NEW THEOREMS ON ALGEBRAIC EQUATION AND ITS APPLICATION TO STATISTICAL PHYSICS

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Three themes are proposed. The first is new rigorous theorems on Ising ferromagnets. The second is new theorems on algebraic equation. The last is its application to antiferromagnetic Ising models.

I. New Rigorous Theorems on Ising Ferromagnets.

A new representation of spin-spin correlation functions of Ising model was discovered. Using that, Lee-Yang theorem[1-4] was extended to correlation functions of Ising ferromagnets.

Definitions 1. Hamiltonian of Ising model:

$$\mathcal{H}_N = -(\vec{\mu}\,\hat{J}\,\vec{\mu}) - \sum_{k=1}^{N} m_k h_k \mu_k \quad ,$$

where $\hat{J}$ is the interaction matrix ($\hat{J}_{ij}=J_{ij}/2$, J_{ij} being the coupling parameter between the i-th and the j-th spins, and $J_{ij}=J_{ji}$), $\vec{\mu}$ the spin variable vecter ($\vec{\mu}=(\mu_1,\mu_2,..,\mu_N)$, μ_k being the spin vaiable at the k-site, and $\mu_k=\pm 1$), $(\cdot\cdot\cdot)$ the scalar product, m_k the magnetic constant, h_k the external field, and N the total number of spins.

2. Generalized Partition Function:

$$F_N(z_1,z_2,..,z_N,\beta,\hat{J}) = \mathrm{Tr}\ e^{-\beta\mathcal{H}_N} = \sum_{\mu_1}\sum_{\mu_2}\cdot\cdot\sum_{\mu_N} e^{\beta(\vec{\mu}\hat{J}\vec{\mu})} \prod_{k=1}^{N} z_k^{\mu_k} \quad ,$$

where z_k^2 is the 'fugacity' of the k-th spin, $z_k=e^{\beta m_k h_k}$.

3. Spin Function of the n-th order:

$$f_n(z_1,z_2,..,z_N,\beta,\hat{J};i_1,i_2,..,i_n) = \mathrm{Tr}\Big(\prod_{k'=1}^{n}\mu_{i_{k'}}\Big)e^{\beta(\vec{\mu}\hat{J}\vec{\mu})}\prod_{k=1}^{N} z_k^{\mu_k}.$$

Lemma Any spin functions of Ising model are generated from the generalized partition function (directly).

Proof: We introduce the fundamental identities such as,

$$\mu_k = -i \times i^{\mu_k} \ , \ i = \sqrt{-1} \ .$$

These can easily be found to hold. Using these, the spin function of the n-th order can be expressed as follows,

$$f_n(z_1,z_2,..z_N,\beta,\hat{J};i_1,i_2,..,i_n) = (-i)^n\, F_N(\tilde{z}_1,\tilde{z}_2,..,\tilde{z}_N,\beta,\hat{J}) \quad ,$$

where $\tilde{z}_k=z_k$ for $k \notin \{i_1,i_2,..,i_n\}$, and $\tilde{z}_{k'}=i\,z_{k'}$ for $k'\in\{i_1,i_2,..i_n\}$.

(Q.E.D.)

Theorem 1 All 'fugacity' zeros of spin functions of Ising ferromagnets (s=1/2) lie on the unit circle of the complex plane. (Generalized Lee-Yang theorem)

Proof: By the well known Lee-Yang lemma, we know that if $F_N(z_1,z_2,..,z_N,\beta,\hat{J})=0$ and $|z_1|\geqq 1, |z_2|\geqq 1,.., |z_N|\geqq 1$ hold, then $|z_1|=|z_2|=\cdot\cdot=|z_N|=1$ for $\hat{J}\geqq 0$, that is, $\forall J_{ij}\geqq 0$. As easily seen, $|\dot{i}\, z|=|z|$. Using these, we immediately find that if $f_n(z,z,..,z,\beta,\hat{J};i_1,i_2,..,i_n)=0$ $(z=e^{\beta mh})$ and $|z|\geqq 1$, then $|z|=1$ for $\hat{J}\geqq 0$. (See the previous Lemma) (Q.E.D.)

Theorem 2 The spin functions, $f_n(z,..,z,\beta,\hat{J};i_1,i_2,..,i_n)$ and $f_{n+1}(z,..,z,\beta,\hat{J};i_1,i_2,..,i_n,i_{n+1})$, have no common 'fugacity' zeros for $\hat{J}\geqq 0$. (Proof: omitted)

Theorem 3 If the interaction matrix, $\hat{J}$, is reducible, then the generalized partition function is also reducible as a function of N variables, $z_1,z_2,..,$and z_N. (Proof: omitted)

Corollary 1 The 'fugacity' zeros of the spin function of the n-th order of Ising ferromagnets distribute for $\beta=0$ (infinite temperature) as follows; the z^2-zeros degenerate at $z^2=1$ (n-fold) or at $z^2=-1$ ((N-n)-fold).

Corollary 2 The 'fugacity' zeros of the spin function of the n-th order of Ising ferromagnets (irreducible) distribute for $\beta=\infty$ (zero temperature) as follows; if n is even, the z^2-zeros are as $z_s^2=e^{i\pi(2s+1)/N}$ (s=1,2,..,N) and that if n is odd, we obtain $z_s^2=e^{i\pi 2s/N}$ (s=1,2,..,N).

Corollary 3 The ordinary spin-spin correlation functions of Ising model under the uniform field are written by using the generalized partition function as,

$$\langle \mu_{i_1}\mu_{i_2}\cdot\cdot\mu_{i_n}\rangle=(-\dot{i})^n\,\frac{F_N(z,..,z,\overset{i_1}{\dot{i}z},z,..,\overset{i_2}{\dot{i}z},z,..,\overset{i_n}{\dot{i}z},z,..,z,\beta,\hat{J})}{F_N(z,z,..,z,\beta,\hat{J})}\;.$$

II. New Theorems on Algebraic Equation.

The algebraic equation was discovered in which the distribution of roots depends only on the 'shape' of coefficients, but not on the degree of equation (Topological invariance of roots about degree). According as the sign of the parameter of coefficients, the topology of roots belong to the Lee-Yang type (on Ising ferromagnets) or not.

Theorem 1 If the polynomial is written as,

$$f^{(2N)}(z,\delta)=(g^{(N)}(z))^2+\delta z(h^{(N-1)}(z))^2\;,$$

where $g^{(N)}(z)$ and $h^{(N-1)}(z)$ are the real polynomials of the N-th and the N-1-th degree respectively, all zeros of whose polynomials lie on the unit circle of the complex plane and separated each by each, and

$g^{(N)}(1)\cdot h^{(N-1)}(1)\neq 0$,then the topology of zeros of $f^{(2N)}(z,\delta)$ is as follows; if $0\geqq\delta\geqq-(g^{(N)}(1)/h^{(N-1)}(1))^2$, all z-zeros lie on the unit circle of the complex plane, that if $\delta>0$, none of zeros lie on the unit circle ,and that if $\delta\leqq-(g^{(N)}(1)/h^{(N-1)}(1))^2$, two zeros lie on the positive real axis, and others on the unit circle. (Proof:omitted)

Theorem 2 If the polynomial is written as,

$$f^{(2N)}(z,\delta) = (g^{(N)}(z))^2+\delta(g_d^{(N)}(z))^2 ,$$

where $g_d^{(N)}(z)=(z\frac{d}{dz}-N/2)g^{(N)}(z)$, $g^{(N)}(1)\neq 0$, and allzeros of the real polynomial, $g^{(N)}(z)$, lie on the unit circle, then the topology of zeros of $f^{(2N)}(z,\delta)$ is as follows; if $\delta\geqq 0$,all zeros lie on the unit circle, and if $\delta<0$, there always exist the zeros, not lying on the unit circle. (Proof: omitted)

Example The following equation satisfies the condition of the theorem 1.

$$f^{(2N)}(z,\{a_k\})=\sum_{k=0}^{2N} a_k z^k=0,\quad a_{2N-k}=a_k,\ \text{and}\ a_k=(1+\delta)k+1\ (0\leqq k\leqq N).$$

III. Application to Antiferromagnetic Ising Models.

Various Ising models are classified through the type of the interaction matrix of the system (after appropriate permutations of lattice sites). We now consider the two cases.

Case 1 The system composed of equivalent ferromagnetic sublattices.

The interaction matrix is, for example, written as,

$$\hat{J}=\begin{pmatrix}\hat{J}_A & \hat{J}_{AB}\\ \hat{J}_{BA} & \hat{J}_B\end{pmatrix}$$

, where $\hat{J}_A$ (or $\hat{J}_B$) is the interaction matrix on the sublattice A (or B) (the sublattice-Hamiltonian as $\mathcal{H}_A$ (or $\mathcal{H}_B$)) and $\hat{J}_{AB}$ (or $\hat{J}_{BA}$) is the one between the sublattices A and B.

If each sublattice has translational symmetry, the partition function of the total system is perturbation-theoretically written as,

$$F_N(z,\beta,\hat{J}) = F_{N_A}(z,\beta,\hat{J}_A)F_{N_B}(z,\beta,\hat{J}_B) + \frac{\beta}{N_AN_B}((\vec{1}_A\hat{J}_{AB}\vec{1}_B)+(\vec{1}_B\hat{J}_{BA}\vec{1}_A))(z\frac{d}{dz}F_{N_A}(z,\beta,\hat{J}_A))(z\frac{d}{dz}F_{N_B}(z,\beta,\hat{J}_B)),$$

where $F_{N_A}(z,\beta,\hat{J}_A)=\mathrm{Tr}\ e^{-\beta\mathcal{H}_A}$, N_A the number of spins on the sublattice A, etc.. $({}^t\vec{1}_A=\underbrace{(1,1,\ldots,1)}_{N_A}$, etc.)

Using the theorem 2 of II, the topology of the z-zeros is characterized as follows; it belongs to the Lee-Yang type if $(\vec{1}_A\hat{J}_{AB}\vec{1}_B)\geqq 0$, and otherwise to the non Lee-Yang one. If the system is composed of three (or more than three) equivalent sublattices, the 'fugacity' zeros of the total partition function may lie on the three (or more than three) curves.

Case 2 The system composed of ferromagnetic sublattice and weak impurity spins.

The interaction matrix is written as,

$$\hat{J} = \begin{pmatrix} \hat{J}_A & \hat{J}_{AI} \\ \hat{J}_{IA} & \hat{J}_I \end{pmatrix}$$

, where A is the ferromagnetic sublattice and I is the lattice of impurities.

Now we only consider the case of $\hat{J}_I=0$. The partition function is perturbation-theoretically written as,

$$F_N(z,\beta,\hat{J}) = F_{N_A}(z,\beta,\hat{J}_A)Z_I+\beta((\vec{1}_A\hat{J}_{AI}\vec{1}_I)+(\vec{1}_I\hat{J}_{IA}\vec{1}_A))\frac{z^2}{N_A N_I}(\frac{dF_{N_A}}{dz})(\frac{dZ_I}{dz})$$

where $Z_I=(z+1/z)^{N_I}$ (N_I being the total number of impurities).

From the above form of the partition function, we can easily find that the topology of 'fugacity' zeros belongs to the Lee-Yang type whether $(\vec{1}_A\hat{J}_{AI}\vec{1}_I) \geqq 0$ or not (within a certain approximation).

The detail will be published later.

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