Institut für Theoretische Physik
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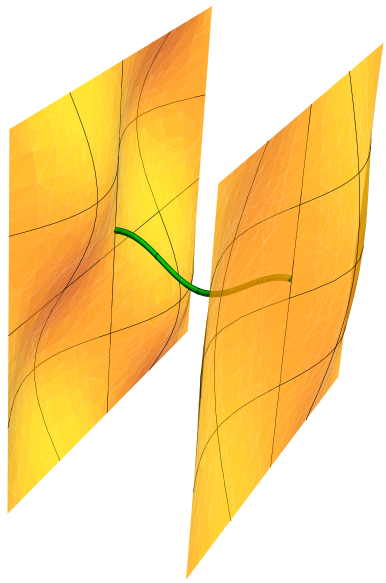
Diplomarbeit

Yang-Mills Configurations on Coset Spaces



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Erklärung

Hiermit versichere ich, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Thorsten Rahn

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Contents

Chapter I

Introduction

In Physics, symmetries are playing a very crucial role. Especially in particle physics so-called local gauge symmetries are fundamentally needed in order to predict interactions of bosons and fermions. Using the techniques of non-abelian gauge field theory, also called Yang-Mills theory, the standard model of particle physics was developed in the last century. One of the major achievements was the unification of the electromagnetic and the weak interaction in one theory, which is done by introducing a $SU(2) \times U(1)$ gauge theory. Via the Higgs mechanism and the idea of spontaneous symmetry breaking the weak gauge bosons get endowed with a mass. This is necessary since such masses were observed in particle accelerators. Since the quarks are living in $SU(3)$ multiplets, the full gauge group of the standard model turns out to be $SU(3) \times SU(2) \times U(1)$. So far almost all predicted particles of the standard model were confirmed by experimental physicists. The only missing piece here is the observation of the Higgs boson which is a scalar particle.

Even though great predictions are formulated by the standard model, there are still aspects physicists are not satisfied with. One could for instance imagine that there might be a further unification of the electroweak and the strong interaction, referred to as grand unified theory (GUT). Most GUTs that have been considered so far are based on higher gauge groups containing the original standard model gauge group, for instance $SU(5)$, $SO(10)$ or the exceptional group E_8. Furthermore, the standard model requires a large amount of parameters that has to be taken from experiments and therefore cannot be deduced from the model. There are about seventeen such parameters containing for instance the quark masses as well as the masses of the leptons.

Some other problems arise by trying to include gravity to the theory. Due to renormalization problems in such quantum field theories it is not possible to include gravity this way at all. But if one supposes that in the classical theory, the elementary particles we see are rather short strings than point particles, one arrives at a resulting theory that naturally comes along with a spin 2 field which can be interpreted as the quantum particle of gravity, i. e. the graviton. Having these hopes in mind, theoretical physicists investigated string theories since 1984 more intensive, even though the idea was born much earlier.

In the framework of string theory, it becomes necessary to consider the world we are living in not to be a four dimensional Lorentzian space-time but rather being a space of higher dimension. Since we do not observe these extra dimensions, we need to introduce these dimensions to our theory in a way such that they are hidden which means for instance that they need to have a very small size and furthermore have to be compact. This procedure is called compactification and is required to yield dimensional reduction to four dimensional space-time.

In order to get a realistic physical theory, string theory has to be equipped with supersymmetry which provides every bosonic particle with a fermionic superpartner and vice versa. In most of the known consistent string theories this requirement of supersymmetry also restricts the overall dimension of the space where the theory lives in to be ten. Some realizations of compactifications providing $\mathcal{N} = 1$ unbroken supersymmetry in the resulting low energy effective action can be realized by assuming the internal six dimensional space to be a Calabi-Yau manifold, i. e. having $SU(3)$ holonomy [1]. This was firstly considered in the mid 1980's as verypromising. However, due to moduli appearance in the low-energy effective action and due to the fact that the specific values of these moduli are not known, predictions about certain quantities such as coupling constants are not possible. Two examples of such moduli are for instance the so-called dilation, determining the string coupling and the radial modulus determining the size of the internal manifold. This problem is usually referred to as the moduli stabilization or moduli space problem. One possibility to solve the moduli stabilization problem is the so-called flux compactification. In such theories certain tensor fields are considered to have non-vanishing background values yielding fluxes that thread cycles of the internal manifold. Such fluxes can be thought of as generalized electromagnetic fluxes that belong to some field strength. For such compactifications, the condition on the internal manifold to be Calabi-Yau is no longer necessary but weaker restrictions to the internal space apply. This yields manifolds referred to as manifolds with an $SU(3)$-structure. The defining property for such manifolds is simply that the structure group of the tangent bundle can be reduced to $SU(3)$, which is a milder condition than the requirement of $SU(3)$-holonomy for Calabi-Yau manifolds.

One way of performing a dimensional reduction of the compact internal manifold is by considering this internal manifold to be a homogeneous coset space. This procedure, referred to as coset space dimensional reduction (CSDR) (see e. g. [2]), is taking advantage of the fact that homogeneous spaces admit isometries. One can then define a gauge theory on the full space and require the fields to depend on the internal coordinates only up to gauge transformations. Doing this the Higgs and the gauge sector are unified naturally which is another nice feature of the theory. Being endowed with additional dimensions, also Yang-Mills theories on such higher dimensional spacetimes become important for certain superstring theories [3], e. g. for heterotic or type IIA string theory [4] as well as for supergravity considerations [5], [6]. Also a subclass of such, namely the so-called nearly Kähler manifolds were investigatet in such frameworks [7], [8], [9], [10], [11], [12], [13], [14]. In the case of six dimensions these nearly Kähler manifolds are contained in a class of coset spaces referred to as non-symmetric coset spaces.

Therefore in this thesis we investigate the structure of $U(k)$ Yang-Mills theories and the corresponding equations of motion as well as their solutions on spaces of the form $\mathbb{R} \times G/H$, where G/H stands for a (non-)symmetric reductive coset space and $\mathbb{R}$ for one of the four flat dimensions we live in. This is a simplification, which could be generalized to four dimensional Minkowski space for instance. All these ansätze are G-equivariant which implements the dimensional reduction of the additional coset variables. The gauge potential of the theory is given by a connection on an associated principal bundle. If we consider the $U(k)$ gauge group to be broken down to $\prod_{i=1}^{m} U(k_i)$, also the gauge potential on the bundle decomposes in such pieces and in general for each block we get a number of Higgs fields that are responsible for the corresponding breakdown [15]. A physical interpretation of this situation is given in the context of type IIA string theory where we can think of these subbundles to be k_i coincident $D2$-branes and the Higgs fields being open string excitations between neighboring blocks of these $D2$-branes [16]. Specifically we will consider Yang-Mills theories with different gauge groups on different homogeneous spaces. Firstly we investigate the general ansatz of a Yang-Mills theory with gauge group G over the base space $\mathbb{R} \times G/H$. In this case we are able to write down the ansatz generically without explicit knowledge about the coset. From the G-equivariant ansatz for the gauge potential we derive the corresponding field strength as well as Yang-Mills equations. Then we distinguish between the two cases of a symmetric or non-symmetric coset spaces which yields two different equations for the Higgs field. We then analyze the equation of the symmetric case, consider BPS as well as non-BPS type solutions yielding instanton configurations as well as sphalerons respectively. The latter can be interpreted as a chain of instanton-anti-instanton pairs.

We do the same considerations for the non-symmetric case which for the BPS-type equations yield configurations that are modifications of bundles [17], [18]. Furthermore we perform a transformation of the metric in order to get a Lorentzian signature and derive the solutions of the corresponding equations of motion. These considerations on almost arbitrary coset spaces are actually generalizations of specific case such as $\mathbb{R} \times S^3$ and $\mathbb{R} \times G$ that were already considered in [19], [20], [21]. Finally, for this ansatz, we derive the Yang-Mills flow equations over the coset space whose solution is a one-parameter family of gauge potentials. This one-parameter family contains also solutions of the Yang-Mills equations at the critical points of the gauge potential,

i. e. points satisfying $\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\tau}\big|_{\tau_{\text{crit}}} = 0$. The solutions to these equations turn out to be bundle modifications similar to those coming from Yang-Mills equations.

Next we will consider $U(n)$ gauge theories for specific n on the symmetric space $\mathbb{C}P^2 = \frac{SU(3)}{SU(2)\times U(1)}$ as well as on the non-symmetric space $Q_3 = \frac{SU(3)}{U(1)\times U(1)}$. Such theories are equivalent to quiver gauge theories and $SU(3)$-equivariant ansätze where derived in [22] explicitly. Here symmetry breaking takes place and the resulting number of Higgs fields is depending on the chosen representation of $SU(3)$ which in our case is also determined by the gauge group. For $\mathbb{C}P^2$ we will take the ansätze for a $U(6)$ and $U(8)$ gauge theory which yields two and four Higgs fields respectively and derive the equation of motion for these fields. For Q_3 we will consider a $U(3)$ gauge theory which involves three Higgs fields and derive the Yang-Mills equations for these using a connection with non-vanishing torsion. In order to obtain solvable equations we need to choose a specific value for the torsion. Lastly we consider the slightly different product space $\mathbb{R}\times\mathbb{C}P^1\times\mathbb{C}P^2$, where we generalize and combine the equivariant ansätze from [19] and [22] to a $U(3(m+1))$ gauge theory, where $2m+1$ Higgs fields are involved. From the gauge potential we derive the gauge field and furthermore the Yang-Mills equations which yield a system of $2m+1$ coupled cubic second order differential equations for the Higgs fields.

The outline of the thesis is as follows. In Chapter II we will introduce basic notions of Riemannian differential geometry containing the notion of a metric, connection and curvature of a manifold. In Chapter III we will have a look at the group structure of manifolds, specifically on Lie groups and the properties of invariant tensor fields. In Chapter IV we take a look at the subclass of differential manifolds, referred to as manifolds with an almost complex structure. There we are going to introduce all notions, needed to define Kähler as well as the more general type of nearly Kähler manifolds which are a subclass of manifolds with $SU(3)$ structure. Chapter V is dedicated to the specific type of manifold we are dealing with, namely the homogeneous spaces. In order to formulate Yang-Mills theories properly we need to understand the notion of a fibre bundle specifically a principal bundle as well as all the generalized notions from Chapter II, i. e. connections and curvature on such bundles. This purely mathematical part of the thesis does not follow one particular book but was motivated and influenced by a wide range of literature, e .g. [23], [24], [25], [26], [27], [28] [29] [30] [31] [32], as well as by lectures on differential geometry and K-theory given by Prof. K. Smoczyk, Instituts für Differentialgeometrie and Prof. E. Schrohe Institut für Analysis at Gottfried Wilhelm Leibniz Universität, Hannover. In Chapter VII we briefly introduce the idea of gauge field theory and then pull together this idea with the geometry of associated principal bundles identifying the corresponding notions following [23] and [33]. The Chapters VIII-XIV are finally dedicated to the explicit calculations of Yang-Mills equations on certain spaces as well as the analysis of their solutions as described above. The results from Chapters VIII-X can be found in [34].

Chapter II

Differential Geometry

II.1 Topological Preliminary

We want to start at the very bottom of the mathematics we are dealing with during the next few chapters. To provide the background for everything that will follow, especially for the concepts of a *manifold*, we need a quite weak but fundamental structure of sets, called *topology*. In this section, we simply want to introduce this and a couple of other definitions and see how we can think of continuous functions between topological spaces.

II.1.1 Topological spaces. A *topological space* (X, T), or briefly X, is defined by a set X and a set of subsets $T = \{U_i, i \in I\}$ of X, wherein I is an arbitrary index set. The elements of T, called the *open sets*, are required to satisfy the following conditions:

$$\begin{aligned} X \in T, &\quad \emptyset \in T, \\ \bigcup_{j \in J} U_j \subset T, &\quad \forall J \subset I, \\ \bigcap_{j=1}^{n} U_j \subset T, &\forall n \in \mathbb{N}\,. \end{aligned} \tag{II.1}$$

Most common spaces are equipped with a topological structure. For instance it is known that an inner product naturally induces a norm, and a norm naturally induces a metric. One can also show that a metric naturally induces a topology, so all these spaces are topological spaces. The reverse generically fails but it may be true for topological spaces satisfying additional constraints. In the following we will see how these conditions arise.

Firstly, we define a topological space X to be *compact* if an arbitrary open cover $\{U_i\}$ of X can be reduced to a finite cover, i. e.

$$X \subset \bigcup_{j \in J} U_j \Rightarrow \exists\ i_1, ..., i_n \in J \text{ s.t. } X \subset \bigcup_{j=i_1}^{i_n} U_j\,.$$

We do not exactly know yet what a manifold is, but it should generalize common flat spaces like the $\mathbb{R}^n$ and hence compactness should not be constraining the manifold. We are actually going to consider a weaker condition here, namely *paracompactness.* For such a space X if $\{U_i, i \in I\}$ is an arbitrary open cover of X, then there is another open cover $\{V_j, j \in J\}$ having the property

$$\forall j \in J,\ \exists i \in I:\ V_j \subset U_i\,,$$

i. e. there is a *refinement* of the open cover satisfying

$$\forall x \in X,\ \exists V_x \in V,\ x \in V_x \text{ and } V_x \cap V_j \neq \emptyset$$

only for a finite number of $j \in J$, i. e. the refinement is *locally finite.* So a topological space is called paracompact if and only if every open cover has a locally finite refinement.

A last condition which is needed to define a manifold is that for every pair of points $x_1 \neq x_2$ in X there must be open sets U_1, U_2 in T, such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. Topological spaces, equipped with this property are called *Hausdorff.* This condition is needed but most spaces in Physics we can think of naturally satisfy this condition anyway.

II.1.2 Continuity. Let (X, T_X) and (Y, T_Y) be topological spaces with given mapping $\varphi : X \to Y$. We define φ to be *continuous* if $\forall\, V \in T_Y$, $\varphi^{-1}(V) \in T_X$. This is another way of saying that the preimage of an arbitrary open set of Y has to be an open set of X. By definition, the continuous maps preserve the topology and therefore topological spaces and continuous maps together form a category and hence bijective continuous maps are called *homeomorphisms.*

As mentioned in the beginning, topological spaces generically do not carry a metric. However, for paracompact Hausdorff spaces, one can always find a homeomorphism to a metric space. Such spaces are called *metrizable.*

II.2 Smooth Manifolds

Now we have everything to define the central object of our mathematical considerations, namely the manifold.

II.2.1 Manifolds. A *topological manifold* M is a topological paracompact Hausdorff space, endowed with a collection of homeomorphisms

$$\{\varphi_i : U_i \longrightarrow \mathbb{R}^n\}\ ,$$

where $\mathcal{U} = \{U_i\}$ is an open cover of M. These maps locally describe the underlying space and are therefore called *charts.* For obvious reasons the set of all possible charts $\{(\varphi_i, U_i)\}$ is called an *atlas* of M. Considering different elements of the atlas we define the *transition functions* t_{ij} by

$$t_{ij} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$
$$t_{ij} := \varphi_i^{-1} \circ \varphi_j\,.$$

In case that all transition functions in an atlas are smooth maps, the atlas is called smooth. A smooth atlas that contains all possible charts satisfying this condition is called a *differentiable structure*. A topological manifold equipped with a differentiable structure is called a *differentiable* or *smooth manifold*. If all the charts in the smooth structure of M are mapping to the same real space $\mathbb{R}^n$, we call n the *real dimension* of M.

II.3 (Co-)Tangent Spaces and Bundles

In the last section we have defined a smooth manifold. One can think of simple examples of manifolds, e. g. curves embedded in $\mathbb{R}^2$. Due to the embedding, we can always imagine what is meant by a tangent vector. For the curve, we just take the derivative of the parameterized position vector and get something we can think of as a velocity vector that spans a one-dimensional vector space, tangent to the corresponding point of the curve. For a 2-dimensional surface, the procedure works similarly and is still intuitive. The only difference here is that we have two curves the latter case, yielding two linearly independent tangent vectors, hence we get a two-dimensional tangent space. These vector spaces are just subspaces of the embedding space of the manifold, here for instance $\mathbb{R}^3$. In this section we want to find a notion of a tangent vector as well as the notion of a tangent space that we can apply to an arbitrary manifold even without any knowledge about a specific embedding. Hence we need something that allows the definition of a vector space at each point of the manifold, satisfying the given requirements. There are many equivalent ways to define it.

II.3.1 Tangent space. Let M be a smooth manifold, $x : U \longrightarrow \mathbb{R}^n$ be a chart on a neighborhood of an element $p \in M$. A map $\gamma : I \longrightarrow M$, $I \in \mathbb{R}$, given by $t \mapsto \gamma(t)$ is called smooth if the composition $x \circ \gamma : I \longrightarrow \mathbb{R}^n$ is a smooth map. Such a map is called a *curve.*

Now let γ and γ' be curves on M, such that $\gamma(0) = \gamma'(0) = p$. We define an equivalence relation $\gamma \sim \gamma'$ for all such curves, by

$$\gamma \sim \gamma' \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} x^i(\gamma(0)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} x^i(\gamma'(0)) \,. \tag{II.2}$$

An equivalence class with respect to this equivalence relation is called a *tangent vector* in p and the set of equivalence classes is called the *tangent space* in p and denoted by T_pM. Although, this definition of the tangent space is quite intuitive, it is not very convenient to work with. Hence, we are going to state another definition, which is equivalent to the one above, but easier to handle.

For this purpose we first need a few more definitions. A map $f : U \to \mathbb{R}$ is called a smooth function if the map $f \circ (x^{-1})$ is a smooth function in the usual sense. The set of all smooth functions defined on U are denoted by $\mathcal{F}(U)$. Defining the sum and the multiplication by a scalar in the canonical way, we see that this set actually forms an $\mathbb{R}$-algebra. We define $\mathcal{F}_p(M)$ to be the set of equivalence classes of smooth functions on a neighborhood of p, where two of them are equivalent if they coincide on some neighborhood of p. Furthermore we define a map

$$\frac{\partial}{\partial x^\mu} : \mathcal{F}(M) \longrightarrow \mathbb{R},$$
$$\text{by } \frac{\partial}{\partial x^\mu}\Big|_p(f) := \partial_\mu f(x^{-1})\Big|_{x(p)}, \tag{II.3}$$

where ∂_μ denotes the usual partial derivative in Euclidian space.

Let $V : \mathcal{F}_p M \to \mathbb{R}$ be an $\mathbb{R}$-linear map, satisfying

$$V(g \cdot f) = f \cdot v(g) + g \cdot v(f), \quad \forall f, g \in \mathcal{F}_p M\,. \tag{II.4}$$

We call V a tangent vector at p and the set of all possible tangent vectors forms a vector space, defined as the *tangent space*. One can show that the maps $\left\{ \frac{\partial}{\partial x^\mu}\Big|_p, \mu = 1, ..., n \right\}$, from (II.3) form a basis of the tangent space, called the *coordinate basis*.

II.3.2 Cotangent space and tensor product. Starting with the definition of the tangent space T_pM of M in p, we can define the *cotangent space* as the dual space of T_pM, the vector space containing all linear maps from T_pM to $\mathbb{R}$. We denote the cotangent space in the usual way by T_p^*M. Since T_pM is finite dimensional vector space, its dual space has the same dimension and hence we can choose a specific basis of the cotangent space, dual to the coordinate basis which is denoted by $\left\{ \mathrm{d}x^\mu\Big|_p, \mu = 1, ..., n \right\}$ and satisfies

$$\frac{\partial}{\partial x^\nu} \lrcorner \mathrm{d}x^\mu := \mathrm{d}x^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \delta^\mu{}_\nu\,. \tag{II.5}$$

The elements of T_p^*M are called *covectors*.

Since we are only dealing with finite-dimensional vector spaces, we can carry over notions and concepts from linear algebra for those spaces, e. g. arbitrary tensor products of T_pM and T_p^*M:

$$T_p{}^r_s M := \underbrace{T_pM \otimes \cdots \otimes T_pM}_{r-\text{times}} \otimes \underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{s-\text{times}} \tag{II.6}$$

Elements of $T_p{}^r_s M$ are called *tensors* and can be expanded with respect to the coordinate basis covering some neighborhood of p. In order to distinguish indices of the tensor, those belonging to a tangent space are written as upper, and those belonging to a cotangent space are written as lower indices. An arbitrary tensor $T \in T_p{}^r_s M$ therefore decomposes as

$$T = T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \frac{\partial}{\partial x^{\mu_1}}\Big|_p \otimes ... \otimes \frac{\partial}{\partial x^{\mu_r}}\Big|_p \otimes \mathrm{d}x^{\nu_1}\Big|_p \otimes ... \otimes \mathrm{d}x^{\nu_s}\Big|_p\,. \tag{II.7}$$

Finally, we want to consider a subspace of $T_{p\,s}^{\,0}M$, namely all tensors carrying only lower indices which are totally antisymmetric in addition, e. g. $T_{\nu_1\nu_2\ldots\nu_s} = -T_{\nu_2\nu_1\ldots\nu_s}$. They are decomposed as

$$T = T_{\nu_1\ldots\nu_s}\mathrm{d}x^{\nu_1}\Big|_p \otimes \ldots \otimes \mathrm{d}x^{\nu_s}\Big|_p, \tag{II.8}$$

and will play a crucial role later on.

It is convenient to write down these tensors in a slightly different way. For this purpose we define the antisymmetric tensor product of covectors by

$$V^* \wedge W^* := V^* \otimes W^* - W^* \otimes V^* . \tag{II.9}$$

Using this notation, equation (II.8) can be written as

$$T = \frac{1}{s!}T_{\nu_1\ldots\nu_s}\mathrm{d}x^{\nu_1}\Big|_p \wedge \ldots \wedge \mathrm{d}x^{\nu_s}\Big|_p . \tag{II.10}$$

Since we can express such kind of tensors with the *wedge product* $\wedge$, this space is denoted by $\Lambda_s T_p^* M$

II.3.3 Vector bundles and sections. So far we have figured out how to attach certain vector spaces to a point $p \in M$. We now want to find a way in which we can deal not only with the vector space at a certain point p but with all the spaces at all other points of the manifold in a direct manner. Such a union of vector spaces is called a *vector bundle*, or a *tensor bundle* in case when the vector space is actually a tensor product of vector spaces. Since we do not want to loose the information that tells us which subspace of the bundle belongs to the point of the manifold, we need a map, called the *projection map*, containing this information. Formally, we define a vector bundle as follows:

Let M be a smooth manifold, also called the *base space* in this context. Let furthermore E be a topological space, called the *total space*, and $\pi : E \to M$ be a smooth map, such that $\pi^{-1}(p) =: E_p$ is a *vector space* of dimension k for all $p \in M$, called the *fibre* over p. The triple (E, π, M) is called a vector bundle if

$$\forall p \in M\ \exists \text{ open } U \ni p, \text{ such that } \pi^{-1}(U) \simeq U \times E_p, \tag{II.11}$$

where an isomorphism between vector bundles is simply the natural generalization of an isomorphism of vector spaces. Furthermore, a smooth map $s : M \to E$, such that $s(p) \in E_p$ is called a *section* in E. The set of sections of a bundle E is denoted by $\Gamma(E)$.

II.3.4 Tangent bundle, cotangent bundle and tensor bundle. Now we want to describe some very important vector bundles of a manifold M as well as their sections. One can show that the space

$$TM := \bigcup_{p\in M} T_pM \tag{II.12}$$

defines a vector bundle over M with the natural projection, mapping tangent vectors of p onto p. This vector bundle is called the *tangent bundle*. The set of sections $\Gamma(TM)$ forms a vector space and its elements are called *vector fields*. A basis of this space is given locally in the coordinates $\{x^\mu, \mu = 1, ..., n\}$ by $\left\{\frac{\partial}{\partial x^\mu}, \mu = 1, ..., n\right\}$. Also the set of all cotangent spaces,

$$T^*M := \bigcup_{p \in M} T_p^*M\,, \tag{II.13}$$

defines a vector bundle over M with the natural projection. It is called the *cotangent bundle* and the space of sections, denoted by $\Gamma(T^*M)$, contains elements called *1-forms*. A basis of 1-forms is locally given by $\{\mathrm{d}x^\mu, \mu = 1, ..., n\}$.

It is quite obvious these definitions can be generalized to an arbitrary tensor product. Hence

$$T_s^r M := \bigcup_{p \in M} T_{p\,s}^{\ r} M \tag{II.14}$$

defines a vector bundle called the *tensor bundle* on M. The space of sections $\Gamma(T_s^r M)$ is locally spanned by the tensor products of the basis vector fields and the basis 1-forms

$$\left\{\frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_r}} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_s}\,, \mu_i, \nu_i = 1, ..., n\right\}\,. \tag{II.15}$$

The sections of the tensor bundle are called *tensor fields*. At last we consider the special case, where the fibre is a wedge product of cotangent spaces, namely the bundle

$$\Lambda_s(TM) := \bigcup_{p \in M} \Lambda_s T_p^* M\,. \tag{II.16}$$

Sections of this bundle are called s-forms and the space they span is denoted by $\Omega_s(M) := \Gamma(\Lambda_s(TM))$. A basis for the space of s-forms is locally given by the wedge product of basis 1-forms, namely

$$\{\mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_s}, 1 \leq \mu_1 \leq ... \leq \mu_s \leq n\}\,. \tag{II.17}$$

II.3.5 Exterior derivative. As a next point, we want to introduce a differential operator that acts on s-forms and maps them to $s+1$ forms. Since the result of the derivative lies in a different space than the s-form that we plugged in, it is called *exterior derivative*. It is denoted by d and defined as follows

$$\begin{aligned}
\mathrm{d} : \Omega_s(M) &\longrightarrow \Omega_{s+1}(M)\,, \\
\omega &\longmapsto \mathrm{d}\omega\,, \\
\mathrm{d}\omega &:= \frac{1}{s!}\frac{\partial \omega_{\mu_1 ... \mu_s}}{\partial x^\rho}\, \mathrm{d}x^\rho \wedge \mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_s}\,.
\end{aligned} \tag{II.18}$$

Let α and ω be an r-form and an s-form, respectively. The exterior derivative has the following properties:

$$\begin{aligned} \mathrm{d}(\lambda\alpha) &= \lambda\mathrm{d}(\alpha) \quad \forall\lambda \in \mathbb{R}, \quad (\mathrm{d} \text{ is } \mathbb{R}\text{-linear}) && \text{(II.19)} \\ \mathrm{d}(\alpha \wedge \omega) &= \mathrm{d}\alpha \wedge \omega + (-1)^r \alpha \wedge \mathrm{d}\omega && \text{(II.20)} \\ \mathrm{d}\mathrm{d}\omega &= 0 && \text{(II.21)} \end{aligned}$$

An s-form ω satisfying $\mathrm{d}\omega = 0$ is called *closed*. If there is an $(s-1)$-form η such that $\mathrm{d}\eta = \omega$, ω is said to be *exact*. According to (II.21) it is obvious that every exact form is also closed.

II.3.6 Holonomic and non-holonomic basis. Let us consider the basis of the tangent space $\left\{\frac{\partial}{\partial x^\mu}, \mu = 1, ..., n\right\}$ and the basis of the cotangent space $\{\mathrm{d}x^\mu, \mu = 1, ..., n\}$, locally given by the coordinates $\{x^\mu, \mu = 1, ..., n\}$ once more. As already mentioned in paragraph II.3.4 and making use of the definition of the Lie bracket, we find

$$\begin{gathered} \frac{\partial}{\partial x_\mu} \lrcorner \mathrm{d}x^\nu = \mathrm{d}x^\nu \left(\frac{\partial}{\partial x_\mu}\right) = \delta^\nu{}_\mu \qquad \text{and} \\ \left[\frac{\partial}{\partial x_\mu}, \frac{\partial}{\partial x_\nu}\right] = 0\,. \end{gathered} \tag{II.22}$$

Since the vector fields contained in this basis commute, we call it a *holonomic* basis. In general it is not necessary that the vector fields forming an arbitrary basis do commute and therefore we also consider the more general case, where they do not. Let $\{e_a, a = 1, ..., n\}$ and $\{e^a, a = 1, ..., n\}$ be a different basis of vector fields and 1-forms satisfying the equations

$$\begin{gathered} e_a \lrcorner e^b = e^b(e_a) = \delta^b{}_a \qquad \text{and} \\ [e_a, e_b] = f_{ab}{}^c e_c\,. \end{gathered} \tag{II.23}$$

Here the elements of the basis do not commute and therefore form a so-called *non-holonomic* basis. Since (II.22) and (II.23) are both bases of the same space, they have to be linearly dependent in each fibre, namely

$$\begin{aligned} e_a = &\; e^\mu{}_a \frac{\partial}{\partial x^\mu}\,, & \frac{\partial}{\partial x^\mu} = &\; e^a{}_\mu e_a \quad \text{(for vector fields)}\,, \\ e^a = &\; e^a{}_\mu \mathrm{d}x^\mu\,, & \mathrm{d}x^\mu = &\; e^\mu{}_a e^a \quad \text{(for 1-forms)}\,. \end{aligned} \tag{II.24}$$

Since the e_a form a non-holonomic basis, it is obvious that the matrices $e^\mu{}_a, e^a{}_\mu$ cannot be constant, otherwise the e_a would commute. In order to see which basis we are using, we will only take indices from the late Greek alphabet to denote components of tensors with respect to the coordinate basis while using letters from the Latin alphabet to denote components with respect to the non-holonomic basis.

II.4 Bundle Metric and Riemannian Manifolds

In the last section, we introduced the tangent space and the tangent bundle, by attaching vector spaces to each point of the manifold. Now we can add some more structure to these spaces, i. e. we want to find a way to equip every tangent space with an inner product, which is later be used to define a metric on the manifold. An *inner product* for an arbitrary vector space $\mathcal{V}$ over $\mathbb{R}$ is defined as a map $\langle\cdot,\cdot\rangle : \mathcal{V}\times\mathcal{V}\to\mathbb{C}$ satisfying three conditions

$$\begin{aligned} \langle V,W\rangle = \langle W,V\rangle\,, \quad \forall V,W\in\mathcal{V} \quad (\text{symmetric})\,,\\ \text{It is } \mathbb{R}\text{-linear in the first argument},\\ \langle V,V\rangle \geq 0 \text{ and } \langle V,V\rangle = 0,\ \text{iff } V=0 \quad (\text{positive definite})\,. \end{aligned} \tag{II.25}$$

In the following we are going to introduce the notion of a metric on the manifold that reduces to such an inner product on every tangent space.

II.4.1 Riemannian metric and Riemannian manifolds. We start by simply defining an object in a straightforward manner by requiring a so-called metric to satisfy the properties of an inner product (II.25) locally. We state these definitions first in a general way and then have a closer look at the case where the vector space is the tangent space and the bundle is the tangent bundle. The generalization of these results back to the general case of a tensor bundle is then straightforward.

Let M be a smooth manifold, p an arbitrary point in M and (U,x) a chart such that $p\in U$. Furthermore, let (E,π,M) be a vector bundle. We define a *bundle metric* g on E to be an element of $\Gamma(E^*\otimes E^*)$ satisfying two conditions

$$g_p(V,W) = g_p(W,V) \quad \text{for all } V,W\in E_p\,, \tag{II.26}$$

$$g_p(V,V)\geq 0 \quad \text{and} \quad g_p(V,V)=0,\ \text{iff } V=0\,. \tag{II.27}$$

Hence if $E=TM$, g maps two tangent vectors to a real number. It is easy to see that one can also consider g as a map of vector fields to smooth functions. Since it is a section, we can express the metric in terms of local coordinates

$$g = g_{\mu\nu}\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu\,. \tag{II.28}$$

The symmetry condition reads then

$$g_{\mu\nu} = g_{\nu\mu}\,. \tag{II.29}$$

A bundle metric defined on the tangent bundle like above, is also called a *Riemannian metric* and a smooth manifold endowed with a Riemannian metric is called a *Riemannian manifold* and denoted by (M,g).

II.4.2 Theorem. *Every smooth manifold M admits a Riemannian metric, which is induced by the smooth structure.*

Considered as a local statement, this theorem is straightforward to prove. For a neighborhood U of p, we have a corresponding diffeomorphism that maps U to some open set in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is already endowed with an inner product, one can simply define the metric on the tangent space to be the pullback of the Euclidian inner product. Since a smooth manifold is also paracompact, one can use a partition of unity to define a metric globally on M. □

II.4.3 Induced metric. Another consequence which is also straightforward to see is that a Riemannian metric on M induces a metric on the cotangent space and on arbitrary products of the tangent space and the cotangent space. Therefore, a Riemannian manifold is also endowed with a metric on the tensor bundle $T^r_s M$ for arbitrary r and s. This will become important in further sections, when we associate some sort of derivative on a bundle with the corresponding bundle metric.

II.5 Lie Derivative

In this section we first want to introduce a notion of a derivative of tensor fields, namely the *Lie derivative*. Later on we will also consider another kinds of derivatives. The idea of this one is quite intuitive. The goal is to describe how a tensor field changes in different directions which can be specified by a vector field. In addition, we need a mechanism that allows us to compare tensors in different points of the manifold, which is not obvious, since the tensor field maps each point of the manifold to a different space.

II.5.1 Local flow and Lie transport. Let M be a manifold, $\gamma : [0, 1] \longrightarrow M$ be a smooth curve. Of course, we can take the derivative of this map and obtain a tangent vector for each $\tau \in I$. These tangent vectors are lying in the corresponding tangent space of $\gamma(\tau)$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\gamma(\tau) \in T_{\gamma(\tau)}M\,. \tag{II.30}$$

Now we can imagine a vector field, mapping the image of the curve to its derivative. It can be shown that for each vector field v and tangent vector $v_p \in T_pM$, where $p \in M$, there is locally exactly one curve γ_v, satisfying

$$\gamma_v(0) = p \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\gamma_v(0) = v_p\,. \tag{II.31}$$

This curve is completely specified by the vector field v and is called an *integral curve* of v. Hence on a neighborhood U of p, we can define a map $F_s : U \longrightarrow M$ by $F_s(p) := \gamma_v(\tau)$. This map is called a *local flow*. Due to its properties

$$F_{\tau_1} \circ F_{\tau_2} = F_{\tau_1+\tau_2}\,, \quad F_{-\tau} = (F_\tau)^{-1} \tag{II.32}$$

it is also referred to as a *one parameter group of transformations.*

Since F_τ is a smooth map from M to M, we can find a corresponding map from the tensor bundle $T^r_s(M)$ to itself, using its pullback F^*_τ:

$$F^*_\tau : T^r_s(M) \longrightarrow T^r_s(M)\,. \tag{II.33}$$

This map allows us to compare tensors in different spaces, e. g. a tensor in $T_{p_1}M$ with a tensor in $T_{p_2}M$, where p_1 and p_2 are elements of the neighborhood U of p. It is called the *Lie transport.*

II.5.2 Lie derivative. Now we have gathered the necessary basics to measure the rate of changing of tensor fields in a given direction. Therefore, the definition of a derivative can be stated in a reasonable way. We define the *Lie derivative* $\mathcal{L}_v t$ of a tensor field $t \in T^r_s(M)$ as the derivative of the pullback of the local flow, evaluated at $\tau = 0$:

$$\mathcal{L}_v t := \frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_{t=0} F^*_\tau t\,. \tag{II.34}$$

It is easy to show that the Lie derivative satisfies all the properties needed for a proper derivative: It is $\mathbb{R}$-linear and satisfies the Leibniz rule, i. e.

$$\mathcal{L}_v(t \otimes t') = \mathcal{L}_v(t) \otimes t' + t \otimes \mathcal{L}_v(t')\,. \tag{II.35}$$

There is one more property, telling us how the Lie derivative acts on a generic tensor fields,

$$\begin{aligned} \mathcal{L}_v(t(v_1, ..., \alpha_1, ..., \alpha_s)) &= (\mathcal{L}_v t)(v_1, ..., \alpha_1, ..., \alpha_s) \\ &+ t(\mathcal{L}_v v_1, ..., \alpha_1, ..., \alpha_s) + ... \\ &+ t(v_1, ..., \mathcal{L}_v \alpha_1, ..., \alpha_s) + ... \\ &+ t(v_1, ..., \alpha_1, ..., \mathcal{L}_v \alpha_s)\,, \end{aligned} \tag{II.36}$$

where $v_1, ..., v_r$ are are vector fields and $\alpha_1, ..., \alpha_s$ are 1-forms.

Using all these properties, we obtain the explicit formula for the Lie derivative of a tensor field t which reads in components

$$\begin{aligned} (\mathcal{L}_v t)^{\mu_1 ... \mu_r}_{\nu_1 ... \nu_s} &= v^\rho \frac{\partial}{\partial x^\rho} t^{\mu_1 ... \mu_r}_{\nu_1 ... \nu_s} + \frac{\partial v^\rho}{\partial x^{\nu_1}} t^{\mu_1 ... \mu_r}_{\rho ... \nu_s} + ... + \frac{\partial v^\rho}{\partial x^{\nu_s}} t^{\mu_1 ... \mu_r}_{\nu_1 ... \rho} \\ &-\frac{\partial v^{\mu_1}}{\partial x^\rho} t^{\rho ... \mu_r}_{\nu_1 ... \nu_s} - ... - \frac{\partial v^{\mu_r}}{\partial x^\rho} t^{\mu_1 ... \rho}_{\nu_1 ... \nu_s}\,. \end{aligned} \tag{II.37}$$

For the special cases, where t is a function, a vector field or a 1-form, (II.37) is written in an especially nice form:

$$\mathcal{L}_v f = v^\rho \frac{\partial}{\partial x^\rho} f = v(f)\,, \quad \forall f \in C^\infty(M)\,, \tag{II.38}$$

$$\mathcal{L}_v w = (v^\rho \frac{\partial}{\partial x^\rho} w^\sigma - w^\rho \frac{\partial}{\partial x^\rho} v^\sigma) \frac{\partial}{\partial x^\sigma} = [v, w]\,, \quad \forall w \in \Gamma(TM)\,, \tag{II.39}$$

$$\mathcal{L}_v \alpha = (v^\rho \frac{\partial}{\partial x^\rho} \alpha_\sigma + \alpha_\rho \frac{\partial}{\partial x^\sigma} v^\rho) \mathrm{d}x^\sigma = (\iota_v \circ \mathrm{d} + \mathrm{d} \circ \iota_v)\alpha\,, \quad \forall \alpha \in \Gamma(T^*M)\,. \tag{II.40}$$

Using the non-holonomic basis defined by equation (II.23) as well as equations (II.38)-(II.40), one can show that the Lie derivative of a tensor which is a tensor again, has the following components in the non-holonomic basis

$$\mathcal{L}_{e_c} t^{a_1 \dots a_r}_{b_1 \dots b_s} = e_c(t^{a_1 \dots a_r}_{b_1 \dots b_s}) + f_{ce}{}^{a_1} t^{e \dots a_r}{}_{b_1 \dots b_s} + \dots + f_{ce}{}^{a_r} t^{a_1 \dots e}{}_{b_1 \dots b_s}$$
$$- f_{cb_1}{}^{e} t^{a_1 \dots a_r}{}_{e \dots b_s} - \dots - f_{cb_s}{}^{e} t^{a_1 \dots a_r}{}_{b_1 \dots e} \,. \tag{II.41}$$

II.5.3 Killing vector fields. As we have stated above, every vector field v gives rise to a local flow $F_t : M \longrightarrow M$ in a neighborhood of some point $p \in M$. Now suppose that M is a Riemannian manifold with the Riemannian metric g. Then it can be shown that there are vector fields whose one-parameter group is an isometry, i. e. it preserves the metric. These vector fields are called *Killing vector fields.* If we consider the Lie derivative of the metric with respect to a Killing vector field we find that

$$\mathcal{L}_v g_{\mu\nu} = v^\rho \frac{\partial}{\partial x^\rho} g_{\mu\nu} + \left(\frac{\partial}{\partial x^\mu} v^\rho \right) g_{\rho\nu} + \left(\frac{\partial}{\partial x^\nu} v^\rho \right) g_{\mu\rho} \,. \tag{II.42}$$

For an isometry this equation simply vanishes,

$$\mathcal{L}_v g_{\mu\nu} = 0 \,. \tag{II.43}$$

Since this equation determines the Killing vector fields, it is called the *Killing equation.* Defining $v_\sigma = v^\tau g_{\tau\sigma}$ and using the Leibniz rule for the partial derivative, we can rewrite (II.42) as

$$\begin{aligned} \mathcal{L}_v g_{\mu\nu} &= v_\tau g^{\tau\rho} \left(\frac{\partial}{\partial x^\rho} g_{\mu\nu} - \frac{\partial}{\partial x^\nu} g_{\mu\rho} - \frac{\partial}{\partial x^\mu} g_{\nu\rho} \right) + \frac{\partial x^\mu}{\partial v_\nu} + \frac{\partial x^\nu}{\partial v_\mu} \\ &=: \frac{\partial x^\mu}{\partial v_\nu} + \frac{\partial x^\nu}{\partial v_\mu} - 2\Gamma^\tau_{\mu\nu} v_\tau \,, \end{aligned} \tag{II.44}$$

wherein

$$\Gamma^\tau_{\mu\nu} := \frac{1}{2} g^{\tau\rho} \left(\frac{\partial}{\partial x^\nu} g_{\mu\rho} + \frac{\partial}{\partial x^\mu} g_{\nu\rho} - \frac{\partial}{\partial x^\rho} g_{\mu\nu} \right) \tag{II.45}$$

are called the *Christoffel symbols.* We will recognize them in a later section, when we will define the so-called *covariant derivative.* Its construction will be done in terms of a so-called connection of the tangent bundle which is quite different from the definition of the Lie derivative.

Another notable property of the Killing vector fields is that the Lie bracket of two Killing vector fields again yields a Killing vector field, in other words, the set of Killing vector fields closes under $[\cdot, \cdot]$. Due to the $\mathbb{R}$-linearity of the Lie derivative, the linear combination of Killing vector fields again results in a Killing vector field. Hence they form a subalgebra of the algebra of vector fields on M which is always finite-dimensional. If it has dimension $n(n+1)/2$, where n denotes the dimension of M, then M is called a *maximally symmetric space.*

II.6 The Levi-Civita Connection

In the previous sections, we were dealing with different kinds of bundles, e. g. with the vector bundle. These bundles are defined by attaching vector spaces to each point of the manifold. But since the manifold does not necessarily have to be flat, the attached spaces do not need to be identical but may be twisted against each other for instance. So we do not have the information in which way elements of different fibres can be compared. For instance, tangent vectors from different tangent spaces cannot be compared straight away, since they are elements of different vector spaces. We discussed a similar thing in section II.5.2, where we used the pullback of the Lie transport to compare tangent vectors with each other. We will therefore define an object which can be used to connect fibres of some vector bundle at different points of the manifold. Such an object is called a connection and is formally defined in the next paragraph.

II.6.1 Connections and the covariant derivative. Let M be a smooth manifold, and E be a vector bundle over M. Furthermore, let $s, s_1, s_2 \in \Gamma(E)$ be sections of E and $v, v_1, v_2 \in \Gamma(TM)$ be vector fields. A *connection* on E is defined as a C^∞-linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) \quad , \tag{II.46}$$

$$s \longmapsto \nabla s \quad , \tag{II.47}$$

which straightforwardly induces a second map, the *covariant derivative* on E, defined by

$$\nabla s : \Gamma(TM) \to \Gamma(E)\,, \tag{II.48}$$

$$v \longmapsto \nabla_v s. \tag{II.49}$$

It is furthermore required to be $\mathbb{R}$-linear and to satisfy the following rule:

$$\nabla_v(fs) = v(f) \cdot s + f\nabla_v(s)\,. \tag{II.50}$$

This is a very generic notion, and in order to define the most canonical, but important Levi-Civita connection, we need to introduce a couple of properties of a connection. From now on we will constrain ourselves to connections on the tangent and cotangent bundle and their tensor products. One can show that if we have a connection on the tangent bundle, this connection induces a connection on the cotangent bundle and also on arbitrary tensor products of those bundles. Therefore we only need to know how the connection acts on basis vectors of the tangent and cotangent bundle.

II.6.2 Connection coefficients. Let ∇ be a connection on the tangent bundle. It is convenient to use the same symbol for the induced connections mentioned above. We define the *connection coefficients* to be the coefficients of the covariant derivative of one basis vector field with respect to another basis vector fields, i. e.

$$\nabla_\mu \frac{\partial}{\partial x^\nu} =: \Gamma^\rho_{\mu\nu} \frac{\partial}{\partial x^\rho}, \qquad \nabla_a e_b =: \Gamma^c_{ab} e_c, \tag{II.51}$$

where

$$\nabla_\mu := \nabla_{\frac{\partial}{\partial x^\mu}}, \qquad \nabla_a := \nabla_{e_a}. \tag{II.52}$$

In this notation, the covariant derivative of the basis 1-forms with respect to the induced connection can be expressed as follows:

$$\nabla_\mu \mathrm{d}x^\nu = -\Gamma^\nu_{\mu\rho} \mathrm{d}x^\rho, \qquad \nabla_a e^b = -\Gamma^b_{ac} e^c \tag{II.53}$$

If we plug the transformation rule (II.24) into the definition of the connection coefficients (II.51), we arrive at the following relations:

$$\Gamma^c_{ab} = e^\mu{}_a \left(\frac{\partial}{\partial x^\mu} e^\rho{}_b + e^\nu{}_b \Gamma^\rho_{\mu\nu}\right) e^c{}_\rho \tag{II.54}$$

$$\Gamma^\rho_{\mu\nu} = e^a{}_\mu \left(-e_a(e^\rho{}_b) + e^\rho{}_c \Gamma^c_{ab}\right) e^b{}_\nu \tag{II.55}$$

The relations (II.54) and (II.55) show that the connection coefficients cannot be coefficients of a tensor, due to an extra term in the transformation law which does not appear in tensor transformations. Since we are free to change the name of silent indices, we can easily see that the spurious piece, i. e. the first summand of the right hand side (II.54) and (II.55), is symmetric in the two lower indices of the connection coefficients. Hence, this term disappears in an object antisymmetric in the connection coefficients with respect to these two indices and we get a proper transformation rule for an object constructed in such a way.

II.6.3 Covariant derivative of tensor fields. In the last paragraph we have shown how to take the covariant derivative of the basis vector fields and 1-forms, stated in equations (II.51) and (II.53). This can be easily generalized to arbitrary tensor fields. Let $T \in \Gamma(T^r_s M)$ be a tensor field on M. Note that using either the holonomic or the non-holonomic basis does not make a difference in our formulae. The difference only appears in the connection coefficients, so we only state the result with respect to the non-holonomic basis. The components of T are defined by

$$T = T^{a_1 \dots a_r}{}_{b_1 \dots b_s} e_{a_1} \otimes \dots \otimes e_{a_r} \otimes e^{b_1} \otimes \dots \otimes e^{b_s}\,. \tag{II.56}$$

Therefore, by taking advantage of equations (II.51), (II.53) and (II.50), we arrive at the following formulae for the components of the covariant derivative of T:

$$\begin{aligned} \nabla_c T^{a_1 \dots a_r}{}_{b_1 \dots b_s} = {} & e_c(T^{a_1 \dots a_r}{}_{b_1 \dots b_s}) \\ & + \Gamma^{a_1}_{ce} T^{e \dots a_r}{}_{b_1 \dots b_s} + \dots + \Gamma^{a_r}_{ce} T^{a_1 \dots e}{}_{b_1 \dots b_s} \\ & - \Gamma^e_{cb_1} T^{a \dots a_r}{}_{e \dots b_s} - \dots - \Gamma^e_{cb_s} T^{a \dots a_r}{}_{b_1 \dots e}\,. \end{aligned} \tag{II.57}$$

II.6.4 Connection 1-form. There is an equivalent way to express how the connection acts on the basis vector fields and 1-forms. We define a matrix valued 1-form, namely the *connection 1-form* with respect to a non-holonomic basis and its connection coefficients:

$$\Theta^c{}_b := \Gamma^c_{ab} e^a \,. \tag{II.58}$$

This formalism allows us to formulate many equations in a different and sometimes more transparent manner.

II.6.5 Metric compatibility, torsion and the Levi-Civita connection. As already mentioned in paragraph II.6.2, we can construct tensors using the connection coefficients, even though they are not tensors themselves. An example is the *torsion tensor* $T : \Gamma(TM \otimes TM) \to \Gamma(TM)$, defined with respect to a given connection as

$$T(v, w) = \nabla_v w + \nabla_w v - [v, w] \,, \quad v, w \in \Gamma(TM) \,. \tag{II.59}$$

It has the components

$$T^\rho{}_{\mu\nu} \;\; := \;\; \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} \,, \tag{II.60}$$

in some holonomic basis, and the components

$$T^a{}_{bc} \;\; := \;\; \Gamma^a_{bc} - \Gamma^a_{cb} - f_{bc}{}^a \,. \tag{II.61}$$

in some non-holonomic basis.

Using the components of the torsion tensor in a non-holonomic basis, we can define the torsion 2-form:

$$T^a := \frac{1}{2} T^a{}_{bc} e^b \wedge e^c. \tag{II.62}$$

There is a relation between the connection 1-form and the torsion 2-form, given by the following equation

$$\mathrm{d}e^a + \Theta^a{}_b \wedge \mathrm{e}^b = T^a \,. \tag{II.63}$$

We now want to restrict our generic connection by a two specific conditions. Firstly we say a connection ∇ on a Riemannian manifold (M, g) is *compatible* with the metric g if $\forall u, v, w \in \Gamma(TM)$

$$\nabla g = 0 \qquad \Leftrightarrow \qquad \nabla_u \left(g(v, w)\right) = 0 \,. \tag{II.64}$$

If we choose local coordinates as in (II.22) and (II.23), we get the following equations:

$$\frac{\partial}{\partial x^\mu} \left(g_{\nu\rho}\right) \;\; = \;\; \Gamma^\sigma_{\mu\nu} \, g_{\sigma\rho} + \Gamma^\sigma_{\mu\rho} \, g_{\nu\sigma} \,, \tag{II.65}$$

$$\mathrm{e}_a \left(g_{bc}\right) \;\; = \;\; \Gamma^d_{ab} \, g_{dc} + \Gamma^d_{ac} \, g_{bd} \,. \tag{II.66}$$

One can show that there is a unique connection which is compatible with the metric and has vanishing torsion and it is called the *Levi-Civita connection*. In local coordinates, the second condition of vanishing torsion is given by (II.60) and (II.61)

$$\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu} = 0, \tag{II.67}$$

$$\Gamma^{c}_{ab} - \Gamma^{c}_{ba} = f_{ab}{}^{c}. \tag{II.68}$$

By virtue of (II.63) the condition of vanishing torsion in terms of the connection 1-form reads

$$\mathrm{d}e^{a} + \omega^{a}{}_{b} \wedge \mathrm{e}^{b} = 0\,. \tag{II.69}$$

In order to distinguish different connections, we will use the symbol ω for the Levi-Civita connection one forms and Θ for arbitrary connections. The two defining properties of the Levi-Civita connection, denoted by ∇^{LC} can be summarized to one defining equation:

$$\begin{aligned} g(\nabla^{\mathrm{LC}}_{u}(v), w) &:= \frac{1}{2}\{u\,(g(v,w)) + v\,(g(u,w)) - w\,(g(u,v))\} \\ &\quad + \frac{1}{2}\{g([u,v],w) - g([u,w],v) - g(u,[v,w])\}\,. \end{aligned} \tag{II.70}$$

Now we can easily read off the connection coefficients from the Levi-Civita connection, by plugging the basis vector fields into this equation and obtain:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left\{\frac{\partial}{\partial x^{\nu}} g_{\mu\sigma} + \frac{\partial}{\partial x^{\mu}} g_{\nu\sigma} - \frac{\partial}{\partial x^{\sigma}} g_{\mu\nu}\right\}, \tag{II.71}$$

$$\begin{aligned} \Gamma^{c}_{ab} &= \frac{1}{2} g^{cd}\{e_a(g_{bd}) + e_b(g_{ad}) - e_d(g_{ab}) + g_{ed}\; f_{ab}{}^{e} - g_{eb}\; f_{ad}{}^{e} - g_{ea}\; f_{bd}{}^{e}\} \\ \Leftrightarrow \Gamma^{c}_{ab} &= \frac{1}{2} g^{cd}\{e_a(g_{bd}) + e_b(g_{ad}) - e_d(g_{ab}) + f_{abd} - f_{adb} - f_{bda}\}\,. \end{aligned} \tag{II.72}$$

The connection coefficients of the Levi-Civita connection are called *Christoffel symbols* and they match with definition (II.45) from paragraph II.5.2.

II.7 Curvature

In this section we are considering the curvature of a manifold. It also depends on how the connection coefficients look like.

II.7.1 Riemann tensor. In the last section, we combined the connection coefficients in such a way that they form an object that transforms as a tensor, namely the torsion tensor. The torsion tensor contained terms of first order in the connection coefficients, i. e. first covariant derivatives. The tensor we introduce in this section will be a combination of second derivatives of vector fields and called the *Riemann curvature tensor* or just *Riemann tensor*. It is defined by

$$R(u,v,w) := \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w\,. \tag{II.73}$$

It tells us how a vector changes under an infinitesimal parallel transport induced by the connection. Parallel transportation here roughly means the transport of a vector under some path such that the vector only changes in the same way, the whole tangent space changes. Since the connection tells us how the tangent space changes from one point of the manifold to another, it also tells us how to perform a parallel transport of a vector. If we plug in the holonomic and non-holonomic basis from equations (II.22) and (II.23), and use equation (II.51) we see that the components of the Riemann tensor in a local basis look like

$$R^{\kappa}{}_{\lambda\mu\nu} = \frac{\partial}{\partial x^\mu}\Gamma^{\kappa}_{\nu\lambda} - \frac{\partial}{\partial x^\nu}\Gamma^{\kappa}_{\mu\lambda} + \Gamma^{\kappa}_{\mu\eta}\Gamma^{\eta}_{\nu\lambda} - \Gamma^{\kappa}_{\nu\eta}\Gamma^{\eta}_{\mu\lambda}\,, \tag{II.74}$$

$$R^{a}{}_{bcd} = e_c(\Gamma^{a}_{db}) - e_d(\Gamma^{a}_{cb}) + \Gamma^{a}_{ce}\Gamma^{e}_{db} - \Gamma^{a}_{de}\Gamma^{e}_{cb} - f_{cd}{}^{e}\Gamma^{a}_{eb}\,. \tag{II.75}$$

It is obvious from the definition that the Riemann tensor is antisymmetric with respect to μ and ν, i. e. in the last two indices.

Considering the Levi-Civita connection and pulling down the first index of the Riemann tensor, using the corresponding metric, we obtain in total six identity equations for the components of the Riemann tensor in the holonomic basis:

$$R_{\kappa\lambda\mu\nu} = -R_{\kappa\lambda\nu\mu}\,, \qquad R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}\,, \tag{II.76}$$

$$R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}\,, \qquad R^{\kappa}{}_{\mu\kappa\nu} = R^{\kappa}{}_{\nu\kappa\mu}\,. \tag{II.77}$$

along with the Bianchi identities:

First Bianchi identity: $$R^{\kappa}{}_{\lambda\mu\nu} + R^{\kappa}{}_{\mu\nu\lambda} + R^{\kappa}{}_{\nu\lambda\mu} = 0\,. \tag{II.78}$$

Second Bianchi identity: $$\nabla_\kappa R^{\xi}{}_{\lambda\mu\nu} + \nabla_\mu R^{\xi}{}_{\lambda\nu\kappa} + \nabla_\nu R^{\xi}{}_{\lambda\kappa\mu} = 0\,. \tag{II.79}$$

In the following, we contract the first with third index of the Riemann tensor. The resulting tensor is called the *Ricci tensor*:

$$\mathrm{Ric}_{\mu\nu} := R^{\kappa}{}_{\mu\kappa\nu} \tag{II.80}$$

Contracting the remaining two indices with each other, we get a smooth function called the *Ricci scalar*:

$$\mathcal{R} := \mathrm{Ric}^{\mu}{}_{\mu} := g^{\mu\nu}\mathrm{Ric}_{\mu\nu} \tag{II.81}$$

II.7.2 Curvature 2-form. In the previous section we defined the vector-valued torsion 2-form in equation (II.62). Analogously, we can define the curvature 2-form with respect to the last two components of the curvature tensor. Due to the antisymmetry in the last two indices of the Riemann tensor, we are able to think of it as of the components of a matrix valued 2-form, the *curvature 2-form*, where we consider the first two indices to be the matrix indices, i. e.

$$R^a{}_b := \frac{1}{2} R^a{}_{bcd}\, e^c \wedge e^d \,. \tag{II.82}$$

By definition, the Riemann tensor depends on the connection. Therefore we can also derive a relation between the curvature 2-form and the connection 1-form, similar to what we did for the torsion 2-form in (II.63):

$$\mathrm{d}\Theta^a{}_b + \Theta^a{}_c \wedge \Theta^c{}_b = R^a{}_b \,. \tag{II.83}$$

The equations (II.63) and (II.83) together are called the *Cartan structure equations.* If we take the exterior derivative of these equations, we obtain precisely the Bianchi identities, already formulated in the holonomic basis in equations (II.78) and (II.79). We get

$$\mathrm{d}T^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b, \tag{II.84}$$

$$\mathrm{d}R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0 \,, \tag{II.85}$$

respectively.

Chapter III

Lie Groups and Lie Algebras

In this chapter we want to introduce the notion of Lie groups, Lie algebras and their relation to each other. We also want to see how a group is able to act not only on a vector space, but also on manifolds.

III.1 Lie Groups and Group Action

III.1.1 Lie groups. First in this paragraph we want to discuss the notion of a group. A *group* G is defined to be a set, endowed with a map $\circ : G \times G \longrightarrow G$ which satisfies three conditions

$$(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3) \qquad \forall g_1, g_2, g_3 \in G\,, \tag{III.1}$$

$$\exists e \text{ such that: } e \circ g = g \qquad \forall g \in G\,, \tag{III.2}$$

$$\forall g \in G\ \exists g^{-1} \in G \text{ such that: } g^{-1} \circ g = e\,. \tag{III.3}$$

A map satisfying equation (III.1) is said to be *associative.* The element e is called the *identity element* and g^{-1} the *inverse* of the element g. We usually suppress the symbol $\circ$ for the group operation. A *topological group* is a group which is also a topological space such that the group operation and the map $g \longmapsto g^{-1}$ are continuous.

We can now define a *Lie group* as a topological group, which is also a smooth manifold. For Lie groups, the group operation and the inverse map are required to be smooth maps.

III.1.2 Group action and representations. Let G be a group and M be a smooth manifold. We define the *action* of G on M as a map $\Theta : M \times G \to M$, fulfilling the following properties:

$$\Theta(\cdot, g) : M \longrightarrow M \quad \text{is smooth}\,, \tag{III.4}$$

$$\Theta(\Theta(x,g),h) = \Theta(x,gh)\,, \quad \forall g,h \in G,\ x \in M\,. \tag{III.5}$$

Now let V be an n-dimensional vector space. Every vector space naturally carries all the structures, needed to define a manifold, and therefore it is sensible to talk about the group action on a vector space. So, let G be a group and Θ be an action of G on V. If the group action is linear in the first argument, we define

$$\begin{aligned} \rho : G &\longrightarrow GL(n, K)\,, \\ g &\longmapsto \Theta(\cdot, g)\,. \end{aligned} \tag{III.6}$$

The tuple (V, ρ) is called an n-dimensional *representation* of G.

Obviously, the group operation of a Lie group defines an action on itself via the given left or right multiplication of a group element.

$$L_\cdot(\cdot) : G \times G \longrightarrow G, \quad L_g(h) = gh\,, \tag{III.7}$$

$$R_\cdot(\cdot) : G \times G \longrightarrow G, \quad R_g(h) = hg\,. \tag{III.8}$$

Clearly, these are not the only ways to define an action of G on G. For instance we can make use of the inverse elements in order to define an action on G by

$$\Theta : G \times G \longrightarrow G, \quad \Theta(g, h) = ghg^{-1}\,. \tag{III.9}$$

This defines a map $\mathrm{ad}_g : G \to G$ in the following way:

$$\begin{aligned} \mathrm{ad}_g &:= \Theta(g, \cdot) \\ \mathrm{ad}_g(h) &:= \Theta(g, h) = ghg^{-1} \end{aligned} \tag{III.10}$$

One can show that this map is a homomorphism called the *adjoint representation* of G. By definition we find that

$$ad_g(e) = e\,. \tag{III.11}$$

Since ad_g is a map from G to G we can define a map on the tangent space of G as follows

$$Ad_g : T_eG \longrightarrow T_eG \tag{III.12}$$

$$Ad_g := ad_{g*}(e)\,. \tag{III.13}$$

This map induces the map Ad: $G \times T_eG \to T_eG$, which we call the *adjoint map* of G. One can show that this map is an action of G on T_eG and it is also called the *adjoint representation* of G.

III.1.3 Left translation. We have shown above that the group defines two group actions on itself in a very simple way. Like the adjoint representation, there are also maps $G \to G$ corresponding to these actions (equations (III.8) and (III.7)). We are considering only the first one, since all the results can be derived for the right action in an analogous manner.

$$\begin{aligned} L_g : G &\longrightarrow G, \quad g \in G, \text{ where} \\ h &\longmapsto gh \quad \forall h \in G \end{aligned} \tag{III.14}$$

This map is called the *left translation* and, as we defined in paragraph III.1.1, it is a smooth and bijective map, i. e. a diffeomorphism. One can observe the following, using the differential which is also called the push-forward: Every diffeomorphism between manifolds induces a homomorphism (i.e. a linear and bijective map) for the tangent bundle, which means a homomorphism of the tangent space for every element of the base. This way the differential of the left translation L_{g*} allows us to map a tangent vector from the tangent space in one point to the tangent space of another point of the manifold. Since L_g is smooth, we will be able to obtain a vector field from each tangent vector of a given tangent space. In the following we display how one can apply push-forward and pull-back maps to both vector fields and 1-forms. For vector fields, we have:

$$\begin{array}{ll} \text{Push-forward:} & \text{Pull-back:} \\ L_{g*} : \Gamma(TG) \to \Gamma(TG)\,, & L_g^* : \Gamma(TG) \to \Gamma(TG),\ L_g^* := (L_g^{-1})_*\,, \\ L_{g*}|_e : T_eG \to T_gG\,. & L_g^*|_e : T_gG \to T_eG\,, \end{array} \tag{III.15}$$

where we made use of the ordinary differential of a smooth map to define the push-forward and defined the pull-back to be the push-forward of the inverse map. For 1-forms, we find:

$$\begin{array}{ll} \text{Pull-back:} & \text{Push-forward:} \\ L_g^* : \Gamma(T^*G) \to \Gamma(T^*G)\,, & L_{g*} : \Gamma(T^*G) \to \Gamma(T^*G), L_{g*} := (L_g^{-1})^*\,, \\ L_g^*|_e : T_g^*G \to T_e^*G\,. & L_{g*}|_e : T_e^*G \to T_g^*G\,, \end{array} \tag{III.16}$$

where of course the pull-back is defined via the ordinary pull-back of 1-forms and the push-forward to be the pull-back of the inverse map.

Having defined push-forward maps for vector fields and 1-forms, it is straightforward to define push-forward maps for arbitrary tensor fields $t \in \Gamma(T^r_sG)$. Hence we can shift an arbitrary tensor t_e in the tensor product of tangent spaces and cotangent spaces of the identity element $T_eG \otimes \ldots T_eG \otimes T_e^*G \ldots \otimes T_e^*G =: T^r_{s,e}G$ to the corresponding tensor product of spaces $T^r_{s,g}G$ at an arbitrary point $g \in G$. That means that we have a map, $L_{g*}|_e : T^r_{s,e}G \to \Gamma(T^r_sG)$ mapping a tensor $t_e \in T^r_{s,e}G$ to a tensor field on G.

$$t(g) := L_{g*}|_e t_e\,. \tag{III.17}$$

As already mentioned in the beginning, any object defined via the left action in this section works out in the same way for a right action as well.

III.2 Invariance of Tensor Fields

Having defined the group action and its application to arbitrary tensor fields, we are now able to introduce the notion of left invariance for these fields.

III.2.1 Left invariance. Let t be a tensor field on a Lie group G, i.e. a smooth map of the manifold to the tensor bundle $t : M \to T^r_s G$, such that $t(p) \in T^r_{s,p}$. We call t *left-invariant*, if:

$$L_{g*}t(h) = t(gh) \quad \forall h \in G\,. \tag{III.18}$$

This notion of left invariance allows us to define a basis of the tangent space which gives us a basis of tensor fields in which all such tensor fields show up in a simple form. For this purpose let $\{E_a\}$ be a basis of the tangent space of G at the identity element e. As we have seen in III.1.2, it is possible to define a unique left-invariant vector field from every tangent vector via performing the push-forward of the left action on it. Therefore we get a set of vector fields $\{e_a\}$ depending on the basis of $T_e G$ which forms a basis of all vector fields. The procedure works in the same way for a basis $\{E^b\}$ of the cotangent space at the identity. Denoting the left-invariant basis of 1-forms on G as $\{e^b\}$ we are able to express arbitrary tensor fields $t \in T^r_s G$ in this basis:

$$t = t^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} e_{a_1} \otimes \ldots e_{a_r} \otimes e^{b_1} \otimes \ldots e^{b_s} \tag{III.19}$$

As shown in [23], the left invariance condition (III.18) for a tensor is equivalent to the tensor having only constant components with respect to the left-invariant basis,

$$t^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} = \text{const}\,, \forall a_1, ..., a_r, b_1, ..., b_s\,. \tag{III.20}$$

Because the left-invariant vector field even form a subalgebra, one can decompose every left-invariant vector field and 1-form in the left-invariant basis and we get that

$$t(v_1, ..., v_r, \alpha_1, ..., a_s) = \text{const}\,, \tag{III.21}$$

for all left-invariant vector fields v_i and left-invariant 1-forms α_j and hence their Lie derivatives along left-invariant vector fields vanish. The fact that the coefficients are constant together with equation (II.41) yields the following condition on the coefficients of t:

$$\begin{aligned} \mathcal{L}_{e_c} t^{a_1 \ldots a_r}_{b_1 \ldots b_s} &= f_{ce}{}^{a_1} t^{e \ldots a_r}{}_{b_1 \ldots b_s} + ... + f_{ce}{}^{a_r} t^{a_1 \ldots e}{}_{b_1 \ldots b_s} \\ &\quad - f_{cb_1}{}^{e} t^{a_1 \ldots a_r}{}_{e \ldots b_s} - ... - f_{cb_s}{}^{e} t^{a_1 \ldots a_r}{}_{b_1 \ldots e} = 0\,. \end{aligned} \tag{III.22}$$

Note that the Lie bracket $[\cdot,\cdot] : \Gamma(TG) \times \Gamma(TG) \to \Gamma(TG)$ of vector fields preserves the left invariance. Plugging in two elements of the left-invariant basis, we obtain a different left-invariant vector field, which is just a linear combination of the left-invariant basis vector fields and has constant components according to equation (III.20):

$$[e_a, e_b] = f_{ab}{}^c e_c. \tag{III.23}$$

The Lie bracket satisfies the Jacobi identity

$$[[u,v],w]+[[w,u],v]+[[v,w],u]=0\,,\quad \forall u,v,w\in\Gamma(TG) \tag{III.24}$$

and therefore the space of left-invariant vector fields forms a so-called finite dimensional Lie algebra.

Additionally, the tangent space becomes a Lie algebra with a Lie bracket induced by the Lie bracket for vector fields:

$$[E_a,E_b]:=[e_a,e_b](e)=f_{ab}{}^{c}E_c. \tag{III.25}$$

The structure constants of these two Lie algebras are of course the same and therefore both are called the *Lie algebra* of G and are denoted by $\mathcal{G}$.

III.3 The Killing Form and The Invariant Metric

As motivated above, the Lie algebra $\mathcal{G}$ of a Lie group G plays a crucial role for the structure of the corresponding Lie group. We will see that one can get information about topological properties such as connectedness and compactness of G by analyzing a special bilinear form on $\mathcal{G}$ which we define in the following.

III.3.1 Killing form and adjoint representation. Let $\mathcal{G}$ be a Lie algebra with generators $\{E_a\}$. The *Killing form* K is defined as a mapping $K:\mathcal{G}\times\mathcal{G}\to\mathbb{R}$ given by

$$K_{ab}=K(E_a,E_b):=\mathrm{tr}(\mathrm{ad}E_a\mathrm{ad}E_b), \tag{III.26}$$

where $\mathrm{ad}E_a$ denotes the adjoint representation of the generator E_a:

$$\mathrm{ad}E_a:\mathcal{G}\to\mathcal{G},\ \mathrm{ad}E_a(E_b):=[E_a,E_b]\,. \tag{III.27}$$

Since the trace is a linear map satisfying the cyclic condition wich allows cyclic permutations of its arguments, e. g. $\mathrm{tr}(ABC)=\mathrm{tr}(CAB)$, it is obvious that (III.26) is a symmetric bilinear form on $\mathcal{G}$. Employing the commutation relation of the Lie algebra, we can write the Killing form in terms of the structure constants. Using the definition of the adjoint representation (III.27) and the commutation relation of the Lie algebra, we get

$$(\mathrm{ad}E_a\mathrm{ad}E_b)(E_c)=[E_a,[E_b,E_c]]=[E_a,f_{bc}{}^{d}E_d]=f_{bc}{}^{d}f_{ad}{}^{f}E_f, \tag{III.28}$$

where for every pair of generators E_a and E_b, we obtain a matrix with components

$$(\mathrm{ad}E_a\mathrm{ad}E_b)^{e}{}_{c}=f_{ad}{}^{e}f_{bc}{}^{d}\,. \tag{III.29}$$

Taking the trace of this matrix, i.e. summing over all diagonal components $e=c$, obtain the Killing form

$$K_{ab}=\mathrm{tr}(\mathrm{ad}E_a\mathrm{ad}E_b)=\mathrm{tr}(f_{ad}{}^{e}f_{bc}{}^{d})=f_{ad}{}^{c}f_{bc}{}^{d}\,. \tag{III.30}$$

III.3.2 Theorem. *A Lie group G is compact, connected and has no conjugation invariant subgroups beside the trivial ones, i. e. G is compact and simple, if and only if its Killing form is negative definite.*

From this theorem and from III.1.2 we can deduce that given a simple compact Lie group, we can always define a metric g_e on $\mathcal{G}$ and therefore a metric g on all of G, which is invariant under the group action:

$$g_{ab}(h) := -L_{h*}\Big| K_{ab}, h \in \Gamma. \tag{III.31}$$

III.4 Maurer-Cartan Form

In this section we want to introduce the Maurer-Cartan forms on a group manifold. We will need the Maurer-Cartan forms as well as its properties later on, for instance in order to derive the geometric structure of so-called coset spaces, which are given by the quotient of a Lie group and some subgroup.

III.4.1 Maurer-Cartan form and equation. Let $\{E_a\}$ and $\{E^a\}$ be bases of the tangent and cotangent space of the identity, respectively, and in addition, let $\{e_a\}$ and $\{e^a\}$ be the left-invariant bases of $\Gamma(TG)$ and $\Gamma(T^*G)$ such that

$$e_a \lrcorner e^b = \delta_a{}^b \,. \tag{III.32}$$

Then it is easy to show that the left-invariant basis of $\Gamma(T^*G)$ satisfies the *Maurer-Cartan equation*

$$\mathrm{d}e^a = -\frac{1}{2} f_{bc}{}^a e^b \wedge e^c \,. \tag{III.33}$$

One can define the *canonical 1-form* or *Maurer-Cartan form* on G as a Lie-algebra-valued 1-form,

$$\omega : T_g G \to T_e G \,, \tag{III.34}$$

$$\omega(X) := (L_{g*})^{-1}\big|_e X, \qquad X \in T_g G \,. \tag{III.35}$$

III.4.2 Theorem. *The Maurer-Cartan form satisfies the following equations:*

$$\omega = e^a \otimes E_a, \tag{III.36}$$

$$0 = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega], \tag{III.37}$$

where we define $[\cdot \wedge \cdot]$ for arbitrary Lie algebra valued 1-forms by

$$[\alpha \wedge \beta] := [E_a, E_b] \otimes \alpha^a \wedge \beta^b \,. \tag{III.38}$$

Since it is quite obvious how to multiply such 1-forms we are going to suppress this notation in the rest of this thesis and simply write the wedge product as usual.

Chapter IV

Manifolds with Almost Complex Structure

This chapter is dedicated to the classification of manifolds with almost complex structure. This is needed later on when we will construct certain bundles over so-called Kähler manifolds in order to obtain a Yang-Mills theory on such spaces.

IV.1 Complex Manifolds and Holomorphic Vector Bundles

In this section we start by introducing manifolds that generalize the idea of a real manifold, which locally looks like $\mathbb{R}^n$ (see definition II.2.1). The generalization simply replaces $\mathbb{R}$ by $\mathbb{C}$ and changes the requirement of the charts in a reasonable way. Then we will see how these changes affect all other quantities and objects we defined for real manifolds in Chapter II.

IV.1.1 Complex manifolds. The definition of these objects is fairly intuitive, if we remind ourself of the definition of smooth manifolds II.2.1. Here, we used a topological space and an atlas consisting of smooth charts. Analogously, we define a *complex manifold* M to be a paracompact topological Hausdorff space, as usual. The only difference to the smooth manifold is that we define the charts as maps $\varphi_i : U_i \longrightarrow \mathbb{C}^n$, where $\mathcal{U} = \{U_i\}$ is an open cover of M. Furthermore, the transition functions are required to be *holomorphic*. As in (II.2.1), we require the atlas to contain all possible charts satisfying this condition and call this atlas a *complex structure*. So a complex manifold is simply a topological manifold equipped with a complex structure. If all charts of the given complex structure are mappings to the complex space $\mathbb{C}^n$ of fixed dimension n, we call n the *complex dimension* of M. Since all holomorphic maps are smooth, it is obvious that every complex manifold M of complex dimension n can also be considered as a smooth manifold, with real dimension equals $2n$.

IV.1.2 Tangent space of a complex manifold. Let M be a complex manifold of complex dimension n. In Chapter II, we defined the tangent space with respect to the local coordinates. Analogously, the tangent space of a complex manifold M should be a complex vector space. This is simply realized by a complexification of the (real) tangent space at a point p of M. It is denoted by $T_p^{\mathbb{C}}M$. Note that elements of the (real) tangent space are mappings from $C^\infty(\mathbb{R})$

to $\mathbb{R}$. Now the elements of the complexified tangent space are maps from the space of the complexified smoth functions $C^\infty(\mathbb{R})^\mathbb{C}$ to $\mathbb{C}$, with

$$C^\infty(\mathbb{R})^\mathbb{C} := \{g + \mathrm{i}h,\ g, h\ \in C^\infty(\mathbb{R})\}\ . \tag{IV.1}$$

Of course, the complexified tangent spaces are still given by the corresponding real tangent spaces. Since M can be considered to be a manifold of real dimension $2n$, local coordinates $\{(x^\mu, y^\mu),\ \mu = 1, ..., n\}$ can be used to define a basis of the tangent and the cotangent spaces of M. The bases are denoted by

$$\left\{(\tfrac{\partial}{\partial x^\mu}, \tfrac{\partial}{\partial y^\mu}), \mu = 1, ..., n\right\}\ , \tag{IV.2}$$

$$\{(\mathrm{d}x^\mu, \mathrm{d}y^\mu), \mu = 1, ..., n\}\ . \tag{IV.3}$$

Since the complex coordinates are given by $z^\mu = x^\mu + \mathrm{i}y^\mu$, this gives rise to the choice of a certain basis in the complexified tangent and cotangent spaces, yielding a basis for the sections of the complexified tangent and cotangent bundle, namely vector fields and 1-forms:

$$\begin{aligned} \tfrac{\partial}{\partial z^\mu} &:= \tfrac{\partial}{\partial x^\mu} - \mathrm{i}\tfrac{\partial}{\partial y^\mu} && \text{(holomorphic vector fields)}\,, \\ \tfrac{\partial}{\partial \bar z^{\bar\mu}} &:= \tfrac{\partial}{\partial x^{\bar\mu}} + \mathrm{i}\tfrac{\partial}{\partial y^{\bar\mu}} && \text{(antiholomorphic vector fields)}\,, \end{aligned} \tag{IV.4}$$

$$\begin{aligned} \mathrm{d}z^\mu &:= \mathrm{d}x^\mu + \mathrm{i}\mathrm{d}y^\mu && \text{(holomorphic 1-forms)}\,, \\ \mathrm{d}\bar z^{\bar\mu} &:= \mathrm{d}x^{\bar\mu} - \mathrm{i}\mathrm{d}y^{\bar\mu} && \text{(antiholomorphic 1-forms)}\,. \end{aligned} \tag{IV.5}$$

Here we denote the indices of the antiholomorphic vector fields and 1-forms by $\bar\mu = 1, ..., n$ in order to distinguish the corresponding components of arbitrary tensor filds. Let $t \in \Gamma(T^r_s M)$ be an arbitrary tensor field. According to (IV.4) and (IV.5), there is the canonical decomposition $\Gamma(T^\mathbb{C} M) = \Gamma(T^+ M) \oplus \Gamma(T^- M)$ and hence we can write t with respect to this basis' as

$$t = t^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \bigotimes_{i=1}^{r} \frac{\partial}{\partial z^{\mu_i}} \bigotimes_{i=1}^{s} \mathrm{d}z^{\nu_i} + \dots + t^{\bar\mu_1 \dots \bar\mu_r}{}_{\bar\nu_1 \dots \bar\nu_s} \bigotimes_{i=1}^{r} \frac{\partial}{\partial \bar z^{\bar\mu_i}} \bigotimes_{i=1}^{s} \mathrm{d}\bar z^{\bar\nu_i}\ , \tag{IV.6}$$

wherein between the two summands there are terms of any combination of holomorphic and antiholomorphic indices. There is a special case of a q-form $\omega \in \Omega^q_\mathbb{C}(M)$, where a lot of coefficients actually vanish and one can write:

$$\omega = \frac{1}{r!\ s!} \omega_{\mu_1 \dots \mu_r \bar\nu_1 \dots \bar\nu_s} \mathrm{d}z^{\mu_1} \wedge \dots \wedge \mathrm{d}z^{\mu_r} \wedge \mathrm{d}\bar z^{\bar\nu_1} \wedge \dots \wedge \mathrm{d}\bar z^{\bar\nu_s}, \tag{IV.7}$$

where $r + s = q$. We call such a q-form to be of *bidegree* (r, s) or simply an (r, s)*-form*. The exterior derivative of an (r, s)-form is given by

$$\begin{aligned} \mathrm{d}\omega &= \frac{\partial}{\partial z^\rho}\omega_{\mu_1\dots\mu_r\bar\nu_1\dots\bar\nu_s}\ \mathrm{d}z^\rho\wedge\mathrm{d}z^{\mu_1}\wedge\dots\wedge\mathrm{d}z^{\mu_r}\wedge\mathrm{d}\bar z^{\bar\nu_1}\wedge\dots\wedge\mathrm{d}\bar z^{\bar\nu_s} \\ &\quad+\frac{\partial}{\partial \bar z^{\bar\rho}}\omega_{\mu_1\dots\mu_r\bar\nu_1\dots\bar\nu_s}\ \mathrm{d}\bar z^{\bar\rho}\wedge\mathrm{d}z^{\mu_1}\wedge\dots\wedge\mathrm{d}z^{\mu_r}\wedge\mathrm{d}\bar z^{\bar\nu_1}\wedge\dots\wedge\mathrm{d}\bar z^{\bar\nu_s}. \end{aligned} \tag{IV.8}$$

We use this splitting of the exterior derivative in a holomorphic and an antiholomorphic part in order to define the so-called *Dolbeault operators* ∂ and $\bar\partial$ by

$$\mathrm{d} = \partial + \bar\partial\,. \tag{IV.9}$$

IV.1.3 Almost complex structure on a complex manifold. If we have a complex structure, we are always able to choose a holomorphic and antiholomorphic basis of the tangent space. Hence we are able to define a map $J : \Gamma(M) \longrightarrow \Gamma(M)$ as follows

$$J := \mathrm{i}\ \mathrm{d}z^\mu \otimes \frac{\partial}{\partial z^\mu} - \mathrm{i}\ \mathrm{d}\bar z^\mu \otimes \frac{\partial}{\partial \bar z^\mu}. \tag{IV.10}$$

The tensor J of type $(1, 1)$ is called an *almost complex structure* on M and in this case it completely specifies the complex structure on M. Obviously it satisfies $J^2 = -\mathrm{id}$. Furthermore, if v is a purely holomorphic vector field and w a purely antiholomorphic vector field, then

$$J(v) = \mathrm{i}v \quad \text{and} \quad J(w) = -\mathrm{i}w\,. \tag{IV.11}$$

IV.2 From Almost Complex to Kähler Manifolds

In the last section we started with a complex manifold and ended up with an almost complex structure, defined by the complex structure. In fact it is not necessary for a smooth manifold of dimension $2n$ to be actually endowed with a complex structure in order to admit an almost complex structure. In other words, there are manifolds with an almost complex structure that do not admit a complex structure. At the very begining of this section we will see what conditions on the almost complex structure are necessary in order to define a corresponding complex structure.

IV.2.1 Almost complex structures. An *almost complex structure* J on a $2n$-dimensional smooth manifold M is defined as an element of $\Gamma(TM \otimes T^*M)$ such that the map $J : TM \to TM$ satisfies the following condition:

$$J(J(v)) = -v \Leftrightarrow J^2 = -id, \tag{IV.12}$$

or in components:

$$J_\mu{}^\nu J_\nu{}^\rho = -\delta_\mu{}^\rho\,. \tag{IV.13}$$

A smooth manifold endowed with an almost complex structure is called an *almost complex manifold*.

IV.2.2 Remark. As the definition already suggests, and as we have seen above, a complex manifold is always a manifold endowed with an almost complex structure induced by the complex structure. In general the reverse is not true. In order for an almost complex manifold to be a complex manifold, one has to impose restrictions on the almost complex structure, which we will state in the following.

IV.2.3 Integrability. Let (M, J) be a manifold with an almost complex structure. We call J *integrable* if

$$\mathrm{N}_{\mu\nu}{}^{\rho} = 0, \tag{IV.14}$$

where the *Nijenhuis torsion tensor* N is defined by:

$$\mathrm{N}_{\mu\nu}{}^{\rho} := J_{\mu}{}^{\sigma}(\nabla_{\sigma} J_{\nu}{}^{\rho} - \nabla_{\nu} J_{\sigma}{}^{\rho}) - J_{\nu}{}^{\sigma}(\nabla_{\sigma} J_{\mu}{}^{\rho} - \nabla_{\mu} J_{\sigma}{}^{\rho}). \tag{IV.15}$$

IV.2.4 Theorem. *An almost complex structure induces a complex structure on a manifold, if and only if it is integrable.*

IV.2.5 Hermitian metric. In the following we are going to introduce a metric in the context of almost complex manifolds. Let (M, g, J) be an almost complex Riemannian manifold. We call the Riemannian metric g *hermitian*, if it is compatible with the almost complex structure, i.e.

$$g_{\rho\tau} J_{\mu}{}^{\rho} J_{\nu}{}^{\tau} = g_{\mu\nu}\,. \tag{IV.16}$$

An almost complex manifold, endowed with a hermitian metric is called an *almost hermitian manifold* and if J is integrable, we call it a *hermitian manifold*. This notion of hermicity is quite intuitive, since it generalizes the notion of a hermicity of a metric on a vector space.

IV.2.6 Theorem. *Every almost complex manifold admits a hermitian metric.*

IV.2.7 Geometry of an hermitian manifold. In the following we want to take a closer look at hermitian manifolds and their corresponding geometry. Let $\{z^{\mu}\}$ be complex coordinate on M. The condition of hermicity yields a couple of constrains for the components of the metric.

$$g_{\mu\nu} = 0 \quad \text{and} \quad g_{\bar{\mu}\bar{\nu}} = 0\,. \tag{IV.17}$$

This follows from the fact that the almost complex structure J acts on the basis vector fields according to (IV.11) and we get an i for each argument. The only non-vanishing components of the hermitian metric are the 'mixed' components and g can locally be expressed as

$$g = g_{\mu\bar{\nu}}\, \mathrm{d}z^{\mu} \otimes \mathrm{d}\bar{z}^{\nu} + g_{\bar{\mu}\nu}\, \mathrm{d}\bar{z}^{\mu} \otimes \mathrm{d}z^{\nu}\,. \tag{IV.18}$$

Considering M as a smooth manifold, we can characterize a connection ∇ by its connection coefficients, introduced in section II.6.2, and find:

$$\nabla_A \frac{\partial}{\partial z^{\mu}} = \Gamma^{B}_{A\mu} \frac{\partial}{\partial z^{B}}, \qquad \nabla_A \frac{\partial}{\partial z^{\bar{\mu}}} = \Gamma^{B}_{A\bar{\mu}} \frac{\partial}{\partial z^{B}}, \tag{IV.19}$$

where the indices A and B take the values of both holomorphic and anti-holomorphic indices. If we suppose that the connection here is compatible with the almost complex structure, which is a reasonable assumption, we would find

$$\nabla_\nu \frac{\partial}{\partial z^\mu} = \Gamma^\rho_{\nu\mu} \frac{\partial}{\partial z^\rho}, \qquad \nabla_{\bar\nu} \frac{\partial}{\partial \bar z^\mu} = \Gamma^{\bar\rho}_{\bar\nu\bar\mu} \frac{\partial}{\partial z^{\bar\rho}}, \tag{IV.20}$$

$$\nabla_{\bar\nu} \frac{\partial}{\partial z^\mu} = 0, \qquad \nabla_\nu \frac{\partial}{\partial \bar z^\mu} = 0. \tag{IV.21}$$

If this is the case, i. e. all connection coefficients with mixed indices are zero, ∇ is called a *hermitian connection*. Straightforwardly we can deduce:

$$\nabla_\nu \mathrm{d}z^\mu = -\Gamma^\mu_{\nu\rho} \mathrm{d}z^\rho, \qquad \nabla_{\bar\nu} \mathrm{d}\bar z^{\bar\mu} = -\Gamma^{\bar\mu}_{\bar\nu\bar\rho} \frac{\partial}{\partial z^{\bar\rho}}. \tag{IV.22}$$

In the following, we want to state all the properties and tensors that we already derived in II.6 in this context. The condition of a hermitian connection to be compatible to the hermitian metric, as stated in (II.64), yields a simple shape of the connection coefficients:

$$\nabla_\mu g_{\nu\bar\rho} = \nabla_{\bar\mu} g_{\nu\bar\rho} = 0 \tag{IV.23}$$

$$\Leftrightarrow \quad \Gamma^\rho_{\mu\nu} = g^{\bar\sigma\rho} \tfrac{\partial}{\partial z^\mu} g_{\nu\bar\sigma}, \quad \Gamma^{\bar\rho}_{\bar\mu\bar\nu} = g^{\sigma\bar\rho} \tfrac{\partial}{\partial \bar z^\mu} g_{\bar\nu\sigma} \quad . \tag{IV.24}$$

Using these properties, one can actually show that the covariant derivative of the almost complex structure is zero with respect to the hermitian connection.

If we plug the connection coefficients of the hermitian connection into equation (II.60) we get the following non-vanishing components for the torsion tensor:

$$T^\rho{}_{\mu\nu} = g^{\bar\sigma\rho} \frac{\partial}{\partial z^\mu} g_{\nu\bar\sigma} - g^{\bar\sigma\rho} \frac{\partial}{\partial z^\nu} g_{\mu\bar\sigma}, \quad T^{\bar\rho}{}_{\bar\mu\bar\nu} = g^{\sigma\bar\rho} \frac{\partial}{\partial \bar z^\mu} g_{\bar\nu\sigma} - g^{\sigma\bar\rho} \frac{\partial}{\partial \bar z^\nu} g_{\bar\mu\sigma}. \tag{IV.25}$$

If we plug the connection coefficients of the hermitian connection into equation (II.74) we get the Riemann tensor for the hermitian connection. Using equation (II.76), we find that the only independent components are

$$R^\rho{}_{\nu\bar\tau\mu} = \frac{\partial}{\partial \bar z^\tau} \Gamma^\rho_{\mu\nu} = \frac{\partial}{\partial \bar z^\tau} \left(g^{\bar\sigma\rho} \frac{\partial}{\partial z^\mu} g_{\nu\bar\sigma} \right),$$

$$R^{\bar\rho}{}_{\bar\nu\tau\bar\mu} = \frac{\partial}{\partial z^\tau} \Gamma^{\bar\rho}_{\bar\mu\bar\nu} = \frac{\partial}{\partial z^\tau} \left(g^{\sigma\bar\rho} \frac{\partial}{\partial \bar z^\mu} g_{\bar\nu\sigma} \right). \tag{IV.26}$$

IV.2.8 Kähler form and Kähler manifold. We can now define a very important class of almost hermitian manifolds. In order to do this, we need a certain 2-form, defined by a hermitian metric. Let (M, J, g) be an almost hermitian manifold with g being the hermitian metric. We define the *Kähler form* Ω as

$$\Omega := \frac{1}{2} \Omega_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \quad \Omega_{\mu\nu} := g_{\mu\rho} J_\nu{}^\rho. \tag{IV.27}$$

This definition is reasonable, because $\Omega_{\mu\nu}$ is antisymmetric in $\mu,\ \nu$ as one can easily see in the following:

$$\begin{aligned}
\Omega_{\mu\nu} &= g_{\nu\tau} J_\mu{}^\tau \\
&= g_{\rho\sigma} J_\nu{}^\rho J_\tau{}^\sigma J_\mu{}^\tau, \text{ since } g \text{ is hermitian} \\
&= g_{\rho\sigma} J_\nu{}^\rho (-\delta_\mu{}^\sigma), \text{ because of (IV.12)} \\
&= -g_{\rho\mu} J_\nu{}^\rho \\
&= -\Omega_{\nu\mu}
\end{aligned} \tag{IV.28}$$

A hermitian manifold is called a *Kähler manifold*, if its Kähler form is closed, i.e.

$$\mathrm{d}\Omega = 0\,. \tag{IV.29}$$

The hermitian metric g of a Kähler manifold is called the *Kähler metric.* Since ∇ is the Levi-Civita connection with respect to the Kähler metric, and hence

$$\mathrm{d}\Omega = \nabla\Omega, \tag{IV.30}$$

we obtain that the condition (IV.29) is equivalent to

$$\nabla_\mu J_\nu{}^\rho = 0\,. \tag{IV.31}$$

We can use the properties of J in order to find an equivalent but firstly weaker looking formulation of (IV.31), namely the condition

$$\nabla_\mu J_\nu{}^\tau - \nabla_\nu J_\mu{}^\tau = 0\,. \tag{IV.32}$$

This is proven in the following:

$$\begin{aligned}
0 &= \nabla_\mu J_\nu{}^\tau - \nabla_\nu J_\mu{}^\tau \\
&= g^{\tau\rho} \left(\nabla_\mu \Omega_{\nu\rho} - \nabla_\mu \Omega_{\nu\rho} + \nabla_\nu \Omega_{\rho\mu} - \nabla_\nu \Omega_{\rho\mu} + \nabla_\rho \Omega_{\mu\nu} - \nabla_\rho \Omega_{\mu\nu}\right) \\
&= g^{\tau\rho} \left(\nabla_\mu \Omega_{\nu\rho} + \nabla_\mu \Omega_{\rho\nu} + \nabla_\nu \Omega_{\rho\mu} + \nabla_\nu \Omega_{\mu\rho} + \nabla_\rho \Omega_{\mu\nu} + \nabla_\rho \Omega_{\nu\mu}\right) \\
&= g^{\tau\rho} \left((\nabla_\mu \Omega_{\nu\rho} + \nabla_\nu \Omega_{\mu\rho}) + (\nabla_\mu \Omega_{\rho\nu} + \nabla_\rho \Omega_{\mu\nu}) + (\nabla_\nu \Omega_{\rho\mu} + \nabla_\rho J_{\nu\mu})\right) \\
&= g^{\tau\rho} \left(2(\nabla_\mu \Omega_{\nu\rho} + \nabla_\mu \Omega_{\rho\nu} + \nabla_\nu \Omega_{\rho\mu})\right) \\
&= g^{\tau\rho} \left(\nabla_\nu \Omega_{\rho\mu}\right) \\
&= \nabla_\mu J_\nu{}^\tau,
\end{aligned}$$

wherein we have used the fact that the Levi-Civita connection is compatible with the metric, and the antisymmetry property of the components of the Kähler form.□

Summarizing, we can say that a hermitian manifold is Kähler if

$$\mathrm{d}\Omega = 0 \quad \Leftrightarrow \quad \nabla_\mu J_\nu{}^\rho = 0 \quad \Leftrightarrow \quad \nabla_{[\mu} J_{\nu]}{}^\tau = 0\,. \tag{IV.33}$$

IV.2.9 Geometry of a Kähler manifold. In order to see how the Kähler metric is explicitly restricted by the defining properties of the Kähler manifold, we use equation (IV.30). As stated in (IV.9), there is a decomposition of the exterior derivative, namely $\mathrm{d} = \partial + \bar{\partial}$. So if we apply this decomposition to the Kähler form Ω, we get two independent equations from (IV.30) for the metric

$$\frac{\partial g_{\mu\bar{\nu}}}{\partial z^{\lambda}} = \frac{\partial g_{\lambda\bar{\nu}}}{\partial z^{\mu}}\,, \qquad \frac{\partial g_{\mu\bar{\nu}}}{\partial \bar{z}^{\lambda}} = \frac{\partial g_{\mu\bar{\lambda}}}{\partial \bar{z}^{\nu}}\,. \tag{IV.34}$$

Formally these equations are the integrability conditions for the metric. Since g satisfies these equations, we can locally find a potential $\mathcal{K}_U \in C^{\infty}(M)$ for g, called the *Kähler potential*,where U is an open neighborhood that belongs to the corresponding chart. Therefore the hermitian mertic is given by

$$g_{\mu\bar{\nu}} = \frac{\partial}{\partial z^{\mu}}\frac{\partial}{\partial \bar{z}^{\nu}}\mathcal{K}_U\,. \tag{IV.35}$$

These results can be used to show that the torsion for a connection defined by the Kähler metric vanishes. The Riemann tensor becomes

$$R^{\kappa}{}_{\bar{\lambda}\mu\bar{\nu}} = -\frac{\partial}{\partial \bar{z}^{\nu}}(g^{\bar{\xi}}\frac{\partial}{\partial z^{\mu}}g_{\bar{\lambda}\bar{\xi}}) \tag{IV.36}$$

$$= R^{\kappa}{}_{\mu\bar{\lambda}\bar{\nu}}\,, \tag{IV.37}$$

and we get the following additional properties:

$$R^{\bar{\kappa}}{}_{\bar{\lambda}\bar{\mu}\nu} = R^{\bar{\kappa}}{}_{\bar{\mu}\bar{\lambda}\nu}\,, \tag{IV.38}$$

$$R^{\kappa}{}_{\lambda\bar{\mu}\nu} = R^{\kappa}{}_{\nu\bar{\mu}\lambda}\,, \tag{IV.39}$$

$$R^{\bar{\kappa}}{}_{\bar{\lambda}\mu\bar{\nu}} = R^{\bar{\kappa}}{}_{\bar{\nu}\mu\bar{\lambda}}\,. \tag{IV.40}$$

IV.2.10 Almost Kähler and nearly Kähler manifold. Suppose that all conditions for a manifold to be Kähler are satisfied besides the one that the almost complex structure induces a complex structure, i.e. (M, J, g) is not necessarily a complex manifold. Then we call such (M, J, g) an *almost Kähler manifold* and the corresponding complex structure an *almost complex structure.*

We now quickly want to introduce another class of almost hermitian manifolds, whose defining condition looks similar to the one of an almost Kähler manifold. Let (M, J, g) be an almost hermitian manifold, with the hermitian metric g. We call (M, J) *nearly Kähler* if J satisfies

$$\nabla_{\mu}J_{\nu}{}^{\rho} + \nabla_{\nu}J_{\mu}{}^{\rho} = 0\,,\ \Leftrightarrow \nabla_{(\mu}J_{\nu)}{}^{\rho} = 0\,. \tag{IV.41}$$

Due to (IV.31) it is obvious that a Kähler manifold is always also a nearly Kähler manifold. In summary, the three types of manifolds corresponding to their almost complex structure are related as follows:

$$T_{\mu\nu}{}^{\rho} = 0 \text{ and } \quad \nabla_{[\mu}J_{\nu]}{}^{\tau} = 0 \quad \Leftrightarrow \; (M, J, g) \text{ is Kähler}\,, \tag{IV.42}$$

$$\nabla_{[\mu}J_{\nu]}{}^{\tau} = 0 \quad \Leftrightarrow \; (M, J, g) \text{ is almost Kähler}\,, \tag{IV.43}$$

$$\nabla_{(\mu}J_{\nu)}{}^{\tau} = 0 \quad \Leftrightarrow \; (M, J, g) \text{ is nearly Kähler}\,. \tag{IV.44}$$

Chapter V

Homogeneous Spaces

The following chapter introduces the notion and properties of so-called homogeneous and symmetric spaces. Such spaces are needed for so-called coset space compactifications in string theory. In later chapters we will consider such spaces and derive their corresponding Yang-Mills theories.

We are going to define these spaces in a formal and quite generic way in the first two sections of this chapter. In the last two sections we are going to write down the consequences that follow for such spaces explicitly and develop the geometry of such spaces. Namely we are going to write down certain connections on the tangent bundle as well as the corresponding torsion and curvature tensors. We end the chapter by stating the simple form of the covariant derivative of left-invariant tensors, which was introduced in the context of Lie groups in (III.20) for instance.

V.1 Group Action and Coset Spaces

In this section we will start and introduce the concept of so-called homogeneous spaces, which are smooth manifolds of special kind. They do not come alone but carry an additional manifold which is a Lie group. This Lie group has a specific action on them which we will explain in detail in the following.

V.1.1 Homogeneous space. Let G be an arbitrary group, acting on a smooth manifold M, and Θ be the corresponding group action. We say that G acts on M *transitively* if:

$$\forall x, y \in M \ \exists \ g \in G : \Theta_g(x) = y. \tag{V.1}$$

In order to define an action, G only needs the structure of a group. In the case where G is even a Lie group, we call M a *homogeneous space* of the Lie group G.

The *stabilizer subgroup* H_x of G is defined as the set of those group elements that leave x invariant:

$$H_x := \{g \in G : \Theta_g(x) = x\} \ . \tag{V.2}$$

One can show that if G acts on M transitively, there exists an isomorphism between H_x and H_y for arbitrary $x, y \in M$. Hence all the stabilizer groups of M are isomorphic. Therefore we denote the unique stabilizer group by H.

One more thing needed in order to write down homogeneous spaces in a nice and convenient way is the definition of so-called coset spaces. In order to define these, we need to know what an equivalence relation means.

Let G be an arbitrary group and H some subgroup of G. One can define an equivalence relation on G in the following way:

$$g_1 \sim g_2, \text{ iff } (g_1)^{-1} g_2 \in H \,. \tag{V.3}$$

This is indeed an equivalence relation, since $\sim$ has the following properties:

1. $\sim$ is reflexive: $g \sim g \Leftrightarrow g^{-1}g = e \in H$
2. $\sim$ is symmetric: $g_1 \sim g_2 \Leftrightarrow g_1^{-1} g_2 = h \in H$
   $\Leftrightarrow H \ni h^{-1} = (g_1^{-1}g_2)^{-1} = g_2^{-1} g_1 \Leftrightarrow g_2 \sim g_1$
3. $\sim$ is transitive: $g_1 \sim g_2 \sim g_3 \Leftrightarrow g_1^{-1} g_2 = h_1, g_2^{-1} g_3 = h_2$
   $\Rightarrow H \ni h_1 h_2 = g_1^{-1} g_2 g_2^{-1} g_3 = g_1^{-1} g_3 \Leftrightarrow g_1 \sim g_3$

The first condition is clearly true, since H is a subgroup and therefore it needs to contain inverse elements to its elements as well as the identity. An equivalence class $[g]$ with respect to this equivalence relation is given by $\{gh : h \in H\} = gH$ which is also denoted by $[gH]$. The set of equivalence classes with respect to this equivalence relation is called the *coset space* G/H.

By definition, the right action $\Theta(x, \cdot) =: \Theta_x : G \to M$ is a continuous map for arbitrary $x \in M$. Therefore the stabilizer group is the preimage of a closed set in M, i. e. $H = H_x = \Theta_x^{-1}(x)$, and due to continuity it is also a closed subgroup of G. It can be shown that in such a case the coset space G/H is equipped with a differential structure and hence a submanifold of G. Furthermore, there is a diffeomorphic map $G/H \to X,\ [gH_x] \longmapsto \Theta(x, g)$ and therefore $X \cong G/H$. This means that we can always consider a homogeneous space to be a coset space with respect to its stabilizer group.

V.2 Symmetric Spaces

As mentioned in the beginning there are less generic homogeneous spaces that are also of interest in the context of coset space compactifications. These spaces are referred to as so-called symmetric spaces and their formal definition along with some important properties are treated in this section.

V.2.1 Symmetric space. Let X be a smooth simply connected Riemannian manifold and g be its Riemannian metric. (X, g) is called a *symmetric space*, if

$$\nabla \mathrm{R}_{\mu\nu\rho\sigma} = 0, \tag{V.4}$$

where ∇ is an affine connection and $\nabla g = 0$, i. e. ∇ is compatible with the metric. From this equation we can deduce:

$$\mathcal{R} = \mathrm{Ric}^{\mu}{}_{\mu} = const, \tag{V.5}$$

$$\mathrm{R}_{\mu\nu\rho\sigma}\mathrm{R}^{\mu\nu\rho\sigma} = const. \tag{V.6}$$

As mentioned in the beginning, symmetric spaces S are always homogeneous. The corresponding group is the

$$\textit{isometry group} := \{f : S \to S \text{ such that } f^* g = g\} \ , \tag{V.7}$$

which is the group of the metric preserving maps. Usually this is only a local property, but since we defined S to be connected, it is true globally as well.

In general we can distinguish between two types of symmetric spaces that do not cover all symmetric spaces, but an important subclass of them, namely the semisimple spaces. Since S is a Lie group, we can talk about all the notions introduced in the previous chapter. The distinction is based on the properties of the Killing form and we define Type I and Type II symmetric spaces in the following way:

- Type I : S is compact, which is equivalent to S being a closed subgroup of $O(n) \Rightarrow$ the Killing form is negative definite,
- Type II: S is not compact $\Rightarrow$ the Killing form is indefinite.

Since we consider only semisimple Lie groups, the Killing form is non-degenerate. Therefore, for type I we can use the push-forward of the left action to define a metric on all of S. One can prove the following statements:

V.2.2 Theorem.

1. *Let G be a simple Lie group and g be a bi-invariant metric on G. Then g is proportional to the Killing metric.*
2. *The Ricci tensor of the Killing metric is proportional to the Killing metric.*

V.3 Explicit Description of Homogeneous Spaces

In the following let G be a Lie group, H a closed subgroup and G/H the corresponding homogeneous space. We are now interested in the explicit shape of all the structure introduced in II and III, namely what the corresponding Lie algebra, connection, torsion and curvature becomes.

V.3.1 Decomposition of the Lie algebra of G. Let $T_eG =: \mathcal{G}$, i. e. $\mathcal{G}$ is the Lie-algebra of G and $\mathcal{H} := T_eH$ is the Lie algebra of H. Then we can find a vector space $\mathcal{K}$ such that

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{K}\,. \tag{V.8}$$

Let $\{E_A\}$ be a basis of $\mathcal{G}$. From this full set of generators of G we choose the elements of $\mathcal{H}$ and $\mathcal{K}$ to be $\{E_i\}$ and $\{E_a\}$, respectively. In the following we always use indices with capital letters to run over the full range of the Lie algebra of G and the indices i, j, k, l, m, n are running over the range of the Lie algebra of H. The indices a, b, c, d, e are meant to run over the remaining values of the Lie algebra $\mathcal{G}$, which do not belong to $\mathcal{H}$. Therefore

$$A, B, C \in \{1, ..., \dim(G)\}\,, \tag{V.9}$$

$$i, j, k \in \{1, ..., \dim(H)\}\,, \quad a, b, c \in \{1, ..., \dim(G/H)\}\,. \tag{V.10}$$

Since $\mathcal{H}$ is an algebra, we stay in this algebra when we multiply elements since it is closed under the group multiplication. Hence we define ${f_{ij}}^k$ to be the structure constants of $\mathcal{H}$. We cannot say anything else for the commutation relations of the other generators and therefore in general we find the following expressions

$$\begin{aligned} [E_i, E_j] &= {f_{ij}}^k E_k, \\ [E_i, E_a] &= {f_{ia}}^b E_b + {f_{ia}}^k E_k, \\ [E_a, E_b] &= {f_{ab}}^c E_c + {f_{ab}}^k E_k\,. \end{aligned} \tag{V.11}$$

In specific cases it is possible to choose a decomposition of the Lie-algebra of G, such that all the components ${f_{ia}}^k$ vanish, which means that the Lie-bracket of any element of $\mathcal{G}$ with an element of $\mathcal{H}$ lies in $\mathcal{H}$ again. Then we call G/H *reductive* and (V.11) becomes

$$\begin{aligned} [E_i, E_j] &= {f_{ij}}^k E_k, \\ [E_i, E_a] &= {f_{ia}}^b E_b \\ [E_a, E_b] &= {f_{ab}}^c E_c + {f_{ab}}^k E_k\,. \end{aligned} \tag{V.12}$$

If G/H is not only homogeneous, but also symmetric, we find that the structure constants ${f_{ab}}^c$ also vanish and hence the Lie-bracket of two elements of $\mathcal{K}$ lies in $\mathcal{H}$. From now we are going to consider the case, where G/H is a reductive homogeneous space.

V.3.2 The Levi-Civita connection on G/H. We are now interested in finding the Levi-Civita connection on the homogeneous space, defined via a metric that is invariant under the group action of G. This means that we are looking for a connection, defined in II.6 by equations

(II.64) and (II.69):

$$\mathrm{d}g_{ab} - \omega^c{}_a g_{cb} - \omega^c{}_b g_{ac} = 0\,, \tag{V.13}$$

$$\mathrm{d}\mathrm{e}^a + \omega^a{}_b \wedge \mathrm{e}^b = 0\,. \tag{V.14}$$

The $\{e^a\}$ from these equations are in this context the left-invariant 1-forms of G/H. Since we want g to be invariant under the group action, it needs to have constant coefficients in this basis and hence $\mathrm{d}g_{ab} = 0$. Let $\{e^i\}$ be a set of one-forms which completes the set of left invariant one-forms on G/H to a basis of one-forms on the full Lie group G. In this notation we can use equation (III.36) to write down the Maurer-Cartan 1-form of G,

$$\Theta := e^A \otimes E_A = e^a \otimes E_a + e^i \otimes E_i\,. \tag{V.15}$$

In order to avoid confusions between the Levi-Civita connection 1-form, we used Θ instead of ω from III.4.1. As stated in theorem III.4.2, we get the equation

$$\begin{aligned}
0 &= \mathrm{d}\Theta + \frac{1}{2}\left[\Theta \wedge \Theta\right] \\
&= \mathrm{d}\Theta + \frac{1}{2}\left([E_A, E_A] \otimes e^A \wedge e^B\right) \\
&\quad \mathrm{d}e^a E_a + \mathrm{d}e^k E_k + \frac{1}{2}\Big([E_a, E_b] \otimes e^a \wedge e^b\Big) \\
&\quad + \frac{1}{2}\left([E_i, E_j] \otimes e^i \wedge e^j\right) + \left([E_a, E_i] \otimes e^a \wedge e^i\right) \\
&= \mathrm{d}e^a E_a + \mathrm{d}e^k E_k + \frac{1}{2}(f_{ab}{}^c E_c + f_{ab}{}^k E_k) \otimes e^a \wedge e^b \\
&\quad + \frac{1}{2}\Big(f_{ij}{}^k E_k \otimes e^i \wedge e^j\Big) + \Big(f_{ai}{}^b E_b \otimes e^a \wedge e^i\Big) \\
&= (\mathrm{d}e^c + \frac{1}{2} f_{ab}{}^c e^a \wedge e^b + f_{ai}{}^c e^a \wedge e^i) \otimes E_c \\
&\quad + (\mathrm{d}e^k + \frac{1}{2} f_{ab}{}^k e^a \wedge e^b + \frac{1}{2} f_{ij}{}^k e^i \wedge e^j) \otimes E_k\,.
\end{aligned} \tag{V.16}$$

Linear independence of E_c and E_k allows to compare their coefficients and hence leads to the following equations for the 1-forms e^a and e^i:

$$\mathrm{d}e^a + \frac{1}{2} f_{cb}{}^a e^c \wedge e^b + f_{kb}{}^a e^k \wedge e^b = 0\,, \tag{V.17}$$

$$\mathrm{d}e^i + \frac{1}{2} f_{jk}{}^i e^j \wedge e^k + \frac{1}{2} f_{bc}{}^i e^b \wedge e^c = 0\,. \tag{V.18}$$

In order to calculate the connection 1-form for the Levi-Civita connection, we use equation (II.72) to calculate the Christoffel symbols first. Note that we used small letters for the non-holonomic basis there, which correspond to capital letters here. We can simplify the expression for the connection coefficients by taking advantage of the fact that the components g_{ab} of the metric are constant with respect to the left-invariant basis $\{e^a\}$. Hence the terms proportional to derivatives of the metric simply vanish and we arrive at

$$\Gamma^C_{AB} = \frac{1}{2}g^{CD}\left\{g([e_A,e_B],e_D) - g([e_A,e_D],e_B) - g(e_A,[e_B,e_D]\right\} . \tag{V.19}$$

In order to avoid confusions with respect to rising and lowering of indices, we chose the form, where all the metric components are still in the equation. We are interested in the components Γ^c_{Ab}, for which we need the Lie-brackets between the non-holonomic basis on G. These are simply given by the corresponding commutators of the generators from equation (V.12):

$$\begin{aligned} [e_i, e_j] &= f_{ij}{}^k e_k, \\ [e_i, e_a] &= f_{ia}{}^b e_b\,, \\ [e_a, e_b] &= f_{ab}{}^c e_c + f_{ab}{}^k e_k\,. \end{aligned} \tag{V.20}$$

The following components of the Christoffel-symbols of G that are going to be relevant for G/H:

$$\Gamma^c_{ab} = \frac{1}{2}g^{cD}\left\{g([e_a,e_b],e_D) - g([e_a,e_D],e_b) - g(e_a,[e_b,e_D]\right\} , \tag{V.21}$$

$$\Gamma^c_{ib} = \frac{1}{2}g^{cD}\left\{g([e_i,e_b],e_D) - g([e_i,e_D],e_b) - g(e_i,[e_b,e_D]\right\} . \tag{V.22}$$

Now if we assume that all components of the metric g_{AB} where $A = a$ and $B = i$, to be zero, i. e. $g_{ai} = 0$ which is true for instance for the Killing metric, we obtain that $g^{ad}g_{db} = \delta^a{}_b$ and hence

$$\begin{aligned} \Gamma^c_{ab} &= \frac{1}{2}g^{cd}\left(f_{ab}{}^e\, g_{ed} - f_{ad}{}^e g_{be} - f_{bd}{}^e g_{ea}\right) \\ &= \frac{1}{2}f_{ab}{}^c + \frac{1}{2}g^{cd}(f_{da}{}^e g_{be} + f_{db}{}^e g_{ab}) \\ &= \frac{1}{2}f_{ab}{}^c + g^{cd} f_{d(a}{}^e\, g_{b)e} \\ &=: \frac{1}{2}f_{ab}{}^c + K_{ab}{}^c\,, \end{aligned} \tag{V.23}$$

along with

$$\Gamma^c_{ib} = g^{cd} f_{ib}{}^e g_{de}\,,$$

which is equivalent to

$$\Gamma^c_{ib} = f_{ib}{}^c\,, \tag{V.24}$$

where we used that $g^{ck} = 0$. In (V.23) also we used the usual notation for the symmetric sum:

$$t_{(ab)} := \frac{1}{2}(t_{ab} + t_{ba})\,.$$

To evaluate the Γ^c_{ib} we used not only the assumption that the metric g_{AB} is block diagonal but also the G-invariance of g_{ab}, which says that

$$f_{ia}{}^c\, g_{cb} + f_{ib}{}^c\, g_{ca} = 0\,. \tag{V.25}$$

Obviously (V.23) shows a decomposition of the Γ^c_{ab} into an antisymmetric and a symmetric part in a, b, where $K_{ab}{}^c$ denotes the symmetric part. So far we have derived the connection coefficients of the full Lie group. It is straightforward to show that these results pretty much hold for the coset space. The difference there is that the invariant 1-forms e^a and e^i are no longer linearly independent and therefore these connection coefficients (V.24) and (V.23) are both contained in the new connection coefficients of the coset. How this looks like in terms of the connection 1-form of the tangent bundle will be the content of the next paragraph.

V.3.3 Connection 1-form and curvature of homogeneous spaces. Due to (V.24) and (V.23) the connection 1-form for the Levi-Civita connection on G/H is given by

$$\begin{aligned} \omega^c{}_b &= \Gamma^c_{Ab} e^A && \text{(V.26)} \\ &= f_{ib}{}^c e^i + (\tfrac{1}{2} f_{ab}{}^c + K_{ab}{}^c) e^a && \text{(V.27)} \\ &= \left(f_{ib}{}^c e^i{}_a + (\tfrac{1}{2} f_{ab}{}^c + K_{ab}{}^c) \right) e^a\,. && \text{(V.28)} \end{aligned}$$

Here we used the decomposition of the left-invariant 1-forms e^i in the basis 1-forms e^a on G/H.

Similar to this Levi-Civita connection we can consider a connection containing a non-vanishing torsion. For this more general case, we get additional terms in the connection coefficients that are symmetric in the lower two indices. Formally we have

$$\omega^c{}_b = \Gamma^c_{Ab} e^A = (T^c{}_{ib} + f_{ib}{}^c)\, e^i \quad + \quad \left(T^c{}_{ab} + \frac{1}{2} f_{ab}{}^c + K_{ab}{}^c \right) e^a\,, \tag{V.29}$$
$$T^c{}_{ib} = T^c{}_{bi} \quad , \quad T^c{}_{ab} = T^c{}_{ba}$$

Following equation (II.83), we can derive the curvature 2-form of this connection:

$$R^a{}_b = \frac{1}{2}(\Gamma^e_{db}\Gamma^a_{ce} - \Gamma^e_{cb}\Gamma^a_{de} - f_{cd}{}^k\Gamma^a_{kb} - f_{cd}{}^e\Gamma^a_{eb}) e^c \wedge e^d\,. \tag{V.30}$$

Inserting the connection 1-form of G/H to equation (II.83), we get the components of Riemannian curvature tensor with respect to the invariant basis as

$$\begin{aligned} R^a{}_{bcd} \;=\; & -f_{cd}{}^k f_{kb}{}^a - \frac{1}{2} f_{cd}{}^e f_{eb}{}^a + \frac{1}{2} f_{b[c}{}^e f_{d]e}{}^a + f_{b[c}{}^e K_{d]e}{}^a \\ & - f_{e[c}{}^a K_{d]b}{}^e - f_{cd}{}^e K_{eb}{}^a + K_{db}{}^e K_{ce}{}^a - K_{cb}{}^e K_{de}{}^a\,. \end{aligned} \tag{V.31}$$

This is the most general result for the components of the curvature tensor for any homogeneous space in the invariant basis. We define the subset of homogeneous spaces, for which the $K_{ab}{}^c$ are equal to zero, as *naturally reductive.* For symmetric spaces, all $f_{ab}{}^c$ are zero and therefore, symmetric spaces are naturally reductive spaces. For them, only the first term of equation (V.31) survives.

Now we can calculate the remaining quantities related to the Riemann tensor, i. e. the Ricci tensor and the Ricci scalar, defined in (II.80) and (II.81). We arrive at

$$\begin{aligned} \mathrm{Ric}_{bd} &= -f_{cd}{}^k f_{kb}{}^c - \frac{1}{2} f_{cd}{}^e f_{eb}{}^c + \frac{1}{2} f_{b[c}{}^e f_{d]e}{}^c + f_{b[c}{}^e K_{d]e}{}^c \\ &\quad -f_{e[c}{}^c K_{d]b}{}^e - f_{cd}{}^e K_{eb}{}^c + K_{db}{}^e K_{ce}{}^c - K_{cb}{}^e K_{de}{}^c , \qquad \text{(V.32)} \\ \mathcal{R} &= -f_{kc}{}^d f^c{}_d{}^k - \frac{1}{2} f_{ab}{}^c f^b{}_c{}^a - \frac{1}{4} f_{ab}{}^c f^{ba}{}_c \qquad \text{(V.33)} \end{aligned}$$

V.4 Invariant Tensors and Covariant Derivative

V.4.1 Covariant derivative. In the last section we have found the explicit form of the Levi-Civita connection and the corresponding curvature. Now we are interested in the covariant derivative with respect to this connection and how it is applied explicitly to arbitrary tensors in the left-invariant basis. Let $t \in \Gamma(T^r_s(G/H)$ be a tensor field on G/H. We can write t with respect to left-invariant basis:

$$t = t^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} e_{a_1} \otimes \ldots \otimes e_{a_r} \otimes e^{b_1} \otimes \ldots \otimes e^{b_s} \qquad \text{(V.34)}$$

Using (II.57) and the formulae for the connection coefficients of G from section V.3.2, we can formulate the covariant derivative by

$$\begin{aligned} \nabla_c t^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} = &+ \Gamma^{a_1}_{cE} t^{E \ldots a_r}{}_{b_1 \ldots b_s} + \ldots + \Gamma^{a_r}_{CE} t^{a_1 \ldots E}{}_{b_1 \ldots b_s} \\ &- \Gamma^{E}_{cb_1} t^{a \ldots a_r}{}_{E \ldots b_s} - \ldots - \Gamma^{E}_{Cb_s} t^{a \ldots a_r}{}_{b_1 \ldots E} . \qquad \text{(V.35)} \end{aligned}$$

V.4.2 Invariant tensor fields. Let t be an arbitrary tensor on G/H, as above, which is additionally invariant under the G-action. As mentioned in III.1.1 we know that the Lie derivative along left invariant vector fields vanishes. Therefore we use equation (III.22) and get

$$\begin{aligned} 0 = &\ f_{ie}{}^{a_1} t^{e \ldots a_r}{}_{b_1 \ldots b_s} + \ldots + f_{ie}{}^{a_r} t^{a_1 \ldots e}{}_{b_1 \ldots b_s} \\ &- f_{ib_1}{}^e t^{a_1 \ldots a_r}{}_{e \ldots b_s} - \ldots - f_{ib_s}{}^e t^{a_1 \ldots a_r}{}_{b_1 \ldots e} . \qquad \text{(V.36)} \end{aligned}$$

Generically the covariant derivative of a left-invariant tensor was given in (V.35) but due to (V.36) it becomes

$$\begin{aligned} \nabla_c t^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} = &+ \Gamma^{a_1}_{ce} t^{e \ldots a_r}{}_{b_1 \ldots b_s} + \ldots + \Gamma^{a_r}_{ce} t^{a_1 \ldots e}{}_{b_1 \ldots b_s} \\ &- \Gamma^{e}_{cb_1} t^{a \ldots a_r}{}_{e \ldots b_s} - \ldots - \Gamma^{e}_{cb_s} t^{a \ldots a_r}{}_{b_1 \ldots e} . \qquad \text{(V.37)} \end{aligned}$$

Here we also used the fact that a left-invariant tensor has constant components with respect to the invariant basis. If we remind ourselves in the explicit shape of the Γ^a_{bc} (V.23), we can already see that if we are dealing with a symmetric space, i. e. $f_{ab}{}^c = 0$, all the Γ^c_{ab} also vanish and therefore all left-invariant tensors are covariantly constant. Let me once again remark that the these Γ^c_{ab} are some of the connection coefficients of G which we used to formulate the covariant derivative on G/H. In later chapters we might change this notation by using the Γ^c_{ab} as connection coefficients of the coset space itself.

Chapter VI

Fibre Bundles

VI.1 Fibre Bundle, Principal Bundle and Associated Bundle

In this section we are going to extend the notion of vector bundles which locally look like the product of a manifolds with a vector space to a more general case. These so-called fibre bundles are formally defined in a similar way than vector bundles were defined before. The difference is that the fibre does not fulfill all conditions of a vector space, but those of a smooth manifold on which a group, called the structure group, is acting from the left.

VI.1.1 Fibre bundle. Let E, M and F be smooth manifolds, called the *total space*, the *base* and the *fibre*, respectively. In addition, let G be a Lie group, called the *structure group*, acting on F, and $\pi : E \to M$ be a surjective smooth map, called the *projection*, satisfying the following conditions:

- $E_p := \pi^{-1}(p)$ is diffeomorphic to F $\forall p \in M$.
- There is an open cover $\{U_i\}$ of M and diffeomorphisms $\phi_i : U_i \times F \to E_{U_i}$ satisfying the condition that $\forall p \in U_i : \phi_i(\{(p, f) : f \in F\}) \subset E_p$.
  Such a collection of maps $\{\phi_i\}$ is called a *local trivialization*.
- The map $\phi_{i,p} := \phi_i\Big|_p : F \to E_p$ has to be a diffeomorphisms as well.
- For mutually intersecting neighborhoods U_i, U_j of p, the transition function $t_{ij}(p) := \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$ is required to be an element of G and a smooth map mapping $U_i \cap U_j$ to G.

Then (E, π, M, F, G) is called a *fibre bundle*. An important subclass of fibre bundles which are mostly used in Physics is formed by those whose fibres F are identical to the their structure group G and they are called *principal bundles* or *(principal) G-bundle*, sometimes also denoted by $\pi : P \to M$. An important property of the principal bundle is that one can define a group action of G on the total space E. This follows from the fact that due to G=F, a right action of G on F can be defined using the local trivialization.

Since it is possible to define an action of G on F from the right (since G=F this is obvious) one can use the local trivialization to define an action of G on all of the total space E. This action is constructed making use of the local trivialization but independent of this specific trivialization anyway.

One can see that the action of G on F is free, i. e. it has trivial stabilizers, and transitive. Therefore following V.1 F is a homogeneous space of G. An important example of such a bundle is the following: Let G be a Lie group. If H is a closed subgroup of G, then we know that G/H is a smooth manifold. For these it is easy to show that $(G, \pi, G/H, H)$ is a principal bundle, where G is the total space, G/H the base, H the fibre and π the natural projection mapping an element of G to its equivalence class in the coset space.

VI.1.2 Functions of type ρ and the associated bundle. Let $\pi : P \longrightarrow M$ be a principal G-bundle. Furthermore, let (V, ρ) be a representation of the Lie-group G. A function

$$\Phi : P \longrightarrow V \tag{VI.1}$$

is called *equivariant* on P, if it satisfies

$$\begin{aligned} \Phi \circ R_g &= \rho^{-1}(g) \circ \Phi \\ \Leftrightarrow \Phi(ug) &= \rho^{-1}(g)\Phi(u), \quad \forall u \in P, \ \forall g \in G\,. \end{aligned} \tag{VI.2}$$

Let us now consider a point $p \in M$ and the corresponding fibre $P_p \subset P$. We define a *function of type* ρ to be an equivariant function that maps P_p to V. One can show that functions of type ρ are related to sections of another bundle over M.

We define the *associated bundle* $\pi' : E \to M$, which is a vector bundle on M endowed with a right action G on E, as follows. The total space is $P \times_\rho V := (P \times V)/\sim$, where for $e_1, e_2 \in P \times V$ we have $e_1 \sim e_2$ if $e_1 = (u, v), e_2 = (ug, \rho(g^{-1})v)$, $u \in P$, and $v \in V$ for some $g \in G$. This means that the total space is just the space of orbits with respect to the right action of G, acting on the elements (u, v) by $(ug, \rho(g)^{-1})$. On every fibre E_p it is straightforward to define vector space operations on the associated bundle by

$$[(u, v)] + \lambda[(u, w)] := [(u, v + \lambda w)]\,.$$

It is obvious that each fibre of the associated principal bundle endowed with this vector space structure is isomorphic to the vector space V itself by the isomorphism $[(u, v)] \longmapsto v$. The transition functions of the associated bundle turn out to be the transition functions of the original principal bundle embedded in the representation (V, ρ), i. e. if U_i and U_j are neighborhoods of $p \in M$ with transition functions t_{ij} for P, then we get the transition functions $\rho(t_{ij})$ for the associated bundle. One can show that the equivariant functions of type ρ defined above are now simply given as sections in the associated bundle, namely

$$\Phi : M \longrightarrow P \times_\rho V\,. \tag{VI.3}$$

VI.2 Connection on Principal Bundles

In this section, the notion of a connection on a principal bundle is introduced. This is important in order to be able to perform parallel transports and define objects like the covariant derivative and curvature of a principal bundle. In gauge theories the corresponding forms, i. e. the connection 1-form and the curvature 2-form, are important, since their pullbacks to the base manifold describe the gauge potential and the field strength of the theory respectively.

VI.2.1 Horizontal and vertical subspaces. Let $\pi : P \to M$ be a principal bundle, $G \simeq F \simeq P_p$ for $p \in M$ be the fibre. In order to define a connection, we consider the tangent bundle TP of the total space P and define a *connection* on a principal bundle to be a decomposition of the tangent space into a vertical subspace, parallel to the fibre, and a horizontal subspace respectively. This decomposition arises from the local trivialization of P, i. e. $P = U_i \times G$, where U_i is a neighborhood around some point $p \in M$. Hence we have coordinates (x, g) and therefore get a basis for the tangent space $(\frac{\partial}{\partial x^\mu}, e_a)$, here e_a is a left invariant basis of TG. The decomposition of the bundle, can be written down as

$$\mathrm{Hor}_u P \oplus \mathrm{Ver}_u P = T_u P, \quad u \in P\,. \tag{VI.4}$$

This is possible since every tangent vector V in $T_u P$ decomposes in a unique way into a horizontal and a vertical part,

$$V = \mathrm{ver}V + \mathrm{hor}V\,. \tag{VI.5}$$

One also requires vector fields on P to decompose in the same way, i. e.

$$\Gamma(TP) = \Gamma(\mathrm{Hor}P) \oplus \Gamma(\mathrm{Ver}P). \tag{VI.6}$$

Obviously, we can already see that the vertical part is locally given by the e_a. However the horizontal part still needs to be specified. Another condition on the connection is that the different horizontal subspaces of the same fibre, as well as the different fibres at the same horizontal space, are obtained by applying the push-forward of the right action, i. e.

$$\mathrm{Hor}_{ug} P = R_{g*}\mathrm{Hor}_u P\,, \quad \mathrm{Ver}_{ug} P = R_{g*}\mathrm{Ver}_u P\,. \tag{VI.7}$$

We also get that

$$\mathrm{Hor} R_{g*} V = R_{g*}\mathrm{Hor} V\,, \quad \mathrm{Ver} R_{g*} V = R_{g*}\mathrm{Ver} V\,. \tag{VI.8}$$

VI.2.2 Connection 1-form. An alternative way of thinking about a connection defined by the decomposition that we introduced in VI.2 is possible by introducing the so-called *connection 1-form* which generalizes the connection 1-form for the tangent bundle to one of fibre bundles. Let $\pi : P \to M$ be a fibre bundle as above and let R_g be the right action of G on P, $X \in \mathcal{G}$. We are considering the following curve on P:

$$g_u(\tau) := R_{\gamma(\tau)} u, \tag{VI.9}$$

where $\gamma(\tau)$ is the one parameter subgroup, namely

$$\gamma(\tau) = e^{\tau X} \,. \tag{VI.10}$$

Then we define

$$\xi_X(u) := \frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_0 R_{\gamma(\tau)} u = \frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_0 g_u(\tau) \tag{VI.11}$$

as the *fundamental* vector field, generated by X. Furthermore, this map can also be considered as a map $\xi_u : \mathcal{G} \to \mathrm{Ver}_u P$. One can show that this map is linear and bijective, therefore we are able to define the corresponding inverse map $\xi_u^{-1} : \mathrm{Ver}_u \to \mathcal{G}$.

Now we have gathered everything that is needed in order to state the definition of the *connection 1-form* $\omega \in \mathcal{G} \otimes \Gamma(T^*P)$ defined by formulae

$$\omega_u := \xi_u^{-1} \circ \mathrm{ver} : T_u P \longrightarrow \mathcal{G}, \tag{VI.12}$$

$$v_u \longmapsto \xi_u^{-1}(\mathrm{ver}\ v_u)\,, \tag{VI.13}$$

$$v_u \in T_u P\,. \tag{VI.14}$$

Hence we can write the connection in terms of the generators of $\mathcal{G}$, namely

$$\omega = E_A \omega^A \,,$$

where E_A denote the generators of $\mathcal{G}$. It is clear that the connection 1-form maps all the horizontal vectors to zero and since ξ_p is a bijection, we even get that a vector is an element of the horizontal space if and only if the connection 1-form maps it to zero. We get following properties of ω:

$$\omega(\mathrm{hor}(v)) = 0, \quad \forall v \in T_u P\,, \tag{VI.15}$$

$$\omega(\xi_X) = X, \quad \forall X \in \mathcal{G}\,, \tag{VI.16}$$

$$R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega, \quad \forall g \in G\,. \tag{VI.17}$$

Here Ad denotes the adjoint representation of the Lie group G as introduced in III.1.2.

One can show that it is always possible first to define the connection 1-form $\omega \in \mathcal{G} \otimes \Gamma(T^*P)$ satisfying conditions VI.12 and VI.14 and then obtain the connection from it. In order to do that, one defines the horizontal part of T_uP to be the kernel of ω and the vertical part to be the complement of the horizontal subspace. Therefore both ways, i. e.

1. defining a connection as the decomposition of the tangent space of P or

2. as the connection 1-form,

turned out to be equivalent.

VI.3 Covariant Derivative and Curvature 2-Form

Since we have defined a connection, one can go on and wonder what curvature may be in this context. We answer that question by defining the curvature of a fibre bundle via a Lie algebra valued 2-form depending on the connection 1-form. This curvature 2-form is of course the generalized curvature 2-form of a vector bundle.

VI.3.1 Covariant derivative. Let V be a vector space and η be a V-valued r-form, i. e. $\eta = v \otimes \sigma$ is an element of $V \otimes \Omega^r(P)$. We define the *covariant derivative* of η by formula

$$\mathrm{D}\eta := v \otimes \mathrm{hor}(\mathrm{d}(\sigma)), \tag{VI.18}$$

where d is just the exterior derivative of r-forms, mapping an r-form to an $(r+1)$-form.

We can apply the covariant derivative for instance to a local section $\Phi : U \longrightarrow P \times_\rho V$ of the associated principal bundle, where $U \subset M$. We get

$$D\Phi = \mathrm{d}\Phi + \rho'(\omega)\Phi\,, \tag{VI.19}$$

where (ρ', V) is a representation of $\mathcal{G}$.

We define the *curvature 2-form* to be the covariant derivative of the connection 1-form:

$$\Omega := D\omega \tag{VI.20}$$

Therefore Ω is an element of $\mathcal{G} \otimes \Omega^2(P)$. Finally one can show that it holds the same equation for the curvature of a principal bundle as the one for the curvature of vector bundles, namely the Cartan structure equation holds

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega\,. \tag{VI.21}$$

Furthermore, the curvature 2-form satisfies the following equations:

$$\mathrm{hor}(\Omega) = \Omega \tag{VI.22}$$

$$R_g^*\Omega = \mathrm{Ad}_{g^{-1}}\Omega \tag{VI.23}$$

$$D\Omega = DD\omega = 0 \tag{VI.24}$$

We may now also take the second derivative of a section in the associated bundle and get

$$DD\Phi = \rho'(\Omega)\Phi, . \tag{VI.25}$$

VI.4 Parallel Transport and Holonomy Group

We introduced the notion of curvature via the Riemannian curvature tensor in paragraph II.7.1. This tensor told us how a vector changes by the parallel transport. Also here we can find a relation between the curvature form and the effect of parallel transport along closed paths on vectors. In order to do that, we first have to define how a fibre is transported along a path on the base in a parallel way. There is a theorem stating that the curvature form of the fibre bundle is related to the Lie algebra of the group of transformations induced by the parallel transport of elements of the fibre. So let us start with the parallel transport.

VI.4.1 Parallel transport. Let $\pi : P \to M$ be a principal bundle and $\gamma : [0,1] \to M$ be a smooth path on the base such that $\gamma(0) = p$ and $\gamma(1) = p'$. We say that a path $\bar{\gamma} : [0,1] \to P$ *lies above* γ if

$$\pi(\bar{\gamma}(t)) = \gamma(t), \ \forall t \in [0,1]. \tag{VI.26}$$

A path $\bar{\gamma}$ lying above γ is called a *horizontal lift* if

$$\bar{\gamma}(t) = u \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\bar{\gamma}(t) \in \mathrm{Hor}_u(P)\,. \tag{VI.27}$$

If we imagine the simple case, where the base is just a curve and the fibre is a straight line in each point, perpendicular to the tangent vector at the point, it is clear that there exist many such horizontal lifts, i. e. all the curves parallel to the base curve. In general one can show that a horizontal lift is unique if one fixes the starting point of $\bar{\gamma}$ in P.

Let $\bar{\gamma}_u : [0,1] \to P$ be the unique lift of γ such that $\bar{\gamma}_u(0) = u$. We define the *parallel transport* of the fibre $\pi^{-1}(p) = P_p$ from p to p' via γ to be the set

$$\left\{u' \in P : u' = \bar{\gamma}_u(1), \ \text{for some } u \in P_p\right\}. \tag{VI.28}$$

VI.4.2 Holonomy group. Having defined parallel transport of elements of the fibre, it is clear that if we perform a parallel transport on such an element along a closed path, we have to end up in the same fibre as we started. But the horizontal lift of a closed path does not necessarily has to be a closed path on the bundle. Hence the transported element is obviously dependent on the path transporting it, which means that the closed paths induce a transformation on the fibre. Therefore we can introduce the notion of a holonomy group for fibre bundles as follows:

Let $\pi : P \to M$ be a principal G-bundle, $u \in P_p,\ p \in M$. We define the *holonomy group* Hol_u of P in u to be the following subgroup of G:

$$\mathrm{Hol}_u = \{h \in G : \exists \text{ closed loop } \gamma : [0,1] \to M \text{ s.t. } \bar{\gamma}_u(1) = uh\} . \tag{VI.29}$$

Chapter VII

Gauge Field Theories and Associated Principal Bundles

In this chapter we want to draw the connection to the field of theoretical particle physics that makes use of the mathematical tools we introduced so far, identifying the physical terms "gauge field" and "gauge potential" with the corresponding mathematical objects. We want to consider gauge theories and the way how to describe them using the theory of fibre bundles. We start with a quick introduction to gauge field strength theory .

VII.1 Gauge Invariance and Yang-Mills Equations

Let us consider the case, in which the so-called *matter field* ϕ is a vector of n real scalar fields $\phi = (\phi_i,\ i = 1, ..., n)$ on a manifold M equipped with some metric g on the cotangent space. The corresponding Lagrangian may be given by

$$\mathcal{L} = g(\mathrm{d}\phi^\top, \mathrm{d}\phi) - m^2\phi^\top\phi \ , \tag{VII.1}$$

where we chose the way of writing $\partial_\mu\phi^\top\partial^\mu\phi = g(\mathrm{d}\phi^\top, \mathrm{d}\phi)$ for a reason which will become clear later on. We define a *global gauge transformation* to be a transformation of the fields which is constant on the whole manifold M. As we can see, this Lagrangian is invariant under the global gauge transformation of the field denoted as B, where $B^\top B = id$. Here the B are meant to be an n-dimensional matrix representation of $SO(n)$. One can easily see the invariance under such transformations, since

$$\phi' = B^{-1}\phi, \quad \phi^\top = B\phi^\top \ , \tag{VII.2}$$

and hence the transformation matrices compose to the identity in both terms in $\mathcal{L}$.

Now we want to generalize the notion of global gauge transformations to such referred to as *local gauge transformations*, i. e. transformations that depend on the position $x \in M$ explicitly. For the $SO(n)$ case these transformations still dissappears in the mass term which is no longer true for the kinetic term:

$$\begin{aligned}(\mathrm{d}\phi^\top)' &= \mathrm{d}(\phi^\top B) = \phi^\top \mathrm{d}(B) + \mathrm{d}\phi^\top B & \text{(VII.3)}\\ (\mathrm{d}\phi)' &= \mathrm{d}(B^{-1}\phi) = \mathrm{d}(B^{-1})\phi + B^{-1}\mathrm{d}\phi\,. & \text{(VII.4)}\end{aligned}$$

From these two equations, it is easy to see that due to the terms containing derivatives of the transformation matrices, the invariance of the Lagrangian is spoiled. Since we still require its invariance, we have to find a way in order to restore this.

The canonical solution is to introduce a new field $\mathcal{A}$, coupled to ϕ which transforms in such a way that the unwanted terms, namely the first terms in equations (VII.3) and (VII.4) disappear.

According to this requirement, the so-called *gauge field* $\mathcal{A}$ has to transform as

$$\mathcal{A}' = B^{-1}\mathcal{A}B + B^{-1}\mathrm{d}B \tag{VII.5}$$

and has therefore to be a $\mathcal{G}$-valued 1-form on M. Hence we obtain a gauge invariant Lagrangian by performing the replacement

$$\mathrm{d} \mapsto \mathrm{d} + \mathcal{A} =: \mathcal{D}\,.$$

$\mathcal{D}$ is then called the *gauge covariant derivative.* To see the invariance, let us see how $\mathcal{D}\phi$ changes under local gauge transformations:

$$\begin{aligned}(\mathcal{D}\phi)' &= \mathrm{d}\phi' + \mathcal{A}'\phi'\\ &= \mathrm{d}(B^{-1}\phi) + (B^{-1}\mathcal{A}B + B^{-1}\mathrm{d}B)B^{-1}\phi\\ &= \mathrm{d}(B^{-1})\phi + B^{-1}\mathrm{d}\phi + B^{-1}\mathcal{A}\phi + B^{-1}\mathrm{d}(B)B^{-1}\phi\\ &= B^{-1}\mathrm{d}\phi + B^{-1}\mathcal{A}\phi + \mathrm{d}(B^{-1})\phi + \left(\mathrm{d}(B^{-1}B)B^{-1} - (\mathrm{d}B^{-1})BB^{-1}\right)\phi\\ &= B^{-1}(\mathrm{d}+\mathcal{A})\phi + \mathrm{d}(B^{-1})\phi - \mathrm{d}(B^{-1})\phi\\ &= B^{-1}\mathcal{D}\phi\,. & \text{(VII.6)}\end{aligned}$$

Similarly we get the transformation rule for the transposed quantity

$$(\mathcal{D}\phi^\top)' = B\mathcal{D}\phi^\top\,. \tag{VII.7}$$

Using these transformation rules, it is trivial that (VII.1) is invariant under local $SO(n)$ gauge transformations.

So far we have managed to preserve the gauge invariance by introducing the gauge potential. However, this new field has a physical meaning, and can be interpreted as a new particle of the corresponding theory. But so far, we only have interaction terms for $\mathcal{A}$ in our Lagrangian. We would like to derive equations of motion for this new particle, either. Hence, we also need to allow dynamical terms for the gauge potential to enter our Lagrangian, i. e. term containing derivatives of the gauge potential $\mathcal{A}$. These terms of course need to be gauge invariant under the gauge group in order to preserve invariance for the whole Lagrangian.

In order to find such a term, we define the *gauge field* or *field strength* to be

$$\mathcal{F} := \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \,. \tag{VII.8}$$

Obviously this is a $\mathcal{G}$-valued 2-form on M. We find the following gauge transformation for the gauge field strength

$$\mathcal{F}' = B^{-1}\mathcal{F}B, \tag{VII.9}$$

which is obviously not an invariant term and not even a scalar. If we remind ourselves of some linear algebra, we remember that traces of matrix products are cyclically invariant. So $\mathcal{F}$ would be an invariant quantity, but we know that $\mathcal{F}$ is also $so(n)$-valued and hence traceless. One can now find that $*\mathcal{F}$, where $*$ is the Hodge operator, transforms in the same way as $\mathcal{F}$ and hence $\mathcal{F} \wedge *\mathcal{F}$ does. This term is not traceless and hence we get the gauge invariant dynamical term for the gauge potential by

$$\mathrm{tr}(\mathcal{F} \wedge *\mathcal{F}) \,, \tag{VII.10}$$

and therefore the new full invariant Lagrangian is

$$\mathcal{L} = g(\mathcal{D}\phi^\top, \mathcal{D}\phi) - m^2\phi^\top\phi \; + \mathrm{tr}(\mathcal{F} \wedge *\mathcal{F}) \,. \tag{VII.11}$$

To sum up, we started with a Lagrangian that was invariant under global gauge transformations. Then by the requirement for the Lagrangian to be also invariant under local transformations, we obtained interaction terms involving both gauge potential and matter fields, as well as the dynamical term for the gauge potential. In these cases, we provided local gauge invariance by replacing all the partial derivatives by gauge covariant derivatives. If all interacting terms of matter fields and gauge bosons in a theory are obtained by such replacements we talk about *minimal coupling*.

VII.2 Gauge Theory and Fibre Bundles

In the last chapter, we introduced the geometry of fibre bundles in a rather general framework. In the following we want to describe a relation between the idea of gauge field theory and the theory of fibre bundles. As already pointed out, certain objects and equations, we introduced in the theory of gauge fields, look similar to corresponding objects and equations in the theory of fibre bundles. For instance it is easy to see that the defining equation of the gauge field strength (VII.8) looks like the Cartan structure equation for the curvature form on a fibre bundle and, indeed, these quantities turn out to be related.

In order to see how this relation arises, consider $\pi : P \longrightarrow M$ to be a principal G-bundle, $s, \hat{s} : U \longrightarrow P$ to be local sections on $U \subset M$, $\omega \in \mathcal{G} \otimes \Gamma(T^*P)$ to be a connection 1-form on P and $\Omega = D\omega$ the corresponding curvature 2-form on P. Moreover let $\Phi : M \longrightarrow P \times_\rho V$ be a section in the associated vector bundle and, for the Lie group G, ω_G be the Maurer-Cartan form. Then the pull-backs under the local sections s and $\hat{s}$ are denoted by

$$A := s^*\omega, \qquad \hat{A} := \hat{s}^*\omega\,, \tag{VII.12}$$

$$F := s^*\Omega, \qquad \hat{F} := \hat{s}^*\Omega\,, \tag{VII.13}$$

$$\phi := s^*\Phi, \qquad \hat{\phi} := \hat{s}^*\Phi\,. \tag{VII.14}$$

One can show that there is a unique way to transform s to $\hat{s}$, namely

$$\hat{s}(x) = R_{T(x)}s(x), \quad \forall x \in U\,, \tag{VII.15}$$

via the unique map $T : U \longrightarrow G$. This map allows us to pull-back the Maurer-Cartan form ω_G on G to the base manifold,

$$\tilde{\omega}_G := S^*\omega_G\,.$$

Our goal is to see how the pulled back objects with respect to the sections s and $\hat{s}$ are related to each other and how they change under the change of a local section. Therefore we consider an arbitrary vector field $v \in \Gamma(TM)$ on the base by s and the push-forward of v by s, which is related to the push-forward by $\hat{s}$ in the following way:

$$(\hat{s}_*v)(u) = R_{T(x)*}(s_*v)\mid_u + \xi_{\tilde{\omega}_G(v)}\mid_u\,, \tag{VII.16}$$

where u is given by $u = \hat{s}(x) = R_{T(x)}s(x)$ while $\xi_{\omega_G(v)} \in \mathrm{vert}(TP)$ is the fundamental vector field generated by $\tilde{\omega}_G(v) \in \mathcal{G}$ as defined in (VI.11). Using (VII.16), we can derive how the change of a section is performed on A, F, and Φ:

$$\begin{aligned}
\hat{A}(v) &= \left(\hat{s}^*(\omega)\right)(v) = \omega(\hat{s}_*v) \\
&= \omega\left(R_{T*}(s_*v) + \xi_{\tilde{\omega}_G(v)}\right) \\
&= \omega\left(R_{T*}(s_*v)\right) + \omega\left(\xi_{\tilde{\omega}_G(v)}\right) \\
&= \left(R_{T*}\omega\right)(s_*v) + \tilde{\omega}_G(v) \\
&= \left(\mathrm{Ad}_{T^{-1}}A\right)(v) + \left(T^*\omega_G\right)(v)\,.
\end{aligned}$$

Here we made use of (VI.15) and (VI.16).

$$\begin{aligned}
\hat{F}(v,w) &= \left(\hat{s}^*(\Omega)\right)(v,w) = \Omega(\hat{s}_*v, \hat{s}_*w) \\
&= \Omega\left(R_{T*}(s_*v), R_{T*}(s_*w)\right) + \Omega\left(\xi_{\tilde{\omega}_G(v)}, \xi_{\tilde{\omega}_G(w)}\right) \\
&= \left(\mathrm{Ad}_{T^{-1}}F\right)(v,w) + 0\,,
\end{aligned}$$

where we made use of the fact that Ω is a horizontal 2-from and the fundamental vector field is purely vertical.

$$\begin{aligned}\hat{\phi}(x) &= (\hat{s}^*\Phi)(x) = \Phi(\hat{s}(x)) \\ &= \Phi(R_{T(x)}s(x)) = \rho\left(T^{-1}(x)\right)\Phi(s(x)) \\ &= \rho\left(T^{-1}(x)\right)(s^*\Phi)(x) \\ &= \rho\left(T^{-1}(x)\right)\phi(x)\,.\end{aligned}$$

Here we made use of the fact that a section in the associated bundle is nothing but a function of type ρ and therefore satisfying (VI.2). Summing up all the results, we have

$$\hat{A} = \left(\mathrm{Ad}_{T^{-1}}A\right)(v) + \left(T^*\omega_G\right)(v) \tag{VII.17}$$

$$\hat{F} = \left(\mathrm{Ad}_{T^{-1}}F\right)(v,w) \tag{VII.18}$$

$$\hat{\phi} = \rho\left(T^{-1}(x)\right)\phi(x)\,. \tag{VII.19}$$

The only thing which remains to be done is to consider the rules of changing sections in the matrix representation (ρ, V) of the structure group G. For this purpose we define B mapping points of the base manifold M to $(n\times n)$-matrices, $n = \dim(V)$ by:

$$B := \rho\circ T\,. \tag{VII.20}$$

Moreover let (ρ', V) be a representation of the Lie algebra $\mathcal{G}$ in V and

$$\mathcal{A} := \rho'(A)\,, \tag{VII.21}$$

$$\mathcal{F} := \rho'(F)\,, \tag{VII.22}$$

meaning that the Lie-algebra part of the Lie-algebra valued forms are for $\mathcal{A}$ and CF given in the representation (ρ', V).

If we plug (VII.21) and (VII.22) into (VII.17) and (VII.18) and use the matrix representation for equation (VII.19), we obtain

$$\hat{\mathcal{A}} = B^{-1}\mathcal{A}B + B^{-1}\mathrm{d}B\,, \tag{VII.23}$$

$$\hat{\mathcal{F}} = B^{-1}\mathcal{F}B\,, \tag{VII.24}$$

$$\hat{\phi} = B^{-1}\phi\,. \tag{VII.25}$$

Since we are only dealing with pull-backs of forms on P, we can define a covariant derivative, acting on such pull-backs by $\mathcal{D}\circ s^* := s^*\circ D$, where D is the covariant derivative of the fibre bundle. Using this, as well as equations (VI.20), (VI.24), (VI.19), (VI.25) and definition (III.38), we find

$$\begin{aligned} \mathcal{F} &= \mathcal{D}\mathcal{A} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}\,, && \text{(VII.26)} \\ \mathcal{D}\mathcal{F} &= \mathrm{d}\mathcal{F} + [\mathcal{F} \wedge \mathcal{A}]\;, && \text{(VII.27)} \\ \mathcal{D}\phi &= \mathrm{d}\phi + \mathcal{A}\phi\,, && \text{(VII.28)} \\ \mathcal{D}\mathcal{D}\phi &= \mathcal{F}\phi\,. && \text{(VII.29)} \end{aligned}$$

Amazingly these are precisely the transformation rules under gauge transformations and the defining properties we derived for the gauge potential, the field strength and the matter fields in section VII.1. This means that the gauge potential is the pull-back of some connection on a principal bundle, the gauge field strength is actually the pull-back of the curvature 2-form of the same principal bundle and the matter fields are the pull-back of a section in the associated principal bundle. What we defined to be a gauge transformation in section VII.1, is a change of the local section in the principal bundle, i. e. $s(x) \mapsto \hat{s}(x) = R_{T(x)}s(x)$. The gauge transformations we called global, refer to a change of sections, wherein $T(x)$ is constant on $U \subset M$ and the local gauge transformations refer to the change of sections, where the $T(x)$ is indeed dependent on $x \in U$.

Chapter VIII

Instantons on Symmetric Coset Spaces

VIII.1 From Yang-Mills on $\mathbb{R} \times G/H$ to the Kink

In this section we want to derive Yang-Mills equations on the product space $\mathbb{R} \times G/H$, where G is a semi-simple Lie group, H a closed subgroup and G/H a reductive homogeneous space.

VIII.1.1 The geometry of $\mathbb{R} \times G/H$. We are going to use the notation and formulas of section V.3. We use the Killing-Cartan metric, induced by the Killing form that we introduced in (III.26). Therefore its components are given by

$$g_{AB} = f^{C}{}_{AD}\, f^{D}{}_{CB} \ . \tag{VIII.1}$$

One can choose the generators of the Lie-algebra in a way such that we have

$$g_{AB} = \delta_{AB} \tag{VIII.2}$$

As in V.3, we are using $A, B, C = 1, ..., \dim(G)$ to be the components of the Lie group, $a, b, c = 1, ..., \dim(G/H)$ those of the coset space and $i, j, k = 1, ...\dim(H)$ to be the components of the subgroup H. Since we are dealing with reductive homogeneous space, (V.12) applies and we obtain the following components of the Killing-Cartan metric

$$g_{ij} = f^{k}{}_{il} f^{l}{}_{kj} + f^{b}{}_{ia} f^{a}_{bj} = \delta_{ij} \ , \quad g_{ia} = 0 \ , \tag{VIII.3}$$

$$g_{ab} = 2 f^{i}{}_{ad} f^{d}_{ib} + f^{c}{}_{ad} f^{d}_{cb} = \delta_{ab} \, . \tag{VIII.4}$$

For the coset space we use the induced metric. We do not want to use the Levi-Civita connection, but a connection with non-vanishing torsion on G/H. The torsion 2-form (II.62) is given by the torsion tensor (II.61). As stated in equation (II.63) the torsion two-form T^a satisfies

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = T^a \, .$$

For our considerations we need the coefficients of the torsion to have the following form:

$$T^a{}_{bc} = \kappa \ f^a{}_{bc} \, . \tag{VIII.5}$$

Using equation (V.29), the torsion-full connection 1-form on the coset space is given by

$$\omega^a{}_b = f^a{}_{ib} e^i + \frac{1}{2}(\kappa+1) f^a{}_{cb} e^c \ =: \ \omega^a_{cb}\ e^c , \tag{VIII.6}$$

which defines the components $\omega^a{}_{bc}$ by

$$\omega^a_{cb} = f^a{}_{ib} e^i{}_c + \frac{1}{2}(\kappa+1) f^a{}_{cb} . \tag{VIII.7}$$

Considering the whole space $\mathbb{R}\times G/H$, we see that all the connection coefficients and therefore all coefficients of the connection 1-form containing indices of $\mathbb{R}$ vanish due to the fact that we use the flat metric for this part of the space, namely

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + \mathrm{d}s^2_{G/H} , \tag{VIII.8}$$

where

$$\mathrm{d}s^2_{G/H} = \delta_{ab} e^a e^b . \tag{VIII.9}$$

VIII.1.2 Yang-Mills equations. Now we collected everything we need to take the covariant derivative of tensor fields on $\mathbb{R}\times G/H$. What we are actually looking for are the Yang-Mills field strength and solutions to the Yang-Mills equations. As we learned in section VII.2, this means we are considering a gauge theory which is given by a principal bundle with some structure group and a connection. We choose the principal G-bundle $P(\mathbb{R}\times G/H, G)$ over $\mathbb{R}\times G/H$. Therefore a gauge potential for this bundle is locally written as

$$\mathcal{A} := \mathcal{A}_0 e^0 + \mathcal{A}_a e^a , \tag{VIII.10}$$

and we choose a gauge in which $\mathcal{A}_0 = 0$. The field strength, depending on the curvature of the principal bundle with respect to the chosen connection, is given by the gauge covariant derivative of the gauge potential (VII.26), namely

$$\begin{aligned} \mathcal{F} &= \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A} \\ &=: \mathcal{F}_{0a} e^0\wedge e^a + \frac{1}{2}\mathcal{F}_{ab} e^a \wedge e^b \end{aligned} \tag{VIII.11}$$

In order to achieve dimensional reduction with respect to this coset space, we choose a G-equivariant ansatz for the gauge potential and therefore obtain a real scalar field $\phi(\tau)$, $\tau\in\mathbb{R}$:

$$\mathcal{A} = e^i E_i + \phi\ e^a\ E_a, \quad \mathcal{A}_a = e^i{}_a E_i + \phi E_a \tag{VIII.12}$$

where E_i and E_a are the generators of H and G/H respectively as used in V.3. If we calculate (VII.26) with respect to the ansatz (VIII.12) we arrive at the following gauge field strength:

$$\mathcal{F} = \dot\phi\ e^0\wedge e^a E_a - \frac{1}{2}\ \left\{(1-\phi^2) f_{bc}{}^i E_i + (\phi-\phi^2) f_{bc}{}^a E_a\right\}\ e^b\wedge e^c , \tag{VIII.13}$$

where $\dot{\phi} := \frac{\mathrm{d}\phi}{\mathrm{d}\tau}$. Hence we read off the corresponding components

$$\mathcal{F}_{0a} = \dot{\phi}\; E_a,\;\; \mathcal{F}_{bc} = -\left\{(1-\phi^2) f^i_{bc} E_i + (\phi - \phi^2) f^a_{bc} E_a\right\}\;, \tag{VIII.14}$$

The Yang-Mills equations on $\mathbb{R} \times G/H$ are given by the condition that the gauge covariant derivative of the gauge field strength vanishes, namely

$$\begin{aligned} \mathcal{D}_a \mathcal{F}^{a0} &:= \nabla^{G/H}_a \mathcal{F}^{a0} + [\mathcal{A}_a, \mathcal{F}^{a0}] \\ &:= e_a\left(\mathcal{F}^{a0}\right) + \omega^a_{ab}\mathcal{F}^{b0} + [\mathcal{A}_a, \mathcal{F}^{a0}] = 0 \end{aligned} \tag{VIII.15}$$

$$\begin{aligned} \mathcal{D}_0 \mathcal{F}^{0b} + \mathcal{D}_a \mathcal{F}^{ab} &= e_0\left(\mathcal{F}^{0b}\right) + \nabla^{G/H}_a \mathcal{F}^{ab} + [\mathcal{A}_a, \mathcal{F}^{ab}] \\ &= e_0\left(\mathcal{F}^{0b}\right) + e_a\left(\mathcal{F}^{ab}\right) + \omega^d_{da}\mathcal{F}^{ab} + \omega^b_{cd}\mathcal{F}^{cd} + [\mathcal{A}_a, \mathcal{F}^{ab}] = 0\,. \end{aligned} \tag{VIII.16}$$

Here we denoted by $\nabla^{G/H}$ the covariant derivative on G/H with respect to the torsion-full connection from (VIII.6). If we plug (VIII.14) in (VIII.16), after pulling down all the indices, which does not matter due to equation (VIII.9), we arrive at the following differential equation for ϕ:

$$\ddot{\phi}\, E_a + \left[(\phi - \frac{1}{2}(\kappa{+}1))(\phi - \phi^2) f_{abc} f_{dbc} + \phi\,(1-\phi^2) f_{aci} f_{dci}\right] E_d = 0\,. \tag{VIII.17}$$

Here we also made use of the Jacobian identity for the structure constants. For further simplifications we constrain ourselves to the case, where we have the following relations for the structure constants:

$$f_{aci} f_{bci} = \frac{1}{2}(1{-}\alpha)\delta_{ab} \qquad \Leftrightarrow \qquad f_{acd} f_{bcd} = \alpha\,\delta_{ab}\,, \alpha \in \mathbb{R}\,, \tag{VIII.18}$$

where α is some real parameter. This simplification still covers a fair amount of interesting spaces (see for instance [35]). Taking this into account, we get

$$\begin{aligned} 2\,\ddot{\phi} &= (1{+}\alpha)\,\phi^3 - \alpha(\kappa{+}3)\,\phi^2 - (1{-}\alpha(\kappa{+}2))\,\phi \\ &= -\phi\,(1-\phi^2) \;+\; \alpha\,\phi\,(1-\phi)(2-\phi) \;+\; \alpha\,\kappa\,\phi\,(1-\phi) \\ &= (1{+}\alpha)\,\phi\,(\phi-1)\left(\phi - \frac{(\kappa+2)\alpha - 1}{\alpha+1}\right). \end{aligned} \tag{VIII.19}$$

If we consider the last line of equation (VIII.19) to be the derivative of some potential $V(\phi)$, namely

$$V'(\phi) = (1{+}\alpha)\,\phi\,(\phi-1)\left(\phi - \frac{(\kappa+2)\alpha - 1}{\alpha+1}\right), \tag{VIII.20}$$

on can recognize

$$\ddot{\phi} = \frac{1}{2} V'(\phi) \tag{VIII.21}$$

to be simply a static equation of motion of some ϕ^4-model.

In the following two sections we want to solve this equation for ϕ choosing certain values of α and κ and therefore obtain solutions $\mathcal{A}$ of the Yang-Mills equations. We are especially interested in the asymptotic behavior of $\mathcal{A}(\tau)$ for $\tau \longrightarrow \pm\infty$.

VIII.2 Yang-Mills Solutions on Symmetric Coset Spaces

VIII.2.1 BPS kink equations for $\alpha = 0$. We start with the simplest case for our equation (VIII.19), where $\alpha = 0$ and hence κ arbitrary. This situation happens in case when G/H is a symmetric space and therefore

$$f^a{}_{bc} = 0\,, \forall a, b, c = 1, ..., \dim\,(G/H)\ . \tag{VIII.22}$$

It follows that $f_{aci} f_{bci} = \frac{1}{2}\delta_{ab}$ and hence (VIII.19) becomes

$$\ddot{\phi} = -\frac{1}{2}\phi(1-\phi)^2\,. \tag{VIII.23}$$

Before focusing our attention to the solution of equation (VIII.23) we want to see how the action functional behaves in this situation. As we have learnt in section (VII.1), the most general invariant term for our gauge field strength is given by (VII.10), namely

$$\mathcal{L} = \mathrm{tr}(\mathcal{F} \wedge *\mathcal{F})\,, \tag{VIII.24}$$

where $*$ is the Hodge operator. Therefore we get the Yang-Mills action functional to be the integrated Lagrangian with a factor:

$$S = -\frac{1}{4}\int_{\mathbb{R}\times G/H} \mathrm{tr}(\mathcal{F} \wedge *\mathcal{F})\,. \tag{VIII.25}$$

Since we are dealing with compact G/H and $\mathcal{F}$ is depending on the coset variables only up to gauge transformations we can easily compute this part of the action functional:

$$\begin{aligned} S &= -\frac{1}{4}\int_{-\infty}^{+\infty} \mathrm{d}\tau \int_{G/H} \mathrm{tr}\,(\mathcal{F}(\tau) \wedge *\mathcal{F}(\tau)) \\ &= -\frac{1}{4}\int_{-\infty}^{+\infty} \mathrm{d}\tau \left(tr(\mathcal{F}(\tau) \wedge *\mathcal{F}(\tau)) \int_{G/H} \right) \\ &= -\frac{1}{4}\mathrm{Vol}(G/H) \int_{-\infty}^{+\infty} \mathrm{d}\tau\ \mathrm{tr}\,(\mathcal{F}(\tau) \wedge *\mathcal{F}(\tau)) \\ &= -\frac{1}{4}\mathrm{Vol}(G/H) \int_{-\infty}^{+\infty} \mathrm{d}\tau \left(\dot{\phi}^2 + \frac{1}{4}(1-\phi^2)^2 \right) \\ &=: -\frac{1}{4}\mathrm{Vol}(G/H) \cdot E\,. \end{aligned} \tag{VIII.26}$$

Here E is precisely the ϕ^4 energy functional. It is easy to see that the energy functional and therefore the Yang-Mills action functional has a lower bound, given by the topological charge of the vacuum

$$q := \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}\phi\,. \tag{VIII.27}$$

In order to see that, we need to integrate (VIII.23) to a first order BPS equation, namely

$$\dot{\phi} = \pm\frac{1}{2}(1-\phi^2)\,. \tag{VIII.28}$$

Let us use this to rearrange the energy functional a little bit:

$$\begin{aligned} E &= \int_{-\infty}^{+\infty} \mathrm{d}\tau \left(\dot{\phi}^2 + \frac{1}{4}(1-\phi^2)^2\right) \\ &= \int_{-\infty}^{+\infty} \mathrm{d}\tau \left(\dot{\phi} \mp \frac{1}{2}(1-\phi^2)\right)^2 \pm \int_{-\infty}^{+\infty} \mathrm{d}\phi\,(1-\phi^2) \\ &\geq \frac{4}{3}|q|\,. \end{aligned} \tag{VIII.29}$$

The vacua of the kink model are given by the asymptotical values of the scalar fields and can therefore be identified with the topological charges q. They can take values in $\{1, 0, -1\}$ since $\phi(-\infty) = \pm 1$ and $\phi(+\infty) = \pm 1$. The corresponding solutions of (VIII.28) and hence of (VIII.23) for either $q = +1$ or $q = -1$ are known as the ϕ^4-kink and the ϕ^4-antikink respectively. The kink is simply given by the positive hyperbolic tangens with obvious asymptotical values and topological charge

$$\phi = \tanh(\frac{\tau}{2}), \quad \phi(\tau) \longrightarrow \pm 1, \tau \longrightarrow \pm\infty, \quad q = +1\,. \tag{VIII.30}$$

The antikink is then just given by the negative kink profile function and therefore has the opposite asymptotical behavior and topological charge

$$\phi = -\tanh(\frac{\tau}{2}), \quad \phi(\tau) \longrightarrow \pm 1, \tau \longrightarrow \mp\infty, \quad q = -1\,. \tag{VIII.31}$$

The inequality (VIII.29) in this case becomes an equality and hence both solutions yield the same energy $E = \frac{4}{3}$.

VIII.2.2 Instantons on $\mathbb{R} \times G/H$. Because we have solved the equation (VIII.23), we also have solved the Yang-Mills equations on $\mathbb{R} \times G/H$. So if we plug (VIII.30) into the ansatz for the gauge potential (VIII.12) and the gauge field strength (VIII.13), we get

$$\mathcal{A} = e^i E_i \;+\; \tanh\frac{\tau}{2}\, e^a E_a\;, \tag{VIII.32}$$

and

$$\mathcal{F} = \frac{1}{2}\cosh^{-2}\frac{\tau}{2}\,\left(e^0 \wedge e^a\, E_a - f^i{}_{ab}\, e^a \wedge e^b\, E_i\right) \tag{VIII.33}$$

If we take a look at the asymptotic behavior of the ϕ and hence of the $\mathcal{A}$, one can see that $\mathcal{A}_\pm := \mathcal{A}(\pm\infty)$ also defines a connection on

$$(G/H)_\pm := G/H \cup \{\pm\infty\} \;, \tag{VIII.34}$$

respectively. If we consider the infinity limit of $\mathcal{F}$ from (VIII.33), we can see that

$$\mathcal{F}_\pm := \mathcal{F}(\pm\infty) = 0\,. \tag{VIII.35}$$

Therefore both gauge potentials $\mathcal{A}_\pm$ describe vacuum configurations. Our gauge potentials can now be written in terms of G-valued functions as

$$\mathcal{A}_- = h_-^{-1}\mathrm{d}h_-, \quad \mathcal{A}_+ = h_+^{-1}\mathrm{d}h_+\,. \tag{VIII.36}$$

Performing the gauge transformation

$$\tilde{\mathcal{A}} := h_-\mathcal{A}h_-^{-1} + h_-\mathrm{d}h_-^{-1}\,, \tag{VIII.37}$$

we find that the corresponding asymptotical connections become

$$\tilde{\mathcal{A}}_- = 0,\; \tilde{\mathcal{A}}_+ = g^{-1}\mathrm{d}g\,, \quad \text{with } g := h_+h_-^{-1} : G/H \to G\,. \tag{VIII.38}$$

One can see that the degree of g is equal to the topological charge of the kink. So (VIII.32) is actually an instanton configuration that acts as a transition function between the two vacua given in (VIII.38).

VIII.2.3 Instanton-anti-instanton chains. The solution ϕ that we found for the differential equation (VIII.23) is actually not unique. The most general solution for this differential equation is actually given by a Jacobi elliptic function. Such a function in general has the following form

$$C(k)\ \mathrm{sn}[b(k)\ \tau; k]\,. \tag{VIII.39}$$

If we take k to be zero, this simply is a sinus. Changing k from zero to one, the sin deforms to a hyperbolic tangens. Therefore for $k = 1$ this is the kink solution we found in (VIII.30). In all other cases, the Jacobi elliptic functions are periodic with period $\mathcal{K}(K)$ which is the complete elliptic integral of the first kind with infinite sum expansion

$$\mathcal{K}(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\ k^{2n} \tag{VIII.40}$$

So one can see that this expression becomes infinite in the limit $k \to 1$ and hence spoils periodicity. Therefore, if we require $\phi(\tau)$ to be periodic in τ, namely

$$\phi(\tau + L) = \phi(\tau)\,, \tag{VIII.41}$$

we can think of the solution to (VIII.30) as of a function on the 1-sphere S^1 with circumference L. For our case this periodicity condition yields the following solution:

$$\phi(\tau;k) = 2k\, b(k)\, \mathrm{sn}[b(k)\tau;k], \text{ where} \qquad \text{(VIII.42)}$$
$$b(k) = (2+2k^2)^{-1/2}, \quad 0 \leq k \leq 1 \,.$$

In the literature, such a solution is known as a sphaleron [36], which are non-BPS solutions unlike the kink and antikink solutions from paragraph VIII.2.1. If we compare (VIII.41) and (VIII.42), we find the following condition on k:

$$b(k)\, L = 4\mathcal{K}(k)\, n \qquad \text{for} \quad n \in \mathbb{N}\,. \qquad \text{(VIII.43)}$$

It follows that as soon as we have a circumference $L \geq 2\pi\sqrt{2}N$, we get a solution

$$\phi(\tau; k(L,n)) \quad \forall n \in \mathbb{N},\ n \leq N\,. \qquad \text{(VIII.44)}$$

The topological charge defined in (VIII.29) equals zero due to the periodicity of the solution. Comparing this periodic solution with the (anti)kink case, we can interpret each half of a full period going from one minimum of $\phi(\tau)$ to a maximum, as a kink and the other half of the period going from a maximum to a minimum as an antikink. Therefore the sphaleron solution describes a chain of kink-antikink pairs winding around the 1-sphere and the subscript n in ϕ_n denotes the number of such pairs.

The value of the energy functional for the sphaleron solutions is given in terms of the complete Jacobi integral of the first as well as of the second kind. The latter one is given by

$$\mathcal{E}(k) = \frac{\pi}{2}\left\{1 - \sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{k^{2n}}{2n-1}\right\}. \qquad \text{(VIII.45)}$$

We arrive at the following energy for the sphaleron

$$E[\phi_n] = \frac{2n}{3\sqrt{2}}\left[8(1+k^2)\,\mathcal{E}(k) \;-\; (1-k^2)(5+3k^2)\,\mathcal{K}(k)\right]. \qquad \text{(VIII.46)}$$

If one takes the circumference L of the circle to be infinity, the energy of the sphaleron configuration is simply given by the sum over the energies of all kinks and antikinks contained in it, namely

$$\lim_{L\to\infty} E[\phi_n] = 2n \cdot \frac{4}{3}\,. \qquad \text{(VIII.47)}$$

In paragraph VIII.2.1, we inserted the kink and antikink solution into the ansatz of the gauge potential and the gauge field strength and arrived at an instanton or anti-instanton configuration. Therefore, if we insert the sphaleron solution (VIII.42) into this ansatz (VIII.13) we obtain a chain of instanton-anti-instanton pairs on the space $S^1_{\mathrm{L}} \times G/H$ which corresponds to a field configuration with finite action. The gauge potential (VIII.13) and the gauge field strength (VIII.13) are given by

$$
\begin{aligned}
\mathcal{A} &= e^i E_i + \phi_n\, e^a E_a\,, \\
\mathcal{F} &= \dot{\phi}_n\, e^0 \wedge e^a\, E_a - \frac{1}{2}\left\{(1-\phi_n^2) f_{bc}{}^i E_i + (\phi_n - \phi_n^2) f_{bc}{}^a E_a\right\}\, e^b \wedge e^c\,.
\end{aligned}
\tag{VIII.48}
$$

Since integrating over the coset gives only a constant factor, these solutions yield a finite value for the action.

Chapter IX

Yang-Mills Configurations on Non-Symmetric Spaces

IX.1 BPS Kink Equation for Non-Vanishing Torsion

Let us now consider the more generic case where G/H is a homogeneous, but not symmetric space. This means that the structure constants ${f_{bc}}^a$ do not all vanish and hence a non-zero α may appear. So the geometric torsion can actually appear in the equations of motion. In the following we are going to choose certain values for the torsion and the parameter α and see how (VIII.19) and the corresponding solutions will change. Let us use the ansatz

$$\phi = \varphi + \beta \,, \tag{IX.1}$$

where β is constant, and insert it into (VIII.19) choosing the following relation between β and κ:

$$\beta = \frac{\kappa{+}3}{3}\,\frac{\alpha}{1{+}\alpha} \quad \text{and} \tag{IX.2}$$

$$\kappa = -3\ , \quad \kappa = \frac{3(1{-}\alpha)}{2\alpha} \quad \text{or} \quad \kappa = \frac{3}{\alpha}\ . \tag{IX.3}$$

This relation makes sure that the coefficient of the φ^2 term and the constant term in (VIII.19) vanish. Furthermore we perform the following rescaling

$$\tau \to \frac{\tau}{\sqrt{1+\alpha}} \qquad \Rightarrow \qquad \ddot{\phi} \to \frac{\ddot{\phi}}{1+\alpha}\,. \tag{IX.4}$$

Now we are able to take explicit values for (β, κ) in order to satisfy (IX.2). Taking the antiderivative of the corresponding $V'(\phi)$ we get

$$
\begin{aligned}
&\text{a)}\quad (\beta,\kappa)=(0,-3) && :\quad 2\ddot{\phi}=-\phi\,(1-\phi^2) \\
& && \Leftrightarrow\quad V=\frac{1}{4}\big(\phi^2-1\big)^2\,, && \text{(IX.5)}\\
&\text{b)}\quad (\beta,\kappa)=(\frac{1}{2},\frac{3(1-\alpha)}{2\alpha}) && :\quad 2\ddot{\phi}=-[\phi-\frac{1}{2}](\frac{1}{4}-[\phi-\frac{1}{2}]^2) \\
& && \Leftrightarrow\quad V=\frac{1}{4}\big([\phi-\frac{1}{2}]^2-\frac{1}{4}\big)^2\,, && \text{(IX.6)}\\
&\text{c)}\quad (\beta,\kappa)=(1,\frac{3}{\alpha}) && :\quad 2\ddot{\phi}=-[\phi-1](1-[\phi-1]^2) \\
& && \Leftrightarrow\quad V=\frac{1}{4}\big([\phi-1]^2-1\big)^2\,. && \text{(IX.7)}
\end{aligned}
$$

Similar to the symmetric case, we can again find the first order (BPS) equations and solutions for these three second order equations. Formally this is done by taking the square root of the potential. We get

$$
\begin{aligned}
&\text{a)}\quad 2\dot{\phi}=\pm(1-\phi^2) && \Leftrightarrow\quad \phi=\pm\tanh\frac{\tau}{2} && \text{(IX.8)}\\
&\text{b)}\quad 2\dot{\phi}=\pm(\frac{1}{4}-[\phi-\frac{1}{2}]^2) && \Leftrightarrow\quad \phi=\frac{1}{2}\pm\frac{1}{2}\tanh\frac{\tau}{4} && \text{(IX.9)}\\
&\text{c)}\quad 2\dot{\phi}=\pm(1-[\phi-1]^2) && \Leftrightarrow\quad \phi=1\pm\tanh\frac{\tau}{2}\,. && \text{(IX.10)}
\end{aligned}
$$

These three solutions can now be written in terms of a new parameter γ as

$$\phi=\beta\pm\gamma\tanh\frac{\gamma\tau}{2}\,, \tag{IX.11}$$

where we now recover the cases a), b), c) at the values $(\beta,\kappa)=(0,1)$, $(\frac{1}{2},\frac{1}{2})$, $(1,1)$. As in the symmetric case these solutions interpolate between two vacua, namely between $\phi(-\infty)=\beta\mp\gamma$ and $\phi(+\infty)=\beta\pm\gamma$, where the upper sign represents the kink and the lower sign the antikink solution. One can also see that $\phi(0)=\beta$ since $\tanh(0)=0$. We denote these three points by $\hat{\phi}$, and they are actually the critical points of the potential V since $V'(\phi)\mid_{-\infty,0,\infty}=0$.

Summarizing we have

case	β	γ	$\phi(-\infty)$	$\phi(0)$	$\phi(+\infty)$
a)	0	1	-1	0	1
b)	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1
c)	1	1	0	1	2

. (IX.12)

IX.2 Modification of Bundles

IX.2.1 Modifications of bundles over G/H. Since we have managed to solve the derived equations for ϕ, we get the corresponding solutions for the Yang-Mills equations on $\mathbb{R}\times G/H$

by inserting (IX.11) to the gauge potential (VIII.12) and the gauge field strength (VIII.13). For the kink type solution we get

$$\begin{aligned}\mathcal{A} &= e^i E_i + (\beta + \gamma \tanh \frac{\gamma\tau}{2})\, e^a E_a\,, \\ \mathcal{F} &= \frac{\gamma^2}{2} \cosh^{-2} \frac{\gamma\tau}{2}\, e^0 \wedge e^a\, E_a \;-\; \frac{1}{2}\big\{h(\tau)\, f^i_{bc} E_i + m(\tau)\, f^a_{bc} E_a\big\}\, e^b \wedge e^c\;,\end{aligned} \tag{IX.13}$$

with

$$\begin{aligned}h(\tau) &:= (1 - \beta - \gamma \tanh \frac{\gamma\tau}{2})\,(1 + \beta + \gamma \tanh \frac{\gamma\tau}{2})\;, \\ m(\tau) &:= (1 - \beta - \gamma \tanh \frac{\gamma\tau}{2})\,(\beta + \gamma \tanh \frac{\gamma\tau}{2})\;.\end{aligned} \tag{IX.14}$$

What we can see from this solution by considering the limit $\tau \to \pm\infty$ is that there always survive constant terms in $\mathcal{F}$ in (IX.13). Therefore the integral over τ in the Yang-Mills action functional is not finite,

$$S = -\frac{1}{4} \int_{\mathbb{R}\times G/H} \mathrm{tr}(\mathcal{F} \wedge *\mathcal{F}) = \infty\,. \tag{IX.15}$$

Obviously this does not happen, if we do not perform the τ-integration, but keep $\tau \in \mathbb{R} \cup \{\pm\infty\}$ fixed. Even though we allow τ to be infinite, the integral

$$S_{G/H} = -\frac{1}{4} \int_{G/H} \mathrm{tr}(\mathcal{F} \wedge *\mathcal{F}) \tag{IX.16}$$

is finite due to compactness of G/H.

The question of interest is, whether this gauge potential with some fixed τ is actually a solution of the Yang-Mills equations on G/H. The difference of these equations to the Yang-Mills equations on $\mathbb{R} \times G/H$ is just that the term $e_0(\mathcal{F}^{0b})$ appearing in (VIII.16) simply does not appear any more. So for an arbitrary choice of τ it cannot be true that we have a solution for the Yang-Mills equations on G/H. One can easily see that the second derivative $\ddot{\phi}$ vanishes for $\tau = 0$ and is asymptotically zero for $\tau = \pm\infty$. Therefore, also $\ddot{A}$ vanishes for these choices of τ and hence we obtain one solution of the Yang-Mills equations on G/H for each such τ in all cases a), b), and c). Let $\hat{\phi} := \phi(\tau_{\mathrm{crit}})$ where $\tau_{\mathrm{crit}} \in \{-\infty,\ 0,\ +\infty\}$ denotes the critical points of V, as above. Then we define

$$\begin{aligned}\mathcal{A}_{\mathrm{crit}} &:= e^i E_i + \hat{\phi}\, e^a E_a, \\ \mathcal{F}_{\mathrm{crit}} &:= -\frac{1}{2}\big\{(1-\hat{\phi})(1+\hat{\phi})\, f^i{}_{bc} E_i + (1-\hat{\phi})\,\hat{\phi}\, f a_{bc} E_a\big\}\, e^b \wedge e^c\,.\end{aligned} \tag{IX.17}$$

In order to introduce matter fields, we already mentioned in section (VII.2) that it is necessary to define an associated bundle with respect to the principal bundle $P(\mathbb{R} \times G/H), G$ over $\mathbb{R} \times G/H$. So let (ρ, V_G) be a representation of G where V_G is a finite-dimensional vector space. For the associated bundle we clearly have the base space $\mathbb{R} \times G/H$ and the total space given by the fibred product $P(\mathbb{R} \times G/H) \times_G V_G$ or if we use the notation of section VI.1, by $P(\mathbb{R} \times G/H) \times_\rho V_G$. In the following, the associate vector bundle is denoted by

$$\mathcal{E} = P(\mathbb{R} \times G/H, G) \times_G V_G \ \rightarrow \ \mathbb{R} \times G/H \ . \tag{IX.18}$$

So far we have defined the associate bundle over $\mathbb{R} \times G/H$, where we have a connection $\mathcal{A}$. But we can fix τ in order to obtain an associate bundle $\mathcal{E}_\tau : P(G/H, G) \rightarrow G/H$ with corresponding connection $\mathcal{A}(\tau)$. Certainly the topology depends on the connection and hence on the choice of τ. In the following table we list all possible values of the ϕ that we are interested in, namely critical points $\hat{\phi}$, the corresponding functions $\hat{h}$ and $\hat{m}$ and corresponding topological properties of the bundle.

value of $\hat{\phi}$	-1	0	$1/2$	1	2
bundle $\mathcal{E}_{\text{crit}}$	$\mathcal{E}_-$	$\mathcal{E}_{\text{can}}$	$\mathcal{E}_0$	$\mathcal{E}_{\text{flat}}$	$\mathcal{E}_+$
topology	irreducible	reducible	irreducible	trivial	irreducible
$\hat{h} = 1-\hat{\phi}^2$	0	1	$3/4$	0	-3
$\hat{m} = \hat{\phi}-\hat{\phi}^2$	-2	0	$1/4$	0	-2

(IX.19)

In these cases where we have an operation that allows to obtain new bundles over G/H out of given ones, we talk about *modifications of bundles*. These modifications in our case are given by the gauge potential (IX.13) and describe

$$\text{a)} \quad \mathcal{E}_- \ \rightarrow \ \mathcal{E}_{\text{can}} \ \rightarrow \ \mathcal{E}_{\text{flat}} \, , \tag{IX.20}$$

$$\text{b)} \quad \mathcal{E}_{\text{can}} \ \rightarrow \ \mathcal{E}_0 \rightarrow \ \mathcal{E}_{\text{flat}} \, , \tag{IX.21}$$

$$\text{c)} \quad \mathcal{E}_{\text{can}} \ \rightarrow \ \mathcal{E}_{\text{flat}} \ \rightarrow \ \mathcal{E}_+ \, . \tag{IX.22}$$

Of course the corresponding gauge field strength and hence the curvature of bundles over G/H are given in (IX.13) by inserting the corresponding values from (IX.19).

The case from section (VIII.2), where we chose α to be zero and hence considered a symmetric space, can actually be described in the same way as we described the non-symmetric counterparts. Of course, the corresponding β needs to vanish. We get the following set of parameters and label this case as d)

$$\text{case d)} \qquad \beta = 0 \ , \quad \gamma = 1 \ , \quad \phi(-\infty) = -1, \quad \phi(0) = 0, \quad \phi(+\infty) = 1 \, , \tag{IX.23}$$

which yields the following values of h and m in the gauge field strength

$$h(\tau) = \cosh^{-2}\frac{\tau}{2}, \quad m(\tau) \equiv 0 \, . \tag{IX.24}$$

As above, the corresponding gauge potential describes a modification of bundles:

$$\mathcal{E}_{\text{flat}} \ \rightarrow \ \mathcal{E}_{\text{can}} \ \rightarrow \ \mathcal{E}_{\text{flat}} \ . \tag{IX.25}$$

The constants β and γ as well as the critical points are the same as for cases a) and d). The only difference lies in the corresponding modification of bundles. In a) we had an interpolation between something irreducible, topologically non-trivial bundle over an irreducible bundle ending up with a trivial one. In d) we start with a trivial bundle, pass a reducible one and end up with a trivial bundle again. This is of course due to the fact that in d) $m(\tau) \equiv 0 \; \forall \tau \in \mathbb{R}$ and in a) we have a non-vanishing value for m at $\tau = -\infty$.

IX.2.2 Chains of bundle modifications. We have previously seen in paragraph VIII.2.3 that there were periodic solutions to the corresponding differential equations for the symmetric case (VIII.23), namely sphaleron solutions. The same is true for the corresponding equations on the nonsymmetric space treated in the last paragraph IX.2.1. Therefore, if we require our solutions to be periodic, we find the following non-BPS solutions for equations (IX.5)-(IX.7):

$$\begin{aligned} &\text{a)} \quad \phi(\tau;k) = \qquad 2k\, b(k)\, \mathrm{sn}[b(k)\tau;k] \\ &\text{b)} \quad \phi(\tau;k) = \frac{1}{2} + \quad k\, b(k)\, \mathrm{sn}[b(k)\frac{\tau}{2};k] \\ &\text{c)} \quad \phi(\tau;k) = 1 + \; 2k\, b(k)\, \mathrm{sn}[b(k)\tau;k] \end{aligned} \tag{IX.26}$$

where $b(k) = (2+2k^2)^{-1/2}$ and $0 \le k \le 1$ as before.

If we insert these solutions into the ansatz of the gauge potential, we obtain a chain of bundle modifications as follows:

$$\begin{array}{llllllllllll} \text{a)} & \cdots \to & \mathcal{E}_- & \to & \mathcal{E}_{\text{can}} & \to & \mathcal{E}_{\text{flat}} & \to & \mathcal{E}_{\text{can}} & \to & \mathcal{E}_- & \to \cdots, \\ \text{b)} & \cdots \to & \mathcal{E}_{\text{can}} & \to & \mathcal{E}_0 & \to & \mathcal{E}_{\text{flat}} & \to & \mathcal{E}_0 & \to & \mathcal{E}_{\text{can}} & \to \cdots, \\ \text{c)} & \cdots \to & \mathcal{E}_{\text{can}} & \to & \mathcal{E}_{\text{flat}} & \to & \mathcal{E}_+ & \to & \mathcal{E}_{\text{flat}} & \to & \mathcal{E}_{\text{can}} & \to \cdots. \end{array} \tag{IX.27}$$

IX.3 Dyons on $\mathbb{R} \times G/H$

So far we have always considered the metric on $\mathbb{R} \times G/H$ having Euclidean signature so that τ is considered to be some spatial variable. But this is not a necessary assumption and we can also consider the $\mathbb{R}$ part of our manifold to play the role of time. The difference is that our metric then has a Lorentzian signature. Formally, we can consider this to be the Wick rotation of $e^0 = \mathrm{d}\tau$

$$t := -\mathrm{i}\tau \qquad \Rightarrow \qquad \tilde{e}^0 := \mathrm{d}t = -\mathrm{i}\mathrm{d}\tau\,, \tag{IX.28}$$

and hence we get the following metric with Lorentzian signature

$$\mathrm{d}s^2 = -\left(\tilde{e}^0\right)^2 + \delta_{ab} e^a e^b\,. \tag{IX.29}$$

If we use the same G-equivariant ansatz (VIII.12) we arrive at the corresponding gauge field strength that looks exactly the same as before with the only difference that the e^0's are replaced by $\tilde{e}^0$'s:

$$\mathcal{F} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\,\tilde{e}^0\wedge e^a E_a - \frac{1}{2}\big\{(1-\phi^2)f^i{}_{bc}E_i + (\phi-\phi^2)f^a{}_{bc}E_a\big\}\,e^b\wedge e^c\ . \tag{IX.30}$$

By performing the Wick rotation (IX.28), we find the following transformation rule for the real scalar field ϕ:

$$\phi(\tau)\to\phi(\mathrm{i}t),\quad \frac{\mathrm{d}}{\mathrm{d}\tau}\phi\to -i\frac{\mathrm{d}}{\mathrm{d}t}\phi,\quad \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\phi\to -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi\,, \tag{IX.31}$$

and hence the potential V corresponding to the Lorentzian version of (VIII.19) transforms as

$$V\to -V\,. \tag{IX.32}$$

If we do the same rescaling of t as we did for τ in (IX.4), namely

$$t\to\frac{t}{\sqrt{1+\alpha}}\ \Rightarrow\ddot{\phi}\to\frac{\ddot{\phi}}{1+\alpha}\,, \tag{IX.33}$$

where the dots now represent t-derivatives, we end up with the following set of differential equations:

$$\tilde{\mathrm{a}})\qquad (\beta,\kappa)=(0,-3):\quad 2\ddot{\phi}=+\phi\,(1-\phi^2)\,, \tag{IX.34}$$

$$\tilde{\mathrm{b}})\qquad (\beta,\kappa)=(\tfrac{1}{2},\tfrac{3(1-\alpha)}{2\alpha}):\quad 2\ddot{\phi}=+[\phi-\tfrac{1}{2}](\tfrac{1}{4}-[\phi-\tfrac{1}{2}]^2)\,, \tag{IX.35}$$

$$\tilde{\mathrm{c}})\qquad (\beta,\kappa)=(1,\tfrac{3}{\alpha}):\quad 2\ddot{\phi}=+[\phi-1](1-[\phi-1]^2)\,. \tag{IX.36}$$

$$\tilde{\mathrm{d}})\qquad (\alpha,\beta,\kappa)=(0,0,\kappa):\quad 2\ddot{\phi}=+\phi\,(1-\phi^2)\,. \tag{IX.37}$$

The solutions of these equations can be summarized using the parameter γ to take the values 1, $\frac{1}{2}$, 1, 1 for ã), $\tilde{\mathrm{b}}$), c̃) and $\tilde{\mathrm{d}}$) respectively, which are the same as in the Euclidean case:

$$\phi(t)=\beta+\sqrt{2}\,\gamma\,\cosh^{-1}\frac{\gamma t}{\sqrt{2}}\,. \tag{IX.38}$$

This configuration is just the bounce in an inverted double-well potential $-V$.

We get the dyon-type configuration for the Yang-Mills theory, if we insert our solution for the scalar field into the ansatz for the gauge potential (VIII.12) and the gauge field strength (VIII.13). For $(\beta,\gamma)=(0,1)$, this is

$$\begin{aligned}\mathcal{A} &= e^iE_i+\sqrt{2}\cosh^{-1}\frac{t}{\sqrt{2}}\,e^aE_a\\ \mathcal{F} &= -\frac{1}{2}\cosh^{-2}\frac{t}{\sqrt{2}}\left(2\sinh\frac{t}{\sqrt{2}}\ \mathrm{d}t\wedge e^a\,E_a+(\sinh^2\frac{t}{\sqrt{2}}-1)\,f^i_{bc}E_i\,e^b\wedge e^c\right)\,.\end{aligned} \tag{IX.39}$$

This solution represents the symmetric case where the ${f_{bc}}^a$ vanish. The non-symmetric case includes another term containing the ${f_{bc}}^a$.

Clearly, the energy functional defined in (VIII.26) given by

$$E = -\frac{1}{4}\int_{G/H} (\mathrm{tr}(\mathcal{F} \wedge *\mathcal{F}))$$

is a function of t proportional to the volume of the coset space and hence finite for all $t \in \mathbb{R}$. It is given by

$$E = -\mathrm{Vol}(G/H) \cdot \mathrm{tr}(2\ \mathcal{F}_{0a}\mathcal{F}_{0a} + \mathcal{F}_{ab}\mathcal{F}_{ab})\,. \tag{IX.40}$$

Due to the asymptotical behavior of $\mathcal{F}$, the Yang-Mills action functional (VIII.26) becomes infinite in the symmetric as well as in the non-symmetric case.

Chapter X

Yang-Mills Flows on G/H

X.1 Reduction of the Yang-Mills Flow Equations

In this chapter we are going to find solutions to the Yang-Mills flow equations on a reductive coset space G/H and compare them to those solutions obtained in Chapter IX from the second order Yang-Mills equations.

The Yang-Mills flow equations are first order differential equations for the gauge potential. Since the gauge potential depends on a parameter they are differential equations with respect to this parameter and therefore represent a family of connections on the coset space. So generically we have the Yang-Mills flow

$$\mathcal{A}(\tau), \quad \tau \in \mathbb{R}, \tag{X.1}$$

satisfying the Yang-Mills flow equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{A}(\tau) = - * \mathcal{D}\left(*\mathcal{F}\right), \tag{X.2}$$

where $\mathcal{D}$ denotes the gauge covariant derivative on the coset space. This differs from the gauge covariant derivative on $\mathbb{R} \times G/H$ only by a derivative in $\mathbb{R}$-direction. We use the same G-equivariant ansatz for the gauge potential as before (VIII.12), namely

$$\mathcal{A} = e^i E_i + \phi\, e^a\, E_a$$

with corresponding gauge field strength

$$\mathcal{F} = -\frac{1}{2}\,\left\{(1-\phi^2) f_{bc}{}^i E_i + (\phi - \phi^2) f_{bc}{}^a E_a\right\}\, e^b \wedge e^c\,. \tag{X.3}$$

Since we are considering the Yang-Mills flow on G/H, there are no components of the gauge field strength pointing in an $\mathbb{R}$-direction as we had in (VIII.13). The Yang-Mills flow equations (X.2) can be written in terms of the left-invariant basis $\{e^a, a = 1, ..., \dim(G/H)\}$ of the coset space as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{A}_a = -\mathcal{D}_b\, \mathcal{F}_{ab}\,. \tag{X.4}$$

Subsituting the gauge field strength and the gauge potential into the Yang-Mills flow equations, we get the following first order differential equation for ϕ:

$$\begin{aligned} 2\,\dot\phi &= (1{+}\alpha)\,\phi^3 - \alpha(\kappa{+}3)\,\phi^2 - (1{-}\alpha(\kappa{+}2))\,\phi \\ &= -\phi\,(1-\phi^2) \;+\; \alpha\,\phi\,(1-\phi)(2-\phi) \;+\; \alpha\,\kappa\,\phi\,(1-\phi) \\ &= (1{+}\alpha)\,\phi\,(\phi-1)\,\big(\phi - \frac{(\kappa+2)\alpha-1}{\alpha+1}\big) = V'(\phi)\,. \end{aligned} \tag{X.5}$$

Here the parameter κ got involved via the covariant derivative on the coset space, as happend before. These equations actually only differ from those obtained by Yang-Mills equations in the order of the derivative. Before, we had a second order differential equation and now we have a first order differential equation. The stable points of the Yang-Mills flow are now given at those points, where the first derivative of ϕ vanishes. These points are the critical points of the potential V introduced in (X.5) as before. They are given by the following values $\hat\phi$ of the field ϕ:

$$\hat\phi \;=\; 0\,, \tag{X.6}$$

$$\hat\phi \;=\; 1\,, \tag{X.7}$$

$$\hat\phi \;=\; \frac{(\kappa+2)\alpha-1}{\alpha+1} \;=:\rho\,. \tag{X.8}$$

If we substitute these critical values into (X.3), we get the following corresponding critical gauge field strengths

$$\left.\begin{aligned} \mathcal{F}^{\text{crit}}_{bc} &= -f_{bc}{}^{i}\,E_i = \mathcal{F}_{\text{can}} \\ \mathcal{F}^{\text{crit}}_{bc} &= 0 = \mathcal{F}_{\text{flat}} \\ \mathcal{F}^{\text{crit}}_{bc} &= -(1-\rho^2) f_{bc}{}^{i} E_i + (\rho-\rho^2) f_{bc}{}^{a} E_a = \mathcal{F}^{\text{crit}}_{bc}(\alpha,\kappa)\, f_{bc}{}^{a}\, E_a \end{aligned}\right\}. \tag{X.9}$$

The third case provide a gauge field strength that depends on the values of the parameters α and κ. If we choose three values of κ from (IX.2) as in the symmetric case, we get the following critical gauge field strengths:

$$\begin{aligned} &\text{a)}\;\; \mathcal{F}^{\text{crit}}_{bc}(\alpha,-3) &&= \; -2 f_{ab}{}^{c} E_c &&= \mathcal{F}_- \\ &\text{b)}\;\; \mathcal{F}^{\text{crit}}_{bc}(\alpha,\frac{3(1-\alpha)}{2\alpha}) &&= \; -\frac{3}{4} f_{bc}{}^{i} E_i - \frac{1}{4} f_{bc}{}^{a} E_a &&= \mathcal{F}_0 \\ &\text{c)}\;\; \mathcal{F}^{\text{crit}}_{bc}(\alpha,\frac{3}{\alpha}) &&= \; \frac{3}{2} f_{bc}{}^{i} E_i + f_{bc}{}^{a} E_a &&= \mathcal{F}_+ \\ &\text{d)}\;\; \mathcal{F}^{\text{crit}}_{bc}(0,\kappa,f_{bc}{}^{a}=0) &&= \; 0 &&= \mathcal{F}_{\text{flat}} \end{aligned} \tag{X.10}$$

X.2 Explicit Solutions

The choice of parameters κ, α we made in a) – d) corresponds to the ρ-values of $-1,\ \frac{1}{2},\ 2,\ -1$ respectively. If we substitute these values into (X.5) and do a rescaling $\tau \to \tau(1+\alpha)^{-1}$, we arrive at the following equations and their solutions

$$\begin{aligned}
&\text{a)}\quad \rho = -1 &&\Rightarrow\quad 2\dot{\phi} = \phi\,(\phi-1)\,\big(\phi+1\big) \Rightarrow &&\phi = \pm(1+\exp\tau)^{-1/2}\\
&\text{b)}\quad \rho = \frac{1}{2} &&\Rightarrow\quad 2\dot{\phi} = \phi\,(\phi-1)\,\big(\phi-\frac{1}{2}\big) \Rightarrow &&\phi = \frac{1}{2} \pm \frac{1}{2}(1+\exp\frac{\tau}{4})^{-1/2}\\
&\text{c)}\quad \rho = 2 &&\Rightarrow\quad 2\dot{\phi} = \phi\,(\phi-1)\,\big(\phi-2\big) \Rightarrow &&\phi = 1 \pm (1+\exp\tau)^{-1/2}\\
&\text{d)}\quad \rho = -1 &&\Rightarrow\quad 2\dot{\phi} = \phi\,(\phi-1)\,\big(\phi+1\big) \Rightarrow &&\phi = \pm(1+\exp\tau)^{-1/2}\,.
\end{aligned} \tag{X.11}$$

We can now insert these solutions into our gauge potential and gauge field strength. Due to their asymptotical behavior, the corresponding Yang-Mills flows describe bundle modifications similar to those we found for the second order equations. The only difference is that these solutions here only give one half of the interpolation we found in (IX.20) and (IX.25). For the lower minus sign and the upper plus sign, we get

$$\begin{aligned}
&\text{a)}\quad \mathcal{E}_{-} \;\to\; \mathcal{E}_{\text{can}} &&\text{and}\qquad \mathcal{E}_{\text{flat}} \;\to\; \mathcal{E}_{\text{can}}\,,\\
&\text{b)}\quad \mathcal{E}_{\text{can}} \;\to\; \mathcal{E}_{0} &&\text{and}\qquad \mathcal{E}_{\text{flat}} \to\; \mathcal{E}_{0}\,,\\
&\text{c)}\quad \mathcal{E}_{\text{can}} \;\to\; \mathcal{E}_{\text{flat}} &&\text{and}\qquad \mathcal{E}_{+} \;\to\; \mathcal{E}_{\text{flat}}\,,\\
&\text{d)}\quad \mathcal{E}_{\text{flat}} \;\to\; \mathcal{E}_{\text{can}} &&\text{and}\qquad \mathcal{E}_{\text{flat}} \;\to\; \mathcal{E}_{\text{can}}\,,
\end{aligned} \tag{X.12}$$

respectively.

X.3 First Order Flow Equations

In this section we finally want to find a first order flow equation meaning an equation which is of lower order than equation (X.2). We have already done such a thing in Chapters VIII and IX, where we took the Yang-Mills equation and integrated those to BPS type equations that were first order differential equations in the scalar fields. Therefore, what we are going to do now is finding a BPS analog for the Yang-Mills flow equations which should be an algebraic equation in the field strength and hence a first order differential equations in the Higgs field ϕ. For the space $\mathbb{R} \times G/H$, where G/H is simply the three-sphere S^3 such equations referred to as Yang-Mills self-duality equations are already known and read

$$\dot{\mathcal{A}}_a = -\frac{1}{2}\,\epsilon_{abc}\,\mathcal{F}_{bc}\ , \tag{X.13}$$

where we sum over the coset indices. The ϵ here denotes the structure constants of $SU(2)$. Some more equations of such kind, namely equations which induce the full Yang-Mills equations were discovered for different spaces already. In [21] for instance such an equation was found for spaces of type $\mathbb{R} \times G$ where G is a semi-simple Lie group. Another generalization of the self duality equations on $\mathbb{R}^d$ that are analogue to the case of $\mathbb{R} \times G$ was done in [37] and [38]. There specifically in [38] also BPS-type equations where considered that do not necessarily solve the full Yang-Mills equation, denoted by B_n and C_n in that paper. Here we want to state a similar first order flow equation by

$$\dot{\mathcal{A}}_a = \mp\lambda\, f_{abc}\, \mathcal{F}_{bc}\,. \tag{X.14}$$

Here the ${f_{ab}}^c$ denote the structure constant that correspond the decomposition of the tangent space of G an in Chapter VIII. Obviously the stable points of the Yang-Mills flow, i. e. $\dot{\mathcal{A}} = 0$ are here given by the algebraic condition

$$f_{abc}\, \mathcal{F}_{bc} = 0\,, \tag{X.15}$$

which is obviously satisfied for the case that G/H is a symmetric coset space. If we insert the components of the G-equivariant ansatz (VIII.12) for the gauge potential as well as the corresponding components of the gauge field strength (VIII.13) to the first order flow equation (X.14) we arrive at the following first order differential equation

$$\dot{\phi} = \pm\lambda\, (\frac{1}{4} - [\phi{-}\frac{1}{2}]^2)\,. \tag{X.16}$$

This equation is precisely the equation we found as BPS type equation for a non-symmetric space with a specific non-vanishing value for the torsion (IX.9). Hence its solution also stated in (IX.9), which is a solution to the Yang-Mills equation for that specific case, is also a solution of the first order flow equation (X.14).

Chapter XI

Yang-Mills on $\mathbb{R} \times \mathbb{C}P^2$ in $C^{(2,0)}$ Quiver Representation

XI.1 Quivers and Higgs fields

In the last chapter we considered a lot of homogeneous spaces at the same time, since we had an ansatz (VIII.12) that was written in a nice generic way. So with a couple of assumptions (VIII.18), we managed to write down the full Yang-Mills equations without any other knowledge about the generators and therefore obtained fairly general results.

It is not so easy to turn to theories where a breakdown of the original gauge symmetry takes place and therefore more scalar fields get involved. These scalar fields are interpreted as Higgs fields that are responsible for the corresponding symmetry breaking. Specifically $SU(3)$-equivariant ansätze that yield the corresponding dimensionally reduced Yang-Mills theory, are equivalent to quiver gauge theories. Such ansätze may become more complicated and hence generic ways of writing them down, although they exist, are much harder to work with than (VIII.12). The explicit shape also depends on the chosen representation in the following way. One considers some highest weight representation $\underline{C}^{k,l}$ of $SU(3)$. The corresponding representation for the subspace H from our homogeneous space G/H is reducible, say

$$\underline{C}^{k,l} \mid_H = \sum_{i=1}^{m} \rho_i \,. \tag{XI.1}$$

The number of Higgs fields in our theory then depends on the so-called *quiver diagram*, containing as many vertices as irreps of H exist, and is determined by the number of maps between these irreps induced by the corresponding lowering operators of $SU(3)$. Therefore a quiver diagram is simply based on the weight diagram of the corresponding $SU(3)$ representation. Hence, the higher the quiver representation we choose, the more Higgs fields come into play.

For the product space $\mathbb{R} \times G/H$ one can generically write the corresponding associated vector bundle as

$$\mathcal{E} = \bigoplus_{i=1}^{m} E_i^{\mathbb{R}} \otimes \mathcal{V}_i \,, \tag{XI.2}$$

where $E_i^{\mathbb{R}}$ is a bundle over $\mathbb{R}$ and $\mathcal{V}_i$ a bundle over G/H having the same rank as the corresponding irrep ρ_i of H. The gauge group for such bundles is given by

$$\prod_{i=1}^{m} U(k_i) \times U(d_i) \,, \tag{XI.3}$$

where k_i corresponds to the rank of the bundle $E_i^{\mathbb{R}}$ over $M_D = \mathbb{R}$ and d_i to the rank of the bundle $\mathcal{V}_i$ over G/H. A G-equivariant gauge potential on (XI.2) is then admitted and given by a block-diagonal part and an off-diagonal one. The block diagonal part can be written as

$$\mathcal{A}^{\text{diag}} = \bigoplus_{i=1}^{m} \mathcal{A}^i \,, \tag{XI.4}$$

where the size of the blocks $\mathcal{A}^i$ depend on the dimensions of the H-irreps as well as of the rank of the bundle over $\mathbb{R}$. In our case we considered the bundle $E_i^{\mathbb{R}}$ over $\mathbb{R}$ to be of rank $k_i = 1$ and the connection A^i on $\mathbb{R}$ to be flat. Therefore, the part of the connection belonging to this bundle vanishes and we find

$$\begin{aligned} \mathcal{A}^i &= A^i \otimes \mathbb{1}_{d_i} + \mathbb{1}_{k_i} \otimes B^i \\ \Rightarrow \mathcal{A}^i &= 0 + 1 \cdot B^i \,, \end{aligned} \tag{XI.5}$$

with B^i denoting the connection on the coset part of the product space. Therefore the $\mathcal{A}^i$ are $d_i \times d_i$ matrices, where d_i is the dimension of the irrep ρ_i.

The off-diagonal part can be written by

$$\mathcal{A}^{\text{off } ij} = (1 - \delta_{ij})\ \Phi_{ij} \,, \tag{XI.6}$$

where (no summing over i, j) $\mathcal{A}^{\text{off } ij}$ is meant to be the (i, j)-th block of the gauge connection for $i \neq j$ and zero on the block diagonal part. The Φ_{ij} on the right hand side denotes the specific map that connects the ith and the jth vertex of the quiver and is the tensor product of maps between the corresponding bundles. This means

$$\Phi_{ij} = \phi_{ij} \otimes \beta_{ij} \,, \tag{XI.7}$$

where β_{ij} are maps connecting the two H-irreps ρ_i and ρ_j that contain the left-invariant basis on the coset space, and ϕ_{ij} are size $k_i \times k_j$ Higgs fields depending on the $\mathbb{R}$-part in $\mathbb{R} \times G/H$. For our consideration, these are just real valued scalar fields, one for each arrow of the quiver.

In order to catch up with the notation used in chapter VIII one can notice that we are considering a G-equivariant associated vector bundle over $\mathbb{R} \times G/H$ with respect to the corresponding representation of $SU(3)$ chosen in (XI.1) by

$$\mathcal{E}^{k,l} = \bigoplus_{i=1}^{m} P(\mathbb{R} \times G/H, U(d_i)) \times_{\rho_i} V^i \to \mathbb{R} \times G/H \,, \tag{XI.8}$$

where V^i is a finite dimensional representation space for the representations ρ_i of the subgroup H. Each term comes along with the structure group $U(d_i)$ and hence the overall structure group is given by

$$U\left(\sum_{i=1}^{m} d_i\right), \tag{XI.9}$$

obviously depending on the chosen representation of $SU(3)$.

The explicit construction of the quivers, their representation and the underlying $SU(3)$-equivariant gauge theories was done in [22]. It includes also the explicit formulae of the gauge potential and the field strength for spaces of the form

$$M_D \times G/H\,, \tag{XI.10}$$

where G/H is either $\mathbb{C}P^2 = \frac{SU(3)}{S(U(2)\times U(1))}$ or $Q_3 = \frac{SU(3)}{U(1)\times U(1)}$. We are going to use these results for the specific case $M_D = \mathbb{R}$ in order to derive the Yang-Mills equations for these quiver gauge theories.

XI.2 Invariant 1-forms on $\mathbb{C}P^2$

Firstly, we want to summarize all the ingredients from [22] that we will need to write down the G-equivariant ansatz. As we have seen in chapter VIII, it is quite convenient to do all the calculations in the invariant basis of the corresponding space because we can choose our metric to have constant coefficients in this basis. Hence the covariant derivative with respect to this metric is only depends on the structure constants of the space (II.72). $\mathbb{C}P^2$ is a complex manifold and therefore we can choose complex coordinates $(y_1, y_2, \bar{y}_1, \bar{y}_2)$ of $\mathbb{C}^2$. Using them one can write down the invariant 1-forms as in [22], namely

$$\bar{\beta} := \begin{pmatrix} \bar{\beta}^{\bar{1}} \\ \bar{\beta}^{\bar{2}} \end{pmatrix} \quad \text{with} \quad \bar{\beta}^{\bar{e}} = \frac{1}{\gamma}\,\mathrm{d}\bar{y}^{\bar{e}} - \frac{\bar{y}^{\bar{e}}}{\gamma^2(\gamma+1)}\,y^d\,\mathrm{d}\bar{y}^{\bar{d}}\,, \tag{XI.11}$$

$$\beta = \begin{pmatrix} \beta^{1} \\ \beta^{2} \end{pmatrix} \quad \text{with} \quad \beta^{e} = \frac{1}{\gamma}\,\mathrm{d}y^{e} - \frac{y^{e}}{\gamma^2(\gamma+1)}\,\bar{y}^{\bar{d}}\,\mathrm{d}y^{d}\,. \tag{XI.12}$$

Obviously, this basis contains two holomorphic and two anti-holomorphic one-forms. As usual, we denote the coset indices by early Latin letters, namely $a, b, c = 1, \bar{1}, 2, \bar{2}$, and the components with respect to the subgroup H by $i, j, k, l = 5, ..., 8$. Furthermore, for convenience we choose the indices d, e to take values only in $\{1, 2\}$. As we have seen in chapter IV, the hermitian metric with respect to this basis only has components with mixed holomorphic and anti-holomorphic indices (IV.17) and therefore we get (IV.18)

$$g^{G/H} = \delta_{\bar{d}e}\,\bar{\beta}^{\bar{d}} \otimes \beta^{e} + \delta_{d\bar{e}}\,\beta^{d} \otimes \bar{\beta}^{\bar{e}} \tag{XI.13}$$

Pulling down indices with this particular metric we obtain that indices get complex conjugated

$$T^a = T_{\bar{a}}\ \delta^{\bar{a}a}\,. \tag{XI.14}$$

Our metric on the product space becomes

$$g = \mathrm{d}\tau \otimes \mathrm{d}\tau + \delta_{\bar{d}e}\ \bar{\beta}^{\bar{d}} \otimes \beta^e + \delta_{d\bar{e}}\ \beta^d \otimes \bar{\beta}^{\bar{e}}\,, \tag{XI.15}$$

which allows us to pull down τ indices without changing the coefficients:

$$T^0 = T_0\ \delta^{00}\,,\ \text{with } \delta^{00} = 1\,. \tag{XI.16}$$

XI.3 The Symmetric $C^{(1,0)}$ Quiver Bundle

Next we want to state the simplest case of a quiver theory for $\mathbb{C}P^2$ in order to see how it works. We will not use this specific ansatz to derive Yang-Mills equations, since considerations with one scalar field were already done in chapter VIII and would lead to similar results here.

One can use the Yang tableaux in order to get the decomposition of the reducible representation of H with respect to the fundamental representation of $SU(3)$ into irreducible representations of H. We have

$$\underline{C}^{1,0}\big|_{SU(2)\times U(1)} = \underline{(1,1)}\ \oplus\ \underline{(0,-2)}\,, \tag{XI.17}$$

where we can already see that there may be only one arrow between the irreps of H and hence we are going to get one scalar field in our gauge potential. We can also see, using equation (XI.9), that the structure group in this example equals $U(3)$. The quiver diagram of this situation is given as

$$\mathbb{R}\otimes\mathcal{V}_{(0,-2)} \xrightarrow{\ \phi\otimes\beta\ } \mathbb{R}\otimes\mathcal{V}_{(1,1)} \tag{XI.18}$$

In the following we are going to write down the G-equivariant connection for the corresponding vector bundle explicitly which requires the explicit form of the generators. For the fundamental 3-dimensional representation of $SU(3)$ the generators corresponding to $G/H = \mathbb{C}P^2$ are given by

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\,, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\,, \tag{XI.19}$$

$$E_{\bar{1}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \qquad E_{\bar{2}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{XI.20}$$

And the generators of $H = S(U(2) \times U(1))$ are given by

$$E_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad E_6 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{XI.21}$$

$$E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad E_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} . \tag{XI.22}$$

If we use the notation of the matrix units for 3×3 matrices, defined by

$$(e_{ij})_{kl} = \delta_{ik}\delta_{jl} \, , \tag{XI.23}$$

these generators can be identified as

$$\begin{aligned} &E_1 = e_{31} \, , \quad E_2 = e_{32} \, , \quad E_{\bar{1}} = e_{13} \, , \quad E_{\bar{2}} = e_{23} \, , \\ &E_5 = e_{12} \, , \quad E_6 = e_{21} \, , \quad E_7 = e_{11} - e_{22} \, , \quad E_8 = e_{11} + e_{22} - 2e_{33} \, . \end{aligned} \tag{XI.24}$$

The Maurer-Cartan form in the fundamental 3-dimensional representation of $SU(3)$ is a flat connection on the trivial bundle $\mathbb{C}P^2 \times \mathbb{C}^3$ over $\mathbb{C}P^2$ and given by

$$A_0 = \begin{pmatrix} B & \bar{\beta} \\ -\beta^\top & -2a \end{pmatrix} \tag{XI.25}$$

where

$$B \;=\; \frac{1}{\gamma^2} \big(- \tfrac{1}{2} \, \mathrm{d}(Y^\dagger Y) \, \mathbb{1}_2 + \bar{Y} \, \mathrm{d}\bar{Y}^\dagger + \Lambda \, \mathrm{d}\Lambda \big) \, , \tag{XI.26}$$

$$a \;=\; -\frac{1}{4\gamma^2} \big(\bar{Y}^\dagger \, \mathrm{d}\bar{Y} - \mathrm{d}\bar{Y}^\dagger \, \bar{Y} \big) \, , \tag{XI.27}$$

along with

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} , \qquad \Lambda \;:=\; \gamma \, \mathbb{1}_2 - \frac{1}{\gamma+1} \, Y Y^\dagger \qquad \text{and} \qquad \gamma \;:=\; \sqrt{1 + Y^\dagger Y} \, . \tag{XI.28}$$

The Maurer-Cartan equation (III.4.1) is satisfied and reads

$$\mathrm{d}A_0 + A_0 \wedge A_0 = 0 \, , \tag{XI.29}$$

which yields

$$\mathrm{d}B + B \wedge B - \bar{\beta} \wedge \beta^\top \;=\; 0 \, , \tag{XI.30}$$

$$\mathrm{d}a - \frac{1}{2} \beta^\dagger \wedge \beta \;=\; 0 \, , \tag{XI.31}$$

$$\mathrm{d}\bar{\beta} + B \wedge \bar{\beta} - 2\bar{\beta} \wedge a \;=\; 0 \, , \tag{XI.32}$$

$$\mathrm{d}\beta^\top + \beta^\top \wedge B - 2a \wedge \beta^\top \;=\; 0 \, . \tag{XI.33}$$

Using this one can extend the flat connection on the trivial bundle to a connection on the bundle over $\mathbb{R}\times G/H$. It is given ((3.50) in [22]) by the following 3×3 matrix:

$$\mathcal{A} = \begin{pmatrix} B_{(1)} + a & \phi\bar{\beta} \\ -\phi\beta^\top & -2a \end{pmatrix} , \tag{XI.34}$$

where we defined the $su(2)$-valued one-instanton field $B_{(1)}$ on $\mathbb{C}P^2$ by the 2×2 matrix

$$B_{(1)} := B - a\mathbb{1}_2 =: \begin{pmatrix} B^{11} & B^{12} \\ -\overline{B^{12}} & -B^{11} \end{pmatrix} . \tag{XI.35}$$

The corresponding gauge field strength is easily calculated using (XI.30)-(XI.33) and takes the form

$$\mathcal{F} = \begin{pmatrix} (1-\phi^2)\ (\bar{\beta}\wedge\beta^\top) & \dot{\phi}\ \mathrm{d}\tau\wedge\bar{\beta} \\ -\dot{\phi}\ \mathrm{d}\tau\wedge\beta^\top & -(1-\phi^2)\ (\beta^\dagger\wedge\beta) \end{pmatrix} , \tag{XI.36}$$

with

$$\begin{pmatrix} \bar{\beta}^1\wedge\beta^1 & \bar{\beta}^1\wedge\beta^2 \\ \bar{\beta}^2\wedge\beta^1 & \bar{\beta}^2\wedge\beta^2 \end{pmatrix} = \bar{\beta}\wedge\beta^\top \tag{XI.37}$$

$$(\bar{\beta}^1\wedge\beta^1 + \bar{\beta}^2\wedge\beta^2) = \beta^\dagger\wedge\beta. \tag{XI.38}$$

Using the explicit form of the generators of the fundamental representation of $SU(3)$, we can write the Maurer-Cartan form as

$$\begin{aligned} A_0 &= -\beta^1\ E_1 - \beta^2\ E_2 + \bar{\beta}^{\bar{1}}\ E_{\bar{1}} + \bar{\beta}^{\bar{2}}\ E_{\bar{2}} \\ &\quad + B^{12}\ E_5 - \overline{B^{12}}\ E_6 + B^{11}\ E_7 + a\ E_8 \end{aligned} \tag{XI.39}$$

and (XI.34) is nothing but

$$\begin{aligned} \mathcal{A} &= \phi\left(-\beta^1 E_1 - \beta^2 E_2 + \bar{\beta}^{\bar{1}} E_{\bar{1}} + \bar{\beta}^{\bar{2}} E_{\bar{2}}\right) \\ &\quad + \overbrace{B^{12}E_5 - \overline{B^{12}}\ E_6 + B^{11}\ E_7 + a\ E_8}^{=:e^i{}_b E_i\ e^b} \end{aligned} \tag{XI.40}$$

where

$$\begin{aligned} &e^1 = \beta^1, \quad e^2 = \beta^2, \quad e^{\bar{1}} = \bar{\beta}^{\bar{1}}, \quad e^{\bar{2}} = \bar{\beta}^{\bar{2}}, \quad \text{and} \\ &e^5{}_b = B^{12}_b, \quad e^6{}_b = -\overline{B^{12}_b}, \quad e^7{}_b = B^{11}_b, \quad e^8{}_a = a_b\,. \end{aligned} \tag{XI.41}$$

We will need (XI.41) later on in order to differentiate the gauge field strength covariantly. From (XI.40) we can see that this ansatz would yield the same results we already deduced in chapter VIII. Getting more scalar fields involved requires higher representations.

XI.4 Symmetric $C^{(2,0)}$ Quiver Bundle

For $\mathbb{C}P^2$ we have seen so far how the quiver bundle looks like for the case of a fundamental representation of $SU(3)$. We now want to use the generalizations to the 6-dimensional quiver representation $\underline{C}^{(2,0)}$. An important point is that (XI.39) actually holds for arbitrary quiver representations by inserting the corresponding higher dimensional generators. Specifically for $\underline{C}^{(2,0)}$, we have the following generators:

$$\begin{aligned} E_1 &= \sqrt{2}\left(e_{41}+e_{64}\right)+e_{52}\,, & E_{\bar{1}} &= \sqrt{2}\left(e_{14}+e_{46}\right)+e_{25}\,,\\ E_2 &= e_{42}+\sqrt{2}\left(e_{53}+e_{65}\right), & E_{\bar{2}} &= e_{24}+\sqrt{2}\left(e_{35}+e_{56}\right),\\ E_5 &= \sqrt{2}\left(e_{12}+e_{23}\right)+e_{45}\,, & E_6 &= \sqrt{2}\left(e_{21}+e_{32}\right)+e_{54}\,,\\ E_7 &= 2(e_1-e_3)+e_4-e_5\,, & E_8 &= 2(e_1+e_2+e_3)-e_4-e_5-4e_6\,. \end{aligned} \tag{XI.42}$$

Using the formalism of Yang tableaux again, we find the following decomposition into irreducible subspaces:

$$\underline{C}^{2,0}\big|_{SU(2)\times U(1)} = \underline{(2,2)} \;\oplus\; \underline{(1,-1)} \;\oplus\; \underline{(0,-4)} \tag{XI.43}$$

The corresponding quiver diagram is then given by

$$\mathbb{R}\otimes\mathcal{V}_{(0,-4)} \xrightarrow{\;\phi_2\otimes\beta_2\;} \mathbb{R}\otimes\mathcal{V}_{(1,-1)} \xrightarrow{\;\phi_1\otimes\beta_1\;} \mathbb{R}\otimes\mathcal{V}_{(2,2)} \tag{XI.44}$$

As we can see one ends up with a G-equivariant connection containing two Higgs fields $\phi_1(\tau)$, $\phi_2(\tau)$ which is a connection on the corresponding associated bundle (XI.8) with structure group $U(6)$. This gauge potential is in general given in (3.108) from [22] and in our case simplifies to the 6×6 matrix

$$\mathcal{A} := \begin{pmatrix} B_{(2)}+2\;a\;\mathbb{1}_3 & \phi_1\bar{\beta}_1 & 0\\ -\phi_1\bar{\beta}_1^\dagger & B_{(1)}-a\;\mathbb{1}_2 & \phi_2\bar{\beta}_2\\ 0 & -\phi_2\bar{\beta}_2^\dagger & -4\;a \end{pmatrix} \tag{XI.45}$$

where the one-instanton connection $B_{(2)}$ in the 3-dimensional irreducible representation is defined as

$$B_{(2)} = \begin{pmatrix} 2B^{11} & \sqrt{2}\,B^{12} & 0\\ -\sqrt{2}\;\overline{B^{12}} & 0 & \sqrt{2}\,B^{12}\\ 0 & -\sqrt{2}\;\overline{B^{12}} & -2B^{11} \end{pmatrix}. \tag{XI.46}$$

The matrices $\bar{\beta}_1$ and $\bar{\beta}_2$ (not to be confused with the invariant one-forms $\bar{\beta}^{\bar{1}}$ and $\bar{\beta}^{\bar{1}}$) are given by

$$\bar{\beta}_1 = \begin{pmatrix} \sqrt{2}\,\bar{\beta}^{\bar{1}} & 0 \\ \bar{\beta}^{\bar{2}} & \bar{\beta}^{\bar{1}} \\ 0 & \sqrt{2}\,\bar{\beta}^{\bar{2}} \end{pmatrix} \qquad \text{and} \qquad \bar{\beta}_2 = \sqrt{2}\begin{pmatrix} \bar{\beta}^{\bar{1}} \\ \bar{\beta}^{\bar{2}} \end{pmatrix} . \tag{XI.47}$$

We also take the gauge field strength from [22], which is obtained from $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$ using the fact that the flat connection, namely the Maurer-Cartan form (XI.39), satisfies the Maurer-Cartan equations. One arrives at the following gauge field strength:

$$\mathcal{F} = \begin{pmatrix} (1-\phi_1^2)\,\bar{\beta}_1\wedge\bar{\beta}_1^\dagger & \dot{\phi}_1\,\mathrm{d}\tau\wedge\bar{\beta}_1 & 0 \\ -\dot{\phi}_1\,\mathrm{d}\tau\wedge\bar{\beta}_1^\dagger & \begin{matrix}(1-\phi_1^2)\,\bar{\beta}_1^\dagger\wedge\bar{\beta}_1 \\ (1-\phi_2^2)\,\bar{\beta}_2\wedge\bar{\beta}_2^\dagger\end{matrix} & \dot{\phi}_2\,\mathrm{d}\tau\wedge\bar{\beta}_2 \\ 0 & -\dot{\phi}_2\,\mathrm{d}\tau\wedge\bar{\beta}_2^\dagger & (1-\phi_2^2)\,\bar{\beta}_2^\dagger\wedge\bar{\beta}_2 \end{pmatrix} \tag{XI.48}$$

where

$$\bar{\beta}_1\wedge\bar{\beta}_1^\dagger = \begin{pmatrix} 2\,\bar{\beta}^1\wedge\beta^1 & \sqrt{2}\,\bar{\beta}^1\wedge\beta^2 & 0 \\ \sqrt{2}\,\bar{\beta}^2\wedge\beta^1 & \bar{\beta}^1\wedge\beta^1+\bar{\beta}^2\wedge\beta^2 & \sqrt{2}\,\bar{\beta}^1\wedge\beta^2 \\ 0 & \sqrt{2}\,\bar{\beta}^2\wedge\beta^1 & 2\,\bar{\beta}^2\wedge\beta^2 \end{pmatrix}, \tag{XI.49}$$

$$\bar{\beta}_1^\dagger\wedge\bar{\beta}_1 = -\begin{pmatrix} 2\,\bar{\beta}^1\wedge\beta^1+\bar{\beta}^2\wedge\beta^2 & \bar{\beta}^1\wedge\beta^2 \\ \bar{\beta}^2\wedge\beta^1 & \bar{\beta}^1\wedge\beta^1+2\,\bar{\beta}^2\wedge\beta^2 \end{pmatrix}, \tag{XI.50}$$

$$\bar{\beta}_2\wedge\bar{\beta}_2^\dagger = 2\begin{pmatrix} \bar{\beta}^1\wedge\beta^1 & \bar{\beta}^1\wedge\beta^2 \\ \bar{\beta}^2\wedge\beta^1 & \bar{\beta}^2\wedge\beta^2 \end{pmatrix}, \tag{XI.51}$$

$$\bar{\beta}_2^\dagger\wedge\bar{\beta}_2 = -2\left(\bar{\beta}^1\wedge\beta^1+\bar{\beta}^2\wedge\beta^2\right) . \tag{XI.52}$$

XI.5 Yang-Mills Equations

Now we collected all the information needed to write down the Yang-Mills equations and are prepared to derive the corresponding differential equations for the scalar fields ϕ_1 and ϕ_2. Due to (II.72) and hence (V.26), we find the following connection 1-form for $\mathbb{C}P^2$ with respect to the invariant metric (XI.13):

$$\omega^c{}_b = f_{ib}{}^c e^i = -f_{bi}{}^c e^i, \quad e^i = e^i{}_a e^a \tag{XI.53}$$

with $e^i{}_a$ from (XI.41). We also used the fact that $\mathbb{C}P^2$ is a symmetric space and hence

$$f_{ab}{}^c = 0 \quad \forall a,b,c\in\{1,2,\bar{1},\bar{2}\} , \tag{XI.54}$$

which can easily be seen by computing the commutation relation of the generators (XI.42) explicitly. We find the following non-vanishing structure constants ${f_{bi}}^c$ of $\mathbb{C}P^2$:

$$\begin{array}{llll} {f_{15}}^2 = 1 & {f_{\bar{1}6}}^{\bar{2}} = -1 & {f_{26}}^1 = 1 & {f_{\bar{2}5}}^{\bar{1}} = -1 \\ {f_{17}}^1 = 1 & {f_{\bar{1}7}}^{\bar{1}} = -1 & {f_{27}}^2 = -1 & {f_{\bar{2}7}}^{\bar{2}} = 1 \\ {f_{18}}^1 = 3 & {f_{\bar{1}8}}^{\bar{1}} = -3 & {f_{28}}^2 = 3 & {f_{\bar{2}8}}^{\bar{2}} = -3\,. \end{array} \tag{XI.55}$$

Again since we use the direct product metric on $\mathbb{R} \times \mathbb{C}P^2$, we have

$$\omega^0_{0b} = \omega^a_{0b} = \omega^0_{cb} = 0\,, \tag{XI.56}$$

and the non-vanishing components are

$$\omega^c_{ab} = -{f_{bi}}^c\, {e^i}_a\, e^a\,. \tag{XI.57}$$

So the Yang-Mills equations read

$$\mathcal{D}_a \mathcal{F}^{a0} = 0\,, \tag{XI.58}$$

$$\mathcal{D}_0 \mathcal{F}^{0b} + \mathcal{D}_a \mathcal{F}^{ab} = 0\,, \tag{XI.59}$$

where $\mathcal{D}_0 := \frac{\mathrm{d}}{\mathrm{d}t}$ and $\mathcal{D}_a \mathcal{F}^{ab} := \mathrm{e}_a(\mathcal{F}^{ab}) + \omega^a_{ac}\mathcal{F}^{cb} + \omega^b_{ac}\mathcal{F}^{ac}$, cf. (VIII.15).

In order to simplify these equations, we use the splitting of the gauge potential in its block-diagonal and off-diagonal parts:

$$\mathcal{A} = \mathcal{A}^{\mathrm{diag}} + \mathcal{A}^{\mathrm{off}}\,. \tag{XI.60}$$

Inserting this splitting of the gauge potential as well as (XI.57) into (XI.59), we get

$$0 = \overbrace{-{e^i}_a \left({f_{ci}}^a\ \mathcal{F}^{c0}\right) + \left[\mathcal{A}^{\mathrm{diag}}_a, \mathcal{F}^{a0}\right]}^{=\ 0,\ \text{trivially}} + \left[\mathcal{A}^{\mathrm{off}}_a, \mathcal{F}^{a0}\right], \tag{XI.61}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{F}^{0b} - {e^i}_a \left({f_{ci}}^a\ \mathcal{F}^{cb} + {f_{ci}}^b\ \mathcal{F}^{ac}\right) + \left[\mathcal{A}^{\mathrm{diag}}_a, \mathcal{F}^{ab}\right] + \left[\mathcal{A}^{\mathrm{off}}_a, \mathcal{F}^{ab}\right]. \tag{XI.62}$$

We find that equation (XI.61) is trivially satisfied and therefore yields no restrictions to the fields. From equation (XI.62) we get

$${e^i}_a \left({f_{ci}}^a\ \mathcal{F}^{cb} + {f_{ci}}^b\ \mathcal{F}^{ac}\right) = \left[\mathcal{A}^{\mathrm{diag}}_a, \mathcal{F}^{ab}\right], \tag{XI.63}$$

and therefore (XI.62) becomes

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{F}^{0b} + \left[\mathcal{A}^{\mathrm{off}}_a, \mathcal{F}^{ab}\right], \tag{XI.64}$$

which for every index b becomes a matrix equation containing two independent differential equations:

$$\begin{aligned} 1) \quad \ddot{\phi}_1 &= -\phi_1 \, \left(3 - (5\phi_1^2 - 2\phi_2^2)\right) && \text{(XI.65)} \\ 2) \quad \ddot{\phi}_2 &= -\phi_2 \, \left(3 - (6\phi_2^2 - 3\phi_1^2)\right) && \text{(XI.66)} \end{aligned}$$

Even though it is not so easy to find a solution to this equation, we can at least recognize that for $\phi_1 = \phi_2 = \phi$ we obtain only one differential equation, similar to (VIII.23), namely

$$\ddot{\phi} = -3\phi \, \left(1 - \phi^2\right) \,, \tag{XI.67}$$

which is solved for instance by

$$\phi(\tau) = \tanh\left(\frac{\sqrt{6}}{2}\tau\right) . \tag{XI.68}$$

In case that we put one of the ϕ_i to zero we get either

$$\ddot{\phi}_1 \; = \; -\phi_1 \, \left(3 - 5\phi_1^2\right) \,, \qquad \phi_2 = 0 \tag{XI.69}$$

or

$$\phi_1 = 0 \,, \qquad \ddot{\phi}_2 = -\phi_2 \, \left(3 - 6\phi_2^2\right) \, . \tag{XI.70}$$

These two equations are also solved by the Jacobi elliptic functions, for instance

$$\phi_1(\tau) = \sqrt{\frac{3}{5}} \tanh\left(\frac{\sqrt{6}}{2}\tau\right), \qquad \phi_2 = 0 \tag{XI.71}$$

and

$$\phi_1 = 0 \,, \qquad \phi_2\,(\tau) = \sqrt{1 - {C_1}^2} \ \mathrm{sn}\left(\sqrt{3}\ C_1\ \tau + C_2, \sqrt{\frac{1 - {C_1}^2}{{C_1}^2}}\right) \,, \tag{XI.72}$$

respectively where C_1 and C_2 are constants. In the first case we chose the constants such that one obtains a hyperbolic tangens. Since we need the second argument of the Jacobi elliptic function $\mathrm{sn}\,(\cdot, \cdot)$ to be 1 in order to obtain such a hyperbolic tangens, this cannot be achieved for the solution ϕ_2 in the case where ϕ_1 vanishes.

Chapter XII

Yang-Mills on $\mathbb{R} \times \mathbb{C}P^2$ in $C^{(1,1)}$ Quiver Representation

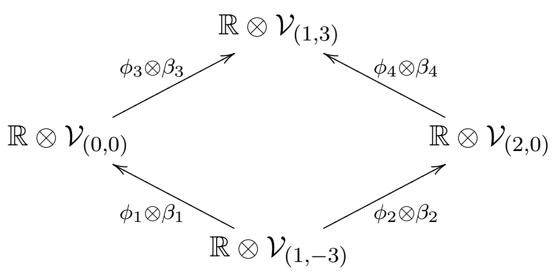
In the last chapter we introduced the notion of a quiver bundle and wrote down the gauge potential and gauge field strength as well as the equations of motion coming from Yang-Mills equations for the $\underline{C}^{2,0}$ representation. The next step here would be to consider a representation that decomposes into more irreps in order to obtain more than two scalar fields. So in the following we are going to quote the gauge connection and gauge field strength again from [22] and use these to derive the corresponding Yang-Mills equations and hence the equations of motion for the Higgs-fields.

XII.1 The Symmetric $C^{(1,1)}$ Quiver Bundle

We are choosing the $\underline{C}^{1,1}$ highest weight representation of $SU(3)$ where we obtain the following decomposition of the subgroup H in this representation:

$$\underline{C}^{1,1}\big|_{SU(2)\times U(1)} = \underline{(1,-3)} \;\oplus\; \underline{(2,0)} \;\oplus\; \underline{(0,0)} \;\oplus\; \underline{(1,3)}\,, \tag{XII.1}$$

We have the following quiver diagramm:



(XII.2)

From this one can already see that there will appear four independent scalar fields in the gauge connection. The generators of this eight-dimensional representation can be written in terms of the matrix units again as

$$E_1 = e_{12} + \sqrt{2}\;(e_{45} + e_{56}) + e_{78}\,, \tag{XII.3}$$

$$E_{\bar{1}} = e_{21} + \sqrt{2}\;(e_{54} + e_{65}) + e_{87}\,, \tag{XII.4}$$

$$E_2 = e_{14} + \sqrt{\frac{3}{2}}\ (e_{23}+e_{37}) + \sqrt{\frac{1}{2}}(e_{25}+e_{57}) + e_{68}\,, \tag{XII.5}$$

$$E_{\bar 2} = e_{41} + \sqrt{\frac{2}{3}}\ (e_{32}+e_{73}) + \sqrt{\frac{1}{2}}(e_{52}+e_{75}) + e_{86}\,, \tag{XII.6}$$

$$E_5 = \sqrt{\frac{3}{2}}(e_{13}-e_{38}) - \sqrt{\frac{3}{2}}(e_{15}-e_{58}) + (e_{47}-e_{26})\,, \tag{XII.7}$$

$$E_6 = \sqrt{\frac{2}{3}}(e_{31}-e_{83}) - \sqrt{\frac{2}{3}}(e_{51}-e_{85}) + (e_{74}-e_{62})\,, \tag{XII.8}$$

$$E_7 = -(e_{11}-e_{22}) + 2(e_{44}-e_{66}) + (e_{77}-e_{88})\,, \tag{XII.9}$$

$$E_8 = 3(e_{11}+e_{22}-e_{77}-e_{88}\,, \tag{XII.10}$$

where as before the generators with subscripts $\{1,2,\bar 1,\bar 2\}$ correspond to the coset space and the others with subscripts $\{5,6,7,8\}$ denote the generators of the subgroup H.

We also see from the decomposition of the subgroup H that the associated vector bundle given in (XI.9) comes with the structure group $U(8)$ in this case. The corresponding G-equivariant connection is then given (equation (3.125) in [22]) by

$$\mathcal{A} = \begin{pmatrix} B_{(1)} + 3a\ \mathbb{1}_2 & \phi_3\ \bar\beta_3 & \phi_4\ \bar\beta_4 & 0 \\ -\phi_3\ \bar\beta_3^\dagger & 0 & 0 & \phi_1\ \bar\beta_1 \\ -\phi_4\ \bar\beta_4^\dagger & 0 & B_{(2)} & \phi_2\ \bar\beta_2 \\ 0 & -\phi_1\ \bar\beta_1^\dagger & -\phi_2\ \bar\beta_2^\dagger & B_{(1)} - 3a\ \mathbb{1}_2 \end{pmatrix}, \tag{XII.11}$$

with $B_{(1)}$, $B_{(2)}$ from (XI.35), (XI.46) and

$$\bar\beta_3 = \sqrt{\frac{3}{2}}\ \begin{pmatrix}\bar\beta^{\bar 1}\\ \bar\beta^{\bar 2}\end{pmatrix}, \qquad \bar\beta_4 = \begin{pmatrix}\bar\beta^{\bar 2} & -\sqrt{\tfrac{1}{2}}\ \bar\beta^{\bar 1} & 0 \\ 0 & \sqrt{\tfrac{1}{2}}\ \bar\beta^{\bar 2} & -\bar\beta^{\bar 1}\end{pmatrix}\ ,$$
$$\bar\beta_1 = \sqrt{\frac{3}{2}}\ (\bar\beta^{\bar 2}\,,\ -\bar\beta^{\bar 1}), \qquad \bar\beta_2 = \begin{pmatrix}\bar\beta^{\bar 1} & 0 \\ \sqrt{\tfrac{1}{2}}\ \bar\beta^{\bar 2} & \sqrt{\tfrac{1}{2}}\ \bar\beta^{\bar 1} \\ 0 & \bar\beta^{\bar 2}\end{pmatrix}\ . \tag{XII.12}$$

We have the following field strength

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$$
$$\begin{pmatrix} \begin{matrix}(1-\phi_3^2)\ \bar\beta_3\wedge\bar\beta_3{}^\dagger \\ +(1-\phi_4^2)\ \bar\beta_4\wedge\bar\beta_4{}^\dagger\end{matrix} & \mathrm{d}\phi_3\wedge\bar\beta_3 & \mathrm{d}\phi_4\wedge\bar\beta_4 & (\phi_3\phi_1-\phi_4\phi_2)\ \bar\beta_3\wedge\bar\beta_1 \\ -\mathrm{d}\phi_3\wedge\bar\beta_3{}^\dagger & (\phi_3^3-\phi_1^3)\ \bar\beta_1\wedge\bar\beta_1^\dagger & (\phi_3\phi_4-\phi_1\phi_2)\ \bar\beta_1\wedge\bar\beta_2{}^\dagger & \mathrm{d}\phi_1\wedge\bar\beta_1 \\ -\mathrm{d}\phi_4\wedge\bar\beta_4{}^\dagger & (\phi_1\phi_2-\phi_3\phi_4)\ \bar\beta_1{}^\dagger\wedge\bar\beta_2 & \begin{matrix}(1-\phi_4^2)\ \bar\beta_4{}^\dagger\wedge\bar\beta_4 \\ +(1-\phi_2^2)\ \bar\beta_2\wedge\bar\beta_2{}^\dagger\end{matrix} & \mathrm{d}\phi_2\wedge\bar\beta_2 \\ (\phi_4\phi_2-\phi_3\phi_1)\ \bar\beta_3{}^\dagger\wedge\bar\beta_1{}^\dagger & -\mathrm{d}\phi_1\wedge\bar\beta_1{}^\dagger & -\mathrm{d}\phi_2\wedge\bar\beta_2{}^\dagger & \begin{matrix}(1-\phi_1^2)\ \bar\beta_1^\dagger\wedge\bar\beta_1 \\ +(1-\phi_2^2)\ \bar\beta_2^\dagger\wedge\bar\beta_2\end{matrix} \end{pmatrix} \tag{XII.13}$$

The wedge products of the different β_i matrices from (XII.12) are given by

$$\bar{\beta}_3 \wedge \bar{\beta}_3{}^\dagger = \frac{3}{2} \begin{pmatrix} \bar{\beta}^{\bar{1}} \wedge \beta^1 & \bar{\beta}^{\bar{1}} \wedge \beta^2 \\ \bar{\beta}^{\bar{2}} \wedge \beta^1 & \bar{\beta}^{\bar{2}} \wedge \beta^2 \end{pmatrix} , \tag{XII.14}$$

$$\bar{\beta}_4 \wedge \bar{\beta}_4{}^\dagger = \begin{pmatrix} \frac{1}{2}\,\bar{\beta}^{\bar{1}} \wedge \beta^1 + \bar{\beta}^{\bar{2}} \wedge \beta^2 & -\frac{1}{2}\,\bar{\beta}^{\bar{1}} \wedge \beta^2 \\ -\frac{1}{2}\,\bar{\beta}^{\bar{2}} \wedge \beta^1 & \bar{\beta}^{\bar{1}} \wedge \beta^1 + \frac{1}{2}\,\bar{\beta}^{\bar{2}} \wedge \beta^2 \end{pmatrix} , \tag{XII.15}$$

$$\bar{\beta}_4{}^\dagger \wedge \bar{\beta}_4 = \begin{pmatrix} \beta^2 \wedge \bar{\beta}^{\bar{2}} & -\sqrt{\frac{1}{2}}\,\beta^2 \wedge \bar{\beta}^{\bar{1}} & 0 \\ -\sqrt{\frac{1}{2}}\,\beta^1 \wedge \bar{\beta}^{\bar{2}} & \frac{1}{2}\left(\beta^1 \wedge \bar{\beta}^{\bar{1}} + \beta^2 \wedge \bar{\beta}^{\bar{2}}\right) & -\sqrt{\frac{1}{2}}\,\beta^2 \wedge \bar{\beta}^{\bar{1}} \\ 0 & -\sqrt{\frac{1}{2}}\,\beta^1 \wedge \bar{\beta}^{\bar{2}} & \beta^1 \wedge \bar{\beta}^{\bar{1}} \end{pmatrix} , \tag{XII.16}$$

$$\bar{\beta}_1 \wedge \bar{\beta}_1{}^\dagger = \tfrac{3}{2}\left(\bar{\beta}^{\bar{1}} \wedge \beta^1 + \bar{\beta}^{\bar{2}} \wedge \beta^2\right) , \tag{XII.17}$$

$$\bar{\beta}_1{}^\dagger \wedge \bar{\beta}_1 = \frac{3}{2} \begin{pmatrix} \beta^2 \wedge \bar{\beta}^{\bar{2}} & -\beta^2 \wedge \bar{\beta}^{\bar{1}} \\ -\beta^1 \wedge \bar{\beta}^{\bar{2}} & \beta^1 \wedge \bar{\beta}^{\bar{1}} \end{pmatrix} , \tag{XII.18}$$

$$\bar{\beta}_2 \wedge \bar{\beta}_2{}^\dagger = \begin{pmatrix} \bar{\beta}^{\bar{1}} \wedge \beta^1 & \sqrt{\frac{1}{2}}\,\bar{\beta}^{\bar{1}} \wedge \beta^2 & 0 \\ \sqrt{\frac{1}{2}}\,\bar{\beta}^{\bar{2}} \wedge \beta^1 & \frac{1}{2}\left(\bar{\beta}^{\bar{1}} \wedge \beta^1 + \bar{\beta}^{\bar{2}} \wedge \beta^2\right) & \sqrt{\frac{1}{2}}\,\bar{\beta}^{\bar{1}} \wedge \beta^2 \\ 0 & \sqrt{\frac{1}{2}}\,\bar{\beta}^{\bar{2}} \wedge \beta^1 & \bar{\beta}^{\bar{2}} \wedge \beta^2 \end{pmatrix} , \tag{XII.19}$$

$$\bar{\beta}_2{}^\dagger \wedge \bar{\beta}_2 = \begin{pmatrix} \beta^1 \wedge \bar{\beta}^{\bar{1}} + \frac{1}{2}\,\beta^2 \wedge \bar{\beta}^{\bar{2}} & \frac{1}{2}\,\beta^2 \wedge \bar{\beta}^{\bar{1}} \\ \frac{1}{2}\,\beta^1 \wedge \bar{\beta}^{\bar{2}} & \frac{1}{2}\,\beta^1 \wedge \bar{\beta}^{\bar{1}} + \beta^2 \wedge \bar{\beta}^{\bar{2}} \end{pmatrix} , \tag{XII.20}$$

$$\bar{\beta}_3 \wedge \bar{\beta}_1 = \tfrac{3}{2}\,\bar{\beta}^{\bar{1}} \wedge \bar{\beta}^{\bar{2}}\ \mathbb{1}_2 , \tag{XII.21}$$

$$\bar{\beta}_1 \wedge \bar{\beta}_2{}^\dagger = \sqrt{\tfrac{3}{2}}\,\left(\bar{\beta}^{\bar{2}} \wedge \beta^1\ ,\ -\sqrt{\tfrac{1}{2}}\,\left(\bar{\beta}^{\bar{1}} \wedge \beta^1 + \bar{\beta}^{\bar{2}} \wedge \beta^2\right)\ ,\ -\bar{\beta}^{\bar{1}} \wedge \beta^2\right) . \tag{XII.22}$$

XII.2 Yang-Mills Equations

In the last section we have collected all the necessary ingredients from [22] in order to derive the equation of motion for the Higgs fields from the Yang-Mills equations for the gauge potential and gauge field strength (XII.11) and (XII.13).

As in the previous chapters, the connection 1-form on the tangent bundle is given by the following formula:

$$\omega^c{}_b = f_{ib}{}^c e^i = -f_{bi}{}^c e^i, \quad e^i = e^i{}_a e^a \tag{XII.23}$$

with $e^i{}_a$ from (XI.41). We still have

$$f_{ab}{}^c = 0 \quad \forall a,b,c \in \{1,2,\bar{1},\bar{2}\}\ . \tag{XII.24}$$

For the calculations we are going to use the structure constants given in (XI.55).

The form of the Yang-Mills equations does not change and is given by

$$\begin{aligned} \mathcal{D}_a\mathcal{F}^{a0} &= 0\,, & \text{(XII.25)}\\ \mathcal{D}_0\mathcal{F}^{0b} + \mathcal{D}_a\mathcal{F}^{ab} &= 0\,, & \text{(XII.26)} \end{aligned}$$

with the same notation as in section XI.5.

Again, we split the gauge potential into its block-diagonal and off-diagonal part as

$$\mathcal{A} =: \mathcal{A}^{\text{diag}} + \mathcal{A}^{\text{off}} \tag{XII.27}$$

If we insert the gauge potential and (XII.23) into (XII.25), we recognize again that the left hand side vanishes and does not restrict our scalar fields.

The other set of equations (XII.26) with a free coset superscript becomes

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{F}^{0b} - e^i{}_a\left(f_{ci}{}^a\ \mathcal{F}^{cb} + f_{ci}{}^b\ \mathcal{F}^{ac}\right) + \left[\mathcal{A}_a^{\text{diag}}, \mathcal{F}^{ab}\right] + \left[\mathcal{A}_a^{\text{off}}, \mathcal{F}^{ab}\right]\ . \tag{XII.28}$$

By inserting (XII.11), (XII.13) as well as (XI.55) and (XI.41) into (XII.28) and after a fair amount of calculations, we find that

$$0 = e^i{}_a\left(f_{ci}{}^a\ \mathcal{F}^{cb} + f_{ci}{}^b\ \mathcal{F}^{ac}\right) - \left[\mathcal{A}_a^{\text{diag}}, \mathcal{F}^{ab}\right]\ , \tag{XII.29}$$

which was a trivial condition for the case of two scalar fields (XI.63), is not trivial here, but restricts our fields by the algebraic constraint

$$\phi_1\ \phi_2 = \phi_3\ \phi_4\,. \tag{XII.30}$$

The remaining part of (XII.28), yields

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{F}^{0b} + \left[\mathcal{A}_a^{\text{off}}, \mathcal{F}^{ab}\right]\ , \tag{XII.31}$$

which for every index b becomes a matrix equation containing six differential equations for the four scalar fields. For $\beta = 1,\ \bar{1}$, four out of six equations turn out to be independent:

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\ \phi_1 &= 3\ \phi_1\left(\frac{3}{2}\ \phi_1^2 - 1 + \frac{1}{2}\ \phi_2^2\right) - 3\ \phi_2\ \phi_3\ \phi_4\\ \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\ \phi_2 &= 3\ \phi_2\left(\frac{5}{6}\ \phi_2^2 - 1 + \frac{2}{3}\ \phi_4^2 + \frac{1}{2}\ \phi_1^2\right) - 3\ \phi_1\ \phi_3\ \phi_4\\ \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\ \phi_3 &= 3\ \phi_3\left(\frac{3}{2}\ \phi_3^2 - 1 + \frac{1}{2}\phi_4^2\right) - 3\ \phi_1\ \phi_2\ \phi_4\\ \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\ \phi_4 &= 3\ \phi_4\left(\frac{5}{6}\phi_4^2 - 1 + \frac{2}{3}\ \phi_2^2 + \frac{1}{2}\phi_3^2\right) - 3\ \phi_1\ \phi_2\ \phi_3\,. \end{aligned} \tag{XII.32}$$

With free coset index 2 or $\bar{2}$, we get the following six independent equations

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\phi_1 &= 3\,\phi_1\left(\frac{3}{2}\,\phi_1^2 - 1 + \frac{1}{6}\,\phi_2^2\right) - 2\,\phi_2\,\phi_3\,\phi_4 \\
\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\phi_2 &= 3\,\phi_2\left(\frac{5}{6}\,\phi_2^2 - 1 - \frac{1}{2}\,\phi_1^2 + \frac{2}{3}\,\phi_4^2\right) \\
\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\phi_2 &= 3\,\phi_2\left(\frac{5}{6}\,\phi_2^2 - 1 + \frac{1}{2}\phi_1^2 + \frac{2}{3}\phi_4^2\right) - 3\,\phi_1\,\phi_3\,\phi_4 \\
\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\phi_3 &= 3\,\phi_3\left(\frac{3}{2}\,\phi_3^2 - 1 + \frac{1}{6}\,\phi_4^2\right) - 2\,\phi_1\,\phi_2\,\phi_4 \\
\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\phi_4 &= 3\,\phi_4\left(\frac{5}{6}\phi_4^2 - 1 - \frac{1}{2}\phi_3^2 + \frac{2}{3}\,\phi_2^2\right) \\
\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\phi_4 &= 3\,\phi_4\left(\frac{5}{6}\,\phi_4^2 - 1 + \frac{1}{2}\,\phi_3^2 + \frac{2}{3}\phi_2^2\right) - 3\,\phi_1\,\phi_2\,\phi_3\,.
\end{aligned} \tag{XII.33}$$

If we make use of the algebraic constraint (XII.30), the system (XII.32) as well as the six equations (XII.33) reduce to the following four coupled differential equations:

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\phi_1 = 3\,\phi_1\left(\frac{3}{2}\,\phi_1^2 - 1 - \frac{1}{2}\,\phi_2^2\right) \tag{XII.34}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\phi_2 = 3\,\phi_2\left(\frac{5}{6}\,\phi_2^2 - 1 - \frac{1}{2}\,\phi_1^2 + \frac{2}{3}\,\phi_4^2\right) \tag{XII.35}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\phi_3 = 3\,\phi_3\left(\frac{3}{2}\,\phi_3^2 - 1 - \frac{1}{2}\phi_4^2\right) \tag{XII.36}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\,\phi_4 = 3\,\phi_4\left(\frac{5}{6}\phi_4^2 - 1 - \frac{1}{2}\phi_3^2 + \frac{2}{3}\,\phi_2^2\right)\,. \tag{XII.37}$$

These equations could be solved in principle and one could do the whole analysis we already did for the case of one scalar field. But we do not want go any further here.

Chapter XIII

Yang-Mills on $\mathbb{R} \times Q_3$ in $C^{(1,0)}$ Quiver Representation

We are now considering the special case of the coset space $G/H = Q_3$ given by

$$Q_3 := \frac{SU(3)}{U(1) \times U(1)}\,, \tag{XIII.1}$$

which, in contrast to the case of $\mathbb{C}P^2$, is a homogeneous but not symmetric space.

XIII.1 Invariant 1-Forms and Generators

As in (XI.2) we firstly want to write down the invariant 1-forms on Q_3 and the G-invariant gauge potential with respect to the fundamental representation of $SU(3)$. These results are taken from [22], and the explicit derivation can be looked up there.

Q_3 is a space with three complex dimensions and therefore we have six linearly independent invariant 1-forms. The explicit form of these 1-forms is given in (3.39) from [22]. They are denoted by

$$\left\{e^1,\ e^2,\ e^3, e^{\bar{1}},\ e^{\bar{2}},\ e^{\bar{3}}\right\} =: \left\{\gamma^1,\ \gamma^2,\ \gamma^3,\ \bar{\gamma}^{\bar{1}},\ \bar{\gamma}^{\bar{2}},\ \bar{\gamma}^{\bar{3}}\right\}\,. \tag{XIII.2}$$

Since the subgroup H is a different than before, we get a different decomposition of the reducible representation of H. This means that if we choose the fundamental representation for $SU(3)$ as the simplest case we get the following decomposition:

$$\underline{C}^{1,0}\big|_{U(1)\times U(1)} = \underline{(1,1)} \ \oplus\ \underline{(-1,1)} \ \oplus\ \underline{(0,-2)}\,. \tag{XIII.3}$$

As one can see here, we already have a decomposition into three irreps of H. The corresponding quiver diagram shows that there will appear three independent scalar fields in the G-equivariant gauge potential of the corresponding associated quiver bundle:

$$\begin{array}{ccc} \mathbb{R}\otimes\mathcal{V}^{Q_3}_{(-1,1)} & \xrightarrow{\ \phi_3\otimes\gamma_3\ } & \mathbb{R}\otimes\mathcal{V}^{Q_3}_{(1,1)} \\ {\scriptstyle \phi_2\otimes\gamma_2}\nwarrow & & \nearrow{\scriptstyle \phi_1\otimes\gamma_1} \\ & \mathbb{R}\otimes\mathcal{V}^{Q_3}_{(0,-2)} & \end{array} \quad . \tag{XIII.4}$$

Due to the fact that each term in (XIII.3) corresponds to a 1-dimensional representation of H, the corresponding structure group for the associated vector bundle is here $U(3)$.

The generators corresponding to Q_3 in this representation are then given by

$$\begin{aligned} &E_1 = e_{31}\,, \qquad &&E_2 = e_{32}\,, \qquad &&E_3 = e_{21}\,, \\ &E_{\bar{1}} = e_{13}\,, \qquad &&E_{\bar{2}} = e_{23}\,, \qquad &&E_{\bar{3}} = e_{12}\,, \end{aligned} \tag{XIII.5}$$

along with the generators of H,

$$E_7 = e_{11} - e_{22}, \qquad E_8 = e_{11} + e_{22} - 2e_{33}\,. \tag{XIII.6}$$

Hence, we have the following structure constants:

$$\begin{array}{llll} f_{ab}{}^c : & f_{\bar{3}\bar{2}}{}^{\bar{1}} = +1 & f_{\bar{3}1}{}^{2} = -1 & f_{\bar{2}1}{}^{3} = +1 \\ & f_{32}{}^{1} = -1 & f_{3\bar{1}}{}^{\bar{2}} = +1 & f_{2\bar{1}}{}^{\bar{3}} = -1 \\ f_{ai}{}^c : & f_{\bar{3}7}{}^{\bar{3}} = -2 & f_{\bar{2}7}{}^{\bar{2}} = +1 & f_{\bar{1}7}{}^{\bar{1}} = -1 \\ & f_{37}{}^{3} = +2 & f_{27}{}^{2} = -1 & f_{17}{}^{1} = +1 \\ & f_{\bar{3}8}{}^{\bar{3}} = 0 & f_{\bar{2}8}{}^{\bar{2}} = -3 & f_{\bar{1}8}{}^{\bar{1}} = -3 \\ & f_{38}{}^{3} = 0 & f_{28}{}^{2} = +3 & f_{18}{}^{1} = +3\,. \end{array} \tag{XIII.7}$$

XIII.2 Connection and Curvature

Next we want to write down the flat connection on the trivial $\mathbb{C}^3$-bundle over Q_3 what we also firstly did for the $\mathbb{C}P^2$ case in (XI.25). The flat connection on the trivial bundle over Q_3 is given in the invariant basis as

$$A_0 = \begin{pmatrix} a_1 & \bar{\gamma}^{\bar{3}} & \bar{\gamma}^{\bar{1}} \\ -\gamma^3 & -a_1 - a_2 & \bar{\gamma}^{\bar{2}} \\ -\gamma^1 & -\gamma^2 & a_2 \end{pmatrix} . \tag{XIII.8}$$

Here, the a_1 and a_2 are $u(1)$-valued connection 1-forms given in equation (3.38) from [22]. The remaining invariant 1-forms of G decompose as 1-forms on G/H and can be written in terms of the $a_1,\ a_2$ where we denote the 1-forms on H by $e^7,\ e^8$. They have the following components:

$$e^7{}_b = \left(a_1 + \frac{1}{2}a_2\right)_b \qquad e^8{}_b = -\frac{1}{2}(a_2)_b\,. \tag{XIII.9}$$

The flat connection satisfies the Maurer-Cartan equations

$$\mathrm{d}A_0 + A_0 \wedge A_0 = 0 \tag{XIII.10}$$

which yields the following equations for the invariant one forms and $u(1)$-valued connection 1-forms:

$$\mathrm{d}a_1 - \bar{\gamma}^1 \wedge \gamma^1 - \bar{\gamma}^3 \wedge \gamma^3 = 0\,, \tag{XIII.11}$$

$$\mathrm{d}a_2 + \bar{\gamma}^1 \wedge \gamma^1 + \bar{\gamma}^2 \wedge \gamma^2 = 0\,, \tag{XIII.12}$$

$$\mathrm{d}\gamma^1 - (a_1 - a_2) \wedge \gamma^1 - \gamma^2 \wedge \gamma^3 = 0\,, \tag{XIII.13}$$

$$\mathrm{d}\gamma^2 + (a_1 + 2a_2) \wedge \gamma^2 + \gamma^1 \wedge \bar{\gamma}^3 = 0\,, \tag{XIII.14}$$

$$\mathrm{d}\gamma^3 - (2a_1 + a_2) \wedge \gamma^3 - \gamma^1 \wedge \bar{\gamma}^2 = 0\,. \tag{XIII.15}$$

The extension to the non-flat connection on the corresponding extended bundle, taken from (3.50) in [22], reads

$$\mathcal{A} = \begin{pmatrix} a_1 & \phi_3\ \bar{\gamma}^{\bar{3}} & \phi_1\ \bar{\gamma}^{\bar{1}} \\ -\phi_3\ \gamma^3 & -a_1 - a_2 & -\phi_2\ \bar{\gamma}^{\bar{2}} \\ -\phi_1\ \gamma^1 & -\phi_2\ \gamma^2 & a_2 \end{pmatrix}\,. \tag{XIII.16}$$

The corresponding gauge field $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is then easily calculated using Maurer-Cartan equations (XIII.11)-(XIII.15):

$$\mathcal{F} = \begin{pmatrix} \begin{matrix}(1-\phi_1^2)\bar{\gamma}^{\bar{1}} \wedge \gamma^1 \\ +\ (1-\phi_3^2)\bar{\gamma}^{\bar{3}} \wedge \gamma^3\end{matrix} & \begin{matrix}\mathrm{d}\phi_3 \wedge \bar{\gamma}^{\bar{3}} \\ +\ (\phi_3 - \phi_1\phi_2)\ \bar{\gamma}^{\bar{1}} \wedge \gamma^2\end{matrix} & \begin{matrix}\mathrm{d}\phi_1 \wedge \bar{\gamma}^{\bar{1}} \\ +\ (\phi_1 - \phi_3\phi_2)\ \bar{\gamma}^{\bar{2}} \wedge \bar{\gamma}^{\bar{3}}\end{matrix} \\ \\ \begin{matrix}-\mathrm{d}\phi_3 \wedge \gamma^3 \\ -\ (\phi_3 - \phi_1\phi_2)\ \gamma^1 \wedge \bar{\gamma}^{\bar{2}}\end{matrix} & \begin{matrix}-(1-\phi_3^2)\ \bar{\gamma}^{\bar{3}} \wedge \gamma^3 \\ +\ (1-\phi_2^2)\ \bar{\gamma}^{\bar{2}} \wedge \gamma^2\end{matrix} & \begin{matrix}\mathrm{d}\phi_2 \wedge \bar{\gamma}^{\bar{2}} \\ +\ (\phi_2 - \phi_3\phi_1)\ \gamma^3 \wedge \bar{\gamma}^{\bar{1}}\end{matrix} \\ \\ \begin{matrix}-\mathrm{d}\phi_1 \wedge \gamma^1 \\ -\ (\phi_1 - \phi_3\phi_2)\ \gamma^2 \wedge \gamma^3\end{matrix} & \begin{matrix}-\mathrm{d}\phi_2 \wedge \gamma^2 \\ -\ (\phi_2 - \phi_3\phi_1)\ \bar{\gamma}^{\bar{3}} \wedge \gamma^1\end{matrix} & \begin{matrix}-(1-\phi_1^2)\ \bar{\gamma}^{\bar{1}} \wedge \gamma^1 \\ -\ (1-\phi_2^2)\ \bar{\gamma}^{\bar{2}} \wedge \gamma^2\end{matrix} \end{pmatrix}\,. \tag{XIII.17}$$

XIII.3 Yang-Mills Equations

The Yang-Mills equations on Q_3 look a little different in this case, since we have another set of non-vanishing structure constants, namely those with coset indices. We can therefore endow Q_3 with a non-vanishing torsion tensor with non-holonomic components

$$T^b_{ac} = \kappa\ f_{ac}{}^b\,. \tag{XIII.18}$$

We end up with the following Yang-Mills equations:

$$\begin{aligned} \mathrm{YM}^b \ :=\ & \frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{F}^{0b} + \frac{(1+\kappa)}{2}\left(f_{ac}{}^b\mathcal{F}^{ac} + f_{ac}{}^a\mathcal{F}^{cb}\right) \\ & -e^i{}_a\left(f_{ci}{}^a\ \mathcal{F}^{cb} + f_{ci}{}^b\ \mathcal{F}^{ac}\right) + \left[\mathcal{A}_a, \mathcal{F}^{ab}\right] = 0\,. \end{aligned} \tag{XIII.19}$$

If we insert (XIII.16) and (XIII.17) into equation (XIII.19), we find an independent differential equation for the scalar fields $\phi_1,\ \phi_2,\ \phi_3$ for each superscript $b = 1,\ 2,\ 3,\ \bar{1},\ \bar{2},\ \bar{3}$. These six equations actually differ via three algebraic conditions on the fields which come from the term containing the coset structure constants in (XIII.19). We can separate the algebraic conditions by adding and subtracting those equations that have conjugated indices from each other, schematically

$$\mathrm{YM}^1 \pm \mathrm{YM}^{\bar{1}}, \qquad \mathrm{YM}^2 \pm \mathrm{YM}^{\bar{2}}, \qquad \mathrm{YM}^3 \pm \mathrm{YM}^{\bar{3}}\,. \tag{XIII.20}$$

By doing that, we arrive at three independent differential equations

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\phi_1 &= 2\phi_1 \left(\phi_1^2 - 1 + \frac{1}{2}\left(\phi_2^2 + \phi_3^2\right)\right) - 2\phi_2\ \phi_3 \\ \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\phi_2 &= 2\phi_2 \left(\phi_2^2 - 1 + \frac{1}{2}\left(\phi_1^2 + \phi_3^2\right)\right) - 2\phi_1\ \phi_3 \\ \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\phi_3 &= 2\phi_3 \left(\phi_3^2 - 1 + \frac{1}{2}\left(\phi_1^2 + \phi_2^2\right)\right) - 2\phi_1\ \phi_2 \end{aligned} \tag{XIII.21}$$

along with the algebraic constraints

$$(\kappa + 1) \cdot (\phi_1 - \phi_2\ \phi_3) = 0 \tag{XIII.22}$$

$$(\kappa + 1) \cdot (\phi_2 - \phi_1\ \phi_3) = 0 \tag{XIII.23}$$

$$(\kappa + 1) \cdot (\phi_3 - \phi_1\ \phi_2) = 0\,. \tag{XIII.24}$$

In order to satisfy the algebraic constraints, we are forced to choose the value $\kappa = -1$ for the torsion. Otherwise these conditions would restrict us to constant fields with value $1,\ 0$ or -1. If we choose the specific value for the torsion, (XIII.21) is again solvable in principle. But as in the case of $\mathbb{C}P^2$ in the last chapter, we do not want to go further in the analysis here but simply finish all our considerations with this result. The next thing one could do would be considering higher representations such as $\underline{C}^{2,0}$ or $\underline{C}^{1,1}$ for $SU(3)$ and would get further decompositions of the subgroup $H = U(1) \times U(1)$ and hence of the gauge potential. Therefore more scalar fields would arise, using the corresponding G-equivariant ansätze, derived and written down in [22].

Chapter XIV

Yang-Mills on $\mathbb{R} \times \mathbb{C}P^1 \times \mathbb{C}P^2$

In this chapter we want to consider the base space $\mathbb{R} \times \mathbb{C}P^1 \times \mathbb{C}P^2$ and a vector bundle over it endowed with the structure group $U(3(m+1))$. Therefore, in contrast to all the other examples, we get a more general ansatz for the corresponding gauge potential on the corresponding associated vector bundle which contains $2m+1$ scalar fields. Here we do not want to fix the specific number of Higgs-fields, but try to derive the equation of motion for this ansatz in general.

XIV.1 Gauge Potential and Gauge Field

XIV.1.1 The ansatz for the gauge potential. We are considering the following ansatz for a $u(3(m+1))$-valued gauge potential (in the temporal gauge $\mathcal{A}_\tau = 0$):

$$\mathcal{A} \;=\; A_m \otimes \mathbb{1}_3 + \mathbb{1}_{m+1} \otimes \begin{pmatrix} B & 0 \\ 0 & -2a \end{pmatrix} + \Psi_m \otimes \begin{pmatrix} 0_2 & \bar{\beta} \\ -\beta^\top & 0 \end{pmatrix} \qquad \text{(XIV.1)}$$

where

$$B := B_{(1)} + a \cdot \mathbb{1}_2\,. \qquad \text{(XIV.2)}$$

As in chapter XI the $B_{(1)}$ denotes the $su(2)$-valued one-instanton field on $\mathbb{C}P^2$ and the $\beta,\ \bar{\beta}$ are row vectors of the invariant basis of 1-forms of $\mathbb{C}P^2$,

$$\Psi_m := \mathrm{diag}(\psi_1, \psi_2, ..., \psi_{m+1})\,, \qquad \text{(XIV.3)}$$

were all the ψ_i are considered to be real scalar fields on $\mathbb{R}$. Furthermore, we have

$$
\begin{aligned}
A_m &:= b_{(m)} + \frac{1}{2}\Phi_m\bar{\gamma} - \frac{1}{2}\Phi^\dagger_m\gamma, && \text{(XIV.4)}\\
b_{(m)} &:= \Upsilon_m b, && \text{(XIV.5)}\\
\Upsilon_m &:= \mathrm{diag}(m, m-2, ..., -m+2, -m), && \text{(XIV.6)}\\
b &:= \frac{1}{2(R^2+y\bar{y})}(\bar{y}\ \mathrm{d}y - y\ \mathrm{d}\bar{y}), && \text{(XIV.7)}\\
\gamma &:= \frac{\sqrt{2}R^2}{R^2+y\bar{y}}\mathrm{d}y, \quad \bar{\gamma} := \frac{\sqrt{2}R^2}{R^2+y\bar{y}}\mathrm{d}\bar{y}\,, && \text{(XIV.8)}\\
\Phi_m &:= \begin{pmatrix} 0 & \phi_1 & \cdots & 0\\ \vdots & 0 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \phi_m\\ 0 & \cdots & \cdots & 0 \end{pmatrix}, && \text{(XIV.9)}
\end{aligned}
$$

where also all the ϕ_i are required to be real scalar fields on $\mathbb{R}$. Here the one form b is the gauge potential on the Dirac one-monopole line bundle over $\mathbb{C}P^1$ and the $(1,0)$-form γ as well as the $(0,1)$-form $\bar{\gamma}$ are the invariant basis of 1-forms on $\mathbb{C}P^1$.

As one can easily see, the invariant 1-forms read

$$
\begin{aligned}
\gamma &= e^{\gamma}{}_y\ \mathrm{d}y, && \text{(XIV.10)}\\
\bar{\gamma} &= e^{\bar{\gamma}}{}_{\bar{y}}\ \mathrm{d}\bar{y}\,, && \text{(XIV.11)}
\end{aligned}
$$

where

$$
\rho := e^{\gamma}{}_y = e^{\bar{\gamma}}{}_{\bar{y}} = \frac{\sqrt{2}R^2}{R^2+y\bar{y}}\,. \qquad \text{(XIV.12)}
$$

The invariant metric g on $\mathbb{C}P^1 \times \mathbb{C}P^2$ is given by the non vanishing components

$$
\begin{aligned}
g_{a\bar{b}} &= \delta_{ab},\ a,b \in \{1,2,\bar{1},\bar{2}\}\,, && \text{(XIV.13)}\\
g_{y\bar{y}} &= \rho^2,\ \text{and hence } g^{\bar{y}y} = \rho^{-2}\,. && \text{(XIV.14)}
\end{aligned}
$$

XIV.1.2 Maurer-Cartan equations and the field strength. We are dealing with invariant 1-forms on symmetric spaces and therefore these 1-forms fulfil the Maurer-Cartan equations (as we have seen in chapter XI), and are easy to calculate for the case of $\mathbb{C}P^1$. The resulting equations for all invariant 1-forms become

$$\begin{aligned} \mathrm{d}b - \frac{1}{2R^2}\,\bar{\gamma}\wedge\gamma &= 0 \\ \mathrm{d}B + B\wedge B - \bar{\beta}\wedge\beta^{\top} &= 0 \\ \mathrm{d}a - \frac{1}{2}\beta^{\dagger}\wedge\beta &= 0 \end{aligned} \tag{XIV.15}$$

$$\mathrm{d}\gamma - 2b\wedge\gamma = 0 \tag{XIV.16}$$

$$\mathrm{d}\bar{\gamma} + 2b\wedge\bar{\gamma} = 0 \tag{XIV.17}$$

$$\begin{aligned} \mathrm{d}\bar{\beta} + B\wedge\bar{\beta} - 2\bar{\beta}\wedge a &= 0 \\ \mathrm{d}\beta^{\top} + \beta^{\top}\wedge B - 2a\wedge\beta^{\top} &= 0\,. \end{aligned}$$

If we insert the ansatz (XIV.1) into the definition $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$ and perform the calculations, written down in the appendix A.1.2, we find the following field strength:

$$\begin{aligned} \mathcal{F} \;=\; & \left(\rho^2\left(\frac{1}{4}\left[\Phi_m^{\top},\Phi_m\right] + \frac{1}{2R^2}\Upsilon_m\right)\otimes\mathbb{1}_3\right)\mathrm{d}\bar{y}\wedge\mathrm{d}y + \frac{\rho}{2}\partial_t\Phi_m\;\mathrm{d}t\wedge\mathrm{d}\bar{y} - \frac{\rho}{2}\partial_t\Phi_m^{\top}\;\mathrm{d}t\wedge\mathrm{d}y \\ & \frac{\rho}{2}\left[\Phi_m,\Psi_m\right]\otimes\begin{pmatrix} 0_2 & \mathrm{d}\bar{y}\wedge\bar{\beta} \\ -\mathrm{d}\bar{y}\wedge\beta^{\top} & 0 \end{pmatrix} - \frac{\rho}{2}\left[\Phi_m^{\top},\Psi_m\right]\otimes\begin{pmatrix} 0_2 & \mathrm{d}y\wedge\bar{\beta} \\ -\mathrm{d}y\wedge\beta^{\top} & 0 \end{pmatrix} \\ & \left(\Psi_m^2 - \mathbb{1}_{m+1}\right)\otimes\begin{pmatrix} -\bar{\beta}\wedge\beta^{\top} & 0 \\ 0 & \beta^{\dagger}\wedge\beta \end{pmatrix} + \partial_t(\Psi_m)\otimes\begin{pmatrix} 0 & \mathrm{d}t\wedge\bar{\beta} \\ -\mathrm{d}t\wedge\beta^{\top} & 0 \end{pmatrix}, \end{aligned} \tag{XIV.18}$$

where

$$\begin{pmatrix} \bar{\beta}^1\wedge\beta^1 & \bar{\beta}^1\wedge\beta^2 \\ \bar{\beta}^2\wedge\beta^1 & \bar{\beta}^2\wedge\beta^2 \end{pmatrix} = \bar{\beta}\wedge\beta^{\top} \tag{XIV.19}$$

$$(\bar{\beta}^1\wedge\beta^1 + \bar{\beta}^2\wedge\beta^2) = \beta^{\dagger}\wedge\beta\,. \tag{XIV.20}$$

XIV.2 Yang-Mills Equations

The Yang-Mills equations on the space $\mathbb{R}\times\mathbb{C}P^1\times\mathbb{C}P^2$ look slightly different, since the dimension is higher then in the previous cases. If we compare this to chapter XI for instance, we can see that, since we consider a product of two projective spaces, we will get one more matrix equation for each additional belonging to $\mathbb{C}P^1$. This means we have

$$\begin{aligned} \mathcal{D}_A\mathcal{F}^{A\delta} &= 0, \qquad \delta\in\{\bar{y},\ y\} \\ \mathcal{D}_A\mathcal{F}^{Aa} &= 0, \qquad a\in\{1,\ 2,\ \bar{1}\ ,\bar{2}\} \end{aligned}$$

which reads

$$\begin{array}{llllllll} \partial_t\mathcal{F}^{t\delta} & +\nabla_\alpha\mathcal{F}^{\alpha\delta} & +\left[\mathcal{A}_\alpha,\mathcal{F}^{\alpha\delta}\right] & +\nabla_c\mathcal{F}^{c\delta} & +\left[\mathcal{A}_c,\mathcal{F}^{c\delta}\right] & = & 0 & \\ \partial_t\mathcal{F}^{td} & +\nabla_\alpha\mathcal{F}^{\alpha d} & +\left[\mathcal{A}_\alpha,\mathcal{F}^{\alpha d}\right] & +\nabla_c\mathcal{F}^{cd} & +\left[\mathcal{A}_c,\mathcal{F}^{cd}\right] & = & 0 & . \end{array} \tag{XIV.21}$$

Here the repeated Greek indices are summed over the components belonging to $\mathbb{C}P^1$, namely $\alpha\in\{y,\ \bar{y}\}$ and the Latin letters are summed over the $\mathbb{C}P^2$ components. The covariant derivatives of the gauge field strength for the projective spaces are given in the canonical way for product spaces.

If we insert (XIV.1) and (VII.8) into (XIV.21), which is written down in detail in appendix A.2, we arrive at the following matrix equations:

$$0 = \partial_t^2\Phi_m - \frac{1}{4}\left[\Phi_m,\left[\Phi_m^\top,\Phi_m\right]\right] + \frac{1}{R^2}\Phi_m - \left[\Psi_m,\left[\Phi_m,\Psi_m\right]\right] \tag{XIV.22}$$

$$0 = \partial_t^2\Psi_m - \frac{1}{2}\left[\Phi_m^\top,\left[\Phi_m,\Psi_m\right]\right] + 3\left(\Psi_m-\Psi_m^3\right). \tag{XIV.23}$$

Inserting Φ_m and Ψ_m into these matrix equations (XIV.22) and (XIV.23), we find the following independent differential equations for our scalar fields ψ_i and ϕ_i:

$$\begin{aligned} 0 &= \partial_t^2\phi_i + \frac{1}{4}(\phi_{i-1}^2 - 2\phi_i^2 + \phi_{i+1}^2)\phi_i + \frac{1}{R^2}\phi_i - \left(\psi_{i+1}^2 - 2\psi_{i+1}\psi_i + \psi_i^2\right)\phi_i \\ 0 &= \partial_t^2\psi_i - \frac{1}{2}\left(\phi_{i-1}^2(\psi_i-\psi_{i-1}) + \phi_i^2(\psi_{i+1}-\psi_i)\right) + 3(\psi_i-\psi_i^3). \end{aligned} \tag{XIV.24}$$

Solutions to similar equations were found in [20] and possibly also solutions for the equations (XIV.24) do exist. Seeking for such solutions could therefore be a further task.

Appendix A

Explicit Calculations for $\mathbb{R}\times\mathbb{C}P^1\times\mathbb{C}P^2$

A.1 Gauge Field on $\mathbb{R}\times\mathbb{C}P^1\times\mathbb{C}P^2$

A.1.1 The ansatz for the gauge potential. We are now considering the $\mathbb{R}\times\mathbb{C}P^1\times\mathbb{C}P^2$ together with the following Ansatz for a $u(3(m+1))$-valued gauge potential (in temporal gauge $\mathcal{A}_\tau=0$):

$$\mathcal{A} \;=\; A_m\otimes\mathbb{1}_3+\mathbb{1}_{m+1}\otimes\begin{pmatrix}B & 0\\ 0 & -2a\end{pmatrix}+\Psi_m\otimes\begin{pmatrix}0_2 & \bar\beta\\ -\beta^\top & 0\end{pmatrix} \tag{A.1}$$

where

$$B := B_{(1)}+a\cdot\mathbb{1}_2\,. \tag{A.2}$$

As in chapter XI the $B_{(1)}$ denotes the $su(2)$-valued one-instanton field on $\mathbb{C}P^2$ and the $\beta,\ \bar\beta$ are row vectors of the invariant basis of 1-forms of $\mathbb{C}P^2$. Furthermore, in (A.1)

$$\Psi_m := \mathrm{diag}(\psi_1,\psi_2,...,\psi_{m+1})\,, \tag{A.3}$$

were all the ψ_i are considered to be real scalar fields on $\mathbb{R}$. We also have

$$A_m := b_{(m)}+\frac{1}{2}\Phi_m\bar\gamma-\frac{1}{2}\Phi_m^\dagger\gamma, \tag{A.4}$$

$$b_{(m)} := \Upsilon_m b, \tag{A.5}$$

$$\Upsilon_m := \mathrm{diag}(m,m-2,...,-m+2,-m), \tag{A.6}$$

$$b := \frac{1}{2(R^2+y\bar y)}(\bar y\ \mathrm{d}y-y\ \mathrm{d}\bar y), \tag{A.7}$$

$$\gamma := \frac{\sqrt{2}R^2}{R^2+y\bar y}\mathrm{d}y,\quad \bar\gamma := \frac{\sqrt{2}R^2}{R^2+y\bar y}\mathrm{d}\bar y \tag{A.8}$$

$$\Phi_m := \begin{pmatrix}0 & \phi_1 & \cdots & 0\\ \vdots & 0 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \phi_m\\ 0 & \cdots & \cdots & 0\end{pmatrix}, \tag{A.9}$$

where all the ϕ_i are required to be real scalar fields on $\mathbb{R}$. Here the one-form b is the gauge potential on the Dirac one-monopole line bundle over $\mathbb{C}P^1$, and the $(1,0)$-form γ together with the $(0,1)$-form $\bar{\gamma}$ are the invariant basis of 1-forms on $\mathbb{C}P^1$.

As one can easily see, the invariant 1-forms are proportional to the complex coordinates, namely

$$\gamma = e^{\gamma}{}_y \, \mathrm{d}y, \tag{A.10}$$

$$\bar{\gamma} = e^{\bar{\gamma}}{}_{\bar{y}} \, \mathrm{d}\bar{y}, \text{ where} \tag{A.11}$$

$$\rho := e^{\gamma}{}_y = e^{\bar{\gamma}}{}_{\bar{y}} = \frac{\sqrt{2}R^2}{R^2 + y\bar{y}}\,. \tag{A.12}$$

The invariant metric g on $\mathbb{C}P^1 \times \mathbb{C}P^2$ is given by the non vanishing components

$$g_{a\bar{b}} = \delta_{ab}, \ a, b \in \{1, 2, \bar{1}, \bar{2}\}\ , \tag{A.13}$$

$$g_{y\bar{y}} = \rho^2, \text{ and hence } g^{\bar{y}y} = \rho^{-2}\,. \tag{A.14}$$

A.1.2 Maurer-Cartan equations and the field strength. We are dealing with invariant 1-forms on symmetric spaces and therefore these 1-forms fulfill the Maurer-Cartan equations which are given in [22] for $\mathbb{C}P^2$ and easy to calculate for the case of $\mathbb{C}P^1$. The resulting equations for all invariant 1-forms become

$$\mathrm{d}b - \frac{1}{2R^2}\,\bar{\gamma} \wedge \gamma = 0 \tag{A.15}$$

$$\mathrm{d}B + B \wedge B - \bar{\beta} \wedge \beta^{\top} = 0 \tag{A.16}$$

$$\mathrm{d}a - \frac{1}{2}\beta^{\dagger} \wedge \beta = 0\,, \tag{A.17}$$

along with

$$\mathrm{d}\gamma - 2b \wedge \gamma = 0 \tag{A.18}$$

$$\mathrm{d}\bar{\gamma} + 2b \wedge \bar{\gamma} = 0 \tag{A.19}$$

$$\mathrm{d}\bar{\beta} + B \wedge \bar{\beta} - 2\bar{\beta} \wedge a = 0 \tag{A.20}$$

$$\mathrm{d}\beta^{\top} + \beta^{\top} \wedge B - 2a \wedge \beta^{\top} = 0\,. \tag{A.21}$$

For convenience we split $\mathcal{A}$ into two pieces,

$$\mathcal{A} =: A_{(m)} \otimes \mathbb{1}_3 + A_{\mathbb{C}P^2}\,,$$

then the field strength $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ becomes

$$\mathrm{d}A_{(m)} \otimes \mathbb{1}_3 + \mathrm{d}A_{\mathbb{C}P^2} + (A_{(m)} \otimes \mathbb{1}_3 + A_{\mathbb{C}P^2}) \wedge (A_{(m)} \otimes \mathbb{1}_3 + A_{\mathbb{C}P^2}).$$

We split the calculations into three big pieces, namely in

$$
\begin{aligned}
\mathcal{F} &= \mathrm{d}A_{(m)} \otimes \mathbb{1}_3 + A_{(m)} \otimes \mathbb{1}_3 \wedge A_{(m)} \otimes \mathbb{1}_3 \\
&\quad + \mathrm{d}A_{\mathbb{C}P^2} + A_{\mathbb{C}P^2} \wedge A_{\mathbb{C}P^2} \\
&\quad + (A_{(m)} \otimes \mathbb{1}_3) \wedge A_{\mathbb{C}P^2} + A_{\mathbb{C}P^2} \wedge (A_{(m)} \otimes \mathbb{1}_3) \\
&= \mathrm{d}A_{(m)} \otimes \mathbb{1}_3 + A_{(m)} \wedge A_{(m)} \otimes \mathbb{1}_3 \qquad \text{(A.22)} \\
&\quad + \mathrm{d}A_{\mathbb{C}P^2} + A_{\mathbb{C}P^2} \wedge A_{\mathbb{C}P^2} \qquad \text{(A.23)} \\
&\quad (A_{(m)} \otimes \mathbb{1}_3) \wedge A_{\mathbb{C}P^2} + A_{\mathbb{C}P^2} \wedge (A_{(m)} \otimes \mathbb{1}_3)\,. \qquad \text{(A.24)}
\end{aligned}
$$

In the following we are going to calculate the terms (A.22)-(A.24) in this order:

A.1.3 Piece A.22 of the field strength.

First the differential of $A_{(m)}$:

$$
\begin{aligned}
\mathrm{d}A_{(m)} = &\quad \mathrm{d}b\Upsilon_m && + \frac{1}{2}\Big(\mathrm{d}\Phi_m \wedge \bar{\gamma} + \Phi_m\ \mathrm{d}\bar{\gamma} - \mathrm{d}\Phi_m^\top \wedge \gamma - \Phi_m^\top\ \mathrm{d}\gamma\Big) \\
= &\quad \frac{1}{2R^2}\Upsilon_m\ \bar{\gamma}\wedge\gamma && + \frac{1}{2}\Big(\partial_t(\Phi_m)\ \mathrm{d}t\wedge\bar{\gamma} - \partial_t(\Phi_m^\top)\ \mathrm{d}t\wedge\gamma\Big) \\
& && + \frac{1}{2}\Big(-\Phi_m\ 2b\wedge\bar{\gamma} - \Phi_m^\top\ 2b\wedge\gamma\Big) \\
= &\quad \frac{1}{2R^2}\Upsilon_m\ \bar{\gamma}\wedge\gamma && + \frac{1}{2}\Big(\partial_t(\Phi_m)\ \mathrm{d}t\wedge\bar{\gamma} - \partial_t(\Phi_m^\top)\ \mathrm{d}t\wedge\gamma\Big) \\
& && - \frac{1}{2}\Big(\Phi_m\ 2b\wedge\bar{\gamma} + \Phi_m^\top\ 2b\wedge\gamma\Big)\ .
\end{aligned}
$$

Now turn to the wedge product

$$
\begin{aligned}
A_{(m)} \wedge A_{(m)} &= \frac{1}{2}\Upsilon_m\ \Phi_m\ b\wedge\bar{\gamma} - \frac{1}{2}\Upsilon_m\ \Phi_m^\top\ b\wedge\gamma \\
&\quad + \frac{1}{2}\Phi_m\ \Upsilon_m\ \bar{\gamma}\wedge b - \frac{1}{4}\Phi_m\ \Phi_m^\top\ \bar{\gamma}\wedge\gamma \\
&\quad - \frac{1}{2}\Phi_m^\top\ \Upsilon_m\ \gamma\wedge b - \frac{1}{4}\Phi_m^\top\ \Phi_m\ \gamma\wedge\bar{\gamma} \\
&= \frac{1}{2}\left([\Upsilon_m, \Phi_m]\, b\wedge\bar{\gamma} - \left[\Upsilon_m, \Phi_m^\top\right] b\wedge\gamma + \frac{1}{2}\left[\Phi_m^\top, \Phi_m\right]\gamma\wedge\bar{\gamma}\right) \\
&= \Phi_m\ b\wedge\bar{\gamma} + \Phi_m^\top\ b\wedge\gamma + \frac{1}{4}\left[\Phi_m^\top, \Phi_m\right]\bar{\gamma}\wedge\gamma\,.
\end{aligned}
$$

In the last step we used that $[\Upsilon_m, \Phi_m] = 2\Phi_m$ as well as $\left[\Upsilon_m, \Phi_m^\top\right] = -2\Phi_m$, and in total we have

$$
\begin{aligned}
\mathrm{d}A_{(m)} + A_{(m)} \wedge A_{(m)} \quad = \quad & \left(\frac{1}{4}\left[\Phi_m^\top, \Phi_m\right] + \frac{1}{2R^2}\Upsilon_m\right)\bar{\gamma}\wedge\gamma \\
& + \Big(\partial_t(\Phi_m)\ \mathrm{d}t\wedge\bar{\gamma} - \partial_t(\Phi_m^\top)\ \mathrm{d}t\wedge\gamma\Big)\ .
\end{aligned}
\qquad \text{(A.25)}
$$

A.1.4 Piece A.23 of the gauge field strength. Using (A.16) and (A.17), we get

$$\begin{aligned}
& \mathrm{d}A_{\mathbb{C}P^2} + A_{\mathbb{C}P^2}\wedge A_{\mathbb{C}P^2} = \\
= \quad & \mathbb{1}_{m+1}\otimes\begin{pmatrix}\mathrm{d}B + B\wedge B & 0\\ 0 & -2\mathrm{d}a\end{pmatrix}\\
& + \partial_t(\Psi_m)\otimes\begin{pmatrix}0 & \mathrm{d}t\wedge\bar\beta\\ 0 & 0\end{pmatrix} - \partial_t(\Psi_m)\otimes\begin{pmatrix}0 & 0\\ \mathrm{d}t\wedge\beta^\top & 0\end{pmatrix}\\
& + \Psi_m\otimes\begin{pmatrix}0 & \mathrm{d}\bar\beta\\ 0 & 0\end{pmatrix} + \Psi_m\otimes\begin{pmatrix}0 & B\wedge\bar\beta\\ 0 & 0\end{pmatrix} + \Psi_m\otimes\begin{pmatrix}0 & -2\bar\beta\wedge a\\ 0 & 0\end{pmatrix}\\
& - \Psi_m\otimes\begin{pmatrix}0 & 0\\ -2a\wedge\beta^\top & 0\end{pmatrix} - \Psi_m\otimes\begin{pmatrix}0 & 0\\ \mathrm{d}\beta^\top & 0\end{pmatrix} - \Psi_m\otimes\begin{pmatrix}0 & 0\\ \beta^\top\wedge B & 0\end{pmatrix}\\
& - \Psi_m\ \Psi_m\begin{pmatrix}\bar\beta\wedge\beta^\top & 0\\ 0 & 0\end{pmatrix} - \Psi_m\ \Psi_m\begin{pmatrix}0 & 0\\ 0 & \beta^\top\wedge\bar\beta\end{pmatrix}\\
= \quad & \mathbb{1}_{m+1}\otimes\begin{pmatrix}\bar\beta\wedge\beta^\top & 0\\ 0 & -\beta^\dagger\wedge\beta\end{pmatrix}\\
& + \partial_t(\Psi_m)\otimes\begin{pmatrix}0 & \mathrm{d}t\wedge\bar\beta\\ 0 & 0\end{pmatrix} - \partial_t(\Psi_m)\otimes\begin{pmatrix}0 & 0\\ \mathrm{d}t\wedge\beta^\top & 0\end{pmatrix}\\
& - \Psi_m\ \Psi_m\otimes\begin{pmatrix}\bar\beta\wedge\beta^\top & 0\\ 0 & 0\end{pmatrix} + \Psi_m\ \Psi_m\otimes\begin{pmatrix}0 & 0\\ 0 & \beta^\dagger\wedge\beta\end{pmatrix}\\
= \quad & \mathbb{1}_{m+1}\otimes\begin{pmatrix}\bar\beta\wedge\beta^\top & 0\\ 0 & -\beta^\dagger\wedge\beta\end{pmatrix}\\
& + \partial_t(\Psi_m)\otimes\begin{pmatrix}0 & \mathrm{d}t\wedge\bar\beta\\ -\mathrm{d}t\wedge\beta^\top & 0\end{pmatrix} + \Psi_m^2\otimes\begin{pmatrix}-\bar\beta\wedge\beta^\top & 0\\ 0 & \beta^\dagger\wedge\beta\end{pmatrix}\\
= \quad & \left(\Psi_m^2 - \mathbb{1}_{m+1}\right)\otimes\begin{pmatrix}-\bar\beta\wedge\beta^\top & 0\\ 0 & \beta^\dagger\wedge\beta\end{pmatrix} + \partial_t(\Psi_m)\otimes\begin{pmatrix}0 & \mathrm{d}t\wedge\bar\beta\\ -\mathrm{d}t\wedge\beta^\top & 0\end{pmatrix}.
\end{aligned} \tag{A.26}$$

In the last as step we have used equation (A.21) as well as equation (A.20).

A.1.5 Piece (A.24) of the gauge field strength.

$$\begin{aligned}
\text{(A.24)} = A_{(m)}\otimes\mathbb{1}_3\wedge\left(\mathbb{1}_{m+1}\otimes\begin{pmatrix}B & 0\\ 0 & -2a\end{pmatrix} + \Psi_m\otimes\begin{pmatrix}0_2 & \bar\beta\\ -\beta^\top & 0\end{pmatrix}\right)\\
+\left(\mathbb{1}_{m+1}\otimes\begin{pmatrix}B & 0\\ 0 & -2a\end{pmatrix} + \Psi_m\otimes\begin{pmatrix}0_2 & \bar\beta\\ -\beta^\top & 0\end{pmatrix}\right)\wedge A_{(m)}\otimes\mathbb{1}_3
\end{aligned}$$

$$
\begin{aligned}
&= \left[A_{(m)}, \Psi_m\right] \wedge \otimes \begin{pmatrix} 0_2 & \bar{\beta} \\ -\beta^\top & 0 \end{pmatrix} \\
&= \left[\Upsilon_m b + \frac{1}{2}\Phi_m \bar{\gamma} - \frac{1}{2}\Phi_m^\top \gamma, \Psi_m\right] \wedge \otimes \begin{pmatrix} 0_2 & \bar{\beta} \\ -\beta^\top & 0 \end{pmatrix} \\
&= \frac{1}{2}\left(\left[\Phi_m, \Psi_m\right]\bar{\gamma} - \left[\Phi_m^\top, \Psi_m\right]\gamma\right) \wedge \otimes \begin{pmatrix} 0_2 & \bar{\beta} \\ -\beta^\top & 0 \end{pmatrix} . \qquad \text{(A.27)}
\end{aligned}
$$

Here we used the commutativity $[\Upsilon_m, \Psi_m] = 0$. It is useful to change the basis for $\mathbb{C}P^1$ from the invariant one to the simple holomorphic complex basis $\{y, \bar{y}\}$ according to equation (A.8). Putting (A.22)-(A.24) together and performing the already mentioned basis transformation, we arrive at

$$
\begin{aligned}
\mathcal{F} \;=\; & \left(\rho^2 \left(\frac{1}{4}\left[\Phi_m^\top, \Phi_m\right] + \frac{1}{2R^2}\Upsilon_m\right) \otimes \mathbb{1}_3\right) \mathrm{d}\bar{y} \wedge \mathrm{d}y + \frac{\rho}{2}\partial_t \Phi_m \, \mathrm{d}t \wedge \mathrm{d}\bar{y} - \frac{\rho}{2}\partial_t \Phi_m^\top \, \mathrm{d}t \wedge \mathrm{d}y \\
& \frac{\rho}{2}\left[\Phi_m, \Psi_m\right] \otimes \begin{pmatrix} 0_2 & \mathrm{d}\bar{y} \wedge \bar{\beta} \\ -\mathrm{d}\bar{y} \wedge \beta^\top & 0 \end{pmatrix} - \frac{\rho}{2}\left[\Phi_m^\top, \Psi_m\right] \otimes \begin{pmatrix} 0_2 & \mathrm{d}y \wedge \bar{\beta} \\ -\mathrm{d}y \wedge \beta^\top & 0 \end{pmatrix} \\
& \left(\Psi_m^2 - \mathbb{1}_{m+1}\right) \otimes \begin{pmatrix} -\bar{\beta} \wedge \beta^\top & 0 \\ 0 & \beta^\dagger \wedge \beta \end{pmatrix} + \partial_t(\Psi_m) \otimes \begin{pmatrix} 0 & \mathrm{d}t \wedge \bar{\beta} \\ -\mathrm{d}t \wedge \beta^\top & 0 \end{pmatrix} ,
\end{aligned}
\qquad \text{(A.28)}
$$

where

$$
\begin{pmatrix} \bar{\beta}^1 \wedge \beta^1 & \bar{\beta}^1 \wedge \beta^2 \\ \bar{\beta}^2 \wedge \beta^1 & \bar{\beta}^2 \wedge \beta^2 \end{pmatrix} = \bar{\beta} \wedge \beta^\top , \qquad \text{(A.29)}
$$

$$
(\bar{\beta}^1 \wedge \beta^1 + \bar{\beta}^2 \wedge \beta^2) = \beta^\dagger \wedge \beta . \qquad \text{(A.30)}
$$

A.2 Yang-Mills Equations for $B \in \{y, \bar{y}\}$

We are now going to write down the Yang-Mills equations and deduce the corresponding equations for the Φ_m, Ψ_m and hence for the ϕ_i, ψ_i explicitly. The Yang-Mills equations are given by

$$
\partial_t \mathcal{F}^{tB} + \nabla_A \mathcal{F}^{AB} + \left[\mathcal{A}_A, \mathcal{F}^{AB}\right] = 0, \quad A, B \in \{y, \bar{y}, 1, 2, \bar{1}, \bar{2}\} .
$$

We are going to start with the case where $B = \delta \in \{y, \bar{y}\}$. We have

$$
\begin{aligned}
\partial_t \mathcal{F}^{t\delta} + \nabla_\alpha \mathcal{F}^{\alpha\delta} + \left[\mathcal{A}_\alpha, \mathcal{F}^{\alpha\delta}\right] + \nabla_c \mathcal{F}^{c\delta} + \left[\mathcal{A}_c, \mathcal{F}^{c\delta}\right] = 0, & \\
c \in \{1, 2, \bar{1}, \bar{2}\} , \; \alpha \in \{y, \bar{y}\} . & \qquad \text{(A.31)}
\end{aligned}
$$

We know that the Φ_m contain only maps from $\mathbb{R}$ to $\mathbb{R}$, so all other derivatives of these fields and hence of the gauge field strength vanish (all but the derivative of components of $\mathcal{F}$ containing

b). If we use

$$\nabla_A \mathcal{F}^{A\delta} = \Gamma^A_{AC} \mathcal{F}^{C\delta} + \Gamma^\delta_{AC} \mathcal{F}^{AC} ,$$

then we get the remaining equation for (A.53):

$$\partial_\alpha \mathcal{F}^{\alpha\delta} + \Gamma^\alpha_{\alpha C} \mathcal{F}^{C\delta} + \Gamma^\delta_{\alpha C} \mathcal{F}^{\alpha C} + \Gamma^c_{cC} \mathcal{F}^{C\delta} + \Gamma^\delta_{cC} \mathcal{F}^{cC} = 0 .$$

We notice, that the Christoffel symbols with mixed indices from $\mathbb{C}P^1$ and $\mathbb{C}P^2$ vanish, and therefore we get the simplification

$$\partial_\alpha \mathcal{F}^{\alpha\delta} + \Gamma^\alpha_{\alpha\epsilon} \mathcal{F}^{\epsilon\delta} + \Gamma^\delta_{\alpha\epsilon} \mathcal{F}^{\alpha\epsilon} + \left[\mathcal{A}_\alpha, \mathcal{F}^{\alpha\delta}\right] + \left[\mathcal{A}_d, \mathcal{F}^{d\delta}\right] + \left[\mathcal{A}_{\bar{d}}, \mathcal{F}^{\bar{d}\delta}\right] + \Gamma^c_{cb} \mathcal{F}^{b\delta} = 0.$$

We can see from (A.28) that the only non-vanishing components $\mathcal{F}^{\alpha\epsilon}$ are $\mathcal{F}^{\bar{y}y}$ and $\mathcal{F}^{y\bar{y}}$. We also know that the only non-vanishing Christoffel symbols are Γ^y_{yy} and $\Gamma^{\bar{y}}_{\bar{y}\bar{y}}$. Hence, it remains

$$\partial_{\bar{\delta}} \mathcal{F}^{\bar{\delta}\delta} + \Gamma^{\bar{\delta}}_{\bar{\delta}\bar{\delta}} \mathcal{F}^{\bar{\delta}\delta} + 0 + \left[\mathcal{A}_{\bar{\delta}}, \mathcal{F}^{\bar{\delta}\delta}\right] + \left[\mathcal{A}_d, \mathcal{F}^{d\delta}\right] + \left[\mathcal{A}_{\bar{d}}, \mathcal{F}^{\bar{d}\delta}\right] + \Gamma^c_{cb} \mathcal{F}^{b\delta} = 0.$$

Here, of course, we do not sum over δ. It is either $\delta = y$ or $\delta = \bar{y}$. The index d is supposed to take values in $\{1, 2\}$ and $b \in \{1, 2, 3, 4\}$.

The last ingredient, we need to calculate the Christoffel symbols on $\mathbb{C}P^1$ and $\mathbb{C}P^2$. On $\mathbb{C}P^1$ we are using the complex basis $\{y, \bar{y}\}$ rather than the invariant basis. Since the only non-vanishing components of the metric are $g_{y\bar{y}} = g_{\bar{y}y}$, we get

$$\begin{aligned} \Gamma^y_{yy} &= g^{\bar{y}y}\, \partial_y\, g_{y\bar{y}} = \rho^{-2}\, \partial_y\, \rho^2 = \rho^{-2}\, \left(-\frac{2\bar{y}}{\sqrt{2}R^2}\right)\rho^3 = -\frac{2\bar{y}}{\sqrt{2}R^2}\rho \\ \Gamma^{\bar{y}}_{\bar{y}\bar{y}} &= \overline{\Gamma^y_{yy}} = -\frac{2y}{\sqrt{2}R^2}\rho . \end{aligned} \tag{A.32}$$

Now we are going to express the Christoffel symbols in terms of the $b_y, b_{\bar{y}}$:

$$b_y = \frac{\bar{y}}{2}\left(\frac{1}{R^2 + y\bar{y}}\right) \cdot \frac{\sqrt{2}R^2}{\sqrt{2}R^2} = \frac{1}{2}\frac{\bar{y}}{\sqrt{2}R^2}\rho = -\frac{1}{4}\Gamma^y_{yy} \tag{A.33}$$

$$b_{\bar{y}} = -\frac{y}{2}\left(\frac{1}{R^2 + y\bar{y}}\right) \cdot \frac{\sqrt{2}R^2}{\sqrt{2}R^2} = -\frac{1}{2}\frac{y}{\sqrt{2}R^2}\rho = \frac{1}{4}\Gamma^{\bar{y}}_{\bar{y}\bar{y}} . \tag{A.34}$$

For $\mathbb{C}P^2$ we are using the invariant basis and can therefore use the convenient shape of the Christoffel symbols, making use of the structure constants:

$$\Gamma^c_{cb} = -e^i{}_c f_{bi}{}^c ,$$

where the $e^i{}_c$ as well as the structure constants can be taken from section XI.3.

It will again turn out to be useful to split the gauge potential into its block-diagonal and off-diagonal parts as done before:

$$\mathcal{A}^{\text{diag}} \;:=\; b_{(m)} \otimes \mathbb{1}_3 + \mathbb{1}_{m+1} \otimes \begin{pmatrix} B & 0 \\ 0 & -2a \end{pmatrix}, \tag{A.35}$$

$$\mathcal{A}^{\text{off}} \;:=\; \frac{\rho}{2}(\Phi_m \mathrm{d}\bar{y} - \Phi_m^\dagger \mathrm{d}y) \otimes \mathbb{1}_3 + \Phi_m \otimes \begin{pmatrix} 0_2 & \bar{\beta} \\ -\beta^\top & 0 \end{pmatrix}. \tag{A.36}$$

We get

$$0 \;=\; \partial_t \mathcal{F}^{t\delta} \tag{A.37}$$

$$+\; \partial_{\bar{\delta}} \mathcal{F}^{\bar{\delta}\delta} \tag{A.38}$$

$$+\; \Gamma^{\bar{\delta}}_{\bar{\delta}\bar{\delta}} \mathcal{F}^{\bar{\delta}\delta} + \left[\mathcal{A}^{\text{diag}}_{\bar{\delta}}, \mathcal{F}^{\bar{\delta}\delta}\right] \tag{A.39}$$

$$+\; \left[\mathcal{A}^{\text{off}}_{\bar{\delta}}, \mathcal{F}^{\bar{\delta}\delta}\right] \tag{A.40}$$

$$+\; \left[\mathcal{A}^{\text{diag}}_{d}, \mathcal{F}^{d\delta}\right] + \left[\mathcal{A}^{\text{diag}}_{\bar{d}}, \mathcal{F}^{\bar{d}\delta}\right] - e^i{}_c f_{di}{}^c \mathcal{F}^{d\delta} - e^i{}_c f_{\bar{d}i}{}^c \mathcal{F}^{\bar{d}\delta} \tag{A.41}$$

$$+\; \left[\mathcal{A}^{\text{off}}_{d}, \mathcal{F}^{d\delta}\right] + \left[\mathcal{A}^{\text{off}}_{\bar{d}}, \mathcal{F}^{\bar{d}\delta}\right] \tag{A.42}$$

which we are going to calculate piecewise in the next few paragraphs.

A.2.1 Piece (A.38) from the Yang-Mills equations. We start evaluating term (A.38). We have to take into account that the metric with respect to the complex basis $\{y, \bar{y}\}$ is not constant and since $\partial_{\bar{\delta}} \rho^{-n} = \frac{n \cdot \delta}{\sqrt{2}R^2} \rho^{-n+1}$, (A.38) becomes

$$\text{(A.38)} \;=\; b_{\bar{\delta}}\; \rho^{-4}\; 4\; \rho^2 \mathcal{F}_{\bar{y}y}\,. \tag{A.43}$$

Here we made use of $b_{\bar{\delta}} = \mp \frac{\delta}{2\sqrt{2}R^2}\; \rho$, where the upper sign stands for $\delta = y$ and the lower one for $\delta = \bar{y}$.

A.2.2 Piece (A.39) from the Yang-Mills equations. In order to calculate this piece of the Yang-Mills equations, we express the Christoffel symbols in terms of $b_{\bar{\delta}}$:

$$\Gamma^{\bar{\delta}}_{\bar{\delta}\bar{\delta}} = \pm 4\; b_{\bar{\delta}}\,.$$

Using this, (A.39) becomes

$$\begin{aligned} \text{(A.39)} \;=\; & \pm 4\; b_{\bar{\delta}} \rho^{-4} \left(\mp \rho^2 \mathcal{F}_{\bar{y}y}\right) \otimes \mathbb{1}_3 + \\ & \mp \rho^{-4} \left[\Upsilon_m b_{\bar{\delta}}, \rho^2 \left(\frac{1}{4}\left[\Phi_m^\top, \Phi_m\right] + \frac{1}{2R^2}\Upsilon_m\right)\right] \otimes \mathbb{1}_3\,. \end{aligned}$$

Since the first two terms inside the second argument of the commutator are diagonal matrices, they commute with Υ_m. So the remaining part of (A.39) is

$$(A.39) \;=\; -4\rho^{-2}\; b_{\bar{\delta}} \mathcal{F}_{\bar{y}y}\,. \tag{A.44}$$

A.2.3 Piece (A.40) from the Yang-Mills equations. We are now considering piece (A.40) from the Yang-Mills equation:

$$(\text{A.40}) = \left(\left[\mathcal{A}^{\text{off}}_{\bar\delta},\mathcal{F}^{\delta\bar\delta}\right]\right) = \left(\rho^{-4}\left[\mathcal{A}^{\text{off}}_{\bar\delta},\mathcal{F}_{\delta\bar\delta}\right]\right) = \rho^{-4}\begin{pmatrix}\left[\mathcal{A}^{\text{off}}_{\bar y},\mathcal{F}_{y\bar y}\right]\\ \left[\mathcal{A}^{\text{off}}_{y},\mathcal{F}_{\bar y y}\right]\end{pmatrix}$$

$$= \rho^{-4}\begin{pmatrix}\left\{\left[\frac{1}{2}\rho\;\Phi_m\otimes\mathbb{1}_3, -\left(\rho^2\;\left(\frac{1}{4}\left[\Phi_m^\top,\Phi_m\right]+\frac{1}{2R^2}\Upsilon_m\right)\right]\right\}\otimes\mathbb{1}_3\\ \left\{\left[-\frac{1}{2}\rho\;\Phi_m^\top\otimes\mathbb{1}_3, +\left(\rho^2\;\left(\frac{1}{4}\left[\Phi_m^\top,\Phi_m\right]+\frac{1}{2R^2}\Upsilon_m\right)\right]\right\}\otimes\mathbb{1}_3\end{pmatrix}$$

$$= -\frac{1}{2}\rho^{-3}\begin{pmatrix}\left\{\left[\Phi_m\otimes\mathbb{1}_3, +\left(\rho^2\;\left(\frac{1}{4}\left[\Phi_m^\top,\Phi_m\right]+\frac{1}{2R^2}\Upsilon_m\right)\right]\right\}\otimes\mathbb{1}_3\\ \left\{\left[+\Phi_m^\top\otimes\mathbb{1}_3, \left(\rho^2\;\left(\frac{1}{4}\left[\Phi_m^\top,\Phi_m\right]+\frac{1}{2R^2}\Upsilon_m\right)\right]\right\}\otimes\mathbb{1}_3\end{pmatrix}$$

$$= -\frac{1}{2}\rho^{-3}\begin{pmatrix}\left\{\frac{1}{4}\rho^2\left[\Phi_m,\left[\Phi_m^\top,\Phi_m\right]\right]+\frac{\rho^2}{2R^2}\left[\Phi_m,\Upsilon_m\right]\right\}\otimes\mathbb{1}_3\\ \left\{-\frac{1}{4}\rho^2\left[\Phi_m^\top,\left[\Phi_m,\Phi_m^\top\right]\right]+\frac{\rho^2}{2R^2}\left[\Phi_m^\top,\Upsilon_m\right]\right\}\otimes\mathbb{1}_3\end{pmatrix}$$

$$= -\frac{1}{2}\rho^{-3}\begin{pmatrix}\left\{\frac{1}{4}\rho^2\left[\Phi_m,\left[\Phi_m^\top,\Phi_m\right]\right]+\frac{\rho^2}{2R^2}(-2)\cdot\Phi_m\right\}\otimes\mathbb{1}_3\\ \left\{-\frac{1}{4}\rho^2\left[\Phi_m^\top,\left[\Phi_m,\Phi_m^\top\right]\right]+\frac{\rho^2}{2R^2}(+2)\cdot\Phi_m^\top\right\}\otimes\mathbb{1}_3\end{pmatrix}$$

$$= -\frac{1}{2}\rho^{-3}\begin{pmatrix}\left\{\frac{1}{4}\rho^2\left[\Phi_m,\left[\Phi_m^\top,\Phi_m\right]\right]-\frac{\rho^2}{R^2}\Phi_m\right\}\otimes\mathbb{1}_3\\ \left\{-\frac{1}{4}\rho^2\left[\Phi_m^\top,\left[\Phi_m,\Phi_m^\top\right]\right]+\frac{\rho^2}{R^2}\Phi_m^\top\right\}\otimes\mathbb{1}_3\end{pmatrix}. \tag{A.45}$$

A.2.4 Piece (A.41) from the Yang-Mills equations. This piece of the Yang-Mills equations dissapears, due to the fact that

$$\left[\begin{pmatrix}B_d & 0\\ 0 & -2a_d\end{pmatrix},\begin{pmatrix}0 & \bar\beta_{\bar d}\\ 0 & 0\end{pmatrix}\right] - e^i{}_c f_{di}{}^c\begin{pmatrix}0 & \bar\beta_{\bar d}\\ 0 & 0\end{pmatrix} = 0 \tag{A.46}$$

and

$$\left[\begin{pmatrix}B_{\bar d} & 0\\ 0 & -2a_{\bar d}\end{pmatrix},\begin{pmatrix}0 & 0\\ \beta_d^\top & 0\end{pmatrix}\right] - e^i{}_c f_{di}{}^c\begin{pmatrix}0 & 0\\ \beta_d^\top & 0\end{pmatrix} = 0\,. \tag{A.47}$$

A.2.5 Piece (A.42) from the Yang-Mills equations.

$$(\text{A.42}) = \left[\mathcal{A}^{\text{off}}_b,\rho^{-2}\mathcal{F}_{\bar b\bar\delta}\right] = \rho^{-2}\left[\mathcal{A}^{\text{off}}_b,-\mathcal{F}_{\bar\delta\bar b}\right]$$

$$= \left[\Psi_m\otimes\begin{pmatrix}0_2 & \bar\beta_{\bar b}\\ -\beta_b^\top & 0\end{pmatrix},\begin{Bmatrix}\rho^{-2}\frac{\rho}{2}\left[\Phi_m,\Psi_m\right]\\ -\rho^{-2}\frac{\rho}{2}\left[\Phi_m^\top,\Psi_m\right]\end{Bmatrix}\otimes\begin{pmatrix}0 & \bar\beta_{\bar b}\\ -\beta_b^\top & 0\end{pmatrix}\right]$$

$$
= \frac{\rho^{-1}}{2} \left\{ \begin{array}{l} +[\Psi_m, [\Phi_m, \Psi_m]] \\ -\left[\Psi_m, \left[\Phi_m^\top, \Psi_m\right]\right] \end{array} \right\} \otimes \begin{pmatrix} -\bar{\beta}_{\bar{d}} \beta_d^\top & 0 \\ 0 & -\beta_d^\top \bar{\beta}_{\bar{d}} \end{pmatrix}
$$

$$
= \frac{\rho^{-1}}{2} \left\{ \begin{array}{l} -[\Psi_m, [\Phi_m, \Psi_m]] \\ +\left[\Psi_m, \left[\Phi_m^\top, \Psi_m\right]\right] \end{array} \right\} \otimes \mathbb{1}_3 \,. \tag{A.48}
$$

The upper (lower) row represents $\delta = y$ ($\delta = \bar{y}$).

A.2.6 Putting all pieces from this part of Yang-Mills equations together. For $B = y$ we arrive therefore at the following two equations for the Φ_m:

$$
\begin{aligned}
0 = & \left(\frac{\rho^{-1}}{2} \partial_t^2 \Phi_m + \rho^{-4}\ b_{\bar{y}}\ \left(4\ \rho^2 \mathcal{F}_{\bar{y}y}\right) + \rho^{-4}\ b_{\bar{y}} \left\{-4\ \rho^2\ \mathcal{F}_{\bar{y}y}\right\} \right. \\
& \left. -\frac{1}{2}\rho^{-3} \left(+\frac{1}{4}\rho^2 \left[\Phi_m, \left[\Phi_m^\top, \Phi_m\right]\right] - \frac{\rho^2}{R^2}\Phi_m \right) \right) \otimes \mathbb{1}_3 \\
& - \frac{\rho^{-1}}{2} [\Psi_m, [\Phi_m, \Psi_m]] \otimes \mathbb{1}_3 \\
= & \left(\frac{\rho^{-1}}{2} \partial_t^2 \Phi_m - \frac{1}{8}\rho^{-1} \left[\Phi_m, \left[\Phi_m^\top, \Phi_m\right]\right] + \frac{\rho^{-1}}{2R^2}\Phi_m \right) \otimes \mathbb{1}_3 \\
& - \frac{\rho^{-1}}{2} [\Psi_m, [\Phi_m, \Psi_m]] \otimes \mathbb{1}_3 \,.
\end{aligned} \tag{A.49}
$$

This is equivalent to

$$
0 = \partial_t^2 \Phi_m - \frac{1}{4} \left[\Phi_m, \left[\Phi_m^\top, \Phi_m\right]\right] + \frac{1}{R^2}\Phi_m - [\Psi_m, [\Phi_m, \Psi_m]] \,. \tag{A.50}
$$

Hence we obtain the following equations for the ϕ_i and ψ_i:

$$
0 = \partial_t^2 \phi_i + \frac{1}{R^2}\phi_i + \frac{1}{4}(\phi_{i-1}^2 - 2\phi_i^2 + \phi_{i+1}^2)\phi_i - \left(\psi_{i+1}^2 - 2\psi_{i+1}\psi_i + \psi_i^2\right)\phi_i \,. \tag{A.51}
$$

A.3 Yang-Mills Equations for $B \in \{1, 2, \bar{1}, \bar{2}\}$

We will now derive the equations for the Φ_m and ϕ_i for the case that the free index in the Yang-Mills-equation is an index $B = d \in \{1, 2, \bar{1}, \bar{2}\}$ of the coset space:

$$
\begin{aligned}
\partial_t \mathcal{F}^{td} + \nabla_\alpha \mathcal{F}^{\alpha d} + \left[\mathcal{A}_\alpha, \mathcal{F}^{\alpha d}\right] + \nabla_c \mathcal{F}^{cd} + \left[\mathcal{A}_c, \mathcal{F}^{cd}\right] = 0 \\
c \in \{1, 2, \bar{1}, \bar{2}\}\,,\ \alpha \in \{y, \bar{y}\}
\end{aligned} \tag{A.52}
$$

which is equivalent to

$$
0 = \partial_t \mathcal{F}^{td} + \nabla_\alpha \mathcal{F}^{\alpha d} + \left[\mathcal{A}_\alpha, \mathcal{F}^{\alpha d}\right] + \nabla_c \mathcal{F}^{cd} + \left[\mathcal{A}_c, \mathcal{F}^{cd}\right] \tag{A.53}
$$

$$\Leftrightarrow 0=\partial_t\mathcal{F}^{td}+\partial_\alpha\mathcal{F}^{\alpha d}+\Gamma^\alpha_{\alpha C}\mathcal{F}^{Cd}+\Gamma^d_{\alpha C}\mathcal{F}^{\alpha C}+\left[\mathcal{A}_\alpha,\mathcal{F}^{\alpha d}\right] \tag{A.54}$$

$$+\Gamma^c_{cC}\mathcal{F}^{Cd}+\Gamma^d_{cC}\mathcal{F}^{cC}+\left[\mathcal{A}_c,\mathcal{F}^{cd}\right]. \tag{A.55}$$

Hence

$$\Leftrightarrow 0 \;=\; \partial_t\mathcal{F}^{td} \tag{A.56}$$

$$+\partial_\alpha\mathcal{F}^{\alpha d}+\Gamma^y_{yy}\mathcal{F}^{yd}+\Gamma^{\bar y}_{\bar y\bar y}\mathcal{F}^{\bar y d}+\left[\mathcal{A}^{\rm diag}_\alpha,\mathcal{F}^{\alpha d}\right] \tag{A.57}$$

$$+\left[\mathcal{A}^{\rm off}_\alpha,\mathcal{F}^{\alpha d}\right] \tag{A.58}$$

$$+\Gamma^c_{cb}\mathcal{F}^{bd}+\Gamma^d_{cb}\mathcal{F}^{cb}+\left[\mathcal{A}^{\rm diag}_c,\mathcal{F}^{cd}\right] \tag{A.59}$$

$$+\left[\mathcal{A}^{\rm off}_c,\mathcal{F}^{cd}\right]. \tag{A.60}$$

After some calculations, one can see that (A.59) vanishes. So in the following we calculate the remaining pieces.

A.3.1 Second piece (A.57) from the Yang-Mills equations.

$$\begin{aligned}
(\text{A.57}) &= \partial_\alpha\left(\rho^{-2}\mathcal{F}_{\bar\alpha\bar d}\right)+\Gamma^y_{yy}\;\rho^{-2}\;\mathcal{F}_{\bar y\bar d}+\Gamma^{\bar y}_{\bar y\bar y}\;\rho^{-2}\;\mathcal{F}_{y\bar d}+\rho^{-2}\;\left[\mathcal{A}^{\rm diag}_\alpha,\mathcal{F}^{\alpha d}\right]\\
&= \rho^{-2}\partial_\alpha\mathcal{F}_{\bar\alpha\bar d}+\mathcal{F}_{\bar\alpha\bar d}\;\partial_\alpha\rho^{-2}-4b_y\;\rho^{-2}\;\mathcal{F}_{\bar y\bar d}+4b_{\bar y}\;\rho^{-2}\;\mathcal{F}_{y\bar d}\\
&\quad+\rho^{-2}\left[b_y\Upsilon_m\otimes\mathbb{1}_3,\frac{\rho}{2}\left[\Phi_m,\Psi_m\right]\otimes\begin{pmatrix}0&\bar\beta\\-\beta^\top&0\end{pmatrix}_{\bar d}\right]\\
&\quad-\rho^{-2}\left[b_{\bar y}\Upsilon_m\otimes\mathbb{1}_3,\frac{\rho}{2}\left[\Phi_m^\top,\Psi_m\right]\otimes\begin{pmatrix}0&\bar\beta\\-\beta^\top&0\end{pmatrix}_{\bar d}\right]\\
&= \rho^{-2}\partial_\alpha\mathcal{F}_{\bar\alpha\bar d}++\mathcal{F}_{\bar\alpha\bar d}\;\frac{2\bar\alpha}{\sqrt{2}R^2}\rho^{-1}-4\rho^{-2}\;\left(b_y\;\mathcal{F}_{\bar y\bar d}-b_{\bar y}\;\mathcal{F}_{y\bar d}\right)\\
&\quad+\rho^{-2}b_y\cdot 2\cdot\frac{\rho}{2}\left[\Phi_m,\Psi_m\right]\otimes\begin{pmatrix}0&\bar\beta\\-\beta^\top&0\end{pmatrix}_{\bar d}\\
&\quad-\rho^{-2}b_{\bar y}\cdot 2\cdot\left(-\frac{\rho}{2}\left[\Phi_m^\top,\Psi_m\right]\otimes\begin{pmatrix}0&\bar\beta\\-\beta^\top&0\end{pmatrix}_{\bar d}\right)\\
&= \rho^{-2}\partial_\alpha\mathcal{F}_{\bar\alpha\bar d}+\rho^{-2}\left(4b_y\;\mathcal{F}_{\bar y\bar d}-4b_{\bar y}\;\mathcal{F}_{y\bar d}\right)-4\rho^{-2}\;\left(b_y\;\mathcal{F}_{\bar y\bar d}-b_{\bar y}\;\mathcal{F}_{y\bar d}\right)\\
&\quad+\rho^{-1}\left(b_y\left[\Phi_m,\Psi_m\right]+b_{\bar y}\left[\Phi_m^\top,\Psi_m\right]\right)\otimes\begin{pmatrix}0&\bar\beta\\-\beta^\top&0\end{pmatrix}_{\bar d}\\
&= \rho^{-2}\partial_\alpha\mathcal{F}_{\bar\alpha\bar d}+\rho^{-1}\left(b_y\left[\Phi_m,\Psi_m\right]+b_{\bar y}\left[\Phi_m^\top,\Psi_m\right]\right)\otimes\begin{pmatrix}0&\bar\beta\\-\beta^\top&0\end{pmatrix}_{\bar d}.
\end{aligned} \tag{A.61}$$

The last thing to do for this piece is to calculate the partial derivative from the first term of (A.61),

$$
\begin{aligned}
&\rho^{-2}\partial_\alpha \mathcal{F}_{\bar{\alpha}\bar{d}} = \\
&= \rho^{-2}\ \partial_y \mathcal{F}_{\bar{y}\bar{d}} + \rho^{-2}\ \partial_y \mathcal{F}_{\bar{y}\bar{d}} \\
&= \rho^{-2}\left(\partial_y \frac{1}{2}\rho\,[\Phi_m, \Psi_m] \otimes \begin{pmatrix} 0 & \bar{\beta} \\ -\beta^\top & 0 \end{pmatrix}_{\bar{d}} - \partial_{\bar{y}} \frac{1}{2}\rho\left[\Phi_m^\top, \Psi_m\right] \otimes \begin{pmatrix} 0 & \bar{\beta} \\ -\beta^\top & 0 \end{pmatrix}_{\bar{d}}\right) \\
&= \rho^{-2}(\underbrace{-\frac{1}{2}\frac{\bar{y}}{\sqrt{2}R^2}}_{=-b_y}\rho\cdot\rho\,[\Phi_m, \Psi_m] + \underbrace{\frac{1}{2}\frac{y}{\sqrt{2}R^2}}_{=-b_{\bar{y}}}\rho\cdot\rho\left[\Phi_m^\top, \Psi_m\right]) \otimes \begin{pmatrix} 0 & \bar{\beta} \\ -\beta^\top & 0 \end{pmatrix}_{\bar{d}} \\
&= \rho^{-1}\left(-b_y\,[\Phi_m, \Psi_m] - b_{\bar{y}}\left[\Phi_m^\top, \Psi_m\right]\right) \otimes \begin{pmatrix} 0 & \bar{\beta} \\ -\beta^\top & 0 \end{pmatrix}_{\bar{d}}.
\end{aligned} \tag{A.62}
$$

Therefore (A.61) vanishes and hence we have

$$
(\text{A.57}) = 0\,. \tag{A.63}
$$

A.3.2 Third piece (A.58) from the Yang-Mills equations.

$$
\begin{aligned}
(\text{A.58}) &= \left[\mathcal{A}_y^{\text{off}}, \rho^{-2}\mathcal{F}_{\bar{y}\bar{d}}\right] + \left[\mathcal{A}_{\bar{y}}^{\text{off}}, \rho^{-2}\mathcal{F}_{y\bar{d}}\right] \\
&= \left[-\frac{1}{2}\rho\Phi^\top \otimes \mathbb{1}_3, \rho^{-2}\cdot\rho\cdot\frac{1}{2}\,[\Phi_m, \Psi_m] \otimes \begin{pmatrix} 0 & \bar{\beta} \\ -\beta^\top & 0 \end{pmatrix}_{\bar{d}}\right] \\
&\quad + \left[\frac{1}{2}\rho\Phi \otimes \mathbb{1}_3, \rho^{-2}\cdot\rho\cdot\frac{1}{2}\left[\Phi_m^\top, \Psi_m\right] \otimes \begin{pmatrix} 0 & \bar{\beta} \\ -\beta^\top & 0 \end{pmatrix}_{\bar{d}}\right] \\
&= -\frac{1}{4}\left(\left[\Phi_m^\top, [\Phi_m, \Psi_m]\right] + \left[\Phi_m, \left[\Phi_m^\top, \Psi_m\right]\right]\right) \otimes \begin{pmatrix} 0 & \bar{\beta} \\ -\beta^\top & 0 \end{pmatrix}_{\bar{d}} \\
&= -\frac{1}{2}\left(\left[\Phi_m^\top, [\Phi_m, \Psi_m]\right]\right) \otimes \begin{pmatrix} 0 & \bar{\beta} \\ -\beta^\top & 0 \end{pmatrix}_{\bar{d}}.
\end{aligned} \tag{A.64}
$$

In the last step, we made use of the fact that $\left[\Phi_m^\top, [\Phi_m, \Psi_m]\right]$ is a diagonal matrix.

A.3.3 Piece (A.60) from the Yang-Mills equations. There are only components of $\mathcal{F}$ with one bar index and one without a bar. Since $d \in \{1, 2, \bar{1}, \bar{2}\}$, we get

$$
(\text{A.60}) = \left[\mathcal{A}_c^{\text{off}}, \mathcal{F}^{cd}\right]
$$

$$
\begin{aligned}
&= \left[\Psi_m \otimes \begin{pmatrix} 0_2 & \bar\beta_c \\ -\beta_c^\top & 0 \end{pmatrix}, \Psi_m^2 \otimes \begin{pmatrix} (-\bar\beta\wedge\beta^\top)_{\bar c\bar d} & 0 \\ 0 & (\beta^\dagger\wedge\beta)_{\bar c\bar d} \end{pmatrix}\right] \\
&\quad - \left[\Psi_m \otimes \begin{pmatrix} 0_2 & \bar\beta_c \\ -\beta_c^\top & 0 \end{pmatrix}, \mathbb{1}_{m+1} \otimes \begin{pmatrix} -(\bar\beta\wedge\beta^\top)_{\bar c\bar d} & 0 \\ 0 & (\beta^\dagger\wedge\beta)_{\bar c\bar d} \end{pmatrix}\right] \\
&\quad \left(\Psi_m^3 - \Psi_m\right) \otimes \left[\begin{pmatrix} 0_2 & \bar\beta_c \\ -\beta_c^\top & 0 \end{pmatrix}, \begin{pmatrix} (-\bar\beta\wedge\beta^\top)_{\bar c\bar d} & 0 \\ 0 & (\beta^\dagger\wedge\beta)_{\bar c\bar d} \end{pmatrix}\right] \\
&= \left(\Psi_m^3 - \Psi_m\right) \otimes (-3) \begin{pmatrix} 0_2 & \bar\beta_{\bar c} \\ -\beta_{\bar c}^\top & 0 \end{pmatrix} \\
&= 3\,\left(\Psi_m - \Psi_m^3\right) \otimes \begin{pmatrix} 0_2 & \bar\beta_{\bar d} \\ -\beta_{\bar d}^\top & 0 \end{pmatrix} . \qquad \text{(A.65)}
\end{aligned}
$$

A.3.4 Putting all pieces from this part of Yang-Mills equations together.

$$
0 \;=\; \partial_t^2\Psi_m - \frac{1}{2}\left(\left[\Phi_m^\top, [\Phi_m, \Psi_m]\right]\right) + 3\,\left(\Psi_m - \Psi_m^3\right). \qquad \text{(A.66)}
$$

This yields the following equations for the $\phi_i,\ \psi_i$:

$$
\begin{aligned}
\partial_t^2\psi_i - \frac{1}{2}\left(\phi_{i-1}^2(\psi_i - \psi_{i-1}) + \phi_i^2(\psi_{i+1} - \psi_i)\right) + 3(\psi_i - \psi_i^3) \;&=\; 0\,, \\
\forall i = 1, ..., m+1\,. \qquad \text{(A.67)}
\end{aligned}
$$

Here as usual, $\phi_{m+1} := 0 =: \phi_0$.

A.4 All Yang-Mills Equations

If we summarize everything, we have the following equations for the $\Phi_m,\ \Psi_m$:

$$
\begin{aligned}
0 \;&=\; \partial_t^2\Phi_m - \frac{1}{4}\left[\Phi_m, \left[\Phi_m^\top, \Phi_m\right]\right] + \frac{1}{R^2}\Phi_m - [\Psi_m, [\Phi_m, \Psi_m]] \\
0 \;&=\; \partial_t^2\Psi_m - \frac{1}{2}\left[\Phi_m^\top, [\Phi_m, \Psi_m]\right] + 3\,\left(\Psi_m - \Psi_m^3\right).
\end{aligned}
$$

which are equivalent to the component equations

$$
\begin{aligned}
0 \;&=\; \partial_t^2\phi_i + \frac{1}{4}(\phi_{i-1}^2 - 2\phi_i^2 + \phi_{i+1}^2)\phi_i + \frac{1}{R^2}\phi_i - \left(\psi_{i+1}^2 - 2\psi_{i+1}\psi_i + \psi_i^2\right)\phi_i\,, \\
0 \;&=\; \partial_t^2\psi_i - \frac{1}{2}\left(\phi_{i-1}^2(\psi_i - \psi_{i-1}) + \phi_i^2(\psi_{i+1} - \psi_i)\right) + 3(\psi_i - \psi_i^3)\,.
\end{aligned}
$$

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