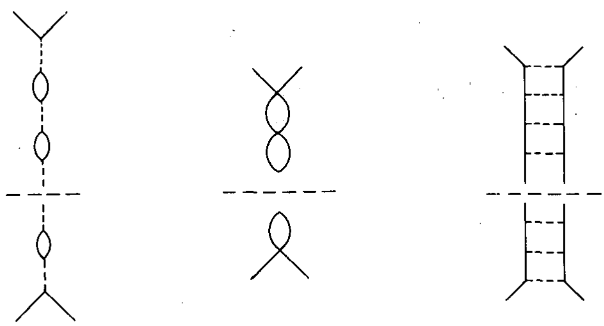
COMPOUND PARTICLE MODELS *

W. THIRRING
INSTITUTE FOR THEORETICAL PHYSICS, UNIVERSITY OF VIENNA,
VIENNA, AUSTRIA

1. INTRODUCTION

In these lectures I would like to talk on bound states and resonances in quantum field theory [1]. I will assume that you are familiar with this problem in potential scattering and will investigate similar problems in field theory. The difficulty is that the only systematic method of calculation in field theory is the perturbation theory. If you look at the analytical properties of single graphs the poles and singularities which correspond to bound states and resonances do not appear. Therefore one has to do something better and I will consider a particular model, the so-called ZACHARIASEN model [2] which is essentially the summing up of the chain-diagrams (Fig. 1a and 1b).



a) b) c)

Fig. 1a, 1b

Typical chain diagrams

Fig. 1c

Typical ladder diagrams

In principle one could think also of the summing up of ladder diagrams (Fig. 2). They are, however, much more complicated and we will at first consider the chain.

The summing up of the chain leads to a unitary S-matrix which is by no means trivial, since one has taken out only special types of diagrams from the whole perturbation expansion. It also satisfies analytically, but not crossing-symmetry.

The model can either be considered as a prescription to select a certain type of diagram in ordinary perturbation theory, or in the framework of

* Text based on notes, by K. Chadan, F. Hadjioannou, A.N. Mitra, H. Mitter, H. Pietschmann, T.K. Radha, E. van der Spuy.

dispersion theory, as a restriction on certain intermediate states, or as the exact solution of a Lagrangian field theory, in which the pairs of particles forming the bubble in Fig. 1a occur as a field with continuous mass spectrum [3]. The scattering amplitude T depends in this model only on one variable or, in other words, we are dealing only with the interaction of particles in a special angular momentum state the moment of which we take to be 0. The only possible justification of the approximation, which this model represents, comes from the field theoretic treatment of the many-body problem, where the bubble-summation leads to physically significant results (plasma oscillations of an electron gas). We shall discuss it in some detail since it reflects several features which are conjectured for a full-fledged field theory.

2. THE SCATTERING AMPLITUDE

In the model we have two particles, one represented by ψ (solid line) with mass $\frac{1}{2}$, the other represented by ϕ (broken line) with mass μ_0, both with spin 0 (the generalization to spin $\frac{1}{2}$ for ψ will be considered later). In the language of a Lagrangian formalism we have to consider the interaction term:

$$L' = -\lambda_0 \psi^4 + g_0 \phi \psi^2 \tag{1}$$

(note that a negative λ_0 means attraction, a positive one repulsion) and have to sum up the diagrams shown in Fig. 2:

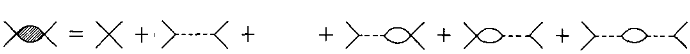


Fig. 2

$$T(s) = \lambda_0 + \frac{g_0^2}{s-\mu_0^2} + \lambda_0 \Delta(s)\lambda_0 + \frac{g_0^2}{s-\mu_0^2}\Delta(s)\lambda_0 + \lambda_0\,\Delta(s)\frac{g_0^2}{s-\mu_0^2} + \frac{g_0^2}{s-\mu_0^2}\,\Delta\,\frac{g_0^2}{s-\mu_0^2}\,.$$

$\Delta(s)$ is an abbreviation for the divergent bubble; which is in co-ordinate space the product of two scalar Feynman propagators with the same argument. This product can be written as a weighted integral over single propagators [4]. In momentum-space this means:

$$\Delta(s) = -\int_1^\infty \frac{W ds'}{s'-s+i\epsilon} \qquad \text{with} \qquad W = \frac{1}{16\pi}\sqrt{\frac{s'-s}{s'}} \tag{3), (4}$$

which is logarithmically divergent. We extract the singular part by making one subtraction as s = 0:

$$\Delta(s) = \Delta(0) + \Delta(s) - \Delta(0) = \Delta(0) - s \int_1^\infty \frac{W ds'}{s'(s'-s)} \quad . \tag{5}$$

$\Delta(0) = -\overline{B}$ is an infinite constant and the remaining integral is convergant (note that $\overline{B}$ is positive).

It can be seen immediately that (2) is a geometrical series which can be summed up to yield:

$$T(s) = [\lambda_0 + g_0^2/(s-\mu_0^2)] \left\{ 1 - [\lambda_0 + g_0^2/(s-\mu_0^2)] \ \Delta(s) \right\}^{-1}. \tag{6}$$

Introducing (5) this can be written in the form:

$$T(s) = \overline{\lambda} / [1 + \overline{\lambda} s \int_1^\infty \frac{ds' W}{s'(s'-s)} + \frac{sR}{\overline{\mu}^2(s-\overline{\mu}^2)}] = \overline{\lambda} / \overline{D}(s) \tag{7}$$

where:

$$\overline{\lambda} = T(0) = \lambda_0 / [1 + B\lambda_0 + g_0^2/(\lambda_0 \mu_0^2 - g_0^2)]$$
$$R = -\overline{\lambda} g_0^2 / \lambda_0^2 \ ; \ \overline{\mu}^2 = \mu_0^2 - g_0^2/\lambda_0 . \tag{8}$$

The form (7) shows explicitly that $T \to \lambda_0$ as $s \to \infty$.

3. POLES OF T(s)

We now want to investigate the explicit form of T(s), specially its poles, which will give us information on bound states and resonances. We therefore evaluate the integral occurring in $\overline{D}(s)$:

$$A \equiv s \int_1^\infty \frac{ds' \sqrt{1-1/s'}}{s'(s'-s)} = 2 - \sqrt{\frac{s-1}{s}} \ \ln \frac{1+\sqrt{s/(s-1)}}{1-\sqrt{s/(s-1)}} \tag{9}$$

for
$s \to -\infty \qquad A \to 2 - \ln 2|s|$
$s \to 0 \qquad A \to \frac{2}{3} s(s+1)$
$s \to 1 \qquad A \to 2 - i\pi\sqrt{s-1} - 2(s-1)$
$s \to +\infty \qquad A \to 2 - \ln 2s - i\pi$.

It should be noticed, that for $s > 1$, A becomes complex, since the denominators have to be taken with small imaginary parts in order to exhibit the properties of Feynman propagators. The real part has a cusp for $s = 1$.

We now discuss the zeros of D and take for granted, that a zero for $s < 0$ means a ghost, for $0 < s < 1$ means a bound state and for $s > 1$ means

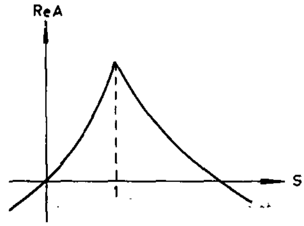


Fig. 3

Re A(s) from Eq. (9)

a resonance. We shall call resonance everything where the phase shift goes through 90°. This will, however, be discussed in more detail later on. We first consider the case where there is no ϕ -field and hence $g_0^2 = R = 0$. It can be seen from the first part of Eq. (8) that, if $\bar{\lambda}$ is to be finite, λ_0 must approach 0 from negative values, i.e. for $B \to \infty$ we have to start with weak attractive interaction.

The situation for negative $\bar{\lambda}$ is plotted in Fig. 4 for two typical cases. Since Re D has a cusp at s = 1 there can be either no bound-state and no resonance ($\bar{\lambda}_1$) or one bound state and one resonance ($\bar{\lambda}_2$). This situation is familiar

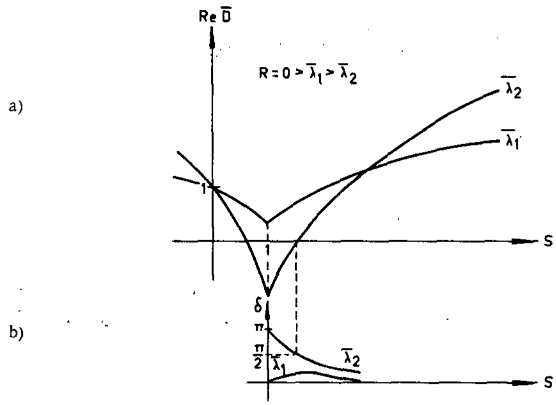


Fig. 4

a) Re $\bar{D}$ for $g_0 = 0$ for two typical negative values of $\bar{\lambda}$.
b) Phase-shift for the same cases

from s-wave attractive potential scattering (for p-waves the situation is different. There one can have two resonances and one bound state). If $\bar{\lambda}$ is bigger than zero, we will get a ghost and a resonance, as is seen from Fig. 5. We will therefore not consider this case.

The situation is similar to potential scattering with weak attractive potential, where the phase shift starts negative, unless a bound state has already been formed, in which case it changes its sign.

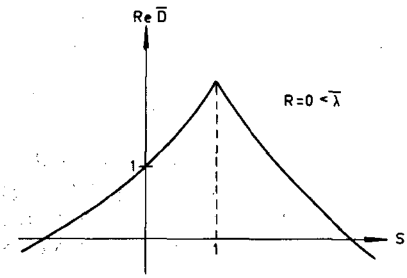


Fig. 5

Re $\bar{D}$ for g_0 = 0 and positive $\bar{\lambda}$ (occurrence of a ghost)

Another possibility is R $\neq$ 0, in which case we also restrict $\bar{\lambda} < 0$, (otherwise we have again a ghost), which implies also R > 0 (compare (8)). Because of the additional term in $\bar{D}(s)$ the real part of $\bar{D}$ has to change sign, since it eventually goes to - ∞ at s = $\bar{\mu}^2$. We have again plotted two typical cases (Fig. 6a). One gets either one resonance ($\bar{\lambda}_1$) or one bound state ($\bar{\lambda}_2$) alone,

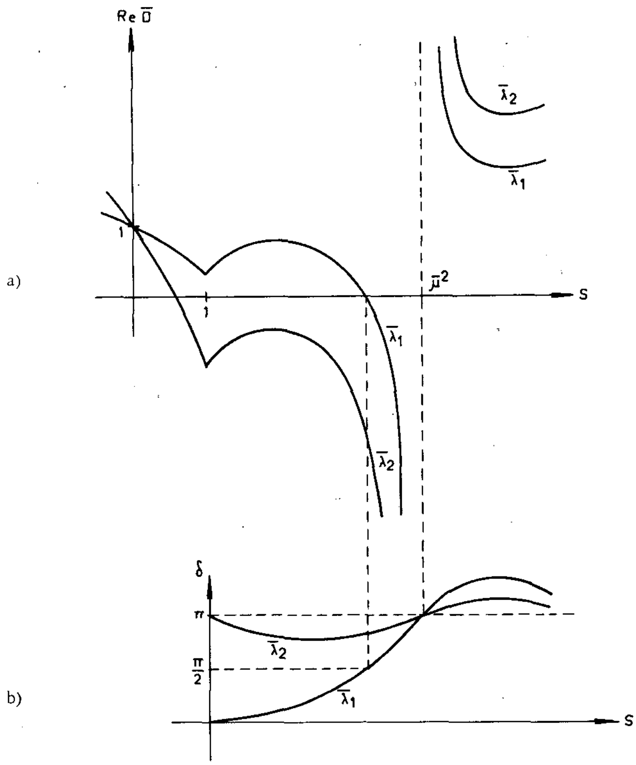


Fig. 6a

Re $\bar{D}$ for g_0 = 0 for two typical negative values of $\bar{\lambda}$.

Fig. 6b

Phase-shift for the same cases

(or with two resonances), depending on the magnitude of $\bar{\lambda}$. $(\lambda_1 < \lambda_2 < 0)$. Finally we investigate the connection with the phase shift

$$\mathrm{tg}\,\delta(s) = (16/\bar{\lambda}) . \sqrt{s(s-1)}\, \mathrm{Re}\, \bar{D}(s). \tag{10}$$

The corresponding plots are (4b) and (6b). The first case again resembles potential scattering. In the second case, we get a slight generalization of the Levinson theorem: in its usual form, it connects the number of bound states to phase shift difference, while here we have [5]

$$\delta(\infty) - \delta(0) = \pi. \text{ (Number of bare particles - number of physical particles)} \tag{11}$$

where physical particle means a discrete point in the mass spectrum, irrespective of whether there is a bare particle associated with this quantum number or not, i. e. all bound states are counted as physical particles. This can easily be seen from (4b) and (6b).

4. RENORMALIZATION AND DEFINITION OF COUPLING CONSTANTS

At first we will discuss the coupling constants attached to the bound states. We will use the usual definition, which states that the coupling constant is the residue of the corresponding pole of the T-matrix.

We rewrite T in the original unsubtracted form

$$T(s) = \lambda_0 (s - \bar{\mu}^2) / [s - \mu_0^2 - \Delta \lambda_0 (s - \bar{\mu}^2)] = \lambda_0 (s - \bar{\mu}^2) / D(s). \tag{12}$$

(Δ is the original expression (3)).

Since we are dealing with bound states only, we suppose

$$D(\mu^2) = 0 \qquad \text{for} \quad 0 < \mu^2 < 1 \tag{13}$$

(this has to be considered as the definition of the physical mass μ).

It is then better to do the subtraction at the point μ^2 so that we now have the infinite constant

$$B = -\Delta(\mu^2) \rightarrow +\infty \tag{14}$$

(13) reads then

$$\mu^2 - \mu_0^2 + B\lambda_0 (\mu^2 - \bar{\mu}^2) = 0. \tag{15}$$

We now subtract (15) from the denominator of (12) and obtain

$$D(s) - D(\mu^2) = D(s) = (s - \mu^2) \left[1 + \lambda_0 \int_1^\infty \frac{ds' W}{(s' - \mu^2)} \cdot \frac{s' - \mu^2}{(s' - s)} \right]$$

$$= (s - \mu^2)[1 + \lambda_0 (B + (\mu^2 - \bar{\mu}^2) I] + \lambda_0 (s - \mu^2) \int_1^\infty \frac{ds' W (s - \bar{\mu}^2)}{(s' - \mu^2)^2 (s' - s)} \tag{16}$$

where

$$I = \frac{d\Delta(\mu^2)}{d\mu^2} = \int_1^\infty \frac{W ds'}{(s' - \mu^2)} = \frac{1}{16\pi^2 \mu^2}\left[\frac{\text{arc tg}\sqrt{\mu^2/(1-\mu^2)}}{\sqrt{\mu^2(1-\mu^2)}} - 1\right] \quad (17)$$

The limiting cases are

$$I \to (2/3)\,(1/16\,\pi^2) \text{ for } \mu \to 0$$

and

$$I \to 1/(32\pi\sqrt{1-\mu^2}) \text{ for } \mu \to 1.$$

Defining now a new renormalized

$$\lambda = \lambda_0 / \left\{1 + \left[B(\mu^2 - \bar{\mu}^2)\right]\lambda_0\right\}. \quad (18)$$

We can write T in a form which exhibits explicitly the singularity as well as the residue:

$$T(s) = \lambda\,(s - \bar{\mu}^2)/(s - \mu^2)\left[1 + \lambda\,(s - \mu^2)\right]. \quad (19)$$

The renormalized coupling constant for the bound state is, as already mentioned, defined by

$$T = g/(s - \mu^2) \quad \text{if} \quad s \to \mu^2.$$

Hence

$$g^2 = \lambda(\mu^2 - \bar{\mu}^2) \quad (21)$$

and one sees that $\bar{\mu}^2$ can be expressed in the same manner by the unrenormalized and renormalized quantities

$$\bar{\mu}^2 = \mu - g_0^2/\lambda_0 = \mu^2 - g^2/\lambda.$$

From this one can infer several interesting points. The first provides us with limits for g. We have

$$1/g^2 = I + (1/g_0^2)\left[(\bar{\mu}^2 - \mu_0^2)/(\bar{\mu}^2 - \mu^2)\right]^2 \geq I. \quad (22)$$

In the case of a pure bound state the equality sign is valid, (case will be shown below), so that in general g^2 is not arbitrary but restricted, its maximum value being $\frac{1}{I}$, and is attained for a bound particle. Furthermore λ is again restricted to be negative otherwise T(s) has a role for $s < \mu^2$. From (21) it then follows that $\bar{\mu}^2 > \mu^2$. After noticing these restrictions on the renormalized quantities, we ask whether the whole procedure is consistent.

Imposing the condition that all renormalized quantities (g, μ, λ) are finite and only B tends to infinity, we end up with a positive g_0^2 tending to zero

$$g_0^2 = g^2 (1 - Ig^2)/(1 - \lambda B - Iq^2) \longrightarrow 0^+$$

and in addition we have

$$\begin{aligned} \mu_0^2 &= \mu^2 - (g^2/\lambda) + (g^2/\lambda)(1 - Ig^2)/(1 - \lambda B - Ig^2) \rightarrow \bar{\mu}^2 \\ \lambda_0 &= \lambda/(1 - \lambda B - Iq^2) \rightarrow 0^- . \end{aligned} \tag{23}$$

5. SPECIAL CASES

Let us consider some special limiting cases of the above: where either λ_0 or $g_0 = 0$. In both cases $\bar{\mu}^2$ tends to infinity and the T matrix (19) in terms of the proper renormalized quantities then reads as follows:

$$T(s) = g^2 \Big/ \left\{ (s - \mu^2) \left[1 + (s - \mu^2) g^2 \int_1^\infty \frac{ds' W}{(s' - \mu^2)(s' - s)} \right] \right\} . \tag{24}$$

A. $\lambda_0 = 0$ (and $\bar{\mu}^2 = \infty$), that is $L' = g_0 \phi \psi^2$ and the renormalized mass value μ^2 will hence be that of an elementary particle of this field. One finds:

$$\begin{aligned} 1/g^2 &= I(\mu) + 1/g_0^2 \\ \text{and} \qquad \mu_0^2 &= \mu^2 + g_0^2 B \end{aligned} \tag{25}$$

so that μ_0^2 tends to infinity as B tends to infinity. Thus we have to start with an infinite mass μ_0^2 and use an infinite mass renormalization to obtain the physical mass μ^2.

B. $g_0 = 0$ or $L' = -\lambda_0 \psi^4$, so that we have no ϕ field but have the possibility of bound states of the ψ field due to the point interaction. Now there is no question of mass renormalization; one finds:

$$g^2 = 1/I(\mu). \tag{26}$$

Formally this has some of the features of the previous case with $g_0 \rightarrow \infty$. This paradox need not be surprising because if we consider the ϕ propagator:

$$g_0^2/(s - \mu_0^2)$$

and let μ_0^2 tend to infinity with g_0^2, we can forget about the s dependence and $g_0^2/(s - \mu_0^2)$ tends to λ_0 a constant and the propagator shrinks to a point which is the present case.

Now from (26) and (17):

$$\frac{g^2}{4\pi} = \frac{1}{4\pi I(s)} = \begin{cases} 6\pi & \text{for} \quad \mu \to 0^+ \\ 8\sqrt{1-\mu^2} & \text{for} \quad \mu \to 1. \end{cases} \tag{27}$$

The first case is that of strong binding; in this limit a strong coupling ($g^2/4\pi = 6\pi$) is obtained irrespective of the coupling constant λ_0 to start with. This may suggest that strong interactions arise because the participating mesons are compound particles.

The latter case corresponds to the weak binding limit. This limit was also studied in Landau's consideration of elementary quantum field theory [6]. From the consideration of the coupling constant in relation to the asymptotic behaviour of the wave functions Landau deduced:

$$g^2/4\pi = 8\sqrt{2\epsilon\,(M_1 + M_2)} \tag{28}$$

where ϵ represents the binding energy and $M_{1,2}$ are the masses. In our case $M_1 = M_2 = \frac{1}{2}$, $\epsilon = 1 - \mu$ so that we get exactly the Landau formula.

If we go to values of $\mu^2 > 1$ the particle becomes unstable; formally, however, the formulae apply also in this case. $\overline{D}(s)$ now develops an imaginary part and does not vanish in the physical sheet. (Compare (7)).

The condition for a resonance at $s = \mu^2$ now has to be defined as:

$$\mathrm{Re}\,\overline{D}(\mu^2) = 0, \tag{29}$$

and consequently one only subtracts $\mathrm{Re}\,\overline{D}(\mu^2)$, (compare (16) for the bound state). Hence the definition (14) for the infinite constant B has to be changed to:

$$B = \mathrm{Re}\int_1^\infty \frac{ds'\,W}{s' - \mu^2 + i\epsilon} = B\int_1^\infty \frac{ds'\,W}{s' - \mu^2}, \tag{30}$$

and similarly for I(s):

$$I(s) = \int_1^\infty \frac{ds'\,W}{(s' - \mu^2 \pm i\epsilon)^2}, \tag{31}$$

where the $\pm\, i\epsilon$ shall indicate that one has to take the integration path once above and once below the singularity and then average. (Which is a sort of generalization of the principal value for higher powers of the denominator).

The coupling constant can now most conveniently be found by means of·

$$\mathrm{Re}\,1/T \to (s - \mu^2)/g^2 \ \text{for} \ s \to \mu^2. \tag{32}$$

It has the following significance in terms of the resonance width Γ (compare also (10)):

$$\mathrm{Re}\,(1/T) = (1/16\pi)\cdot\sqrt{(s-1)/s}\cdot\cot\delta = (1/16\pi)\cdot\sqrt{(s-1)/s}\cdot(s-\mu^2)/\Gamma \tag{33}$$

which is a relativistic Breit - Wigner formula.
Here

$$\Gamma = (g^2/16\pi)\cdot\sqrt{(\mu^2-1)/\mu^2}. \tag{34}$$

6. ANALYTICAL PROPERTIES AND RIEMANNIAN SHEETS

Now we investigate the analytical properties of T(s) and observe (one sees easily that Im T^{-1} (x + iy) $\neq$ 0 unless y = 0) that it has not poles in the complex s-plane unless one continues through the branch-line s = 1 to ∞. Even this branch-line is only a consequence of the choice of the variable s and one can get rid of it by considering T as a function of g $=\sqrt{s-1}$ which is half the momentum in the centre-of-mass system.

$$\frac{16\pi^2}{T} = \frac{16\pi^2}{\lambda} + s\int_1^\infty \frac{ds'\sqrt{1-1/s'}}{s'(s'-s)} + \frac{s-R}{s-\bar{\mu}^2}$$

$$= C - \frac{q}{\sqrt{1+q^2}}\ln\frac{q+\sqrt{1+q^2}}{q-\sqrt{1+q^2}} + \frac{\bar{R}(q^2+1)}{q^2+1-\bar{\mu}^2}, \tag{35}$$

where

$$C = 2 + 16\pi^2/\bar{\lambda} \tag{36}$$

is a convenient abbreviation for subsequent discussions and

$$\bar{R} = -16\pi^2 g_0^2/\bar{\mu}^2\lambda_0 .$$

The integral over s' in (35) behaves like $\sqrt{s-1}$ for s $\to$ 1 as can be seen from (9) whereas no such root is present in the second, q-dependent, part. The mapping of the complex s-plane into the complex q-plane is shown in Fig. 7. The crosses are two complex conjugate points in the s-plane which are thus mapped into two points symmetrical with respect to the imaginary q-axis.

From (35) one can further obtain the relations

$$T(s^*) = T^*(s), \qquad T(-q^*) = T^*(q). \tag{37}$$

The fact that T(s) has no singularities in the physical s-plane means that there are no singularities in the upper q-plane; except for possible bound-states which correspond to poles on the positive imaginary q-axis between O and i.

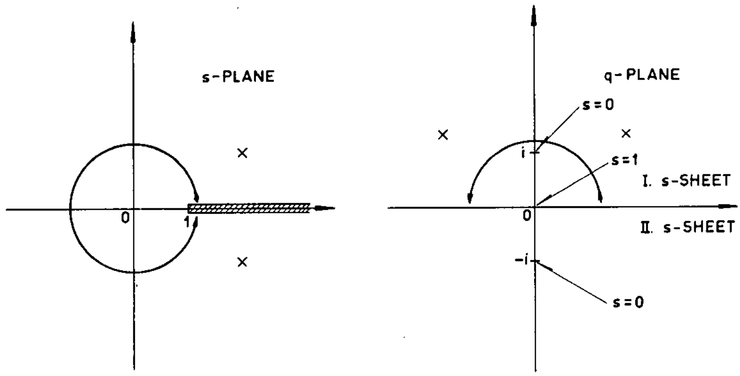


Fig. 7

Mapping of the complex s-plane in the complex q-plane

In the second s-sheet (the lower half of the q-plane) we have

$$(T^{II}(s))^{-1} = (T^{I}(s))^{-1} + 2\pi i\sqrt{(s-1)/s} \tag{38}$$

or

$$T(-q)^{-1} = \bar{T}(q)^{-1} + 2\pi iq/\sqrt{1+q^2}.$$

This means an additional cut from $-\infty$ to 0 in the second s-sheet, corresponding to a branch-line extending from $q = -i$ to $-i\infty$ in the lower half q-plane. These singularities are shown in Figs. 8 and 9.

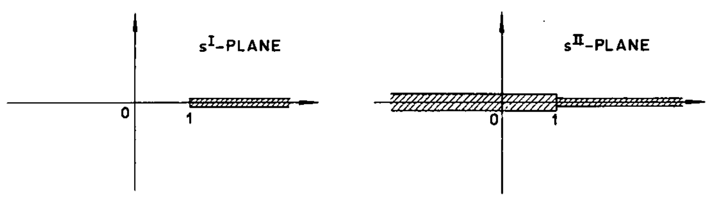


Fig. 8

Singularities in the s-plane

Because of the logarithmic character of T^{-1} there are an infinite number of additional Riemannian sheets and the problem thus arises how to continue to the various sheets. If, in the second s-sheet, one does not cross the additional cut from $-\infty$ to 0, one does not touch any cut in the q-plane and therefore reaches the first sheet again by crossing the main cut from 1 to ∞. A possible path of this type is shown in Fig. 10a. On the other hand, by crossing the $-\infty$ -0 cut in s, one crosses the q-cut and hence steps down to the third s-sheet when one crosses the main cut. (See Fig. 10b).

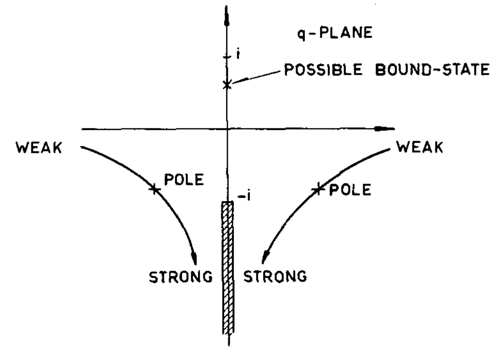


Fig. 9

Singularities in the q-plane

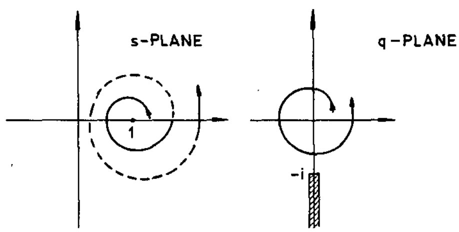


Fig. 10a

Particular singularities in the s, q-planes

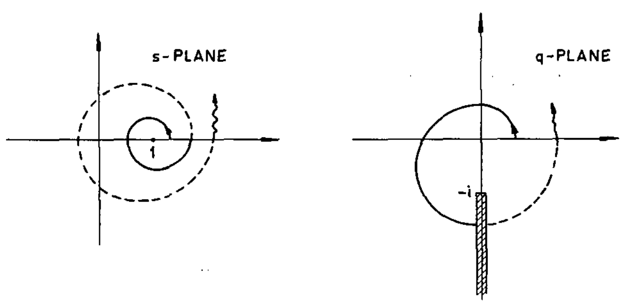


Fig. 10b

Particular singularities in the s, y-planes

7. SPECIAL CASES

As before we now consider the two typical cases in the q-plane.

A. $g_0 = 0$ and hence $\overline{R} = 0$. (no ϕ-field).

To find the poles of T(q) corresponding to bound states, and therefore lying on the imaginary axis, put:

$$q = ik.$$

As can be seen from (35), the condition for such a pole turns out to be

$$C = \frac{\pi k}{\sqrt{1-k^2}} - \frac{2k}{\sqrt{1-k^2}} \arctan \frac{k}{\sqrt{1-k^2}} \begin{cases} = 2 \text{ for } k = 1 \\ \rightarrow \frac{2\pi k}{1-k^2} \text{ for } k \rightarrow -1. \end{cases} \quad (39)$$

These two limits 2 and $-\infty$ for C correspond respectively to $\bar{\lambda} \rightarrow -\infty$ and 0^-, as the cases of very strong and weak binding. Therefore as we turn on the interaction, the pole moves from q = -i to +i, along the imaginary axis.

As $q \rightarrow 0$ the S-matrix assumes the simple form:

$$S = (q + iC/\pi)/(-q + iC/\pi)$$

where C now determines the scattering length.

For q outside the domain -i to +i on the imaginary axis the pole would correspond to a ghost, which could occur if $\bar{\lambda}$ were not negative.

B. $\lambda_0 = 0$, $\bar{R} > 0$ (presence of an elementary particle corresponding to ϕ). Here one pole moves as before but just somewhat displaced. We have now, however, two other poles due to the last term in (35) which equals $R(q^2 + 1)$ when $\bar{\mu}^2$ tends to infinity. It turns out that the signs of the imaginary parts of the logarithm and this latter term are opposite only in the lower half plane and thus the additional poles have to be located there. They move as indicated in Fig. 9, corresponding to the ϕ-particle which becomes less and less stable when one turns on the interaction.

8. DISPERSION RELATION

Because of its analytic properties one has the simple dispersion relation for T(s) which has no left hand cut because our model has no crossing symmetry:

$$T(s) = \frac{g^2}{s - \mu^2} + \int_1^\infty \frac{ds' W}{s' - s} |T(s')|^2 . \quad (40)$$

It has been mentioned in the introduction that the dispersion relation together with elastic unitarity provides us with a different starting point for our model.

9. PROPAGATOR IN THE ZACHARIASEN MODEL

In the previous sections we investigated the S-matrix in the Zachariasen model. Now, I would like to investigate some other field theoretical quantities like the propagator and form-factor in the same model. Now, one can make a complete field theory out of the Zachariasen model. Therefore, one would expect that those features of the propagator and form-factors which one could deduce from general principles of field theory should also hold in the Zachariasen model. However, there is one feature in the Zachariasen model which violates the general principles of field theory: there is no crossing symmetry in the S-matrix. This can be traced back to a failure of satisfying the asymptotic condition. However, this does not disturb our finding of the propagator and form-factors; and for these quantities the model provides an interesting illustration of various general conjectures and general theorems about the propagator and form factors. Before going to these quantities, let us make a short digression on the failure of the asymptotic condition.

One can formulate the Zachariasen model in the following way: one takes as the basic diagrams of the model shown in Fig. 11.

Fig. 11

Basic diagrams

The pairs of particles can be represented by a field with a continuous mass distribution. This corresponds to a quadratic Hamiltonian which can be diagonalized exactly. Now the question is why this field theory does not satisfy the asymptotic condition. The reason is that the particles always occur in pairs. Therefore, a single particle cannot be projected out. This means that the single-particle states are not coupled in the Zachariasen model. Yet what we are interested in is the scattering of these particles and there is no crossing symmetry in the S-matrix because of the lack of the asymptotic condition. If the asymptotic condition were true one would have automatically the crossing symmetry as it is obvious from the definition of the S-matrix in terms of the asymptotic fields in the L. S. Z. formalism.

Because of this also we cannot trace Regge pole trajectories. We do not have any momentum-transfer dependence in the S-matrix and therefore there is only one angular momentum involved.

To calculate the propagator, let us start as in the old-fashioned way by summing diagrams. These are shown in Fig. 12.

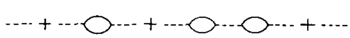


Fig. 12

Diagram for the propagator calculations

Actually, a partial summation of these has been done a long time ago by Dyson who showed that the complete unrenormalized propagator has the form:

$$\Delta_u = 1/(s - \mu_0^2 + \pi). \tag{41}$$

π is the proper self-energy part and is the sum of the diagrams, represented in Fig. 13.

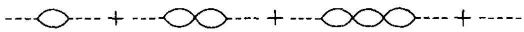


Fig. 13

Diagrams related to the proper self energy part

i.e., the diagrams which cannot be separated in two parts by cutting a dashed line. This is again a geometrical series and we can sum up the whole series. We get:

$$\pi = g_0^2 \Delta/(1 - \lambda_0 \Delta), \tag{42}$$

and

$$\Delta_u = 1/[s - \mu_0^2 + g^2\Delta/(1 - \lambda_0\Delta)] = (1 - \lambda_0\Delta)/[(s - \mu_0^2)(1 - \lambda_0\Delta) + g^2 \Delta]. \tag{43}$$

Let us now study the poles and various other properties of this propagator. To do this we first write Δ_u in the following form:

$$\Delta_u = [1/(s - \bar{\mu}^2)][1 - \lambda_0\Delta + (s - \mu^2)/(s - \bar{\mu}^2) - (s - \mu^2)/(s - \bar{\mu}^2)]/[(s - \mu_0^2)/(s - \bar{\mu}^2) - \lambda_0\Delta]$$

$$= 1/(s - \bar{\mu}^2) + (\mu_0^2 - \bar{\mu}^2)/(s - \bar{\mu}^2)\lambda_0 [(s - \mu_0^2)/\lambda_0(s - \bar{\mu}^2) - \Delta]$$

$$= 1/(s - \bar{\mu}^2) + (\mu_0^2 - \bar{\mu}^2)/(s - \bar{\mu}^2)^2 \; T^{-1}(s)\lambda_0. \tag{44}$$

We see therefore that this contains T(s) and other known factors; so we know essentially the poles of the propagator. They are identical with those of the T-matrix. If we introduce the function D, introduced before, formula (15), we get:

$$\Delta_u = 1/(s - \bar{\mu}^2) + (\mu_0^2 - \bar{\mu}^2)\lambda/(s - \bar{\mu}^2)(s - \mu^2)\lambda_0 D(s),$$
$$D(\mu^2) = 1. \tag{45}$$

This expression is valid provided there is a stable particle $\mu^2 < 1$. Therefore, Δ_u has a pole at $s = \mu^2$. One might think that there is also one pole at $s = \bar{\mu}$, but this is only apparent because $T(\bar{\mu}^2) = 0$. There is only one pole corresponding to one stable particle and we can renormalize the expression according to the usual prescription:

$$\Delta_u \xrightarrow[s \to \mu^2]{} Z_3/(s - \mu^2). \tag{46}$$

This gives:

$$Z_3 = (\lambda/\lambda_0)[(\mu_0^2 - \bar{\mu}^2)/(\mu^2 - \bar{\mu}^2)] = 1 - Ig^2 = D(\bar{\mu}^2) \qquad (47)$$

where I is given by (17).

As we have seen before, one has always

$$g^2 < 1/I,$$

except in the case of a compound particle ($g_0 = 0$). Therefore, we see that, in general:

$$0 < Z_3 < 1,$$

which means that the theory is a good field theory. The renormalized propagator defined by:

$$\Delta_\pi = \Delta_M / Z_3 \qquad (48)$$

is given by:

$$\Delta_\pi = [1/(s - \bar{\mu}^2)][1/D(\bar{\mu}^2)] + (\mu^2 - \bar{\mu}^2)/(s - \bar{\mu}^2)(s - \mu^2), \qquad (49)$$

or, in a better form:

$$\Delta_\pi = [1/(s - \mu^2)]\,[1/D(s)] + [1/(s - \mu^2)]\,[1/D(\bar{\mu}^2) - 1/D(s)]. \qquad (50)$$

As we shall see later, this last form of the renormalized propagator is appropriate for the introduction of the spectral representation.

The value $g^2 = 1/I$ is obtained in the case of a compound particle. In this case $Z_3 = 0$.

If one starts with $\lambda_0 = 0$ and $g_0^2 \neq 0$, one gets:

$$I = g^2 / g_0^2. \qquad (51)$$

A bound particle can also be considered as the limit where one has only g_0^2 and $\mu_0^2 \to \infty$, which means $g_0^2/(s - \mu_0^2) \to \lambda_0$.

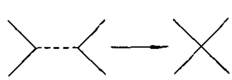


Fig. 13a

Diagrams related to $g_0^2/(s - \mu_0^2) \to \lambda_0$

In this limit $Z_3 = 0$ and this is clear if we remember the physical significance of Z_3. Let us assume that there is a vacuum in the theory and apply to it the Heisenberg operator $\phi(0)$. This generates a state which we represent by $|1)$:

$$\phi(0)|0\rangle \approx |1)$$

up to a normalization factor. This state is neither a physical one-particle state nor a bare one-particle state. Let us call it the undressed one-particle state. Then Z_3 is just the probability on finding a physical one-particle state in the undressed one-particle state. Now, this undressed particle is not an eigenstate of the energy but has a mass distribution and one can show that this mass distribution is centred around the bare mass μ_0. In fact, the bare mass is just the average of this mass distribution. However, the physical mass is μ. If we keep μ^2 fixed and let $\mu_0^2 \to \infty$, the probability of finding the state $|1\rangle$ in the state $|1)$ becomes smaller and smaller and therefore $Z_3 \to 0$.

As far as the bound particle is concerned, one can also argue in the following way. If we start with $g_0 = 0$, the field ϕ is no longer coupled to the field ψ and therefore does not come into the game. However, we can consider ψ^2 (pairs of particles) instead of ϕ. In this way we can also define a propagator of ψ^2 rather than of ϕ and see if it is possible to define a reasonable renormalization constant in this way. This means that we consider only the diagrams shown in Fig. 14.

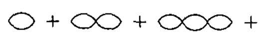


Fig. 14

Diagrams related to the case $g_0 = 0$

In this case we find:

$$\Delta_u = \Delta/(1 - \lambda_0 \Delta) \tag{52}$$

and again one can look for the poles and define Z_3. We get:

$$Z_3 = g^2 \tag{53}$$

which is no longer less than unity. This fact is not in contradiction with any fundamental principle of the Q.F.T. because the bound unity for Z_3 was derived from the canonical commutation relations for the field ϕ. Now, ψ^2 does not satisfy similar canonical commutation relations and therefore Z_3 is no longer bounded by one.

10. SPECTRAL REPRESENTATION OF THE PROPAGATOR

After this digression about Z_3, let us see what the spectral representation of the propagator looks like. First, because of the analytic properties of $D(s)$, $D^{-1}(s)$ can be written in the form:

$$D^{-1}(s) = 1 - g^2 \int_1^\infty \frac{ds'}{s'-s} \frac{(s-\mu^2)(s'-\mu^2)}{(s'-\mu^2)^2 |D(s')|^2}. \qquad (54)$$

By inserting this we get the spectral representation of $\Delta_\pi(s)$. This is of interest if we want to see what happens if there is no stable particle but only a resonance. The question is what happens to the pole. In fact, one would conjecture the following: in the case of one stable particle the spectral function looks like the curves shown in Fig. 15.

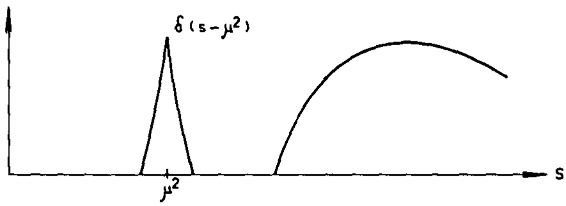


Fig. 15

Spectral function

If the particle becomes unstable, then the continuum moves down, goes below μ^2 and the δ-function disappears. However, a bump will be left on the continuum and the width of it will be related to the lifetime of the unstable particle. That this actually happens in our theory can be seen if we go back to T(s) and

$$\Delta_u = 1/(s-\bar{\mu}^2) + (\mu_0^2 - \bar{\mu}^2)\, T(s)/(s-\bar{\mu}^2)\,\lambda_0 .$$

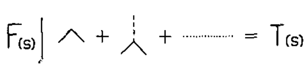
In this form we see that what happens to the pole in the propagator is exactly the same that happens to the pole in T(s): the pole moves to the second sheet of the Riemann surface (the unphysical sheet) and its effects show up by a strong peak in the spectral function.

11. FORM-FACTORS IN THE ZACHARIASEN MODEL

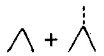
The form factor of the pion is defined by

$$F(s) = \langle 2\pi | j | 0 \rangle = g \langle 2\pi | \psi^2 | 0 \rangle; \qquad (55)$$

As shown in the diagram one has



This is very closely related to the S-matrix. In fact, if we stick to F(s), the following pieces

F(s) |  = T(s),

we reproduce the T-matrix. In other words, we have

$$F(s)[\lambda_0 + g_0^2/(s - \mu_0^2)] = g\,T(s). \tag{56}$$

Therefore

$$F(s) = gT(s)/\lambda_0(s - \mu_0^2) = gT(s)(s - \mu_0^2)/(s - \bar{\mu}^2). \tag{57}$$

Remembering that $T(\bar{\mu}^2) = 0$, the last equation shows that the poles of F(s) are just those of T(s). From this fact immediately one draws the conclusion that, if there is a resonance, it manifests also in the form-factor; i.e. a bump on the spectral representation of F(s).

12. MORE REALISTIC MODELS

Till now we have worked only with scalar particles. This means that we can get only an S-wave resonance. Indeed this summing of the bubbles gives just the S-wave dominant solution of Chew and Mandelstam. But we know that such a resonance has not been found in nature. To get a p-wave resonance one can try to generalize the theory. For instance, one can produce a p-wave resonance by introducing a vector particle.

Another generalization is to assume that ψ corresponds to nucleons and see whether one can produce a bound-state of nucleon-antinucleon by summing up the chain diagrams shown in Fig. 16.

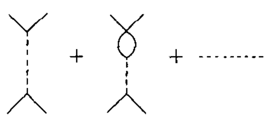


Fig. 16

Chain diagrams to be summed when ψ corresponds to nucleons

In both cases, one meets with divergences. In fact, the degree of divergency is increased by one compared to the simple case we had before. Whereas we had before one infinite constant, here everything diverges by one power more and therefore everything is more ambiguous in the model. In fact, if ψ is a spinor, the calculations go more or less in the same way as before. For instance, if we have a γ_5 interaction we have to take the trace around the bubbles. In this case the square root we had before goes to

$$W \longrightarrow (1/4\pi^2)\sqrt{1 - 1/s}\,(s + 1/4) \tag{58}$$

Therefore, the expression I diverges logarithmically. If we use a cut-off Λ, then

$$I \sim \log \Lambda .$$

This has been done by Nambu who gets

$$g^2 = 2\pi \sqrt{2}/\sqrt{\log \Lambda} .$$

If we identify the bound object with the pion, we obtain the pion nucleon coupling constant. Actually, it turns out that the same kind of formula also holds if we sum up not the bubbles but the ladder diagrams shown in Fig. 17.

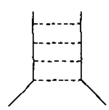


Fig. 17

Ladder diagrams

However, this is a more difficult problem and what one can do is to sum this up in the extreme relativistic limit where the masses involved are neglected compared with moments which are involved. In fact, one uses the Bethe-Salpeter equation which has a very simple spectral representation in the relativistic limit.

The problem of finding whether or not the pion is composed of nucleon-antinucleon can be solved in the following way. As we have seen before, there is a distinction in our theory between the case where there is an elementary particle behind it and the case where there is no elementary particle behind it. This just goes via the Levinson theorem. In particular, if we have no bare particle, then we can have for $R_e T^{-1}$ a behaviour like Fig. 4.

If there is a bare particle, $R_e T^{-1}$ has a behaviour similar to Fig. 6. If there is an additional resonance then the phase shift goes back to 90° as shown in Fig. 4b.

So, what one can do is to calculate the phase-shift supposing the pion is a composite particle. Then one looks where the phase-shift goes to 90° and sees whether one can find it experimentally. This sounds nice theoretically, but in practice it does not seem feasible.

If we sum the chain diagrams, after obtaining the renormalization constant and the mass of the pion the rest is then cut off independent and finite. So one can calculate the place where the phase-shift goes to 90° and the width of it if we call it a resonance. Numerically, it turns out that the mass is:

$$M = 2\, M_p + 940 \text{ MeV}.$$

M_p is the mass of the proton and the width is 450 MeV. This means a very broad S-wave nucleon-antinucleon resonance. This is physically very hard to observe because in this energy region the S-wave cross-section is very small and also there are inelastic channels and they will obscure the picture further.

Let us now make some remarks about the influence of the inelastic channels on the whole structure of the theory. For simplicity, let us assume first that $g_0 = 0$ and keep only the direct coupling. We assume that there are two kinds of particles:

A-particle : —— mass $\frac{1}{2}$
B-particle : ---- mass m

The interactions are of the type shown in Fig. 18,

Fig. 18

Types of interactions

and the basic diagrams are those shown in Fig. 19,
and similarly for the exchange scattering T_{AB} and for T_{BB}. Now, we are

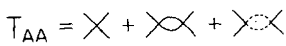


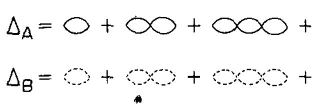
Fig. 19

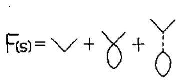
Basic diagrams

dealing once again with geometrical series which can be summed up rather easily. If we use matrix notations (2 X 2 matrices) then the coupling could be written in the form:

$$\lambda = \begin{pmatrix} \lambda_{AA} & \lambda_{AB} \\ \lambda_{BA} & \lambda_{BB} \end{pmatrix} \tag{60}$$

and Δ, the sum of the bubbles:





in the form:

$$\Delta = \begin{pmatrix} \Delta_A & 0 \\ 0 & \Delta_B \end{pmatrix} \qquad (61)$$

Then, the T-matrix is given by:

$$T = \lambda + \lambda \Delta T. \qquad (62)$$

Δ is diagonal, but λ has non-diagonal elements, so that T is non-diagonal. We can easily solve this equation and the solution is:

$$T^{-1} = \lambda^{-1} - \Delta. \qquad (63)$$

Again, we have a rather simple expression for the T-matrix and the analysis goes as before.

This model in itself is interesting to study, for instance, the cusps in one cross-section at the threshold of a second channel.

Let me make a last remark as to what happens to the Levinson theorem if we have a two-channel reaction. For instance, let us suppose that the diagram shown in Fig. 20

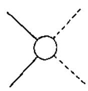


Fig. 20

Diagram corresponding to nucleon-antinucleon annihilation into two pions

corresponds to the annihilation of nucleon-antinucleon into two pions and let us assume that we have also these kinds of inelastic contributions. How does this change our conclusion that the phase-shift goes back at 90° at certain energy? It turns out that in this model the Levinson theorem still holds in an analogous form to the case we had before. Here, however, the phase-shifts are complex but the real part goes back to 90°. So what one has to do is to look at Reδ and see if it goes back to 90°. Practically, this is very difficult because the elastic cross-section is a small part of σ_{total} . If there were a small bump in $\sigma_{elastic}$ it will be overshadowed by $\sigma_{inelastic}$.

One can say that, in principle, in the framework of field theory there may be an exact criterion to distinguish whether a particle is elementary or composite. However, in practice, it will take quite a long time until one can really make this test experimentally.

13. GENERALIZATION TO MORE REALISTIC CASES

Up to now we have been confined to the Zachariasen model which neglects the crossing symmetry as well as the contribution from the inelastic channels. To incorporate crossing symmetry also we have to include the left hand out so that the general dispersion relation for the partial wave scattering amplitude T(s) reads as

$$T(s) = \left\{ \int_{-\infty}^{0} + \int_{4\mu^2}^{\infty} \right\} ds' \; \frac{\operatorname{Im} T(s' + i\epsilon)}{\pi(s' - s)}$$

where $+i\epsilon$ indicates that we are integrating above the real axis. The lower limit of integration $4\mu^2$ indicates the physical threshold (as we had in the Zachariasen model). However, superimposed on it we will have inelastic thresholds corresponding to branch cuts starting at s . The unitarity condition in the form we had so far holds only in the elastic region, i.e. $s_i <$ $< s < 4\mu^2$,

$$\operatorname{Im} T(s + i\epsilon) = \rho\,(s + i\epsilon)\,|T|^2 \tag{65}$$

where ρ, the phase space factor is given by

$$\rho\,(s) = \sqrt{(s - 4\mu^2)/s} \tag{66}$$

Above the inelastic threshold the situation becomes more complicated and has been dealt with by B. Lee (see his lecture notes). We therefore confine ourselves to the region below production threshold. Cf. also [9]. In writing the dispersion relation (64), we have assumed certain analyticity properties which we have not proved. But this follows in particular from the Mandelstam representation, and we shall not bother about it here. Again T is considered only as a function of s which holds for any partial wave amplitude. However the unitarity relation (65) holds only for S waves but the generalization can easily be done. The relations (37) follow from the dispersion relation (64). The analytical continuation through the branch cut in the S plane extensively dealt with in lecture II can be done only between $4\mu^2$ and s_i . ρ (s) has two branch points at $s = 4\mu^2$ and $s = 0$, and as before we locate the branch cut from $-\infty$ to 0 and $4\mu^2$ to $+\infty$. It then follows $\rho\,(s) = -[\rho(s)]^*$ since in this region ρ (s) is purely imaginary.

From (65) we obtain

$$\operatorname{Im} T^{-1}\,(s + i\epsilon) = -\operatorname{Im} T(s + i\epsilon)/|T|^2 = -\rho(s + i\epsilon) = \rho\,(s - i\epsilon) \tag{67}$$

and therefore get the analytic continuation to the second sheet of the inverse amplitude.

$$T_I^{-1}\,(s + i\epsilon) = T_I^{-1}\,(s - i\epsilon) + 2\,i\,\rho\,(s - i\epsilon) = T_{II}^{-1}\,(s - i\epsilon) \tag{68}$$

or if we invert this relation (68) we obtain

$$T_{II} = T_I\,/(1 + 2i\rho\; T_I\,). \tag{69}$$

Again we see there may be poles in the second sheet due to the vanishing of the denominator in (69)

(i. e.) $$1 + 2i\rho T = 0$$

and these poles correspond to resonances. The location of the branch cuts in the s and q plane are given in Fig. 21.

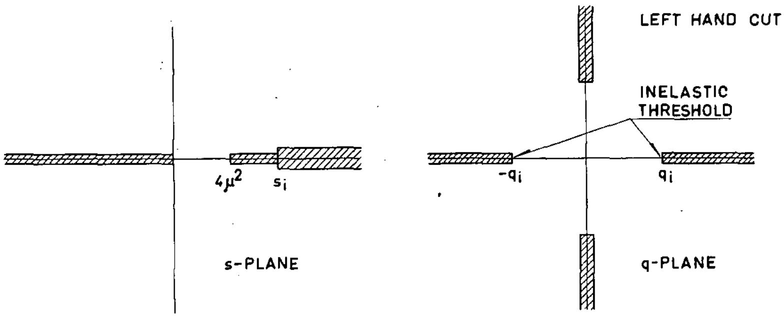


Fig. 21

Location of branch cuts in the s and q-plane

Concluding this section, we list a few formulae decomposing T into dispersive and absorptive parts [10]:

$$T = d + i\rho a$$
$$T_{II} = d - i\rho a \tag{70}$$

and inverting we get

$$a = T T_{II}$$
$$d = T_{II} \; (1 + i\rho T). \tag{71}$$

The advantage of this decomposition is that even in the s plane the elastic branch cut is absorbed into the phase space factor ρ (s).

In particular in the elastic region

(i. e.) $$4\mu^2 < s < s_i$$
we have
$$a = \frac{1}{\rho} \operatorname{Im} T \tag{72}$$
$$d = \operatorname{Re} T.$$

The partial wave S matrix now takes the simple form

$$S = 1 + 2i\,\rho\, T_I = T_I / T_{II} = T(q)/T(-q). \tag{73}$$

Comparing (69) with (73) we see that the resonances correspond to the zeros of the S matrix (See also [11])

14. FORM FACTORS

We will now investigate whether the same resonances show up in the form factors also. We will distinguish the two cases with and without anomalous threshold.

A. No anomalous threshold

The graphical representation for the form factor is shown in Fig. 22 where the solid line represents the particle whose scattering we have con-

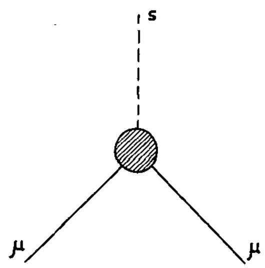


Fig. 22

Graphical representation of the form factor

sidered before. The connection between the scattering amplitude and the form factor can be seen from Fig. 23

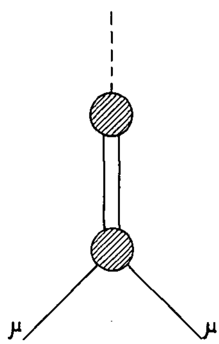


Fig. 23

Decomposition of the form factor

where we have decomposed the bubble taking out the lowest intermediate state. This immediately implies that there is a branch cut starting at $4\mu^2$. Hence we obtain

$$F(s) = 1 + \frac{s}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\operatorname{Im} F(s' + i\epsilon)}{s'(s' - s)}$$

This formula again implies the analyticity and boundedness properties of F(s) which is also normalized to unity at s = 0.

To apply the dispersion relation (74) we require some explicit statements about the imaginary part of F(s). In the Zachariasen model we saw that it is directly related to the scattering amplitude. We will now show that a similar relation holds also here. By the general Lehmann - Symanzik- Zimmermann technique [12].

F(s) is related to $\langle \pi | j | \pi \rangle$ or to $\langle 0 | j | 2\pi_{in} \rangle$. Applying the time reversal operation (assuming the current j has definite properties under this operation) we obtain

$$F(s) \sim \langle 0 | j | 2\pi_{in} \rangle = [\langle 0 | j | 2\pi_{out} \rangle]^*. \tag{75}$$

Since the in- and out- states are related by the S-matrix and since $\langle 0 | j^*$ is a state of definite angular momentum (implying a diagonalized S-matrix) we obtain

$$F(s) = F^*(s) e^{2i\delta} \tag{76}$$

which leads to

$$\operatorname{Im} F = \sin \delta \, e^{-id} F = \rho T^* F. \tag{77}$$

Because of the simple diagonalization of the S-matrix this holds only in the elastic region. All quantities in (77) have to be taken on the upper lip of the branch cut and because of (68) and (69) we have

$$\operatorname{Im} F = \rho F T/(1 + 2i\rho T). \tag{78}$$

Since we now know the imaginary part of F, we can continue F analytically into the second sheet by

$$F_{II} = F_I - 2i \operatorname{Im} F_I = F(1 - 2i\rho T^*) \tag{79}$$

and because of (78) and the unitarity of the S-matrix we end up with

$$F_{II} = F_I/(1 + 2i\rho T). \tag{80}$$

Therefore in the second sheet F has the same poles as T, corresponding to the same resonances.

Similar considerations can be made in the q-plane.

B. ANOMALOUS THRESHOLDS [13]

In this more complicated case we have to consider a special contribution to the graph of Fig. 22 which leads to the anomalous branch point (Fig. 24). This is the form factor of the particle m and as a physical example we consider the form factor of the Σ; i.e. $\mu \longleftrightarrow \pi$, $M \longleftrightarrow \Lambda$ and $m \longleftrightarrow \Sigma$.

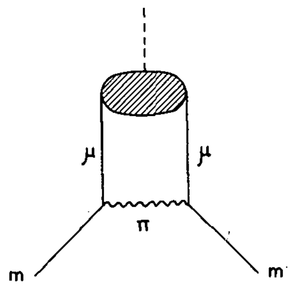


Fig. 24

Graph, leading to anomalous branch point

The anomalous thresholds appear only for special relations among the masses M, m and μ and to investigate this problem we first consider the diagram in Fig. 25,

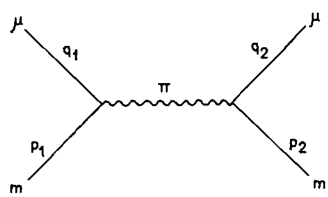


Fig. 25

Scattering diagram, leading to anomalous threshold

where the four vectors p_i and q_i are related by energy-momentum conservation.

We choose the usual three invariant variables

$$\begin{aligned} s &= (q_1 + q_2)^2 \\ t &= (p_1 + q_1)^2 \\ \bar{t} &= (p_1 + q_2)^2 \end{aligned} \tag{81}$$

In the centre-of-mass system the energy-momentum vectors are given by

$$q_1 = \begin{pmatrix} -\sqrt{q'^2 + \mu^2} \\ -\vec{q}' \end{pmatrix} \qquad q_2 = \begin{pmatrix} \sqrt{q'^2 + \mu^2} \\ \vec{q}' \end{pmatrix}$$
$$p_1 = \begin{pmatrix} -\sqrt{q^2 + \mu^2} \\ -\vec{q}' \end{pmatrix} \qquad p_2 = \begin{pmatrix} \sqrt{q^2 + m^2} \\ -\vec{q} \end{pmatrix} \tag{82}$$

and the following relations hold:

$$\begin{aligned} s &= 4(q'^2 + \mu^2) = 4(q^2 + m^2) \\ t &= -q^2 - q'^2 + 2qq' \cos\theta \\ \bar{t} &= -q^2 - q'^2 - 2q_1 \cos\theta \end{aligned} \tag{83}$$

where θ is the scattering angle.

In the usual way, the poles correspond to t or $\bar{t} = M^2$. However, if we project out a partial wave in the s-channel, that is

$$T_\ell(s) = \int_{-1}^{1} d\eta P_\ell(\eta) T(s, \eta) \qquad \eta \equiv \cos\theta \tag{84}$$

the pole is converted into a branch-line with the end points corresponding to $\cos\theta = \pm 1$. From (83) we thus get the branch points at

$$(q \pm q')^2 = -M^2 . \tag{85}$$

Solving (83) and (85) we get

$$q'_a = \pm i \frac{m^2 + M^2 - \mu^2}{2M} \tag{86}$$

or

$$s = -\frac{1}{M^2}\{m^2 - (M+\mu)^2\}\{m^2 - (M-\mu)^2\} = g(m^2) \tag{87}$$

where g is, of course, a function of M and μ also. If we plot it, however, as a function of m^2, we obtain the location of the branch-points as shown in Fig. 26.

The points $(M+\mu)^2$ and $(M-\mu)^2$ correspond to Σ and Λ respectively, becoming instable with the decay processes $\Sigma \to \Lambda + \pi$ and $\Lambda \to \Sigma + \pi$. Between these two points all particles are stable. We can now draw the branch-cuts in the q'-plane for two typical cases (Fig. 27).

where the two branch-lines approach each other when $m^2 = M^2 + \mu^2$. It is clear how this looks in the s-plane for $m^2 = M^2 + \mu^2$ the left branch cut of the first and second sheet meet each other at the threshold of the right-hand

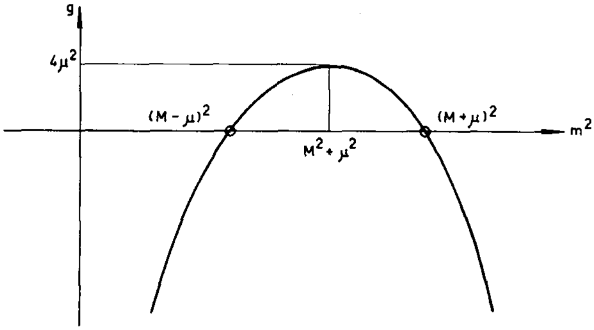


Fig. 26

$g(m^2)$ - Eq.(87)- as a function of m^2.

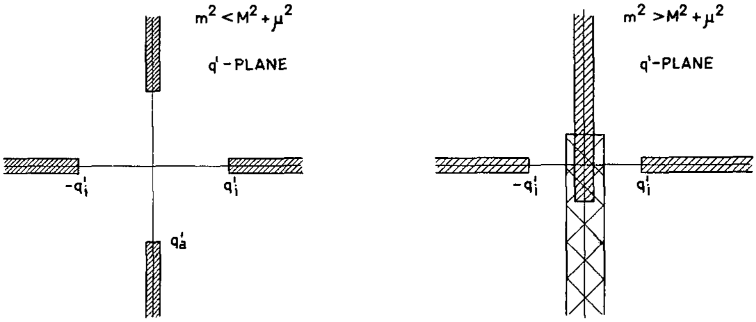


Fig. 27

Branch-cuts in q'-plane for two typical cases

cut. For $m^2 > M + \mu^2$ the lower branch point moves out into the physical sheet, leading to an anomalous threshold. The situation is indicated in Fig. 28.

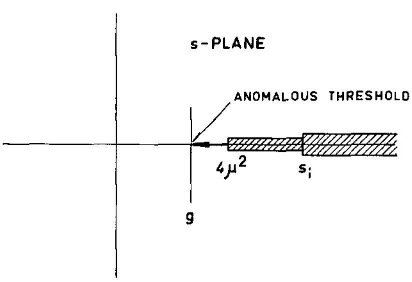


Fig. 28

Anomalous threshold in the s-plane

For the form factor the situation is entirely similar except there is no left-hand cut in the physical sheet.

We therefore have to modify the lower limit of the integral in equation (74) from $4\mu^2$ to $g(m^2)$.

It is well-known that in the configuration space a dispersion relation corresponds just to the superposition of Yukawa potentials. (In the centre-of-mass system).

$$f(x) = \int_g^\infty ds \frac{\mathrm{Im}F(s) e^{-|\bar{x}|\sqrt{s}}}{4\pi |\bar{x}|} . \qquad (88)$$

Hence $g(m^2)$ provides us with the Yukawa potential of longest range, given by

$$\frac{e^{-|x|\sqrt{g}}}{4\pi |\bar{x}|}$$

where $|x| \sqrt{g}$ can be calculated from (87) to be

$$|\bar{x}|\sqrt{g} = |\bar{x}| \cdot 2(M+\mu)/M\sqrt{2\epsilon M\mu/(M+\mu)} = 2 \ r \quad 2\epsilon M$$

where ϵ is the binding energy, defined by

$$m = M + \mu - \epsilon$$

and r_e is the relative co-ordinate, say of the $\Lambda\ \pi$ system forming the ϵ. M_r is the reduced mass. We therefore conclude, that physically an anomalous threshold corresponds to a spreading of the compound system which is greater then the Compton wave-length of the particles into which it can decay virtually.

15. CONCLUSION

We have been considering only a particular model and a slight extension, where one removes some of the restrictions. It is, of course, to be hoped that all this can one day directly be deduced from axiomatic field theory, but up to now it was not possible, and one has to be content to illustrate the situation by simple examples. Nevertheless, these examples show that the situation in field theory ties on directly to the situation one has in potential scattering.

REFERENCES

[1] THIRRING, W., Nuovo Cimento 23 (1962) 1064.

[2] ZACHARIASEN, F., Phys. Rev. 121 (1961) 1851.

[3] GELL-MANN, M. and ZACHARIASEN, F., Phys. Rev. 124 (1961) 953.

[4] THIRRING, W., Phys. Rev. to appear in Vol. 126.

[5] THIRRING, W., The Principles of Quantum Electrodynamics, New York, 1958, App. II.
[6] IDA, M., Progr. of Theor. Phys. 20 (1958) 979.
[7] LANDAU, JETP 12 (1961) 1294.
[8] BJORKEN, J. D., Phys. Rev. Lett. 4 (1960) 473.
[9] OEHME, R., Herzegnovi-Lectures (1961).
[10] OEHME, R., The Compound Structure of Elementary Particles, Celebration Volume for Heisenberg's 60th Birthday (1961).
[11] LEVY, M., Nuovo Cimento 13 (1959) 115.
[12] LEHMANN, H., SYMANZIK, K., ZIMMERMANN, W., Nuovo Cimento 1 (1955) 205 and 6 (1957) 319.
[13] KARPLUS, R., SOMMERFIELD, C. and WICHMAN, E., Phys. Rev. 111 (1958) 1187.