Revisiting chameleon gravity–thin-shells and no-shells with appropriate boundary conditions

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Abstract

We derive analytic solutions of a chameleon scalar field ϕ that couples to a non-relativistic matter in the weak gravitational background of a spherically symmetric body, paying particular attention to a field mass m_A inside of the body.

1 Introduction

Recently there have been a lot of efforts to understand the origin of dark energy under the framework of modified gravity theories (see Refs. [1] for reviews). In modified gravity models of dark energy, it is crucially important to appropriately study the compatibility of couplings with local gravity experiments as well as with a late-time acceleration of the Universe preceded by a standard matter era. Interesting attempt to reconcile large coupling models with local gravity constraints is to use "chameleon" scalar fields whose masses depend on the environment they are in [2, 3]. The analysis in Refs. [2, 3] assumes that the field is frozen ($\phi = \phi_A$) in the regime $0 < r < r_1$ without explicitly taking into account the field mass m_A. Here and [4], we consider m_A.

The action we study is given by

$$S \quad = \quad \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right] - \int \mathrm{d}^4x\,\mathcal{L}_m(\Psi_m^{(i)}, g_{\mu\nu}^{(i)})\,, \tag{1}$$

where $\Psi_m^{(i)}$ are matter fields that couple to a metric $g_{\mu\nu}^{(i)}$ related with the Einstein frame metric $g_{\mu\nu}$ via $g_{\mu\nu}^{(i)} = e^{2Q_i\phi}g_{\mu\nu}$. Here Q_i are the strength of couplings for each matter field. An example is

$$V(\phi) = M^{4+n}\phi^{-n}\,, \tag{2}$$

where M has a unit of mass and n is a constant. In the context of $f(R)$ gravity, Hu and Sawicki [5] and Starobinsky [6] proposed models that can be consistent with cosmological and local gravity constraints. We consider

$$V(\phi) = V_0\left[1 - C(1 - e^{-2Q\phi})^p\right]\,, \tag{3}$$

where $V_0 > 0,\ C > 0,\ 0 < p < 1$ as a generalization of the potential in $f(R)$ gravity.

2 Chameleon mechanism

The trace of the i-th matter is given by $T^{(i)} \equiv g_{(i)}^{\mu\nu}T_{\mu\nu}^{(i)} = -\tilde{\rho}_i$ for a non-relativistic fluid, where $\tilde{\rho}_i$ is an energy density. It is more convenient to introduce the quantity $\rho_i = \tilde{\rho}_i\, e^{3Q_i\phi}$, which is conserved in the Einstein frame. In a spherically symmetric background, we obtain

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{\mathrm{d}V_{\rm eff}}{\mathrm{d}\phi}\,, \tag{4}$$

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where $V_{\rm eff}(\phi) \equiv V(\phi) + \sum_i \rho_i e^{Q_i \phi}$. In the following, we shall consider the case $Q_i = Q$ and $\rho_i = \rho$. We assume that the body has a homogeneous density $\rho = \rho_A$ and that the density is homogeneous with a value $\rho = \rho_B$ outside of the body. The mass of this body is given by $M_c = (4\pi/3)\rho_A r_c^3$, where r_c is a radius of the body. The effective potential $V_{\rm eff}$ has minima at field values ϕ_A and ϕ_B. The former corresponds to the interior of the body that gives $m_A^2 \equiv \frac{{\rm d}^2 V_{\rm eff}}{{\rm d}\phi^2}(\phi_A)$, whereas the latter to the exterior of the body with $m_B^2 \equiv \frac{{\rm d}^2 V_{\rm eff}}{{\rm d}\phi^2}(\phi_B)$. We impose $\frac{{\rm d}\phi}{{\rm d}r}(r=0) = 0\,, \quad \phi(r \to \infty) = \phi_B$.

In the region $0 < r < r_1$, the solution is

$$\phi(r) = \phi_A + \frac{A(e^{-m_A r} - e^{m_A r})}{r} \qquad (0 < r < r_1). \tag{5}$$

This automatically satisfies the boundary condition: $\frac{{\rm d}\phi}{{\rm d}r}(r=0) = 0$.

In the region $r_1 < r < r_c$, we obtain

$$\phi(r) = \frac{1}{6} Q \rho_A r^2 - \frac{C}{r} + D \qquad (r_1 < r < r_c), \tag{6}$$

where C and D are constants. In the limit $r_1 \to 0$ the coefficient C is required to be zero.

In the region outside of the body, the solution is given by

$$\phi(r) = \phi_B + E \frac{e^{-m_B(r - r_c)}}{r} \qquad (r > r_c). \tag{7}$$

We match three solutions (5), (6) and (7) by imposing continuous conditions for ϕ and ${\rm d}\phi/{\rm d}r$ at $r = r_1$ and $r = r_c$. Especially when the conditions, $m_B r_c \ll 1$ and $m_A \gg m_B$, are satisfied so that the contribution of the m_B-dependent terms are negligible, we have

$$\phi(r) = \phi_A - \frac{1}{m_A(e^{-m_A r_1} + e^{m_A r_1})} \left[\phi_B - \phi_A + \frac{1}{2} Q \rho_A (r_1^2 - r_c^2) \right] \frac{e^{-m_A r} - e^{m_A r}}{r} \qquad (0 < r < r_1) \tag{8}$$

$$\begin{aligned} \phi(r) &= \phi_B + \frac{1}{6} Q \rho_A (r^2 - 3r_c^2) + \frac{Q \rho_A r_1^3}{3r} \\ &\quad - \left[1 + \frac{e^{-m_A r_1} - e^{m_A r_1}}{m_A r_1 (e^{-m_A r_1} + e^{m_A r_1})} \right] \left[\phi_B - \phi_A + \frac{1}{2} Q \rho_A (r_1^2 - r_c^2) \right] \frac{r_1}{r} \qquad (r_1 < r < r_c), \end{aligned} \tag{9}$$

$$\begin{aligned} \phi(r) &= \phi_B - \left[r_1(\phi_B - \phi_A) + \frac{1}{6} Q \rho_A r_c^3 \left(2 + \frac{r_1}{r_c} \right) \left(1 - \frac{r_1}{r_c} \right)^2 \right. \\ &\quad \left. + \frac{e^{-m_A r_1} - e^{m_A r_1}}{m_A(e^{-m_A r_1} + e^{m_A r_1})} \left\{ \phi_B - \phi_A + \frac{1}{2} Q \rho_A (r_1^2 - r_c^2) \right\} \right] \frac{e^{-m_B(r - r_c)}}{r} \qquad (r > r_c) \end{aligned} \tag{10}$$

The radius r_1 is determined by the following condition $m_A^2 \left[\phi(r_1) - \phi_A \right] = Q \rho_A$, which translates into

$$\phi_B - \phi_A + \frac{1}{2} Q \rho_A (r_1^2 - r_c^2) = \frac{6 Q \Phi_c}{(m_A r_c)^2} \frac{m_A r_1 (e^{m_A r_1} + e^{-m_A r_1})}{e^{m_A r_1} - e^{-m_A r_1}}\,, \tag{11}$$

where $\Phi_c = M_c/(8\pi r_c) = \rho_A r_c^2/6$ is a gravitational potential at the surface of the body. Under this relation the field profile (10) outside of the body can be written as

$$\phi(r) = \phi_B - 2 Q \Phi_c r_c \left[1 - \frac{r_1^3}{r_c^3} + 3 \frac{r_1}{r_c} \frac{1}{(m_A r_c)^2} \left\{ \frac{m_A r_1 (e^{m_A r_1} + e^{-m_A r_1})}{e^{m_A r_1} - e^{-m_A r_1}} - 1 \right\} \right] \frac{e^{-m_B(r - r_c)}}{r} \qquad (r > r_c)\,. \tag{12}$$

3 Thin-shell and no-shell solutions

3.1 Thin-shell solutions $((r_c - r_1) \ll r_c)$

3.1.1 The massive case $(m_A r_c \gg 1)$

As we see below, thin-shell solutions originally derived in Refs. [2, 3] can be recovered by taking the limit $m_A r_1 \gg 1$, together with the thin-shell condition given by $\Delta r_c \equiv r_c - r_1 \ll r_c$. Expanding Eq. (11) in terms of small parameters $\Delta r_c / r_c$ and $1/m_A r_c$, we obtain $\epsilon_{\rm th} \simeq \frac{\Delta r_c}{r_c} + \frac{1}{m_A r_c}$.

From Eq. (12) the field profile outside of the body is given by

$$\phi(r) \simeq \phi_B - 2Q_{\rm eff}\frac{GM_c}{r}e^{-m_B(r-r_c)}\,, \tag{13}$$

where $Q_{\rm eff}$ is the effective coupling given by $Q_{\rm eff} \simeq 3Q\left(\frac{\Delta r_c}{r_c}+\frac{1}{m_A r_c}\right) = 3Q\epsilon_{\rm th}$. As long as both $\Delta r_c/r_c$ and $1/(m_A r_c)$ are much smaller than unity so that $\epsilon_{\rm th} \ll 1$, it is possible to satisfy local gravity constraints.

3.1.2 The light mass case ($m_A r_c \ll 1$)

From Eq. (11) we get $\epsilon_{\rm th} \simeq \frac{\Delta r_c}{r_c} + \frac{1}{(m_A r_c)^2} \simeq \frac{1}{(m_A r_c)^2}$, which gives the relation $\epsilon_{\rm th} \gg 1$. Even if the body has a thin-shell, the parameter $\epsilon_{\rm th}$ is much larger than unity under the condition $m_A r_c \ll 1$. The field profile (12) in the region $r > r_c$ reduces to

$$\phi(r) \simeq \phi_B - 2Q\left[1-\frac{1}{15}(m_A r_c)^2\right]\frac{GM_c}{r}e^{-m_B(r-r_c)}\,. \tag{14}$$

This means that the coupling is of the order of Q as in the thick-shell case. Hence it is not possible to be compatible with local gravity constraints for $|Q| = \mathcal{O}(1)$.

3.2 No-shell solutions ($r_1 = r_c$)

We require $m_A^2\left[\phi(r_c) - \phi_A\right] \le Q\rho_A$, which is equivalent to

$$\epsilon_{\rm th} \le \frac{e^{m_A r_c} + e^{-m_A r_c}}{m_A r_c (e^{m_A r_c} - e^{-m_A r_c})}\,. \tag{15}$$

3.2.1 The massive case ($m_A r_c \gg 1$)

If the field ϕ is massive such that $m_A r_c \gg 1$, the solution outside of the body is

$$\phi(r) \simeq \phi_B - 6Q\frac{GM_c}{r}\epsilon_{\rm th}\left(1-\frac{1}{m_A r_c}\right)e^{-m_B(r-r_c)}\,. \tag{16}$$

This shows that the effective coupling is given by $Q_{\rm eff} \simeq 3Q\epsilon_{\rm th}$. It is possible to satisfy local gravity constraints provided that $\epsilon_{\rm th} \ll 1$. Equation (15) gives the following constraint

$$\epsilon_{\rm th} \le \frac{1}{m_A r_c}\,. \tag{17}$$

The opposite inequality, $\epsilon_{\rm th} > 1/(m_A r_c)$, holds for the thin-shell case in the massive limit ($m_A r_c \gg 1$).

3.2.2 The light mass case ($m_A r_c \ll 1$)

When the field is almost massless such that $m_A r_c \ll 1$, the solution in the region $r > r_c$ is

$$\phi(r) \simeq \phi_B - 2Q\frac{GM_c}{r}\epsilon_{\rm th}(m_A r_c)^2 e^{-m_B(r-r_c)}\,. \tag{18}$$

From Eq. (15) we get $\epsilon_{\rm th} \le \frac{1}{(m_A r_c)^2}$. As long as $\epsilon_{\rm th}$ is much smaller than $1/(m_A r_c)^2$, it is possible to make the effective coupling $Q_{\rm eff} = Q\,\epsilon_{\rm th}(m_A r_c)^2$ small. However, the mass m_A is generally heavy to satisfy the condition $\epsilon_{\rm th}(m_A r_c)^2 \gg 1$ in concrete models that satisfy local gravity constraints.

4 Concrete models

The fifth force induced by the field $\phi(r)$ leads to the acceleration of a point particle given by $a_\phi = |Q_{\rm eff}\phi(r)|$ [3]. This then gives rise to a difference for free-fall accelerations of the moon ($a_{\rm Moon}$) and the Earth ($a_\oplus$) toward the Sun. Using the present experimental bound, $2|a_{\rm Moon} - a_\oplus|/(a_{\rm Moon}+a_\oplus) < 10^{-13}$, we obtain

$$|\phi_{B,\oplus}| < 3.7 \times 10^{-15}\,. \tag{19}$$

4.1 Inverse-power law potential

Let us consider the inverse power-law potential (2). In this case we have

$$\phi_{B,\oplus} = \left[\frac{n}{Q} \frac{M_{\rm pl}^4}{\rho_B} \left(\frac{M}{M_{\rm pl}} \right)^{n+4} \right]^{\frac{1}{n+1}} M_{\rm pl} \,. \tag{20}$$

On using the bound (19) with n and Q of the order of unity, we get the following constraint

$$M \lesssim 10^{-\frac{15n+130}{n+4}} M_{\rm pl} \,. \tag{21}$$

This shows that $M \lesssim 10^{-2}\,{\rm eV}$ for $n = 1$ and $M \lesssim 10^{-4}\,{\rm eV}$ for $n = 2$, which are consistent with the bound derived in Ref. [3].

4.2 The potential motivated by $f(R)$ gravity

The next example is the potential (3), which covers viable $f(R)$ models that satisfy local gravity constraints [5, 6]. The field value $\phi_{B,\oplus}$ is given by

$$|\phi_{B,\oplus}| = \frac{1}{2|Q|} \left(\frac{2pCV_0}{\rho_B} \right)^{\frac{1}{1-p}} M_{\rm pl} \,. \tag{22}$$

Employing (19), one can derive the bound $p > 14/9$.

The quantity $(m_A r_c)^2$ for the Earth is

$$(m_A r_c)^2 = 6|Q|(1-p) \frac{\Phi_{c,\oplus}}{|\phi_{B,\oplus}|} \left(\frac{\rho_A}{\rho_B} \right)^{\frac{1}{1-p}} \,. \tag{23}$$

Under the constraint (19) we get

$$m_A r_c \gtrsim [|Q|(1-p)]^{1/2} \cdot 10^{\frac{31-6p}{2(1-p)}} \,. \tag{24}$$

When $p > 0.65$ this condition correspond to $m_A r_c \gtrsim 10^{39}$, which means that the field is extremely massive inside of the body.

5 Conclusions

In this paper we have derived analytic solutions of a chameleon scalar field ϕ in the background of a spherically symmetric body by taking into account a field mass m_A about the potential minimum at $\phi = \phi_A$ inside of the body. We have shown that the chameleon mechanism works in a robust way provided that the field mass inside of the body satisfies the condition $m_A r_c \gg 1$ with appropriate boundary conditions.

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