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KAPPA-DEFORMATIONS: HISTORICAL DEVELOPMENTS AND RECENT RESULTS

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Abstract. I shall recall in historical perspective some results from nineties and show further how κ -deformed symmetries and κ -Minkowski space inspired DSR (Doubly of Deformed Special Relativity) approach proposed after 2000. As very recent development I shall show how to describe quantum-covariant κ -deformed phase spaces by passing from Hopf algebras to Hopf algebroids (arXiv:1507.02612) and I will briefly describe the κ -deformations of $AdS_5 \times S^5$ superstring target spaces (arXiv:1510.030.83).

1. Introduction

Transition from classical to quantum physics leads to the appearance of noncommutative algebraic structures. In standard quantum mechanics (QM) the canonical quantum phase space is described by Heisenberg algebra (HA) ($i,j=1,2,3$)

$$\text{HA in} \quad [\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 \quad (1a)$$

$$\text{standard QM} \quad [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij} \quad (1b)$$

The noncommutativity (1b) of positions \hat{x}_i and momenta \hat{p}_i can be used to derive the Heisenberg uncertainty relations which introduce the bounds on accuracy of simultaneous measurements of positions and momenta. Besides it follows from (1a) that in standard QM one can localize separately the positions or momenta with arbitrary accuracy, what is reflected in the use in QM of classical geometry, with commutative space and time.

The canonical nonrelativistic HA (1a–1b) is changed however if (\hat{x}_i, \hat{p}_i) describe a dynamical system, e.g. point particles moving in a field-theoretic background. In particular, in the presence of electromagnetic (EM) fields the canonical momenta are becoming noncommutative (NC). Nonrelativistic EM background described by magnetic field $\vec{H} = \text{rot} \vec{A}$ ($H_i = \frac{1}{2} \varepsilon_{ijk} F_{jk}$ where $F_{jk} = \partial_j A_k - \partial_k A_j$), leads to the following modification of algebra (1a–1b) [1, 2]

$$\text{HA in the presence} \quad [\hat{x}_i, \hat{x}_j] = 0 \quad [\hat{p}_i, \hat{p}_j] = ie F_{ij} \quad (2a)$$

$$\text{of magnetic field H} \quad [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij} \quad (2b)$$

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In relativistic theories one uses the Poincaré algebra as describing the group of motions in Minkowski space-time. The presence of a constant relativistic EM background ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; $\mu, \nu = 0, 1, 2, 3$) leads to the deformation of Poincaré algebra into Maxwell algebra [3]–[5], with noncommutative fourmomenta generators P_μ

$$\begin{array}{ccc} [P_\mu, P_\nu] = 0 & \xrightarrow{F_{\mu\nu} \neq 0} & [P_\mu, P_\nu] = iZ_{\mu\nu} \\ \text{(Poincaré algebra)} & & \text{(Maxwell algebra)} \end{array} \quad (3)$$

where new six Abelian generators $Z_{\mu\nu} = eF_{\mu\nu}$ describe the tensorial central charges.

Further let us consider quantum-mechanical system in the presence of gravitational field ($g_{\mu\nu}(x)$ or $e^a_\mu(x)$, where $g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu$; $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$). It has been argued [6, 7] that one gets in quantum regime for such a system the deformed canonical phase space with noncommutative positions sector, because the quantum fluctuations of gravitational field does not allow the localization of positions (space-time coordinates) with arbitrary accuracy. Such restriction of measurements has been physically interpreted in the framework of Einstein gravity as caused by the formation at Planckian distances ($\lambda_p \simeq 10^{-33}$ cm) of mini-black holes, screening experimental visibility of subPlanckian distances ($\lambda < \lambda_p$)².

Summarizing, if we consider quantized dynamical systems interacting with gravity (QG) one should use new type of quantum phase spaces, with noncommuting coordinates, in relativistic quantum theories described by deformed noncommutative Minkowski spaces. We get

$$\begin{array}{ccc} [\hat{x}_\mu, \hat{x}_\nu] = 0 & \xrightarrow{\text{nonvanishing}} & [\hat{x}_\mu, \hat{x}_\nu] \neq 0 \\ \text{space-time without} & \text{QG effects} & \text{at Planckian} \\ \text{QG effects} & & \text{distances} \end{array} \quad (4)$$

One can point out that noncommutative (quantum) space-time coordinates represent [new geometric paradigma in theoretical physics, providing the description of quantum dynamical systems with inclusion of quantum gravity effects](#).

Let me report briefly on the appearance of noncommutative geometry, quantum spaces and particularly quantum space-times in mathematical physics. First noncanonical noncommutative structures were introduced around 1980 as so-called quantum algebras describing the algebraic and geometric properties of quantum integrable system [9]–[11]. These algebro-geometric structures were introduced as one-parametric deformations of universal enveloping algebra for some low-dimensional complex Lie algebras, and supplemented with additional Hopf-algebraic structure, i.e. with coalgebras (coproducts), coinverses (antipodes) and counits³. The deformed quantum Hopf algebras were proposed as noncommutative and noncocommutative generalizations of the notions of classical Lie groups G and Lie algebras \hat{g} . First classification of deformed simple complex Lie algebras \hat{g} (the Hopf deformations of enveloping Lie algebras $U(\hat{g})$) was given by Drinfeld [12] and Jimbo [15], who introduced standard quantum deformations, called as well Drinfeld-Jimbo (DJ) or q -deformation (q - complex deformation parameter). In eighties, in parallel way, there were also introduced the quantum groups as Hopf-algebraic deformations of matrix Lie groups G , described by functions on matrix groups with suitably introduced noncommutative matrix entries [16, 17, 13]. Subsequently, using Hopf-algebraic duality, one gets the description of quantum symmetries by pairs of dual Hopf algebras generalizing dual pairs of classical matrix Lie groups and classical Lie algebras, linked by exponential map (passage from Lie algebras to Lie group) or differentiation of Lie group elements at group unit (passage from Lie groups to Lie algebras).

² Such mechanism was firstly qualitatively predicted by Bronstein in 1936 [8]. He predicted that the geometry describing all interactions should incorporate three fundamental constants: c, \hbar and Newton constant G ; in place of G there was later introduced the Planck mass $m_{\text{pl}} = (\frac{\hbar c}{G})^{1/2} \simeq 10^{-5} g$.

³ For Hopf-algebraic description of deformed groups and Lie algebras see e.g. [12]–[14].

Quantum symmetries are described in their global (finite) and infinitesimal versions as follows:

$$\begin{array}{ccc}
 \begin{array}{c} \text{infinitesimal} \\ \text{quantum symmetries} \\ \text{(quantum Lie} \\ \text{algebras)} \end{array} & \xleftrightarrow[\text{duality}]{\text{Hopf-algebraic}} & \begin{array}{c} \text{finite} \\ \text{quantum symmetries} \\ \text{(quantum matrix} \\ \text{groups)} \end{array}
 \end{array} \quad (5)$$

We add that quantum matrix groups, with noncommutative group elements, were also introduced as the linear transformation groups describing the quantum symmetries of various noncommutative spaces (see e.g. [17]).

The basic problem at the end of eighties was the consistent quantum deformation of relativistic symmetries: $D = 4$ Lorentz and Poincaré algebras. The explicit q -deformation of Lorentz algebra was easily introduced as particular application of general DJ deformation framework [18]–[21]. However, to obtain quantum deformation of non-semisimple Poincaré algebra a new way of deriving Hopf-algebraic quantum deformations was needed⁴.

The quantum deformations of important class of nonsemisimple classical Lie algebras, described by semidirect product of Abelian and simple Lie subalgebras, were obtained by the quantum modification of known Wigner-Inonu (WI) contraction procedure [22, 23]. Firstly such quantum WI contraction was applied to the derivation of quantum κ -deformation $U_\kappa(e(2))$ of $D=2$ Euclidean inhomogeneous algebra $e(2) = O(2) \times T^2$ [24]

$$U_q(su(2)) \xrightarrow[R \rightarrow \infty]{q(R)} U_\kappa(e(2)) \quad (6)$$

where R denotes the WI contraction parameter, $U_q(su(2))$ describes DJ deformation of enveloping algebra $U(su(2))$ and $\xrightarrow[R \rightarrow \infty]{q(R)}$ denotes the quantum WI contraction limit $R \rightarrow \infty$ with $q(R) \rightarrow 1$, which requires the special choice of R -dependence of $q(R)$, provided by the following asymptotic expansion

$$q(R) = 1 + \frac{1}{\kappa R} + O\left(\frac{1}{(\kappa R)^2}\right) \xrightarrow[R \rightarrow \infty]{} 1 \quad (7)$$

The particular R -dependence given by (7) leads to finite but nonclassical (deformed) contraction limit, defining κ -deformed Hopf algebra $U_\kappa(e(2))$. Because for space-time symmetries R has the length dimension $[L]$ and $q(R)$ should be dimensionless, it follows that in (7) the parameter κ has the dimensionality $[L^{-1}]$, what (after putting $c = \hbar = 1$) implies the mass dimensionality of κ . In such a way there were introduced quantum-deformed $D=2$ inhomogeneous Euclidean symmetries with mass-like deformation parameter κ , which has been further identified with the Planck mass m_{pl} .

Our presentation reviews briefly three periods of research activity.

- i) last decade of XXth century, when the description of κ -deformed quantum Hopf-algebraic symmetries was proposed and well established,
- ii) first decade of present century, with the development of rather semi-phenomenological approach to the deformation of special relativity, called Doubly Special Relativity or later Deformed Special Relativity (DSR),
- iii) recent years, when several new applications of κ -deformations were proposed.

The plan of our presentation is the following:

⁴ Few years later it was shown that DJ q -deformation of $D=4$ Lorentz algebra can be as well extended to q -deformed $D=4$ Poincaré algebra, but in the framework of nonstandard braided Hopf algebras [21], with coalgebras defined with the use of nonstandard braided tensor product.

In Sect.2 I recall the introduction in 1991–92 of κ -deformed D=4 Poincaré algebra [25]–[27], obtained by quantum Wigner-Inonu contraction procedure (see (6–7)) applied to suitable Lie algebras of rank 2. Further it is presented briefly the problem of choice of algebraic basis for κ -deformed Poincaré-Hopf algebra, and we provide two important explicit choices: Majid-Ruegg bicrossproduct basis [28, 29] and classical Poincaré algebra basis [30]–[32].

In Sect.3 I describe the notion of duality of Hopf algebras and define κ -deformed Poincaré-Hopf group as dually determined by the κ -deformed Poincaré-Hopf algebra (see e.g. [33, 34]). It will be shown how by using κ -deformed Poincaré algebra in a selected basis one can obtain the finite κ -deformed quantum Poincaré group transformations.

In Sect.4 I consider briefly the postulates of DSR theories [35]–[37] and expose their link with earlier descriptions of κ -deformed Poincaré symmetries. One can show that basic two formulae of DSR theories providing the deformation of mass-shell condition and quantum deformation of finite Lorentz transformations were already present in earlier Hopf algebraic description of κ -deformed quantum symmetries (see e.g. [38]).

Further I shall recall subsequent development of DSR theories, with coalgebraic part of κ -deformed formalism represented as curved structures of classical momentum spaces [39]–[42]. I will comment as well on other deformations which introduce mass-like deformation parameter, in particular the oldest one, the Snyder deformation of space-time [43], which introduces mass-like deformation parameter (e.g. Planck mass) without violating the classical Lorentz symmetry.

In Sect.5 I shall describe briefly two recent applications of κ -deformations: the introduction of quantum-covariant κ -deformed phase space as an example of quantum space with Hopf algebroid structure [44, 45] and the insertion of κ -deformation into the Yang-Baxter (YB) sigma model defined on $\frac{SU(2,2;4)}{O(4,1) \times O(5)}$ coset space, which leads to the κ -deformation of D=10 GS superstring target space geometry [46].

Finally, in Sect. 6, an outlook is presented. The κ -deformation is an example of a global, nondynamical deformation of space-time algebra, which only approximates “physical” noncommutative structures in QG. More general deformations, local and of dynamical origin, should relate the parametrization of noncommutativity with dynamical degrees of freedom of quantum gravity + quantum matter systems. New formalism determining such relations will provide in future the final fundamental quantum theory, describing interacting QG with the quantized matter fields.

2. κ -deformed Poincaré-Hopf algebras

Drinfeld-Jimbo (DJ) q -deformation scheme [12] has been proposed only for complex simple Lie algebras. Because space-time groups of motions are real and nonsemisimple, in order to obtain e.g. quantum-deformed D=4 Poincaré algebra $\hat{p}_{3,1}$, which is a nonsemisimple cross product of real Lorentz and translation (Abelian) subalgebras ($\hat{p}_{3,1} = \hat{o}(3,1) \bowtie T_4$), one should apply the quantum WI contraction procedure. In four dimensions in place of (6) we perform the contraction⁵

$$U_q(\hat{o}(3,2)) \xrightarrow[R \rightarrow \infty]{q(R)} U_\kappa(\hat{p}_{3,1}) \quad (8)$$

where $q(R)$ has asymptotic expansion (7).

To perform explicitly the contraction (8) we should extend DJ deformation of Cartan-Chevalley (CC) basis of complexified $\hat{o}(3,2)$ algebra ($Sp(4; c) \simeq o(5; c)$), defined for 6 generators $h_i, e_{\pm i}$, ($i = 1, 2$), to Cartan-Weyl (CW) Lie-algebraic basis (10 generators extending $\hat{o}(3,2)$ CC basis by four generators $e_{\pm 3}, e_{\pm 4}$ [47, 48]). We resolve in such a way the q -deformed Serre

⁵ The contraction (8) provides undeformed $\hat{o}(3)$ rotation. If we apply quantum WI contraction to $U_q(o(4,1))$ we obtain κ -deformation of D=4 Poincaré algebra with undeformed $\hat{o}(2,1)$ subalgebra (see [47], table 1.)

relations formulated in DJ deformation scheme⁶. Further, because the D=4 physical space-time symmetry algebras ($\hat{p}_{3,1}, \hat{o}(3,2), \hat{o}(4,1), \hat{o}(4,2)$ etc.) are real, with Hermitean or antiHermitean generators, one should study all possible reality conditions, and consider corresponding real \star -Hopf algebras. There were used in literature mainly the following two types of real \star -Hopf algebras:

i) Standard \star -Hopf algebras, with involutive anticonjugation $(x, y)^\star = y^\star x^\star$, which maps tensor products in nonflipped way $((x \otimes y)^\star = x^\star \otimes y^\star)$ [27, 28]. Such real q -deformed \star -Hopf algebra $U_q(\hat{o}(3,2))$ with q real was used to obtain by quantum WI contraction the κ -deformed Poincaré algebra as standard real \star -Hopf algebra (for notion of standard \star -Hopf algebras see e.g. [49], [50]).

ii) nonstandard \star -Hopf algebra, with involutive conjugation $(xy)^\star = x^\star \cdot y^\star$ and flipped conjugation of tensor products $(x \otimes y)^\star = y^\star \otimes x^\star$, which is for example used in q -deformed models with q described by roots of unity [51, 52]. Such nonstandard reality conditions was also used in first derivations of κ -deformed Poincaré algebra [25, 26] obtained from q -deformed AdS algebra $\hat{o}(3,2)$ with $|q| = 1$.

The quantum WI contraction should be performed on all Hopf-algebraic operations, i.e.

$$\mathbb{H}_q = (A_q = U_q(\hat{o}(3,2)), m, \Delta_q, \epsilon, S_q) \xrightarrow[R \rightarrow \infty]{q(R)} \mathbb{H}_\kappa = (A_\kappa = U_\kappa(\hat{p}_{3,1}), m, \Delta_\kappa, \epsilon, S_\kappa) \quad (9)$$

where $m : A \otimes A \rightarrow A$ denotes multiplication, $\Delta : A \rightarrow A \otimes A$ the coproduct and $S : A \rightarrow A$ the antipode (coinverse).

The κ -deformation of Poincaré algebra obtained by (9) contains undeformed $\hat{o}(3) \oplus \hat{T}_3$ subalgebra describing nonrelativistic D=3 Euclidean group of motions.

Using three-dimensional notation for D=4 Lorentz generators $M_{\mu\nu} = (M_i, N_i)$ ($i=1,2,3$) and fourmomenta $P_\mu = (P_i, P_0)$, there were obtained in [25]–[27] the following deformations

- modified commutators $[N_i, N_j]$ and $[P_i, N_j]$
- nonprimitive coproducts $\Delta(N_i), \Delta(P_i)$
- modified antipodes (in undeformed case $S(P_\mu) = -P_\mu, S(M_{\mu\nu}) = -M_{\mu\nu}$)

Subsequently it was realized that the suitable change of linear Poincaré algebra generators

$$M_{\mu\nu} \rightarrow M'_{\mu\nu}(M_{\mu\nu}, P_\mu) \quad P_\mu \rightarrow P'_\mu = P'_\mu(P_\mu) \quad (10)$$

can shift some deformations from algebra to coalgebra. The following two bases became very useful and subsequently used:

i) Majid-Ruegg basis [28]

In such a basis the κ -Poincaré-Hopf algebra is described as follows:

a) Algebra

- Lorentz algebra remains classical, i.e. described by $o(3,1)$ Lie algebra
- Space-time translations generators P_μ commute (i.e. also remain classical)
- The only κ -deformed Poincaré algebra commutator looks as follows:

⁶ We point out that q -deformations of Cartan-Weyl bases for rank two Lie algebras and superalgebras has been firstly presented in explicit form in [48].

$$[N_i, P_j] = i\delta_{ij} \left[\frac{\kappa}{2} (1 - e^{-\frac{2P_0}{\kappa}}) + \frac{1}{2\kappa^2} \vec{P}^2 \right] + \frac{1}{\kappa} P_i P_j \quad (11)$$

b) Coalgebra

The coproducts $\Delta(P_0)$ and $\Delta(M_i)$ remained primitive (classical), the ones which were deformed consistently with (11) are the following

$$\begin{aligned} \Delta P_i &= P_i \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes P_i \\ \Delta N_i &= N_i \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes N_i + \frac{1}{\kappa} \epsilon_{ijk} P_j \otimes M_k \end{aligned} \quad (12)$$

c) Antipodes (coinverses)

$$\begin{aligned} S(M_i) &= -M_i & S(N_i) &= -N_i + \frac{3i}{\kappa} P_i \\ S(P_i) &= -e^{-\frac{P_0}{\kappa}} P_i & S(P_0) &= -P_0 \end{aligned} \quad (13)$$

d) Casimir operators

d1) The κ -deformed mass Casimir, defining κ -deformed mass-shell condition, takes the form

$$C_2 = \vec{P}^2 e^{\frac{P_0}{\kappa}} - (2\kappa \sin \frac{P_0}{\kappa})^2 \quad (14)$$

d2) The κ -deformed spin-square Casimir looks as follows

$$C_4 = (\cosh \frac{P_0}{\kappa} - \frac{\vec{P}^2}{4\kappa^2}) W_0^2 - \vec{W}_\kappa^2 \quad (15)$$

where the component W_0 is undeformed⁷

$$W_0 = \vec{P} \vec{M} \quad (16)$$

but three-vector \vec{W}_κ is κ -dependent

$$\vec{W}_\kappa = \kappa \vec{M} \sinh \frac{P_0}{\kappa} + \vec{P} \times \vec{N} \quad (17)$$

ii) *classical Poincaré basis* [30]–[32]

In such a basis the whole Poincaré algebra is not deformed, and all deformations are incorporated into quite complicated coproducts (in particular the coproduct for energy generator $\Delta(P_0)$ becomes nonprimitive).

The κ -deformed Poincaré-Hopf algebras in Majid-Ruegg and classical Poincaré bases differ only by nonlinear change of the fourmomenta generators. In both bases the algebraic and coalgebraic sectors are described as κ -deformed bicrossproduct Hopf algebras. The choice of bicrossproduct basis appears to be convenient in the study of duality relations between κ -deformed Poincaré algebra and κ -deformed Poincaré group (see Sect. 3).

⁷ The fourvector $W_\mu = (W_0, W_i)$ was introduced in undeformed case by Lubanski [53]. It is called Pauli-Lubanski fourvector.

It is known that leading $\frac{1}{\kappa}$ deformation term in the coproducts for quantum-deformed Lie algebras is provided for large class of bases by means of general formula (\hat{g} describes Lie algebra generators)

$$\Delta(\hat{g}) = \Delta^0(\hat{g}) + \frac{1}{\kappa} \left[\Delta^{(0)}(\hat{g}), \hat{r} \right] + O\left(\frac{1}{\kappa^2}\right) \quad (18)$$

where $\hat{r} \in \hat{g} \otimes \hat{g}$ denotes the classical r -matrix which satisfies the classical Yang-Baxter equation (CYBE)

$$[[\hat{r}, \hat{r}]] \equiv [\hat{r}_{12}, \hat{r}_{13}] + [\hat{r}_{12}, \hat{r}_{23}] + [\hat{r}_{13}, \hat{r}_{23}] = \hat{\Omega}_3 \equiv \Omega_3^{ijk} I_i \wedge I_j \wedge I_k \quad (19)$$

In (19) the expression $[[\hat{r}, \hat{r}]]$ describes Schouten bracket (see e.g. [14]), and $\Omega_3 \in \hat{g} \otimes \hat{g} \otimes \hat{g}$ is a \hat{g} -invariant 3-form. One gets the following two classes of deformations:

a) satisfying standard CYBE with $\Omega_3 \equiv 0$.

In such a case it can be shown that the deformation (quantization) of Lie algebra \hat{g} is determined by twist function $\hat{F} \in U(\hat{g}) \otimes U(\hat{g})$, determining the deformed coproducts as follows:

$$\Delta(\hat{g}) = \hat{F}(\hat{g}) \circ \Delta^{(0)}(\hat{g}) \circ \hat{F}(\hat{g}) \quad (20)$$

where $D^0(\hat{g}) = \hat{g} \otimes 1 + 1 \otimes \hat{g}$ and $A \circ B = A_{(1)}B_{(1)} \otimes A_{(2)}B_{(2)}$ where $A = A_{(1)} \otimes A_{(2)}$ etc.

b) satisfying modified CYBE with $\Omega_3 \neq 0$.

It appears that for D=4 Poincaré algebra we get unique choice of \hat{g} -invariant 3-form, which may occur in CYBE namely

$$\Omega_3(P_\mu, M_{\mu\nu}) = t P_\mu \wedge P_\nu \wedge M^{\mu\nu} \quad t - \text{ complex parameter} \quad (21)$$

The κ -deformed Poincaré algebras obtained in [27, 28] are generated by the following classical r -matrix (see (18))

$$\hat{r} = N_i \wedge P_i \quad (a \wedge b \equiv a \otimes b - b \otimes a) \quad (22)$$

In a middle of nineties (see e.g. [54]) there was introduced the generalization of κ - deformations depending on constant fourvector a_μ , generated by the following generalization of classical r -matrix (22)

$$\hat{r}(a_\mu) = \frac{1}{\kappa} P_\mu \wedge M^{\mu\nu} a_\nu \quad (23)$$

CYBE for the generalized r -matrix (23) takes the following form

$$[[\hat{r}(a_\mu), \hat{r}(a_\nu)]] = a_\mu a^\mu \cdot \Omega_3(P_\mu, M_{\mu\nu}) \quad (24)$$

For Lorentzian signature ($a_\mu a^\mu = \vec{a}^2 - a_0^2$) one obtains three different types of κ -deformations, which can not be related by any change of Poincaré algebra basis. We get namely that

1) If $a_\mu a^\mu = 1$ (one can choose $a_\mu = (1, 0, 0, 0)$) the generalized classical r -matrix (23) reduces to (22). We obtain in such a case standard or time-like κ -deformations (see e.g. (11)–(17)).

2) If $a_\mu a^\mu = -1$ (one can choose $a_\mu = (0, 1, 0, 0)$) one gets tachyonic κ -deformation [55].

3) If $a_\mu a^\mu = 0$ (one can choose light-like vector $a_\mu = (1, 1, 0, 0)$) one obtains the light-cone κ -deformation [56]. It follows from (24) that for light-cone κ -deformation the classical r -matrix (23) describes the solution of standard CYBE, with $\Omega_3 = 0$. Such quantization can be realized by twist factor, which has been explicitly calculated [57, 58].

3. Quantum κ -Poincaré group from Hopf-algebraic duality

If we consider classical symmetries in classical mechanics the finite symmetry transformations are described by elements g of classical matrix groups G ($g \in G$), and infinitesimal ones are calculated by using the matrix realizations of corresponding Lie algebra \hat{g} . The theory of quantum symmetries was analogously developed in two-fold way. Firstly, the classical matrix groups G after their consistent q -deformation were used to construct the Hopf algebras $\tilde{\mathbb{H}}$ of functions $F(\hat{G}_q)$, with noncommutative elements of quantum-deformed matrix group \hat{G}_q . In second way, Lie algebras \hat{g} and their enveloping algebras $U(\hat{g})$ were deformed into the quantum enveloping algebras $U_q(\hat{g})$. After supplementing consistently the Hopf-algebraic operations (coproducts Δ , antipodes S and counits ϵ) one gets quantum Lie-Hopf algebra \mathbb{H} . In previous Section we presented \mathbb{H} as κ -Poincaré-Hopf algebra describing deformed relativistic space-time symmetries; below we shall introduce quantum κ -Poincaré group $\tilde{\mathbb{H}}$ as its dual Hopf-algebra.

One says that the Hopf algebra $\mathbb{H} = (A = U_q(\hat{g}), m, \Delta, S, \epsilon)$ is in duality with Hopf algebra $\tilde{\mathbb{H}} = (A^* = F(\hat{G}_q), m^*, \Delta^*, S^*, \epsilon^*)$ if there is a nondegenerate pairing $\langle \cdot, \cdot \rangle: A \otimes A^* \rightarrow \mathbb{C}$ such that ($a, b \in A, c, d \in A^*$)

$$\langle ab, c \rangle = \langle a \otimes b, \Delta^*(c) \rangle \equiv \langle a, c_{(1)} \rangle \langle b, c_{(2)} \rangle \quad (25a)$$

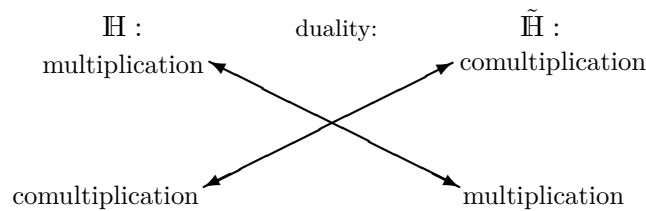
$$\langle \Delta(a), c \otimes d \rangle = \langle a, cd \rangle \equiv \langle a_{(1)}, c \rangle \langle a_{(2)}, d \rangle \quad (25b)$$

where $\Delta(a) = a_{(1)} \otimes a_{(2)}$ and $\Delta^*(c) = c_{(1)} \otimes c_{(2)}$. Further we postulate that

$$\langle S(a), c \rangle = \langle a, S^*(c) \rangle \quad (26)$$

$$\langle a, 1_{A^*} \rangle = \epsilon(a) \quad \langle 1_A, c \rangle = \epsilon^*(c)$$

The duality relations (25) link the multiplication (products) in $\mathbb{H}(\tilde{\mathbb{H}})$ and comultiplication (coproducts) in $\tilde{\mathbb{H}}(\mathbb{H})$. One obtains the following diagram



Two Hopf algebras \mathbb{H} and $\tilde{\mathbb{H}}$ act one on another - we get the formulae for right action \triangleright and left action \triangleleft (see e.g. [14])

$$a \triangleright c = c_{(1)} \langle a, c_{(2)} \rangle \quad (27)$$

$$a \triangleleft c = \langle a_{(1)}, c \rangle a_{(2)}$$

The action of A on products of elements of A^* is given by the Hopf-algebraic formula

$$a \triangleright cd = (a_{(1)} \triangleright c)(a_{(2)} \triangleright d) \quad (28)$$

and similarly for the left action on the product in A . We say that A^* is a right \mathbb{H} module, and A is a left $\tilde{\mathbb{H}}$ -module.

Using the relations (25b) one can get from the coproducts of standard κ -deformed Poincaré-Hopf algebra the algebra of elements $\tilde{e}^B \in A^*$ ($A = 1, \dots, 10$) spanning the orthonormal dual linear basis, satisfying the relation

$$\langle e_A, \tilde{e}^B \rangle = \delta_A^B \quad (29)$$

where $e_A = (P_\mu, M_{\mu\nu}) \in A$, $\tilde{e}^B = (\hat{x}^\mu, \hat{\Lambda}^{\mu\nu}) \in \tilde{A}$. One gets explicitly [33, 28, 29]

$$[\hat{x}^\mu, \hat{x}^\nu] = -\frac{i}{\kappa}(\hat{x}^\mu \delta_0^\nu - \hat{x}^\nu \delta_0^\mu) \quad (30a)$$

$$[\hat{x}^\mu, \hat{\Lambda}_\rho^\nu] = \frac{i}{\kappa}(\Lambda_0^\nu - \delta_0^\nu)\Lambda_\rho^\mu + \eta^{\mu\nu}(\hat{\Lambda}_\rho^0 - \delta_\rho^0) \quad (30b)$$

$$[\hat{\Lambda}_\nu^\mu, \hat{\Lambda}_\tau^\rho] = 0 \quad (30c)$$

The relation (30a) describes the algebra of κ -deformed D=4 Minkowski space-time corresponding to the choice $a_\mu = (1, 0, 0, 0)$ of the constant fourvector introduced in formula (23). It should be also observed that due to the rhs of relation (30b) the formulae (30a-c) do not describe Lie algebra with linear commutators.

Similarly, using relations (25a) one obtains the following coproduct formulae in $\tilde{\mathbb{H}}$:

$$\begin{aligned} \Delta(\hat{x}_\mu) &= \hat{x}^\mu \otimes 1 + \hat{\Lambda}_\nu^\mu \otimes \hat{x}^\nu \\ \Delta(\hat{\Lambda}_\mu^\nu) &= \hat{\Lambda}_\mu^\rho \otimes \hat{\Lambda}_\rho^\nu \end{aligned} \quad (31)$$

The coproducts (31) describe the undeformed Poincaré group transformations. Indeed, if we put $\hat{b}^\mu = \hat{x}_\mu \otimes 1$, $\hat{\alpha}_\nu^\mu = \hat{\Lambda}_\mu^\nu \otimes 1$ and $\Delta(\hat{x}_\mu) = \hat{x}'_\mu$, $1 \otimes \hat{x}_\mu = \hat{x}_\mu$ one gets the standard transformation laws

$$\begin{aligned} \hat{x}'_\mu &= \hat{b}_\mu + \hat{\alpha}_\mu^\nu \hat{x}_\nu \\ \Lambda'^\mu{}_\nu &= \hat{\alpha}_\mu^\rho \hat{\Lambda}_\rho^\nu \end{aligned} \quad (32)$$

It should be observed that following [13, 16] the classical transformation laws (coproducts) remain valid after the quantum deformation (quantization) $G \rightarrow \hat{G}_q$ of any classical matrix group G . Denoting noncommutative matrix entries $\hat{G}_b^a \in \hat{G}_q$, we get general coproduct formula for quantum matrix groups

$$\Delta(\hat{G}_b^a) = \hat{G}_c^a \otimes \hat{G}_b^c \quad (33)$$

The pair of dual Hopf algebras can be calculated as well for generalized κ -deformations, described by the classical r -matrix (23). In particular one gets the following κ -deformed Minkowski space algebra depending on constant fourvector a_μ

$$\begin{array}{l} \text{generalized} \\ \kappa\text{-deformed} \\ \text{Minkowski space:} \end{array} \quad [\hat{x}^\mu, \hat{x}^\nu] = \frac{i}{\kappa}(\hat{x}^\mu a^\nu - \hat{x}^\nu a^\mu) \quad (34)$$

It appears that the noncommutativity in quantum space-time (34) is present in only one quantum coordinate $\hat{x}^\mu a_\mu$; if we introduce three linearly independent fourvectors $b_\mu^{(i)}$ ($i = 1, 2, 3$) which are orthogonal to a^μ (i.e. $a^\mu b_\mu^{(i)} = 0$), the remaining three quantum coordinates $\hat{x}^{(i)} = b_\mu^{(i)} \hat{x}^\mu$ describe three-dimensional commutative manifold, i.e. $[\hat{x}^{(i)}, \hat{x}^{(j)}] = 0$ for any i, j .

The dual pair of Hopf algebras $\mathbb{H}, \tilde{\mathbb{H}}$ is useful in describing the algebraic quantum-deformed phase spaces. For that purpose one introduces cross-products of two dual Hopf algebras (see e.g. [14])

$$\mathcal{H} = \mathbb{H} \ltimes \tilde{\mathbb{H}} \quad (\text{e.g. } U_\kappa(\hat{g}) \in \mathbb{H}, F(\hat{G}_\kappa) \in \tilde{\mathbb{H}}) \quad (35)$$

which is called Heisenberg double algebra \mathcal{H} . The cross product multiplication rule for $a \in \mathbb{H}$ and $c \in \tilde{\mathbb{H}}$ is given by formula ($\mathcal{H} = \mathcal{A} \otimes \mathcal{A}^*$)

$$a \cdot c \equiv (a \otimes 1) \cdot (1 \otimes c) = c_{(1)} \langle a_{(1)}, c_{(2)} \rangle a_{(2)} \quad (36)$$

what permits to calculate in \mathcal{H} as well the cross commutators between \mathbb{H} and $\tilde{\mathbb{H}}$. A simple application of Heisenberg double construction is provided by relativistic quantum-mechanical Heisenberg algebra. Such simple Heisenberg double is obtained from two dual Abelian Hopf algebras describing commuting space-time coordinates $\hat{x}_\mu \in \tilde{\mathbb{H}}_x$ and fourmomenta $\hat{p}_\mu \in \mathbb{H}_p$; the Heisenberg double $\mathcal{H}_{4,4}^{(0)} = \mathbb{H}_p \ltimes \tilde{\mathbb{H}}_x$ is characterized by the cross-product relations derived from (36) (we put $\hbar = 1$)

$$[\hat{x}_\mu, \hat{p}^\nu] = i \delta_\mu^\nu \quad (37)$$

i.e. we get relativistic quantum-mechanical phase space. Subsequently, various quantum-deformed phase spaces can be treated as consistent deformations of the classical Heisenberg double $\mathcal{H}_{4,4}^{(0)}$.

The Heisenberg doubles describe quantum spaces, however without Hopf-algebraic structure. In Sect. 5 we shall show that Heisenberg doubles \mathcal{H} define \mathbb{H} -covariant quantum-deformed phase spaces, which belong to the algebraic category of quantum spaces with Hopf algebroid structure.

4. From κ -deformed Hopf-algebraic symmetries to DSR approach and curved momentum space

In search for an explanation of possible high energy effects of QG (see e.g. [59]) there was proposed around year 2000 the generalization of Einstein special relativity, with two observer-independent parameters - light velocity c and Planck length $L_p \simeq 10^{-33}$ cm - named Doubly Special Relativity (DSR) [60]–[63]⁸.

In first formulation of DSR framework [60, 63] the basic formulae were coincident with the ones which follow from the formulation of κ -deformed Poincaré algebra in bicrossproduct basis, with supplementary identification of mass-like deformation parameter κ as the Planck mass m^9 .

The basic two notions of DSR framework are the following

i) Deformed energy-momentum dispersion relation, which was called in [60] “key characteristic of DSR, both conceptually and phenomenologically”. The deformed mass shell condition presented in [60] was earlier described by the formula for κ -deformed mass Casimir in bicrossproduct basis [28, 29]. Further it can be shown (see e.g. [64]) that various κ -deformed mass-shell conditions in DSR framework were directly related with κ -deformed Poincaré-Hopf algebra written in different algebra bases.

ii) Deformation of classical linear Lorentzian boost transformations of fourmomenta into the ones described by nonlinear formulae [65]. We add that DSR framework, by following the description of particle kinematics in scattering theory, is formulated in commutative fourmomentum space, with commuting space-time coordinates introduced by standard Fourier transform of the

⁸ Later the notation “DSR” was used as well in more general sense, with the meaning “Deformed Special Relativity”.

⁹ For $\hbar = c = 1$ one gets $m_p = (L_p)^{-1}$.

momentum-dependent functions (see e.g. [66])¹⁰. It was shown however [68, 69] that the modified finite Lorentz transformations calculated in [65] can be also obtained by using the Hopf algebra structure of κ -deformed Poincaré algebra in bicrossproduct basis. The general Hopf-algebraic expression for finite κ -deformed boosts in arbitrary basis is given by the following formula (see [70], Sect. 2d)¹¹.

$$P_\mu(\alpha) = ad_{e^{i\alpha N_3}} P_\mu \equiv \sum_{k=0}^{\infty} \frac{i^k}{k!} (ad_{\alpha N_3} (ad_{\alpha N_3} \dots (ad_{\alpha N_3} P_\mu) \dots)) \quad (38)$$

where quantum adjoint action $ad_Y X$ defined as follows

$$ad_Y X = Y_{(1)} X S(Y_{(2)}) \quad (\Delta(Y) = Y_{(1)} \otimes Y_{(2)}) \quad (39)$$

can be expressed in Majid-Ruegg bicrossproduct basis of $\mathcal{U}_\kappa(\hat{p}_{3,1})$ by known “classical” formula

$$P_\mu(\alpha) = e^{i\alpha N_3} P_\mu e^{-i\alpha N_3} \quad (40)$$

The differential equation following from formula (40) was used in [65] in order to calculate explicitly the nonlinear boost transformations in DSR theory.

Important question addressed as well by DSR approach is κ -deformed addition of fourmomenta describing modified conservation law. In Hopf-algebraic approach such addition is determined by the coproducts and from quantum algebra/quantum group duality follows the appearance of noncommutative quantum space-times. In DSR approach the choice of coproduct was rather ambiguous, because the symmetric and nonsymmetric coproducts, both allowed by deformed mass Casimirs, were used. In the case of symmetric addition law (see e.g. [63], where the postulates of DSR theory are discussed) the deformed relativistic symmetries of fourmomenta are described by the standard special relativity rules, however expressed by nonlinearly transformed classical fourmomenta generators (see [68], [71]–[73]). If we employ however the deformed Poincaré algebra as quantum group¹² the coproducts are necessarily nonsymmetric and dual space-time becomes noncommutative. It appears that for some large class of κ -Poincaré algebra bases the κ -deformed quantum space-time is described by κ -Minkowski space (see also (32a)).

In DSR formalism instead of full Hopf-algebraic description of κ -deformed relativistic symmetries one restricts usually the framework to the so-called DSR algebras [74]–[77] describing κ -deformation of classical semidirect product $R_{3,1} \rtimes \hat{p}_{3,1}$, where $R_{3,1}$ denotes Minkowski space and $\hat{p}_{3,1}$ the classical Poincaré algebra¹³. DSR algebra is spanned by generators $(\hat{X}_\mu, \hat{P}_\mu, \hat{M}_{\mu\nu})$, with \hat{X}_μ describing κ -deformed Minkowski space and $(\hat{P}_\mu, \hat{M}_{\mu\nu})$ given by quantum-deformed Poincaré algebra which can be endowed with Hopf algebra structure. In such algebraic (not Hopf-algebraic!) approach the deformations of DSR algebra are present in the commutators $[\hat{X}_\mu, \hat{P}_\nu]$ and $[\hat{X}_\mu, \hat{M}_{\rho,\tau}]$ (see (47)).

Because fourmomenta P_μ are Abelian, one can look also for the realizations of DSR algebra in terms of classical Heisenberg algebra generators \hat{p}_μ, \hat{x}_μ , satisfying the relations (37). If we use

¹⁰ Analogous naive way of introducing space-time coordinates was used in early formalism of κ -deformed relativistic symmetries, in 1991–1993 [26, 27, 29, 67]. Firstly it was clearly stressed in [28] that the modules of κ -deformed Poincaré-Hopf algebra, e.g. describing κ -Minkowski spaces, should be necessarily noncommutative.

¹¹ We choose boosts along the third space axis.

¹² See Drinfeld’ definition of quantum group in [12], described as noncocommutative Hopf algebras with nonsymmetric coproducts $(\Delta(x) = \Delta_{(1)}(x) \otimes \Delta_{(2)}(x) \neq \Delta_{(2)}(x) \otimes \Delta_{(1)}(x))$.

¹³ Such algebras were also called Heisenberg-Poincaré algebras [78].

the Schrödinger realization of the algebra (37)

$$\hat{p}_\mu = p_\mu \quad \hat{x}_\nu = x_\nu \equiv i \frac{\partial}{\partial p^\nu} \quad (41)$$

one can express the κ -Minkowski coordinates \hat{X}_μ by using of momentum-dependent tetrad $E_\mu^\alpha(p)$

$$\hat{X}_\mu = \hat{x}_\alpha E_\mu^\alpha(p) \quad (42)$$

with $E_\mu^\alpha(p)$ determined by the commutator $[\hat{X}_\mu, \hat{P}_\mu]$. Embedding of κ -deformed quantum symmetry algebras with Hopf algebra structure into enveloping classical Heisenberg algebra has been also used, however such procedure is questionable because such a realization does not preserve the Hopf-algebraic structure¹⁴.

The non-Abelian fourmomenta coproducts, if applied to the description of two fourvectors p_μ , and dp_μ indicate the curved structure of fourmomentum space (see e.g. [80, 81]). In DSR approach instead of using coproducts and algebraic methods of Hopf algebra theory, the curved momentum space techniques are used as representing the non-Abelian fourmomenta addition law. It was realized that coproducts of Poincaré-Hopf algebras with associative composition law of fourmomenta describe curved Cartan-Riemann fourmomentum spaces, with nonvanishing torsion and vanishing curvature. The examples of such curved spaces are provided by group manifolds. Indeed, for κ -deformed Poincaré-Hopf algebra it was shown [82]–[84] that corresponding curved fourmomentum space is described by a four-dimensional Lie group $A \cdot N(3)$, entering into the Iwasawa decomposition of $SO(1,4)$ group manifold.

The use of nonlinear momentum space with nontrivial metric which leads to non-Abelian composition law leads to nonstandard description of dynamical systems. Such framework helped to introduce the notion of relative locality [85] providing space-time as derived concept which is described effectively by the interactions of particle probes in momentum space. It can be also added that recently the curved momentum space has been incorporated into dynamical curved phase space framework describing so-called meta-string theory (see e.g. [86, 87]).

5. Recent applications of κ -deformations - two examples

The applications of κ -deformed symmetries to physical models (e.g. deformed QFT) are somewhat limited because for κ -deformed Poincaré-Hopf algebra the universal R -matrix is not known. We shall provide two recent results in the framework of κ -deformations which use the κ -deformed Heisenberg double (see [88]) and classical κ -Poincaré r -matrices (see formulae (22),(23)).

5.1. κ -deformed phase space as Hopf bialgebroid

It has been recently observed [89]–[92], [44, 45] that quantum phase spaces can be supplemented with coalgebra structure in the framework of Hopf algebroids [93, 94]. It is known already a long-time that large class of quantum-deformed phase spaces, with coordinate sector described by Hopf-algebraic quantum group, can be identified as Heisenberg double [88]. Further, exploiting Liu theorem¹⁵ that finite-dimensional Heisenberg double has the structure of a Hopf algebroid [94], there was shown [44, 45] that quantum κ -deformed phase space is the quantum space with Hopf algebroid structure.

¹⁴ The inconsistency follows from the fact that Heisenberg algebra (37) is not endowed with bialgebra structure (see Sect. 5.1). An example of realization of Hopf algebra as embedding into enveloping Heisenberg algebra can be found e.g. in [79].

¹⁵ This theorem is proved rigorously for finite-dimensional Hopf algebras, but formally it can be applied to infinite-dimensional case [45, 92].

While Hopf algebras are quantum analogues of groups, Hopf algebroids are quantum analogues of groupoids [95]. Hopf algebroids are bialgebroids supplemented with the antipode map and additional structures in algebra sector: base subalgebra B of total algebra A and two maps ($h \in A, b \in B$)

i) Source algebra map $s: b \cdot h = s(b)h \quad s(b) \in h$

ii) target antialgebra map $t: h \cdot b = t(b)h \quad t(b) \in h$

which commute, i.e.

$$[s(b), t(b')] = 0 \quad b, b' \in B \quad (43)$$

The Hopf algebroid \mathbb{H} is specified as follows:

$$\mathbb{H} = (A, m; B, s, t; \Delta, s, \epsilon) \quad (44)$$

where B in the case of Hopf algebra reduces to the unity element $\mathbb{1}$ of algebra A , i.e. disappears.

The role of base algebra plays essential role in defining coalgebraic sector of bialgebroid: one defines coproducts Δ using new tensor product $A \otimes_B A$ over generally noncommutative base ring B [96, 97]. The coproduct for bialgebroid can be also realized in terms of standard tensor products $A \otimes A$. If we introduce left ideal in algebra A defined in terms of source and target maps (s, t) as follows

$$I_L = s(b) \otimes 1 - 1 \otimes t(b) \quad (45)$$

the tensor product $A \otimes_B A$ is defined by the following equivalence classes of standard tensor products $A \otimes A$ (see e.g. [97])

$$h \otimes_B h' \simeq h \otimes h' \quad \text{iff} \quad I_L \circ (h \otimes h') = s(b)h \otimes h' - h \otimes t(b)h' = 0 \quad (46)$$

for all $b \in B$. The ideal I_L generates in terms of standard tensor product the nonuniqueness, which is consistent with the homomorphism property of coalgebra reproducing the algebraic structure of A . In [44] such a freedom was called the coproduct gauge.

The simplest example of Hopf bialgebroid is a standard quantum phase space with canonical Heisenberg relations (see (37)), which is given as a Heisenberg double of a pair of Abelian and co-Abelian Hopf algebras $\tilde{\mathbb{H}}_x$ and \mathbb{H}_p , describing classical coordinates x_μ and classical momenta p^μ . The cross-product multiplication rule (36) applied to Heisenberg double $\mathbb{H}_p \times \mathbb{H}_x$ leads to the relation (37), i.e. results in the quantization of Poisson structure on classical phase space. If we calculate the Heisenberg double of κ -deformed Poincaré-Hopf algebra \mathbb{H}_κ and dual quantum κ -Poincaré group $\tilde{\mathbb{H}}_\kappa$, with generators $(P_\mu, M_{\mu\nu}, \hat{x}_\mu, \hat{\Lambda}_{\mu\nu})$, we obtain the following cross-commutators (see [88])

i) κ -deformed relations containing \hat{x}_μ

$$[P_\mu, \hat{x}_\rho] = -i\eta_{\mu\rho} - \frac{i}{\kappa}(\eta_{\rho 0}P_\mu - \eta_{\mu 0}P_\rho) \quad (47)$$

$$[M_{\mu\nu}, \hat{x}_\rho] = i(\eta_{\mu\rho}\hat{x}_\nu - \eta_{\nu\rho}\hat{x}_\mu) + \frac{i}{\kappa}(\eta_{0\mu}M_{\nu\rho} - \eta_{0\nu}M_{\mu\rho})$$

ii) nondeformed relations containing

$$[P_\mu, \hat{\Lambda}_{\nu\rho}] = -i(\eta_{\mu\nu}P_\rho - \eta_{\nu\rho}P_\nu) \quad (48)$$

$$[M_{\mu\nu}, \hat{\Lambda}_{\rho\tau}] = i(\eta_{\mu\rho}\hat{\Lambda}_{\nu\tau} - \eta_{\nu\rho}\hat{\Lambda}_{\mu\tau}) - (\rho \leftrightarrow \tau)$$

The relations (47,48) supplemented with the algebraic relations of κ -deformed Poincaré algebra $\hat{\mathbb{H}}_\kappa$ (see (11)) and κ -deformed Poincaré group \mathbb{H}_κ (see (30)) describe generalized κ -deformed quantum phase space $\mathcal{P}^{10;10}$. It can be shown that $\mathcal{P}^{10;10}$ is a Hopf algebra $\hat{\mathbb{H}}_\kappa$ -module, or equivalently $\mathcal{P}^{10;10}$ is κ -covariant.

One can distinguish the following three subalgebras of $\mathcal{P}^{10;10}$.

- i) κ -deformed DSR algebra $\mathcal{P}^{10;4}$, with base $(P_\mu, M_{\mu\nu}; \hat{x}_\nu)$, describing noncommutative κ -Minkowski space with acting covariantly the κ -deformed Poincaré-Hopf algebra \mathbb{H}_κ
- ii) κ -deformed DSR group $\mathcal{P}^{4;10}$, with base $(\hat{P}_\mu; \hat{x}^\mu, \hat{\Lambda}^\mu_\nu)$, which is the quantum κ -Poincaré group \mathbb{H}_κ with \hat{p}_μ acting in covariant way
- iii) κ -deformed standard phase space $\mathcal{P}^{4;4}$, with base $(\hat{x}_\mu, \hat{p}_\nu)$, where \hat{x}_μ satisfy the algebra (32a) and commutative fourmomenta \hat{p}_μ have the following cross relations: ($\mu = 0, i; i = 1, 2, 3$)

$$\begin{aligned} [\hat{p}_i, \hat{x}_j] &= -i \delta_{ij} & [\hat{p}_i, \hat{x}_0] &= 0 \\ [\hat{p}_0, \hat{x}_0] &= -i & [\hat{p}_0, \hat{x}_i] &= \frac{i}{\kappa} \hat{p}_i \end{aligned} \quad (49)$$

We see that only last relation in (49) is κ -deformed, in comparison with standard relativistic Heisenberg algebra (37).

The generalized phase space $\mathcal{P}^{10;10}$ (with Lorentz spin sector $(M_{\mu\nu}, \hat{\Lambda}_\mu^\nu)$) as well as $\mathcal{P}^{4;4}$ are endowed with Hopf algebroid structure. The base algebra B is identified with quantum group algebra part $(\hat{x}_\mu$ for $\mathcal{P}^{4;4}$ and $(\hat{x}_\mu, \hat{\Lambda}_\mu^\nu)$ for $\mathcal{P}^{10;10}$).

The explicit Hopf algebroid structure for standard κ -deformed quantum phase space $\mathcal{P}^{4;4}$ has been obtained by purely algebraic considerations in [44, 45]. Following Lu theorem (see [94]) we have chosen the primary coproduct on base algebra as $\Delta(\hat{x}_\mu) = 1 \otimes \hat{x}_\mu$, with nonuniqueness described by coproduct gauge, e.g.

$$\Delta(\hat{x}^\mu) \xrightarrow[\text{gauge}]{\text{coproduct}} \Delta(\hat{x}^\mu) + \Lambda^\mu(\hat{x}, \hat{p}) = \hat{x}_\mu \otimes f_\mu^\nu(\hat{p}) \quad (50)$$

where $f_\mu^\nu(\hat{p})$ can be calculated by using for coproducts the algebraic structure (49). One gets

$$\begin{aligned} \Lambda_i(\hat{x}, \hat{p}) &= \hat{x}_i \otimes e^{\frac{F_0}{\kappa}} - 1 \otimes \hat{x}_i \\ \Lambda_0(\hat{x}, \hat{p}) &= \hat{x}_0 \otimes 1 + \frac{1}{\kappa} \hat{x}_i \otimes e^{\frac{F_0}{\kappa}} \hat{x}_i - 1 \otimes \hat{x}_0 \end{aligned} \quad (51)$$

The coproduct gauge (51) can be further generalized, in accordance with the homomorphism of coproducts with the algebraic relations in $\mathcal{P}^{4;4}$ (see [44]).

I would like to add that explicit Hopf algebroid structure of $\mathcal{P}^{10;10}$, with the description of coproduct gauges, is now under considerations.

5.2. κ -deformation of string target (super)spaces

The dynamics of (super) strings can be described by twodimensional σ -models, on suitable (super)group or (super)coset manifolds.

In particular the action describing D=10 Green-Schwarz superstring has been described as D=2 σ -model with D=10 N=2 super-Poincaré target manifold [98]. If one replaces in target superspace the D=10 Poincaré manifold by $AdS_5 \times S_5$, with S_5 describing an internal sector, one should consider D=2 σ -model on supercoset \mathbb{K} [99].

$$AdS_5 \times S_5 \quad D = 10 \quad \text{superstring} \quad \longleftrightarrow \quad \begin{array}{l} D=2 \quad \sigma\text{-model} \\ \text{supercoset } \mathbb{K} = \frac{PSU(2,2;4)}{O(4,1) \times O(5)} \end{array} \quad (52)$$

The σ -model action for strings with target space described by group manifold G or coset $\frac{G}{H}$ can be expressed by bilinears in D=2 Lie algebra - valued currents $\mathbb{J}_\alpha = J_\alpha^a I^a$ ($\alpha = 1, 2$), where I^a describe the Lie algebra generators

$$S = -\frac{T}{2} \int d^2\xi (h^{\alpha\beta} - \epsilon^{\alpha\beta}) Tr[\mathbb{J}_\alpha P \mathbb{J}_\beta] \quad (53)$$

The trace is taken over matrix values of the generators I^a and P describes the projection operator $G \rightarrow \frac{G}{H}$. In the presence of quantum deformations described by classical r -matrix $\hat{r} = \hat{a}_i \wedge \hat{b}_i \in \hat{g} \otimes \hat{g}$ one can introduce the deformation of string action (53) by introducing Yang-Baxter YB σ -model [100, 101] with particular choice of deforming YB kernel $(1 - \xi R)^{-1}$

$$S_{v_B} = -\frac{T}{2} \int d^2\xi (h^{\alpha\beta} - \epsilon^{\alpha\beta}) Tr(\mathbb{J}_\alpha \frac{1}{1 - \xi R} P \mathbb{J}_\beta) \quad (54)$$

The operator $\mathbb{R} = R(I_k) \cdot I_k$ is expressed by classical r -matrix as follows

$$R(I_k) = \sum \hat{a}_i \langle b_i, I_k \rangle = Str(\hat{r} \circ (1 \otimes I_k)) \quad (55)$$

and satisfies modified CYBE in Semenov-Tien-Shansky form [102, 103].

$$[R(I_j), R(I_k)] - R[R(I_j), I_k] + [I_j, R(I_k)] = c[I_j, I_k] \quad (56)$$

We add that if we consider superstrings, the generators \mathbb{J}_α describe both sets of bosonic (even) and fermionic (odd) currents, i.e. $\mathbb{J}_\alpha \rightarrow (J_\alpha^B, J_\alpha^F)$, and the trace Tr should be replaced in formulae (53–54) by supertrace Str . Further if we consider GS superstring with supercoset (52) as target space, one should introduce four projection operators, corresponding to Z_4 -graded decomposition of $PSU(2, 2; 4)$ supergroup generators [104]–[106].

Because the conformal algebra $SU(2, 2)$ contains as subalgebras the commuting fourmomenta as well as D=4 Poincaré algebra, one can embed D=4 Minkowski space-time into the coset \mathbb{K} (see (52)). One can ask further how the target superspace \mathbb{K} will be modified and which NS-NS and R-R fluxes will be obtained if we insert in YB σ -model the classical r -matrices (22) or (23). Using suitable computer program such calculation have been performed ([107]; see also [108]). We calculated YB Minkowski space deformations for standard κ -deformation (see (23), with $a^2 > 0$), for tachyonic deformation ($a^2 < 0$) and for light-cone deformation ($a^2 = 0$). We obtained the following results (modulo Abelian T -duality transformations [109, 110]):

type of κ -def.	space-time geometry	remarks
$a^2 > 0$ (standard)	D=4 de-Sitter	κ^{-1} describes dS radius
$a^2 < 0$ (tachyonic)	D=4 AdS	κ^{-1} describes AdS radius
$a^2 = 0$ (light-cone)	$p - p$ wave background	x_+ - dependent ¹⁶ nonvanishing Ricci curvature; $R = 0$

¹⁶ $x_+ = x_0 + x_3$ describes light-cone time.

It can be added that only in light-cone case one gets nonvanishing antisymmetric tensor field B (NS-NS 2-form), however in the form of a total derivative.

An interesting issue is the classification of integrable deformations of $AdS_5 \times S^5$ superstring. In [105] it has been shown that the generalized a_μ -dependent κ -deformations are integrable. For three types of κ -deformations we obtained the explicit formulae for Lax pairs, generating via Lax equations the infinite number of conserved currents. In particular it has been shown that zero curvature conditions for Lax pairs leads to the equations of motion for κ -deformed YB σ -model.

An important application of YB σ -models is the generalization of standard IIB SUGRA theory [111]. It appears that κ -invariance of D=10 IIB strings embrace as well some more extensions of IIB SUGRA solutions, which can be obtained from YB σ -models [112].

6. Final remarks

I would like to stress that there were presented only some aspects of the theoretical developments related with κ -deformations of relativistic space-time symmetries and we regret that the list of references is far from being complete. Let me mention some other important developments:

i) SUSY extension of κ -deformed framework.

It was shown already in 1993 [113, 114] that κ -deformation of N=1 Poincaré superalgebra can be calculated by considering quantum WI contraction of q -deformed $OSp(4|1)$ superalgebra.

ii) κ -deformed QM.

It has been realized that quantum-deformed particle model in NC phase space $(\hat{x}_\mu, \hat{p}_\mu)$ with classical momenta sector $(\hat{p}_\mu = p_\mu)$ is equivalent via noncanonical map $\hat{x}_\mu = \hat{F}_\mu(x, p)$ to the free particle model in standard quantum phase space (x_μ, p_μ) (for κ -deformed model see e.g. [115]). For standard time-like κ -deformations such noncanonical map is well-known from early nineties $(\hat{x}_i = x_i, \hat{x}_0 = x_0 - \frac{1}{\kappa} \vec{p} \vec{x})$.

iii) κ -deformed classical and quantum field theory.

If one introduces \star -product describing the algebra of NC functions on κ -Minkowski space in terms of classical fields [116] one can only obtain in such a way the κ -deformed classical theory. In quantum κ -deformed QFT the algebra of field oscillators is however κ -deformed as well [117]–[119] i.e. in construction of κ -deformed QFT one should introduce NCQFT with new notion of \star -product which takes into consideration as well the κ -deformation of oscillators algebra (see e.g. [120]).

iv) κ -deformed gravity action.

Seiberg-Witten map [121] applied to κ -deformed Einstein gravity, with local diffeomorphisms as local gauge group, produced higher order curvature corrections [122, 123]. The lowest nonvanishing order in $\frac{1}{\kappa}$ is second, i.e. one gets terms proportional to $\frac{1}{\kappa^2}$. If we put $\kappa = m_p$, these second order corrections are beyond observability limits in present experiments.

At present still the final choice of QG model is not established. Breaking through will arrive if we shall find experimentally some observable QG effects, what however still did not happened. At present we are therefore rather at early stage of the construction procedure of theoretically sound and phenomenologically confirmed quantum gravity model.

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