from above and (7.1.2) is well-defined. We start with the following

Result:[3] Assume the cluster decomposition property in its weakest form

$$| \langle \Omega, B_1 B_2 \Omega\rangle - \langle \Omega, B_1\Omega\rangle\langle\Omega, B_2\Omega\rangle | \leq c\frac{1}{\tau^2}, \tag{7.1.3}$$

where B_1, B_2 are bounded local operators, localized in space-time regions $\mathcal{O}_1, \mathcal{O}_2$, τ denotes the spacelike distance between $\mathcal{O}_1$ and $\mathcal{O}_2$, and Ω denotes the vacuum.
Given an irreducible representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$ and a vacuum state $\Omega \in \mathcal{H}$.

1. $\tilde{\phi}(f) = \mathrm{w} - \lim_{V\to\infty} \tilde{\phi}_V(f)$ exists.[4]
2. $\tilde{\phi}(f)$ is a c-number.
3. $$\tilde{\phi}(f) = \hat{\phi}(f). \tag{7.1.4}$$

Argument:
The set $\pi(\mathcal{A})\Omega$ is dense in $\mathcal{H}$. We choose arbitrary vectors $\psi_A = \pi(A)\Omega$, $\psi_B = \pi(B)\Omega$, where $A, B \in \mathcal{A}$ are observables with bounded localization in space-time. Then we obtain

$$\begin{aligned}\langle\psi_A, \tilde{\phi}_V\psi_B\rangle &= \frac{1}{V}\int_V d^3\bar{x}\langle\psi_A, \pi(\alpha_{\bar{x}}(\phi(f)))\psi_B\rangle \\ &= \frac{1}{V}\int_V d^3\bar{x}\langle\Omega, \pi(A^*)[\pi(\alpha_{\bar{x}}(\phi(f))), \pi(B)]\Omega\rangle \\ &+\frac{1}{V}\int_V d^3\bar{x}\langle\Omega, \pi(A^*)\pi(B)\pi(\alpha_{\bar{x}}(\phi(f)))\Omega\rangle.\end{aligned} \tag{7.1.5}$$

[3] Though we have no doubt that both this statement and our argument is known to a group of mathematical physicists, we present it in detail since we did not find it in the literature.
[4] Here, w − lim denotes the limit in the weak topology defined by the $\langle .,.\rangle$- seminorms. As long as it is not stated otherwise, all limits used in this chapter are limits with respect to the weak topology.

Our aim is to give estimates for the two terms in (7.1.5). In the first term of (7.1.5), the commutator is only non-zero, if $\alpha_{\bar{x}}(\phi(f))$ lies in the causal shadow of B.[5] This is only the case for a finite volume V_f and hence we write

$$| \frac{1}{V}\int_V d^3\bar{x}\langle\Omega, \pi(A^*)[\pi(\alpha_{\bar{x}}(\phi(f))), \pi(B)]\Omega\rangle | \leq \frac{1}{V}c_1 V_f =: C_1(V), \qquad (7.1.6)$$

where c_1 is a finite real constant. To reformulate the second term in (7.1.5), we use the cluster decomposition property (7.1.3) in the form: $|\langle\Omega, \pi(A_1)\pi(\alpha_{\bar{x}}(A_2))\Omega\rangle - \langle\Omega, \pi(A_1)\Omega\rangle\langle\Omega, \pi(A_2)\Omega\rangle|$ $< c_2\frac{1}{|\bar{x}|^2}$ for $A_1, A_2 \in \mathcal{A}$ and c_2 a finite constant. This allows us to write

$$\begin{aligned} &\tfrac{1}{V}\int_V d^3\bar{x}\langle\Omega, \pi(A^*)\pi(B)\pi(\alpha_{\bar{x}}(\phi(f)))\Omega\rangle \\ &= \tfrac{1}{V}\int_V d^3\bar{x}\langle\Omega, \pi(A^*)\pi(B)\Omega\rangle\langle\Omega, \pi(\alpha_{\bar{x}}(\phi(f)))\Omega\rangle + C_2(V). \qquad (7.1.7)\end{aligned}$$

Here $C_2(V)$ is a correction term which we may estimate using the cluster decomposition property and a set of increasing concentric spheres with radii R for the volumes V,

$$| C_2(V) | \leq c_3\frac{1}{R^3} + c_4\frac{1}{R^3}\int_1^R 4\pi r^2\frac{1}{r^2}dr, \qquad (7.1.8)$$

where c_3 and c_4 are finite constants. The first term comes from integrating in (7.1.7) over $| \bar{x} | < 1$, the second term from $| \bar{x} | > 1$. Using the space translation invariance of the vacuum, $\frac{1}{V}\int_V d^3\bar{x}\langle\Omega, \pi(\alpha_{\bar{x}}(\phi(f)))\Omega\rangle = \langle\Omega, \pi(\phi(f))\Omega\rangle$, we obtain from (7.1.5)-(7.1.8)

$$\langle\psi_A, \tilde{\phi}_V\psi_B\rangle = \langle\Omega, \pi(A^*)\pi(B)\Omega\rangle\langle\Omega, \pi(\phi(f))\Omega\rangle + C_1(V) + C_2(V). \qquad (7.1.9)$$

Since the first term in (7.1.9) has no volume dependence, it follows that

$$\begin{aligned} &\lim_{V\to\infty} C_1(V) = 0 \quad ; \quad \lim_{V\to\infty} C_2(V) = 0, \\ &\lim_{V\to\infty}\langle\psi_A, \tilde{\phi}_V(f)\psi_B\rangle = \langle\psi_A, \psi_B\rangle\langle\Omega, \pi(\phi(f))\Omega\rangle, \qquad (7.1.10)\end{aligned}$$

for all ψ_A,ψ_B of a dense subset of $\mathcal{H}$. Hence, $\tilde{\phi}(f)$ exists as a weak limit of $\tilde{\phi}_V(f)$ and $\tilde{\phi}(f) = \langle\Omega, \phi(f)\Omega\rangle \cdot I$, i.e., $\tilde{\phi}(f)$ is a c-number in all irreducible representations. Also, from (7.1.10) and the time translation invariance of the vacuum, it follows that

$$\langle\psi_A, \hat{\phi}(f)\psi_B\rangle = \lim_{T\to\infty}\langle\psi_A, \hat{\phi}_T(f)\psi_B\rangle = \langle\psi_A, \tilde{\phi}(f)\psi_B\rangle, \qquad (7.1.11)$$

where we have used $\hat{\phi}_T(f) = \frac{1}{2T}\int_{-T}^{T} dt\alpha_t(\tilde{\phi}(f))$ for finite T.

[5] The causal shadow of an element B of the local field algebra $\mathcal{A}$ is the unification of the forward and backward cones of those points in space-time in which B is localized.

In irreducible representations $(\mathcal{H}, \pi)$, $\hat{\phi}(f)$ is a complex-valued constant. For a more general class of representations, it is an element in the center of the bicommutant of $\pi(\mathcal{A})$ as can be seen from the following

Result: Assume the cluster decomposition property in its weakest form (7.1.3). Given a direct integral representation[6] $(\mathcal{H}, \pi)$ of $\mathcal{A}$ with a vacuum state $\Omega \in \mathcal{H}$.

1. $\hat{\phi}(f)$ exists as a weak limit.
2. $\hat{\phi}(f) \in \mathcal{Z}(\pi(\mathcal{A})'')$, the center of the bicommutant of $\pi(\mathcal{A})$.[7]

Argument:

For direct integral representations (cf. (7.1.13)), the counterpart of (7.1.5) reads

$$\begin{aligned}\langle\psi_A, \check{\phi}_V(f)\psi_B\rangle = &\int_\Lambda d\mu(\lambda)\langle\psi(\lambda), \pi_\lambda(A^*)\pi_\lambda(B)\psi(\lambda)\rangle\langle\psi(\lambda), \pi_\lambda(\phi(f))\psi(\lambda)\rangle \\ &+\int_\Lambda d\mu(\lambda)\langle\psi(\lambda), \psi(\lambda)\rangle(C_1(V) + C_2(V))\end{aligned} \tag{7.1.14}$$

Using this expression, the argument following (7.1.9) goes through with obvious modifications. We obtain $\langle\psi_A, \hat{\phi}(f)\psi_B\rangle = \langle\psi_A, \check{\phi}(f)\psi_B\rangle$ and

$$\hat{\phi}(f) = \int_\Lambda d\mu(\lambda)\langle\psi(\lambda), \pi_\lambda(\phi(f))\psi(\lambda)\rangle I_\lambda, \tag{7.1.15}$$

where I_λ is the identity on the subspace $\mathcal{H}_\lambda$. (7.1.15) lies in the commutant of all decomposable operators. Hence it lies in the center of the bicommutant of $\pi(\mathcal{A})$ by the von Neumann bicommutant theorem [53].

[6] We consider direct integral representations $(\mathcal{H}, \pi)$ of the algebra $\mathcal{A}$ with

$$\begin{aligned}\mathcal{H} &= \int_\Lambda d\mu(\lambda)\mathcal{H}_\lambda, \\ \pi &= \int_\Lambda d\mu(\lambda)\pi_\lambda,\end{aligned} \tag{7.1.12}$$

obtained from the cyclic translationally invariant state

$$\psi = \int_\Lambda d\mu(\lambda)\psi(\lambda).$$

Here, the integrals over Λ are direct integrals (i.e. generalizations of direct sums), Λ is a compact measure space with Borel measure μ, $\psi(\lambda)$ is a vacuum vector of the irreducible space $\mathcal{H}_\lambda$ for $\pi_\lambda(\mathcal{A})$ and the scalar product on $\mathcal{H}$ is defined as

$$\langle\psi_A, \pi(D)\psi_B\rangle = \int_\Lambda d\mu(\lambda)\langle\psi_A(\lambda), \pi_\lambda(D)\psi_B(\lambda)\rangle. \tag{7.1.13}$$

[7] The bicommutant of $\pi(\mathcal{A})$ is the set of all elements of $\mathcal{B}(\mathcal{H})$ which commute with the commutant $\pi(\mathcal{A})'$ of $\pi(\mathcal{A})$.

Gauge-invariant order parameters

In this section, we briefly discuss the question whether a similar algebraic characterization can be obtained for order parameters in a gauge field theory. The problem is that the vacuum expectation value $\hat{\phi}(f)$ in (7.1.1) does not reveal the phase structure of the theory if $\phi(x)$ is coupled to a gauge field. In this case, $\hat{\phi}(f)$ is gauge-dependent, and a different, gauge-invariant order parameter has to be used. In general, such order parameters are (volume averages over) non-local objects (e.g. two point functions [44, 45]), made gauge-invariant by gauge strings.

Here, we draw attention to an order parameter introduced by Palma [74] in his analysis of a $SU(2)$-Higgs model on a 4-dimensional Euclidean lattice Λ. This order parameter may be written as the $|\Lambda| \to \infty$ (infinite volume) limit Φ of a block spin variable

$$\Phi(x) = a^4 \sum_{z\in\Lambda} C(x,z)\phi(z) \qquad ; \qquad C(x,z) = \sum_{\omega : z\to x} \rho(\omega) U(\omega), \tag{7.1.16}$$

where $U(\omega)$ is the parallel transporter (gauge string) along the path ω from z to some arbitrarily chosen block center x and $\rho(\omega)$ is a properly normalized weight for each path ω.

Our motivation for mentioning (7.1.16) is two-fold: Firstly, Palma calculates Φ perturbatively to $O(g^2)$ in the gauge coupling [74]. This calculation shows that to arbitrary order in g, for each additional summation $\sum_{z\in\Lambda}$ over lattice fields (defined on a single point of the lattice), there comes exactly one damping factor $\frac{1}{|\Lambda|}$. Assuming that this structure persists in the corresponding continuum QFT, arguments similar to those employed in the case of a scalar QFT may be applied to show that Φ lies in the center of the field algebra. The second reason for mentioning the order parameter Φ is that in the analysis of non-local interaction terms given in the next section, certain objects involving gauge strings are found to lie in the center of the field algebra. Since that analysis does not deal properly with certain technical difficulties, it is reassuring that similar mathematical features arise in other approaches too.

2. Using central elements to simplify interaction terms

In this section, we firstly recall Haag's discussion of the BCS-model [48]. This uses the fact that certain polynomials in the fields lie in the center of the field algebra in order to simplify a certain interaction term. Then, we try to parallel his approach for gauge field theories by choosing an unconventional interaction term which at least formally allows a similar treatment.

Haag's discussion of the BCS-model

Haag's investigation concerns the Hamiltonian[8][48] $K = K_0 + K_I$. This is the infinite volume limit of the Hamiltonians $K_0(V)$ and $K_I(V)$, defined for finite volume V:

$$K_0(V) = \sum_{\alpha=1,2} \int_V d^3x \psi_\alpha^*(x)[\frac{1}{2m}\hat{p}^2 - \mu]\psi_\alpha(x),$$

$$K_I(V) = \frac{1}{V}\int \psi_1^*(x)\psi_2^*(x+z)v(z,z')\psi_2(x'+z')\psi_1(x')d^3xd^3x'd^3zd^3z'. \tag{7.2.1}$$

Here, ψ_α, ψ_α^*[9] denote annihilation and creation operators for electron states with spin up ($\alpha = 1$) or down ($\alpha = 2$), satisfying time zero anticommutation relations,

$$[\psi_\alpha(x), \psi_\beta(y)]_+ = 0 \qquad ; \qquad [\psi_\alpha(x), \psi_\beta^*(y)]_+ = \delta_{\alpha\beta}\delta^{(3)}(x-y). \tag{7.2.2}$$

$\hat{p}$ is the three-dimensional momentum operator in the presence of a magnetic field and the chemical potential μ is a free parameter of the theory. The function $v(z,z')$, which characterizes the attractive interaction, is assumed to satisfy $\int | v(z,z') | d^3zd^3z' < \infty$. The remark of Haag on which we focus in what follows is that

$$\Delta(z) = \lim_{V\to\infty} \frac{1}{V}\int\int v(z,z')\psi_2(x'+z')\psi_1(x')d^3x'd^3z' \tag{7.2.3}$$

lies in the center of the field algebra and is a c-number in all irreducible representations. This allows us to simplify $K_I(V)$ by working out the commutators of K_I with $\psi(f)$ and $\psi^*(f)$:

$$\lim_{V\to\infty} [K_I(V), \psi_1(y)] = -\int \Delta(z)\psi_2^*(y+z)dz. \tag{7.2.4}$$

Similarly, the commutators of K_I with ψ_2, ψ_1^* and ψ_2^* are always linear in ψ or ψ^*. Hence, in all representations, K_I can be replaced by the 'dynamically equivalent' expression

$$K'_I = \int [\Delta(z)\psi_1^*(x)\psi_2^*(x+z) + \Delta^*(z)\psi_2(x+z)\psi_1(x)]d^3xd^3z, \tag{7.2.5}$$

which leads to the same commutators with arbitrary fields and hence to the same time evolution. The Fourier transform of this expression reads

$$K'_I = \frac{1}{(2\pi)^3}\int [\Delta(p)\psi_1^*(-p)\psi_2^*(p) + \Delta^*(p)\psi_2(p)\psi_1(-p)]d^3p. \tag{7.2.6}$$

[8]Here, we do not discuss the question whether this is a realistic model of superconductivity. Clearly, the interaction term (7.2.1) shows a global $U(1)$ gauge invariance while the local gauge invariance is broken explicitly. We shall be interested in the mathematical properties of this model only.

[9]The Fermi fields

$$\psi(f) = \int \psi(x)f(x)d^3x,$$

smeared with square integrable test functions f, are bounded operators. Hence, in contrast to (7.1.1), no additional assumption is needed.

Remarks:

- (7.2.1) is invariant under the global gauge transformations

$$\begin{aligned}\psi &\longrightarrow e^{i\alpha}\psi,\\ \psi^* &\longrightarrow e^{-i\alpha}\psi^*.\end{aligned}\tag{7.2.7}$$

  But in all irreducible representations, this gauge automorphism is not implemented, as may be seen by taking in (7.2.6) $\triangle(p)$, $\triangle(p)^*$ to be constants. The gauge symmetry is broken by the specification of an irreducible representation.

- Despite the breaking of the global gauge symmetry, no Goldstone bosons arise. The reason is that for non-local interactions as (7.2.1), Goldstone's theorem does not apply [47].

Experimenting with non-local interaction terms

In the last sections, we have observed that

1. vacuum expectation values correspond to elements in the center of the field algebra in scalar field theories as well as in gauge field theories.

2. elements in the center of the field algebra can arise in a particular way of simplifying (non-local) globally gauge-invariant interaction terms. This may be symptomatic of spontaneous symmetry breaking.

This has motivated us to seek locally gauge-invariant expressions which allow for a similar treatment. Here, we present an unconventional, locally gauge-invariant 'interaction term', built up from the fields of the Glashow-Salam-Weinberg (GSW) standard model. A formal application of Haag's analysis reveals spontaneous symmetry breaking and leads to a 'dynamically equivalent' interaction term which resembles the mass terms of the GSW-model. Because of the tentative character of these results, we hasten to remark that our discussion has serious deficiencies:

1. the interaction term is non-local and will be difficult to reconcile with the locality requirements of a relativistic theory. Also, it is non-renormalizable.

2. from the mathematical viewpoint, our calculation does not survive the smearing of the fields with test functions.

3. the chosen interaction term has no obvious physical motivation.

Despite these obvious drawbacks, we consider our attempt interesting enough for a short presentation on the next few pages.

We consider the interaction term H_I of a Hamiltonian $H = H_0 + H_I$, where H_0 denotes the massless part of the Hamiltonian of the Glashow-Weinberg standard model, $H_0 = \int d^3x\mathcal{H}_0$,

$$\mathcal{H}_0 = \frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\ell}_L i\gamma_\mu D^\mu_L \ell_L + \bar{e}_R i\gamma_\mu D^\mu_R e_R$$
$$+\bar{q}_L i\gamma_\mu D^\mu_L q_L + \bar{u}_R i\gamma_\mu D^\mu_R u_R + \bar{d}_R i\gamma_\mu D^\mu_R d_R. \qquad (7.2.8)$$

We restrict ourselves to one lepton family only with up and down quarks u and d, electrons e and electron neutrinos ν. The subscripts L and R denote the left- and right-handed parts. The left-handed doublets $\ell_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$, $q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$ transform under $SU(2)\times U(1)$, which is reflected in their covariant derivative $D_{L\mu} = \partial_\mu + igA^a_\mu \frac{1}{2}\sigma^a + ig'\frac{Y}{2}B_\mu$. The Pauli matrices σ^a generate $SU(2)$ transformations, g and g' are the gauge couplings and Y is the hypercharge. The $U(1)$-field B_μ corresponds to the field tensor $F_{\mu\nu}$ while the A^a_μ correspond to the field tensors $G^a_{\mu\nu}$. The covariant derivative for the right-handed singlets, transforming under $U(1)$, is given by $D_{R\mu} = \partial_\mu + ig'\frac{Y}{2}B_\mu$. For notational convencience, we introduce the gauge field combinations

$$C_\mu = A^a_\mu \frac{1}{2}\sigma^a + \frac{g'}{g}\frac{Y}{2}B_\mu. \qquad (7.2.9)$$

Furthermore, we introduce the right-handed doublet $\ell_R = \begin{pmatrix} \nu_R \\ e_R \end{pmatrix}$ which transforms under $U(1)$ only, the fermion bilinears $\phi = \begin{pmatrix} \bar{\nu}_L \nu_R \\ \bar{e}_L e_R \end{pmatrix}$, and the gauge-invariant string

$$S(x, x+R) = \bar{\phi}(x)\hat{P}e^{[ig\int_{\Gamma_{x,x+\vec{R}}} C_\mu(z(s))\frac{dz(s)^\mu}{ds}ds]}\phi(x+\vec{R}). \qquad (7.2.10)$$

Here, $\vec{R}$ is a 3-dimensional vector, $\hat{P}$ denotes the path-ordering operator and the exponent is integrated along the straight line from x to $x + \vec{R}$, excluding the endpoints. H_I is now given as the isotropic $N \to \infty$-limit $H_I = \lim_{N\to\infty} H_I(N)$ of the finite volume average over string fields S in N different directions $\hat{e}_k$,

$$H_I(N) = -c\frac{1}{N}\sum_{k=1}^{N}\int_V [S(x, x + R\hat{e}_k) + S^\dagger(x, x + R\hat{e}_k)]d^3x. \qquad (7.2.11)$$

Here, c is a coupling constant, the strings S are of a fixed finite length R and the unit vectors

$$\hat{e}_k(\theta_k, \varphi_k) = \begin{pmatrix} \sin\theta_k \cos\varphi_k \\ \sin\theta_k \sin\varphi_k \\ \cos\theta_k \end{pmatrix}$$

are parametrized by pairs of spherical coordinates (θ_k, φ_k). The infinite volume limit of $H_I(N)$ plays no important rôle in what follows.

The analysis

In this subsection, we want to show that a formal application of Haag's analysis leads to a 'dynamically equivalent' local Hamiltonian for (7.2.11). The analysis of the interaction term H_I is carried out in the temporal gauge, and we use the following equal time commutation and anticommutation relations:

$$[e_i(x), \bar{e}_j(y)]_+ = \gamma_0 \delta_{ij} \delta^{(3)}(x-y), \tag{7.2.12}$$

where γ_0 denotes a Dirac matrix, $i, j = L, R$ and all other anti-commutators between two Fermi-fields vanish and

$$[A^a_\mu(x), \dot{A}^b_\nu(y)] = -i\delta_{ab} g_{\mu\nu} \delta^{(3)}(x-y) \qquad ; \qquad [B_\mu(x), \dot{B}_\nu(y)] = -i g_{\mu\nu} \delta^{(3)}(x-y), \tag{7.2.13}$$

where $g_{\mu\nu} = (1, -1, -1, -1)$. All other commutators with at least one gauge field vanish. We rely on the formal Taylor expansion of a path-ordered product in orders of g:

$$\begin{aligned} &\hat{P} \exp[ig \int_{\Gamma_{x,x+R\hat{e}}} C_\mu(z(s)) \frac{dz^\mu(s)}{ds} ds] \\ &= \sum_{n=0}^{\infty} (ig)^n \int_0^1 ds_1 \int_0^{s_1} ds_2 ... \int_0^{s_{n-1}} ds_n C_\mu(z(s_1)) \frac{dz^\mu(s_1)}{ds_1} ... C_\nu(z(s_n)) \frac{dz^\nu(s_n)}{ds_n}, \end{aligned} \tag{7.2.14}$$

where the path $\Gamma_{x,x+R\hat{e}}$, $z(s) \in \Gamma_{x,x+R\hat{e}}$, has been parametrized by $s \in [0,1]$. For simplicity, we discuss H_I with the additional constraint $\nu_R = 0$, i.e. we consider strings with fermion bilinears $\phi = \bar{e}_L e_R \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Our analysis proceeds by working out the commutators of H_I and then replacing H_I by a Hamiltonian $H_I{}'$ which is dynamically equivalent in the sense that it leads to the same commutators. We start by investigating the commutator of $H_I(N)$ with an arbitrary fermion field $e_L(y)$ say, defined at the point $y \in V$,

$$\begin{aligned} &[H_I(N), e_L(y)] \\ &= -c\frac{1}{N} \sum_{k=1}^{N} \gamma_0 e_R(y)(0,1) \hat{P} e^{[ig\int_{y,y+R\hat{e}_k} C_\mu(z) dz^\mu]} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{e}_R(y+R\hat{e}_k) e_L(y+R\hat{e}_k) \\ &-c\frac{1}{N} \sum_{k=1}^{N} \gamma_0 e_R(y)(0,1) \hat{P} e^{[-ig\int_{y-R\hat{e}_k,y} C_\mu(z) dz^\mu]} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{e}_R(y-R\hat{e}_k) e_L(y-R\hat{e}_k). \end{aligned} \tag{7.2.15}$$

Consider now an arbitrary field $F(y')$, defined at a point $y' \in V$, $y' \neq y$. We obtain

$$\lim_{N\to\infty} [[H_I(N), e_L(y)], F(y')] = 0, \tag{7.2.16}$$

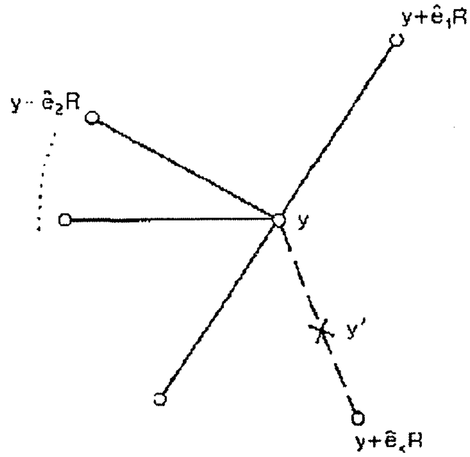

Figure 4: Depiction of the N strings contributing to the commutator $[H_I(N), e_L(y)]$. Only the k-th string denoted by a dashed line, contributes to the double commutator $[[H_I(N), \dot{B}_\nu(y)], F(y')]$.

since the two points y, y' determine a direction in V and only strings passing through both points can contribute to the double commutator (7.2.16). Their contribution, however, is multiplied by $\frac{1}{N}$ and vanishes in the $N \to \infty$-limit. This implies that the double commutator behaves like a commutator of $F(y')$ with $\gamma_0 e_R(y)$. Hence, in all irreducible representations

$$\lim_{N\to\infty} [H_I(N), e_L(y)] = c_e \gamma_0 e_R(y), \tag{7.2.17}$$

where c_e is a c-number. Similarly, we work out the commutators of $H_I(N)$ with $\overline{e}_L(y)$, $e_R(y)$ and $\overline{e}_R(y)$. In the $N \to \infty$-limit, the same commutators can be obtained from

$$H'_{I,e} = \int d^3y \{c_e \overline{e}_L(y) e_R(y) + c_e^* \overline{e}_R(y) e_L(y)\}. \tag{7.2.18}$$

This calculation can be understood in a simple pictorial way (cf. Figure 4 where we denote the sum of the strings $S_i(x, x + R\hat{e}_k) + S_i^\dagger(x, x + R\hat{e}_k)$ by a straight line between the points x and $x + R\hat{e}_k$). A similar analysis is possible for the commutators and double commutators of $H_I(N)$ with the gauge fields. Details of this calculation are given in Appendix A. Here, we mention just that a similar pictorial representation of the calculation exists (cf. Figure 5). One concludes that in all irreducible representations, the Hamiltonian

$$H_{IG}'(N) = -\frac{1}{N}\sum_{k=1}^{N} \int_V d^3y \sum_{n=1}^{\infty} i^n g^n c_n(R)(0,1)[C_i(y)\hat{e}_k^i]^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7.2.19}$$

leads to identical commutators for all gauge fields in the isotropic $N \to \infty$-limit. Here, the $c_n(R)$ are c-numbers corresponding to the elements of the center of the field algebra. Obviously, H'_{IG} and $H'_{I,e}$ denote local interaction terms.

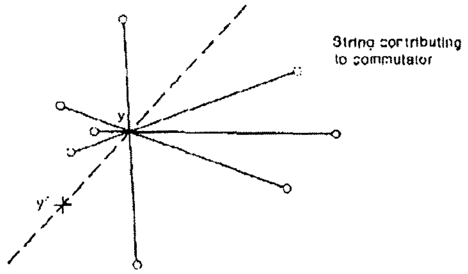

Figure 5: Some of the strings contributing to $[H_I(N), \dot{B}_\nu]$. Only strings on the dashed line contribute to $[[H_I(N), \dot{B}_\nu], F(y')]$.

Remark:[10]

- The interaction terms H'_{IG} and $H'_{I,\epsilon}$ may be seen to formally resemble to $O(g^4)$ the mass terms of the GSW-standard model. Namely, as shown in Appendix B $H'_{IG} = \lim_{N\to\infty} H'_{IG}(N)$ can be written as

$$H'_{IG} = -\sum_{n=1}^{\infty}(-1)^n g^{2n} c_{2n}(R) \frac{8\pi^2}{2n+1} \int d^3y (0,1)\left(C_1^2(y) + C_2^2(y) + C_3^2(y)\right)^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (7.2.20)$$

A Lorentz-invariant formulation which reduces to (7.2.20) in the temporal gauge $C_0(x) = 0$, can be obtained by substituting

$$C_1^2(y) + C_2^2(y) + C_3^2(y) \longrightarrow -C_\mu(y)C^\mu(y). \quad (7.2.21)$$

With this substitution, let us examine expression (7.2.20) to $O(g^4)$. We start from

$$H_{IG}'(R) = -g^2 c_2(R) \frac{8\pi^2}{5} \int_V (0,1) C_\mu(y) C^\mu(y) \begin{pmatrix} 0 \\ 1 \end{pmatrix} d^3y + O(g^4). \quad (7.2.22)$$

Using (7.2.9), the integrand of this expression can be written in the following form:

$$\begin{aligned} &-g^2\, c_2(R)\, (0,1) C_\mu(y) C^\mu(y) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \; -g^2 c_2(R)[A^1_\mu(y)A^{1\mu}(y) + A^2_\mu(y)A^{2\mu}(y) + A^3_\mu(y)A^{3\mu}(y) \\ &\quad -\frac{g'}{g} Y(A^3_\mu(y)B^\mu(y) + B_\mu(y)A^{3\mu}(y)) + \frac{g'^2}{g^2} Y^2 B_\mu(y) B^\mu(y)]. \end{aligned} \quad (7.2.23)$$

[10] The tentative calculations presented in this subsection may lead to some speculation about a mechanism of mass generation without additional scalar fields. We restrict ourselves to the following remark: due to Gauss' law, operators creating charged fields cannot have local support. In fact, they are expected to have string-like structures or to be localized in spacelike cones [23]. In a heuristic sense, a string (7.2.10), applied to the vacuum, creates a charged particle together with its antiparticle at spacelike separation $\vec{R}$, both connected by a flux line. If one wants to stick to a formulation in terms of local fields but incorporate the intrinsic non-locality of Gaussian charges nonetheless, one may have the heuristic idea to make an energetically favourable field configuration out of string-like fields by adding (7.2.11) to the Hamiltonian H_0. Roughly speaking, this should imply that the local fermion and gauge boson fields evolve dynamically in a potential which glues them together. At an early stage of this work, the appearance of 'mass terms' in such a formal attempt has led us to speculations whether the Higgs field in the GSW-standard model may be seen as a purely technical tool, compensating for the non-locality of the charges in this theory and not being associated with a physical particle.

For $Y = 1$, the hypercharge of the composite ϕ, this expression denotes nothing but the gauge boson mass terms in the electroweak theory. Defining

$$W_\mu^+ = \frac{1}{\sqrt{2}}\left(A_\mu^1 - iA_\mu^2\right) \quad ; \quad W_\mu^- = \frac{1}{\sqrt{2}}\left(A_\mu^1 + iA_\mu^2\right),$$
$$Z_\mu = A_\mu^3 \cos\theta_W - B_\mu \sin\theta_W, \quad ; \quad \tilde{A}_\mu = A_\mu^3 \sin\theta_W + B_\mu \cos\theta_W. \quad (7.2.24)$$

with the Weinberg angle θ_W, $\tan\theta_W = \frac{g'}{g}$, we obtain the physical fields with mass terms:

$$M_W^2 = \frac{16\pi^2}{5} g^2 c_2(R) \quad ; \quad M_Z^2 = \frac{16\pi^2}{5} c_2(R)\left(g^2 + g'^2\right) \quad ; \quad M_A^2 = 0. \quad (7.2.25)$$

Similarly, (7.2.18) resembles a fermion mass term $m_e^2 = 2c_e$. Here, the central elements c_e, $c_2(R)$ formally play the rôle which in the GSW-standard model is played by the vacuum expectation value of the Higgs field.

- If one wants to stretch the formal analogy , one may even obtain 'quark mass terms' by introducing an interaction term (7.2.11) with bilinears $\phi' = \begin{pmatrix} \overline{u}_L u_R \\ \overline{d}_L d_R \end{pmatrix}$.

Appendix A: Commutators and double commutators of $H_I(N)$ with gauge fields

We start with an investigation of the commutator $[H_I(N), \dot{B}_\nu(y)]$. For notational convenience, we introduce $H_I(\hat{e}_k)$ and $S_i^{(n)}(x, x + R\hat{e}_k)$, defined by

$$H_I(N) = \frac{1}{N}\sum_{k=1}^{N} H_I(\hat{e}_k);$$
$$S_i(x, x + R\hat{e}_k) = \sum\nolimits_n (ig)^n S_i^{(n)}(x, x + R\hat{e}_k). \quad (7.2.26)$$

In a first step, we compute $[H_I(V, \hat{e}_1), \dot{B}_\nu(y)]$ up to $O(g^2)$ for a special spatial direction $\hat{e}_1 = (1, 0, 0)$ say. Specifying $\hat{e}_1$, we are able to simplify the directed path-ordered exponential:

$$\int d^3x S(x, x + R\hat{e}_1) = \int d^3x \sum_{n=0}^{\infty} (ig)^n \int_0^R dR_1 \int_0^{R_1} dR_2 ... \int_0^{R_{n-1}} dR_n$$
$$\times \overline{\phi}(x) C_1(x + R_1\hat{e}_1)\hat{e}_1 ... C_1(x + R_n\hat{e}_1)\hat{e}_1 \phi(x + R\hat{e}_1)$$
$$= \int d^3x \sum_{n=0}^{\infty} (ig)^n S^{(n)}(x, x + R\hat{e}_1), \quad (7.2.27)$$

where the subscript i on $S_i^{(n)}$ has been omitted for notational convenience. Using(7.2.9) and the commutation relations (7.2.12), (7.2.13), this allows us to calculate

$$\int d^3x [S^{(0)}(x, x + R\hat{e}_1), \dot{B}_\nu(y)] = 0, \quad (7.2.28)$$

$$\begin{aligned}
&\int d^3x\, (ig)[S^{(1)}(x, x + R\hat{e}_1), \dot{B}_\nu(y)] \\
&= (ig) \int d^3x \int_0^R dR_1 \frac{g'}{g}\frac{Y}{2}\overline{\phi}_2(x)[B_1(x, x + R_1\hat{e}_1)\hat{e}_1, \dot{B}_\nu(y)]\phi_2(x + R\hat{e}_1) \\
&= g \int d^3x \int_0^R dR_1 \frac{g'}{g}\frac{Y}{2}\hat{e}_1\delta_{1\mu}\delta(x^1 + R\hat{e}_1 - y^1)\delta(x^2 - y^2)\delta(x^3 - y^3)\overline{\phi}_2(x)\phi_2(x + R\hat{e}_1) \\
&= g\delta_{1\mu}\frac{g'}{g}\frac{Y}{2}\hat{e}_1 \int_0^R dR_1 \overline{\phi}_2(y - R_1\hat{e}_1)\phi_2(y + (R - R_1)\hat{e}_1) \qquad (7.2.29)
\end{aligned}$$

and

$$\begin{aligned}
&(ig)^2 \int d^3x [S^{(2)}(x, x + R\hat{e}_1), \dot{B}_\nu(y)] \\
&= (ig)^2 \int d^3x \int_0^R dR_1 \int_0^{R_1} dR_2 \overline{\phi}_2(x)[C_1(x + R_1\hat{e}_1)\hat{e}_1 C_1(x + R_2\hat{e}_1)\hat{e}_1, \dot{B}_\nu(y)]\phi_2(x + R\hat{e}_1) \\
&= g^2 \frac{g' i}{g}\frac{Y}{2}\hat{e}_1 \int_0^R dR_1 \int_0^{R_1} dR_2 \{\overline{\phi}_2(y - R_2\hat{e}_1)C_1(y + (R_1 - R_2)\hat{e}_1)\hat{e}_1\phi_2(y + (R - R_2)\hat{e}_1) \\
&+ \overline{\phi}_2(y - R_1\hat{e}_1)C_1(y + (R_2 - R_1)\hat{e}_1)\hat{e}_1\phi_2(y + (R - R_1)\hat{e}_1)\}. \qquad (7.2.30)
\end{aligned}$$

The commutator $[H_I(N), \dot{B}_\nu(y)]$ is given to $O(g^2)$ as the sum of the terms (7.2.28), (7.2.29), (7.2.30) and the contributions corresponding to the second term in (7.2.11), summed over the N different spatial directions $\hat{e}_k$. Similarly to (7.2.16), we obtain for $y' \neq y$

$$\lim_{N\to\infty} [[H_I(N), \dot{B}_\nu(y)], F(y')] = 0. \qquad (7.2.31)$$

The reason is that the points y and y' specify one direction $\hat{e}_{yy'}$ in space. Hence, the double commutator (7.2.31) has to vanish since it receives contributions from $H_I(\hat{e}_{yy'})$ only, but includes a damping factor $\frac{1}{N}$. This holds true to all orders. Here, in all irreducible representations, $[H_I(N), \dot{B}_\nu(y)]$ can be written for any order $O(g^n)$ as a function of fields at the point y times a c-number. Similar results can be obtained for the commutator of $H_I(N)$ with $\dot{A}^a_\mu(y)$. We conclude that in the $N \to \infty$-limit of all irreducible representations, the Hamiltonian

$$H_{IG}'(N) = -\frac{1}{N}\sum_{k=1}^{N} \int_V d^3y \sum_{n=1}^{\infty} i^n g^n c_n(R)(0,1)[C_i(y)\hat{e}_k^i]^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (7.2.32)$$

is dynamically equivalent to $H_I(N)$ with respect to the gauge fields. The $c_n(R)$ are c-numbers corresponding to elements of the center of the field algebra.

Appendix B: The isotropic $N \to \infty$-limit

Here, we calculate the $N \to \infty$-limit of $H'_{IG}(N)$ where the countably many directions $\hat{e}_k$ are chosen so that $H'_{IG} = \lim_{N\to\infty} H'_{IG}(N)$ becomes an isotropic expression. Considering the $C_i(y)$ as coefficients of the unit vectors $\hat{e}^i_k$ in (7.2.32), we calculate H'_{IG} by substituting in (7.2.32) the sum over the N directions by an integral over the 2-sphere:

$$\frac{1}{N}\sum_{k=1}^{N} \longrightarrow \int_0^{\pi}\int_0^{2\pi} 2\pi \sin\theta d\theta d\varphi. \tag{7.2.33}$$

Parametrizing the unit vectors $\hat{e}_k$ by spherical coordinates, we obtain

$$\begin{aligned}
&C_i(y)\hat{e}^i_k \longrightarrow \hat{C}(y),\\
&\hat{C}(y) = C_1(y)\sin\theta\sin\varphi + C_2(y)\sin\theta\cos\varphi + C_3(y)\cos\theta,\\
&\frac{1}{N}\sum_{k=1}^{N}[C_i(y)\hat{e}^i_k]^n \longrightarrow \int_0^{\pi}\int_0^{2\pi}\hat{C}^n(y)2\pi\sin\theta d\theta d\varphi,
\end{aligned} \tag{7.2.34}$$

which we substitute in (7.2.32). Using the formulae

$$\begin{aligned}
\int_0^{\frac{\pi}{2}}\sin^{2m}x dx = \int_0^{\frac{\pi}{2}}\cos^{2m}x dx &= \frac{(2m-1)!!}{(2m)!!}\frac{\pi}{2},\\
\int_0^{\frac{\pi}{2}}\sin^{2m+1}x dx = \int_0^{\frac{\pi}{2}}\cos^{2m+1}x dx &= \frac{(2m)!!}{(2m+1)!!},
\end{aligned} \tag{7.2.35}$$

a lengthy but straightforward calculation shows that

$$\begin{aligned}
\int_0^{\pi}\int_0^{2\pi}\hat{C}^{2n}(y)2\pi\sin\theta d\theta d\varphi &= \sum_{(k,r,s)}\frac{(2n)!}{(2k)!(2r)!(2s)!}8\pi^2 C_1^{2k}(y)C_2^{2r}(y)C_3^{2s}(y)\times\\
&\left(\sum_{l=0}^{s}\binom{s}{l}(-1)^l\frac{(2(k+r+l))!!}{(2(k+r+l)+1)!!}\right)\left(\sum_{l'=0}^{r}\binom{r}{l'}(-1)^{l'}\frac{(2(k+l')-1)!!}{(2(k+l'))!!}\right),
\end{aligned} \tag{7.2.36}$$

whereas the odd powers of $\hat{C}(y)$ vanish. Here, $\sum_{(k,r,s)}$ goes over triples (k,r,s) such that $n = k+r+s$. To simplify this expression, one proves by induction that

$$\begin{aligned}
\sum_{l=0}^{r}\binom{r}{l}(-1)^l\frac{(2(k+l)-1)!!}{(2(k+l))!!} &= \frac{(2k-1)!!}{(2(k+r))!!}(2r-1)!!,\\
\sum_{l=0}^{s}\binom{s}{l}(-1)^l\frac{(2(k+l))!!}{(2(k+l)+1)!!} &= \frac{(2k)!!}{(2(k+s)+1)!!}(2s-1)!!.
\end{aligned} \tag{7.2.37}$$

Inserting (7.2.37), one obtains

$$\int_0^{\pi}\int_0^{2\pi}\hat{C}^{2n}(y)2\pi\sin\theta d\theta d\varphi = \frac{8\pi^2}{2n+1}\left(C_1^2(y)+C_2^2(y)+C_3^2(y)\right)^n. \tag{7.2.38}$$

Hence, the isotropic $N\to\infty$-limit H'_{IG} of $H'_{IG}(N)$ is (7.2.20).

References

[1] N.P. Landsman and U.A. Wiedemann . Massless Particles, Electromagnestism and Rieffel Induction. *DESY-94-109, submitted to Rev. Math. Phys.*

[2] U.A. Wiedemann. Non-differentiability of the effective potential . *Nucl. Phys.*, B 406:808–822, 1993.

[3] Mischa I. Düwermann. *Poems for Tia I. Neuthal.* Molino Press, Ticino, 1992.

[4] C.A. Hurst. Dirac's Theory of Constraints (dedicated to Prof. Walter Thirring on the occasion of his 60th birthday. In H. Mitter and L. Pittner, editors, *Recent Developments in Mathematical Physics*, Heidelberg, 1987. XXVI th Int. Universitätswochen für Kernphysik, Schladming, Austria, Springer Verlag.

[5] K. Sundermeyer. *Constrained Dynamics.* Springer Verlag, Berlin, Heidelberg, New York, 1982.

[6] H.B.G.S. Grundling and C.A. Hurst. Algebraic Quantization of Systems with a Gauge Degeneracy. *Commun. Math. Phys.*, 98:369 – 390, 1985.

[7] H.B.G.S. Grundling and C.A. Hurst . Algebraic Structures of degenerate systems and the indefinite metric. *Jour. Math. Phys.*, 28:559 – 572, 1987.

[8] H.B.G.S. Grundling . Systems with Outer Constraints. Gupta-Bleuler Electromagnetism as an Algebraic Field Theory. *Commun. Math. Phys.*, 114:69 – 91, 1988.

[9] N.P. Landsman. Rieffel induction as generalized quantum Marsden-Weinstein reduction. *Preprint DAMTP-93-22, to appear in J. Geom. Phys.*, 1994.

[10] P.A.M. Dirac. *Lectures on Quantum Mechanics.* Yeshiva University, New York, 1964.

[11] H.B.G.S. Grundling and C.A. Hurst . A note on Regular States and Supplementary Conditions. *Lett. Math. Phys.*, 15:205 – 212, 1988.

[12] H. Narnhofer and W. Thirring . Covariant QED without indefinite metric. *Rev. Math. Phys. (Special Issue dedicated to R. Haag)*, Dec. 1992:193–211.

[13] S.N. Gupta. Theory of longitudinal photons in quantum electrodynamics. *Proc. Phys. Soc. Lond.*, A 63:681–691, 1950.

[14] K. Bleuler. Eine neue Methode zur Behandlung der longitudinalen und skalaren Photonen. *Helv. Phys. Acta*, 23:567–586, 1950.

[15] F. Strocchi. Gauge problem in quantum field theory . *Phys. Rev.*, 162:1429–1438, 1967.

[16] S.S. Horuzhy. *Introduction to Algebraic Quantum Field Theory*. Reidel, Dordrecht, 1990.

[17] L.D. Faddeev and V.N. Popov . Feynman Diagrams for the Yang-Mills Field. *Phys. Lett.*, 25 B:29–30, 1967.

[18] F. Strocchi and A.S. Wightman . Proof of the charge superselection rule in local relativistic quantum field theory . *Jour. Math. Phys.*, 15:2198–2224, 1974.

[19] A.L. Carey, J.M. Gaffney and C.A. Hurst. A C^*-algebra Formulation of the Quantization of the Electromagnetic Field . *Jour. Math. Phys.*, 18:629 – 640, 1977.

[20] A.L. Carey and C.A. Hurst. A Note on the Fermi Quantization of Landau Gauge . *Lett. Math. Phys.*, 2:227 – 234, 1978.

[21] A.L. Carey, J.M. Gaffney and C.A. Hurst. A C^*-algebra Formulation of Gauge Transformations of the Second Kind for the Electromagnetic Field . *Rep. Math. Phys.*, 13:419 – 436, 1978.

[22] C.A. Hurst. Quantum Theory of the Free Electromagnetic Field . In B. Gruber and R. Lenczewski, editors, *Symmetries in Science II*. Plenum Publishing Corporation, 1986.

[23] R. Haag. *Local Quantum Physics*. Springer-Verlag, Berlin, 1992.

[24] M.J. Gotay, J. Isenberg, J.E. Marsden, R. Montgomery (with the collaboration of J. Śniatycki and P.B. Yasskin). *Momentum Maps and Classical Relativistic Fields*. Springer Verlag, New York, to appear.

[25] E.P. Wigner. On Unitary Representations of the Inhomogeneous Lorentz Group. *Ann. Math.*, 40:149–204, 1939.

[26] A.O. Barut and R. Raczka. *Theory of Group Representations and Applications*. World Scientific, Singapore, 1986.

[27] M.A. Rieffel. Induced Representations of C^*-algebras. *Adv. Math.*, 13:176–257, 1974.

[28] G.K. Pedersen. *C^*-Algebras and their Automorphism Groups*. Academic Press, London, 1979.

[29] R.F. Streater. Outline of axiomatic relativistic quantum field theory. *Rep. Prog. Phys.*, 38:771–846, 1975.

[30] J. Goldstone, A. Salam and S. Weinberg. Broken Symmetries . *Phys. Rev.*, 127:965–970, 1962.

[31] G. Jona-Lasinio. Relativistic Field Theories with Symmetry-breaking solutions. *Nuovo Cimento*, 34:1790–1795, 1964.

[32] Y. Nambu. S-Matrix in Semiclassical Approximation. *Phys. Lett.*, 26B:626–629, 1968.

[33] S. Coleman and E. Weinberg . Radiative Corrections as the Origin of Spontaneous Symmetry Breaking . *Phys. Rev. D*, 7:1888–1910, 1973.

[34] R. Jackiw. Functional Evaluation of the effective potential. *Phys. Rev. D*, 9:1686–1701, 1974.

[35] S. Norimatsu, K. Yamamoto and A. Tanaka. What is the effective potential? . *Phys. Rev. D*, 35:2009–2012, 1987.

[36] M. Hindmarsh and D. Johnston . Convexity of the Effective Potential . *Jour. Phys. A*, 19:141–157, 1986.

[37] Y. Fujimoto, L. O'Raifeartaigh and G. Parravicini. Effective Potential for Non-Convex Potentials. *Nucl. Phys. B*, 212:268–300, 1983.

[38] L.S. Brown. *Quantum Field Theory*. Cambridge University Press, Cambridge, 1992.

[39] K. Tabata and I. Umemura . Convexity of the Effective Potential . *Prog. Theor. Phys.*, 74:1360–1363, 1985.

[40] C. Itzykson and J.-B. Zuber. *Quantum Field Theory*. McGraw-Hill, Inc., Singapore, 1985.

[41] A.S. Wightman. Fundamental Problems in Gauge Field Theory - Introduction to the Problems. In G. Velo and A.S. Wightman, editors, *Fundamental Problems in Gauge Field Theory*, New York, 1986. Erice 1985, Plenum Publishing Corporation.

[42] S. Elitzur. Impossibility of spontaneously breaking local gauge symmetries. *Phys. Rev.*, D 12:3978–3982, 1975.

[43] J. Fröhlich, G. Morchio and F. Strocchi. Higgs Phenomenon without Symmetry Breaking Order Parameter. *Nucl. Phys.*, B 190:553–582, 1981.

[44] T. Kennedy and C. King . Spontaneous Symmetry Breakdown in the Abelian Higgs Model. *Commun. Math. Phys.*, 104:327–347, 1986.

[45] C. Borgs and F. Nill . Symmetry Breaking in Landau Gauge. A comment to a paper by T. Kennedy and C. King. *Commun. Math. Phys.*, 104:349–352, 1986.

[46] M. Creutz and T.N. Tudron . Higgs mechanism in the temporal gauge. *Phys. Rev.*, D 17:2619–2623, 1978.

[47] G. Morchio and F. Strocchi. Infrared Problem, Higgs Phenomenon and Long Range Interactions. In G. Velo and A.S. Wightman, editors, *Fundamental Problems in Gauge Field Theory*, New York, 1986. Erice 1985, Plenum Publishing Corporation.

[48] R. Haag. The Mathematical Structure of the Bardeen-Cooper-Schrieffer Model. *Nuovo Cimento*, XXV:287–299, 1962.

[49] O. Bratteli and D.W. Robinson. *Operator Algebras and Quantum Statistical Mechanics II*. Springer-Verlag, Berlin, 1981.

[50] S. Weinberg. The Quantum Theory of Massless Particles . In Johnson e.al., editor, *Lectures on Particles and Field Theory, Vol. II*, Englewood Cliffs, N.J., 1965. Brandeis Summer Institute in Theoretical Physics, Prentice-Hall, Inc.

[51] A. Guichardet. *Symmetric Hilbert Spaces and Related Topics*. Springer Verlag, Berlin, Heidelberg, New York, 1972.

[52] G.B. Folland. *Harmonic Analysis in Phase Space*. Princeton University Press, Princeton, N.J., 1989.

[53] O. Bratteli and D.W. Robinson. *Operator Algebras and Quantum Statistical Mechanics I, 2nd ed.* Springer-Verlag, Berlin, 1987.

[54] M. Reed and B. Simon. *Methods of Modern Mathematical Physics, Vol II.* Academic Press, London, 1975.

[55] J. Glimm and A. Jaffe. *Quantum Physics.* Springer Verlag, New York, 1987.

[56] H.B.G.S Grundling. A group algebra for inductive limit groups. continuity problems of the canonical commutation relations. *Adelaide preprint 1994.*

[57] A.V. Skohorod. *Integration in Hilbert Spaces.* Springer-Verlag, Berlin, 1974.

[58] J.D. Bjorken and S.D. Drell. *Relativistic Quantum Fields.* McGraw-Hill, Inc., New York, 1965.

[59] J. Hertle and R. Honegger. Limiting Gibbs states and dynamics for thermal photons. *Jour. Math. Phys*, 31:143 – 151, 1992.

[60] U.A. Wiedemann. The effective potential for gauge theories and at finite temperature. In F.C. Khanna, R. Kobes, G. Kunstatter, H. Umezawa, editor, *Banff/CAP Workshop on Thermal Field Theory*, Singapore, 1994. World Scientific.

[61] J. Fröhlich and B. Simon . Pure States for general $P(\phi)_2$-theories: Construction, Regularity and Variational Equality. *Ann. Math.*, 105:493–526, 1977.

[62] F. Guerra, L. Rosen and B. Simon . Boundary Conditions for the $P(\phi)_2$ Euclidean Field Theory. *Ann. Inst. Henri Poincaré*, 25A:231–334, 1976.

[63] G. Jona-Lasinio. Large Fluctuations of Random Fields and the Renormalization group. In J. Föhlich, editor, *Scaling and Self-Similarity in Physics*, Boston, Basel, Stuttgart, 1983. Birkhäuser.

[64] G. Slade. The Loop Expansion for the Effective Potential in the $P(\phi)_2$ Quantum Field Theory. *Commun. Math. Phys.*, 102:425–462, 1985.

[65] R.T. Rockafellar. *Convex Analysis.* Princeton University Press, N.J., 1970.

[66] J. Glimm, A. Jaffe and T. Spencer . Phase Transitions for ϕ_2^4 Quantum Fields. *Commun. Math. Phys.*, 45:203–216, 1975.

[67] R. Fukuda. The Form of the Effective Potential. *Prog. Theor. Phys.*, 56:258–269, 1976.

[68] U.A. Wiedemann. Phase Transitions and the Wilson Recursion Formula. In J. Trân Thanh Vân, editor, *'93 Electroweak Interactions and Unified Theories*. Rencontres de Moriond, Edition Frontiers, 1993.

[69] G. Slade. The Effective Potential as an Energy Density: the One Phase Region. *Commun. Math. Phys.*, 104:573–580, 1986.

[70] G.L. Sewell. *Quantum Theory of Collective Phenomena*. Clarendon Press, Oxford, 1986.

[71] P. Pfeifer. *Chiral Molecules - A Superselection Rule Induced by the Radiation Field* . Gotthard S+D AG, Zürich, Thesis, ETH-Zürich No. 6551, 1980.

[72] A. Amann. Ground States of a Spin-Boson Model . *Ann. Phys.*, 208:414–448, 1991.

[73] H. Spohn. Ground state(s) of the spin-boson hamiltonian. *Commun. Math. Phys.*, 113:277–304, 1989.

[74] G. Palma. Renormalized Loop Expansion for the Gauge Covariant constraint Effective Higgs Potential on the Lattice. *Thesis*, DESY-92-31.