Von Neumann Algebras.

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Chapter 1

Introduction.

The purpose of these notes is to provide a rapid introduction to von Neumann algebras which gets to the examples and active topics with a minimum of technical baggage. In this sense it is opposite in spirit from the treatises of Dixmier [], Takesaki[], Pedersen[], Kadison-Ringrose[], Stratila-Zsido[]. The philosophy is to lavish attention on a few key results and examples, and we prefer to make simplifying assumptions rather than go for the most general case. Thus we do not hesitate to give several proofs of a single result, or repeat an argument with different hypotheses. The notes are built around semester-long courses given at UC Berkeley and Vanderbilt though they contain more material than could be taught in a single semester.

The notes are informal and the exercises are an integral part of the exposition. These exercises are vital and mostly intended to be easy.

Chapter 2

Background and Prerequisites

2.1 Hilbert Space

A Hilbert Space is a complex vector space $\mathcal{H}$ with inner product $\langle , \rangle : \mathcal{H} \text{x} \mathcal{H} \to \mathbb{C}$ which is linear in the first variable, satisfies $\overline{\langle \xi, \eta \rangle} = \langle \eta, \xi \rangle$, is positive definite, i.e. $\langle \xi, \xi \rangle > 0$ for $\xi \neq 0$, and is complete for the norm defined by test $||\xi|| = \sqrt{\langle \xi, \xi \rangle}$.

Exercise 2.1.1. *Prove the parallelogram identity :*

$$||\xi - \eta||^2 + ||\xi + \eta||^2 = 2(||\xi||^2 + ||\eta||^2)$$

and the Cauchy-Schwartz inequality:

$$|\langle \xi, \eta \rangle| \leq ||\xi|| \, ||\eta||.$$

Theorem 2.1.2. *If C is a closed convex subset of $\mathcal{H}$ and ξ is any vector in $\mathcal{H}$, there is a unique $\eta \in C$ which minimizes the distance from ξ to C, i.e. $||\xi - \eta'|| \leq ||\xi - \eta|| \quad \forall \eta' \in C$.*

Proof. This is basically a result in plane geometry.

Uniqueness is clear—if two vectors η and η' in C minimized the distance to ξ, then ξ, η and η' lie in a (real) plane so any vector on the line segment between η and η' would be strictly closer to ξ.

To prove existence, let d be the distance from C to ξ and choose a sequence $\eta_n \in C$ with $||\eta_n - \xi|| < d + 1/2^n$. For each n, the vectors ξ, η_n and η_{n+1} define a plane. Geometrically it is clear that, if η_n and η_{n+1} were not close, some point on the line segment between them would be closer than d to ξ. Formally, use the parallelogram identity:

$$||\xi - \frac{\eta_n + \eta_{n+1}}{2}||^2 = ||\frac{\xi - \eta_n}{2} + \frac{\xi - \eta_{n+1}}{2}||^2$$

$$= 2(||\frac{\xi - \eta_n}{2}||^2 + ||\frac{\xi - \eta_{n+1}}{2}||^2 - 1/8||\eta_n - \eta_{n+1}||^2)$$
$$\leq (d + 1/2^n)^2 - 1/4||\eta_n - \eta_{n+1}||^2$$

Thus there is a constant K such that $||\eta_n - \eta_{n+1}||^2 < K/2^n$ or $||\xi - \frac{\eta_n + \eta_{n+1}}{2}||^2$ would be less than d^2.

Thus (η_n) is Cauchy, its limit is in C and has distance d from ξ.

□

Exercise 2.1.3. *If $\phi \in \mathcal{H}^*$ (the Banach-space dual of $\mathcal{H}$ consisting of all continuous linear functionals from $\mathcal{H}$ to $\mathbb{C}$), kerϕ is a closed convex subset of $\mathcal{H}$. Show how to choose a vector ξ_ϕ orthogonal to* $\ker \phi$ *with $\phi(\eta) = \langle \xi_\phi, \eta \rangle$ and so that $\phi \mapsto \xi_\phi$ is a conjugate-linear isomorphism from $\mathcal{H}^*$ onto $\mathcal{H}$.*

We will be especially concerned with *separable* Hilbert Spaces where there is an *orthonormal basis*, i.e. a sequence $\{\xi_1, \xi_2, \xi_3, ...\}$ of unit vectors with $\langle \xi_i, \xi_j \rangle = 0$ for $i \neq j$ and such that 0 is the only element of $\mathcal{H}$ orthogonal to all the ξ_i.

We will use the abbreviation ONB for orthonormal basis.

Exercise 2.1.4. *Show that an ONB always exists (e.g. Gram-Schmidt) and that if $\{\xi_i\}$ is an ONB for $\mathcal{H}$ then the linear span of the $\{\xi_i\}$ is dense in $\mathcal{H}$.*

A trivial but useful observation. If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces with vectors $\xi_i \in \mathcal{H}$ and $\psi_k \in \mathcal{K}$ respectively then if

$$\langle \xi_i, \xi_j \rangle = \langle \psi_i, \psi_j \rangle \quad \forall i, j$$

then the map $\xi_i \mapsto \psi_i$ extends to a linear $\langle , \rangle$-preserving bijection from the closure of the subspace spanned by the ξ_i to the closure of the subspace spanned by the ψ_i.

A linear map (operator) $a : \mathcal{H} \to \mathcal{K}$ is said to be bounded if there is a number K with $||a\xi|| \leq K||\xi|| \quad \forall \xi \in \mathcal{H}$. The infimum of all such K is called the *n*orm of a, written $||a||$. The set of all bounded operators from $\mathcal{H}$ to $\mathcal{K}$ is written $\mathcal{B}(\mathcal{H}, \mathcal{K})$ and if $\mathcal{H} = \mathcal{K}$ we use $\mathcal{B}(\mathcal{H})$. Boundedness of an operator is equivalent to continuity.

To every bounded operator a between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, by exercise 2.1.3 there is another, a^*, between $\mathcal{K}$ and $\mathcal{H}$, called the *a*djoint of a which is defined by the formula $\langle a\xi, \eta \rangle = \langle \xi, a^*\eta \rangle$.

Exercise 2.1.5. *Prove that*

$$||a|| = \sup_{||\xi|| \leq 1, ||\eta|| \leq 1} |\langle a\xi, \eta \rangle|$$
$$= ||a^*|| = ||a^*a||^{1/2}.$$

Some definitions:

The identity map on $\mathcal{H}$ is a bounded operator denoted 1.

An operator $a \in \mathcal{B}(\mathcal{H})$ is called *self-adjoint* if $a = a^*$.

An operator $p \in \mathcal{B}(\mathcal{H})$ is called a *projection* if $p = p^2 = p^*$.

An operator $a \in \mathcal{B}(\mathcal{H})$ is called *positive* if $\langle a\xi, \xi \rangle \geq 0 \quad \forall \xi \in \mathcal{B}(\mathcal{H})$. We say $a \geq b$ if $a - b$ is positive.

An operator $u \in \mathcal{B}(\mathcal{H})$ is called an *isometry* if $u^*u = 1$.

An operator $v \in \mathcal{B}(\mathcal{H})$ is called a *unitary* if $uu^* = u^*u = 1$.

An operator $u \in \mathcal{B}(\mathcal{H})$ is called a *partial isometry* if u^*u is a projection.

The last three definitions extend to bounded linear operators between different Hilbert spaces.

If $S \subseteq \mathcal{B}(\mathcal{H})$ then the *commutant* S' of S is $\{x \in \mathcal{B}(\mathcal{H}) | xa = ax \quad \forall a \in S\}$. Also $S'' = (S')'$.

Exercise 2.1.6. A word on matrices. *If e_i is an ONB of $\mathcal{H}$ then $e_i \mapsto \xi_i$ (the characteristic function of $\{i\}$) defines a unitary from $\mathcal{H}$ to $\ell^2(\mathbb{N})$. So for any $\xi \in \mathcal{H}$,*

$$\xi = \sum_{i=1}^{\infty} \langle \xi, e_i \rangle e_i$$

the sum being convergent in the norm of $\mathcal{H}$.

If $a \in \mathcal{B}(\mathcal{H})$ we define the matrix of a wrt the ONB to be $a_{i,j} = \langle ae_i, e_j \rangle$. For fixed i, $j \mapsto a_{i,j}$ is in ℓ^2 and for fixed j, $i \mapsto a_{i,j}$ is in ℓ^2. And $a(e_i) = \sum_j a_{i,j} e_j$. Thus the matrix of a determines a and if $b \in \mathcal{B}(\mathcal{H})$ has matrix $b_{i,j}$ then

$$ab_{i,j} = \sum_k a_{i,k} b_{k,j}$$

the sum being absolutely convergent.

Exercise 2.1.7. *Show that every $a \in \mathcal{B}(\mathcal{H})$ is a linear combination of two self-adjoint operators.*

Exercise 2.1.8. *A positive operator is self-adjoint.*

Exercise 2.1.9. *Find an isometry from one Hilbert space to itself that is not unitary.* (The unilateral shift on $\mathcal{H} = \ell^2(\mathbb{N})$ is a fine example. There is an obvious orthonormal basis of $\mathcal{H}$ indexed by the natural numbers and the shift just sends the $nth.$ basis element to the $(n+1)th.$)

Exercise 2.1.10. *If $\mathcal{K}$ is a closed subspace of $\mathcal{H}$ show that the map $P_{\mathcal{K}}: \mathcal{H} \to \mathcal{K}$ which assigns to any point in $\mathcal{H}$ the nearest point in $\mathcal{K}$ is linear and a projection.*

Exercise 2.1.11. *Show that the correspondence $\mathcal{K} \to P_\mathcal{K}$ of the previous exercise is a bijection between closed subspaces of $\mathcal{H}$ and projections in $\mathcal{B}(\mathcal{H})$.*

If S is a subset of $\mathcal{H}$, $S^\perp$ is by definition $\{\xi \in \mathcal{H} : \langle \xi, \eta \rangle = 0 \quad \forall \eta \in S\}$. Note that $S^\perp$ is always a closed subspace.

Exercise 2.1.12. *If $\mathcal{K}$ is a closed subspace then $\mathcal{K}^{\perp\perp} = \mathcal{K}$ and $P_{\mathcal{K}^\perp} = 1 - P_\mathcal{K}$.*

Exercise 2.1.13. *If u is a partial isometry then so is u^*. The subspace $u^*\mathcal{H}$ is then closed and called the* initial domain *of u, the subspace $u\mathcal{H}$ is also closed and called the* final domain *of u. Show that a partial isometry is the composition of the projection onto its initial domain and a unitary between the initial and final domains.*

The commutator $[a, b]$ of two elements of $\mathcal{B}(\mathcal{H})$ is the operator $ab - ba$.

Exercise 2.1.14. *If $\mathcal{K}$ is a closed subspace and $a = a^*$ then*

$$a\mathcal{K} \subseteq \mathcal{K} \quad iff \quad [a, P_\mathcal{K}] = 0.$$

In general $(a\mathcal{K} \subseteq \mathcal{K} \quad \text{and} \quad a^*\mathcal{K} \subseteq \mathcal{K}) \iff [a, P_\mathcal{K}] = 0.$

2.2 The Spectral Theorem

The *spectrum* $\sigma(a)$ of $a \in \mathcal{B}(\mathcal{H})$ is $\{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible}\}$.

Exercise 2.2.1. (Look up proofs if necessary.) *Show that $\sigma(a)$ is a non-empty closed bounded subset of $\mathbb{C}$ and that if $a = a^*$, $\sigma(a) \subseteq [-||a||, ||a||]$ with either $||a||$ or $-||a||$ in $\sigma(a)$.*

The spectral theorem takes a bit of getting used to and knowing how to prove it does not necessarily help much. If one cannot "see" the spectral decomposition of an operator it may be extremely difficult to obtain—except in a small finite number of dimensions where it is just diagonalisation. But fortunately there is nothing like a course in operator algebras, either C^* or von Neumann, to help master the use of this theorem which is the heart of linear algebra on Hilbert space. The book by Reed and Simon, "Methods of mathematical physics" vol. 1, Functional Analysis, contains a treatment of the spectral theorem which is perfect background for this course. We will make no attempt to prove it here—just give a vague statement which will establish terminology.

The spectral theorem asserts the existence of a *projection valued measure* from the Borel subsets of $\sigma(a)$ (when $a = a^*$ or more generally when a is *normal* i.e. $[a,a^*]=0$) to projections in $\mathcal{B}(\mathcal{H})$, written symbolically $\lambda \to E(\lambda)$, such that

$$a=\int \lambda dE(\lambda).$$

This integral may be interpreted as a limit of sums of operators (necessitating a topology on $\mathcal{B}(\mathcal{H})$), as a limit of sums of vectors: $a\xi=\int \lambda dE(\lambda)\xi$ or simply in terms of measurable functions $\langle \xi, a\eta\rangle=\int \lambda d\langle \xi, E(\lambda)\eta\rangle$. The projections $E(B)$ are called the *spectral projections* of a and their images are called the *spectral subspaces* of a.

Given any bounded Borel complex-valued function f on $\sigma(a)$ one may form $f(a)$ by $f(a)=\int f(\lambda)dE(\lambda)$.

Exercise 2.2.2. *If μ is a sigma-finite measure on X and $f\in L^\infty(X,\mu)$, the operator $M_f: L^2(X,\mu)\to L^2(X,\mu), (M_f g)(x)=f(x)g(x)$, is a bounded (normal) operator with $||M_f||=\text{ess-sup}_{x\in X}(|f(x)|)$. If f is real valued then M_f is self adjoint. Find $\sigma(f)$ and the projection-valued measure $E(\lambda)$.*

Exercise 2.2.3. *If $\dim(\mathcal{H})<\infty$ find the spectrum and projection-valued measure for a (which is a Hermitian matrix).*

The example of exercise 2.2.2 is generic -that any self-adjoint operator is of the from M_f is another version of the spectral theorem.

Exercise 2.2.4. *(A "visible" spectral decomposition.) Let $\mathcal{H}=\ell^2(\mathbb{Z}$ and let $u\in\mathcal{B}(\mathcal{H})$ be translation by 1, i.e. $uf(n)=f(n+1)$. Then $u+u^*$ is self-adjoint. Use the Fourier transform to exhibit a measure space (X,μ) and a function f on it so that $(\mathcal{H},u+u^*)$ is unitarily equivalent to $(L^2(X,\mu),M_f)$.*

Proofs of the spectral theorem use the following general scheme:

If $\xi\in\mathcal{H}$ is any vector and $a=a^*\in\mathcal{B}(\mathcal{H})$, let $\mathcal{K}$ be the closed linear span of the $\{a^n\xi : n=0,1,2,3,...\}$, then a defines a self-adjoint operator on $\mathcal{K}$ and one tries to find a finite measure μ on the spectrum $\sigma(a)$ such that $(\mathcal{K},a)$ is isomorphic in the obvious sense to $(L^2(\sigma(a),\mu)$, multiplication by x). It is not to hard to see how this should go-if a is supposed to be multiplication by x, then we know that, for each $n\in\mathbb{N}$,

$$\int_{\sigma(a)} x^n d\mu=\langle a^n\xi,\xi\rangle.$$

Moreover by the classical Weierstrass theorem polynomials are dense in the continuous functions on the compact set $\sigma(a)$. So we appeal to some classical result from measure theory that says there is an appropriate measure. But the inner product between $a^n\xi$ and $a^m\xi$ is then $\int_{\sigma(a)} x^{n+m}d\mu$ so we may define a unitary from $\mathcal{K}$ to $L^2(\sigma(a),\mu)$ by sending $a^m\xi$ to the function x^m on $\sigma(a)$.

Note how it is the real numbers $\langle a^n\xi,\xi\rangle$ that determine everything.

So if you cannot "see" the spectral decomposition of an explicit operator a, all is not lost but you should sniff around the "moments" $\langle a^n\xi,\xi\rangle$.

Continuing such an argument by restricting to $\mathcal{K}^\perp$ one obtains a full spectral theorem.

Exercise 2.2.5. *Show that a self-adjoint operator a is the difference $a_+ - a_-$ of two positive commuting operators called the positive and negative parts of a, obtained as functions of a as above.*

2.3 Polar decomposition

Exercise 2.3.1. *Show that every positive operator a has a unique positive square root $a^{1/2}$.*

Given an arbitrary $a \in \mathcal{B}(\mathcal{H})$ we define $|a| = (a^*a)^{1/2}$.

Exercise 2.3.2. *Show that there is a partial isometry u such that $a = u|a|$, and that u is unique subject to the condition that its initial domain is $ker(a)^\perp$. The final domain of this u is $\overline{Im(a)} = ker(a^*)^\perp$.*

2.4 Tensor product of Hilbert Spaces.

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces one may form their algebraic tensor product $\mathcal{H} \otimes_{alg} \mathcal{K}$ (in the category of complex vector spaces). On this vector space one defines the sesquilinear form $\langle,\rangle$ by:

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \langle \xi, \xi' \rangle \langle \eta, \eta' \rangle$$

and observes that this form is positive definite. The Hilbert space tensor product $\mathcal{H} \otimes \mathcal{K}$ is then the completion of $\mathcal{H} \otimes_{alg} \mathcal{K}$. It is easy to see that if $a \in \mathcal{B}(\mathcal{H}), b \in \mathcal{B}(\mathcal{K})$, there is a bounded operator $a \otimes b$ on $\mathcal{H} \otimes \mathcal{K}$ defined by $a \otimes b(\xi \otimes \eta) = a\xi \otimes b\eta$.

Exercise 2.4.1. *Let $L^2(X,\mathcal{H},\mu)$ be the Hilbert space of measurable square integrable functions (up to null sets) $f : X \to \mathcal{H}$, with $\mathcal{H}$ a separable Hilbert space. For each $\xi \in \mathcal{H}$ and $f \in L^2(X,\mu)$ let $f_\xi \in L^2(X,\mathcal{H},\mu)$ be defined by $f_\xi(x) = f(x)\xi$. Show that the map $\xi \otimes f \mapsto f_\xi$ defines a unitary from $\mathcal{H} \otimes L^2(X,\mu)$ onto $L^2(X,\mathcal{H},\mu)$.*

Chapter 3

The definition of a von Neumann algebra.

3.1 Topologies on $\mathcal{B}(\mathcal{H})$

1. The *norm* or *uniform* topology is given by the norm $||a||$ defined in the previous chapter.

2. The topology on $\mathcal{B}(\mathcal{H})$ of pointwise convergence on $\mathcal{H}$ is called the *strong operator topology.* A basis of neighbourhoods of $a \in \mathcal{B}(\mathcal{H})$ is formed by the

$$N(a, \xi_1, \xi_2, ..., \xi_n, \epsilon) = \{b : ||(b - a)\xi_i|| < \epsilon \quad \forall i = 1, \cdots, n\}$$

3. The *weak operator topology* is formed by the basic neighbourhoods

$$N(a, \xi_1, \xi_2, ..., \xi_n, \eta_1, \eta_2, .., \eta_n, \epsilon) = \{b : |\langle (b-a)\xi_i, \eta_i\rangle| < \epsilon \quad \forall i = 1, \cdots, n\}$$

Note that this weak topology is the topology of pointwise convergence on $\mathcal{H}$ in the "weak topology" on $\mathcal{H}$ defined in the obvious way by the inner product.

The unit ball of $\mathcal{H}$ is compact in the weak topology and the unit ball of $\mathcal{B}(\mathcal{H})$ is compact in the weak operator topology. These assertions follow easily from Tychonoff's theorem by embedding the unit balls in products of discs using Hilbert space duality.

Exercise 3.1.1. *Show that we have the following ordering of the topologies (strict in infinite dimensions).*

(weak operator topology) < (strong operator topology) < (norm topology)

Note that a weaker topology has less open sets so that if a set is closed in the weak topology it is necessarily closed in the strong and norm topologies.

3.2 The bicommutant theorem.

We will now prove the von Neumann "density" or "bicommutant" theorem which is the first result in the subject. We prove it first in the finite dimensional case where the proof is transparent then make the slight adjustments for the general case.

Theorem 3.2.1. *Let M be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ containing* 1, *with* $\dim(\mathcal{H}) = n < \infty$. *Then $M = M''$.*

Proof. It is tautological that $M \subseteq M''$.

So we must show that if $y \in M''$ then $y \in M$. To this end we will "amplify" the action of M on $\mathcal{H}$ to an action on $\mathcal{H} \otimes \mathcal{H}$ defined by $x(\xi \otimes \eta) = x\xi \otimes \eta$. If we choose an orthonormal basis $\{v_i\}$ of $\mathcal{H}$ then $\mathcal{H} \otimes \mathcal{H} = \oplus_{i=1}^n \mathcal{H}$ and in terms of matrices we are considering the $n \times n$ matrices over $\mathcal{B}(\mathcal{H})$ and embedding M in it as matrices constant down the diagonal. Clearly enough the commutant of M on $\mathcal{H} \otimes \mathcal{H}$ is the algebra of all $n \times n$ matrices with entries in M' and the second commutant consists of matrices having a fixed element of M'' down the diagonal.

Let v be the vector $\oplus_{i=1}^n v_i \in \oplus_{i=1}^n \mathcal{H}$ and let $V = Mv \subseteq \mathcal{H} \otimes \mathcal{H}$. Then $MV \subseteq V$ and since $M = M^*$, $P_V \in M'$ (on $\mathcal{H} \otimes \mathcal{H}$) by exercise 2.1.14. So if $y \in M''$ (on $\mathcal{H} \otimes \mathcal{H}$), then y commutes with P_V and $yMv \subseteq Mv$. In particular $y(1v) = xv$ for some $x \in M$ so that $yv_i = xv_i$ for all i, and $y = x \in M$. □

Theorem 3.2.2. *(von Neumann) Let M be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ containing* 1. *Then $M'' = \overline{M}$ (closure in the strong operator topology).*

Proof. Commutants are always closed so $\overline{M} \subseteq M''$.

So let $a \in M''$ and $N(a, \xi_1, \xi_2, ..., \xi_n, \epsilon)$ be a strong neighbourhood of a. We want to find an $x \in M$ in this neighbourhood. So let $v \in \oplus_{i=1}^n \mathcal{H}$ be $\oplus_{i=1}^n \xi_i$ and let $\mathcal{B}(\mathcal{H})$ act diagonally on $\oplus_{i=1}^n \mathcal{H}$ as in the previous theorem. Then the same observations as in the previous proof concerning matrix forms of commutants are true. Also M commutes with $P_{\overline{Mv}}$ which thus commutes with a (on $\oplus_{i=1}^n \mathcal{H}$). And since $1 \in M$, $av = \oplus a\xi_i$ is in the closure of Mv so there is an $x \in M$ with $||x\xi_i - a\xi_i|| < \epsilon$ for all i. □

Corollary 3.2.3. *If $M = M^*$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ with $1 \in M$, then the following are equivalent:*

1. *$M = M''$*
2. *M is strongly closed.*
3. *M is weakly closed.*

Definition 3.2.4. *A subalgebra of $\mathcal{B}(\mathcal{H})$ satisfying the conditions of corollary 3.2.3 is called a von Neumann algebra.*

(A self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the *norm* topology is called a C^*-algebra.)

3.3 Examples.

Example 3.3.1. *Any finite dimensional *-subalgebra of $\mathcal{B}(\mathcal{H})$ containing* 1.

Example 3.3.2. $\mathcal{B}(\mathcal{H})$ *itself.*

Exercise 3.3.3. *Let (X, μ) be a finite measure space and consider $A = L^\infty(X, \mu)$ as a *-subalgebra of $\mathcal{B}(L^2(X, \mu))$ (as multiplication operators as in exercise 2.2.2). Show that $A = A'$, i.e. A is maximal abelian and hence a von Neumann algebra. (Hint: if $x \in A'$ let $f = x(1)$. Show that $f \in L^\infty$ and that $x = M_f$.)*

Example 3.3.4. If $S \subseteq \mathcal{B}(\mathcal{H})$, we call $(S \cup S^*)''$ the von Neumann algebra generated by S. It is, by theorem 3.2.2 the weak or strong closure of the *-algebra generated by 1 and S. Most constructions of von Neumann algebras begin by considering some family of operators with desirable properties and then taking the von Neumann algebra they generate. But is is quite hard, in general, to get much control over the operators added when taking the weak closure, and all the desirable properties of the generating algebra may be lost. (For instance any positive self-adjoint operator a with $||a|| \leq 1$ is a weak limit of projections.) However, if the desirable properties can be expressed in terms of matrix coefficients then these properties will be preserved under weak limits since the matrix coefficients of a are just special elements of the form $\langle \xi, a\eta \rangle$. We shall now treat an example of this kind of property which is at the heart of the subject and will provide us with a huge supply of interesting von Neumann algebras quite different from the first 3 examples.

Let Γ be a discrete group and let $\ell^2(\Gamma)$ be the Hilbert space of all functions $f : \Gamma \to \mathbb{C}$ with $\sum_{\gamma \in \Gamma} |f(\gamma)|^2 < \infty$ and inner product $\langle f, g \rangle = \sum_{\gamma \in \Gamma} \overline{f(\gamma)} g(\gamma)$. An orthonormal basis of $\ell^2(\Gamma)$ is given by the $\{\varepsilon_\gamma\}$ where $\varepsilon_\gamma(\gamma') = \delta_{\gamma,\gamma'}$ so that $f = \sum_{\gamma \in \Gamma} f(\gamma)\varepsilon_\gamma$ in the ℓ^2 sense. For each $\gamma \in \Gamma$ define the unitary operator u_γ by $(u_\gamma f)(\gamma') = f(\gamma^{-1}\gamma')$. Note that $u_\gamma u_\rho = u_{\gamma\rho}$ and that $u_\gamma(\varepsilon_\rho) = \varepsilon_{\gamma\rho}$. Thus $\gamma \mapsto u_\gamma$ is a unitary group representation called the *left regular representation.* The u_γ are linearly independent so the algebra they generate is isomorphic to the group algebra $\mathbb{C}\Gamma$. The von Neumann algebra generated by the u_γ goes under various names, $U(\Gamma)$, $\lambda(\Gamma)$ and $L(\Gamma)$ but we will call it $vN(\Gamma)$ as it is the "group von Neumann algebra" of Γ.

To see that one can control weak limits of linear combinations of the u_γ, consider first the case $\Gamma = \mathbb{Z}/n\mathbb{Z}$. With basis $u_0, u_1, u_2, \cdots, u_{n-1}$, the element u_1 is represented by the matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & . & . \\ 0 & 0 & 1 & 0 & 0 & . \\ 0 & . & 0 & 1 & 0 & 0 \\ 0 & . & . & 0 & 1 & 0 \\ 0 & 0 & . & . & 0 & 1 \\ 1 & 0 & 0 & . & . & 0 \end{bmatrix}$$

which is a matrix constant along the "diagonals". Clearly all powers of u_1 and all linear combinations of them have this property also so that an arbitrary element of the algebra they generate will have the matrix form (when $n = 6$):

$$\begin{bmatrix} a & b & c & d & e & f \\ f & a & b & c & d & e \\ e & f & a & b & c & d \\ d & e & f & a & b & c \\ c & d & e & f & a & b \\ b & c & d & e & f & a \end{bmatrix}$$

(Such matrices are known as circulant matrices but to the best of our knowledge this term only applies when the group is cyclic.) If $\mathbb{Z}/n\mathbb{Z}$ were replaced by another finite group the same sort of structure would prevail except that the "diagonals" would be more complicated, according to the multiplication table of the group.

Now let Γ be an infinite group. It is still true that the (γ, ρ) matrix entry of a finite linear combination of the u_γ's will depend only on $\gamma^{-1}\rho$. As observed above, *the same must be true of weak limits of these linear combinations, hence of any element of* $vN(\Gamma)$.

We see that the elements of $vN(\Gamma)$ have matrices (w.r.t. the basis ε_γ) which are constant along the "diagonals" : $\{(\gamma, \rho) : \gamma\rho^{-1} \text{ is constant}\}$.

Exercise 3.3.5. *Check whether it should be $\gamma^{-1}\rho$ or $\gamma\rho^{-1}$ or some other similar expression.....*

Using the number c_γ on the diagonal indexed by γ we can write, formally at least, any element of $vN(\Gamma)$ as a sum $\sum_{\gamma\in\Gamma} c_\gamma u_\gamma$. It is not clear in what sense this sum converges but certainly $\sum_{\gamma\in\Gamma} c_\gamma u_\gamma$ must define a bounded linear operator. From this we deduce immediately the following:

(i) The function $\gamma \mapsto c_\gamma$ is in ℓ^2. (Apply $\sum_{\gamma\in\Gamma} c_\gamma u_\gamma$ to ε_{id}.)

(ii) $(\sum_{\gamma\in\Gamma} c_\gamma u_\gamma)(\sum_{\gamma\in\Gamma} d_\gamma u_\gamma) = \sum_{\gamma\in\Gamma}(\sum_{\rho\in\Gamma} c_\rho d_{\rho^{-1}\gamma})u_\gamma$

where the sum defining the coefficient of u_γ on the right hand side converges since $\rho \mapsto c_\rho$ and $\rho \mapsto d_{\rho^{-1}\gamma}$ are in ℓ^2.

Exactly what functions $\gamma \mapsto c_\gamma$ define elements of $vN(\Gamma)$ is unclear but an important special case gives some intuition.

Case 1. $\Gamma = \mathbb{Z}$.

It is well known that the map $\sum c_n\varepsilon_n \to \sum c_n e^{in\theta}$ defines a unitary V from $\ell^2(\Gamma)$ to $L^2(\mathbb{T})$. Moreover $Vu_nV^{-1}(e^{ik\theta}) = Vu_n(\varepsilon_k) = V\varepsilon(k+n) = e^{in\theta}e^{ik\theta}$ so that Vu_nV^{-1} is the multiplication operator $M_{e^{in\theta}}$. Standard spectral theory shows that $M_{e^{in\theta}}$ generates $L^\infty(\mathbb{T})$ as a von Neumann algebra, and clearly if $M_f \in L^\infty(\mathbb{T})$, $\quad V^{-1}M_fV = \sum c_n\varepsilon_n$ where $\sum c_n e^{in\theta}$ is the Fourier series of f. We see that, in this case, the functions $\gamma \mapsto c_\gamma$ which define elements of $vN(\mathbb{Z})$ are precisely the Fourier series of L^∞ functions. In case we forget to point it out later on when we are in a better position to prove it, one way to characterise the functions which define elements on $vN(\Gamma)$ is as <u>all</u> functions which define bounded operators on $\ell^2(\Gamma)$. This is not particularly illuminating but can be useful at math parties.

At the other extreme we consider a highly non-commutative group, the free group on n generators, $n \geq 2$.

Case 2. $\Gamma = F_n$.

"Just for fun" let us compute the centre $Z(vN(\Gamma))$ of $vN(F_n)$, i.e. those $\sum c_\gamma u_\gamma$ that commute with all $x \in vN(\Gamma)$. By weak limits of linear combinations, for $\sum c_\gamma u_\gamma$ to be in $Z(vN(\Gamma))$ it is necessary and sufficient that it commute with every u_γ. This clearly is the same as saying $c_{\gamma\rho\gamma^{-1}} = c_\rho \quad \forall\gamma,\rho$, i.e. the function c is constant on conjugacy classes. But in F_n all conjugacy classes except that of the identity are infinite. Now recall that $\gamma \mapsto c_\gamma$ is in ℓ^2. We conclude that $c_\gamma = 0 \quad \forall\gamma \neq 1$, i.e. $Z(vN(\Gamma)) = \mathbb{C}1$.

Note that the only property we used of F_n to reach this conclusion was that every non-trivial conjugacy class is infinite (and in general it is clear that $Z(vN(\Gamma))$ is in the linear span of the u_γ with γ in a finite conjugacy class.) Such groups are called <u>i.c.c.</u> groups and they abound. Other examples include S_∞ (the group of finitely supported permutations of an infinite set), $PSL(n,\mathbb{Z})$ and $\mathbb{Q} \rtimes \mathbb{Q}^*$.

Unsolved problem in von Neumann algebras:

$$\textit{Is } vN(F_n) \cong vN(F_m) \textit{ for } n \neq m \quad (\textit{for } n \textit{ and } m \geq 2)?$$

Note that it is obvious that the group algebras $\mathbb{C}F_n$ and $\mathbb{C}F_m$ are not isomorphic. Just consider algebra homomorphisms to $\mathbb{C}$. But of course these homomorphisms will *not* extend to $vN(\Gamma)$.

Definition 3.3.6. *A von Neumann algebra whose centre is* $\mathbb{C}1$ *is called a* factor*.*

Exercise 3.3.7. *Show that* $\mathcal{B}(\mathcal{H})$ *is a factor.*

Exercise 3.3.8. *Suppose* $\mathcal{H} = \mathcal{K}_1 \otimes \mathcal{K}_2$ *and let* $M = \mathcal{B}(\mathcal{K}_1) \otimes 1$ *Show that* $M' = 1 \otimes \mathcal{B}(\mathcal{K}_2)$ *so that* M *and* M' *are factors.*

This exercise is supposed to explain the origin of the term "factor" as in this case M and M' come from a tensor product factorisation of $\mathcal{H}$. Thus in general a factor and its commutant are supposed to correspond to a bizarre "factorisation" of the Hilbert space.

The factor we have constructed as $vN(\Gamma)$ is of an entirely different nature from $\mathcal{B}(\mathcal{H})$. To see this consider the function $tr : vN(\Gamma) \to \mathbb{C}$ defined by $tr(a) = \langle a\varepsilon_1, \varepsilon_1\rangle$, or $tr(\sum c_\gamma u_\gamma) = c_1$. This map is clearly linear, weakly continuous, satisfies $tr(ab) = tr(ba)$ and $tr(x^*x) = \sum_\gamma |c_\gamma|^2 \geq 0$ (when $x = \sum_\gamma c_\gamma u_\gamma$). It is called a *trace* on $vN(\Gamma)$. If $\Gamma = \mathbb{Z}$ it obviously equals $\frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$ under the isomorphism between $vN(\mathbb{Z})$ and $L^\infty(\mathbb{T})$.

Exercise 3.3.9. *(i)Suppose* $\dim \mathcal{H} < \infty$*. If* $tr : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ *is a linear map with* $tr(ab) = tr(ba)$*, show that there is a constant* K *with* $tr(x) = Ktrace(x)$*.*

(ii) There is no non-zero weakly continuous linear map $tr : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ *satisfying* $tr(ab) = tr(ba)$ *when* $\dim(\mathcal{H}) = \infty$*.*

(iii) There is no non-zero linear map $tr : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ *satisfying* $tr(ab) = tr(ba)$ *and* $tr(x^*x) \geq 0$ *when* $\dim(\mathcal{H}) = \infty$*.*

(iv) (harder) There is no non-zero linear map $tr : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ *satisfying* $tr(ab) = tr(ba)$ *when* $\dim(\mathcal{H}) = \infty$*.*

Thus our factors $vN(\Gamma)$ when Γ is i.c.c. are infinite dimensional but seem to have more in common with $\mathcal{B}(\mathcal{H})$ when $\dim \mathcal{H} < \infty$ than when $\dim \mathcal{H} = \infty$! They certainly do not come from honest tensor product factorisations of $\mathcal{H}$.

Let us make a couple of observations about these factors.

1)They contain no non-zero finite rank operators, for such an operator cannot be constant and non-zero down the diagonal. (Take x^*x if necessary.)

2)They have the property that $tr(a) = 0 \Rightarrow a = 0$ for a positive element a (a positive operator cannot have only zeros down the diagonal).

3)They have the property that $uu^* = 1 \Rightarrow u^*u = 1$ (i.e. they contain no non-unitary isometries).

Proof. If $u^*u = 1$, uu^* is a projection so $1 - uu^*$ is too and $tr(1 - uu^*) = 1 - tr(u^*u) = 0$. □

Exercise 3.3.10. *Show that in $vN(\Gamma)$, $ab = 1 \Rightarrow ba = 1$. Show that if F is any field of characteristic 0, $ab = 1 \Rightarrow ba = 1$ in $F\Gamma$.*

Hints: 1) You may use elementary property 8 of the next chapter.

2) Only finitely many elements of the field are involved in ab and ba in $F\Gamma$.

As far as I know this assertion is still open in non-zero characteristic. The above exercise is a result of Kaplansky.

The next observation is a remarkable property of the set of projections.

4) If $\Gamma = F_n$, $\{tr(p) : p$ a projection in $vN(\Gamma)\} = [0, 1]$.

Proof. It is clear that the trace of a projection is between 0 and 1. To see that one may obtain every real number in this interval, consider the subgroup $\langle a \rangle$ generated by a single non-zero element. By the coset decomposition of F_n the representation of $\langle a \rangle$ on $\ell^2(F_n)$ is the direct sum of countably many copies of the regular representation. The bicommutant of u_a is then, by a matrix argument, $vN(\mathbb{Z})$ acting in an "amplified" way as block diagonal matrices with constant blocks so we may identify $vN(\mathbb{Z})$ with a subalgebra of $vN(\Gamma)$. Under this identification the traces on the two group algebras agree. But as we have already observed, any element $f \in L^\infty(0, 2\pi)$ defines an element of $vN(\mathbb{Z})$ whose integral is its trace. The characteristic function of an interval is a projection so by choosing intervals of appropriate lengths we may realise projections of any trace. □

We used the bicommutant to identify $vN(\mathbb{Z})$ with a subalgebra of $vN(\Gamma)$. It is instructive to point out a problem that would have arisen had we tried to use the weak or strong topologies. A vector in $\ell^2(\Gamma)$ is a square summable sequence of vectors in $\ell^2(\mathbb{Z})$ so that a basic strong neighbourhood of a on $\ell^2(\Gamma)$ would correspond to a neighbourhood of the form $\{b : \sum_{i=1}^\infty ||(a - b)\xi_i||^2 < \epsilon\}$ for a sequence (ξ_i) in $\ell^2(\mathbb{Z})$ with $\sum_{i=1}^\infty ||\xi_i||^2 < \infty$. Thus strong convergence on $\ell^2(\mathbb{Z})$ would not imply strong convergence on $\ell^2(\Gamma)$. This leads us naturally to define two more topologies on $\mathcal{B}(\mathcal{H})$.

Definition 3.3.11. *The topology defined by the basic neighbourhoods of a, $\{b : \sum_{i=1}^\infty ||(a - b)\xi_i||^2 < \epsilon\}$ for any ϵ and any sequence (ξ_i) in $\ell^2(\mathcal{H})$ with $\sum_{i=1}^\infty ||\xi_i||^2 < \infty$, is called the* ultrastrong *topology on $\mathcal{B}(\mathcal{H})$.*

The topology defined by the basic neighbourhoods

$$\{b : \sum_{i=1}^{\infty} |\langle (a-b)\xi_i, \eta_i \rangle| < \epsilon\}$$

for any $\epsilon > 0$ *and any sequences* (ξ_i), (η_i) *in* $\ell^2(\mathcal{H})$ *with*

$$\sum_{i=1}^{\infty} ||\xi_i||^2 + ||\eta_i||^2 < \infty$$

is called the ultraweak *topology on* $\mathcal{B}(\mathcal{H})$.

Note that these topologies are precisely the topologies inherited on $\mathcal{B}(\mathcal{H})$ if it is amplified infinitely many times as $\mathcal{B}(\mathcal{H}) \otimes 1_{\mathcal{K}}$ with $\dim \mathcal{K} = \infty$.

Exercise 3.3.12. *Show that the ultrastrong and strong topologies coincide on a bounded subset of* $\mathcal{B}(\mathcal{H})$ *as do the weak and ultraweak topologies. That they differ will be shown in 5.1.4.*

Exercise 3.3.13. *Repeat the argument of the von Neumann density theorem (3.2.2) with the ultrastrong topology replacing the strong topology.*

Here are some questions that the inquisitive mind might well ask at this stage. All will be answered in succeeding chapters.

Question 1) If there is a weakly continuous trace on a factor, is it unique (up to a scalar multiple)?

Question 2) If there is a trace on a factor M is it true that $\{tr(p) : p \text{ a projection in } M\} = [0,1]$?

Question 3) Is there a trace on any factor not isomorphic to $\mathcal{B}(\mathcal{H})$?

Question 4) Are all (infinite dimensional) factors with traces isomorphic?

Question 5) If M is a factor with a trace, is M' also one? (Observe that the commutant of a factor is obviously a factor.)

Question 6) Is $vN(\Gamma)'$ the von Neumann algebra generated by the right regular representation?

Question 7) If $\phi : M \to N$ is a *-algebra isomorphism between von Neumann algebras on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is there a unitary $u : \mathcal{H} \to \mathcal{K}$ so that $\phi(a) = uau^*$ for $a \in M$?

3.4 Elementary properties of von Neumann algebras.

Throughout this chapter M will be a von Neumann algebra on a Hilbert space $\mathcal{H}$.

EP1) If $a = a^*$ is an element of M, all the spectral projections and all bounded Borel functions of a are in M. Consequently M is generated by its projections.

According to one's proof of the spectral theorem, the spectral projections $E(\lambda)$ of a are constructed as strong limits of polynomials in a. Or the property that the spectral projections of a are in the bicommutant of a may be an explicit part of the theorem. Borel functions will be in the bicommutant.

EP2) Any element in M is a linear combination of 4 unitaries in M.

Proof. We have seen that any x is a linear combination of 2 self-adjoints, and if a is self-adjoint, $||a|| \leq 1$, let $u = a + i\sqrt{1-a^2}$. Then u is unitary and $a = \frac{u+u^*}{2}$. □

EP3) M is the commutant of the unitary group of M' so that an alternative definition of a von Neumann algebra is the commutant of a unitary group representation.

This follows from EP2)

Exercise 3.4.1. *Show that multiplication of operators is jointly strongly continuous on bounded subsets but not on all of $\mathcal{B}(\mathcal{H})$.*

Show that $* : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is weakly continuous but not strongly continuous even on bounded sets.

The following result is well known and sometimes called Vigier's theorem.

Theorem 3.4.2. *If $\{a_\alpha\}$ is a net of self-adjoint operators with $a_\alpha \leq a_\beta$ for $\alpha \leq \beta$ and $||a_\alpha|| \leq K$ for some $K \in \mathbb{R}$, then there is a self-adjoint a with $a = lim_\alpha a_\alpha$, convergence being in the strong topology. Furthermore $a = lub(a_\alpha)$ for the poset of self-adjoint operators.*

Proof. A candidate a for the limit can be found by weak compactness of the unit ball. Then $\langle a_\alpha \xi, \xi \rangle$ is increasing with limit $\langle a\xi, \xi\rangle$ for all $\xi \in \mathcal{H}$ and $a \geq a_\alpha \quad \forall \alpha$. So $lim_\alpha \sqrt{a - a_\alpha} = 0$ in the strong topology. Now multiplication is jointly strongly continuous on bounded sets so $s{-}lim\, a_\alpha = a$. □

Note that we have slipped in the notation $s\!-\!lim$ for a limit in the strong topology (and obviously $w\!-\!lim$ for the weak topology).

If a and (a_α) are as in 3.4.2 we say the net (a_α) is *monotone convergent* to a.

EP4) M is closed under monotone convergence of self-adjoint operators.

The projections on $\mathcal{B}(\mathcal{H})$ form an ortholattice with the following properties:

$$p \le q \iff p\mathcal{H} \subseteq q\mathcal{H}$$
$$p \wedge q = \text{orthogonal projection onto } p\mathcal{H} \cap q\mathcal{H}$$
$$p^\perp = 1 - p$$
$$p \vee q = (p^\perp \wedge q^\perp)^\perp = \text{orthogonal projection onto } \overline{p\mathcal{H} + q\mathcal{H}}.$$

Exercise 3.4.3. *Show that* $p \wedge q = s\!-\!lim_{\,n\to\infty}(pq)^n$.

The lattice of projections in $\mathcal{B}(\mathcal{H})$ is complete (i.e. closed under arbitrary sups and infs) since the intersection of closed subspaces is closed.

EP5) The projections in M generate M as a von Neumann algebra, and they form a complete sublattice of the projection lattice of $\mathcal{B}(\mathcal{H})$.

Proof. If S is a set of projections in M then finite subsets of S are a directed set and $F \to \bigvee_{p\in F} p$ is a net in M satisfying the conditions of 3.4.2. Thus the strong limit of this net exists and is in M. It is obvious that this strong limit is $\bigvee_{p\in S} p$, the sup being in $\mathcal{B}(\mathcal{H})$. □

Easier proof. For each projection $p \in M$, $p\mathcal{H}$ is invariant under each element of M'. Thus the intersection of these subspaces is also. □

EP6) Let A be a *-subalgebra of $\mathcal{B}(\mathcal{H})$. Let $\mathcal{W}$ be $\bigcap_{a\in A} ker(a)$ and $\mathcal{K} = \mathcal{W}^\perp$. Then $\mathcal{K}$ is invariant under A and if we let $B = \{a|_\mathcal{K} : a \in A\}$, then $1_\mathcal{K}$ is in the strong closure of B, which is thus a von Neumann algebra. Moreover on $\mathcal{K}$, B'' is the strong (weak, ultrastrong, ultraweak) closure of B.

Proof. By the above, if p and q are projections $p \vee q = 1 - (1-p) \wedge (1-q)$ is in the strong closure of the algebra generated by p and q. By spectral theory, if $a = a^*$ the range projection $P_{ker(a)^\perp}$ is in the strong closure of the algebra generated by a so we may apply the argument of the proof of EP5) to conclude that $\bigvee_{a\in A} P_{\ker(a)^\perp}$ is in the strong closure of A. But this is $1_\mathcal{K}$.

Finally, on $\mathcal{K}$, let C be the algebra generated by 1 and B. Clearly $C' = B'$ and just as clearly the strong closure of B is the same as the strong closure of C. So B'' is the strong closure of B by the bicommutant theorem. $\square$

Thus if we were to define a von Neumann algebra as a weakly or strongly closed subalgebra of $\mathcal{B}(\mathcal{H})$, it would be unital as an abstract algebra but its identity might not be that of $\mathcal{B}(\mathcal{H})$ so it would not be equal to its bicommutant. However on the orthogonal complement of all the irrelevant vectors it would be a von Neumann algebra in the usual sense.

EP7) If M is a von Neumann algebra and $p \in M$ is a projection, $pMp = (M'p)'$ and $(pMp)' = M'p$ as algebras of operators on $p\mathcal{H}$. Thus pMp and $M'p$ are von Neumann algebras.

Proof. Obviously pMp and $M'p$ commute with each other on $p\mathcal{H}$. Now suppose $x \in (M'p)' \subseteq \mathcal{B}(p\mathcal{H})$ and define $\tilde{x} = xp (= pxp) \in \mathcal{B}(\mathcal{H})$. Then if $y \in M'$, $y\tilde{x} = yxp = ypxp = (xp)(yp) = xpy = \tilde{x}y$, so $\tilde{x} \in M$ and $x = p\tilde{x}p$. Thus $(pM')' = pMp$ which is thus a von Neumann algebra.

If we knew that $M'p$ were a von Neumann algebra on $p\mathcal{H}$ we would be done but a direct attempt to prove it strongly or weakly closed fails as we would have to try to extend the limit of a net in $M'p$ on $p\mathcal{H}$ to be in M'.

So instead we will show directly that $(pMp)' \subseteq M'p$ by a clever extension of its elements to $\mathcal{H}$. By EP2 it suffices to take a unitary $u \in (pMp)'$. Let $\mathcal{K} \subseteq \mathcal{H}$ be the closure of $Mp\mathcal{H}$ and let q be projection onto it. Then $\mathcal{K}$ is clearly invariant under M and M' so $q \in Z(M)$. We first extend u to $\mathcal{K}$ by

$$\tilde{u} \sum x_i \xi_i = \sum x_i u \xi_i$$

for $x_i \in M$ and $\xi_i \in p\mathcal{H}$. We claim that $\tilde{u}$ is an isometry:

$$||\tilde{u} \sum x_i \xi_i||^2 = \sum_{i,j} \langle x_i u \xi_i, x_j u \xi_j \rangle$$

$$= \sum_{i,j} \langle p x_j^* x_i p u \xi_i, u \xi_j \rangle$$

$$= \sum_{i,j} \langle u p x_j^* x_i p \xi_i, u \xi_j \rangle$$

$$= ... = || \sum x_i \xi_i ||^2$$

This calculation actually shows that $\tilde{u}$ is well defined and extends to an isometry of $\mathcal{K}$. By construction $\tilde{u}$ commutes with M on $Mp\mathcal{H}$,hence on $\mathcal{K}$. So $\tilde{u}q \in M'$ and $u = \tilde{u}qp$. Hence $(pMp)' = M'p$. $\square$

Corollary 3.4.4. *If M is a factor, pMp is a factor on $p\mathcal{H}$, as is pM'. Moreover the map $x \mapsto xp$ from M' to $M'p$ is a weakly continuous *-algebra isomorphism.*

Proof. As in the proof of the previous result, the projection onto the closure of $Mp\mathcal{H}$ is in the centre of M, hence it is 1 . So if $xp = 0$ for $x \in M'$, $xmp\xi = mxp\xi = 0$ for any $m \in M$, $\xi \in \mathcal{H}$. Hence the map $x \mapsto px$ is an injective $*$-algebra map and pM' is a factor. So by the previous result $(pMp)'$ is a factor and so is pMp. Continuity and the is obvious. $\square$

Corollary 3.4.5. *If M is a factor and $a \in M$ and $b \in M'$ then $ab = 0$ implies either $a = 0$ or $b = 0$.*

Proof. Let p be the range projection of b and apply the previous corollary. $\square$

Exercise 3.4.6. *Show that if M is a von Neumann algebra generated by the self-adjoint, multiplicatively closed subset S, then pSp generates pMp (if p is a projection in M or M'). Show further that the result fails if S is not closed under multiplication.*

Exercise 3.4.7. *Show that if M is a factor and V and W are finite dimensional subspaces of M and M' respectively then the map $a \otimes b \mapsto ab$ defines a linear isomorphism between $V \otimes W$ and the space VW spanned by all vw with $v \in V$ and $w \in W$.*

EP8) If $a \in M$ and $a = u|a|$ is the polar decomposition of a then $u \in M$ and $|a| \in M$.

Proof. By the uniqueness of the polar decomposition both $|a|$ and u commute with every unitary in M'. $\square$

EP9) None of the topologies (except $|| * ||$) is metrizable on $\mathcal{B}(\mathcal{H})$ but they all are on the unit ball (when $\mathcal{H}$ is separable) and $\mathcal{B}(\mathcal{H})$ is separable for all except the norm topology.

Proof. First observe that a weakly convergent sequence of operators is bounded. This follows from the uniform boundedness principle and 2.1.5 which shows how to get the norm from inner products.

Here is the cunning trick. Let $\{\eta_i, i = 1, \cdots \infty\}$ be an orthonormal basis of $\mathcal{H}$ and let e_i be projection onto $\mathbb{C}\eta_i$. Consider the family $\{e_m + me_n : m, n = 1, \cdots \infty\}$. Let V a basic ultrastrong neighbourhood of 0 defined by ϵ and $\{\xi_i : \sum ||\xi_i||^2 < \infty\}$ and let $| - |_V$ be the corresponding seminorm, then writing

$\xi_i = \sum_j \xi^i_j \eta_j$ we have $\sum_{i,j} |\xi^i_j|^2 < \infty$. Now choose m so that $\sum_i |\xi^i_m|^2 < \epsilon^2/4$ and n so that $\sum_i |\xi^i_n|^2 < \epsilon^2/4m^2$. Observing that $||e_n(\xi_i)||^2 = |\xi^n_i|^2$ we have

$$\begin{aligned} |e_m + me_n|_V &\le |e_m|_V + m|e_n|_V \\ &= \sqrt{\sum_i ||e_m\xi||^2} + m\sqrt{\sum_i ||e_n\xi||^2} \\ &\le \epsilon/2 + \epsilon/2 \end{aligned}$$

so that $e_m + me_n \in V$.

On the other hand no subsequence of $\{e_m + me_n : m, n = 1, \cdots \infty\}$ can tend even weakly to 0 since it would have to be bounded in norm which would force some fixed m to occur infinitely often in the sequence, preventing weak convergence! So by the freedom in choosing m and n to force $e_m + me_n$ to be in V, there can be no countable basis of zero for any of the topologies (except of course the norm).

If we consider the unit ball, however, we may choose a dense sequence ξ_i of unit vectors and define $d(a, b) = [\sum_i 2^{-i}||(a - b)\xi_i||^2]^{1/2}$ which is a metric on the unit ball defining the strong topology. (Similarly for the weak topology.)

Separability of $\mathcal{B}(\mathcal{H})$ for separable $\mathcal{H}$ follows from looking at finitely support rational matrices for some onb. (Separability of the unit ball is more subtle.)

We leave non-separability of $\mathcal{B}(\mathcal{H})$ in the norm topology as an exercise. □

EP10) An Abelian von Neumann algebra on a separable Hilbert space is generated by a single self-adjoint operator.

Proof. The unit ball of $\mathcal{B}(\mathcal{H})$ is a compact metrizable space in the weak topology, hence it is separable. So the unit ball of a von Neumann algebra is a separable metric space. So let $\{e_0, e_1, e_2, \cdots\}$ be a sequence of projections that is weakly dense in the set of all projections in the Abelian von Neumann algebra A. Let $a = \sum_{n=0}^{\infty} \frac{1}{3^n} e_n$. The sum converges in the norm topology so $a \in A$. The norm of the self-adjoint operator $a_1 = \sum_{n=1}^{\infty} \frac{1}{3^n} e_n$ is obviously at most $1/2$ so that the spectral projection for the interval $[3/4, 2]$ for a is e_0. Continuing in this way we see that all the $e'_n s$ are in $\{a\}''$. □

This relegates the study of Abelian von Neumann algebras to the spectral theorem. One can show that any Abelian von Neumann algebra on a separable Hilbert space is isomorphic to either $\ell^\infty(\{0, 1, \cdots, n\})$ (where $n = \infty$ is allowed) or $L^\infty([0, 1], dx)$ or a direct sum of the two. This is up to abstract algebra isomorphism. To understand the action on a Hilbert space, multiplicity must be taken into account.

Chapter 4

Finite dimensional von Neumann algebras and type I factors.

4.1 Definition of type I factor.

The first crucial result about factors (remember a factor is a von Neumann algebra with trivial centre) will be the following "ergodic" property.

Theorem 4.1.1. *If M is a factor and p and q are non-zero projections in M there is an $x \in M$ with $pxq \neq 0$. Moreover x can be chosen to be unitary.*

Proof. Suppose that for any unitary $u \in M$, $puq = 0$. Then $u^*puq = 0$ and $\left(\bigvee_{u \in M} u^*pu\right) q = 0$. But clearly $\bigvee_{u \in M} u^*pu$ commutes with all unitaries $u \in M$ so is the identity. □

The reason we have called this an "ergodic" property is because of a pervasive analogy with measure-theoretic dynamical systems (and it will become much more than an analogy). A transformation $T : (X, \mu) \to (X, \mu)$ preserving the measure μ is called *ergodic* if $T^{-1}(A) \subseteq A$ implies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$ for a measurable $A \subseteq X$. If T is invertible one can then show that there is, for any pair $A \subset X$ and $B \subset X$ of non-null sets, a power T^n of T such that $\mu(T^n(A) \cap B) \neq 0$. Or, as operators on $L^2(X, \mu)$, $AT^N B \neq 0$ where we identify A and B with the multiplication operators by their characteristic functions. The proof is the same—the union of all $T^n(A)$ is clearly invariant, thus must differ from all of X by a set of measure 0.

Corollary 4.1.2. *Let p and q be non-zero projections in a factor M. Then there is a partial isometry u ($\neq 0$) in M with $uu^* \leq p$ and $u^*u \leq q$.*

p and q had been swapped around in the proof, so we interchanged uu^* and u^*u in the statement.

Proof. Let u be the partial isometry of the polar decomposition of pxq for x such that $pxq \neq 0$. □

Definition 4.1.3. *If M is a von Neumann algebra, a non-zero projection $p \in M$ is called* minimal*, or an* atom*, if* $(q \leq p) \Rightarrow (q = 0 \quad or \quad q = p)$.

Exercise 4.1.4. *Show that p is minimal in M iff $pMp = \mathbb{C}p$.*

Definition 4.1.5. *A factor with a minimal projection is called a* type I factor*.*

4.2 Classification of all type I factors

We will classify all type I factors quite easily. We begin with the model, which we have already seen.

Let $\mathcal{B}(\mathcal{H}) \otimes 1$ be the constant diagonal matrices on $\mathcal{H} \otimes \mathcal{K}$. Its commutant $1 \otimes \mathcal{B}(\mathcal{K})$ will be our model. It is the algebra of all matrices defining bounded operators with every matrix entry being a scalar multiple of the identity matrix on $\mathcal{H}$. A matrix with a single 1 on the diagonal and zeros elsewhere is obviously a minimal projection.

Theorem 4.2.1. *If M is a type I factor of a Hilbert space $\mathcal{L}$, there are Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and a unitary $u : \mathcal{L} \to \mathcal{H} \otimes \mathcal{K}$ with $uMu^* = \mathcal{B}(\mathcal{H}) \otimes 1$.*

Proof. Let $\{p_1, p_2, ...\}$ be a maximal family of minimal projections in M such that $p_i p_j = 0$ for $i \neq j$. (We assume for convenience that $\mathcal{L}$ is separable.) Our first claim is that $\bigvee_i p_i = 1$ so that $\mathcal{L} = \oplus_i p_i \mathcal{L}$. For if $1 - \bigvee_i p_i$ were nonzero, by corollary 4.1.2 there would be a $u \neq 0$ with $uu^* \leq p_1$ and $u^*u \leq 1 - \bigvee_i p_i$. By minimality uu^* is minimal and hence so is u^*u contradicting maximality of the p_i. Now for each i choose a non-zero partial isometry e_{1i} with $e_{1i}e_{1i}^* \leq p_1$ and $e_{1i}^* e_{1i} \leq p_i$. By minimality $e_{1i}e_{1i}^* = p_1$ and $e_{1i}^* e_{1i} = p_i$. Then M is generated by the e_{1i}'s, for if $a \in M$ we have $a = \sum_{i,j} p_i a p_j$ the sum converging in the strong topology. Moreover for each i and j, $e_{1i} a e_{1j}^* = p_1 e_{1i} a e_{1j}^* p1$ so by minimality it is $\lambda_{i,j} p_1$ for $\lambda_{i,j} \in \mathbb{C}$. So $p_i a p_j = e_{1i}^* e_{1i} a e_{1j}^* e_{1j} = \lambda_{i,j} e_{1i}^* e_{1j}$. And so

$$a = \sum_{i,j} \lambda_{ij} e_{1i}^* e_{1j}.$$

(The details of the convergence of the sum are unimportant—we just need that a be in the strong closure of finite sums.)

If n is the cardinality of $\{p_i\}$, let $X = \{1, 2, ..., n\}$ and define the map $u : \ell^2(X, p_1\mathcal{L}) \to \mathcal{L}$ by

$$uf = \sum_i e_{1i}^* f(i).$$

Observe that u is unitary and $u^*e_{1i}u$ is a matrix on $\ell^2(X, p_1\mathcal{L})$ with an identity operator in the $(1, i)$ position and zeros elsewhere. The algebra generated by these matrices is $\mathcal{B}(\ell^2(X)) \otimes 1$ on $\ell^2(X) \otimes p_1\mathcal{L}$ and we are done. □

Remark 4.2.2. *The importance of being spatial.*

We avoided all kinds of problems in the previous theorem by constructing our isomorphism using a unitary between the underlying Hilbert spaces. In general given von Neumann algebras M and N generated by S and T respectively, to construct an isomorphism between M and N it suffices to construct (if possible !!!) a unitary u between their Hilbert spaces so that T is contained in uSu^*. To try to construct an isomorphism directly on S could be arduous at best. ter

4.3 Tensor product of von Neumann algebras.

If M is a von Neumann algebra on $\mathcal{H}$ and N is a von Neumann algebra on $\mathcal{K}$ we define $M \otimes N$ to be the von Neumann algebra on $\mathcal{H} \otimes \mathcal{K}$ generated by $\{x \otimes y : x \in M, y \in N\}$.

Exercise 4.3.1. *Show that $M \otimes N$ contains the algebraic tensor product $M \otimes_{alg} N$ as a strongly dense *-subalgebra.*

Definition 4.3.2. *Let M be a von Neumann algebra. A* system of matrix units *(s.m.u.) of size n in M is a family $\{e_{ij} : i, j = 1, 2, ..., n\}$ ($n = \infty$ allowed) such that*

(i) $e_{ij}^* = e_{ji}$.

(ii) $e_{ij}e_{kl} = \delta_{j,k}e_{il}$

(iii) $\sum_i e_{ii} = 1$.

Exercise 4.3.3. *Show that if $\{e_{ij}; i, j = 1, ..., n\}$ is an s.m.u. in a von Neumann algebra M, then the e_{ij} generate a type I factor isomorphic to $\mathcal{B}(\ell^2(\{1, 2, ..., n\}))$ and that M is isomorphic (unitarily equivalent to in this instance) to the von Neumann algebra $e_{11}Me_{11} \otimes \mathcal{B}(\ell^2(\{1, 2, ..., n\}))$.*

4.4 Multiplicity and finite dimensional von Neumann algebras.

Theorem 4.2.1 shows that type I factors on Hilbert space are completely classified by two cardinalities (n_1, n_2) according to:

$n_1 =$ rank of a minimal projection in M, and
$n_2 =$ rank of a minimal projection in M'.

We see that the isomorphism problem splits into "abstract isomorphism" (determined by n_2 alone), and "spatial isomorphism", i.e. unitary equivalence. A type I_n factor is by definition one for which $n = n_2$. It is abstractly isomorphic to $\mathcal{B}(\mathcal{H})$ with $\dim \mathcal{H} = n$. The integer n_1 is often called the *multiplicity* of the type I factor.

We will now determine the structure of all finite dimensional von Neumann algebras quite easily. Note that in the following there is no requirement that $\mathcal{H}$ be finite dimensional.

Theorem 4.4.1. *Let M be a finite dimensional von Neumann algebra on the Hilbert space $\mathcal{H}$. Then M is abstractly isomorphic to $\oplus_{i=1}^k M_{n_i}(\mathbb{C})$ for some positive integers $k, n_1, n_2, ..., n_k$. ($M_n(\mathbb{C})$ is the von Neumann algebra of all $n \times n$ matrices on n-dimensional Hilbert space.) Moreover there are Hilbert spaces $\mathcal{K}_i$ and a unitary $u : \oplus_i \ell^2(X_i, \mathcal{K}_i) \to \mathcal{H}$ (with $|X_i| = n_i$) with $u^*Mu = \oplus_i \mathcal{B}(\ell^2(X_i)) \otimes 1$.*

Proof. The centre $Z(M)$ is a finite dimensional abelian von Neumann algebra. If p is a minimal projection in $Z(M)$, pMp is a factor on $p\mathcal{H}$. The theorem follows immediately from theorem 4.2.1 and the simple facts that $Z(M) = \oplus_{i=1}^k p_i\mathbb{C}$ where the p_i are the minimal projections in $Z(M)$ (two distinct minimal projections p and q in $Z(M)$ satisfy $pq = 0$), and $M = \oplus_i p_i M p_i$. □

The subject of finite dimensional von Neumann algebras is thus rather simple. It becomes slightly more interesting if one considers *subalgebras* $N \subseteq M$. Let us deal first with the factor case of this. Let us point out that the identity of M is the same as that of N.

Theorem 4.4.2. *If M is a type I_n factor, its type I_m factors are all uniquely determined, up to conjugation by unitaries in M, by the integer (or ∞) $k > 0$ such that pMp is a type I_k factor, p being a minimal projection in the subfactor N and $mk = n$.*

Proof. Let N_1 and N_2 be type I_m subfactors with generating s.m.u.'s $\{e_{ij}\}$ and $\{f_{ij}\}$ respectively. If k is the integer (in the statement of the theorem) for N_1 then $1 = \sum_1^m e_{ii}$ and each e_{ii} is the sum of k mutually orthogonal minimal projections of M, hence $n = mk$. The same argument applies to N_2. Build a partial isometry u with $uu^* = e_{11}$ and $u^*u = f_{11}$ by adding together partial isometries between maximal families of mutually orthogonal projections less than e_{11} and f_{11} respectively. Then it is easy to check that $w = \sum_i e_{j1} u f_{1j}$ is a unitary with $w f_{kl} w^* = e_{kl}$. So $wN_2w^* = N_1$. □

Now we can do the general (non-factor) case. If $N = \oplus_{i=1}^{n} M_{k_i}(\mathbb{C})$ and $M = \oplus_{j=1}^{m} M_{r_j}(\mathbb{C})$ and $N \subseteq M$ as von Neumann algebras, let p_j be minimal central projections in M and q_i be those of N. Then for each (i, j), $p_j q_i M q_i p_j$ is a factor and $p_j q_i N$ is a subfactor so we may form the matrix $\Lambda = (\lambda_{ij})$ where λ_{ij} is the integer associated with $p_j q_i N \subseteq p_j q_i M q_i p_j$ by theorem 4.4.2.

Exercise 4.4.3. *Show that the integer λ_{ij} defined above is the following: if e_i is a minimal projection in the factor $q_i N$, λ_{ij} = trace of the matrix $p_j e_i \in M_{r_j}\mathbb{C}$.*

Example 4.4.4. *Let $M = M_5(\mathbb{C}) \oplus M_3(\mathbb{C})$ and N be the subalgebra of matrices of the form:*

$$\left(\begin{smallmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & z \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} X & 0 \\ 0 & z \end{smallmatrix}\right)$$

where $z \in \mathbb{C}$ and X is a 2×2 matrix. Then N is isomorphic to $M_2(\mathbb{C}) \oplus \mathbb{C}$ and if $p_1 = 1 \oplus 0$, $q_1 = 1 \oplus 0$, etc., we have

$$\Lambda = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right).$$

The matrix Λ is often represented by a bipartite graph with the number of edges between i and j being λ_{ij}. The vertices of the graph are labelled by the size of the corresponding matrix algebras. Thus in the above example the picture would be:

This diagram is called the *Bratteli diagram* for $N \subseteq M$.

Exercise 4.4.5. *Generalise the above example to show that there is an inclusion $N \subseteq M$ corresponding to any Bratteli diagram with any set of dimensions for the simple components of N.*

4.5 A digression on index.

If $N \subseteq M$ are type I factors we have seen that there is an integer k (possibly ∞) such that M is the algebra of $k \times k$ matrices over N. If $k < \infty$, M is thus a free left N-module of rank k^2. It seems reasonable to call the number k^2 the *index* of N in M and write it $[M : N]$. This is because, if $H < G$ are groups and $\mathbb{C}H \subseteq \mathbb{C}G$ their group algebras, the coset decomposition of G shows that $\mathbb{C}G$ is a free left $\mathbb{C}H$-module of rank $[G : H]$.

Chapter 5

Kaplansky Density Theorem.

5.1 Some simple but telling results on linear functionals.

We begin with a result about linear functionals of independent interest.

Theorem 5.1.1. *Let V be a subspace of $\mathcal{B}(\mathcal{H})$ and let $\phi : V \to \mathbb{C}$ be a linear functional. The following are equivalent:*

(i) There are vectors in $\mathcal{H}$, $\xi_1, \xi_2, ..., \xi_n$ and $\eta_1, \eta_2, ..., \eta_n$ with

$$\phi(x) = \sum_{i=1}^{n} \langle x\xi_i, \eta_i \rangle$$

(ii) ϕ is weakly continuous.

(iii) ϕ is strongly continuous.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious, so suppose ϕ is strongly continuous. One may use the seminorms $\sqrt{\sum_{i=1}^{n} ||a\xi_i||^2}$ as $\{\xi_1, \xi_2, ..., \xi_n\}$ ranges over all finite subsets of $\mathcal{H}$ to define the strong topology. Srong continuity implies that there is an $\epsilon > 0$ and $\{\xi_1, \xi_2, ..., \xi_n\}$ such that $\sqrt{\sum_{i=1}^{n} ||a\xi_i||^2} < \epsilon$ implies $|\phi(a)| \leq 1$. But then if $\sqrt{\sum_{i=1}^{n} ||a\xi_i||^2} = 0$ then multiplying a by large scalars implies $\phi(a) = 0$. Otherwise it is clear that $|\phi(a)| \leq \frac{1}{\epsilon}\sqrt{\sum_{i=1}^{n} ||a\xi_i||^2}$.

Now let $\xi = \xi_1 \oplus ...\xi_n \in \oplus_i \mathcal{H}$ and let $\mathcal{K} = \overline{(V \otimes 1)(\xi)}$. Then define $\tilde{\phi}$ on $V \otimes 1(\xi)$ by $\tilde{\phi}(\oplus_i x\xi_i) = \phi(x)$. Observe that $\tilde{\phi}$ is well-defined and continuous so extends to $\mathcal{K}$ which means there is a vector $\eta = \oplus \eta_i \in \mathcal{K}$ with $\phi(x) = \tilde{\phi}(x \otimes 1)(\eta) = \langle (x \otimes 1)(\xi), \eta \rangle$. □

Exercise 5.1.2. *Replace weak and strong by ultraweak and ultrastrong, and the finite sequences of vectors by ℓ^2-convergent ones in the previous theorem.*

Corollary 5.1.3. *If C is a convex subset of $\mathcal{B}(\mathcal{H})$, its weak and strong closures coincide.*

Proof. Two locally convex vector spaces with the same continuous linear functionals have the same closed convex sets. This is a consequence of the Hahn-Banach theorem to be found in any text on functional analysis. □

Corollary 5.1.4. *If* $\dim \mathcal{H} = \infty$ *the strong and ultrastrong topologies differ on* $\mathcal{B}(\mathcal{H})$.

Proof. Let (ξ_i) be an orthonormal basis of $\mathcal{H}$ and let $\omega(x) = \sum_i \frac{1}{n^2}\langle x\xi_i, \xi_i\rangle$. Then ω is ultraweakly continuous but not strongly continuous. For if it were weakly continuous it would be of the form $\sum_{i=1}^n \langle x\nu_i, \eta_i\rangle$ and $\omega(p) = 0$ where p is the projection onto the orthogonal complement of the vector space spanned by the ν_i. But by positivity $\omega(p) = 0$ forces $p(\xi_i) = 0$ for all i. □

5.2 The theorem

In our discussion of $vN(\Gamma)$ we already met the desirability of having a *norm-bounded* sequence of operators converging to an element in the weak closure of a *-algebra of operators. This is not guaranteed by the von Neumann density theorem. The Kaplansky density theorem fills this gap.

Theorem 5.2.1. *Let A be a *-subalgebra of $\mathcal{B}(\mathcal{H})$. Then the unit ball of A is strongly dense in the unit ball of the weak closure M of A, and the self-adjoint part of the unit ball of A is strongly dense in the self-adjoint part of the unit ball of M.*

Proof. By EP6) we may assume $1 \in M$ and the worried reader may check that we never in fact suppose $1 \in A$. We may further suppose that A is norm-closed, i.e. a C*-algebra. Consider the closure of A_{sa}, the self-adjoint part of A. The * operation is weakly continuous so if x_α is a net converging to the self-adjoint element $x \in M$, $\frac{x_\alpha + x_\alpha^*}{2}$ converges to x so the weak closure of A_{sa} is equal to M_{sa}. Since A_{sa} is convex, the strong closure is also equal to M_{sa} by 5.1.3.

Let us now prove the second assertion of the theorem. Let $x = x^* \in M$, $||x|| < 1$, and $\xi_1, ..., \xi_n, \epsilon > 0$ define a strong neighbourhood of x. We must come up with a $y \in A_{sa}$, $||y|| < 1$, with $||(x-y)\xi_i|| < \epsilon$. The function $t \to \frac{2t}{1+t^2}$ is a homeomorphism of $[-1, 1]$ onto itself. So by the spectral theorem we may choose an $X \in M_{sa}$ with $||X|| \le 1$, so that $\frac{2X}{1+X^2} = x$. Now by strong density choose $Y \in A_{sa}$ with

$$||Yx\xi_i - Xx\xi_i|| < \epsilon, \text{ and } ||\frac{Y}{1+X^2}\xi_i - \frac{X}{1+X^2}\xi_i|| < \epsilon/4.$$

Put $y = \frac{2Y}{1+Y^2}$ and note that $||y|| \leq 1$.

Now consider the following equalities:

$$y - x = \frac{2Y}{1+Y^2} - \frac{2X}{1+X^2}$$

$$= 2(\frac{1}{1+Y^2}(Y(1+X^2) - (1+Y^2)X)\frac{1}{1+X^2})$$

$$= 2(\frac{1}{1+Y^2}(Y-X)\frac{1}{1+X^2} + \frac{Y}{1+Y^2}(X-Y)\frac{X}{1+X^2})$$

$$= \frac{2}{1+Y^2}(Y-X)\frac{1}{1+X^2} + \frac{1}{2}y(X-Y)x.$$

By the choice of Y, we see that $||(y-x)\xi_i|| < \epsilon$. This proves density for the self-adjoint part of the unit ball.

Now consider a general $x \in M$ with $||x|| \leq 1$. The trick is to form $\left(\begin{smallmatrix} 0 & x \\ x^* & 0 \end{smallmatrix}\right) \in M \otimes M_2(\mathbb{C})$. Strong convergence of a net $\left(\begin{smallmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is equivalent to strong convergence of the matrix entries so $A \otimes M_2(\mathbb{C})$ is strongly dense in $M \otimes M_2(\mathbb{C})$. Moreover if $\left(\begin{smallmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 0 & x \\ x^* & 0 \end{smallmatrix}\right)$ strongly then b_α tends strongly to x. And $||b_\alpha|| \leq 1$ follows from $||\left(\begin{smallmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{smallmatrix}\right)|| \leq 1$ and $\langle b_\alpha \xi, \eta\rangle = \langle \left(\begin{smallmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 \\ \xi \end{smallmatrix}\right), \left(\begin{smallmatrix} \eta \\ 0 \end{smallmatrix}\right)\rangle$. □

Corollary 5.2.2. *If M is a *-subalgebra of $\mathcal{B}(\mathcal{H})$ containing 1 then M is a von Neumann algebra iff the unit ball of M is weakly compact.*

Proof. The unit ball of $\mathcal{B}(\mathcal{H})$ is weakly compact, and M is weakly closed.

Conversely, if the unit ball of M is weakly compact, then it is weakly closed. Let x be in the weak closure of M. We may suppose $||x|| = 1$. By Kaplansky density there is a net x_α weakly converging to x with $||x_\alpha|| \leq 1$. Hence $x \in M$. □

Chapter 6

Comparison of Projections and Type II_1 Factors.

6.1 Order on projections

Definition 6.1.1. *If p and q are projections in a von Neumann algebra M we say that $p \preceq q$ if there is a partial isometry $u \in M$ with $uu^* = p$ and $u^*u \leq q$. We say that p and q are equivalent, $p \approx q$ if there is a partial isometry $u \in M$ with $uu^* = p$ and $u^*u = q$.*

Observe that $\approx$ is an equivalence relation.

Theorem 6.1.2. *The relation $\preceq$ is a partial order on the equivalence classes of projections in a von Neumann algebra.*

Proof. Transitivity follows by composing partial isometries. The issue is to show that $e \preceq f$ and $f \preceq e$ imply $e \approx f$. Compare this situation with sets and their cardinalities.

Let u and v satisfy $uu^* = e, u^*u \leq f$ and $vv^* = f, v^*v \leq e$. Note the picture:

We define the two decreasing sequences of projections $e_0 = e, e_{n+1} = v^*f_nv$ and $f_0 = f, f_{n+1} = u^*e_nu$. The decreasing property follows by induction since $p \to v^*pv$ gives an order preserving map from projections in M less than f to projections in M less than e and similarly interchanging the roles of e and f, v and u. Let $e_\infty = \bigwedge_{i=0}^{\infty} e_i$ and $f_\infty = \bigwedge_{i=0}^{\infty} f_i$. Note that $v^*f_\infty v = e_\infty$ and $f_\infty vv^*f_\infty = f_\infty$ so that $e_\infty \approx f_\infty$. Also $e = (e - e_1) + (e_1 - e_2) + \cdots + e_\infty$ and $f = (f - f_0) + (f_1 - f_2) + \cdots + f_\infty$ are sums of mutually orthogonal projections. But for each even i, $u^*(e_i - e_{i+1})u = f_{i+1} - f_{i+2}$ so $e_i - e_{i+1} \approx f_{i+1} - f_{i+2}$, and

$v^*(f_i - f_{i+1})v = e_{i+1} - e_{i+2}$ so one may add up, in the strong topology, all the relevant partial isometries to obtain an equivalence between e and f. $\square$

Note that if we had been dealing with $vN(\Gamma)$ this argument would have been unnecessary as we could have used the trace:

$$tr(v^*v) \leq tr(e) = tr(uu^*) = tr(u^*u) \leq tr(f) = tr(vv^*) = tr(v^*v)$$

so that $tr(e - v^*v) = 0$ which implies $e = v^*v$. However in general it is certainly possible to get a projection equivalent to a proper subprojection of itself. Just take the unilateral shift on $\mathcal{B}(\ell^2(\mathbb{N}))$ which exhibits an equivalence between 1 and the projection onto the orthogonal complement of the first basis vector. This is analogous to the notion of an infinite set—one which is in bijection with a proper subset of itself.

Definition 6.1.3. *A projection p in a von Neumann algebra M is called* infinite *if $p \approx q$ for some $q < p$, $p \neq q$. Otherwise p is called* finite. *A von Neumann algebra is called* finite *if its identity is finite, and it is called* purely infinite *if it has no finite projections other than* 0. *A factor is called* infinite *if its identity is infinite.*

We will show that purely infinite von Neumann algebras exist though it will not be easy.

Remark 6.1.4. *If $\dim \mathcal{H} = \infty$ then $\mathcal{B}(\mathcal{H})$ is infinite.*

Remark 6.1.5. *A factor with a trace like $vN(\Gamma)$ is finite.*

Remark 6.1.6. *Every projection in a finite von Neumann algebra is finite. Or, more strongly, if $p \leq q$ and q is finite then p is finite.*

For if $p \approx p'$, $\quad p' < p$, $\quad p \neq p'$ then $p + (q - p) \approx p' + (q - p) \neq q$.

Remark 6.1.7. *If M is any von Neumann algebra, 1 is an infinite projection in $M \otimes \mathcal{B}(\mathcal{H})$ if $\dim \mathcal{H} = \infty$.*

Theorem 6.1.8. *If M is a factor and p, q are projections in M, either $p \preceq q$ or $q \preceq p$.*

Proof. Consider the family of partial isometries u with $uu^* \leq p, u^*u \leq q$. This set is partially ordered by $u \leq v$ if $u^*u \leq v^*v$ and $v = u$ on the initial domain $u^*u\mathcal{H}$ of u. This partially ordered set satisfies the requirements for Zorn's lemma so let u be a maximal element in it. If $u^*u = q$ or $uu^* = p$ we are done so suppose $q - u^*u$ and $p - uu^*$ are both non-zero. Then by 4.1.1 there is a $v \neq 0$ with $v^*v \leq q - u^*u$ and $vv^* \leq p - uu^*$. But then $u + v$ is larger than u which was supposed maximal. $\square$

Exercise 6.1.9. *Show that two equivalent projections p and q in a finite factor M are unitarily equivalent, i.e. there is a unitary $u \in M$ with $upu^* = q$.*

We see that the equivalence classes of projections in a factor form a totally ordered set. It is known that, on a separable Hilbert space, the possible isomorphism types for this set are:

1)	$\{0, 1, 2, ..., n\}$ where $n = \infty$ is allowed.	"type I_n"
2)	$[0, 1]$	"type II_1"
3)	$[0, \infty]$	"type II_∞"
4)	$\{0, \infty\}$	"type III"

Strictly speaking this is nonsense as type III is the same as type I_1 and II_1 is the same as II_∞. We mean not only the order type but whether 1 is infinite or not.

Observe that the type II_1 case certainly exists. We saw that $vN(F_2)$ has projections of any trace between 0 and 1. By the previous theorem it is clear that the trace gives an isomorphism between the ordered set of equivalence classes of projections and the unit interval. We will proceed to prove a statement generalising this considerably.

Definition 6.1.10. *A* type II_1 factor *is an infinite dimensional factor M on $\mathcal{H}$ admitting a non-zero linear function $tr : M \to \mathbb{C}$ satisfying*

(i) $tr(ab) = tr(ba)$

(ii) $tr(a^*a) \geq 0$

(iii) tr is ultraweakly continuous.

The trace is said to be normalised if $tr(1) = 1$.

Definition 6.1.11. *In general a linear functional ϕ on a *-algebra A is called* positive *if $\phi(a^*a) \geq 0$ (and $\phi(a^*) = \overline{\phi(a)}$ though this is redundant if A is a C^*-algebra), and* faithful *if $\phi(a^*a) = 0 \Rightarrow a = 0$. A positive ϕ is called a* state *if $1 \in A$ and $\phi(1) = 1$. A linear functional ϕ is called* tracial *(or a trace) if $\phi(ab) = \phi(ba)$.*

It is our job now to show that a II_1 factor has a unique ultraweakly continuous tracial state, which is faithful. First a preliminary result on ideals.

Theorem 6.1.12. *Let $\mathcal{M}$ be an ultraweakly closed left ideal in a von Neumann algebra M. Then there is a unique projection $e \in M$ such that $\mathcal{M} = Me$. If $\mathcal{M}$ is 2-sided, e is in $Z(M)$.*

Proof. $\mathcal{M} \cap \mathcal{M}^*$ is an ultraweakly closed *-subalgebra so it has a largest projection e. Since $e \in \mathcal{M}$, $Me \subseteq \mathcal{M}$. On the other hand if $x \in \mathcal{M}$ let

$x = u|x|$ be its polar decomposition. Since $u^*x = |x|$, $|x| \in \mathcal{M} \cap \mathcal{M}^*$. Hence $|x|e = |x|$ and $x = u|x| = u|x|e \in Me$. So $\mathcal{M} = Me$.

Uniqueness follows easily since $f = xe \Rightarrow f \leq e$.

Moreover if $\mathcal{M}$ is 2-sided, for any unitary $u \in M$, $u\mathcal{M} = \mathcal{M} = u\mathcal{M}u^* = Me = Mueu^*$ so $ueu^* = e$ by uniqueness. Hence $e \in Z(M)$. □

Corollary 6.1.13. *An ultraweakly continuous positive non-zero trace Tr on a II_1 factor is faithful.*

Proof. Let $\mathcal{M} = \{x \in M : Tr(x^*x) = 0\}$. Then since $x^*a^*ax \leq ||a||^2 x^*x$, $\mathcal{M}$ is a left ideal and since $Tr(ab) = Tr(ba)$, $\mathcal{M}$ is a 2-sided ideal. Moreover by the Cauchy Schwarz inequality $Tr(x^*x) = 0$ iff $Tr(xy) = 0 \quad \forall y \in M$. Thus $\mathcal{M}$ is ultraweakly closed, being the intersection of the kernels of ultraweakly continuous functionals. Thus $\mathcal{M} = Me$ for some central projection. And e must be zero since M is a factor. □

Corollary 6.1.14. *If M is a type II_1 factor on $\mathcal{H}$ and $p \in M$ is a non-zero projection, pMp is a type II_1 factor on $p\mathcal{H}$.*

Proof. This is clear—a trace on M restricts to a trace on pMp which is non-zero by faithfulness and all the other properties are immediate. Since a minimal projection in pMp would be minimal in M, pMp is infinite dimensional. □

The uniqueness of tr will follow easily once we have gathered some facts about projections in a II_1 factor.

Theorem 6.1.15. *There are non-zero projections in a type II_1 factor of arbitrarily small trace.*

Proof. Let $d = inf\{tr(p) : p \in M, p^2 = p^* = p \neq 0\}$. Suppose $d > 0$. Let p be a projection with $tr(p) - d < d$. Then p is not minimal since we have seen that M is not isomorphic to $\mathcal{B}(\mathcal{H})$. So there is a non-zero projection $q < p$. But then we have $tr(p - q) = tr(p) - tr(q) \leq tr(p) - d < d$. This is a contradiction. So $d = 0$. □

Theorem 6.1.16. *Let M be a type II_1 factor with an ultraweakly continuous positive non-zero trace tr. Then $\{tr(p) : p \in M, p^2 = p^* = p\} = [0, tr(1)]$.*

Proof. For $r \in [0, tr(1)]$ consider $S = \{p : p$ a projection in M and $tr(p) \leq r\}$. Then S is a partially ordered set and if p_α is a chain in S, $p = \bigvee_\alpha p_\alpha \in M$ and p is in the strong closure of the p_α so p is in S. So by Zorn, S has a maximal element, say q. If $tr(q)$ were less than r, then by 6.1.8, $q \prec p$. So choose $q' \cong q, \quad q' < p$. Applying 6.1.14 to $p - q'$ we find a projection strictly between q' and p. □

Corollary 6.1.17. *The map tr gives an isomorphism between the totally ordered set of equivalence classes of projections on a type* II_1 *factor and the interval* $[0, tr(1)]$*.*

Proof. By 6.1.16 it suffices to show that the equivalence class of a projection is determined by its trace. This is immediate from 6.1.8. □

Exercise 6.1.18. *Let* M *be a type* II_1 *factor. Then for each* $n \in \mathbb{N}$ *there is a subfactor* $N \subseteq M$ *with* $N \cong M_n(\mathbb{C})$*.*

Corollary 6.1.19. *Any two non-zero ultraweakly continuous normalised traces on a type* II_1 *factor are equal.*

Proof. By the elementary facts it suffices to prove that two such traces Tr and tr agree on projections. We may assume one of them, say tr, is positive. By the previous exercise, 6.1.17, and the uniqueness of the trace on a matrix algebra, tr and Tr are equal on projections for which tr is rational. Given a projection for which $tr(p)$ is irrational build an increasing sequence e_i of subprojections as follows:

Suppose we have already constructed e_i with $tr(e_i) = Tr(e_i)$ and $tr(p) - tr(e_i) < 1/i$. Then $(p - e_i)M(p - e_i)$ is a type II_1 factor so tr and Tr agree on projections in it whose tr is arbitrarily close to $tr(p - e_i)$. So choose in it a projection e_{i+1} between e_i and p, on which tr and Tr agree and with $tr(p) - tr(e_{i+1}) < \frac{1}{i+1}$. Then tr and Tr agree on $\bigvee_i e_i$ which is equal to p by the faithfulness of tr. □

We shall see that a positive trace on a type II_1 factor is norm-continuous and a self-adjoint operator is actually a norm-limit of linear combinations of its spectral projections so in fact an apparently weaker property than ultraweak continuity is all we used in the previous corollary—namely that the trace of the supremum of an increasing net of projections is the supremum of the traces.

Corollary 6.1.20. *Let* M *be a von Neumann algebra with a positive ultraweakly continuous faithful normalised trace* tr*. Then* M *is a type* II_1 *factor iff* $Tr = tr$ *for all ultraweakly continuous normalised traces* Tr*.*

Proof. We just have to show that $Z(M)$ is trivial. But if it were not, choose by faithfulness a projection $p \in Z(M)$ with $0 < tr(p) < 1$. Define $Tr(x) = (\frac{1}{tr(p)})tr(xp)$. Then Tr is an ultraweakly continuous normalized trace different from tr on $1 - p$. □

Exercise 6.1.21. *Let* a *be a non-zero positive self adjoint operator. Show that there is a bounded piecewise smooth function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $af(a)$ *is a non-zero projection.*

Exercise 6.1.22. *A type* II_1 *factor is algebraically simple. (Hint—use the previous exercise to show that a 2-sided ideal contains a projection, then add projections to obtain the identity.)*

6.2 The GNS construction

Thus uniqueness of the trace implies factoriality. This suggests another interesting way to construct a type II_1 factor. If $A = M_2(\mathbb{C})$, A is embedded in $A \otimes A$ as diagonal matrices: $a \hookrightarrow a \otimes 1$. Iterate this procedure to form an increasing sequence A_n of *-algebras with $A_1 = A$ and $A_{n+1} = A_n \otimes A$, and consider the *-algebra $A_\infty = \cup_n A_n$ which could also be called $\otimes_{alg,n=1}^{\infty} A_n$. If we normalise the matrix trace on all matrix algebras so that $tr(1) = 1$ then $tr(a \otimes 1) = tr(a)$ so that tr defines a positive faithful normalised trace on A_∞. Elements of A_∞ can be thought of as linear combinations of tensors of the form $a_1 \otimes a_2 \otimes a_3 \otimes \cdots \otimes 1 \otimes 1 \otimes 1 \otimes \cdots$, on which the trace is just the product of the traces of the a_i's. We now turn A_∞ into a von Neumann algebra.

Define an inner product on A_∞ by $\langle x, y\rangle = tr(y^*x)$. Then A_∞ is a pre-Hilbert space and let $\mathcal{H}$ be its completion. Note that $M_n(\mathbb{C})$ is a von Neumann algebra so $tr(y^*x^*xy) \leq ||x||^2 tr(y^*y)$. This means that the operator L_x on A_∞, $L_x(y) = xy$, satisfies $||L_x(\xi)|| \leq ||x|| \cdot ||\xi||$ (where $||x||$ is the operator norm of the matrix x and $||\xi||$ is the Hilbert space norm of ξ) and so extends uniquely to a bounded operator also written L_x on $\mathcal{H}$. One checks that $(L_x)^* = L_{x^*}$ so $x \to L_x$ defines a faithful (=injective) representation of the *-algebra A_∞ on $\mathcal{H}$. Let M be the von Neumann algebra on $\mathcal{H}$ generated by the L_x and identify A_∞ with a subalgebra of M.

The trace on A_∞ is defined by $tr(a) = \langle a\xi, \xi\rangle$ where ξ is the element $1 \in A_\infty$ considered as a vector in $\mathcal{H}$. So tr extends to a trace on M which is ultraweakly continuous, positive and normalised. It is also unique with these properties by the uniqueness of the trace on the ultraweakly dense subalgebra A_∞ of M. If we can show that tr is faithful on M then it follows that M is a type II_1 factor. It is important to note that this does not follow simply from the faithfulness of tr on A. In fact it is true but we need to do something to prove it.

When we showed that L_x was bounded, the same calculation, with $tr(ab) = tr(ba)$, would have shown that R_x, right multiplication by x, is also bounded. Associativity shows that L_x and R_y commute on A_∞, hence on $\mathcal{H}$. Thus M commutes with R_y for each $y \in A_\infty$. Now we can show faithfulness: if

$tr(x^*x) = 0$ for $x \in M$ then for each $a \in A_\infty$ we have

$$||x(a)||^2 = ||xR_a(\xi)||^2 = ||R_a x(\xi)||^2 \leq ||R_a||^2||x\xi||^2 = ||R_a||^2 tr(x^*x) = 0.$$

Since A_∞ is dense, this means $x = 0$. So tr is faithful on M which is thus a type II_1 factor.

Exercise 6.2.1. *Let F_n be the Fibonacci numbers. Show that there is a unique (up to you to figure out in what sense) unital embedding of $M_{F_n}(\mathbb{C}) \oplus M_{F_{n+1}}(\mathbb{C})$ inside $M_{F_{n+1}}(\mathbb{C}) \oplus M_{F_{n+2}}(\mathbb{C})$ for $n \geq 3$. Thus one may form the *-algebra*

$$F_\infty = \cup_{n=1}^{\infty} M_{F_n}(\mathbb{C}) \oplus M_{F_{n+1}}(\mathbb{C}).$$

Show that there is a unique C^*-norm and unique positive trace on F_∞ so we may repeat the procedure above to obtain another type II_1 factor.

Many points are raised by this example. The easiest to deal with are the properties of the vector ξ which played a prominent role. We used both $\overline{M\xi} = \mathcal{H}$ and $\overline{M'\xi} = \mathcal{H}$.

Definition 6.2.2. *Let M be a von Neumann algebra on $\mathcal{H}$. A vector $\xi \in \mathcal{H}$ is called* cyclic *for M if $\overline{M\xi} = \mathcal{H}$ and* separating *for M if $(x\xi = 0) \Rightarrow (x = 0)$ for all $x \in M$.*

Proposition 6.2.3. *With notation as above, ξ is cyclic for M iff ξ is separating for M'.*

Proof. ($\Rightarrow$) Exercise—in fact done in the discussion of A_∞ above.

($\Leftarrow$) Let p be the projection onto the closure of $M\xi$. Then $p \in M'$. But $(1 - p)\xi = 0$ so $p = 1$. □

The construction of M from A_∞ is a special case of what is known as the GNS construction (Gelfand-Naimark-Segal). Given a positive linear functional ϕ satisfying $\phi(a^*) = \overline{\phi(a)}$ on a *-algebra A we let N_ϕ be $\{x \in A : \phi(x^*x) = 0\}$. We also define a sesquilinear form $\langle , \rangle_\phi$ on A by $\langle x, y\rangle_\phi = \phi(y^*x)$. This form is positive semidefinite but this is enough for the Cauchy-Schwartz inequality to hold so that N is the same as $\{x : \langle x, y\rangle_\phi = 0 \quad \forall y \in A\}$ so that N is a subspace and $\langle , \rangle_\phi$ defines a pre-Hilbert space structure on the quotient A/N. Under favourable circumstances, left multiplication by x, L_x defines a bounded linear operator on it. Favourable circumstances are provided by C^*-algebras.

Exercise 6.2.4. *If ϕ is a linear functional on a C^*-algebra satisfying $\phi(a^*a) \geq 0$ show that $\phi(a^*) = \overline{\phi(a)}$. Moreover if A is unital show that ϕ is norm-continuous and in fact $||\phi|| = \phi(1)$.*

Remark 6.2.5. *It is a standard elementary fact in C^*-algebras that one may always adjoin an identity to a C^*-algebra.*

Proposition 6.2.6. *If A is a unital C^*-algebra and $\phi : A \to \mathbb{C}$ is a positive linear functional then*

$$\phi(y^*x^*xy) \leq ||x||^2\phi(y^*y)$$

Proof. Let $\tilde{\phi}(a) = \phi(y^*ay)$. Then $\tilde{\phi}$ is positive so by the exercise $\tilde{\phi}(x^*x) \leq ||x||^2\tilde{\phi}(1)$. □

It follows immediately that, given a positive linear functional ϕ on a unital C^*-algebra, each $x \in A$ determines a bounded linear operator $\pi_\phi(x)$ on the Hilbert space $\mathcal{H}_\phi$ of the GNS construction via left multiplication: $\pi_\phi(x)(y) = xy$. Moreover $||\pi_\phi(x)|| \leq ||x||$ and $\pi_\phi(x^*) = \pi_\phi(x)^*$ since $\langle \pi_\phi(x)y, z\rangle = \phi(z^*xy) = \langle y, \pi_\phi(x^*)z\rangle$. Note that $\phi(x) = \langle \pi_\phi(x)1, 1\rangle$.

To sum up we have the following:

Definition 6.2.7. *If A is a C^*-algebra and ϕ is a positive linear functional on A, the Hilbert space of the GNS construction is written $\mathcal{H}_\phi$ and the representation π_ϕ by left multiplication is called the GNS representation.*

Proposition 6.2.8. *If A is a C^*-algebra on $\mathcal{H}$ and $\xi \in \mathcal{H}$, define $\omega_\xi(a) = \langle a\xi, \xi\rangle$. Then ω_ξ is a positive linear functional and $a \mapsto a\xi$ defines a unitary $u : \mathcal{H}_{\omega_\xi} \to \overline{A\xi}$ such that $u\pi_{\omega_\xi}(a)u^* = a$.*

Proof. Obvious. □

If A is actually a von Neumann algebra, $\pi_\phi(A)$ will not in general be one on $\mathcal{H}_\phi$. This difficulty will be resolved in the next section.

Chapter 7

Normality, complete additivity.

7.1 Normal states.

In quantum mechanics if ξ is a unit vector in a Hilbert space it defines a "state" ϕ. In particular this means that if an observable is given by the self-adjoint operator a then the average value of repeated observations of the system in the state ϕ is $\langle a\xi, \xi\rangle$. For this reason one calls a positive linear functional ϕ a "state" on a unital C*-algebra provided $\phi(1) = 1$.

Definition 7.1.1. *If A is a C^*-algebra on $\mathcal{H}$ and ϕ is a state on A we say ϕ is a* vector state *if there is a unit vector $\xi \in \mathcal{H}$ with $\phi = \omega_\xi$, i.e. $\phi(a) = \langle a\xi, \xi\rangle$ for all $a \in A$.*

Not all states are vector states but our goal in this chapter is to show that on von Neumann algebras there is a natural class of states which are automatically vector states provided one amplifies the Hilbert space.

Definition 7.1.2. *(i) If M is a von Neumann algebra a positive linear functional ϕ is called* completely additive *if*

$$\phi(\bigvee_\alpha p_\alpha) = \sum_\alpha \phi(p_\alpha)$$

whenever p_α is a family of mutally orthogonal projections.

(ii) A positive linear map $\Phi : A \to B$ between von Neumann algebras is called normal *if*

$$\Phi(\bigvee_\alpha a_\alpha) = \bigvee_\alpha \Phi(a_\alpha)$$

for any increasing net (a_α) of self-adjoint operators in A.

Our goal in this chapter is to show the following:

Theorem 7.1.3. *If ϕ is state on a von Neumann algebra M on $\mathcal{H}$ the following are equivalent:*
(1) ϕ is normal.
(2) ϕ is completely additive
(3) ϕ is a vector state on $\mathcal{H} \otimes \ell^2(\mathbb{N})$
(4) ϕ is ultraweakly continuous.

The only implication that is not obvious from what we have done is (2) $\implies$ (3). To prove it we will put together some results. The first couple actually establish (4) $\implies$ (3) by 5.1.2.

Lemma 7.1.4. *Let A be a C^*-algebra on $\mathcal{H}$ containing* 1*. If ψ is a positive linear functional on A and $\xi \in \mathcal{H}$ is a vector with $\psi \leq \omega_\xi$ (i.e. $\omega_\xi - \psi$ is positive), then there is a $s \in A'$ with $\psi = \omega_{s\xi}$.*

Proof. Define a sesquilinear form $(,)$ on $A\xi$ by $(a\xi, b\xi) = \psi(b^*a)$. Cauchy-Schwarz and $\psi \leq \phi_\xi$ give that $|(a\xi, b\xi)| \leq ||a\xi||||b\xi||$ so $(,)$ is well-defined and there is a bounded positive operator t on $\overline{A\xi}$ with $\langle a\xi, tb\xi\rangle = \psi(b^*a)$. But $\langle a\xi, tbc\xi\rangle = \psi(c^*b^*a) = \langle b^*a\xi, tc\xi\rangle = \langle a\xi, btc\xi\rangle$ so that $t \in A'$ on $\overline{A\xi}$. If $p = p_{\overline{A\xi}}$, tp is a positive operator in A' and if $s = \sqrt{t}$, $\psi(a) = \langle a\xi, t\xi\rangle = \langle as\xi, s\xi\rangle = \omega_\xi(a)$. □

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Corollary 7.1.5. *If ξ and η are vectors such that $\omega(a) = \langle a\xi, \eta\rangle$ is positive (on a C^*-algebra A on $\mathcal{H}$) then there is a vector ν with $\omega = \omega_\nu$.*

Proof. For $a \geq 0$,

$$\begin{aligned}\langle a\xi, \eta\rangle &= 1/4(\langle a(\xi+\eta), \xi+\eta\rangle - \langle a(\xi-\eta), \xi-\eta\rangle)\\ &\leq 1/4\omega_{\xi+\eta}(a).\end{aligned}$$

□

Now we begin to show that complete additivity means that two states cannot disagree too erratically.

Lemma 7.1.6. *Let ϕ_1 and ϕ_2 be completely additive. Suppose $p \in M$ is a projection and $\phi_1(p) < \phi_2(p)$. Then there is a projection $q \leq p$, for which $\phi_1(x) < \phi_2(x) \quad \forall x \geq 0$ with $qxq = x$.*

Proof. Choose a maximal family of mutually orthogonal "bad" projections $e_\alpha \leq p$ for which $\phi_1(e_\alpha) \geq \phi_2(e_\alpha)$. By complete additivity $\bigvee_\alpha e_\alpha$ is bad so let $q = p - \bigvee_\alpha e_\alpha$. By maximality $\phi_1(f) < \phi_2(f)$ for all projections $f \leq q$ and since α is norm continuous, by the spectral theorem $\phi_1(x) < \phi_2(x) \quad \forall x \geq 0$ with $qxq = x$.

□

Next we get vector state behaviour for ϕ on some small projection.

Lemma 7.1.7. *There exists $p > 0$ and $\xi \in \mathcal{H}$ for which*

$$\phi(x) = \langle x\xi, \xi\rangle \quad \forall x \in pMp$$

Proof. Choose $\xi \in \mathcal{H}$ with $\phi(1) = 1 < \langle \xi, \xi\rangle$. Then by the previous lemma there is a $p > 0$ for which $\phi(x) \leq \langle x\xi, \xi\rangle \quad \forall x \in pMp$. By 7.1.4 we are done. □

Now we put together all the little parts and prove that (3) $\implies$ (4) in 7.1.3. So let ϕ be a completely additive state on a von Neumann algebra M acting on $\mathcal{H}$. Let p_α be a maximal family of pairwise orthogonal projections admitting a vector $\xi_\alpha \in p_\alpha\mathcal{H}$ with $\phi(x) = \langle x\xi_\alpha, \xi_\alpha\rangle$ on $p_\alpha M p_\alpha$. Then by the previous lemma $\bigvee_\alpha p_\alpha = 1$. And obviously $||\xi_\alpha||^2 = \phi(p_\alpha)$. Since $\phi(p_\alpha)$ can only be non-zero for countably many α we can assume the set of α's is countable.

By Cauchy-Schwarz, for *any* $x \in M$,

$$|\phi(xp_\alpha)| \leq \phi(p_\alpha x^* x p_\alpha)^{1/2}\phi(p_\alpha)^{1/2} = ||x\xi_\alpha||\phi(p_\alpha)^{1/2}.$$

So the linear functional $x\xi_\alpha \mapsto \phi(xp_\alpha)$ is well-defined and bounded on $M\xi_\alpha$ which means there is a vector η_α, $||\eta_\alpha||^2 = \phi(p_\alpha)$, with

$$\phi(xp_\alpha) = \langle x\xi_\alpha, \eta_\alpha\rangle.$$

Moreover, also by Cauchy-Schwarz, $|\phi(x) - \sum_{\alpha\in F}\phi(xp_\alpha)|$ can be made arbitrarily small by choosing the finite set F sufficiently large since ϕ is completely additive. We conclude that there exist ξ_α, η_α, each of norm $\leq \phi(\alpha)^{1/2}$ with

$$\phi(x) = \sum_\alpha \langle x\xi_\alpha, \eta_\alpha\rangle$$

which is the same as saying that $\phi(x) = \langle (x\otimes 1)\xi, \eta\rangle$ for some $\xi, \eta \in \ell^2(\mathbb{N}, \mathcal{H})$. By corollary 7.1.5 we have proved theorem 7.1.3.

Sadly this proof doesn't work. There's a mistaken Cauchy Schwarz at some point. The projections can not be made to add up in this way. One needs to do a maximality argument with another Cauchy Schwarz.

Corollary 7.1.8. *If ϕ is a normal state on the von Neumann algebra M then the GNS representation π_ϕ is ultraweakly continuous onto a von Neumann algebra on $\mathcal{H}_\phi$.*

Proof. We saw in the last theorem that $\phi(x) = \langle x \otimes 1(\nu), \nu \rangle$ on $\mathcal{H} \otimes \ell^2(\mathbb{N})$. The map $x \mapsto x \otimes 1$ is ultraweakly continuous. By 6.2.8 we have that π_ϕ is ultraweakly continuous since the reduction to $\overline{M \otimes 1(\nu)}$ is ultraweakly continuous. So the kernel of π_ϕ is an ultraweakly closed 2-sided ideal, hence of the form Me for some e in the centre of M. It follows that π_ϕ is injective on $M(1-e)$ and since the norm of an operator x is determined by the spectrum of x^*x, the unit ball of the image of M is the image of the unit ball which is weakly compact so by 5.2.2 we are done. □

We record a corollary that is used often without explicit mention:

Corollary 7.1.9. *Let M be a von Neumann algebra and let A be a weakly dense *-subalgebra of M generated by some self-adjoint set X. Suppose ϕ is a faithful normal state on M and N is another von Neumann algebra with faithful normal state ψ. If $\theta : X \to N$ is a function, multiplicatively extend θ to words $w(x_1, x_2, \cdots x_n)$. Then if $\psi(w(\theta(x_1), \theta(x_2), \cdots \theta(x_n)) = \phi(w(_1, x_2, \cdots x_n))$, θ extends uniquely to a von Neumann algebra isomorphism from M to $\theta(X)''$.*

Proof. Faithfulness of the states ϕ and ψ means that the extension of θ to linear combinations of words is a well-defined *-isomorphism from A to the *-subalgebra $\theta(A)$ of N which sends ϕ to ψ. This further extends to a unitary between the GNS constructions for ϕ and $\psi|_{\theta(A)}$ which intertwines the actions of A and $\theta(A)$. We are done by 7.1.8. □

7.2 Isomorphisms are spatial.

Recall that an isomorphism $\Phi : M \to N$ between von Neumann algebras on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively is called *spatial* if there is a unitary $u : \mathcal{H} \to \mathcal{K}$ such that $\Phi(x) = uxu^*$ for all $x \in M$. Though the title of this section is not literally true, it becomes true on amplification as a result of theorem 7.1.3:

Theorem 7.2.1. *Given an isomorphism $\Phi : M \to N$ between von Neumann algebras on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, there is a Hilbert space $\mathcal{W}$ and a unitary $u : \mathcal{H} \otimes \mathcal{W} \to \mathcal{K} \otimes \mathcal{W}$ with $\Phi(x) \otimes 1 = u(x \otimes 1)u^*$ for all $x \in M$.*

Proof. If $\xi \in \mathcal{H}$ defines the vector state ϕ on M,then since normality (or complete additivity) is defined by algebra, the state $\phi \circ \Phi^{-1}$ is also a vector state on $\mathcal{K} \otimes \ell^2(\mathbb{N})$ given by the vector η. This means that there is a unitary from the closure of $M\xi$ to the closure of $N\eta$ intertwining the actions of x and $\Phi(x) \otimes 1$. One may exhaust $\mathcal{H}$ in this way to obtain an isometry

$u : \mathcal{H} \to \oplus_\alpha \mathcal{K} \otimes \ell^2(\mathbb{N})$ intertwining the actions of M. For a big enough $\mathcal{W}$, $\oplus_\alpha \mathcal{K} \otimes \ell^2(\mathbb{N})$ is $\mathcal{K} \otimes \mathcal{W}$ and tensoring again by $\mathcal{W}$ we get an intertwining isometry $u : \mathcal{H} \otimes \mathcal{W} \to \mathcal{K} \otimes \mathcal{W}$. Now consider the action of M on $(\mathcal{H}\otimes\mathcal{W}) \oplus (\mathcal{K}\otimes\mathcal{W})$ defined in terms of matrices by $\left(\begin{smallmatrix} x\otimes 1 & 0 \\ 0 & \Phi(x)\otimes 1 \end{smallmatrix}\right)$. To say that u intertwines the actions is precisely the same as saying that $\left(\begin{smallmatrix} 0 & 0 \\ u & 0 \end{smallmatrix}\right)$ is in M'. So $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \preceq \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in M'. Applying this to Φ^{-1} as well we see by theorem 6.1.2 that these two projections are equivalent in M'. But any partial isometry witnessing their equivalence has the form $\left(\begin{smallmatrix} 0 & 0 \\ w & 0 \end{smallmatrix}\right)$ with w a *unitary* between $\mathcal{H}$ and $\mathcal{K}$ intertwining the actions. (Note that we never assumed that M was more than a unital *-algebra on $(\mathcal{H} \otimes \mathcal{W}) \oplus (\mathcal{K} \otimes \mathcal{W})$).

□

7.3 Exercises on two projections.

Let p and q be projections onto closed subspaces $\mathcal{H}$ and $\mathcal{K}$ of the Hilbert space $\mathcal{U}$ respectively. Let $M = \{p, q\}''$.

Exercise 7.3.1. *Show that* $\mathcal{U} = (\mathcal{H}\cap\mathcal{K}) \oplus (\mathcal{H}^\perp\cap\mathcal{K}^\perp) \oplus (\mathcal{H}\cap\mathcal{K}^\perp) \oplus (\mathcal{H}^\perp\cap\mathcal{K}) \oplus \mathcal{W}$ *for some* $\mathcal{W}$ *and this decomposition is invariant under* p *and* q.

Exercise 7.3.2. *Show that, on* $\mathcal{W}$, p *and* q *are in "general position", i.e.* $p \wedge q = 0$, $p \vee q = 1$, $(1-p) \wedge q = 0$ *and* $(1-p) \vee q = 1$.

Exercise 7.3.3. *Show that if* $a \in \mathcal{B}(\mathcal{H})$, $0 \le a \le 1$, $\begin{pmatrix} a & \sqrt{a(1-a)} \\ \sqrt{a(1-a)} & 1-a \end{pmatrix}$ *is a projection on* $\mathcal{H} \oplus \mathcal{H}$. *When is it in general position with* $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$?

Exercise 7.3.4. *Let* $a = (p-q)^2$ *and* $A = \{a\}''$. *Show that* $a \in Z(M)$ *and that* $\{a_0 + a_1 p + a_2 q + a_3 pq + a_4 qp : a_i \in A\}$ *is a *-algebra (which is necessarily weakly dense in* M*).*

Exercise 7.3.5. *Show that* pMp *is abelian, generated by* pqp.

>From now on suppose p and q are in general position.

Exercise 7.3.6. *Show that* $p \cong q$ *in* M. *(Hint: consider the polar decomposition of* pq.*)*

Exercise 7.3.7. *Show there is a* 2×2 *system of matrix units* $(e_{ij}) \in M$ *with* $p = e_{11}$.

Exercise 7.3.8. *Show that* M *is spatially isomorphic to* $B \otimes M_2(\mathbb{C})$ *for some abelian von Neumann algebra* B *generated by* $b, 0 \le b \le 1$, *with* p *corresponding to* $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *and* q *corresponding to* $\begin{pmatrix} b & \sqrt{b(1-b)} \\ \sqrt{b(1-b)} & 1-b \end{pmatrix}$

Now drop the hypothesis that p and q are in general position.

Exercise 7.3.9. *Show that* $p \vee q - p \cong q - p \wedge q$ *in* M

Exercise 7.3.10. *Show that if* e *and* f *are finite projections in a factor* M *then* $p \vee q$ *is also finite. (In fact it's true for a non-factor as well.)*

(Here's how to do it: use the previous exercise to reduce it to the case where e *and* f *are orthogonal. Then one can assume* $f = e^{\perp}$*. Suppose we could find an infinite projection* p *so that* $e \wedge p \preceq e^{\perp} \wedge p^{\perp}$*. Then*

$$\begin{aligned} p &= e \wedge p + p - e \wedge p \\ &\preceq e^{\perp} \wedge p^{\perp} + p \vee e - e \text{ (remembering that } e^{\perp} \wedge p^{\perp} = (e \vee p)^{\perp}) \\ &\leq e^{\perp} \end{aligned}$$

so that p *is finite.*

To find such an e *and* p*, construct an infinite projection* p *equivalent to* $1 - p$ *("halving"). Then either* $e \wedge p \preceq e^{\perp} \wedge p^{\perp}$ *in which case we are done, or* $e^{\perp} \wedge p^{\perp} \preceq e \wedge p$ *in which case we simply switch to* $e^{\perp}$ *and* $p^{\perp}$*.)*

Alternative approach using group representations.

Exercise 7.3.11. *Show that* $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})$ *(infinite dihedral group).*

Exercise 7.3.12. *Classify all unitary representations of* $\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})$*. (Hint—use the spectral theorem for unitaries.)*

Exercise 7.3.13. *Observe that* $2p - 1$ *and* $2q - 1$ *are self-adjoint unitaries.*

Exercise 7.3.14. *Obtain the structure of 7.3.8 using the last 3 exercises.*

Chapter 8

The Predual

An ultraweakly continuous linear functional ϕ on a von Neumann algebra M is norm continuous so defines an element of M^*. Our goal in this chapter is to show that the set of all such ϕ is a closed subspace M_* of M^* and that the duality between M_* and M makes M equal to the Banach space dual of M_*. We will first establish this in the special case $M = \mathcal{B}(\mathcal{H})$.

8.1 Trace class and Hilbert Schmidt operators.

The material in this section is standard so we will only prove results as it suits us, otherwise referring any unproved assertions to Reed and Simon.

Lemma 8.1.1. *If $a \in \mathcal{B}(\mathcal{H})$ is positive and (ξ_i) and (η_i) are two orthonormal bases of $\mathcal{H}$, then*

$$\sum_i \langle a\xi_i, \xi_i \rangle = \sum_i \langle a\eta_i, \eta_i \rangle$$

(where ∞ is a possible value for the sum).

Proof. We have

$$\sum_i \langle a\xi_i, \xi_i \rangle = \sum_i ||\sqrt{a}\xi_i||^2$$

$$= \sum_i (\sum_j |\langle \sqrt{a}\xi_i, \eta_j \rangle|^2)$$

$$= \sum_j (\sum_i |\langle \sqrt{a}\eta_j, \xi_i \rangle|^2)$$

$$= \sum_j ||\sqrt{a}\eta_j||^2$$

$$= \sum_j \langle a\eta_j, \eta_j \rangle$$

where every number is positive so the order of the sum is immaterial. □

The number $\sum_i \langle a\xi_i, \xi_i \rangle$ of the previous theorem is called the trace of a, written $Trace(a)$.

Definition 8.1.2. *An element $a \in \mathcal{B}(\mathcal{H})$ is said to be of* trace class *if $Trace(|a|)$ is finite.*

If a is trace class and (ξ_i) is an orthonormal basis, the sum

$$\sum_i \langle a\xi_i, \xi_i \rangle$$

converges absolutely and is called the trace, $Trace(a)$, of a.

Theorem 8.1.3. *The trace class operators on $\mathcal{H}$ form a self-adjoint ideal of compact operators, I_1, in $\mathcal{B}(\mathcal{H})$. The function $|a|_1$ defined by $|a|_1 = Trace(|a|)$ defines a norm on I_1 for which it is complete. Moreover $||a|| \leq |a|_1$.*

Proof. The only thing not proved in Reed and Simon is completeness. For this observe that if a_n is a Cauchy sequence in $| - |_1$, it is Cauchy in $|| - ||$ so what we have to do is show that the norm limit of a $| - |_1$-Cauchy sequence (a_n) is trace class and that the sequence tends to that limit in $| - |_1$. So suppose $\epsilon > 0$ is given. Then for m and n large enough

$$\sum_{i=1}^{\infty} \langle |a_n - a_m| \xi_i, \xi_i \rangle < \epsilon.$$

So for any N,

$$\sum_{i=1}^{N} \langle |a_n - a_m| \xi_i, \xi_i \rangle < \epsilon.$$

Now if b_n tends in norm to b, then $|b_n|$ tends in norm to $|b|$ (obviously $b_n^* b_n \to b^* b$, and approximate the square root function by polynomials on an interval) so for each fixed i,

$$\lim_{n \to \infty} |a_n - a_m| \xi_i = |a - a_m| \xi_i.$$

So $\sum_{i=1}^{N} \langle |a - a_m| \xi_i, \xi_i \rangle < \epsilon$ and letting N tend to ∞ we see that $a \in I_1$ since I_1 is a vector space, and also that $a_n \to a$ in $| - |_1$. □

The trace is independent of the orthonormal basis and if a is trace class and $b \in \mathcal{B}(\mathcal{H})$, $Tr(ab) = Tr(ba)$.

We see that each $h \in I_1$ determines a linear functional ϕ_h on $\mathcal{B}(\mathcal{H})$ by $\phi_h(x) = Trace(xh)$.

Definition 8.1.4. *The trace-class matrix as above is called the* <u>*density matrix*</u> *for the state* ϕ_h*.*

Proposition 8.1.5. *Each* ϕ_h *is ultraweakly continuous and its norm as an element of* $\mathcal{B}(\mathcal{H})^*$ *is* $|h|_1$*.*

Proof. Since h is compact, choose an orthonormal basis (ξ_i) of eigenvectors of $|h|$ with eigenvalues λ_i and let $h = u|h|$ be the polar decomposition. Then

$$\phi_h(x) = \sum_{i=1}^{\infty} \langle xu|h|\xi_i, \xi_i\rangle$$

so ultraweak continuity is apparent, and

$$\phi_h(x) \leq \sum_{i=1}^{\infty} ||x|| \; || \, |h|\xi_i||$$

$$= ||x|| \sum_{i=1}^{\infty} \lambda_i$$

$$= ||x|| \, |h|_1.$$

Moreover evaluating ϕ_h on u^* gives $||\phi_h|| = |h|_1$. □

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, a bounded operator $x : \mathcal{H} \to \mathcal{K}$ is called *Hilbert-Schmidt* if x^*x is trace class, i.e. $\sum_{i=1}^{\infty} ||x\xi_i||^2 < \infty$ for some (hence any) orthonormal basis (ξ_i) of $\mathcal{H}$. The set of all Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{K}$ is written $\ell^2(\mathcal{H}, \mathcal{K})$ and if x is Hilbert-Schmidt, so is x^*, and x is compact.

Theorem 8.1.6. *If* $a \in \mathcal{B}(\mathcal{H})$*,* $b \in \mathcal{B}(\mathcal{K})$ *and* $x \in \ell^2(\mathcal{H}, \mathcal{K})$ *then* $bxa \in \ell^2(\mathcal{H}, \mathcal{K})$*. If* $x \in \ell^2(\mathcal{H}, \mathcal{K})$ *and* $y \in \ell^2(\mathcal{K}, \mathcal{H})$ *then* yx *is trace class. With the inner product* $\langle x, y\rangle = Trace(y^*x)$*,* $\ell^2(\mathcal{H}, \mathcal{K})$ *is a Hilbert space in which the finite rank operators are dense.*

Proof. See Reed and Simon. □

Exercise 8.1.7. *Prove all the assertions made above about trace-class and Hilbert-Schmidt operators.*

Exercise 8.1.8. *If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces construct a natural map from $\mathcal{K} \otimes \mathcal{H}^*$ to $\ell^2(\mathcal{H}, \mathcal{K})$ and show that it is unitary.*

Let $|x|_2$ be the Hilbert space norm on Hilbert-Schmidt operators.

Lemma 8.1.9. *If $x \in \ell^2(\mathcal{H}, \mathcal{K})$ and $y \in \ell^2(\mathcal{K}, \mathcal{H})$ then $Trace(xy) = Trace(yx)$.*

Proof. First note that the result is true if we suppose that $|x|$ is trace class. For then let $x = u|x|$ be the polar decomposition, choose an orthonormal basis (ξ_i) of the final domain of u and extend it to an orthonormal basis of $\mathcal{K}$. Also extend $(u^*\xi_i)$ to an orthonormal basis of $\mathcal{H}$ by vectors in $\ker(|x|)$. Then

$$Trace(xy) = \sum_i \langle u|x|y\xi_i, \xi_i\rangle$$

$$= \sum_i \langle |x|yuu^*\xi_i, u^*\xi_i\rangle$$

$$= Trace(|x|(yu))$$

$$= Trace((yu)|x|)$$

$$= Trace(yx.)$$

Now suppose only that x is Hilbert-Schmidt. Let $\epsilon > 0$ be given and choose x' of finite rank with $|x - x'|_2 < \epsilon$. Then

$$|Trace(xy) - Trace(yx)| = |Trace((x - x')y) - Trace(y(x - x'))|$$

which by Cauchy-Schwartz is $\leq 2\epsilon|y|_2$. □

Corollary 8.1.10. *If ω is an ultraweakly continuous linear functional on $\mathcal{B}(\mathcal{H})$, there is a trace class h so that $\omega = \phi_h$.*

Proof. By 5.1.2 there are (ξ_i) and (η_i) in $\ell^2(\mathbb{N}, \mathcal{H})$ so that $\omega(x) = \sum_i \langle x\xi_i, \eta_i\rangle$. Then if we define a and b from $\ell^2(\mathbb{N})$ to $\mathcal{H}$ by $a(f) = \sum_i f(i)\xi_i$ and $b(f) = \sum_i f(i)\eta_i$, a and b are Hilbert Schmidt and $\omega(x) = Trace(b^*xa)$ which is $Trace(xab^*)$ by the previous result. □

Putting everything together so far, we have identified the image of the Banach space I_1 under the map $h \mapsto \phi_h$ with the closed subspace of $\mathcal{B}(\mathcal{H})^*$ consisting of ultraweakly continuous linear functionals. To close the loop we only need to show that the Banach space dual of I_1 is $\mathcal{B}(\mathcal{H})$.

Theorem 8.1.11. *If $\alpha : I_1 \to \mathbb{C}$ is linear and bounded for $|-|_1$, there is an $x \in \mathcal{B}(\mathcal{H})$ so that $\alpha(a) = \phi_a(x)$, and $||\alpha|| = ||x||$.*

Proof. This is rather routine. Two vectors ξ and η define an element x of I_1 by $x(v) = \langle v, \xi \rangle \eta$ so one may define a sesquilinear form on $\mathcal{H}$ by $(\xi, \eta) = \alpha(x)$. Boundedness of x follows from that of α so there is an appropriate $x \in \mathcal{B}(\mathcal{H})$. To show that the norm of x as an element of the dual of I_1 is actually $||x||$, suppose $||x|| = 1$ and choose a unit vector ξ with $||x\xi||$ almost equal to 1. Then $Tr(hx)$ is almost 1 if h is the partial isometry which sends $v \in \mathcal{H}$ to $\langle v, x\xi \rangle \frac{\xi}{||x\xi||}$. □

Exercise 8.1.12. *Fill in the missing details in the previous proof.*

Now we pass to von Neumann algebras though in fact these results work for any ultraweakly closed subspace of $\mathcal{B}(\mathcal{H})$.

Theorem 8.1.13. *If V is an ultraweakly closed subspace of $\mathcal{B}(\mathcal{H})$ then $V = V^{\perp\perp}$ in the sense that if $\phi(x) = 0$ for every ultraweakly continuous ϕ for which $\phi(V) = 0$ then $x \in V$.*

Proof. This is a simple application of the Hahn-Banach theorem—if $x \notin V$ construct an ultraweakly continuous functional which is zero on V and non-zero on x. □

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Exercise 8.1.14. *Exhibit a non-zero trace class operator on $\ell^2(\Gamma)$ which is orthogonal to $vN(\Gamma)$.*

Theorem 8.1.15. *If V is an ultraweakly closed subspace of $\mathcal{B}(\mathcal{H})$ then it is canonically the dual Banach space of V_* which is defined as the space of ultraweakly continuous linear functionals on V. Moreover the ultraweak topology on V is the weak-* topology on V as the dual of V_*.*

Proof. If B is a Banach space with dual B^* and V is a weak-* closed subspace of B^* then V is the dual of $B/V^\perp$ (surjectivity of the natural map from V to the dual of $V/B^\perp$ is a result of the previous theorem), so V is a dual space. So we just have to identify the Banach space $B/V^\perp$ with the space of weak-* continuous (as elements of B^{**}) linear functionals on V. This is a simple exercise. Putting $B = I_1$ we are done. □

Exercise 8.1.16. *If V is an ultraweakly closed subspace of $\mathcal{B}(\mathcal{H})$, show that V_* is a separable Banach space if $\mathcal{H}$ is a separable Hilbert space.*

8.2 A technical lemma.

Let us prove a lemma which shows what the techniques developed so far can be good for. It will be crucial in our treatment of Tomita-Takesaki theory. It is a "Radon-Nikodym" type theorem inspired by one due to Sakai([]).

find reference

Lemma 8.2.1. *Let $\lambda \in \mathbb{R}^+$ be given and let ϕ be a faithful ultraweakly continuous state on a von Neumann algebra M. Let $\psi \in M_*$ be such that $|\psi(y^*x)| \leq \sqrt{\phi(x^*x)}\sqrt{\phi(y^*y)}$. Then there is an $a \in M_{1/2}$ (elements of norm $\leq 1/2$) so that*

$$\psi(x) = \lambda\phi(ax) + \lambda^{-1}\phi(xa).$$

Proof. For $a \in M$ let $\theta_a(x) = \phi(\lambda ax + \lambda^{-1}xa)$. Then the map $\alpha : M \to M_*$, $\alpha(a) = \theta_a$, is continuous for the topologies of duality between M and M_*. But we know that this topology on M is the ultraweak topology so that $\alpha(M_1)$ is a compact convex set. By contradiction suppose that ψ is not in $\alpha(M)$.

Then by Hahn-Banach there is an $h \in M$ with $\Re(\psi(h)) > D$ where $D = \sup_{a \in M_{1/2}} \Re(\theta_a(h))$. But if $h = u|h| = |h^*|u$ is the polar decomposition of h, we have

$$\theta_{u^*/2}(h) = 1/2(\lambda\phi(|h|) + \lambda^{-1}\phi(|h^*|))$$

so that

$$2D \geq \lambda\phi(|h|) + \frac{1}{\lambda}\phi(|h^*|) \geq 2\sqrt{\phi(|h|)}\sqrt{\phi(|h^*|)}.$$

But also $D < |\psi(h)| = |\psi(u|h|^{1/2}|h|^{1/2})| \leq \sqrt{\phi(|h|)}\sqrt{\phi(u|h|u^*)}$, a contradiction. □

Chapter 9

Standard form of a II_1 factor and II_∞ factors.

9.1 Standard form.

In this section M will be a von Neumann algebra with an ultraweakly continuous faithful normalized trace tr and $L^2(M, tr)$ will be abbreviated to $L^2(M)$.

In section 6.2 we learned how to construct a von Neumann algebra from a C^*-algebra and a positive linear functional on it. If we apply this construction to $L^\infty(X, \mu)$ (when $\mu(X) < \infty$) with trace given by $\int f d\mu$, the Hilbert space would be $L^2(X, d\mu)$. For this reason, if M is a type II_1 factor we write $L^2(M, tr)$ for the GNS Hilbert space obtained from the trace. In fact one can define L^p spaces for $1 \leq p \leq \infty$ using the L^p norm $||x||_p = tr(|x|^p)^{1/p}$. A noncommutative version of the Holder inequality shows that $|| - ||_p$ is a norm and $L^p(M)$ is the completion. We set $L^\infty(M) = M$ and we shall see that $L^1(M)$ is the predual M_*.

Let us fix on the notation Ω for the vector in $L^2(M)$ which is the identity of M.

Proposition 9.1.1. *If M is as above the $|| - ||$-unit ball of M is a complete metric space for $|| - ||_2$ and the topology defined by $|| - ||_2$ on the unit ball is the same as the strong (and ultrastrong and *-strong) topology.*

Proof. If x_n is Cauchy in $|| - ||_2$ then for each $a \in M$, $x_n a$ is also since $||x_n a||_2 \leq ||a|| \, ||x_n||_2$. So we can define x on the dense subspace $M\Omega$ of $L^2(M)$ by $x(a\Omega) = lim_{n\to\infty} x_n a\Omega$. Since $||x|| \leq 1$, we have $||x\xi|| \leq ||\xi||$ for $\xi \in M\Omega$ so x extends to a bounded operator on $L^2(M)$ which is obviously in M, and $x\Omega = x = \lim_{n\to\infty} x_n$ in $|| - ||_2$.

The strong topology is obviously no stronger than $||-||_2$ since the single seminorm $a \mapsto ||a\Omega||$ defines the $||-||_2$ topology. Moreover $||xa\Omega|| \le ||x||_2||a||$ shows that $||-||_2$ controls the strong topology on the unit ball.

Finally note that in the statement of the theorem it does not matter what representation of M is used to define the strong topology on the unit ball as the ultrastrong topology does not change under the manipulations that we used to get the GNS construction from a II_1 factor on an arbitrary Hilbert space. □

The action of M on $L^2(M, tr)$ is called the *standard form* of M. Note that $vN(\Gamma)$ on $\ell^2(\Gamma)$ is already in standard form. (We see that we could have obtained our first example of a II_1 factor by applying the GNS construction to the group algebra $\mathbb{C}\Gamma$ with the trace $tr(\sum_\gamma c_\gamma u_\gamma) = c_{id}$.)

We now want to determine the commutant M' when M is in standard form. It will be more convenient to adopt the clearly equivalent situation where M is acting on a Hilbert space $\mathcal{H}$ and Ω is a cyclic and separating vector in $\mathcal{H}$ with $\langle x\Omega, \Omega\rangle = tr(x)$ for $x \in M$.

Definition 9.1.2. *Let $J : \mathcal{H} \to \mathcal{H}$ be the antilinear unitary involution which is the extension to $\mathcal{H}$ of the antiunitary isometry*

$$J(x\Omega) = x^*\Omega.$$

Lemma 9.1.3. *For x, a in M, and ξ, η in $\mathcal{H}$*

(i) $\langle J\xi, J\eta\rangle = \langle \eta, \xi\rangle$
(ii) $JxJ(a\Omega) = ax^*\Omega$

Proof. (i) If $\xi = a\Omega$ and $\eta = b\Omega$, $\langle J\xi, J\eta\rangle = tr(ba^*) = \langle \eta, \xi\rangle$.
(ii) $JxJ(a\Omega) = J(xa^*\Omega) = ax^*\Omega$.
□

Corollary 9.1.4. *For M on $\mathcal{H}$, $JMJ \subseteq M'$.*

Proof. Left and right multiplication commute. □

Lemma 9.1.5. *For M on $\mathcal{H}$, if $x \in M'$, $Jx\Omega = x^*\Omega$.*

Proof. Take $a \in M$, then

$$\langle Jx\Omega, a\Omega\rangle = \langle Ja\Omega, x\Omega\rangle$$

$$= \langle a^*\Omega, x\Omega\rangle$$

$$= \langle \Omega, xa\Omega \rangle$$
$$= \langle x^*\Omega, a\Omega \rangle.$$

□

Theorem 9.1.6. *For M on $\mathcal{H}$, $JMJ = M'$.*

Proof. We begin by showing that $x \mapsto \langle x\Omega, \Omega \rangle$ is a trace on M':
For $x, y \in M'$,

$$\begin{aligned} \langle xy\Omega, \Omega \rangle &= \langle y\Omega, x^*\Omega \rangle \\ &= \langle y\Omega, Jx\Omega \rangle \\ &= \langle x\Omega, Jy\Omega \rangle \\ &= \langle x\Omega, y^*\Omega \rangle \\ &= \langle yx\Omega, \Omega \rangle. \end{aligned}$$

let us call Tr this trace on M'. Then clearly the (M', Tr, Ω) satisfy the hypotheses we have been using so if $K(x\Omega) = x^*\Omega$ is extended to $\mathcal{H}$ it satisfies $KM'K \subseteq M'' = M$. But by the previous lemma K coincides with J on the dense subspace $M'\Omega$. Hence $JM'J \subseteq M$ and we are done.

□

We see that the commutant of the left regular representation of Γ on $\ell^2(\Gamma)$ is the von Neumann algebra generated by the right regular representation since $Ju_\gamma J\varepsilon_{\gamma'} = \varepsilon_{\gamma'\gamma^{-1}}$. And more generally the commutant of the left action of M on $L^2(M)$ is the *-algebra of right multiplication operators. In particular the commutant of a type II_1 factor M on $L^2(M)$ is also a type II_1 factor. This is *not* the case for M on an arbitrary Hilbert space. For instance we could consider $M \otimes 1$ on $L^2(M) \otimes \mathcal{H}$ for some infinite dimensional $\mathcal{H}$. Then the commutant of $M \otimes 1$ would be $JMJ \otimes \mathcal{B}(\mathcal{H})$—infinite matrices over JMJ.

Definition 9.1.7. *A II_∞ factor is a factor of the form $M \otimes \mathcal{B}(\mathcal{H})$ with M a type* II_1 *factor and* $\dim \mathcal{H} = \infty$.

Proposition 9.1.8. *Let M be an infinite factor with a projection $p \in M$ so that pMp is a type* II_1 *factor. Then M is a II_∞ factor.*

Proof. Choose a maximal family $\{p_\alpha\}$ of mutually orthogonal projections in M with $p_\alpha \cong p$ $\forall\alpha$. If it were the case that $1 - \sum_\alpha p_\alpha \succeq p$ then we could contradict the maximality of the family $\{p_\alpha\}$. So write $1 = q + \sum_\alpha p_\alpha$ with $q \preceq p$. By 7.3.10 the set of indices $\{\alpha\}$ is infinite so we may choose a bijection with itself minus α_0 and write $1 = q + \sum_\alpha p_\alpha \preceq p_{\alpha_0} + \sum_{\alpha \neq \alpha_0} p_\alpha \preceq$

1. We conclude that $\sum_\alpha p_\alpha$ is equivalent to 1 so we may suppose it equal to 1. We may then construct a system of matrix units by using partial isometries implementing the equivalences between the p_α to obtain the result from exercise 4.3.3.

□

It could conceivably happen that, given a II_∞ factor M, the type II_1 factor of the form pMp depends on p (obviously only up to equivalence). We now introduce the trace on a II_∞ factor which will make this issue more clear.

If M is a type II_1 factor, define the map tr from $(M \otimes \mathcal{B}(\mathcal{H}))_+$ (the set of positive elements of $M \otimes \mathcal{B}(\mathcal{H})$), to $[0, \infty]$ by

$$tr((x_{ij})) = \sum_{i=1}^{\infty} tr(x_{ii})$$

where we have chosen a basis of the infinite dimensional Hilbert space $\mathcal{H}$ to identify $M \otimes \mathcal{B}(\mathcal{H})$ with certain matrices over M.

Theorem 9.1.9. *Let M be as above.*
(i) $tr(\lambda x) = \lambda tr(x)$ *for* $\lambda \geq 0$.
(ii) $tr(x + y) = tr(x) + tr(y)$.
(iii) If (a_α) *is an increasing net of positive operators with* $\bigvee_\alpha a_\alpha = a$ *then* $tr(\bigvee_\alpha a_\alpha) = \lim_\alpha tr(a_\alpha)$.
(iv) $tr(x^*x) = tr(xx^*)$ $\forall x \in M \otimes \mathcal{B}(\mathcal{H})$.
(v) $tr(uxu^*) = tr(x)$ *for any unitary* $u \in M \otimes \mathcal{B}(\mathcal{H})$ *and any* $x \geq 0$ *in* $M \otimes \mathcal{B}(\mathcal{H})$.
(vi) If p is a projection in $M \otimes \mathcal{B}(\mathcal{H})$ *then p is finite iff* $tr(p) < \infty$.
(vii) If p and q are projections with p finite then $p \preceq q$ *iff* $tr(p) \leq tr(q)$.
(viii) $p(M \otimes \mathcal{B}(\mathcal{H}))p$ *is a type* II_1 *factor for any finite projection p.*

Proof. The first two assertions are immediate. For (iii), note that the diagonal entries of positive matrices are ordered as the matrices, and all numbers are positive in the sums. (iv) Is obvious using matrix multiplication. (v) follows from (iv) via $uxu^* = (u\sqrt{x})(\sqrt{x}u^*)$. For (vi), if $tr(p) < \infty$ but p is infinite, there is a proper subprojection of p having the same trace as p. The difference would be a projection of trace zero which is clearly impossible. If $tr(p) = \infty$ then if q is a projection of finite trace, $q \preceq p$ and if $q \leq p$ then $tr(p - q) = \infty$ so one may construct an infinite sequence of mutually orthogonal equivalent projections less than p. Using a bijection with a proper subsequence, p dominates an infinite projection so is infinite itself. (vii) follows easily as in the case of a type II_1 factor. For (viii) simply observe that

$tr(p) < \infty$ means that $p \preceq q$ for some q whose matrix is zero except for finitely many 1's on the diagonal. And obviously qMq is a type II_1 factor for such a q. □

Corollary 9.1.10. *Let M be a II_∞ factor on a separable Hilbert space and tr be the trace supplied by a decomposition $II_1 \otimes \mathcal{B}(\mathcal{H})$. Then tr defines an isomorphism of the totally ordered set of equivalence classes of projections in M to the interval* $[0, \infty]$.

Proof. Given the previous theorem, we only have to prove that any infinite projection is equivalent to the identity. But if p is infinite choose u with $uu^* = p$ and u^*u strictly less than p. Then $(u^*)^n u^n$ are a strictly decreasing sequence of equivalent projections so we may write p as an orthogonal sum $p = p_\infty + \sum_{i=1}^{\infty} p_i$ with all the p_i equivalent for $i \geq 1$. Now write the identity as a countable orthogonal sum of projections all $\preceq p_1$ (using the decomposition $\mathrm{II}_1 \otimes \mathcal{B}(\mathcal{H})$ if necessary). We see that $1 \preceq p$. □

Unlike the II_1 case, or for that matter the $\mathcal{B}(\mathcal{H})$ case, the trace cannot be normalised (by $tr(1) = 1$ in the type II_1 factor case or the trace of a minimal projection being 1 in the $\mathcal{B}(\mathcal{H})$ case). This allows for the possibility of an automorphism α of M with $tr(\alpha(x)) = \lambda tr(x)$ for $x \geq 0$ and $\lambda > 0$, $\lambda \neq 1$.

Exercise 9.1.11. *Show that the trace on a II_∞ factor is unique with properties (i) to (vi), up to a scalar.*

Exercise 9.1.12. *If $\alpha : M \to N$ is a *-homomorphism from a type* II_1 *factor onto another, then α is an isomorphism, strongly continuous on the unit ball.*

Chapter 10

The Coupling Constant

We want to compare actions of a given II_1 factor on (separable) Hilbert spaces. We will show that they are parameterized by a single number in $[0, \infty]$.

Definition 10.0.13. *If M is a type II_1 factor, by M-*module *we will mean a Hilbert space $\mathcal{H}$ together with an ultraweakly continuous unital *-homomorphism from M to a type II_1 factor acting on $\mathcal{H}$. Thus M acts on $\mathcal{H}$ and we will write that action simply as $x\xi$ for $x \in M$ and ξ in $\mathcal{H}$.*

In fact the ultraweak continuity condition is superfluous. The identity map makes the Hilbert space on which M is defined into an M-module. Given M on $\mathcal{H}$ and another Hilbert space $\mathcal{K}$, $x \mapsto x \otimes id$ makes $\mathcal{H} \otimes \mathcal{K}$ into an M-module. The GNS representation makes $L^2(M)$ into an M-module. (The notion of $M-M$ bimodule is defined similarly as two commuting actions of M on some Hilbert space, $L^2(M)$ being the first example.) There is an obvious notion of direct sum of M-modules. We will compare a given M-module $\mathcal{H}$ with $L^2(M)$ by forming the direct sum of it $\mathcal{H}$ and infinitely many copies of $L^2(M)$.

10.1 Definition of $\dim_M \mathcal{H}$

Theorem 10.1.1. *Let M be a type II_1 factor and $\mathcal{H}$ a separable M-module. Then there is an isometry $u : \mathcal{H} \to L^2(M) \otimes \ell^2(\mathbb{N})$ such that $ux = (x \otimes 1)u$ (i.e. u is M-linear).*

Proof. Form the M-module $\mathcal{K} = \mathcal{H} \oplus L^2(M) \otimes \ell^2(\mathbb{N})$. Let $p = id \oplus 0 \in \mathcal{B}(\mathcal{K})$ be the projection onto $\mathcal{H}$ and $q = 0 \oplus id$ be the projection onto $L^2(M) \otimes \ell^2(\mathbb{N})$. Both p and q are in M' (on $\mathcal{K}$) which is a II_∞ factor since q is clearly infinite in

M' and if e is a rank one projection in $\mathcal{B}(\ell^2(\mathbb{N}))$ then $(0\oplus(1\otimes e))M(0\oplus(1\otimes e))$ is a type II_1 factor, being the commutant of M on $L^2(M)$.

Since q is an infinite projection in M', by 9.1.10 there is a partial isometry in M' with $u^*u = p$ and $uu^* \leq q$. Using the obvious matrix notation for operators on $\mathcal{K}$, let u be represented by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

Then calculating $u^*u = p$ and $uu^* \leq q$ gives $b^*b+d^*d = 0$ and $aa^*+bb^* = 0$ so that

$$u = \left(\begin{smallmatrix} 0 & 0 \\ w & 0 \end{smallmatrix}\right)$$

for some isometry $w : \mathcal{H} \to L^2(M) \otimes \ell^2(\mathbb{N})$.
Moreover the fact that u commutes with $\tilde{M}$ is equivalent to $wx = (x \otimes 1)w$ $\forall x \in M$. □

Corollary 10.1.2. *The commutant of a type* II_1 *factor is either a type* II_1 *factor or a type* II_∞ *factor.*

Proof. We leave the proof as an exercise. □

Proposition 10.1.3. *If* $u : \mathcal{H} \to L^2(M) \otimes \ell^2(\mathbb{N})$ *is an* M*-linear isometry then* $uu^* \in M'$ *on* $L^2(M) \otimes \ell^2(\mathbb{N})$ *and* $tr(uu^*)$ *is independent of* u.

Proof. If v were another M-linear isometry then $uu^* = uv^*vu^*$ so by 9.1.9 $tr(uu^*) = tr((vu^*)(uv^*)) = tr(vv^*)$. □

Observe that if M were replaced by $\mathbb{C}$ in the above construction the number $tr(uu^*)$ would be the dimension of $\mathcal{H}$.

Definition 10.1.4. *For a type* II_1 *factor (or the* $n \times n$ *matrices) and an* M*-module* $\mathcal{H}$*, the number* $tr(u^*u)$ *defined by the two previous results is called* $\dim_M \mathcal{H}$*, or the* coupling constant *or the* M-dimension *of* $\mathcal{H}$.

Put another way, any action of M on $\mathcal{H}$ is unitarily equivalent to $p(L^2(M)\otimes \ell^2(\mathbb{N}))$ for some $p \in (M \otimes 1)'$. $\dim_M(\mathcal{H})$ is then the trace in $(M \otimes 1)'$ where the trace is normalised so that $tr(1 \otimes q) = 1$ for a rank one projection in $\mathcal{B}(\ell^2(\mathbb{N}))$.

Simply by reducing by projections in $(M \otimes 1)'$ one obtains Hilbert spaces whose M-dimension is any number in $[0, \infty]$.

Trivial examples
(i) $\dim_M L^2(M) = 1$.
(ii) $\dim_M(L^2(M) \otimes \ell^2(\mathbb{N})) = \infty$

Note that the calculation of the trace in $M \otimes \mathcal{B}(\mathcal{H})$ is actually the calculation of a trace in $\mathcal{B}(\mathcal{H})$ in the usual sense of trace-class operators: (put $\mathcal{H} = \ell^2(\mathbb{N})$)

Proposition 10.1.5. *Let q be the projection on $L^2(M) \otimes \mathcal{H}$ defined by $\tilde{q} \otimes 1$ where $\tilde{q}$ is the projection from $L^2(M)$ onto $\mathbb{C}\Omega$. Then if $a \in M \otimes \mathcal{B}(\mathcal{H})$ (or $JMJ \otimes \mathcal{B}(\mathcal{H})$),*

$$Tr_{M \otimes \mathcal{B}(\mathcal{H})}(a) = Trace(qaq)$$

Proof. This is very easy- just extend Ω to an orthonormal basis of $L^2(M)$ and tensor with a basis of $\ell^(\mathbb{N})$ to obtain a basis for the whole Hilbert space. Then write out the sum for $Tr_{L^2(M)\otimes\mathcal{H}}(qaq)$ for this basis and notice that q kills all terms not involving Ω. The remaining sum is the definition of $Tr_{M\otimes\mathcal{B}(\mathcal{H})}(a)$.

□

10.2 Elementary properties of $\dim_M \mathcal{H}$

Theorem 10.2.1. *With notation as above,*

(i) $\dim_M(\mathcal{H}) < \infty$ *iff* M' *is a type* II_1 *factor.*
(ii) $\dim_M(\mathcal{H}) = \dim_M(\mathcal{K})$ *iff* M *on* $\mathcal{H}$ *and* M *on* $\mathcal{K}$ *are unitarily equivalent (= spatially isomorphic).*
(iii) If $\mathcal{H}_i$ *are (countably many)* M*-modules,*

$$\dim_M(\oplus_i \mathcal{H}_i) = \sum_i \dim_M \mathcal{H}_i.$$

(iv) $\dim_M(L^2(M)q) = tr(q)$ *for any projection* $q \in M$.
(v) If p *is a projection in* M, $\dim_{pMp}(p\mathcal{H}) = tr_M(p)^{-1} \dim_M(\mathcal{H})$.

For the next two properties we suppose M' *is finite, hence a type* II_1 *factor with trace* $tr_{M'}$.

(vi) If p *is a projection in* M', $\dim_{Mp}(p\mathcal{H}) = tr_{M'}(p) \dim_M \mathcal{H}$.
(vii) $(\dim_M \mathcal{H})(\dim_{M'} \mathcal{H}) = 1$.

Proof. Using an M-linear isometry u we see that M on $\mathcal{H}$ is unitarily equivalent to M on $uu^*L^2(M) \otimes \ell^2(\mathbb{N})$. This makes (i) and (ii) obvious.

To see (iii), choose M-linear isometries u_i from $\mathcal{H}_i$ to $L^2(M) \otimes \ell^2(\mathbb{N})$ and compose them with isometries so that their ranges are all orthogonal. Adding we get an M-linear isometry u with $uu^* = \sum u_i u_i^*$. Taking the trace we are done.

For (iv), choose a unit vector $\xi \in \ell^2(\mathbb{N})$ and define $u(v) = v \otimes \xi$. Then uu^* is $JqJ \otimes e$ where e is a rank one projection.

(v) Let us first prove the relation in the case $\mathcal{H} = L^2(M)q$ where q is a projection in M with $q \leq p$.

Then $pxp\Omega \mapsto p(x\Omega)p$ is a unitary from $L^2(pMp)$ to $pL^2(M)p$ which intertwines the left and right actions of pMp. Hence pMp on $pL^2(M)q$ is unitarily equivalent to pMp on $L^2(pMp)q$. So by (iv), $\dim_{pMp}(p\mathcal{H}) = tr_{pMp}(q) = tr_M(p)^{-1}tr_M(q) = tr_M(p)^{-1}\dim_M \mathcal{H}$.

Now if $\mathcal{H}$ is arbitrary, it is of the form $e(L^2(M)\otimes\ell^2(\mathbb{N}))$ for $e \in (M\otimes 1)'$. But e is the orthogonal sum of projections all equivalent to ones as in (iv) with $q \leq p$.

(vi) We may suppose $\mathcal{H} = e(L^2(M)\otimes\ell^2(\mathbb{N}))$ so $M' = e(JMJ\otimes\mathcal{B}(\ell^2(\mathbb{N}))e$ and p defines the isometry in the definition of $\dim_M(p\mathcal{H})$. But p is a projection less than e in a II∞ factor so by uniqueness of the trace, $\dim_M(p\mathcal{H}) = tr_{(M\otimes 1)'}(p) = tr_{(M\otimes 1)'}(p)/tr_{(M\otimes 1)'}(e)\dim_M(\mathcal{H}) = tr_{M'}(p)\dim_M(\mathcal{H})$.

(vii) Observe that, on $L^2(M)$, $\dim_M(\mathcal{H})\dim_{M'}(\mathcal{H}) = 1$ so by (v) and (vi) the result is true for M-modules of the form $L^2(M)p$. Also if one forms $\mathcal{K} = \oplus_{i=1}^k\mathcal{H}$ then $\dim_{M\otimes 1}(\mathcal{K}) = k\dim\mathcal{H}$ and $\dim_{(M\otimes 1)'}\mathcal{K} = k^{-1}\dim_{M'}$ by (v). But any $\mathcal{H}$ can be obtained from $L^2(M)$ as $\oplus_{i=1}^k L^2(M)p$ for suitable k and p. □

Example 10.2.2. If $\Gamma_0 < \Gamma$ are icc groups, $vN(\Gamma_0)$ acts on $\ell^2(\Gamma)$. And if $\gamma \in \Gamma$ the unitary $\rho(\gamma)$ of the right regular representation gives a $vN(\Gamma_0)$-linear unitary between $\ell^2(\Gamma_0)$ and $\ell^2(\Gamma_0\gamma^{-1})$. Hence by the coset decomposition, $\dim_{vN(\Gamma_0)}(\ell^2(\Gamma)) = [\Gamma : \Gamma_0]$.

Example 10.2.3. (Due to Atiyah and Schmidt.)

Discrete series representations of locally compact groups.

Reduction by a finite projection in the commutant of a type II$_1$ factor occurs in the representation theory of locally compact groups. If a discrete series representation is restricted to an icc lattice it generates a type II$_1$ factor. The coupling constant is given by the ratio of the "formal dimension" and the covolume of the lattice.

We illustrate in the case of $PSL(2,\mathbb{R})$ which is the group of transformations of the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$, $z \mapsto \dfrac{az+b}{cz+d}$ defined by invertible real 2×2 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. It is well known that there is a fundamental domain D for the action of the subgroup $\Gamma = PSL(2,\mathbb{Z})$ illustrated below:

DO FIGURE

The set D and all its translates under $PSL(2,\mathbb{Z})$ cover $\mathbb{H}$ and are disjoint apart from boundaries which are of Lebesgue measure 0. Thus if μ is an invariant measure equivalent to Lebesgue measure, $L^2(\mathbb{H}, d\mu)$ gives a unitary representation of Γ which is unitarily equivalent to the left regular representation tensored with the identity on $L^2(D, d\mu)$, making $L^2(\mathbb{H}, d\mu)$ into a $vN(\Gamma)$-module whose $vN(\Gamma)$ dimension is infinite.

The measure $\dfrac{dxdy}{y^2}$ is Γ-invariant but we want to vary this procedure slightly. For each $n \in \mathbb{N}$ consider $\dfrac{dxdy}{y^{2-n}}$. This measure is not invariant but we can make the action of $PSL(2,\mathbb{R})$ unitary on $L^2(\mathbb{H}, \dfrac{dxdy}{y^{2-n}})$ by the formula

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) f(z) = \frac{1}{(cz+d)^n} f\left(\frac{az+b}{cz+d}\right)$$

(with perhaps an inverse matrix...—exercise as usual). This changes nothing as far as how the representation looks to $PSL(2,\mathbb{Z})$ so we obtain (unitarily equivalent) $vN(\Gamma)$-modules $\mathcal{H}_n = L^2(\mathbb{H}, \dfrac{dxdy}{y^{2-n}})$ for each n.

The commutant of $vN(\Gamma)$ on $\mathcal{H}_n$ is a II_∞ factor. But as is well known, holomorphic functions form a closed subspace of L^2 functions which is manifestly invariant under $PSL_2(\mathbb{R})$. The ensuing unitary representation is known to be irreducible and in the *discrete series* of $PSL_2(\mathbb{R})$. It can be shown to be a finite projection in Γ'. Thus we have a concrete example of a $vN(\Gamma)$-module with finite $vN(\Gamma)$-dimension or coupling constant.

In general, if G is a locally compact group with Haar measure dg, the discrete series representations are precisely those irreducible unitary representations π that are direct summands of the left regular representation on $L^2(G, dg)$. So if Γ is a discrete subgroup with a fundamental domain D so that G is covered by the $\gamma(D)$ which are disjoint up to measure zero sets, we may apply the same analysis as above to obtain a $vN(\Gamma)$ module. The obvious question is to calculate its coupling constant. This turns out to be quite simple because of a key property of discrete series representations.

See [ref robert] for the proof that if $\mathcal{H}$ is a Hilbert space affording a discrete series representation π of G, then the functions $g \mapsto \langle \pi_g \xi, \eta \rangle$, the so-called *coefficients* of π are in $L^2(G, dg)$, and that the following argument is justified. We may then fix a unit vector $\eta \in \mathcal{H}$ and consider the map $T : \mathcal{H} \to L^2(G)$ defined by

$$T(\xi)(g) = \langle \pi_g \xi, \eta \rangle$$

This map is obviously G-linear so T^*T commutes with G on $\mathcal{H}$ and is thus a multiple of an isometry. Hence there is a constant d_π such that

$$d_\pi \int_G \langle \pi_g \xi, \eta \rangle \langle \eta, \pi_g \xi' \rangle dg = \langle \xi, \xi' \rangle.$$

If G is compact and Haar measure is normalized so that G has measure 1, d_π is the dimension of the vector space $\mathcal{H}$. In general d_π depends on the choice of Haar measure but obviously the product of d_π with the covolume $\int_D dg$ does not. The coefficients give an explicit embedding of $\mathcal{H}$ in $L^2(G, dg)$.

Theorem 10.2.4.

$$\dim_{vN(\Gamma)}(\mathcal{H}) = d_\pi \, covolume(\Gamma).$$

Proof. Realize $\mathcal{H}$ as $pL^2(G, dg)$ for some projection on $L^2(G, dg)$ commuting with the left regular action λ_g.

Observe first that since D is a fundamental domain (for the left action of Γ on G, the following unitary w gives an explicit Γ-linear isomorphism from $\ell^2(\Gamma) \otimes L^2(D)$ to $L^2(G, dg)$:

$$w(\epsilon_\gamma \otimes f)(g) = f(\gamma^{-1} g).$$

Noting that $\ell^2(\Gamma) = L^2(vN(\Gamma)$, we shall identify the operator $w^{-1}qw$ on $L^2(G)$ and then apply 10.1.5. (Where q is the projection of 10.1.5.)

But $wf = \sum_\gamma \epsilon_\gamma \otimes (f \circ \lambda_\gamma|_D)$ so that $w^{-1}qw$ is nothing but orthogonal projection χ_D from $L^2(G)$ to $L^2(D)$. Hence by 10.1.5 we have

$$\dim_{vN(\Gamma)}(L^2(G, dg)) = Trace(\chi_D p \chi_D.)$$

We now have to calculate something invoking the definition of the formal dimension. So let η be a unit vector in $pL^2(G)$ and consider the constant function $g \mapsto ||\lambda_g \eta||^2$. If we choose an orthonormal basis ξ_n for $L^2(D)$ then $\{\lambda_\gamma \xi_n\}$ is a basis of $L^2(G)$. Thus

$$||\lambda_g \eta||^2 = \sum_{\gamma, n} |\langle \lambda_g \eta, \lambda_\gamma(\xi_n) \rangle|^2$$

Now integrate over the fundamental domain D to obtain

$$covolume(\Gamma) = \sum_{\gamma, n} \int_D |\langle \lambda_g \eta, \lambda_\gamma(\xi_n) \rangle|^2$$

$$= \sum_n (\sum_\gamma \int_D |\langle \lambda_{\gamma g} \eta, \xi_n \rangle|^2)$$

$$= \sum_n (\sum_\gamma \int_{\gamma D} |\langle g\eta, \xi_n\rangle|^2)$$

$$= \sum_n (\int_G |\langle \lambda_g \eta, p\xi_n\rangle|^2)$$

and by the formula defining the formal dimension this is

$$= \frac{1}{\text{(formal dimension)}} \sum_n (||\eta||^2 ||p\xi_n||^2)$$

$$= \frac{1}{\text{(formal dimension)}} \dim_{vN(\Gamma)}(\mathcal{H})$$

Where we can extend the basis of $L^2(D, dg)$ to a basis of $L^2(G, dg)$.

□

see also [1] pp. 142-148.

Proposition 10.2.5. *If M is a type* II_1 *factor on $\mathcal{H}$ then*

(a) M has a separating vector if $\dim_M(\mathcal{H}) \geq 1$.

(b) M has a cyclic vector if $\dim_M(\mathcal{H}) \leq 1$.

Proof. Both assertions follow immediately by comparing $\mathcal{H}$ to $L^2(M)p$ or a direct sum of copies of it. □

In fact both conditions in the last proposition are iff. For that one needs to control arbitrary vectors in $L^2(M)$. In fact the original definition of the coupling constant by Murray and von Neumann was as follows. Let M on $\mathcal{H}$ be a type II_1 factor whose commutant is a type II_1 factor. Choose any nonzero vector $\xi \in \mathcal{H}$ and let p and q be projections onto the closures of $M\xi$ and $M'\xi$ respectively. Then $p \in M'$ and $q \in M$ and using the normalised traces the coupling constant was defined as the ratio $\dfrac{tr_M(q)}{tr'_M(p)}$, the hard part being to show that this ratio is independent of ξ. Assuming this last statement it is trivial to identify the Murray-von Neumann coupling constant with our $\dim_M(\mathcal{H})$ but at this stage we have nothing to offer in the way of a simplified proof of why this number does not depend on ξ.

Example 10.2.6. *(due to M. Rieffel)* If (X, μ) is a measure space and Γ is a countable group acting by measure preserving transformations on (X, μ) so that Γ acts by unitaries u_γ on $L^2(X, \mu)$ in the obvious way. We say that a measurable subset $D \subseteq X$ is a *fundamental domain* for Γ if $X = \cup_\gamma \gamma(D)$ and $\mu(D\gamma(D)) = 0$ for all $\gamma \in \Gamma$, $\gamma \neq id$. (One may clearly suppose the

$\gamma(D)$ are disjoint by removing a set of measure zero.) In this situation the abelian von Neumann algebra $L^\infty(X)^\Gamma$ of Γ-invariant L^∞ functions may be identified with the space $L^\infty(D)$.

Now suppose Γ and Λ are two groups acting on X as above with fundamental domains D and E respectively. We may consider the von Neumann algebra $M_{\Gamma,\Lambda}$ on $L^2(X,\mu)$ defined as $\{\{u_\gamma : \gamma \in \Gamma\} \cup L^\infty(X)^\Lambda\}''$.

Chapter 11

The Crossed Product construction.

Perhaps the most useful way of producing von Neumann algebras from others is the crossed product. In pure algebra, if G is a group acting by automorphisms on an algebra A we form the vector space of finite formal sums

$$\sum_{g \in G} a_g u_g$$

with the $a_g \in A$. We multiply the sums with the rules $u_g u_h = u_{gh}$ (and $u_1 = 1$) and $u_g a u_g^{-1} = g(a)$ reminiscent of the semidirect product of groups- we use the notation $A \rtimes G$ for this algebra, called the "crossed product". It is obviously universal for "covariant representations", i.e. whenever A acts on a vector space V and $g \to v_g$ is a representation of G on V with $v_g a v_{g^{-1}}$ then the action of A extends to one of $A \rtimes G$ with u_g acting via v_g.

From our experience with group algebras we expect the von Neumann algebra version to be neither so simple nor universal (for an icc group, almost no group representations extend to the von Neumann algebra).

We begin by defining a very general notion of von Neumann algebraic crossed product about which there is not a lot to say, but then examine it carefully in special cases.

11.1 Group actions.

Let M be a von Neumann algebra and G a group. An *action* of G on M is a homomorphism $g \mapsto \alpha_g$ from G to the automorphism group $AutM$ of M (where automorphisms may be assumed ultraweakly continuous if necessary). The algebra of fixed points for the action is denoted M^G and is a von

Neumann algebra. A special case of some importance is when the action is a unitary group representation $g \mapsto u_g$ with $u_g M u_g^* = M \ \forall g \in G$. In that case setting $\alpha_g(x) = u_g x u_g^*$ defines an action of G on M (and M'). We say that the action α is *implemented* by the unitary representation u_g. If the u_g are actually in M, we say that the action is *inner* as an inner automorphism of M is by definition one of the form $\operatorname{Ad} u(x) = uxu^*$ for u a unitary in M. An automorphism is called *outer* if it is not inner.

Actions are not always implementable though the notion depends on the Hilbert space on which M acts.

Exercise 11.1.1. *If (X, μ) is a measure space and T is a bijection of X which preserves the measure class of μ (i.e. $\mu(A) = 0 \Leftrightarrow \mu(T^{-1}(A)) = 0$.) show how T defines an automorphism α_T of $L^\infty(X, \mu)$. Show further that this automorphism is implemented by a unitary u on $L^2(X, \mu)$.*

A bijection T as above is called *ergodic* if $T(A) = A$ for a measurable subset $A \subseteq X$ implies either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Proposition 11.1.2. *With notation as above T is ergodic iff the only fixed points for α_T are constant functions.*

Proof. ($\Rightarrow$) Let $f \in L^\infty$ and $\alpha_T(f) = f$. After throwing away a null set we may assume that $f(x) = f(T(x))$ for all $x \in X$. Then for every $\epsilon > 0$, by the definition of the essential supremum, $\mu(\{x : ||f|| - |f(x)| < \epsilon\} \neq 0$. But this set is invariant under T so it is equal to X up to a set of measure 0. Letting ϵ tend to 0 we see that $\mu(\{x : |f(x)| \neq ||f||\}) = 0$. So we may assume $f(x) = e^{ig(x)}$ for some measurable g taking values in $[0, 2\pi)$. Repeating the argument for g gives f constant almost everywhere.

($\Leftarrow$) If A is a measurable invariant set then its characteristic function is fixed by α in L^∞ iff A is invariant. □

Exercise 11.1.3. *Let $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ and $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ be the Pauli spin matrices. Show that $Ad\, u_x$, $Ad\, u_y$ and $Ad\, u_z$ define an action of the group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ on the two by two matrices which is not implementable for $M_2(\mathbb{C})$ on $\mathbb{C}^2$.*

Exercise 11.1.4. *Show that* any *group action is implementable for a type* II_1 *factor in standard form and more generally any automorphism group preserving a faithful normal state is implementable in the GNS representation.*

Exercise 11.1.5. *Show that every automorphism of $\mathcal{B}(\mathcal{H})$ is inner.*

Exercise 11.1.6. *Show that the automorphism of $vN(F_2)$ coming from the group automorphism which exchanges the 2 generators is outer.*

If G is a topological group there are many possible notions of continuity. The most useful is that of pointwise *-strong convergence, i.e. we assume that the map $g \mapsto \alpha(g)(x)$ is *-strong continuous for any $x \in M$. Typically many other notions of continuity will be equivalent to that and even a measurability assumption can be enough to ensure this continuity.

We will always assume pointwise *-strong continuity when referring to an action of a topological group.

Exercise 11.1.7. *Is the action by translation of $\mathbb{R}$ on $L^\infty(\mathbb{R})$ pointwise norm continuous? pointwise strongly continuous? pointwise *-strong continuous?*

Actions of a given group on von Neumann algebras are easy to construct but actions of a group on a given von Neumann algebra may be hard to come by.

Definition 11.1.8. *An action of G on M is said to be* ergodic *if $M^G = \mathbb{C}id$.*

Exercise 11.1.9. *Show that if G acts preserving μ on (X, μ) then the resulting action of G on $L^\infty(X, \mu)$ is ergodic iff the only measurable subsets $A \subseteq X$ which satisfy $\mu(g(A)\Delta A) = 0\, \forall g \in G$ satisfy either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.*

(Here $A\Delta B$ means $A \setminus B \cup B \setminus A$.)

The following question is an intriguing open problem:

Does $SU(3)$ have any ergodic action on a type II_1 factor?

It is shown in [] that $SU(2)$ has no such action and it is shown in [] that if a compact group acts ergodically on a von Neumann algebra then that von Neumann algebra has a faithful normal trace.

11.2 The crossed product

Suppose α is an action of the locally compact group G with Haar measure dg on the von Neumann algebra M with Hilbert space $\mathcal{H}$. Form the Hilbert space $\mathcal{K} = L^2(G, \mathcal{H}) = L^2(G) \otimes \mathcal{H}$ and let G act on $\mathcal{K}$ by $u_g = \lambda_g \otimes 1$, λ being the left regular representation. Further, let M act on $\mathcal{K}$ by

$$(\tilde{x}f)(g) = \alpha_{g^{-1}}(f(g))$$

.

Exercise 11.2.1. *Show that $x \mapsto \tilde{x}$ is an ultraweakly continuous *-isomorphism of M onto a von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$.*

Exercise 11.2.2. *Show that* $u_g \tilde{x} u_g^* = \widetilde{\alpha_g(x)}$.

Note that this gives another way of making a group action implementable, at least when it is locally compact.

Definition 11.2.3. *If* M*,* $\mathcal{H}$*,* G *and* α *are as above, the* crossed product $M \rtimes_\alpha G$ *is the von Neumann algebra on* $\mathcal{K}$ *generated by* $\{u_g : g \in G\}$ *and* $\{\tilde{x} : x \in M\}$.

From now on we will drop the $\tilde{}$ and identify M with $\tilde{M}$. Note that finite linear combinations $\sum_g x_g u_g$ form a dense *-subalgebra of $M \rtimes_\alpha G$. Moreover the u_g are linearly independent over M in the sense that $\sum_g x_g u_g = 0 \Rightarrow x_g = 0$ for each g in the sum. This dense subalgebra could be called the *algebraic* crossed product.

There is a well-developed theory of $M \rtimes_\alpha G$ when G is compact or abelian, but we shall be mostly interested in the case where G is discrete as then we may replay the matrix element game that we played for $vN(\Gamma)$ to gain control of weak limits of elements in the algebraic crossed product. (In fact of course $vN(\Gamma)$ is the special case of the crossed product when $M = \mathbb{C}$ and the action is trivial.) Indeed we see immediately as in 3.3.4 that if G is discrete, any element of $M \rtimes_\alpha G$ defines a function $g \mapsto x_g$ so that the sum $\sum_g x_g u_g$ stands for a certain matrix of operators on $\mathcal{K} = \mathcal{H} \otimes \ell^2(G)$. Moreover any matrix of this form which defines a bounded operator on $\mathcal{K}$ is in $M \rtimes_\alpha G$. This is because the sum converges pointwise at least on the dense set of functions of finite support from G to $\mathcal{H}$. In the case where the crossed product is a II_1 factor we know that the commutant consists of right multiplication by elements of $M \rtimes_\alpha G$ so a weakly dense subalgebra of $(M \rtimes_\alpha G)'$ preserves this dense subspace of vectors and on that subspace $\sum_g x_g u_g$ and right multiplication by u_g and $x \in M$ commute. We will return to the general case later on.

Moreover the formulae

$$(\sum x_g u_g)^* = \sum \alpha_g(x_{g^{-1}}) u_g$$

and

$$(\sum x_g u_g)(\sum y_g u_g) = \sum_g \{\sum_h x_h \alpha_h(y_{h^{-1}g})\} u_g$$

are justified by matrix multiplication.

We shall now provide some sufficient conditions for $M \rtimes_\alpha G$ to be a factor—always assuming G is discrete.

Definition 11.2.4. *An action* α *of* G *on* M *is called* outer *if the only* g *in* G *for which* α_g *is inner is the identity.*

Proposition 11.2.5. *If G is a discrete group and α is an outer action of G on the factor M then $M \rtimes_\alpha G$ is a factor with $M' \cap M \rtimes_\alpha G = \mathbb{C}1$.*

Proof. If $x = \sum x_g u_g \in Z(M)$ then equating coefficients in the expression that x commutes with M gives us $y x_g = x_g \alpha_g(y)$ $\forall y \in M, g \in G$. By the next lemma this implies $x_g = 0$ for any $g \neq 1$. Thus $x \in M$. Since M is a factor we are done. □

Lemma 11.2.6. *Let $\alpha \in Aut\, M$ for a factor M. Suppose there is an $x \in M$, $x \neq 0$, with*

$$yx = x\alpha(y) \qquad \forall\, y \in M.$$

Then α is inner.

Proof. If x were unitary this would be obvious. So take the adjoint of the relation to obtain $x^* y = \alpha(y) x^*$ $\forall y \in M$. Thus $yxx^* = x\alpha(y)x^* = xx^*y$ and $xx^* \in Z(M)$. Similarly $x^*x \in Z(M)$. But xx^* and x^*x always have the same spectrum so since M is a factor both xx^* and x^*x are equal to the same positive number λ. Dividing by $\sqrt{\lambda}$ converts x into a unitary and we are done. □

These two results prompt the following definition.

Definition 11.2.7. *An automorphism α of a von Neumann algebra M is called free if*

$$yx = x\alpha(y) \qquad \forall\, y \in M \;\; \Rightarrow \;\; x = 0.$$

An action α is called free if α_g is free for every $g \neq id$.

The argument of proposition 11.2.5 shows in fact that if α is a free action on a von Neumann algebra M then $Z(M \rtimes_\alpha G) \subseteq M$, in fact that $M' \cap M \rtimes_\alpha G \subseteq M$.

Theorem 11.2.8. *If α is a free ergodic action of G on a von Neumann algebra M, then $M \rtimes_\alpha G$ is a factor.*

Proof. This follows immediately from the preceding remark. □

To understand the meaning of freeness for automorphisms of the form α_T we need to make a hypothesis on (X, μ) as otherwise one could envisage a T which is non-trivial on X but for which α_T is the identity. So we will suppose from now on that (X, μ) is countably separated. This means there is a sequence B_n of measurable sets with $\mu(B_n) > 0$ for which, if $x \neq y$, there is an n with $x \in B_n$ but $y \notin B_n$. Obviously $\mathbb{R}^n$ is countably separated.

Exercise 11.2.9. *Show that $\alpha_T = id$ means that $Tx = x$ almost everywhere.*

Hint-look at the proof of the next result.

Proposition 11.2.10. *If T is a transformation of (X, μ) then α_T is free iff $\mu(\{x : T(x) = x\}) = 0$.*

Proof. ($\Rightarrow$)If A is any measurable set on which $T = id$ then $\chi_A f = \alpha_T(f)\chi_A$ for all $f \in L^\infty$.

($\Leftarrow$) First throw away any fixed points of T. Then suppose $f_1\alpha_T(f_2) = f_2 f_1 \quad \forall f_2 \in L^\infty$. Let A be the support of f_1. Then since T has no fixed points, $A = \cup_n(A \cap B_n \setminus T^{-1}(B_n))$. If f_1 were non-zero in L^∞, we could thus choose an n for which $\mu(A \cap B_n \setminus T^{-1}(B_n)) > 0$. Set $f_2 = \chi_{B_n}$. Then for any $x \in A \cap B_n \setminus T^{-1}(B_n)$ we have $f_1(x)f_2(x) \neq 0$ but $f_1(x)f_2(Tx) = f_1(x)\chi_{B_n}(Tx) = 0$ since $x \notin T^{-1}(B_n)$. Thus $f_1\alpha_T(f_2) \neq f_2 f_1$ in L^∞. So the measure of A must be zero. $\square$

We conclude that if Γ is a countable group acting freely and ergodically on a measure space (X, μ), preserving the class of μ, then the crossed product $L^\infty(X, \mu) \rtimes \Gamma$ is a factor.

Note that if Γ is abelian, ergodic implies free.

Exercise 11.2.11. *Show that freeness of the action actually proves that $L^\infty(X, \mu)$ is maximal abelian in the crossed product.*

The crossed product $M \rtimes \Gamma$ when M is abelian and Γ is discrete is called the *group measure space construction*. Here are several examples.

Example 11.2.12. $X = \mathbb{Z}$, $\Gamma = \mathbb{Z}$ acting by translation, $\mu =$ counting measure.

The action is free and ergodic and $L^\infty(X, \mu) \rtimes \Gamma = \mathcal{B}(\ell^2(\mathbb{Z}))$.

Example 11.2.13. The irrational rotation algebra-von Neumann algebra version.

$(X, \mu) = (\mathbb{T}^1, d\theta)$, $\Gamma = \mathbb{Z}$ generated by the transformation T where $T(z) = e^{i\alpha}z$ and $\alpha/2\pi$ is irrational.

Exercise 11.2.14. *Use Fourier series to show that this T is ergodic.*

Example 11.2.15. Let H be a finite abelian group and $\Gamma = \bigoplus_{n\in\mathbb{N}} H$ be the countable group of sequences (h_n) with h_n eventually the identity. Put $X = G = \prod_{n\in\mathbb{N}} H$ (the set of *all* sequences) with the product topology. Then G is a compact group so has a Haar measure μ. Γ acts on X by left translation. The action is clearly free and ergodic as we shall now argue.

There is a particularly von Neumann algebraic way to view this example without even constructing the space (X, μ) !

Let $A = L^\infty(H) = \mathbb{C}\hat{H}$ be the group algebra of the dual group $\hat{H}$, with its usual trace. As in section 6.2, form the algebraic tensor product $\otimes_{alg,n\in\mathbb{N}}A$ with product trace tr. Then perform the GNS construction with respect to tr to obtain an abelian von Neumann algebra. It may be identified with $L^\infty(G,\mu)$ so the Hilbert space $\mathcal{H}$ of the GNS construction is $L^2(X,\mu)$. But it is clear that an orthornormal basis of $\mathcal{H}$ is given by finite sequences (χ_n) of elements of $\hat{H}$ which define elements $\chi_1\otimes\chi_2\otimes\cdots\otimes 1\otimes 1\otimes 1\cdots$ in $\otimes_{alg,n\in\mathbb{N}}A$. The point is that these basis vectors are eigenvectors for the action of Γ on $L^2(X,\mu)$:

$$(h_n)(\chi_1\otimes\chi_2\otimes\cdots\otimes 1\cdots) = (\prod_n \chi_n(h_n))\ \chi_1\otimes\chi_2\otimes\cdots\otimes 1\cdots .$$

Ergodicity follows easily since the only basis element which is fixed by all the (h_n) is the one with all χ_n equal to 1.

Exercise 11.2.16. *Show that if $H = \mathbb{Z}/2\mathbb{Z}$ in this example then the subalgebra of the crossed product generated by $\otimes_{alg,n\in\mathbb{N}}A$ and Γ is the algebraic infinite tensor product of $M_2(\mathbb{C})$.*

Both of the last two examples are special cases of a more general one: X is a compact group with its Haar measure and Γ is a countable dense subgroup acting (freely) by left translation. The Peter Weyl theorem shows that this action is ergodic.

Example 11.2.17. Bernoulli shift.

If Γ is any infinite group and $A = \mathbb{Z}/2\mathbb{Z}$ we may form the tensor product indexed by Γ of a copy of A for each $\gamma\in\Gamma$. The von Neumann algebra thus obtained is once again the L^∞ space of the infinite product measure space, this time with the set indexing the product being Γ. As in the previous example we can obtain a basis of L^2 indexed by functions from Γ to the set $\{0,1\}$ which are almost always 0. These functions are the same as finite subsets of Γ and the action of Γ on the Hilbert space is by permuting the basis in the obvious way. Ergodicity follows from the fact that the orbit of any non-empty subset is infinite.

One could also chose another trace than the usual one and modify the orthonormal basis of A accordingly. The measures are the obvious ones unless specified.

We give a few more examples of free ergodic actions without supplying proofs of ergodicity.

Example 11.2.18. $SL(2,\mathbb{Z})$ acts on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ via the linear action on $\mathbb{R}^2$.

Example 11.2.19. $PSL(2, \mathbb{Z})$ acts on $\mathbb{R} \cup \{\infty\}$ by linear fractional transformations.

Example 11.2.20. $SL(2, \mathbb{Z})$ acts on $\mathbb{R}^2$ by linear transformations.

Example 11.2.21. $\mathbb{Q}$ acts on $\mathbb{R}$ by translation.

There are two fairly easy ways to see that this action is ergodic. The first is to reduce it to a dense subgroup of a compact group by observing that an L^∞ function on $\mathbb{R}$ which is invariant under translation by $\mathbb{Z}$ defines an L^∞ function on the quotient $\mathbb{T}$. Then use Fourier series.

bullshit

The second way is a direct attack which should generalise to show that translation by any countable dense subgroup of a locally compact group is ergodic. If $f \in L^\infty(\mathbb{R})$ is invariant under $\mathbb{Q}$, set things up so that there are sets A and B both of nonzero measure, so that $g(A) \cap g(B) = \emptyset$. Cover A and B with intervals of the same width with rational endpoints. Some of these must intersect A and B in non-nul sets. But all these intervals are all translates of each other so g cannot be invariant up to sets of measure zero.

Example 11.2.22. The "$ax + b$" group $\mathbb{Q} \rtimes \mathbb{Q}^*$ acts on $\mathbb{R}$

Example 11.2.23. Same as example 11.2.13 with $H = \mathbb{Z}/2\mathbb{Z}$ but using a normalised trace on $\mathbb{C}H$ which is different from the usual one. Such a trace is specified by its values on the minimal projections of $\mathbb{C}H$ which we could call p and $1 - p$ for $0 < p < 1$. The product measure is not absolutely continous with respect to Haar measure, and it is not preserved by group translation so this example is perhaps most easily approached by the von Neumann algebra construction where one can *implement* the action of $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ by unitaries. These unitaries come from ones on $L^2(H)$ which exchange two points of unequal weight so they must be correctly scaled.

Exercise 11.2.24. *Work out the details of example 11.2.23*

In the examples we see four different kinds of free ergodic actions:
Type I : Γ acts transitively.11.2.12
Type II_1 : Γ preserves a finite measure. 11.2.13,11.2.15,11.2.17,11.2.18
Type II_∞ : Γ preserves an infinite measure.11.2.20,11.2.21
Type III : Γ preserves no measure equivalent to μ.11.2.19,11.2.22,11.2.23

11.3 The type of the crossed product.

We adopt the notations and conventions of the previous section. The map $E_m : M \rtimes_\alpha \Gamma \to M$ which assigns a_{id} to the element $\sum_{\gamma \in \Gamma}$ is destined to play a big role in the theory. It is called the *conditional expectation* onto M and obviously satisfies the following contitions:

(i) $E_M^2 = E_M$.
(ii) $E_M(x)^* = E_M(x^*)$, $E_M(1) = 1$, $E_M(x^*x) = 0 iff x = 0$.
(iii) $E_M(x^*x) \geq E_M(x^*)E_M(x)$, $||E(x)|| \leq ||x||$.
(iv) $E_M(axb) = aE_M(x)b$ for $a, b \in M$.
(v) E_M is ultraweakly continuous.

So E_M is a projection of norm one in the Banach space sense. The condition (iv) says that E_M is an $M - M$-bimodule map.

Theorem 11.3.1. *If Γ acts non-transitively, freely and ergodically, preserving the finite measure μ then $L^\infty(X, \mu) \rtimes \Gamma$ is a II_1 factor. If Γ preserves the infinite σ-finite measure μ then $L^\infty(X, \mu) \rtimes \Gamma$ is a II_∞ factor unless Γ acts transitively in which case $L^\infty(X, \mu) \rtimes \Gamma$ is type I.*

Proof. (i) It is clearer to prove a more general statement (in the case where Γ preserves μ and $\mu(X) = 1$). So suppose Γ preserves the faithful positive ultraweakly continuous trace tr on the von Neumann algebra A and that its action is free and ergodic. Then we claim $M = A \rtimes \Gamma$ is a type II_1 factor (or a finite dimensional factor). By previous results we need only show that it has an ultraweakly continous positive trace. So define $Tr = tr \circ E_A$ on M. Ultraweak continuity and positivity are obvious so by continuity and linearity it suffices to prove $Tr(au_\gamma bu_\eta) = Tr(bu_\eta au_\gamma)$. For either side of the equation to be non-zero means $\eta = \gamma^{-1}$ amd then the left hand side is $tr(a\alpha_\gamma(b)) = tr(\alpha_\gamma^{-1}(a\alpha_\gamma(b))) = tr(b\alpha^{-1}(a_))$ which is equal to $Tr(bu_\eta au_\gamma)$.

(ii) If μ is infinite and Γ does not act transitively then there are no atoms hence there are subsets Y of X of arbitrary positive measure. Let Y have finite non-zero measure and let ξ be the function $\xi(\gamma) = \delta_{\gamma,id}\, \chi_Y$. Then $\langle au_\gamma\xi, \xi\rangle = \omega_\xi(au_\gamma) = \delta_{id,\gamma} \int_Y a(x)d\mu(x)$. One easily checks that $\omega_\xi((pau_\gamma p)(pbu_\eta p)) = \omega_\xi((pbu_\eta p)(pau_\gamma p))$ so by 3.4.6 ω_ξ defines a positive ultraweakly continuous trace on $p(A \rtimes \Gamma)p$ which is a type II_1 factor. But $A \rtimes \Gamma$ is not itself a type II_1 factor since A contains an infinite family of equivalent mutually orthogonal projections. By 9.1.8 we are done.

(iii) If Γ acts transitively then $(X, \mu) = (\Gamma, \text{counting measure})$ and the characteristic function of a set with one element is a minimal projection in $L^\infty(X, \mu) \rtimes \Gamma$. □

Exercise 11.3.2. *If Γ acts ergodically on (X, μ) preserving the σ-finite measure μ then any other invariant equivalent measure is proportional to μ.*

We now want to show that there are factors that are neither of type I nor type II. Suppose that $M = L^\infty(X, \mu) \rtimes \Gamma$ is a type I or II factor. Then it has a trace $tr : M_+ \to [0,\infty]$. We would like to define an invariant measure on X, absolutely continous with respect to μ, by reversing the procedure of theorem 11.3.1 and defining the measure $\sigma(A)$ to be $tr(\xi_A)$ ($\xi_A \in L^\infty(X,\mu) \subseteq M$). Invariance of the measure σ is no problem. The snag is that $tr(\chi_A)$ could be infinite for every non-null set A. We will show that this is not the case. To this end the concept of lower semicontinuity will be useful.

Definition 11.3.3. *If X is a topological space we say that $f : X \to \mathbb{R}$ is* lower semicontinous *if for every $x \in X$ and $\epsilon > 0$ there is an open set $U \subseteq X$ such that $f(u) > f(x) - \epsilon$ for all $u \in U$.*

Exercise 11.3.4. *Prove that if f is lower semicontinous then*
(a)$f^{-1}((-\infty, K]))$ is closed for every $K \in \mathbb{R}$.
(b)f attains its minimum on any compact subset of X.

Exercise 11.3.5. *If $\mathcal{H}$ is a Hilbert space and $\xi \in \mathcal{H}$, the function $a \mapsto ||a\xi||$ from $\mathcal{B}(\mathcal{H})$ to $\mathbb{R}$ is* weakly *lower semicontinuous.*

Exercise 11.3.6. *If f_α are lower semicontinous then $\vee_\alpha f_\alpha$ is lower semicontinous if it exists.*

Lemma 11.3.7. *Let M be a type I or II factor and $tr : M^+ \to [0,\infty]$ be Trace in type I, as in 9.1.9 in type II_∞ and the trace in type II_1. Then for each $K \geq 0$, $M_{1,K} = \{x : tr(x^*x) \leq K\}$ is weakly compact.*

Proof. Clear in the II_1 case. In a decomposition $M \cong N \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ on $\mathcal{H}$ with N a type II_1 factor or $\mathbb{C}$ we may assume by 10.2.5 that there is a vector $\xi \in e_{11}\mathcal{H}$ with ω_ξ a trace on $e_{11}Me_{11}$. So if $\xi_i = e_{i1}\xi$ we have, up to a scalar, that

$$tr(x) = \sum_{i=1}^{\infty} \langle x\xi_i, \xi_i \rangle.$$

By the previous exercises and weak compactness of the unit ball, we are done. □

Proposition 11.3.8. *With notation as above, for $x \in M_{1,K}$ let $W(x)$ be the weak closure of the convex set of finite sums $\{\sum_i \lambda_i u_i x u_i^* : \sum_i \lambda_i = 1, \lambda_i > 0, u_i$ unitary in $L^\infty(X,\mu)\}$. Then $W(x) \subseteq M_{1,K}$ and if $\phi(y) = tr(y^*y)$ for $y \in W(x)$ then ϕ attains its minimum at a unique point $\mathcal{E}(x)$ of $W(x)$.*

Proof. Note first that $\{z \in M : tr(z^*z) < \infty\}$ is a vector space on which $||z|| = tr(z^*z)$ defines a prehilbert space structure. (Since $(a+b)^*(a+b) \leq 2(a^*a + b^*b)$ as operators, and the parallelogram identity passes to the potentially infinite sum defining tr.) Moreover $W(x)$ is a weakly compact subset of M so by lower semicontinuity ϕ attains its minimum at a point which is unique by two dimensional Euclidean geometry as in 2.1.2. □

Proposition 11.3.9. *Suppose that $M = L^\infty(X,\mu) \rtimes \Gamma$ is a type I or II factor for a free ergodic action of Γ on $L^\infty(X,\mu)$. Let tr be as above and p be a projection in M with $tr(p) < \infty$. Then*

$$\mathcal{E}(p) = E_{L^\infty(X,\mu)}(p)$$

and $0 < tr(\mathcal{E}(p)^2) \leq tr(p)$.

Proof. Let $E = E_{L^\infty(X,\mu)}$. By the uniqueness of $\mathcal{E}(p)$ it commutes with every unitary in L^∞ so it is in L^∞ by 11.2.11. On the other hand $E(y) = E(p)$ for all $y \in W(p)$ by the bimodule linearity of the conditional expectation and its ultraweak continuity. So $E(\mathcal{E}(p)) = E(p) = \mathcal{E}(p)$. But $\phi(\mathcal{E}(p) \leq \phi(p) = tr(p)\infty$. Finally $E(p) = E(p^2)$ which is a positive non-zero self-adjoint operator and hence has non-zero trace. □

Theorem 11.3.10. *Let Γ act freely and ergodically on the countably separated σ-finite measure space (X,μ) so that there is no σ-finite Γ-invariant measure on X absolutely continuous with respect to μ. Then $L^\infty(X,\mu) \rtimes \Gamma$ is a factor not of type I or II.*

Proof. If the crossed product were of type I or II, define the measure ρ on X by $\rho(A) = tr(\chi_A)$. By the previous result $\rho(A)$ would have to be finite and non-zero for some A since the L^∞ function$f = E(p)^2$ must dominate a multiple of χ_A for some A (e.g. let A be those x with $f(x)$ sufficiently close to $||f||$). But then by ergodicity $X = \cup_{\gamma\in\Gamma}\gamma(A)$ (up to null sets) so that ρ is σ-finite. It is automatically absolutely continuous wrt μ. Invariance of ρ under Γ follows from $tr(u_\gamma x u_\gamma^{-1}) = tr(x)$ for $x \geq 0$. □

Definition 11.3.11. *A factor not of type I or II is called a* type III factor.

Example 11.2.22 provides a type III factor since the subgroup $\mathbb{Q}$ acts ergodically so the only possible invariant measure is a multiple of dx by exercise 11.3.2 and this is not invariant under multiplication!

Note that the above technique works in somewhat greater generality than actions of groups on measure spaces.

Exercise 11.3.12. *Adapt the proofs of the results just obtained to show that $M \rtimes_\alpha \mathbb{Z}$ is a type III factor if the action α is generated by a single automorphism of the II_∞ factor scaling the trace by a factor $\lambda \neq 1$.*

11.4 A wrinkle: 2-cohomology.

In a purely algebraic setting it is possible to "twist" the crossed product construction with a 2-cocycle. So suppose G (with identity 1) acts on the unital algebra A. Call C the abelian group of central invertible elements of A and let $\mu : G \times G \to C$be a function satisifying

11.4.1.
$$\mu(g,h)\mu(gh,k) = \alpha_g(\mu(h,k))\mu(g,hk)$$

Then one may define the algebra $A \rtimes_{\alpha,\mu}$ of formal (finite) sums as for the crossed product but with multiplication defined by $(au_g)(bu_h) = a\alpha_g(b)\mu(g,h)u_{gh}$. Then the cocycle condition ensures that this multiplication is associative. (The same twisiting is possible for the semidirect product of groups.) In order for u_1 to be the identity for this algebra we need the normalisation condition $\mu(1,g) = 1 = \mu(g,1) \quad \forall g \in G$. It also helps things along if we assume further that $\mu(g,g^{-1}) = 1$.

Note immediately that such a cocycle can dramatically alter the crossed product. The simplest case of this is for a finite abelian group G with the algebra M just being $\mathbb{C}$. Then if $\mu : G \times G \to \mathbb{T}^1$ is antisymmetric and bilinear (thinking additviely), it satisfies the cocycle condition 11.4.1 with trivial action.

Exercise 11.4.2. *Find a bilinear μ as above on $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for which $\mathbb{C} \rtimes_\mu G$ is isomorphic to $M_n(\mathbb{C})$.*

This makes the μ-twisted crossed product quite different from the untwisted one, which is abelian.

A trivial way to obtain 2-cocyles is to define $\mu(g,h) = \nu(g)\alpha_g(\nu(h))$ for some function $\nu : G \to C$. Such a cocycle is called a coboundary and the twisted crossed product by a coboundary can be untwisted by multiplying the u_g's by $\nu(g)^{-1}$ to obtain an isomorphism with the untwisted crossed product.

The 2-cocycles form a group under pointwise multiplication and the coboundaries are a subgroup. The quotient is called the second cohomology group $H^2(G,C)$.

To make sense of this in the von Neumann algebra setting one begins with the data for the usual crossed product, namely a von Neumann algebra M on $\mathcal{H}$ with an action α of the discrete group G on M. The 2-cocycle will then be a function μ from $G \times G$ to the unitary group of $Z(M)$ satisfying 11.4.1 and normalisation conditions. One then lets M act on $\ell^2(G,\mathcal{H})$ as for the usual crossed product but one defines unitaries $(u_g f)(h) = \mu(g,h)f(g^{-1}h)$ instead of the left regular representation.

Exercise 11.4.3. *Find out the correct version of this formula so that the cocycle condition implies $u_g u_h = \mu(g,h)u_{gh}$.*

Definition 11.4.4. *The twisted crossed product $M \rtimes_{\alpha,\mu} G$ is the von Neumann algebra on $\ell^2(G,\mathcal{H})$ generated by M and the u_g defined above.*

One may also consider twistings by non-central elements but then one is led into actions modulo inner automorphisms and the cocycles do not form a group.

11.5 More on the group-measure space construction $A \rtimes G$, $A = L^\infty(X, \mu)$.

If G is a countable discrete group acting freely on the probability space (X, μ) preserving μ we may identify the Hilbert space of the crossed product, $\ell^2(G, L^2(X, \mu))$ in the obvious way with $\mathcal{H} = L^2(X \times G)$ (with the product of counting measure and μ).

The operators $a \in L^\infty(X, \mu)$ and u_g defining the crossed product then act on $L^2(X \times G)$ as follows:

$$(af)(x, h) = a(hx)f(x, h), \text{ and } (u_g f)(x, h) = f(x, g^{-1}h)$$

The function $1(h, x) = \delta_{h,e}$ is a cyclic and separating trace vector for $A \rtimes G$ which is thus embedded in $\mathcal{H}$ as follows:

$$\text{If } a = \sum_g a_g u_g \text{ then } (a1)(x, h) = a_h(hx).$$

So if $b = \sum_g b_g u_g$ we have, using this embedding,

11.5.1.

$$(ab)(x, h) = \sum_g a_g(hx) b_{g^{-1}h}(g^{-1}hx)$$

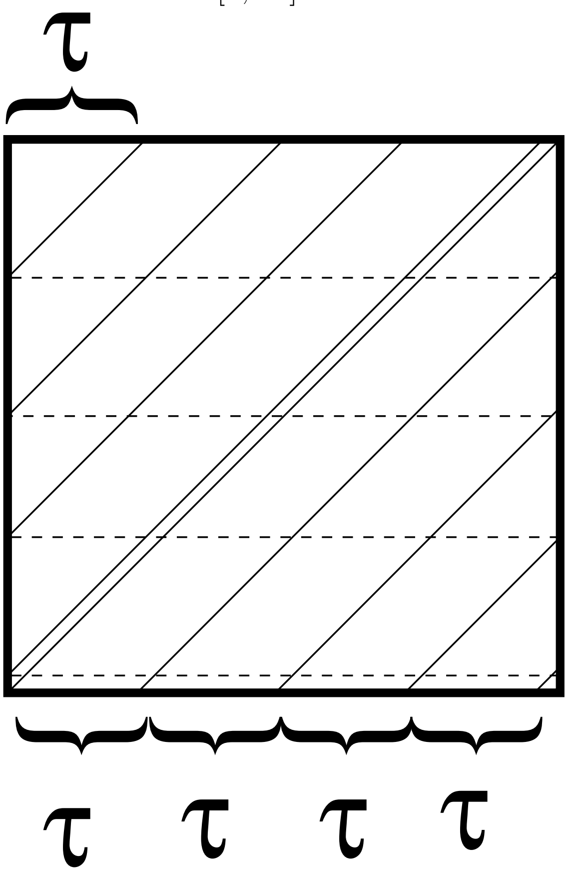
Moreover since the action is free we may identify $G \times X$ with a subset, necessarily measurable, of $X \times X$ via $(x, g) \mapsto (x, gx)$. This subset is nothing but the *graph* $\Gamma(\sim)$ of the equivalence relation on X defined by the *orbits* of G : $x \sim y$ iff $y = gx$ for some (unique) $g \in G$.. Thus each element $a = \sum_g a_g u_g \in A \rtimes g$ defines a function on $\Gamma(\sim)$ by $a(x, y) = a_h(hx)$ for $y = hx$. This all sounds like abstract nonsense until one observes that the multiplication 11.5.1 becomes

11.5.2.

$$(ab)(x, y) = \sum_{z \sim x} a(x, z) b(z, y)$$

from which the group action has disappeared and been replaced entirely by the orbits it defines! In particular if G_1 and G_2 are countable discrete groups acting freely on (X_1, μ_1) and (X_2, μ_2) respectively then any measurable isomorphism from X_1 to X_2 which sends the orbits for G_1 to the orbits for G_2 will define an isomorphism between $L^\infty(X_1, \mu_1) \rtimes G_1$ and $L^\infty(X_2, \mu_2) \rtimes G_2$.

The graphs of these equivalence relations can be interesting subsets of $X \times X$. Here is a picture giving five points in the equivalence class $[x]$ for all x in the case of the irrational rotation by τ on the circle (which is identified with the interval $[0, 2\pi]$:



Here the horizontal dotted lines just denote the identification of one point with another mod 2π. Clearly if one continued one would see that the graph of $\sim$ is dense in $X \times X$.

This led to the development of the now obvious notion of *orbit equivalence* of actions of groups which is outside von Neumann algebras. The first major result was that of Dye [] which states that two ergodic measure preserving actions of $\mathbb{Z}$ are orbit equivalent. This was extended to actions of amenable groups in [] and to non measure-preserving actions in []. Perhaps not surprisingly, the III_λ classification of Connes is reproduced.

Another development whose motivation is clear from the above is that of the study of measurable equivalence relations with countable orbits. The definitive treatment is that of Feldman and Moore ([],[]). They construct a von Neumann algebra from a suitably measurable equivalence relation $\sim$ on (X, μ) with the property that the equivalence classes are all countable. They give $\Gamma(\sim)$ the measure coming from counting measure vertically and μ horizontally and consider the Hilbert space $L^2(\Gamma(\sim))$. Functions on $\Gamma(\sim)$ which have *finite vertical support* for each $x \in X$ form a *-algebra under the multiplication 11.5.2. This algebra acts on $L^2(\Gamma(\sim))$ and the "crossed product" is the von Neumann algebra generated by this algebra. Everything is done in great generality so the type III case is also covered. There are notions of measure-class preserving, measure-preserving and ergodic for equivalence relations, and even a notion of 2-cohomology which allows one to do a twisted version.

Technically, everything depends on being able to show that the graph of the equivalence relation admits measurable local sections so that it looks somewhat like our picture for the irrational rotation. In particular Feldman and Moore show that any of their equivalence relations is in fact the orbit space for a group. It was open for a long time as to whether that group could be assumed to act freely but a counterexample was found in []. (Note that equivalence relations behave well with respect to restricting to subsets which gives them an advantage over group actions.)

In [], Connes vastly extended the equivalence relation construction so that it works in the context of "measured groupoids" where the equivalence classes are not necessarily discrete and the ordered pair (x, y) is generalised to a morphism from the object x to the object y. As his main new example, Connes used smooth foliations where the morphisms are holonomy classes of smooth paths joining two points in a leaf. The leaves in a foliation (such as the flow lines of a vector field) can exhibit ergodic properties which make Connes' von Neumann algebra into a factor.

11.6 The normaliser-the full group.

How much of G and its action on M can be recovered from M inside $M \rtimes G$ for a free action? One thing that is canonically defined is the *normaliser* $\mathcal{N}(M) = \{u \text{ unitary in } M \rtimes G | uMu^* = M\}$ This group obviously contains the unitary group $U(M)$ as a normal subgroup. There are two extreme cases.
(i) If M is a factor. Suppose $u = \sum_g a_g u_g$ is in $\mathcal{N}(M)$, then there is an automorphism β of M so that $ux = \beta(x)u \quad \forall x \in M$. That is

$$\sum_g a_g \alpha_g(x) u_g = \sum_g \beta(x) a_g u_g \quad \forall x \in M$$

. So for each $g \in G$ we have $a_g \alpha_g \beta^{-1}(x) = x a_g$. By 11.2.6 there can be only one g for which a_g is different from 0 and for that g, a_g is unitary. We see that the quotient $\mathcal{N}(M)/U(M)$ is in fact G itself. So we recover G and its action (up to inner automorphisms) on M.

(ii) If $M = L^\infty(X, \mu)$ the situation is different and somewhat richer. As before, if $\sum_g a_g \alpha_g(x) u_g \in \mathcal{N}(M)$ there is a β such that

$$\sum_g a_g \alpha_g(x) u_g = \sum_g \beta(x) a_g u_g \quad \forall x \in M.$$

But now freeness is less strong. For a given g we have $a_g \alpha_g(x) = \beta(x) a_g$ for all x as before. Thus on the support of a_g $\alpha_g(x) = \beta(x)$ for all L^∞ functions x. So if the support of a_g and a_h intersect in a set of non-zero measure then, arguing as in 11.2.10 the transformations defined by g and h would agree on that set which is not allowed by freeness. After throwing away sets of measure zero we may thus conclude that the supports of the a_g's are *disjoint*! Moreover since $\sum_g a_g \alpha_g(x) u_g$ is unitary, $\sum_g a_g a_g^* = 1$ so that the a_g are all characteristic functions of subsets S_g which form a partition of X. And on S_g, the transformation determined by β agrees with α_g.

We thus have the remarkable structure of the transformations of X determined by $\mathcal{N}(L^\infty(X))$:
there is a partition of X into measurable subsets, on each of which the transformation agrees with some element of G. It is just as clear from the above calculation that such a transformation is implemented by a unitary in $\mathcal{N}(L^\infty(X))$. Playing freely and easily with sets of measure zero we define:

Definition 11.6.1. *If G is a discrete group of automorphisms of $L^\infty(X, \mu)$, the* full group *of G is the group of all automorphisms T for which there is a partition $X = \bigcup_{g \in G} C_g$ into disjoint sets with $T = g$ on C_g.*

It is perhaps not immediately obvious that the full group contains any elements besides G itself. But if G acts ergodically then every subset is spread all over the place so a maximality argument shows that one can extend any partially defined element to an isomorphism. Note that the elements of the full group preserve orbits under G. It can be shown that any orbit-preserving isomorphism of G is in the full group.

Chapter 12

Unbounded Operators and Spectral Theory

There are many naturally arising examples of unbounded operators, some of the most fundamental being multiplication by x and differentiation, the position and momentum operators of quantum mechanics. Our immediate motivation for studying unbounded operators here is to facilitate the study of arbitrary von Neumann algebras acting on GNS Hilbert spaces. Here we establish the necessary preliminaries on unbounded operators. The material closely follows Reed and Simon [2].

12.1 Unbounded Operators

Definition 12.1.1. *An* operator T *on a Hilbert space* $\mathcal{H}$ *consists of a linear subspace* $D(T)$*, the* domain *of* T*, and a linear map from* $D(T)$ *to* $\mathcal{H}$*.*

Example 12.1.2.

(i) M_x*, multiplication by* x *on* $\mathrm{L}^2(\mathbb{R})$*.*

$$D(M_x) = \left\{ f \in \mathrm{L}^2(\mathbb{R}) : \int_{\mathbb{R}} x^2 |f(x)|^2 \mathrm{d}\, x < \infty \right\}.$$

(ii) $T = \frac{\mathrm{d}}{\mathrm{d}\, x}$ *on* $\mathrm{L}^2([0,1])$*.* $D(T) = C^1[0,1]$*.*

In order to do some analysis we want to restrict our attention a little so as not to consider completely arbitrary linear maps.

Definition 12.1.3. *Let* T *be an operator on* $\mathcal{H}$*. The* graph *of* T *is*

$$\Gamma(T) = \{(\xi, T\xi) : \xi \in D(T)\} \subset \mathcal{H} \oplus \mathcal{H}.$$

T *is* closed *if* $\Gamma(T)$ *is closed in* $\mathcal{H} \oplus \mathcal{H}$*.*

Remark 12.1.4. *Note that if T is closed and $D(T) = \mathcal{H}$ then T is bounded by the Closed Graph Theorem.*

Lemma 12.1.5. *A linear subspace $\Gamma \subset \mathcal{H} \oplus \mathcal{H}$ is the graph of an operator iff $(0, \eta) \in \Gamma$ implies $\eta = 0$.*

Proof. Straightforward. □

Many operators are not closed, but can be extended to a closed operator.

Definition 12.1.6. *Let S, T be operators on $\mathcal{H}$. T is an* extension *of S, denoted $S \subset T$, if $\Gamma(S) \subset \Gamma(T)$. Equivalently $D(S) \subset D(T)$ and $T|_{D(S)} = S$.*

Definition 12.1.7. *An operator T is* preclosed *(or* closable*) if it has a closed extension.*

Lemma 12.1.8. *Suppose T is preclosed. Then T has a smallest closed extension $\overline{T}$. $\Gamma(\overline{T}) = \overline{\Gamma(T)}$.*

Proof. Take a closed extension A of T. $\Gamma(A)$ is closed and contains $\Gamma(T)$ so $\overline{\Gamma(T)} \subset \Gamma(A)$. $\overline{\Gamma(T)}$ is the graph of an operator (call it $\overline{T}$) because:

$$(0, \eta) \in \overline{\Gamma(T)} \subset \Gamma(A) \Rightarrow \eta = A(0) = 0.$$

$\overline{T}$ is the smallest closed extension because for all closed extensions A, $\Gamma(\overline{T}) = \overline{\Gamma(T)} \subset \Gamma(A)$. □

Definition 12.1.9. *$\overline{T}$ is called the* closure *of T.*

Remark 12.1.10. *We thus obtain two equivalent definitions of a preclosed operator:*

(i) $(0, \eta) \in \overline{\Gamma(T)} \Rightarrow \eta = 0$.

(ii) ($\xi_n \in D(T)$, $\xi_n \to 0$ *and* $T\xi_n$ *converges*) $\Rightarrow T\xi_n \to 0$.

Exercise 12.1.11.

(i) Define S on $L^2(\mathbb{R})$ by $D(S) = C_0^\infty(\mathbb{R})$ (infinitely differentiable functions with compact support), $Sf = f'$. Show that S is preclosed.

(ii) Define T from $L^2(\mathbb{R})$ to $\mathbb{C}$ by $D(T) = L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $T(f) = \int_{\mathbb{R}} f$. Show that T is not preclosed.

Definition 12.1.12. *Suppose T is a closed operator. A* core *for T is a linear subspace $D_0 \subset D(T)$ such that $\overline{T|_{D_0}} = T$.*

We can perform basic arithmetic operations with (unbounded) operators as follows: $S+T$ is the operator with domain $D(S+T) = D(S) \cap D(T)$ and $(S+T)\xi = S\xi + T\xi$. ST is the operator with domain $D(ST) = \{\xi \in D(T) : T\xi \in D(S)\}$ and $(ST)\xi = S(T\xi)$. Of particular importance is the adjoint.

Definition 12.1.13. *Let T be a densely defined operator on $\mathcal{H}$. Let*

$$\begin{aligned} D(T^*) &= \{\eta \in \mathcal{H} : \exists \sigma \in \mathcal{H} \text{ such that } \langle T\xi, \eta\rangle = \langle \xi, \sigma\rangle \forall \xi \in D(T)\} \\ &= \{\eta \in \mathcal{H} : \exists C > 0 \text{ such that } |\langle T\xi, \eta\rangle| \leq C||\xi|| \ \forall \xi \in D(T)\}. \end{aligned}$$

For $\xi \in D(T^*)$ note that the element σ is unique (by the density of $D(T)$) and define $T^*\xi = \eta$.

Remark 12.1.14. *Note that if $S \subset T$ then $T^* \subset S^*$.*

Exercise 12.1.15. *Give an example to show that the domain of the adjoint need not be dense. [In fact it can be $\{0\}$].*

Proposition 12.1.16. *Let T be a densely defined operator. Then*

1. *T^* is closed.*
2. *$D(T^*)$ is dense iff T is preclosed. In that case $\overline{T} = T^{**}$.*
3. *If T is preclosed then $(\overline{T})^* = T^*$.*

Proof. Note that $(\eta, \sigma) \in \Gamma(T^*)$ iff $< T\xi, \eta >=< \xi, \sigma >$ for all $\xi \in D(T)$ iff $< (-T\xi, \xi), (\eta, \sigma) >= 0$. Hence

$$\Gamma(T^*) = \{(-T\xi, \xi) : \xi \in D(T)\}^\perp = (u\Gamma(T))^\perp = u\Gamma(T)^\perp,$$

where $u : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ is the unitary operator $u(\xi, \eta) = (-\eta, \xi)$. Now:

1. Orthogonal complements are closed, hence $\Gamma(T^*)$ is closed.
2. $\overline{\Gamma(T)} = (\Gamma(T)^\perp)^\perp = u^*\Gamma(T^*)^\perp$, so

   $$\begin{aligned} (0, \xi) \in \overline{\Gamma(T)} &\Leftrightarrow (-\xi, 0) \in \Gamma(T^*)^\perp \\ &\Leftrightarrow 0 =< (-\xi, 0), (\eta, T^*\eta) >= - < \xi, \eta > \text{ for all } \eta \in D(T^*) \\ &\Leftrightarrow \xi \in D(T^*)^\perp. \end{aligned}$$

   Hence T is preclosed iff $D(T^*)^\perp = \{0\}$ iff $D(T^*)$ is dense.

   In that case $\Gamma(T^{**}) = u\Gamma(T^*)^\perp = u^2\Gamma(T)^{\perp\perp} = -\overline{\Gamma(T)} = \Gamma(\overline{T})$, so $T^{**} = \overline{T}$.
3. $T^* = \overline{T^*} = T^{***} = (\overline{T})^*$.

Definition 12.1.17. *An operator T is* symmetric *if $T \subset T^*$. Equivalently $< T\xi, \eta >=< \xi, T\eta >$ for all $\xi, \eta \in D(T)$. T is* self-adjoint *if $T = T^*$. A self-adjoint operator T is* positive *if $< T\xi, \xi >\geq 0$ for all $\xi \in D(T)$.*

12.2 Spectral Theory for Unbounded Operators

Definition 12.2.1. *Let T be a closed operator on $\mathcal{H}$. The* resolvent *of T is*

$$\rho(T) = \{\lambda | \lambda 1 - T : D(T) \to \mathcal{H} \text{ is a bijection}\}.$$

The spectrum *of T is $\sigma(T) = \mathbb{C} \backslash \rho(T)$.*

Remark 12.2.2. *Note that if $\lambda 1 - T : D(T) \to \mathcal{H}$ is a bijection then $(\lambda 1 - T)^{-1}$ is bounded by the Closed Graph Theorem.*

Exercise 12.2.3. *The spectrum is highly dependent on the domain. Let $AC[0,1]$ denote the set of absolutely continuous functions on $[0,1]$. Let $T_1 = \frac{\mathrm{d}}{\mathrm{d}x}$, $T_2 = \frac{\mathrm{d}}{\mathrm{d}x}$, with*

$$\begin{aligned} D(T_1) &= \{f \in AC[0,1] : f' \in \mathrm{L}^2([0,1])\} \\ D(T_2) &= \{f \in AC[0,1] : f' \in \mathrm{L}^2([0,1]), f(0) = 0\}. \end{aligned}$$

Show that T_1 and T_2 are closed. Show that $\sigma(T_1) = \mathbb{C}$ while $\sigma(T_2) = \emptyset$.

Proposition 12.2.4. *Let (X, μ) be a finite measure space and F a measureable, real-valued, a.e. finite function on X. Let $D(M_f) = \{g \in \mathrm{L}^2(X,\mu) : fg \in \mathrm{L}^2(X,\mu)\}$ and let $M_f g = fg$. Then M_f is self-adjoint and $\sigma(M_f) = \text{ess.range}(f) = \{\lambda \in \mathbb{C} : \mu(\{x : |\lambda - f(x)| < \epsilon\}) > 0 \ \forall \epsilon > 0\}$.*

Exercise 12.2.5. *Prove Prop 12.2.4.*

Theorem 12.2.6 (Spectral Theorem - Multiplier Form)**.** *Let A be a self-adfoint operator on $\mathcal{H}$ with dense domain. Then there exists a finite measure space (X, μ), a real-valued a.e. finite function f on X and a unitary operator $u : \mathcal{H} \to \mathrm{L}^2(X, \mu)$ such that $uAu^* = M_f$*

Proof. See [2]. □

Remark 12.2.7 (Borel Functional Calculus)**.** *Note that the Spectral Theorem allows us to define a Borel functional calculus for self adjoint operators. Given a Borel function h on the spectrum of A, define $h(A) = u^* M_{h \circ f} u$.*

12.3 Polar Decomposition

Theorem 12.3.1. *Let $A : \mathcal{H} \to \mathcal{K}$ be a closed, densely defined operator. Then:*

(i) A^*A *and* AA^* *are positive self-adjoint operators (hence* $(A^*A)^{1/2}$ *and* $(AA^*)^{1/2}$ *exist).*

(ii) There exists a partial isometry with initial space $\overline{\text{Range}}(A^*A)^{1/2}$ *and final space* $\overline{\text{Range}}(A)$ *and*

$$A = v(A^*A)^{1/2}.$$

(iii) If $A = uB$ *for some positive* B *and partial isometry* v *with initial space* $\overline{\text{Range}}(B)$ *then* $u = v$ *and* $B = (A^*A)^{1/2}$.

(iv) In addition $A = (AA^*)^{1/2}v$.

Proof. (i) Since $\Gamma(A)$ is closed, it is a Hilbert space. Let $P : \Gamma(A) \to \mathcal{H}$ be projection onto the first component. Since A is an operator $\text{Ker}(P) = \{0\}$ and hence $\text{Range}(P^*)$ is dense in $\Gamma(A)$ (so $PP^*\mathcal{H}$ is a core for A). Let $\xi \in \mathcal{H}$, $P^*\xi = (\eta, A\eta)$. Then, for all $\sigma \in D(A)$,

$$\begin{aligned} & <\xi, \sigma> = < P^*\xi, (\sigma, A\sigma) > = <\eta, \sigma> + < A\eta, A\sigma> \\ \Rightarrow\ & <\xi - \eta, \sigma> = < A\eta, A\sigma> \\ \Rightarrow\ & A\eta \in D(A^*) \text{ and } A^*A\eta = \xi - \eta. \end{aligned}$$

Thus $D(A^*A) \supset PP^*\mathcal{H}$ which is a core for A. In addition $\text{Range}(A^*A + 1) = \mathcal{H}$.

It is easy to see that A^*A is symmetric, so $A^*A + 1 \subset (A^*A + 1)^*$. Let $\xi \in D((A^*A + 1)^*)$. Since $\text{Range}(A^*A + 1) = \mathcal{H}$ there exists $\tilde{\xi} \in D(A^*A + 1)$ with $(A^*A + 1)^*\xi = (A^*A + 1)\tilde{\xi} (= (A^*A + 1)^*\tilde{\xi})$. $\text{Ker}((A^*A + 1)^*) = \{0\}$ because $\text{Range}(A^*A + 1) = \mathcal{H}$, and hence $\xi = \tilde{\xi} \in D(A^*A+1)$. Thus $(A^*A+1)^* = A^*A+1$ and so $(A^*A)^* = A^*A$.

Finally, for $\xi \in D(A^*A)$, $< A^*A\xi, \xi > = < A\xi, A\xi > \geq 0$ so A^*A is positive, i.e. $\sigma(A^*A) \subset [0, \infty)$ (just use the Spectral Theorem).

(ii) As we noted above, $D(A^*A)$ is a core for A. $D(A^*A)$ is also a core for $|A| = (A^*A)^{1/2}$ (use spectral theory). Thus $\overline{AD(A^*A)} = \overline{\text{Range}A}$ and $\overline{|A|D(A^*A)} = \overline{\text{Range}|A|}$. Note that for $\xi \in D(A^*A)$,

$$|||A|\xi||^2 = < A^*A\xi, \xi > = < A\xi, A\xi > = ||A\xi||^2,$$

so that the map $v : |A|\xi \mapsto A\xi$, $\xi \in D(A^*A)$, extends to a partial isometry with initial space $\overline{|A|D(A^*A)} = \overline{\text{Range}|A|}$ and final space $\overline{AD(A^*A)} = \overline{\text{Range}A}$.

For $\xi \in D(|A|)$ take $\xi_n \in D(A^*A)$ with $(\xi_n, |A|\xi_n) \to (\xi, |A|\xi)$. Then $A\xi_n = v|A|\xi_n \to v|A|\xi$ and, as A is closed, $\xi \in D(A)$ and $A\xi = v|A|\xi$.

For $\xi \in D(A)$ take $\xi_n \in D(A^*A)$ with $(\xi_n, A\xi_n) \to (\xi, A\xi)$. Then

$$|A|\xi_n = v^*v|A|\xi_n = v^*A\xi_n \to v^*A\xi.$$

Since $|A|$ is closed, $\xi \in D(|A|)$.

Hence $D(A) = D(|A|)$ and $A = v|A|$.

(iii) If $A = uB$ then $A^* = B^*u^* = Bu^*$. $A^*A = Bu^*uB = B^2$ since u^*u is projection onto $\overline{\text{Range}}(B)$. By uniqueness of the positive square root of a positive operator (Exercise 12.3.3), $(A^*A)^{1/2} = B$. Thus the initial space of u is $\overline{\text{Range}}(|A|)$ and $u|A| = A = v|A|$ so $u = v$.

(iv) $A = v(A^*A)^{1/2}$ so $A^* = (A^*A)^{1/2}v^*$ and hence $AA^* = v(A^*A)^{1/2}(A^*A)^{1/2}v^* = v(A^*A)v^*$ (Exercise 12.3.3). Thus v implements the unitary equivalence of $AA^*|_{\overline{\text{Range}(A)}}$ and $A^*A|_{\overline{\text{Range}(A^*)}}$. Hence $(AA^*)^{1/2} = v(A^*A)^{1/2}v^*$ and then $A = v(A^*A)^{1/2} = (AA^*)^{1/2}v$.

□

Remark 12.3.2. *Note that it was very important in (i) to establish that $D(A^*A)$ contained a core for A and hence was dense. It was not clear a priori that $D(A^*A)$ contained* any *elements other than* 0*.*

Exercise 12.3.3. *(i) Let T be a positive operator. Show that $T^{1/2}T^{1/2} = T$.*

(ii) Show that a positive operator has a unique positive square-root.

12.4 Unbounded operators affiliated with a von Neumann algebra.

If M is a von Neumann algebra on $\mathcal{H}$, an element $a \in \mathcal{B}(\mathcal{H})$ is in M iff $au = ua$ for every unitary in M'. This inspires the following.

Definition 12.4.1. *If $T : D(T) \to \mathcal{H}$ is a linear operator on the Hilbert space $\mathcal{H}$ and M is a von Neumann algebra on $\mathcal{H}$ we say that T is* affiliated with *M, written $T\eta M$ if, for any unitary $u \in M'$,*

$$uD(T) = D(T) \quad \textit{and}$$

$$uT\xi = Tu\xi \;\; \forall \xi \in D(T).$$

Lemma 12.4.2. *If T is preclosed with closure $\overline{T}$ then $\overline{T}\eta M$ if $T\eta M$.*

Proof. It is clear that $T\eta M$ iff $u\Gamma(T) = \Gamma(T)$ for all unitaries in M'. But this property passes to the closure of the graph. □

Lemma 12.4.3. *If T is a closed operator affiliated with M then*

1. *The projection onto $\Gamma(T)$ is a 2×2 matrix of operators in M.*
2. *If $T = u|T|$ is the polar decomposition of T then $u \in M$ and $f(|T|) \in M$ for any bounded Borel function of $|T|$.*

Proof. 1. is obvious from the characterisation of affiliation given in the proof of the previous lemma.

2. follows from uniqueness of the polar decomposition and the bicommutant theorem.

□

Chapter 13

Tomita-Takesaki theory.

In chapter 9 we showed that the GNS construction on M using a faithful normal trace produces a perfectly symmetric Hilbert space $\mathcal{H}_{tr}$ with respect to M and its commutant. This is because the map J, which is the extension to $\mathcal{H}_{tr}$ of the * operation on M, is an isometry. So $x \mapsto JxJ$ is the extension to $\mathcal{H}_{tr}$ of right multiplication by x^*. Unfortunately if we use a (normal) non-tracial state ϕ the * operation is no longer an isometry and there is no reason to expect either it or right multiplication by elements of M to have bounded extensions to $\mathcal{H}_\phi$. But as we shall see, the * operation is actually preclosed in the sense of unbounded operators and if $S = J\Delta^{1/2}$ is the polar decomposition of its closure S, we will show that $JMJ = M'$. Quite remarkably, the operator $\Delta^{1/2}$ will satisfy $\Delta^{it}M\Delta^{-it} = M$ so that a state actually gives rise to a dynamics – a one parameter automorphism group of M (and M').

We will prove these results using a method of van Daele for which we have followed some notes of Haagerup ([],[]). But before getting started on this difficult theory it is essential to do some elementary calculations to see how it all works out in the 2×2 matrices.

Exercise 13.0.4. *Let M be $M_2(\mathbb{C})$. Show that any state ϕ on M is of the form $\phi(x) = Trace(hx)$ for some positive h of trace 1. And that ϕ is faithful iff h is invertible. Thus with respect to the right basis,*

$$\phi(x) = Trace(x \begin{pmatrix} \frac{1}{1+\lambda} & 0 \\ 0 & \frac{\lambda}{1+\lambda} \end{pmatrix})$$

for some λ, $0 \leq \lambda \leq 1$.

Exercise 13.0.5. *With notation as in the previous exercise, suppose ϕ is faithful and let S be the * operation on the GNS Hilbert space $\mathcal{H}_\phi$. Calculate* *the polar decomposition* $S = J\Delta^{1/2}$ *and show that* $SMS = JMJ = M'$. *Show that* $\Delta^z M \Delta^{-z} = M$ *for* $z \in \mathbb{C}$ *so that* $\sigma_z^\phi(x) = \Delta^z x \Delta^{-z} = M$ *defines a representation of* $\mathbb{C}$ *as automorphisms of* M *which are* **-automorphisms iff* $z \in i\mathbb{R}$.

Exercise 13.0.6. *Generalize the above to the* $n \times n$ *matrices and in fact any finite dimensional von Neumann algebra.*

13.1 S,F and their graphs.

Throughout this section M will be a von Neumann algebra on $\mathcal{H}$ and $\Omega \in \mathcal{H}$ a cyclic and separating vector for M and hence M'. (The same data as a faithful normal state.) Let S_0 and F_0 be the conjugate linear operators with domains $M\Omega$ and $M'\Omega$ defined by $S_0(x\Omega) = x^*\Omega$ and $F_0(x\Omega) = x^*\Omega$ for $x \in M$ and M' respectively.

Lemma 13.1.1. *In the sense of unbounded operators* $F_0 \subseteq S_0^*$ *and* $S_0 \subseteq F_0^*$ *so that* S_0 *and* F_0 *have densely defined adjoints and hence are preclosed.*

Proof. To show $S_0^*(a'\Omega)$ is defined if $\langle S_0(a\Omega), a'\Omega\rangle$ extends to a bounded conjugate linear map on all of $\mathcal{H}$. But $\langle S_0(a\Omega), a'\Omega\rangle = \langle (a')^*\Omega, a\Omega\rangle$ which is bounded as a function of $a\Omega$ by Cauchy-Schwartz. Hence $a'\Omega$ is in the domain of S_0^* and $S_0^*(a'\Omega) = (a')^*\Omega = F_0(a'\Omega)$. Interchanging S_0 and F_0 we get the other inclusion. □

Definition 13.1.2. *Let* S *and* F *be the closures of* S_0 *and* F_0 *respectively. Let* $S = J\Delta^{1/2}$ *be the polar decomposition of* S.

Observe that $S_0 = S_0^{-1}$ so S is injective and $S^2 = 1$ in the sense of unbounded operators. Thus $\Delta^{1/2}$ has zero kernel, $J^2 = 1$ and $J\Delta^{1/2}J = \Delta^{-1/2}$. The same goes for F and its polar decomposition, but we shall now see that $F = S^*$.

Theorem 13.1.3. *(Takesaki,[].)* $S^* = F$, $F^* = S$ *and the graph of* S *is the set of all* $(c\Omega, c^*\Omega)$ *where* c *is a closed densely defined operator affiliated with* M *and* $\Omega \in D(c) \cap D(c^*)$.

Proof. Let $(\xi, F^*\xi)$ be in the graph of F^*. By the definition of F we know that $\langle \xi, (a')^*\Omega\rangle = \langle a'\Omega, F^*\xi\rangle$. Now define operators a and b with domain $M'\Omega$ by $ax'\Omega = x'\xi$ and $bx'\Omega = x'F^*\xi$. Then a and b are closable for if x' and y' are in M' we have

$$\langle a(x'\Omega), y'\Omega\rangle = \langle x'\xi, y'\Omega\rangle = \langle \xi, (x')^*y'\Omega\rangle$$

$$= \langle (y')^* x'\Omega, F^*\xi\rangle = \langle x'\Omega, y'F^*\xi\rangle = \langle x'\Omega, b(y'\Omega)\rangle$$

so that as before $a \subseteq b^*$ and $b \subseteq a^*$.

Let c be the closure of a. Then $c\Omega = a\Omega = \xi$ and $c^* = a^* \supseteq b$ so $c^*\Omega = F^*\xi$. Now by construction the domain of a is invariant under the unitary group of M' and on it a commutes with the unitaries in M'. This means that c is affiliated with M. At this stage we have shown that the graph of F^* consists of all $(c\Omega, c^*\Omega)$ where c is a closed densely defined operator affiliated with M and $\Omega \in D(c) \cap D(c^*)$.

We now want to show that the graph of F^* is contained in the graph of S. This is not hard. Let c be as above and $c = \sqrt{c^*c}$ be its polar decomposition. Then if $f_n(t) = t$ for $0 \le t \le n$ and $f_n(t) = 0$ for $t > n$ we have that $f_n(\sqrt{c^*c}) \to \sqrt{c^*c}$ on any vector in the domain of c, and since c is affiliated with M, $f_n(\sqrt{c^*c}) \in M$ so that $uf_n(\sqrt{c^*c})\Omega$ is in the domain of S and tends to ξ. Moreover $f_n(\sqrt{c^*c})u^*\Omega$ tends to $c^*\Omega = F^*\xi$ so $(\xi, F^*\xi)$ is in the graph of S.

Thus $F^* \subseteq S$ and we have already observed that $S \subseteq F^*$. Hence $S = F^*$ and $S^* = F$. □

Corollary 13.1.4. *The polar decomposition of F is $J\Delta^{-1/2}$.*

We now prove a crucial result connecting M and M'.

Lemma 13.1.5. *Let $\lambda \in \mathbb{R}^+$ be given. Then for $a' \in M'$ there is an $a \in M$ with $a\Omega$ in the domain of F and $a'\Omega = (\lambda S + \lambda^{-1}F)a\Omega$.*

Proof. Assuming $||a'|| \le 1$ we may apply theorem 8.2.1 to the ψ defined by $\psi(x) = \langle x\Omega, a'\Omega\rangle$ and $\phi(x) = \langle x\Omega, \Omega\rangle$ to obtain the existence of an $a \in M$ with

$$\langle x\Omega, a'\Omega\rangle = \lambda\langle ax\Omega, \Omega\rangle + \lambda^{-1}\langle xa\Omega, \Omega\rangle$$

$$= \lambda\langle x\Omega, a^*\Omega\rangle + \lambda^{-1}\langle a\Omega, x^*\Omega\rangle.$$

Provided $a\Omega$ is in the domain of F this equation reads $a'\Omega = (\lambda S + \lambda^{-1}F)a\Omega$.

On the other hand rearranging the equation gives

$$\langle a\Omega, x^*\Omega\rangle = \lambda\langle x\Omega, a'\Omega - \lambda a^*\Omega\rangle$$

so by Cauchy Schwartz $a\Omega$ is in the domain of $F = S^*$. □

Corollary 13.1.6. *For each $\xi \in D(\Delta^{1/2}) \cap D(\Delta^{-1/2})$ there is a sequence $a_n \in M$ with $a_n\Omega \to \xi$, $\Delta^{1/2}a_n\Omega \to \Delta^{1/2}\xi$ and $\Delta^{-1/2}a_n\Omega \to \Delta^{-1/2}\xi$.*

Proof. Set $\eta = (S+F)\xi$ and choose a sequence $a'_n \in M'$ with $a'_n \to \eta$. By the previous lemma there are $a_n \in M$ with $(S+F)a_n\Omega = a'_n\Omega$. But $S + F = J(\Delta^{1/2} + \Delta^{-1/2})$ has bounded inverse (in the usual sense of unbounded operators) so put $\xi_n = (S+F)^{-1}(a'_n\Omega)$. So $a_n\Omega = (S+F)^{-1}a'_n\Omega \to \xi$. Moreover

$$\Delta^{1/2}a_n\Omega = \Delta^{1/2}(\Delta^{1/2} + \Delta^{-1/2})^{-1}Ja'_n\Omega$$

and $\Delta^{1/2}(\Delta^{1/2} + \Delta^{-1/2})^{-1}$ is bounded by spectral theory. So $\Delta^{1/2}a_n\Omega \to \Delta^{1/2}(S+F)^{-1}(S+F)\xi = \Delta^{1/2}\xi$. In the same way $\Delta^{-1/2}a_n\Omega \to \Delta^{-1/2}\xi$. □

We put everything together with a lemma linking M and M' on a dense subspace to which many functions of Δ can be applied.

Lemma 13.1.7. *If ξ and η are in $D(S) \cap D(F)$, a', λ and a as in 13.1.5, then*

$$\lambda\langle SaS\xi, \eta\rangle + \lambda^{-1}\langle FaF\xi, \eta\rangle = \langle a'\xi, \eta\rangle.$$

Proof. By moving one S and F to the other side of the inner products, we see by the previous lemma that we may assume ξ and η are $x\Omega$ and $y\Omega$ respectively, both in $D(F)$, for x and y in M. But on $M\Omega$, SaS acts by right multiplication by a^* so $\langle SaS\xi, \eta\rangle = \langle xa^*\Omega, y\Omega\rangle = \langle Sa\Omega, x^*y\Omega\rangle$. On the other hand, systematically using $F^* = S$ we obtain $\langle FaFx\Omega, y\Omega\rangle = \langle y^*x\Omega, a\Omega\rangle = \langle Sx^*y\Omega, a\Omega\rangle = \langle Fa\Omega, x^*y\Omega\rangle$. Combining these two we see

$$\lambda\langle SaS\xi, \eta\rangle + \lambda^{-1}\langle FaF\xi, \eta\rangle = \langle(\lambda Sa + \lambda^{-1}Fa)\Omega, x^*y\Omega\rangle.$$

But by 13.1.5 this is $\langle a'\Omega, x^*y\Omega\rangle = \langle a'\xi, \eta\rangle$. □

13.2 Proof of the main theorem.

We begin with an easy exercise in contour integration.

Exercise 13.2.1. *Let S be the strip $\{z \in \mathbb{C} : -1/2 \le \Re(z) \le 1/2\}$. Suppose f is continuous and bounded on S and analytic on the interior of S. Then*

$$f(0) = \int_{-\infty}^{\infty} \frac{f(1/2 + it) + f(-1/2 + it)}{2\cosh \pi t} dt$$

Hint: Integrate $\dfrac{f(z)}{\sin \pi z}$ around rectangular contours in S tending to the boundary of S.

Proposition 13.2.2. *With notation as in the previous section*

$$a = \int_{-\infty}^{\infty} \lambda^{2it} \frac{\Delta^{it} Ja'J\Delta^{-it}}{2\cosh \pi t} dt$$

Proof. Since $J\Delta^{1/2}J = \Delta^{-1/2}$ we have $J(D(S) \cap D(T)) = D(S) \cap D(T)$ so after a little rearrangement the formula of 13.1.7 reads

$$\langle Ja'J\xi, \eta\rangle = \lambda \langle a\Delta^{-1/2}\xi, \Delta^{1/2}\eta\rangle + \lambda^{-1}\langle a\Delta^{1/2}\xi, \Delta^{-1/2}\eta\rangle.$$

Now let $\mathcal{H}_0$ be the dense subspace of all vectors in $\mathcal{H}$ which is the union of all $\xi_{[a,b]}(\Delta$ for $0 < a < b < \infty$. Certainly $\mathcal{H}_0 \subseteq D(S) \cap D(F)$, $\mathcal{H}_0$ is invariant under J and Δ^z for $z \in \mathbb{C}$, and moreover for $\xi \in \mathcal{H}_0$, $z \mapsto \Delta^z \xi$ is an entire function of z.

For $\xi, \eta \in \mathcal{H}_0$ define the analytic function

$$f(z) = \lambda^{2z} \langle a\Delta^{-z}\xi, \Delta^{\overline{z}}\eta\rangle.$$

Then f is bounded in the strip S of the previous lemma and $f(0) = \langle a\xi, \eta\rangle$. Also $f(1/2 + it) = \langle \Delta^{it}\Delta^{1/2}\xi, \eta\rangle$ so that

$$f(1/2 + it) + f(-1/2 + it) = \lambda^{2it}\langle \Delta^{it} Ja'J\Delta^{-it}\xi, \eta\rangle.$$

So by the previous lemma we are done. □

Theorem 13.2.3. *Let M be a von Neumann algebra on $\mathcal{H}$ and Ω a cyclic and separating vector for M. Suppose S is the closure of $x\Omega \mapsto x^*\Omega$ on $M\Omega$. Let $\Delta = S^*S$, and J be the antiunitary of the polar decomposition $S = J\Delta^{1/2}$. Then*

(i) $JMJ = M'$

(ii) $\Delta^{it} M \Delta^{-it} = M \, \forall t \in \mathbb{R}$

Proof. If $a' \in M'$ we know that

$$\int_{-\infty}^{\infty} \lambda^{2it} \frac{\Delta^{it} Ja'J\Delta^{-it}}{2\cosh \pi t} dt \in M.$$

Conjugating by a unitary $u \in M'$ and writing $\lambda = e^{\frac{i\theta}{2}}$ we see that the Fourier transforms of the strongly continuous rapidly decreasing functions $\dfrac{\Delta^{it} Ja'J\Delta^{-it}}{2\cosh \pi t}$ and $u\dfrac{\Delta^{it} Ja'J\Delta^{-it}}{2\cosh \pi t}u^*$ are equal. Hence $\Delta^{it}Ja'J\Delta^{-it} \in M$ for all real t since it commutes with every unitary $u \in M'$. (Take inner products with vectors if you are not happy with Fourier transforms of operator valued functions.)

Putting $t = 0$ we see $JM'J \subseteq M$ and by symmetry $JMJ \subseteq M'$. Hence $JMJ = M'$ and we are done. □

Definition 13.2.4. *The operator J of the previous result is called the* modular conjugation *and the strongly continuous one-parameter group of automorphisms of M defined by $\sigma_t^\phi(x) = \Delta^{it} x \Delta^{-it}$ is called the* modular automorphism group *defined by ϕ.*

13.3 Examples.

Example 13.3.1. *ITPFI*

The acronym ITPFI stands for "infinite tensor product of finite type I". These von Neumann algebras are formed by taking the *-algebra A_∞ as the union A_∞ of tensor products $A_m = \bigotimes_{k=1}^{m} M_{n_k}(\mathbb{C})$, the inclusion of A_m in A_{m+1} being diagonal. The state ϕ on A_∞ is then the tensor product of states on each M_{n_k}. One may then perform the GNS construction with cyclic and separating vector Ω given by $1 \in A_\infty$, to obtain the von Neumann algebra $M = \bigotimes_{k=1}^{\infty} M_{n_k}(\mathbb{C})$ as the weak closure of A_∞ acting on $\mathcal{H}_\phi$. The case where all the n_k are equal to 2 and all the states are the same is called the "Powers factor" and the product state the "Powers state" as it was R.Powers who first showed that they give a continuum of non-isomorphic type III factors.

A slight snag here is that we do not know that Ω defines a faithful state on M. But if we proceed anyway to construct what have to be J and Δ we will soon see that the state is indeed faithful, i.e. Ω is cyclic for $M'\Omega$.

Recall from exercise 13.0.6 that, for $M_n(\mathbb{C})$, and $\phi_h(x) = trace(xh)$ where h is the diagonal matrix (density matrix) with $h_{ii} = \mu_i, \sum \mu_i = 1, \mu_i > 0$, then $J_n(e_{ij}) = \sqrt{\frac{\mu_j}{\mu_i}} e_{ji}$ and $\Delta_n(e_{ij}) = \frac{\mu_i}{\mu_j} e_{ij}$ (where dependence on h has been suppressed).

To diagonalise the modular operators on $\mathcal{H}_\phi$ completely it is most convincing to choose an orthonormal basis d_i of the diagonal matrices, with $d_1 = 1$. Then a basis for the Hilbert space $\mathcal{H}_\phi$ is formed by tensors $\otimes_{k=1}^\infty v_k \Omega$ where $v_k = 1$ for large k, and is otherwise a d_i or an e_{ij} with $i \neq j$.

We can guess that J is, on each basis vector, the tensor product of the J's coming from the matrix algebras. Defining it as such it is clearly an isometry on $A_\infty\Omega$ and thus extends to all of $\mathcal{H}_\phi$. But then, for any $x \in A_\infty$, JxJ is in M' by the finite dimensional calculation! But the linear span of these $JxJ\Omega$ is dense so Ω is cyclic for M' and hence separating for M. We are hence in a position to apply Tomita-Takesaki theory. Each of the basis elements is in $M\Omega$ so $S(\otimes_{k=1}^\infty v_k \Omega) = \otimes_{k=1}^\infty w_k \Omega$ where w_k is v_k if v_k is diagonal, and e_{ji} if

$v_k = e_{ij}$. So JS is diagonal and hence essentially self-adjoint. We conclude that

$$J(x\Omega) = J_m(x)\Omega \quad \text{and} \quad \Delta(x\Omega) = \Delta_m(x)\Omega \qquad \text{for } x \in A_m,$$

and

$$\sigma_t^\phi = \bigotimes_{k=1}^{\infty} \sigma^{\phi_{h_k}} \quad .$$

Example 13.3.2. *Group-measure-space construction.*

Let Γ be a discrete group acting on the finite measure space (X, μ) preserving the class of the finite measure μ. The Hilbert space of the crossed product $L^\infty(X, \mu)$ is $L^2(X, \mu) \otimes \ell^2(\Gamma)$ and as we saw in chapter 11 the vector $1 \otimes \varepsilon_{id}$ is a cylic and separating vector Ω for $M = L^\infty(X, \mu) \rtimes \Gamma$.

Since the class of μ is preserved by the $\gamma \in \Gamma$ the Radon Nikodym theorem guarantees positive L^1 functions h_γ so that $\phi(h_\gamma \alpha_\gamma(y)) = \phi(x)$ where $\phi(y) = \int_X y d\mu$. We know that, if $x \in L^\infty(X, \mu)$ then $S(u_\gamma x) = x^* u_{\gamma^{-1}}$. In general we will not be able to completely diagonalise Δ but the same argument as in the previous example gives that the domain of Δ is

$$\{f : \Gamma \to L^2(X, \mu) : \sum_\gamma \int_X |h_\gamma(x) f(x)|^2 d\mu(x) < \infty\}$$

on which

$$(\Delta f)(\gamma) = h_\gamma f(\gamma),$$

and

$$(Jf)(\gamma) = h_\gamma^{-1/2} f(\gamma).$$

We can now finally answer the question as to which sums $\sum_\gamma x_\gamma u_\gamma$ define elements of $M = L^\infty(X, \mu) \rtimes \Gamma$.

Theorem 13.3.3. *With notation as above, if the function $\gamma \mapsto x_\gamma \in L^\infty(X, \mu)$ is such that $\sum_\gamma x_\gamma u_\gamma$, interpreted as a matrix of operators as in section 11.2, defines a bounded operator, then that operator is in $M = L^\infty(X, \mu) \rtimes \Gamma$.*

Proof. By 13.2.3 it suffices to show that $\sum_\gamma x_\gamma u_\gamma$ commutes with $Jxu_\gamma J$ for all $x \in L^\infty(X, \mu)$ and $\gamma \in \Gamma$. And for this it suffices to check that the commutation holds on functions of the form $f \otimes \varepsilon_\gamma$ for $f \in L^2$. This is just a routine computation. □

Exercise 13.3.4. *Show that example 13.3.1 is in fact a special case of this group-measure-space example in which $L^\infty(X, \mu)$ is provided by the tensor products of the diagonal elements and the group Γ is a restricted infinite Cartesian product of cyclic groups, constructed from the non-diagonal e_{ij}'s. Conclude by the method of 11.2.15 that ITPFI algbras are factors.*

This example brings to light a significant inadequacy of our treatment of Tomita-Takesaki theory. We would like to treat the case where the measure of the space is infinite. Although of course we could choose an equivalent finite measure, this choice may not be natural. To do this we would have to consider the theory of "weights" which are to states as the trace on a II_∞ factor is to the trace on a type II_1 factor. We need the same notion in order to understand the origin of the term "modular" used above as coming from the modular function on a non-unimodular locally compact group. But a serious treatment of weights would take many pages so we simply refer the reader to Takesaki's book [3].

Example 13.3.5. *Hecke algebras à la Bost-Connes.*

If G is a finite group let u_g and v_g be the unitaries of the left and right regular representations respectively. If H is a subgroup, the projection $p_H = \frac{1}{|H|}\sum_{h\in H} v_h$ projects from $\ell^2(G)$ onto functions that are right translation invariant under H, i.e. functions on the quotient space G/H. Thus the so-called "quasi-regular" representation of G on G/H is a direct summand of the left regular representation and we have from EP7 of chapter 3.4 that the commutant of the action of G on $\ell^2(G/H)$ is $p_H\rho(G)p_H$ where $\rho(G)$ is the algebra generated by the right regular representation (of course isomorphic to $\mathbb{C}$). This commutant is spanned by the $p_H v_g p_H$ which, thought of as functions on G, are multiples of the characteristic functions of the double cosets HgH which form the double coset space $H\backslash G/H$. The subalgebra of $\rho(G)$ spanned by these double cosets is the space of $H-H$ bi-invariant functions and we see it is the commutant of G on $\ell^2(G/H)$. It is known as the *Hecke algebra* for the pair (G, H) and has a considerable role to play in the representation theory of finite groups. A famous example is the case where G is the general linear group over a finite field and H is the group of upper triangular matrices. The coset space is then the so-called "flag variety" and the Hecke algebra in this case leads to a lot of beautiful mathemtatics. See Bourbaki [].

Nothing could better indicate how differently things work for infinite discrete groups than how the Hecke algebra works. Far from being direct summands, the quasiregular representations can be totally different from the left regular representations and can even generate type III factors! These Hecke algebras give nice examples where the modular operators can be calculated explicitly.

Definition 13.3.6. *A subgroup H of the discrete group G is called* almost normal *if either of the two equivalent conditions below is satisfied.*

(a) $gHg^{-1} \cap H$ is of finite index in H for all $g \in G$.

(b) Each double coset of H is a finite union of left cosets of H (i.e. the orbits of H on G/H are all finite).

If H is almost normal in G one may construct operators in the commutant of the quasiregular representation of G on $\ell^2(G/H)$ as follows:

Given an element x of G/H let ε_x be the characteristic function of x. These functions form an orthonormal basis of $\ell^2(G/H)$. Moreover each vector ε_x is cyclic for the action of G hence separating for the commutant. If D is a double coset of H define T_D by the matrix

$$(T_D)_{x,y} = \begin{cases} 1 & \text{if } y^{-1}x = D; \\ 0 & \text{otherwise.} \end{cases}$$

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Clearly T_D is bounded since H is almost normal and it obviously commutes with the action of G. From the definition we have

$$T_D^* = T_{D^{-1}}.$$

It is also easy to check that

$$T_D T_E = \sum_F n_{D,E}^F T_F$$

where the structure constants are defined by

$$n_{D,E}^F = \begin{cases} \#(E/H) & \text{if } F \subseteq ED; \\ 0 & \text{otherwise.} \end{cases} x$$

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We will call the von Neumann algebra generated by the T_D's the *Hecke-von Neumann algebra of the pair $H \subseteq G$* and write it $HvN(G,H)$. The vector state ϕ defined on $HvN(G,H)$ by ε_H is faithful and normal, and $\langle T_D\varepsilon_H, T_{D'}\varepsilon_H\rangle = 0$ unless $D = D'$ so that the T_D's are orthogonal. It is thus easy to calculate the operators for the modular theory on $\mathcal{H}_\phi$ (note that this is <u>not</u> $\ell^2(G/H)$). We guess as usual that $J(T_D\Omega) = (constant)T_{D^{-1}}\Omega$ and by counting cosets in double cosets (or sizes of orbits of H on G/H) we find that the constant has to be $(\#(D/H))^{1/2}(\#(H\backslash D))^{-1/2}$. Thus as before JS is diagonal on the basis $T_D\Omega$ of H_ϕ so essentially self-adjoint and

$$\Delta(T_D\Omega) = \frac{\#(H\backslash D)}{\#(D/H)} T_D\Omega$$

with the obvious domain. Thus

$$\sigma_t^\phi(T_D) = \left(\frac{\#(H\backslash D)}{\#(D/H)}\right)^{it} T_D.$$

Examples of almost normal subgroups are not hard to find. The classical example of Hecke himself is the case where $G = SL(2, \mathbb{Q})$ and $H = SL(2, \mathbb{Z})$. In this case the Hecke algebra is abelian. Bost and Connes in [4] examined the case of the $ax+b$ group over the rationals with the subgroup being integer translations. They showed that $HvN(G, H)$ in this case is a type III factor and made a connection with prime numbers.

13.4 The KMS condition.

In the examples of the previous section the operators of the modular group were easy to calculate explicitly, including the domain of Δ. One can imagine that this is not always so. If we are particularly interested in the modular group σ_t^ϕ it would be useful to be able to guess it and show that the guess is right without bothering about the domain of Δ. The KMS (Kubo-Martin-Schwinger) condition from quantum statistical mechanics allows us to do just that. The modular group came from the non-trace-like property of a state and the KMS condition allows us to correct for that. Let us do a formal calculation assuming that the modular group can be extended to complex numbers (remember that Ω is fixed by S, J and Δ):

$$\begin{aligned}
\phi(xy) &= \langle y\Omega, x^*\Omega\rangle \\
&= \langle y\Omega, J\Delta^{-1/2}\Delta x\Delta^{-1}\Omega\rangle \\
&= \langle \Delta x\Delta^{-1}\Omega, Sy\Omega\rangle \\
&= \langle y\Delta x\Delta^{-1}\Omega, \Omega\rangle.
\end{aligned}$$

We conclude that

$$\phi(xy) = \phi(y\sigma_i^\phi(x)).$$

Thus the trace is commutative provide we operate by the modular group.

Exercise 13.4.1. *If M is finite dimensional and ϕ is a faithful state, show that $\phi \circ \sigma_t^\phi = \phi$ and that for each x and y in M there is an entire function $F(z)$ with, for $t \in \mathbb{R}$,*

$$\begin{aligned}
F(t) &= \phi(\sigma_t^\phi(x)y) \qquad \textit{and} \\
F(t+i) &= \phi(y\alpha_t(x)).
\end{aligned}$$

If M is infinite dimensional we would not expect the function $F(z)$ of the previous exercise to be entire.

Definition 13.4.2. *Let α_t be a strongly continuous one parameter automorphism group of a von Neumann algebra M, and ϕ be a faithful normal state on M. We say that* <u>*α satisfies the KMS condition for ϕ*</u> *if $\phi \circ \alpha_t = \phi$ and , for each x and y in M, there is a function F, continuous and bounded on the strip $\{z : 0 \leq \Im m(z) \leq 1\}$, analytic on the interior of the strip and such that for $t \in \mathbb{R}$,*

$$\begin{aligned} F(t) &= \phi(\sigma_t^\phi(x)y) \qquad \textit{and} \\ F(t+i) &= \phi(y\alpha_t(x)). \end{aligned}$$

Theorem 13.4.3. *If ϕ is a faithful normal state on a von Neumann algebra M then σ_t^ϕ is the unique one parameter automorphism group satisfying the KMS condition for ϕ.*

This chapter has been heavily technical so we defer the proof, which is by approximation on dense subspaces of the domain of Δ to which the previous calculations can be applied, to an appendix. We content ourselves here with an interesting corollary, identifying a part or M on which ϕ behaves as a trace.

Corollary 13.4.4. *For $a \in M$ the following are equivalent:*

1. *$\phi(ax) = \phi(xa)$ for all $x \in M$.*
2. *$\sigma_t^\phi(a) = a$ for all $t \in \mathbb{R}$.*

Proof. ($1 \Rightarrow 2$) Observe that for $x \in M$, $\langle x^*\Omega, a\Omega\rangle = \langle \Omega, xa\Omega\rangle = \langle \Omega, ax\Omega\rangle$ (by 1). So $\langle Sx\Omega, a\Omega\rangle = \langle a^*\Omega, x\Omega\rangle$ so that $a\Omega \in D(S^*)$ and $S^*(a\Omega) = \Omega^*$. So $\Delta(a\Omega) = a\Omega$, $\Delta^{it}a\Omega = a\Omega$ and finally $\sigma_t^\phi(a) = a$ for all $t \in \mathbb{R}$.

($2 \Rightarrow 1$) $\phi(\sigma_t^\phi(x)a) = \phi(\sigma_t^\phi(xa)) = \phi(xa)$ so that $F(t)$ is constant. Use the Schwarz reflection principle to create a holomorphic function, constant on $\mathbb{R}$, in the union of the strip with its complex conjugate. Thus F is constant on the strip and $\phi(xa) = \phi(ax)$. □

Definition 13.4.5. *The von Neumann subalgebra of M defined by either of the conditions of the previous corollary is called the* <u>*centraliser*</u> *of the state ϕ.*

Chapter 14

Connes' theory of type III factors.

14.1 The Connes unitary cocycle Radon-Nikodym theorem.

This result will allow us to extract information from the modular group of a state which is independent of the state.

Theorem 14.1.1. *Let ϕ and ψ be faithful normal states on a von Neumann algebra M. Then there is a strongly continous map $t \to u_t$ from $\mathbb{R}$ to the unitary group of M so that*

$$\sigma_t^\phi = Adu_t\sigma_t^\psi \qquad \forall\, t \in \mathbb{R}.$$

Morevoer u_t satisfies the cocycle condition $u_t\sigma_t^\psi(u_s) = u_{t+s}$.

Proof. We define the faithful normal state Φ on $M \otimes M_2(\mathbb{C})$ by $\Phi((x)_{ij}) = \frac{1}{2}(\phi(x_{11}) + \psi(x_{22}))$. The projection $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is fixed by σ^Φ by 13.4.4. So σ^Φ defines a one parameter automorphism group of $pM \otimes M_2(\mathbb{C})p$ which satisfies the KMS condition for ϕ. Hence $\sigma_t^\Phi(x \otimes e_{11}) = \sigma_t^\phi(x) \otimes e_{11}$. Similarly $\sigma_t^\Phi(x \otimes e_{22}) = \sigma_t^\psi(x) \otimes e_{22}$. Let $V_t = \sigma_t^\Phi(1 \otimes e_{21})$. Then $V_tV_t^* = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $V_t^*V_t = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Hence $V_t = \left(\begin{smallmatrix} 0 & 0 \\ v_t & 0 \end{smallmatrix}\right)$ for some unitary $v_t \in M$. Routine computations give the rest. □

Corollary 14.1.2. *If M is a factor and σ_t^ϕ is outer for any ϕ and t then M is of type III.*

Proof. By the previous result it suffices to exhibit a single faithful normal state on a type II factor with inner modular group. In the II_1 case use the trace and in the II_∞ case choose a faithful normal state ϕ on $\mathcal{B}(\mathcal{H})$ and use $tr \otimes \phi$, using the KMS condition (if necessary) to very that the modular group for the tensor product is the tensor product of the modular groups. □

Corollary 14.1.3. *The subgroup of all $t \in \mathbb{R}$ for which σ_t^ϕ is inner is independent of the faithful normal state ϕ.*

Definition 14.1.4. *The subgroup of the previous corollary, which is an invariant of M, is called $T(M)$.*

We shall now calculate $T(M)$ for the Powers factor R_λ where this refers to the ITPFI factor with all $n_k = 2$ and all states having the same density matrix $h = \begin{pmatrix} \frac{1}{1+\lambda} & 0 \\ 0 & \frac{\lambda}{1+\lambda} \end{pmatrix}$.

Theorem 14.1.5.

$$T(R_\lambda) = \frac{2\pi}{\log \lambda}\mathbb{Z}$$

.

Proof. By the formula for the modular group $\sigma^\phi_{\frac{2\pi}{\log\lambda}} = id$ so $\frac{2\pi}{\log\lambda}\mathbb{Z} \subseteq T(R_\lambda)$. For the other direction it suffices to show that an automorphism α of the form

$$\alpha = \otimes_{k=1}^\infty \text{Ad}u$$

is outer whenever the unitary u is not a scalar.

For this first define $u_k = \otimes_1^k u$ and observe that if $\alpha = \text{Ad}v$ then $(u_k \otimes 1)^{-1}v = id$ on the matrix algebra $A_k = \otimes_1^k M_2(\mathbb{C})$. By exercise 4.3.3 this means that $v = u_k \otimes w$. Now it is clear from our basis that we can choose $\otimes_{j=1}^p x_i \otimes 1\Omega$ with non-zero inner procuct with $v\Omega$. But then fixing p and letting k tend to infinity we see that

$$\langle(\otimes_{j=1}^p x_i \otimes 1)\Omega, v\Omega\rangle = \prod_{j=1}^p \langle x_i, u\rangle\langle 1, u\rangle^{k-p}\langle 1, w\rangle.$$

The left hand side does not depend on k and $|\langle 1, w\rangle| \leq 1$ so we must have $|\langle 1, u\rangle| = 1$ which means that u is a scalar multiple of 1 by the Cauchy-Schwarz inequality. □

We conclude that the Powers factors R_λ are type III factors, mutually non-isomorphic for different values of λ.

14.2 Type III$_\lambda$.

The spectrum of the modular operator Δ is easy to calculate for an ITPFI factor. It is simply the closure of the set of all ratios $\frac{\mu_i}{\mu_j}$ as μ varies over all the density matrices defining the product state. Apart from being closed under the inverse operation this set of non-negative real numbers has no particular structure and can be altered easily by making changes in finitely many density matrices which of course do not change the factor.

Definition 14.2.1. *If M is a von Neumann algebra the invariant $S(M)$ is the intersection over all faithful normal states ϕ of the spectra of their corresponding modular operators Δ_ϕ.*

Theorem 14.2.2. *A factor M is of type III iff $0 \in S(M)$.*

Theorem 14.2.3. *(Connes-van Daele) $S(M) \setminus \{0\}$ is a closed subgroup of the positive real numbers.*

There are only three kinds of closed subgroups of $\mathbb{R}^+$.

Definition 14.2.4. *A factor M is called type III$_\lambda$ for $0 \leq \lambda \leq 1$ if*

$$\begin{aligned} \lambda = 0 \quad &: \quad S(M) = \{0\} \cup \{1\} \\ 0 < \lambda < 1 \quad &: \quad S(M) = \{0\} \cup \{\lambda^n : n \in \mathbb{Z}\} \\ \lambda = 1 \quad &: \quad S(M) = \{0\} \cup \mathbb{R}^+ \end{aligned}$$

Theorem 14.2.5. *The Powers factor R_λ is of type III$_\lambda$.*

In his thesis, Connes showed that every type III$_\lambda$ factor for $0 < \lambda < 1$ is canonically isomorphic to the crossed product of a type II$_\infty$ factor with an action of $\mathbb{Z}$ whose generator scales the trace by λ. Connes thesis

If A is a locally compact abelian group with an action α on a von Neumann algebra M, there is an action $\hat{\alpha}$ of the Pontryagin dual $\hat{A}$ on the crossed product $M \rtimes_\alpha A$ satisfying

$$\begin{aligned} \hat{\alpha}_a(x) &= x && \text{for } x \in M \\ \hat{\alpha}_{\hat{a}}(u_a) &= \hat{a}(a)u_a && \text{if } u_a \text{ are the unitaries defining the crossed product.} \end{aligned}$$

The existence of the so-called "dual action" $\hat{\alpha}$ is trivial proved since it is implemented by the obvious unitary representation of $\hat{A}$ on $L^2(A)$.

Exercise 14.2.6. *If A is finite consider the projection $p = \sum_a u_a \in M \rtimes A$. Show that $pM \rtimes Ap = M^A p$ and thus show that $(M \rtimes_\alpha A) \rtimes_{\hat{\alpha}} \hat{A}$ is isomorphic to $M \otimes M_{|A|}(\mathbb{C})$.*

Observe that the crossed product of a von Neumann algebra M on $\mathcal{H}$ by the modular group σ^ϕ does not depend, up to isomorphism, on the faithful normal state ϕ. This follows from theorem 14.1.1 by defining the unitary V on $L^2(\mathbb{R}, \mathcal{H})$ by

$$Vf(t) = u_t f(t)$$

where ψ is another faithful normal state with unitary one-cocycle u_t. Conjugating the operators that generate $M \rtimes_{\sigma^\phi} \mathbb{R}$ by V one obtains the generators of $M \rtimes_{\sigma^\psi} \mathbb{R}$.

Theorem 14.2.7. *The crossed product of M by the modular group admits a trace satisfying the properties of 9.1.9*

Definition 14.2.8. *The action of $\hat{\mathbb{R}}$ on $Z(M \rtimes_{\sigma^\phi} \mathbb{R})$ is called the "flow of weights" of M.*

Theorem 14.2.9. *(Takesaki duality) The crossed product*

$$(M \rtimes_{\sigma^\phi} \mathbb{R}) \rtimes_{\widehat{\sigma^\phi}} \hat{\mathbb{R}}$$

is isomorphic to the tensor product $M \otimes \mathcal{B}(\mathcal{H})$ for $\mathcal{H} = L^2(\mathbb{R})$.

Thus if M is a factor the flow of weights is ergodic.

Theorem 14.2.10. *If M is a factor of type III_λ the flow of weights is*

III_1: The trivial flow on a one point set if M is III_1.

III_λ: The transitive flow on the circle with period $\frac{2\pi}{\lambda}$ if M is of type III_λ, $0 < \lambda < 1$.

III_0: Ergodic non-transitive if M is of type III_0.

Moreover any ergodic non-transitive flow arises as the flow of weights for some type III_0 factor.

Chapter 15

Hyperfiniteness

Definition 15.0.11. *A von Neumann algebra M on a separable Hilbert space is called* hyperfinite *if there is an increasing sequence A_n of finite dimensional *-subalgebras of M which generates M as a von Neumann algebra.*

15.1 The hyperfinite type II_1 factor R

The first main result about hyperfiniteness was proved by Murray and von Neumann in []. We will use R to denote the hyperfinite II_1 factor whose uniqueness they proved.

Theorem 15.1.1. *Up to abstract isomorphism there is a unique hyperfinite II_1 factor.*

Sketch of proof. One works with the norm $||x||_2 = tr(x^*x)^{1/2}$ on M. It is not hard to show that a von Neumann subalgebra N of M is strongly dense in M iff it is dense in $||-||_2$. Given a subalgebra A of M and a subset $S \subseteq M$ one says

$$S \underset{\varepsilon}{\subseteq} A$$

if for each $x \in S$ there is a $y \in A$ with $||x - y||_2 < \varepsilon$.

The hyperfiniteness condition then implies:

*For every finite subset $S \subseteq M$ and every $\varepsilon > 0$ there is a finite dimensional *-subalgebra A of M with*

$$S \underset{\varepsilon}{\subseteq} A.$$

The next step is to show that the A in the preceeding condition can be chosen to be the $2^n \times 2^n$ matrices for some (possibly very large) n. This part uses what might be described as "II_1 factor technique". One begins with A and approximates all its minimal projections $\{e_i\}$ by ones whose traces are numbers of the form $k/2^n$. The matrix units of A can be changed a little bit in $|| - ||_2$ so that, together with matrix units conecting projections of trace $1/2^n$ less than the e_i, they generate a $2^n \times 2^n$ matrix algebra containing, up to ε, the matix units of A. Perturbation of the matrix units will involve results of the form:

If $u \in M$ satisfies $||(uu^*)^2 - uu^*||_2 < \epsilon$ then there is a partial isometry $v \in M$ with $||v - u||_2 < F(\epsilon)$
(for some nice function f with $f(0) = 0$).

or:

If p and q are projections with $||pq||_2 < \epsilon$ then there is a projection q' with $pq' = 0$ and $||q - q'|| < F(\epsilon)$.

or:

If f_{ij} are "almost $n \times n$ matrix units", i.e.

(a) $||f_{ij} - f_{ji}||_2 < \epsilon$

(b) $||f_{ij}f_{kl} - \delta_{j,k}f_{il}||_2 < \epsilon$

(c) $||1 - \sum_{i=1}^{n} fii||_2 < \epsilon$

then there are $n \times n$ matrix units e_{ij} with $||e_{ij} - f_{ij}|| < F(\epsilon)$ where F depends only on n and $F(0) = 0$.

Such results are proved by a skilful use of the polar decomposition and spectral theorem.

Thus one shows that in a hyperfinite type II_1 factor one has:

Property $*$: *For every finite subset $S \subseteq M$ and every $\varepsilon > 0$ there is a $2^n \times 2^n$ matrix subalgebra of M with*

$$S \underset{\varepsilon}{\subseteq} A.$$

One may now proceed to prove the theorem by choosing a $|| - ||_2$-dense sequence x_k in M and inductively constructing an <u>increasing</u> sequence of $2^{n_k} \times 2^{n_k}$ matrix algebras A_k with the property that

$$\text{For each } k = 1, 2, 3, ..., \qquad \{x_1, x_2, ..., x_k\} \underset{1/k}{\subseteq} A_k \; .$$

The union of the A_k's is clearly dense in $||-||_2$. This is enough to prove the theorem since the A_k's can be used to give an isomorphism of M with the type II_1 factor $\otimes^\infty M_2(\mathbb{C})$ constructed in section 6.2.

To construct A_{k+1} from A_k one needs to arrange for the new algebra to contain the one already constructed. So to the elements $x_1, x_2, ..., x_{k+1}$, add matrix units e_{ij} for A_{k+1}. Now use property * to obtain a B almost containing the x_i and the matrix units, with ϵ small enough to control sums over the matrix units e_{ij}. In B we know there are approximate matrix units close to the e_{ij} so by a technical lemma, exact matrix units f_{ij} close to the e_{ij}. Now choose a unitary close to the identity which conjugates the f_{ij} to the e_{ij} and use this unitary to conjugate B to a superalgebra of A_k. This superalgebra is A_{k+1} and it contains the x_i up to epsilon since u is close to the identity.

This completes the sketch of the proof. The technique involved is considered standard in von Neumann algebras and the details can be found in . dixmier

Corollary 15.1.2. *If S_∞ is the group of finitely supported permutations of a countably infinite set then $vN(S_\infty) \cong \otimes^\infty M_2(\mathbb{C})$.*

Proof. The subgroups of S_∞ permuting an increasing sequence of finite sets show that $vN(S_\infty)$ is hyperfinite. □

It is surprising at first sight that the type II_1 factor $L^\infty(X,\mu) \rtimes \mathbb{Z}$ obtained from an ergodic measure-preserving transformation T is hyperfinite. This can be shown by Rokhlin's tower theorem which asserts that, for each $n \in \mathbb{N}$ and each $\epsilon > 0$ there is a measurable subset $A \subseteq X$ with

(1) $T^i(A) \cap T^j(A) = \emptyset$ for $1 \le i < j \le n$, and

(2) $\mu(X \setminus \cup_{i=0}^n T^i(A)) < \epsilon$.

The unitary u_1 of the crossed product and the characteristic function of A can be combined, with a little perturbation to get the identity, to get a $n \times n$ matrix algebra. Careful application of the tower theorem will allow one to get any element of $L^\infty(X,\mu)$, and u_1, in this matrix algebra up to some ϵ. This was first proved by Henry Dye in who went on to prove that in fact all Dye groups of polynomial growth give hyperfinite II_1 factors in this way.

The ultimate result in this direction is the celebrated "Injective factors" theorem of Connes who showed that hyperfiniteness for a von Neumann algebra M on $\mathcal{H}$ is equivalent to "injectivity" which means there is a projection in the Banach space sense of norm one from $\mathcal{B}(\mathcal{H})$ onto M. This theorem, whose proof is a great, great tour de force, has a raft of corollaries, many of which were open questions. Let us just mention the fact that it follows easily that any subfactor of R which is infinite dimensional is in fact isomorphic to R. It also implies that $vN(\Gamma)$, as well as $L^\infty(X,\mu) \rtimes \Gamma$ is hyperfinite as soon as Γ is amenable.

15.2 The type III case.

Connes actions

Connes Injective factors

The complete classification of injective(=hyperfinite) factors is a triumph of 20th. century mathematics. Connes showed in that there is only one trace-scaling automorphism of $R \otimes \mathcal{B}(\mathcal{H})$ for each scaling factor $\lambda \neq 1$ up to conjugacy. Together with this shows that for each λ with $0 < \lambda < 1$ there is a unique injective factor of type III_λ.

krieger

injective

Using results of Krieger in , his thesis and , Connes showed that hyperfinite type III_0 factors are classified by their flow of weights (up to conjugacy of flows, not orbit equivalence). This means that there is a rather large number of III_0 factors but their classification is in the realm of ergodic theory rather than von Neumann algebras.

uffeIIIone

The remaining case of injective type III_1 factors was solved by Haagerup in . There is just one such factor and a hyperfinite factor is “generically” of type III_1.

Chapter 16

Central Sequences.

16.1 Generalities.

Definition 16.1.1. *If M is a type II_1 factor, a central sequence in M is a norm bounded sequence (x_n) with $\lim_{n\to\infty} ||[x_n, a]||_2 = 0$. A central sequence is said to be trivial if $\lim_{n\to\infty} ||x_n - tr(x_n)id||_2 = 0$. M is said to have property Γ if there is a central*

The collection of all central sequences is clearly a C*-subalgebra of $\ell^\infty(\mathbb{N}, M)$. If ω is a free ultrafilter on $\mathbb{N}$, the subalgebra $\mathcal{I}_\omega$ of $\ell^\infty(\mathbb{N}, M)$ consisting of sequences with $\lim_{n\to\omega} ||x_n||_2 = 0$ is a 2-sided ideal of $\ell^\infty(\mathbb{N}, M)$. Note also that M is embedded in $\ell^\infty(\mathbb{N}, M)$ as constant sequences.

Definition 16.1.2. *With notation as above, the ultraproduct of M along ω is the quotient of $\ell^\infty(\mathbb{N}, M)$ by $\mathcal{I}_\omega$. It is written M^ω. The algebra of (ω-)central sequences is the centraliser $M_\omega = M' \cap M^\omega$ of M in $\ell^\infty(\mathbb{N}, M)$.*

By compactness, the trace on M defines a trace on M^ω by

$$tr((x_n)) = \lim_{n\to\omega} tr(x_n)$$

and by definition it is faithful on M^ω.

Exercise 16.1.3. *Show that the unit ball (in the C* norm) of M^ω is complete in $|| - ||_2$ so that M^ω and M_ω are von Neumann algebras.*

16.2 Central sequences in R

All models for R exhibit central sequences in abundance. The most obvious situation is that of $\otimes^\infty M_2(\mathbb{C})$. Fixing $x \in M_2(\mathbb{C})$ we can define the sequence

$x_n = 1 \otimes 1 \otimes 1 ... x \otimes 1 \otimes 1 ...$ with the x in the nth slot in the tensor product. For large enough n, x_n will commute with any element in the algebraic tensor product so by the obvious (in the II_1 case!) inequality $||[x_n, a]|| \leq 2||x_n|| \, ||a||_2$ we see that (x_n) is central and clearly non-trivial if x is not a scalar. Just as clearly the central sequence algebra is non-commutative as we only need to choose x and y that do not commute and construct the sequences (x_n) and (y_n) as above. In fact it is not hard to show that R_ω is a factor.

Theorem 16.2.1. *The central sequence algebra R_ω is a type II_1 factor.*

finish proof!

Proof. If (x_n) represents an element $X \in R_\omega$, □

16.3 Type II_1 factors without property Γ.

Theorem 16.3.1. *Let Γ be an icc group possessing a subset Δ not containing the identity and three elements α, β and γ such that*

(a)$\Gamma = \{1\} \cup \Delta \cup \alpha \Delta \alpha^{-1}$

(b)$\Delta, \beta \Delta \beta^{-1}$ *and* $\gamma \Delta \gamma^{-1}$ *are mutually disjoint.*

then for $x \in vN(\Gamma)$,

$$||x - tr(x)id||_2 \leq 14 \max\{||[x, u_\alpha]||_2, ||[x, u_\beta]||_2, ||[x, u_\gamma]||_2\}.$$

Proof. Write x as $\sum_{\nu \in \Gamma} x_\nu u_\nu$. We will frequently use the formula

$$||[x, u_\rho]||_2^2 = ||u_{\rho^{-1}} x u_\rho - x||^2 = \sum_{\nu \in \Gamma} |x_\nu - x_{\rho \nu \rho^{-1}}|^2.$$

By replacing x by $x - tr(x)id$ it suffices to prove the result if $tr(x) = 0$ and we may suppose $||x||_2 = 1$ so that for such an x we must show $1 \leq 14 \max\{||[x, u_\alpha]||_2, ||[x, u_\beta]||_2, ||[x, u_\gamma]||_2\}$.

We first make a simple estimate. If Λ is any subset of Γ and $\rho \in \Gamma$ then

$$\begin{aligned} |\sum_{\nu \in \Lambda} |x_\nu|^2 - \sum_{\nu \in \Lambda} |x_{\rho \nu \rho^{-1}}|^2| &= \sum_{\nu \in \Lambda} (|x_\nu| + |x_{\rho \nu \rho^{-1}}|)||x_\nu| - |x_{\rho \nu \rho^{-1}}|| \\ &\leq \sum_{\nu \in \Lambda} (|x_\nu| + |x_{\rho \nu \rho^{-1}}|)(|x_\nu - x_{\rho \nu \rho^{-1}}|) \\ &\leq 2||x||_2 (\sum_{\nu \in \Lambda} |x_\nu - x_{\rho \nu \rho^{-1}}|^2)^{1/2} \end{aligned}$$

so that if $\rho \in \{\alpha, \beta, \gamma\}$ we have

$$|\sum_{\nu \in \Lambda} |x_\nu|^2 - \sum_{\nu \in \Lambda} |x_{\rho\nu\rho^{-1}}|^2| \leq 2\epsilon$$

where $\epsilon = \max\{||[x, u_\alpha]||_2, ||[x, u_\beta]||_2, ||[x, u_\gamma]||_2\}$.

Let us now first overestimate $||x||^2 = 1$:

$$\begin{aligned} 1 &\leq \sum_{\nu \in \Delta} |x_\nu|^2 + \sum_{\nu \in \Delta} |x_{\alpha\nu\alpha^{-1}}|^2 \\ &\leq 2\sum_{\nu \in \Delta} |x_\nu|^2 + 2\epsilon. \end{aligned}$$

Now underestimate it:

$$\begin{aligned} 1 &\geq \sum_{\nu \in \Delta} |x_\nu|^2 + \sum_{\nu \in \Delta} |x_{\beta\nu\beta^{-1}}|^2 + \sum_{\nu \in \Delta} |x_{\gamma\nu\gamma^{-1}}|^2 \\ &\geq 3\sum_{\nu \in \Delta} |x_\nu|^2 - 4\epsilon. \end{aligned}$$

Let $y = \sum_{\nu \in \Delta} |x_\nu|^2$ and eliminate y from the inequalities $1 \leq 2y + 2\epsilon$ and $1 \geq 3y - 4\epsilon$ to obtain

$$\epsilon \geq 1/14$$

as desired. □

It is easy to come up with groups having subsets as in the previous theorem. For instance if $G = F_2$, free on generators g and h, let Δ be the set of all reduced words ending in a non-zero power of g. Let $\alpha = g$, $\beta = h$ and $\gamma = h^{-1}$. The same works for more than two generators. We conclude:

Theorem 16.3.2. *The type* II_1 *factor* $vN(F_n)$ *for* $n \geq$ *does not have property* Γ.

Chapter 17

Bimodules and property T

Chapter 18

Fermions and Bosons:CAR and CCR

According to physics lore, the states of a quantum system are given by (the one-dimensional subspaces of) a Hilbert space $\mathcal{H}$ and if two systems have state spaces $\mathcal{H}$ and $\mathcal{K}$, the joint system has state space $\mathcal{H} \otimes \mathcal{K}$. Fermions are particles such that "the wave function of several fermions is antisymmetric" which means that it is the antisymmetric tensor product $\Lambda^n\mathcal{H}$ which describes n identical fermions. Bosons are particles whose wave functions are symmetric so it is the symmetric tensor power $S^n\mathcal{H}$ which describes n identical bosons. In order to treat families with an unlimited number of fermions and Bosons we need the fermionic and bosonic Fock spaces (which are Hilbert space direct sums):

$$\mathcal{F}(\mathcal{H}) = \oplus_{n=0}^{\infty}\Lambda^n\mathcal{H}$$

and

$$S(\mathcal{H}) = \oplus_{n=0}^{\infty}S^n\mathcal{H}.$$

The passage from $\mathcal{H}$ to $\mathcal{F}(\mathcal{H})$ or $S(\mathcal{H})$ is known as "second quantisation'.

We will not attempt to explain the physics above but will define properly these two Fock spaces and how they give rise to interesting von Neumann algebras related to physics.

Both these Fock spaces are subspaces of the "full Fock space" or tensor algebra

$$\mathcal{T}(\mathcal{H}) = \oplus_{n=0}^{\infty} \otimes^n \mathcal{H}$$

$\mathcal{T}(\mathcal{H})$ is related to quantum physics also though so far in a less fundamental way through the large N behaviour of random $N \times N$ matrices and Voiculescu's free probability.

18.1 The Fock spaces.

18.1.1 Full Fock space

Definition 18.1.2. *If $\mathcal{H}$ is a real or complex Hilbert space the* full Fock space *$\mathcal{T}(\mathcal{H})$ is the Hilbert space direct sum $\oplus_{n=0}^{\infty} \otimes^n \mathcal{H}$. By definition $\otimes^0\mathcal{H}$ is one dimensional, spanned by the "vacuum" vector Ω.*

Even when $\mathcal{H}$ is real one complexifies $\mathcal{T}(\mathcal{H})$ so that it is a complex Hilbert space.

For each n and $f \in \mathcal{H}$ the operator $\ell(f) : \otimes^n\mathcal{H} \to \otimes^{n+1}\mathcal{H}$ given by

$$\ell(f)(\xi_1 \otimes \xi_2 \cdots \xi_n) = f \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n$$

is clearly bounded by $||f||$ so extends to an operator we will call $\ell(f)$ on all of full Fock space.

Exercise 18.1.3. *(i) Show that*

$$\ell(f)^*(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \langle \xi_1, f\rangle \xi_2 \otimes \xi_3 \cdots \xi_n,$$

$$\ell(f)^*(\xi) = \langle \xi, f\rangle\Omega \text{ for } \xi \in \otimes^1\mathcal{H},$$

$$\text{and } \ell(f)^*\Omega = 0.$$

(ii) Show that

$$\ell(f)^*\ell(g) = \langle g, f\rangle$$

Proposition 18.1.4. *The action of the $\ell(f)$ and $\ell(f)^*$ on full Fock space is irreducible.*

Proof. It suffices to show that any non-zero vector in $\mathcal{T}(\mathcal{H})$ is cyclic. The vacuum vector Ω is obviously cyclic. Note that the linear span of the images of the $\ell(f)\ell(f)^*$ is the orthogonal complement $\Omega^\perp$. The projection onto $\Omega^\perp$ is thus in $\{\ell(f), \ell(f)^*\}''$. If ξ is any vector we are thus done if $\langle \xi, \Omega\rangle \neq 0$. Otherwise $\langle \xi, f_1 \otimes f_2 \cdots f_n\rangle$ must be non-zero for some $f_i \in \mathcal{H}$. But then $\langle \ell(f_1)\ell(f_2)\cdots\ell(f_n)\Omega, \xi\rangle \neq 0$ and the vector $(\ell(f_1)\ell(f_2)\cdots\ell(f_n))^*\xi$, which can be reached from ξ, projects non-trivially onto the vacuum and is thus cyclic. □

One may also consider the right creation operators $r(\xi)$ defined by

$$r(f)(\xi_1 \otimes \xi_2 \cdots \xi_n) = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \otimes f.$$

They satisy the same relations as the $\ell(f)$ and almost commute with them. To be precise

18.1.5.
$$\ell(f)r(g) = r(g)\ell(f)$$

and
$$\ell(f)r(g)^* - r(g)^*\ell(f) = -\langle f, g\rangle p_\Omega$$

where p_Ω is projection onto the one dimensional subspace spanned by the vacuum.

The $r(f)$'s and $r(f)^*$'s act just as irreducibly as the ℓ's.

18.1.6 Fermionic Fock space.

Given a Hilbert space $\mathcal{H}$, the $nth.$ exterior or antisymmetric power of $\mathcal{H}$ is the Hilbert space $\Lambda^n\mathcal{H} = p(\otimes^n\mathcal{H})$ where p is the antisymmetrisation projection

$$p(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma \xi_{\sigma(1)} \otimes \xi_{\sigma(1)} \otimes \xi_{\sigma(2)} \cdots \otimes \xi_{\sigma(n)}$$

Definition 18.1.7. *The fermionic Fock space of $\mathcal{H}$ is the Hilbert space direct sum*
$$\mathcal{F}(\mathcal{H}) = \oplus_{n=0}^{\infty}\Lambda^n\mathcal{H}$$

.

Given $\xi_1, ..\xi_n \in \mathcal{H}$ we set

$$\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n = \sqrt{n!} \quad p(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n)$$

Exercise 18.1.8. *Show that $\langle \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n, \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_n\rangle$ is the determinant of the matrix whose (i, j) entry is $\langle \xi_i, \eta_j\rangle$.*

Obviously if $\sigma \in S_n$,

$$\xi_{\sigma(1)} \wedge \xi_{\sigma(1)} \wedge \xi_{\sigma(2)} \cdots \wedge \xi_{\sigma(n)} = (-1)^\sigma \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n.$$

Exercise 18.1.9. *For $f \in \mathcal{H}$ define $A(f) : \otimes^n\mathcal{H} \to \Lambda^{n+1}\mathcal{H}$ by*

$$A(f)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \frac{1}{\sqrt{n!}} f \wedge \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n,$$

show that $A(f)(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n) = f \wedge \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n$.

Exercise 18.1.10. *The previous exercise shows that for each $f \in \mathcal{H}$ there is a bounded linear map from $\Lambda^n\mathcal{H}$ to $\Lambda^{n+1}\mathcal{H}$ defined by:*

$$a(f)(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n) = f \wedge \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n.$$

Show that

$$a(f)^*(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \xi_1 \wedge \cdots \hat{\xi_i} \cdots \wedge \xi_{n+1}$$

We have sloppily left out the n on our operators $a(f)$. But we can put them all together to form the densely defined operators $a(f)$ and $a(f)^*$ on $\mathcal{F}(\mathcal{H})$ whose domain is the algebraic direct sum of the $\Lambda^n\mathcal{H}$.

Exercise 18.1.11. *Show that these densely defined operators satisfy the CAR relations.*

Exercise 18.1.12. *Show that $||a(f)\xi|| \leq ||\xi||$ for ξ in the domain of $a(f)$ so $a(f)$ an $a(f)^*$ extend to bounded operators on $\mathcal{F}(\mathcal{H})$ which are one another's adjoints and satisfy the CAR relations.*

Exercise 18.1.13. *Imitate 18.1.4 to show that the *-algebra generated by the $a(f)$ acts irreducibly on fermionic Fock space.*

18.2 CAR algebra, CCR algebra and the (extended) Cuntz algebra.

18.2.1 CAR algebra

Definition 18.2.2. *If $\mathcal{H}$ is a complex Hilbert space the CAR (canonical anticommuation relations) algebra $CAR(\mathcal{H})$ is the unital *-algebra with generators $a(f)$ for each $f \in \mathcal{H}$ subject to the following relations:*
(i) The map $f \mapsto a(f)$ is linear.
(ii) $a(f)a(g) + a(g)a(f) = 0 \quad \forall f, g \in \mathcal{H}$.
(iii) $a(f)a(g)^* + a(g)^*a(f) = \langle f, g \rangle \quad \forall f, g \in \mathcal{H}$.

(the identity is implicit on the right hand side of (iii))

We already know a non-trivial representation of the CAR algebra on Fermionic Fock space.

Exercise 18.2.3. *Show that if $\dim \mathcal{H} = 1$, $CAR(\mathcal{H})$ is isomorphic to $M_2(\mathbb{C})$.*

Proposition 18.2.4. *If* $\dim \mathcal{H} = n < \infty$*, the Fock space representation is irreducible and faithful so* $CAR(\mathcal{H}) \cong M_{2^n}(\mathbb{C})$.

Proof. Irreducibility was already shown. This means that the dimension of the image of the representation is 2^{2n}. But words in the $a(f)$ may be rearranged without changing their linear span so that all $a(f)$'s come before all $a(g)^*$'s. Moreover the order of the $a(f)$'s in a word only matters up to a sign so that, after choice of a orthogonal basis, the CAR algebra is linearly spanned by words given by pairs of subsets of the basis-one for the $a(f)$'s and one for the $a(f)^*$'s. Thus the dimension of the CAR algebra is $\leq 2^{2n}$ and the Fock space represenation is bijective.

□

Corollary 18.2.5. *If* $\mathcal{K}$ *is a subspace of* $\mathcal{H}$*, the obvious inclusion map of* $CAR(\mathcal{K})$ *into* $CAR(\mathcal{H})$ *is injective.*

Proof. If $\mathcal{K}$ is finite dimensional this follows from the simplicity of a matrix algebra. In general an element of $CAR(\mathcal{H})$ is a finite linear combination of words on $a(f)$'s and $a(f)^*$'s so in CAR of a finite dimensional subspace.

□

Corollary 18.2.6. *There is a unique* C^**-norm and a unique normalised trace on* $CAR(\mathcal{H})$.

Proof. This follows from the uniqueness of the norm and trace on a matrix algebra as in 18.2.5 □

We will see an explicit formula for the trace on words in the $a(f)$ and $a(f)^*$ later-it is a "quasi-free" state.

Exercise 18.2.7. *Show that* $||a(f)|| = ||f||$.

This shows that, if we choose an orthonormal basis ξ_i of $\mathcal{H}$ (supposed separable), the *-algebra generated by $\{a(\xi_i)|i = 1, 2, \cdots \infty\}$ is dense in the C^*-completion of $CAR(\mathcal{H})$. Thus this C^*-algebra is in fact isomorphic to the inductive limit of $2^n \times 2^n$ matrices and one obtains the hyperfinite II_1 factor as its GNS completion.

From now on we will abuse notation by using $CAR(\mathcal{H})$ for the C^*-algebra completion.

Exercise 18.2.8. *A unitary* u *on* $\mathcal{H}$ *obviously defines an automorphism* α_u *of* $CAR(\mathcal{H})$ *(sometimes called a Bogoliubov automorphism) by functorially extending* $\alpha_u(a(f)) = a(uf)$*. In particular choosing* $u = -1$ *makes* $CAR(\mathcal{H})$ *into a* $\mathbb{Z}/2\mathbb{Z}$*-graded algebra. Define a notion of graded product* $A \otimes_{\mathbb{Z}/2\mathbb{Z}} B$ *for* $\mathbb{Z}/2\mathbb{Z}$*-graded algebras* A *and* B*. Show that if* V *and* W *are orthogonal Hilbert subspaces of* $\mathcal{H}$ *then* $CAR(V \oplus W) \cong CAR(V) \otimes_{\mathbb{Z}/2\mathbb{Z}} CAR(W)$.

18.3 Vacuum expectation value

The vacuum vector Ω defines a state ω_Ω on $CAR(\mathcal{H})$ as usual via $\langle x\Omega, \Omega\rangle$ which as we know would reconstruct Fock space via the GNS construction. The following formula is clear:

18.3.1.

$$\omega_\Omega\big(a(f_m)^* a(f_{m-1})^* \cdots a(f_m)^* a(g_1)a(g_2)\cdots a(g_n)\big) = \delta_{m,n} \det\left(\langle g_i, f_j\rangle\right)$$

We know that states on matrix algebras are given by $tr(h)$ so we would like to know what ω_Ω looks like in this picture. For this we will construct an *explicit* isomorphism between $CAR(\mathcal{H})$ and $M_{2^n}(\mathbb{C})$. To do this it suffices to exhibit a family of n commuting 2×2 matrix algebras. If we choose an orthonormal basis ξ, each $a(\xi)$ will give a 2×2 matrix algebra but they won't quite commute. But this can be fixed up by unitaries which implement the Bogoliubov automorphism corresponding to -1.

So let $\mathcal{H}$ be a Hilbert space and let $\{\xi_i \mid i = 1, 2, ...\infty\}$ be an orthonormal basis. Set

$$a_i = a(\xi_i) \in CAR(\mathcal{H})$$

and $v_i = 1 - 2a_i^* a_i$. The v_i commute among themselves so put

$$u_k = \prod_{i=1}^{k} v_i.$$

Exercise 18.3.2. *Show:*
$v_i^2 = 1 = u_i^2$, $u_j a_i u_j = -a_i$ *for* $i \le j$ *and* $u_j a_i u_j = a_i$ *for* $i > j$.

If we put $e_{12}^k = u_k a_k$, *then* $[e_{12}^k, e_{12}^j] = 0 = [e_{12}^k, (e_{12}^j)^*]$ *for all* j, k *so that the* e_{12}^k *generate mutually commuting* 2×2 *matrix units with* $e_{11}^k = a_k a_k^*$ *and* $e_{22}^k = a_k^* a_k$. *This gives an isomorphism between the* CAR *algebra of the linear span of* $\xi, \xi_2 \cdots ..., \xi_n$ *with* $M_{2^n}(\mathbb{C})$ *and hence* $CAR(\mathcal{H})$ *with* $\otimes_1^\infty M_2(\mathbb{C})$. *Observe that* u_k *implements the Bogoliubov automorphism for* -1 *and thus depends on the basis only up to a sign.*

Now we can see what the vacuum expectation value looks like:

$$\omega_\Omega(e_{ij}^k) = \begin{cases} 0 & i \ne j \text{ or } i = 1 \\ 1 & i = j = 2 \end{cases}$$

Thus in the matrix picture if $h = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ then ω_Ω is the product state

$$\omega_\Omega(\otimes_{i=1}^\infty x_i) = \prod_{i=1}^{\infty} tr(hx_i).$$

18.4 Quasi-free states

We will now generalise formula 18.3.1 to what are called quasi-free states. The operator a in the theorem below is called the "covariance" of the state.

Theorem 18.4.1. *For each self-adjoint a on $\mathcal{H}$, $0 \le a \le 1$ there is a state ϕ_a on $CAR(\mathcal{H})$ defined by*

$$\phi_a\big(a(f_m)^* a(f_{m-1})^* \cdots a(f_1)^* a(g_1) a(g_2) \cdots a(g_n)\big) = \delta_{m,n} \det\left(\langle a g_i, f_j\rangle\right)$$

Proof.

Lemma 18.4.2. *Theorem 18.4.1 is true if a is a projection p and* $\dim \mathcal{H} = n < \infty$.

Proof. Choose a basis $\xi_1, \xi_2, ..., \xi_k$ for $p\mathcal{H}$ and $\eta_1, \eta_2, ..., \eta_{n-k}$ for $(1-p)\mathcal{H}$. We claim that if $v = \eta_1 \wedge \eta_2 \wedge \cdots \eta_{n-k}$ the the vector state ω_v is the required state. For this note that it suffices to prove the formula

$$\omega_v\big(a(f_m)^* a(f_{m-1})^* \cdots a(f_1)^* a(g_1) a(g_2) \cdots a(g_n)\big) = \delta_{m,n} \det\left(\langle p g_i, f_j\rangle\right)$$

when the f's and g's are basis vectors since both sides are suitably multilinear. If any of the f's and g's is in $(1-p)\mathcal{H}$ both sides are zero. If all the f's and g's are in $p\mathcal{H}$ the left hand side is $\langle g_1 \wedge \cdots \wedge g_n \wedge v, f_1 \wedge \cdots f_m \wedge v\rangle$ which is 0 unless $m = n$ in which case it is the determinant:

$$\begin{pmatrix} \det(\langle g_i, f_j\rangle) & 0 \\ 0 & \det(\langle \eta_i, \eta_j\rangle) \end{pmatrix} = \det\left(\langle p g_i, f_j\rangle\right)$$

□

Lemma 18.4.3. *Theorem 18.4.1 is true if p is a projection.*

Proof. Choose bases $\xi_1, \xi_2, \cdots$ for $p\mathcal{H}$ and $\eta_1, \eta_2, \cdots$ for $(1-p)\mathcal{H}$ and let V_k be the subspace of $\mathcal{H}$ spanned by $\{\xi_i, \eta_j | 1 \le i, j \le k\}$. Then for each k there is a state on $CAR(V_k)$ satisfying the formula and these states are coherent with the inclusions $CAR(V_k) \subset CAR(V_{k+1})$. By density and continuity of states they extend to a state on $CAR(\mathcal{H})$ still satisfying the formula of the theorem. □

To end the proof of theorem 18.4.1 we form $\mathcal{H} \oplus \mathcal{H}$ and consider the projection

$$p = \begin{pmatrix} a & \sqrt{a(1-a)} \\ \sqrt{a(1-a)} & 1-a \end{pmatrix}$$

. Obviously the quasifree state ϕ_p on $CAR(\mathcal{H} \oplus \mathcal{H})$ restricts to a state on $CAR(\mathcal{H}) \oplus 0$ satisfying the formula of the theorem. □

Exercise 18.4.4. *Show that if a is diagonalisable with eigenvalues λ_i then using the basis of eignvectors to identify $CAR(\mathcal{H})$ with $\otimes^\infty M_2(\mathbb{C})$, the quasi-free state with covariance a becomes a product state with $h_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & 1-\lambda_i \end{pmatrix}$.*

Let us think a little more about a quasi-free state whose covariance is a projection p, first in finite dimensions. The vector of the GNS construction has been identified with $\eta_1 \wedge \eta_2 \cdots \wedge \eta_k$ where the η are an orthonormal basis for $(1-p)\mathcal{H}$. Physicists think of this $\eta_1 \wedge \eta_2 \cdots \wedge \eta_k$ as a new "vacuum" in which the "states" $\eta_1, \cdots \eta_k$ have been filled. There is no particular reason not to use this notation when $dim(1-p)\mathcal{H} = \infty$ so the vacuum is $\eta_1 \wedge \eta_2 \wedge \eta_3 \cdots$ which represents Dirac's "sea" and a state $\eta_1 \wedge \eta_2 \cdots \wedge \check{\eta_i} \wedge \cdots$ represents an excitation of the vacuum by a "hole" or antiparticle η_i.

One may make of this what one likes but there is a particularly significant mathematical consequence. If u_t is a one-parameter group of unitaries on $\mathcal{H}$ which commutes with p then the corresponding group of Bogoliubov automorphisms α_t preserves ϕ_p and so defines a one parameter unitary group $U_t(\pi_{\phi_p}(x)\Omega_p) = \pi_{\phi_p}(\alpha_t(x))\Omega_p$ of the GNS space $\mathcal{H}_p$ with vacuum vector Ω_p for ϕ_p. The map from $(1-p)\mathcal{H}$ to $\mathcal{H}$, $\eta \mapsto a(\eta)^*\Omega_p$ is *anti*-linear so the sign of t will be reversed. More concretely suppose for simplicity that u_t has an orthonomal basis of eigenvectors $\{\xi_k | k \in \mathbb{Z}\}$ with

$$u_t \xi_k = e^{iE_k t}\xi_k,$$

and that p is the projection onto the space spanned by the ξ_k with $E_k < 0$. Then following through the definition of U_t, we see that

$$U_t\xi_k = \begin{cases} e^{iE_k t}\xi_k & \text{if } E_k \geq 0 \\ e^{-iE_k t}\xi_k & \text{if } E_k < 0. \end{cases}$$

Physically this is remarkable. If we start with a Hamiltonian inadmissible because of its negative energy eigenvalues, second quantisation with the appropriate quasi-free state turns all the negative energies into positive ones!

18.5 Complex structure

One may obtain the existence of the quasi-free states without basis calculations by changing the complex structure on $\mathcal{H}$.

Definition 18.5.1. *If $\mathcal{K}$ is a* real *Hilbert space, a complex structure on $\mathcal{K}$ is an orthogonal transformation J with $J^2 = -1$.*

Lemma 18.5.2. *A real Hilbert space $\mathcal{K}$ with inner product $(,)$ and a complex structure J can be made into a complex Hilbert space by defining the action of $\mathbb{C}$ as $(x+iy)\xi = x\xi + yJ\xi$ and the inner product*

$$\langle \xi, \eta \rangle = (\xi, \eta) - i(J\xi, \eta)$$

Proof. The vector space structure is routine as is sesquilinearity of $\langle , \rangle$. But $\langle \xi, \xi \rangle = (\xi, \xi) - i(J\xi, \xi)$ and $(J\xi, \xi) = -(\xi, J\xi) = -(J\xi, \xi)$ which is therefore zero. Hence $<,>$ is positive definite and defines the same norm as $(,)$ so completeness is unchanged. □

Definition 18.5.3. *The Clifford algebra of a real Hilbert space $\mathcal{K}$ is the (complex) *-algebra generated by $c(f)$ subject to:*
(i) The map $f \mapsto c(f)$ is real linear.
(ii) $c(f) = c(f)^* \quad \forall f \in \mathcal{K}$.
(iii) $\{c(f), c(g)\} = 2(f, g) \quad \forall f, g \in \mathcal{H}$.

(clearly $c(f)^2 = (f, f)$ is equivalent to (iii)).

Proposition 18.5.4. *If $\mathcal{K}$ is a real Hilbert space with complex structure J, mapping $a(f)$ to $\frac{1}{2}(c(f) - ic(Jf))$ defines an isomorphism of the CAR algebra of the complex Hilbert space onto the Clifford algebra of $\mathcal{K}$. The inverse map is given by $c(f) \mapsto a(f) + a(f)^*$.*

Proof. It is routine that $f \mapsto a(f)$ is complex-linear and satisfies the CAR relations, so the map extends to all of CAR. Also $a(f) + a(f)^*$ satisfies the Clifford algebra relations and is an inverse to $a(f) \mapsto \frac{1}{2}(c(f) - ic(Jf))$ on the generators hence on all of the Clifford algebra. □

Thus given a complex Hilbert space $\mathcal{H}$ and another complex structure J on the underlying real Hilbert space, there is an isomorphism of complex *-algebras $\chi_J : CAR(\mathcal{H}) \to CAR(\mathcal{H}_J)$ and hence a representation of $CAR(\mathcal{H})$ every time we have one of $CAR(\mathcal{H}_J)$, in particular the Fock representation of $\mathcal{H}_J$ gives a representation of $CAR(\mathcal{H})$. Explicitly if we follow the isomorphism through the Clifford algebra we obtain

18.5.5.

$$\chi_J(a(f)) = \frac{1}{2}\Big(A(f - Jif) + A(f + Jif)^*\Big)$$

where we have used $A(f)$ for the generators of $CAR(\mathcal{H}_J)$.

The simplest J if given by choosing a projection $p \in \mathcal{B}(\mathcal{H})$ and changing of i on $p\mathcal{B}(\mathcal{H})^\perp$, thus $J = ip - i(1-p)$ (which is actually $\mathbb{C}$-linear).

Exercise 18.5.6. *Show that the state induced on $CAR(\mathcal{H})$ by ξ and the Fock vacuum for $\mathcal{H}_J$ with J as above is quasi-free of covariance p.*

The question thus becomes: what does $\mathcal{H}_J$, and hence its Fock space, look like? If $\mathcal{H}$ is a Hilbert space let $\overline{\mathcal{H}}$ be the dual Hilbert space of $\mathcal{H}$ and $\xi \mapsto \overline{\xi}$ be the canonical antilinear map from $\mathcal{H}$ to its dual.

Proposition 18.5.7. *Let $q = (1-p)$. Then the map $\xi \mapsto p\xi \oplus \overline{q\xi}$ is a $\mathbb{C}$-linear isomorphism from $\mathcal{H}_J$ to $p\mathcal{H} \oplus \overline{(1-p)\mathcal{H}}$.*

Exercise 18.5.8. *If the Hilbert space $\mathcal{H}$ is the direct sum $\mathcal{K} \oplus \mathcal{L}$, show that $\mathcal{F}(\mathcal{H})$ is canonically isomorphic to*

$$\oplus_{n=0}^{\infty} \oplus_{i+j=n} \left(\Lambda^i \mathcal{K} \otimes \Lambda^j \mathcal{L}\right)$$

Thus $\mathcal{F}(\mathcal{H}_J) \cong \oplus_{n=0}^{\infty} \oplus_{i+j=n} \left(\Lambda^i p\mathcal{H} \otimes \Lambda^j \overline{(1-p)\mathcal{H}}\right)$ on which $CAR(\mathcal{H})$ acts according to 18.5.5.

Exercise 18.5.9. *Show how the general quasi-free states are related to arbitrary complex structures on a complex Hilbert space.*

18.5.10 CCR algebra

18.5.11 Cuntz algebra

Definition 18.5.12. *Given the complex Hilbert space $\mathcal{H}$, let the* extended Cuntz algebra *of $\mathcal{H}$, $\mathcal{C}(\mathcal{H})$, be the unital $*$-algebra with generators $\ell(f)$ for each $f \in \mathcal{H}$ subject to the following relations:*
(i) The map $f \mapsto \ell(f)$ is linear.
(ii) $\ell(f)^*\ell(g) = \langle g, f\rangle \quad \forall f, g \in \mathcal{H}$.

The $\ell(f)$ defined on full Fock space show that this algebra is non-trivial.

Exercise 18.5.13. *Show that the representation of $\mathcal{C}(\mathcal{H})$ on full Fock space is faithful.*

This means that there is a C^*-norm on $\mathcal{C}(\mathcal{H})$ so we may consider it as a C* algebra.

Exercise 18.5.14. *If $\xi_1, \xi_2, ..., \xi_n$ are orthogonal unit vectors then $\ell(\xi_i)$ are isometries with orthogonal ranges, and the projection*

$$\sum_{i=1}^{n} \ell(\xi_i)\ell(\xi_i)^*$$

depends only on the space spanned by $\xi_1, \xi_2, ..., \xi_n$.

If $\mathcal{H}$ is finite dimensional and ξ_i is an orthonormal basis we see that the projection $p = 1 - \sum_{i=1}^n \ell(\xi_i)\ell(\xi_i)^*$ doesn't depend on anything. We may take the quotient C* algebra by the two sided ideal generated by this projection. This quotient is THE Cuntz algebra discovered by Cuntz in []. Note that in the representation on full Fock space p is the projection onto the vacuum that we used to prove irreducibility.

The case $\dim \mathcal{H} = 1$ is already interesting. The full Fock space is $\ell^2(\mathbb{N})$ and if ξ is a unit vector, $\ell(\xi)$ is the unilateral shift. $\mathcal{C}(\mathcal{H})$ in this case is known as the Toeplitz algebra and there is an exact sequence $0 \to k(\ell^2(\mathbb{N})) \mapsto \mathcal{C}(\mathcal{H}) \mapsto C(S^1)$ where $k(\ell^2(\mathbb{N}))$ is the ideal generated by $1 - \ell(\xi)\ell(\xi)^*$ which is the compact operators.

If $\dim \mathcal{H} > 1$ it is known that the Cuntz algebra is simple ([]).

We refer to [] for a development of the notion of quasi-free states on the extended Cuntz algebra. Most important is of course the vacuum state $\phi = \omega_\Omega$. It is obvious that $\mathcal{C}(\mathcal{H})$ is spanned by products of the form $\ell(f_1)\ell(f_2)\cdots\ell(f_m)\ell(g_1)^*\cdots\ell(g_n)^*$ and the vacuum expectation value of this word is 0 unless $m = n = 0$.

Given a subspace V of $\mathcal{H}$, $\mathcal{C}(V)$ is naturally included in $\mathcal{C}(\mathcal{H})$.

Definition 18.5.15. *Let $\ell(V)''$ be the von Neumann algebra generated by $\mathcal{C}(V)$ on $\mathcal{T}(\mathcal{H})$.*

Proposition 18.5.16. *Let $x \in \ell(V)''$ be such that $\phi(x) = 0$. Then there is a sequence x_i with $||x_i|| \leq ||x||$ of linear combinations of products of the form $\ell(f_1)\ell(f_2)\cdots\ell(f_m)\ell(g_1)^*\cdots\ell(g_n)^*$ (with m or n different from zero) such that x_i tends strongly to x.*

Proof. Use Kaplansky density to get x_i's in $\mathcal{C}(V)$ then subtract $\phi(x_i)$ times the identity. Since ϕ is continuous the correction tends to zero. □

Lemma 18.5.17. *The state ϕ has the following "freeness" property:*
let V_1 and V_2 be orthogonal subspaces of $\mathcal{H}$ and suppose $x_1x_2\cdots x_n$ is a product in $\ell(\mathcal{H})''$ such that
(i) $\phi(x_i) = 0 \quad \forall i$
(ii) Each x_i is in $\ell(V_1)''$ or $\ell(V_2)''$ and $x_i \in \ell(V_1)'' \iff x_{i\pm1} \in \ell(V_2)''$, then

$$\phi(x_1x_2\cdots x_n) = 0$$

.

Proof. Applying the previous proposition we can work in the $\mathcal{C}(V)$'s where the result is obvious from orthogonality. □

Observe that the result works just as well for any family of mutually orthogonal subspaces and appropriate words. Note that the "free" terminology comes from $vN(F_n)$ where the algebras generated by the generators of F_n have this property with ϕ replaced by the trace (by essentially the same reasoning).

Definition 18.5.18. *If A is a complex unital *-algebra with a state ϕ, two unital *-subalgebras A_1 and A_2 will be called* ϕ-free *if*
$\phi(x_1 x_2 \cdots x_n) = 0$ whenever $x_1 x_2 \cdots x_n$ is a product in A such that
(i) $\phi(x_i) = 0 \quad \forall i$
(ii) Each x_i is in A_1 or A_2 and $x_i \in A_1 \iff x_{i\pm 1} \in A_2$.

make sure defintion of state applies to a general *-algebra

The analogue of the Clifford algebra generators would be $c(f) = \ell(f) + \ell(f)^*$. Taking commutators reveals nothing interesting but considering $\mathcal{C}(\mathcal{H})$ on full Fock space where we have the right creation operators and we may form $d(f) = r(f) + r(f)^*$.

Proposition 18.5.19. $[c(f), d(f)] = \langle g, f\rangle - \langle f, g\rangle$

Proof. See 18.1.5 □

We see that $c(f)$ and $d(f)$ commute if $\langle f, g\rangle$ is real.

Definition 18.5.20. *A real subspace of $\mathcal{H}$ on which $\langle , \rangle$ is real will be called isotropic A* real structure *on $\mathcal{H}$ is one of the following equivalent notinons.*
(i) An antilinear involution σ on $\mathcal{H}$.
(ii) An isotropic subspace V of $\mathcal{H}$ with $\mathcal{H} = V + iV$.

The subspace V is the fixed points for the involution σ.

Definition 18.5.21. *If V is an isotropic subspace of $\mathcal{H}$, call $c(V)$ the von Neumann algebra generated by the $c(f)$ for $f \in V$ on $\mathcal{T}(\mathcal{H})$.*

Lemma 18.5.22. *If V is an isotropic subspace of $\mathcal{H}$ then ϕ is a trace on $c(V)$.*

Proof. By continuity it suffices to show that $\phi(wc(f)) = \phi(c(f)w)$ for all $f \in V$ any word w on the $c(g)$'s. But

$$\begin{aligned}
\langle wc(f)\Omega, \Omega\rangle &= \langle wf, \Omega\rangle && (18.1)\\
&= \langle wd(f)\Omega, \Omega\rangle && (18.2)\\
&= \langle d(f)w\Omega, \Omega\rangle && (18.3)\\
&= \langle w\Omega, d(f)\Omega\rangle && (18.4)\\
&= \langle w\Omega, c(f)\Omega\rangle && (18.5)\\
&= \langle c(f)w\Omega, \Omega\rangle && (18.6)
\end{aligned}$$

□

We will write tr for the restriction of ϕ to $c(V)$.

Lemma 18.5.23. *If V is a real structure on $\mathcal{H}$, Ω is cyclic and separating for $c(V)$.*

Proof. By symmetry with the $d(f)$'s it suffices to prove that Ω is cyclic for $c(V)$. By induction on n suppose $c(V)\Omega$ contains $\oplus_{i=0}^{n} \otimes^i \mathcal{H}$. Then for $v \in \otimes^n \mathcal{H}$, $c(f)v = f \otimes v + x$ with $x \in \otimes^{n-1}\mathcal{H}$. Hence $c(V)\omega$ contains $f \otimes (\otimes^n \mathcal{H})$ and since $\mathcal{H} = V + iV$ we are done. □

We see that $c(V)$ is a finite von Neumann algebra in standard form on $\mathcal{T}(\mathcal{H})$. We will see that for $\dim h > 1$ it is a type II_1 factor by showing it is isomorphic to $vN(F_n)$ where $n = \dim \mathcal{H}$, but let us begin by understanding the one dimensional case. Any unit vector ξ spans a real structure and $\ell(\xi)$ is unitarily equivalent to the unilateral shift so that $c(\xi)$ is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \cdots \end{pmatrix}$$

Lemma 18.5.24. *$c(\xi)$ has no eigenvalues.*

Proof. If the eigenvalue were λ then it would have to be real. Let the eigenvector be (x_n) with $n \geq 0$. $\lambda = 0$ is easily excluded so $x_{n+1} = \lambda x_n - x_{n-1}$ for $n \geq 1$ and $x_1 = \lambda x_0$. Thus $x_n = A\sigma^n + B\sigma^{-n}$ with both A and B different from 0. So (x_n) is not square summable. □

Although this lemma is enough to obtain our type II_1 factor result, let us complete the spectral analysis of $c(\xi)$ by obtaining the moments, i.e. the traces or vacuum expectation values of $c(\xi)^n$ for $n \geq 0$. Our method will be a bit long-winded but adapted to further calculations.

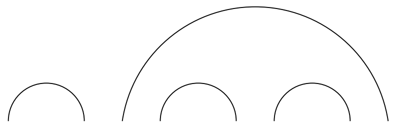
Lemma 18.5.25. *We have*

$$tr(c(\xi)^n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{m+1}\binom{2m}{m} & \text{if } n = 2m \end{cases}$$

Proof. Let $x = c(\xi)$. Then we want to calculate

$$\langle (x + x^*)(x + x^*) \cdots (x + x^*)\Omega, \Omega \rangle.$$

That this is zero for odd n is obvious, so put $n = 2m$. Expand the product into 2^n terms, each a word on x and x^*. We want to enumerate those which give a non-zero contribution to trace. There must be as many x's as x^*'s and the word must end in x. We proceed to reduce the word by the following algorighm: the last occurrence of x^* is followed by an x so use $x^*x = 1$ to eliminate the pair. The new word must also end in x so continue until only $\langle \Omega, \Omega \rangle$ remains. We may record the sequence of eliminations of (x^*, x) pairs by pairing them as indicated below for a typical word:



$x^* \quad x \quad x \quad x^* \quad x \quad x^* \quad x \quad x.$

The diagram above the word is known as a Temperley-Lieb diagram or non-crossing pairing or planar pairing. It consists of m smooth non-intersecting arcs joining the letters in the word. Thus for every such picture up to isotopy there is a contribution of 1 to the trace. It remains only to count such Temperley-Lieb diagrams. Let t_n be the number of such diagrams, with t_0 set equal to 1. Then by considering the letter to which the first letter of the word is connected, it is obvious that

$$t_{n+1} = \sum_{j=0}^{n} t_j t_{n-j} \text{ for } n \geq 0.$$

Multiplying both sides by z^{n+1} and summing over n we get

$$\Phi(z) - 1 = z\Phi(z)^2$$

where $\Phi(z) = \sum_{n=0}^{\infty}$ is the generating function for the t_n. So

$$\Phi(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

and if we expand using the binomial formula we get the answer. □

Corollary 18.5.26. *For* $-2 \leq x \leq 2$ *let* $d\mu = \frac{1}{2\pi}\sqrt{4 - x^2}dx$*. Then there is a trace preserving isomorphism of* $c(\xi)''$ *onto* $L^\infty([-2, 2], d\mu)$ *sending* $c(\xi)$ *onto the operator of multiplication by* x*.*

Proof. By7.1.9 it suffices to prove that

$$\frac{1}{2\pi}\int_{-2}^{2} x^n \sqrt{4 - x^2}dx = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{m+1}\binom{2m}{m} & \text{if } n = 2m \end{cases}$$

We leave this as an exercise. □

Now return to showing that $c(V)'' \cong vN(F_n)$ for $n = \dim \mathcal{H}$. We will do this when $n = 2$, leaving the general case as a straightforward generalisation. So let $\mathcal{H}$ be a two dimensional complex vector space with real structure V and let V_1 and V_2 be the subspaces of V spanned by orthonormal vectors f_1 and f_2 respectively. Then by lemma 18.5.17 we see that $c(V)$ is generated by two abelian subalgebras $c(V_1)$ and $c(V_2)$ with the property that $tr(x_1 x_2 \cdots x_n) = 0$ whenever $tr(x_i) = 0 \quad \forall i$ and the x_i are in $c(V_1)$ or $c(V_2)$ depending only on i mod 2. But then if $w = x_1 x_2 \cdots x_n$ is any such product without imposing $tr(x_i) = 0$ we may in a universal way calculate the trace of w by writing $x_i = (x_i - tr(x_i)) + tr(x_i)$. The result depends only on the traces of the x_i. So if M is any other finite von Neumann algebra with faithful normal trace tr generated by two abelian subalgebras A_1 and A_2 having the same property, we can construct an isomorphism between M and $c(V)$ as soon as we are given tr-preserving isomorphisms from A_1 to $c(V_1)$, and A_2 to $c(V_2)$ respectively.

Let us record this more formally.

Theorem 18.5.27. *Let (A, A_1, A_2, ϕ) and (B, B_1, B_2, ψ) be algebras and states as in definition 18.5.18, with A_1 and A_2 free with respect to ϕ and B_1 and B_2 free with respect to ψ. Suppose θ_i are unital *-isomorphisms from A_i to B_i for $i = 1, 2$, taking ϕ to ψ. Then there is a unique *-isomorphism from the algebra generated by A_1 and A_2 onto the algebra generated by B_1 and B_2 extending θ_1 and θ_2.*

Proof. By faithfulness it suffices to show that

$$\phi(y_1 y_2 \cdots y_n) = \psi(\theta(y_1)\theta(y_2) \cdots \theta(y_n))$$

whenever each y_i is in either A_1 or A_2 and θ is θ_1 or θ_2 accordingly. We will prove this assertion by induction on n. We may clearly assume successive y_i's belong to different A_i's since otherwise we can reduce the length of the word using the properties of the θ_i and apply the inductive hypothesis. But then write $x_i = y_i - \phi(y_i)$ so that $y_i = \phi(y_i) + x_i$. Expanding $(\phi(y_1) + x_1)(\phi(y_2) + x_2) \cdots (\phi(y_n) + x_n)$ we see $x_1 x_2 \cdots x_n$ plus a linear combination of words of length less than n with coefficients the same as those expanding $(\psi(\theta(y_1)) + \theta(x_1))(\psi(\theta(y_2)) + \theta(x_2)) \cdots (\psi(\theta(y_n)) + \theta(x_n))$ in the same way. The freeness condition and the inductive hypothesis imply the desired equality. □

Corollary 18.5.28. *Let $\mathcal{H}$ be a Hilbert space of dimension n with complex structure V. Then $c(V)'' \cong vN(F_n)$.*

Proof. If F_n is free on generators a_i and x_i is an orthonormal basis in V for $\mathcal{H}$, then by 18.5.26,both $\{u_{a_i}\}''$ and $c(\mathbb{R}x_i)$ are L^∞ of a standard atomless probability space so there are trace preserving isomorphisms between them. We are done by 7.1.9 and the previous theorem (with 2 replaced by n).

□

We can generalise 18.5.25 immediately to $\dim \mathcal{H} > 1$ as follows.

Proposition 18.5.29. *Let $f_1, f_2, ..., f_k$ be vectors in $\mathcal{H}$. Then*

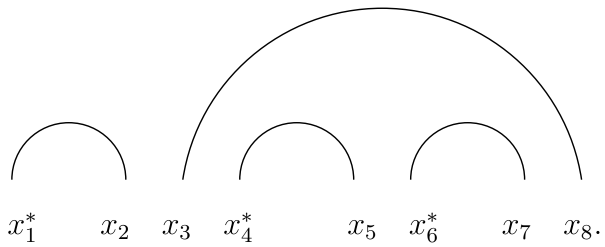
$$\langle c(f_1)c(f_2)...c(f_k)\Omega, \Omega\rangle = \sum \prod_{i,\sigma(i)} \langle f_i, f_{\sigma(i)}\rangle$$

where the sum is over all planar pairings σ of $(1, 2, 3, \cdots, k)$, with $i < \sigma(i)$.

Proof. The same argument as in 18.5.25 applies. □

Remark 18.5.30. *We may form the *-algebra $\mathbb{C}\langle X_1, X_2, \cdots X_n\rangle$ of polynomials in n non-commuting self-adjoint variables. The previous work may be considered as defining a trace on this algebra by sending X_i to $c(\xi_i)$ for an orthonormal basis $\{\xi_i\}$ of V.*

Thus the trace of a word $x_1x_2x_3\cdots x_k$, where each of the x_i is one of the X_i is the number of Temperley Lieb diagrams as below for which $x_j = x_j$ if they are joined by a curve in the diagram:



We call this trace the *Voiculescu trace* on $\mathbb{C}\langle X_1, X_2, \cdots X_n\rangle$. An explicit formula like that of 18.5.25 is not so clear and it can be difficult to work with a scalar product for which the words are not orthogonal. This can be corrected by using the obvious orthonormal basis of Fock space as tensor products of the ξ_i. Multiplication in this basis is more complicated but not much more so:

Exercise 18.5.31. *Define multiplication on $\mathbb{C}\langle X_1, X_2, \cdots X_n\rangle$ as follows: Let $x_1x_2\cdots x_p$ and $y_1y_2...y_q$ be words on $X_1, X_2, \cdots X_n$. Then*

$$x_1x_2\cdots x_p \star y_1y_2...y_q = \sum_{i=0}^{\min(p,q)} \delta_{x_p,y_1}\delta_{x_{p-1},y_2}\cdots\delta_{x_{p-i+1},y_i} x_1x_2\cdots x_{p-i}y_{i+1}y_{i+2}\cdots y_q$$

Thus for instance

$$X_1^2X_2X_3 \star X_3X_2X_1X_2 = X_1^2X_2X_3^2X_2X_1X_2 + X_1^2X_2^2X_1X_2 + X_1^3X_2 + X_1X_2$$

We would like to show how the Voiculescu trace arises in the study of large random matrices. For this we will use Wick's theorem concerning jointly Gaussian random variables. A *complex* (centred) Gaussian random variable is a sum $A + iB$ of two independent identically distributed real centred Gaussian random variables. The variance of $A + iB$ is $\sqrt{E(A^2) + E(B^2)}$, and $E((A + iB)^2) = 0$. Suppose $Z_1, Z_2 \cdots Z_n$ are complex centred jointly Gaussian random variables with $E(Z_i Z_j) = a_{ij}$.

Theorem 18.5.32.

$$E(Z_1 Z_2 \cdots Z_n) = \sum_{\sigma} \prod_{i<\sigma(i)} a_{i\sigma(i)}$$

where the sum is over all *pairings* σ *of* $\{1, 2, \cdots n\}$.

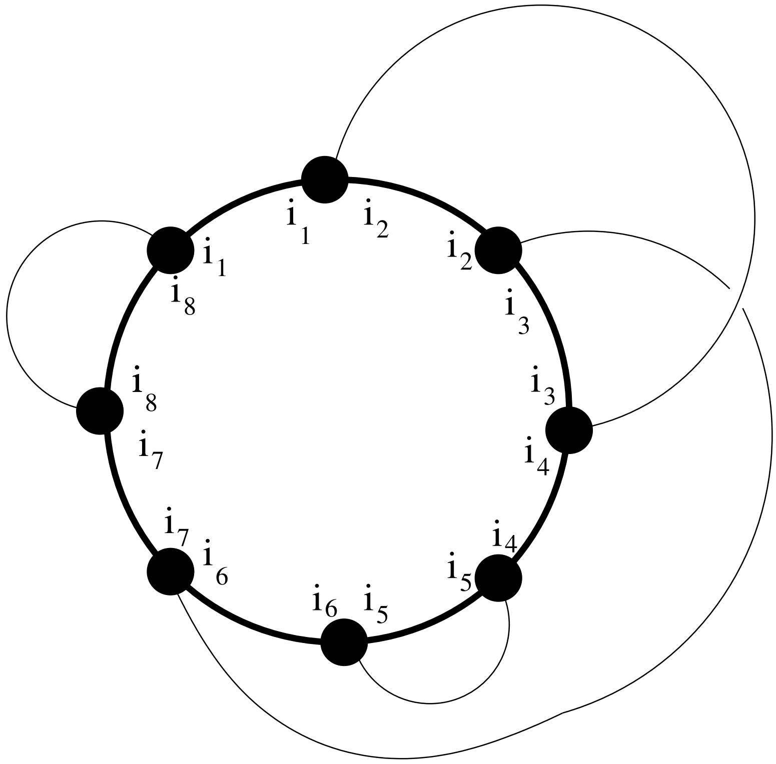
Now let $X = X_{ij}$ be a self-adjoint $N \times N$ random matrix. This means that the X_{ij} are jointly Gaussian complex random variables with

$$X_{ij} = \overline{X_{ji}} \text{ for } i \neq j \text{ and } X_{ii} \text{is real,}$$

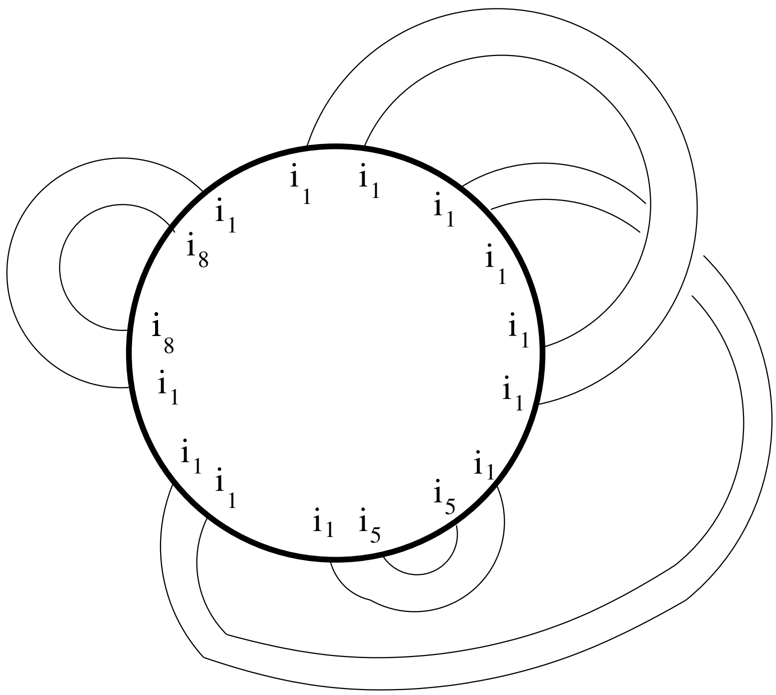
and all other matrix entries are independent. Suppose $E(|X_{ij}|^2) = d$.We want to consider $E(Trace(X^k))$. Writing this out in full we get

$$\sum_{i_1, i_2, \cdots i_k} E(X_{i_1 i_2} X_{i_2 i_3} X_{i_3 i_4} \cdots X_{i_k i_1}).$$

The individual terms in this sum can each be expanded using Wick's formula. In the figure below we have represented a typical term in the expansion, each black dot being an occurrence of X and the pairing is indicated by curves outside the circle. We have used a circle rather than a straight line segment to emphasize the cyclic aspect of the trace.



Because of the independence of the Gaussians we will only get a non-zero condition when k is even and the indices at one end of the pairing are the same as at the other end, but in the opposite order. In order to get a non-zero contribution, In the figure above this forces $i_1 = i_4$, $i_4 = i_6$, $i_6 = i_3$, $i_3 = i_2$ and $i_7 = i_1$. So in fact there are only 3 freely varying indices, i_1, i_5 and i_8 each of which gives a contribution to the total sum of d^3. We represent each such contribution below where we have thickened the curves defining the pairing into (flat) ribbons. Observe that the indices i_1, i_5 and i_8 extend to the boundary components of the surface obtained by gluing the ribbons to a central disc. There are N^3 ways to assign the indices and once assigned, each term contributes $d^{k/2}$. So the total contribution of all terms with the given pairing is $N^3 d^{k/2}$.



Now consider a general pairing and proceed in the same way. If we glue in (abstract) discs along the boundary components we get an orientable surface whose Euler characteristic is "V-E+F" which in general will be $1 - k/2 + F$ where F is the number of discs glued in, i.e. the number of freely varying indices for the given pairing. If g is the genus of the surface, we have $2-2g = F + 1 - k/2$ which gives

$$F = k/2 + 1 - 2g.$$

So the total contribution of all terms with the given pairing is $N^F d^{k/2}$. We see that if $d = \frac{1}{\sqrt{N}}$ then this contribution will be N^{1-2g} so that $\frac{1}{N}E(Trace(X^k))$ will tend, as $N \to \infty$, to the number of pairings with $g = 0$. But if the pairing is planar, obviously $g = 0$ and if $g = 0$ we know from the classification of surfaces that we get a 2-sphere, from which it is clear that the partition is planar! Hence we have shown:

$$\lim_{N\to\infty} \frac{1}{N}E(Trace(X^k)) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{1}{m+1}\binom{2m}{m} & \text{if } k = 2m \end{cases}$$

The above argument works equally well with n random $N \times N$ matrices $X_1, X_2, \cdots X_n$ each of which has entries with covariance as above and for which entries in different random matrices are independent. We see we have proved the following:

Theorem 18.5.33. *If w is a word on the random matrices $X_1, X_2, \cdots X_n$ as above then $\lim_{N\to\infty} \frac{1}{N} E(Trace(w))$ exists and is equal to the Voiculescu trace of the same word viewed as an element of $\mathbb{C}\langle X_1, X_2, \cdots X_n\rangle$.*

This result, together with 18.5.28 gave Voiculescu a remarkable new insight into the $vN(F_n)$ and he was able to prove some spectacular isomorphisms between them -[].

Chapter 19

Subfactors

19.1 Warmup. Finite Groups.

Let G be a finite group with an outer action α on the type II_1 factor M. Let $N = M^G$ be the fixed point algebra. We continue the notational conventions from chapter 11 on the crossed product.

A covariant representation of (M, α) is an action of M on some Hilbert space $\mathcal{H}$ together with a unitary representation v_g on $\mathcal{H}$ with $v_g x v_g^* = \alpha_g(x)$ for $g \in G$ and $x \in M$.

Proposition 19.1.1. *For finite groups the crossed product is universal for covariant unitary representations. In fact any covariant representation of (M, α) extends to an isomorphism from $M \rtimes G$ onto $\{M, \{v_g\}\}''$ by sending u_g to v_g.*

Proof. Define $\pi : M \rtimes G \to \{M, \{v_g\}\}''$ by $\pi(\sum_g a_g u_g) = \sum_g a_g v_g$. π is obviously ultraweakly continuous so its image is a von Neumann algebra. But that image contains M and the v_g. And a type II_1 factor is simple. □

A canonical way to obtain a covariant representation is to extend the action of G on M to $L^2(M)$. We call these unitaries w_g. We see that, for finite groups only, another model for the crossed product is the von Neumann algebra on $L^(M)$ generated by M and the w_g.

Exercise 19.1.2. $\dim_M M \rtimes G = |G|$.

Proposition 19.1.3. *The extension to $L^2(M)$ of the conditional expectation $E_N : M \to N$ is $e_N = \frac{1}{|G|}\sum_g w_g$.*

Proof. Obvious. □

Theorem 19.1.4.

$$JN'J = \{M \cup \{w_g\}\}'' = \{M \cup \{e_N\}\}''$$

Proof. Clearly J commutes with the w_g and e_N so the assertion is the same as $N' = \{M' \cup \{w_g\}\}'' = \{M' \cup \{e_N\}\}''$. Both M' and the w_g's are in N' so it suffices to prove that $N' \subseteq \{M' \cup \{e_N\}\}''$ or equivalently $\{M' \cup \{e_N\}\}' \subseteq N$ which follows from the assertion:

$$x \in M \text{ and } [x, e_N] = 0 \implies x \in N.$$

For this just evaluate xe_N and $e_N x$ on the identity inside $L^2(M)$.

□

Remark 19.1.5. *There is actually quite a bit of content here. How you would write an individual w_g for instance as an element of $\{M \cup \{e_N\}\}''$?*

Corollary 19.1.6. *If G is a finite group acting by outer automorphisms on a type II_1 factor M then M^G is a subfactor with trivial centraliser, $\dim_{M^G}(L^2(M)) = |G|$ and $(M^G)' \cap M \rtimes G = \mathbb{C}G$.*

Proof. N is the commutant of a type II_1 factor inside a type II_1 factor, hence a type II_1 factor. And $N' \cap M = (M')' \cap \{M' \cup \{v_g\}\}''$ which is the scalars by 19.1.1 and 11.2.5. For the dimension calculation note that by 11.2.5 we obtain $M \subseteq M \rtimes G$ from any covariant representation. In particular we can start with the crossed product on its own L^2 space and reduce by a projection of trace $|G|^{-1}$ in its commutant. Thus by the formulae governing the behaviour of $\dim_M$, $\dim_{\{M,\{w_g\}\}''} L^2(M) = |G|^{-1}$ and the result follows from 19.1.4. The last assertion is a trivial caclulation.

□

Exercise 19.1.7. *If α is an outer action of the finite group G on the type II_1 factor M and $\xi : G \to \mathbb{T}$ is a one dimensional character, show there is a unitary $u \in M$ with*

$$\alpha_g(u) = \xi(g)u \quad \forall g \in G$$

.

Hint: try a 2x2 matrix argument, changing the action $\alpha \otimes 1$ by Adv_g, v_g being the unitary $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \xi(g) \end{smallmatrix}\right)$.

The group $\hat{G}$ of all 1-dimensional characters $\xi : G \to \mathbb{T}$ acts on $M \rtimes G$ via the formula

$$\hat{\alpha}_\xi(\sum_g a_g u_g) = \sum_g \xi(g) a_g u_g$$

This is called the dual action.

Exercise 19.1.8. *Show that the dual action (even for infinite groups G) is outer.*

If G is abelian one may form the crossed product

$$(M \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$$

Exercise 19.1.9. *Show that if G is finite, the second dual action of G on $(M \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ is conjugate to the "stabilised" action*

$$\alpha \otimes Ad\ell_g$$

on $M \otimes \mathcal{B}(L^2(G))$ (where ℓ_g is the left regular representation).

The result of the previous exercise remains true for locally compact abelian groups and motivates an alternative *definition* of the crossed product as the fixed points for the stabilised action.

19.2 Index.

Inspired by the above and 10.2.2 we make the following:

Definition 19.2.1. *If $N \subseteq M$ are II_1 factors, the* index $[M : N]$ *of N in M is the real number* $\dim_N L^2(M)$.

Exercise 19.2.2. *Show that $[M : N] = 1$ implies $N = M$.*

Proposition 19.2.3. *(i) If M acts on $\mathcal{H}$ so that $\dim_N \mathcal{H} < \infty$ then*

$$[M : N] = \frac{\dim_N \mathcal{H}}{\dim_M \mathcal{H}}.$$

(ii) If $[M : N] < \infty$ and p is a projection in $N' \cap M$ then set $[M : N]_p = [pMp : pN]$, then

$$[M : N]_p = tr_{N'}(p) tr_M(p)[M : N].$$

(for any action of M on $\mathcal{H}$ for which $N'$is a type II_1 factor.) (iii) If $\{p\}$ is a partition of unity in $N' \cap M$ then

$$[M : N] = \sum_p \frac{[M : N]_p}{tr(p)}.$$

(iv) If $N \subseteq P \subseteq M$ are type II_1 factors then

$$[M : N] = [M : P][P : Q].$$

(v) If M acts on $\mathcal{H}$ such that $\dim_N \mathcal{H} < \infty$ then

$$[M : N] = [N' : M']$$

Proof. (i) Certainly M' (on $\mathcal{H}$) is a type II_1 factor since N' is and taking the direct sum of finitely many copies of $\mathcal{H}$ will not change the ratio $\frac{\dim_N \mathcal{H}}{\dim_M \mathcal{H}}$. So we may assume $\dim_M \mathcal{H} \geq 1$ which means there is a projection p in M' with $p\mathcal{H} \cong L^2(M)$ as an M module. But the trace of this p in N' is the same as the trace in M' by uniqueness of the trace. Hence by the properties of the coupling constant, $\frac{\dim_N \mathcal{H}}{\dim_M \mathcal{H}}$ does not change under reduction by this p.
(ii) This follows immediately from (i) and properties of the coupling constant.
(iii) Just sum $\frac{[M:N]_p}{tr_M(p)}$ over p.
(iv) The only case of interest is when $[M : N] < \infty$. Then the result follows immediately from (i).
(v) Immediate from (i). □

Corollary 19.2.4. *If* $N' \cap M \neq \mathbb{C}id$ *then* $[M : N] \geq 4$.

Definition 19.2.5. *We call a subfactor* irreducible *if* $N' \cap M = \mathbb{C}id$.

Definition 19.2.6. *A subfactor* $N \subseteq M$ *is called* locally trivial *if* $[M : N]_p = 1$ *for any minimal projection in* $N' \cap M$.

Exercise 19.2.7. *Show that* $\dim(N' \cap M) \leq [M : N]$.

Here is a list of what might be called the "classical" subfactors- ones whose existence owes nothing to the dedicated development of subfactor theory.

Example 19.2.8. *The trivial subfactors.*
If M is a type II_1 factor, so is $M \otimes M_k(\mathbb{C})$ for any integer $k > 0$. We can embed M in $M \otimes M_k(\mathbb{C})$ by $x \mapsto x \otimes 1$. It is clear that $L^2(M \otimes M_k(\mathbb{C}))$ is the direct sum of k^2 copies of $L^2(M)$ so $[M \otimes M_k(\mathbb{C}) : M] = k^2$.

Example 19.2.9. *Continuously varying index.*
Choose a projection of trace d in the hyperfinite type II_1 factor R. Then pRp and $(1-p)R(1-p)$ are isomorphic by hyperfiniteness so choose a von Neumann algebra isomorphism $\theta : pRp \to (1-p)R(1-p)$. Let M be R and N be the subalgebra $\{x + \theta(x) | x \in pRp\}$. It is clear that $pMp = Np$ and $(1-p)M(1-p)$ so by lemma 19.2.3,

$$[M : N] = \frac{1}{d} + \frac{1}{1-d}.$$

As d varies between 0 and 1, this index takes all real values ≥ 4.

Observe though that $N' \cap M$ contains p so the subfactor is reducible. The set of index values for irreducible subfactors of R is not understood though for other type II_1 factors it may be the interval $[4, \infty]$

Example 19.2.10. *Group-subgroup subfactors.*
If G is a discrete group acting by outer automorphisms on the type II_1 factor M, and H is a subgroup of G, it is clear that $M \otimes H$ is a subfactor of $M \otimes G$ of index $[G : H]$.

If G is finite we may consider $M^G \subseteq M^H$ which also has index $[G : H]$ by 19.1.2 and 19.2.3

Example 19.2.11. *Making the trivial non-trivial.*

Definition 19.2.12. *An action of a compact group on a factor M is called* minimal *if $(M^G)' \cap M = \mathbb{C}id$.*

If G has a minimal action α on M and ρ is an irreducible unitary representation of G on $\mathbb{C}^k$ we may take the action $\alpha \otimes Ad\rho$ on $M \otimes M_k(\mathbb{C})$. One then defines the "Wassermann subfactor"

$$(M \otimes 1)^G \subseteq (M \otimes M_k(\mathbb{C}))^G.$$

The point is that the commutant of $(M \otimes 1)^G$ in $M \otimes M_k(\mathbb{C})$ is already just $M_k(\mathbb{C})$ by minimality of the action. So the fixed points are indeed factors and the Wassermann subfactor is irreducible.

Already for finite groups this provides lots of examples. If G is infinite there is a simple way to construct minimal actions. Just take a finite dimensional unitary representation ρ and consider $\otimes_1^\infty Ad\rho$ on R. The group S_∞ is contained in the fixed points via its (inner) action permuting the tensor product factors. Moreover if we choose an orthonormal basis $\{x_i | i = 1, 2, ...k^2\}$ for $M_k(\mathbb{C})$ with $x_1 = 1$, an orthonormal basis of R is formed by tensors $\otimes_{j=1}^\infty x_{i(j)}$ indexed by functions $i : \mathbb{N} \to \{1, 2, \cdots, k^2\}$ with $i(j) = 1$ for sufficiently large j. The action of S_∞ on this basis has only one finite orbit-that of the identity. So the only fixed points on in $L^2(R)$ are the scalar multiplies of the identity.

Example 19.2.13. *Finitely generated discrete groups.*
This example shows that finite index subfactors can be infinite objects in disguise. Let $\Gamma = \langle \gamma_1, \gamma_2 \cdots \gamma_k \rangle$ be a finitely generated discrete group. We have seen that Γ can act in lots of ways, in particular outer, on type II_1 factors. Choose any action on M and for each x in M define the matrix $d(x) = x_{i,j}$ over M by

$$x_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ \gamma_i(x) & \text{if } i = j \end{cases}$$

Then consider the subfactor

$$D(M) = \{d(x)|x \in M\} \subseteq M \otimes M_k(\mathbb{C}).$$

This subfactor is locally trivial so its index is k^2 and one may think of it as a "twisted" version of the trivial subfactor of index k^2.

Exercise 19.2.14. *Show that* $\dim(D(M)' \cap M \otimes M_k(\mathbb{C})) = k$ *iff* $\gamma_i^{-1}\gamma_j$ *is outer whenever* $i \neq j$.

In fact one may easily extract the image of Γ modulo inner automorphisms from the subfactor $D(M)$.

We now want to consider an entirely arbitrary subfactor. For this the following "basic construction" is important. We have already seen its usefulness for finite group actions.

Proposition 19.2.15. *Let* $N \subseteq M$ *be a type* II_1 *factors acting on* $L^2(M)$ *and let* e_N *be the extension to* L^2 *of the trace-preserving conditional* E_N *expectation onto* N*. Then*

$$JN'J = (JNJ)' = \{M, e_N\}''.$$

Proof. Already done in 19.1.4. □

Definition 19.2.16. *The von Neumann algebra* $\langle M, e_N \rangle = \{M, e_N\}''$ *of the previous result is said to be the "basic construction" for* $N \subseteq M$.

Here are the most important facts about the basic construction. It will be convenient from now on to use τ for $[M : N]^{-1}$. Since $\langle M, e_N \rangle$ is a type II_1 factor its trace is unique and its restriction to M is the trace of M. So we just use tr for it.

Proposition 19.2.17.
(i) For $x \in M, [x, e_N] = 0$ *iff* $x \in N$.
(ii) $e_N x e_N = E_N(x) e_N$ *for* $x \in M$.
(iii) $[M : N] < \infty$ *iff* $\langle M, e_N \rangle$ *is a type* II_1 *factor, in which case*

$$[\langle M, e_N \rangle : M] = [M : N].$$

(iv) $M + Me_N M$ *is a weakly dense *-subalgebra of* $\langle M, e_N \rangle$.
(v) $e_N \langle M, e_N \rangle e_N = N e_N$
(vi) $tr(e_N) = [M : N]^{-1}$
(vii) For $x \in M$, $tr(e_N x) = \tau tr(x)$

Proof. (i) was done in 19.1.4.
(ii) is a consequence of the bimodule property of E_N on the dense subspace M of $L^2(M)$.
(iii) is immediate from proposition 19.2.15.
(iv) Closure of $M + Me_N M$ under multiplication follows from (ii). It contains M and e_N hence is dense.
(v) Follows immediately from (ii) and (iv).
(vi) Follows from (v) and the behaviour of the coupling constant under reduction by projections-note that $e_N(L^2(M)) = L^2(N)$.
(vii) $tr(xe_N) = tr(e_N xe_N) = tr(e_N xe_N) = tr(E_N(x)e_N) = \tau(E_N(x)$ where we deduce the last equality from uniqueness of the trace on the type II_1 factor N. Since the conditional expectation preserves the trace, we are done. □

From now on we will use τ for $[M : N]^{-1}$.

Corollary 19.2.18. *There is no subfactor $N \subseteq M$ with $1 < [M : N] < 2$.*

Proof. By the uniqueness of the trace we see that $tr_{N'}(e_N) = \tau$. Thus $tr_{N'}(1 - e_N) = 1 - \tau$. Hence $[(1 - e_N)\langle M, e_N\rangle(1 - e_N) : N(1 - e_N)] = (1 - \tau)^2(1/\tau)^2$ which is less than 1 if $1/2 < \tau < 1$. □

If we suppose $[M : N] < \infty$ we see we may do the basic construction for $M \subseteq \langle M, e_N\rangle$. In the type II_1 factor $\langle\langle M, e_N\rangle, e_M\rangle$ we have the two projections e_M and e_N.

Proposition 19.2.19.

$$e_M e_N e_M = \tau e_M \text{ and } e_N e_M e_N = \tau e_N$$

.

Proof. For the first relation we must show that $E_M(e_N) = \tau id$. But this is just another way of saying (vii) of 19.2.17.
To prove the second relation, by (iv) of 19.2.17 it suffices to apply each side to elements of the form $x + ye_N z \in L^2(\langle M, e_N\rangle)$ for $x, y, z \in M$. To do this note that e_N acts by left multiplication. □

Corollary 19.2.20. *If $[M : N] \neq 1$ then*

$$e_M \vee e_N = \frac{1}{1 - \tau}(e_N + e_M - e_M e_N - e_N e_M)$$

Proof. The relations show that e_N and e_M generate a 4-dimensional non-commutative algebra. By our analysis of two projections its identity must be a multiple of $(e_M - e_N)^2$. The normalisation constant can be obtained by evaluating the trace. □

Note that the special case $e_N \vee e_M = 1$ (which is equivalent to $\tau = 1/2$ or index 2) means that e_N and e_M satisfy an algebraic relation.

Exercise 19.2.21. *Use this relation to prove that, in index two, $\langle\langle M, e_N\rangle, e_M\rangle$ is the crossed product of $\langle M, e_N\rangle$ by an outer action of $\mathbb{Z}/2\mathbb{Z}$. Use duality to deduce Goldman's theorem ([]): a subfactor of index 2 is the fixed point algebra for an outer $\mathbb{Z}/2\mathbb{Z}$ action.*

Let ϕ be the golden ratio $\frac{1+\sqrt{5}}{2}$.

Corollary 19.2.22. *There is no subfactor $N \subseteq M$ with $2 < [M : N] < \phi^2$.*

Proof. We see that e_N and e_M are equivalent in the algebra they generate so their traces are equal wherever they are. Thus $tr_{\langle\langle M,e_N\rangle,e_M\rangle}(e_N \vee_M) = tr_{N'}(e_N \vee_M) = 2\tau$ and

$$[(1 - e_N \vee e_M)\langle\langle M, e_N\rangle, e_M\rangle(1 - e_N \vee e_M) : (1 - e_N \vee e_M)N] = (1 - 2\tau)^2\tau^{-3}$$

This is less than 1 if $\phi^{-2} < \tau < 1/2$. □

If we did yet another basic construction in the same way and calculated the trace of the supremum of the three conditional expectations we would conclude that there is no subfactor with index between ϕ^2 and 3. But it is high time to systematise the process.

19.3 The tower of type II_1 factors and the e_i's.

Definition 19.3.1. *Let $N \subseteq M$ be a subfactor of finite index τ^{-1}. Set $M_0 = N, M_1 = M$ and define inductively the tower of type II_1 factors*

$$M_{i+1} = \langle M_i, e_{M_{i-1}}\rangle.$$

Set $e_i = e_{M_{i-1}}$ for $i = 1, 2, 3, \cdots$.

Proposition 19.3.2. *The e_i's enjoy the following properties.*
(i) $e_i^2 = e_i^* = e_i$
(ii) $e_ie_j = e_je_i$ if $|i - j| \geq 2$
(iii) $e_ie_{i\pm1}e_i = \tau e_i$
(iv) $tr(we_{i+1}) = \tau tr(w)$ for any word w on $\{e_1, e_2, \cdots e_i\}$.

Proof. These are all trivial consequences of the 19.2.17 and 19.2.20. Note that the trace in (iv) is unambiguous by uniqueness of the trace on a type II_1 factor. □

The relations of proposition 19.3.2 were discovered, albeit in a slightly disguised form, in statistical mechanics in [], and were presented in almost the above form in [] although property (iv) does not appear. With a beautiful insight they were given a diagrammatic form in []. They are now universally known, in whatever form, as the Temperley-Lieb relations or the Temperley-Lieb algebra. We present Kauffman's diagrammatics in the appendix A.

There is a lot of interesting combinatorics going with the Temperley-Lieb algebra but we want to get directly to the results on index for subfactors. Here are some exercises to familiarise the reader with these relations.

Exercise 19.3.3. *Any word w on $e_1, e_2, \cdots e_n$ which is reduced in the obvious sense with respect to the relations 19.3.2 contains e_n (and e_1) at most once.*

Exercise 19.3.4. *The dimension of the algebra generated by 1 and $e_1, e_2, \cdots e_n$ is at most*

$$\frac{1}{n+2}\binom{2n+2}{n+1}$$

(This exercise is the first hint that there might be some connection between subfactors and random matrices-see 18.5.25.)

19.4 Index restrictions

It is clear from the restrictions we have obtained so far that we should be interested in the trace of the sup of the first n e_i's.

Definition 19.4.1. *Let $P_n(\tau)$ be the polynomials defined by $P_0 = 1, P_1 = 1$ and*

$$P_{n+1} = P_n - \tau P_{n-1}$$

Thus $P_2 = 1 - \tau = tr(1 - e_1)$, $P_3 = 1 - 2\tau = tr(1 - e_1 \vee e_2)$ and $P_4(\tau) = 1 - 3\tau + \tau^2$.

Exercise 19.4.2. *Define q by $\tau^{-1/2} = q + q^{-1}$. Show that $P_n(\tau)$ is essentially the "quantum integer" $[n+1]_q = \dfrac{q^{n+1} - q^{-n-1}}{q - q^{-1}}$, to be precise*

$$P_n(\tau) = \frac{[n+1]_q}{([2]_q)^n}$$

Definition 19.4.3. *Put $f_0 = 1$ and for each $n = 1, 2, 3, \cdots$ let*

$$f_n = 1 - e_1 \vee e_2 \vee \cdots \vee e_n$$

.

Note that the f_n are *decreasing*.

Theorem 19.4.4. *If $f_n \neq 0$ then*

$$tr(f_{n+1}) = P_{n+2}(\tau)$$

Proof. Observe that the assertion is true for $n = 0$. Now suppose it is true up to n. For convenience set $s_n = 1 - f_n = e_1 \vee e_2 \vee e_3 \cdots \vee e_n$. We want to calculate $tr(s_n \vee e_{n+1})$ and we know $tr(s_n)$ and $tr(e_n)$. So it suffices to calculate $tr(s_n \wedge e_{n+1})$. To do this note that $e_{n+1}s_n e_{n+1} = E_{M_n}(s_n)e_{n+1}$ by 19.2.17, and $E_{M_n}(s_n)$ is in the algebra generated by $\{1, e_1, e_2, \cdots e_{n-1}\}$ by 19.3.3 and (iv) of 19.3.2. But by the bimodule property for a condional expectation $e_i E_{M_n}(s_n) = E_{M_n}(s_n)e_i = e_i$ for $i \leq n-1$. So $s_n E_{M_n}(s_n)$ is the identity for the algebra generated by $\{e_1, e_2, \cdots e_{n-1}\}$ and $E_{M_n}(s_{n-1}) = s_{n-1} + (1 - s_{n-1})E_{M_n}(s_n)$. However $1 - s_{n-1}$ is a minimal and central projection in this algebra so

$$E_{M_n}(s_n) = s_n + \lambda(1 - s_n)$$

for some constant λ. Obviously $0 \leq \lambda \leq 1$ because conditional expectations do not increase norms. But if λ were equal to 1, we would have $E_{M_n}(s_n) = 1$ which implies $s_n = 1$, i.e. $f_n = 0$ by faithfulness of the conditional expectation. Thus $\lambda < 1$ and taking the limit as $k \to \infty$ of $(e_{n+1}s_n e_{n+1})^k$,

$$e_{n+1} \wedge s_n = e_{n+1}s_{n-1}$$

Taking the trace we see that $tr(e_{n+1} \wedge s_n) = \tau tr(s_{n-1})$.

Finally $tr(s_{n+1}) = tr(s_n) + \tau - \tau tr(s_{n-1})$ and $tr(f_{n+1}) = tr(f_n) - \tau tr(f_{n-1})$. By induction and the definition of the P_n we are through. □

The formula of the next theorem is due to Wenzl in [] which contains complete information about families of projections on Hilbert space satisfying (i),(i) and (i)

Theorem 19.4.5. *If $f_n \neq 0$ then*

$$f_{n+1} = f_n - \frac{P_n(\tau)}{P_{n+1}(\tau)} f_n e_{n+1} f_n$$

Proof. It is easy to check for $n = 1$ and $n = 2$ for good measure.

So suppose $f_n \neq 0$. Then by the previous result $P_{n+1}(\tau) \neq 0$ and we may consider the element $x = f_n - \dfrac{P_n(\tau)}{P_{n+1}(\tau)} f_n e_{n+1} f_n$. Obviously $e_i x = 0 = x e_i$

for $i \leq n$ and $e_{n+1}x = e_{n+1}f_n - \frac{P_n(\tau)}{P_{n+1}(\tau)} E_{M_n}(f_n)e_{n+1}f_n$. By induction and the definition of P_n,

$$E_{M_n}(f_n) = \frac{P_{n+1}(\tau)}{P_n(\tau)} f_{n-1}$$

Since the f_n are decreasing we get $e_{n+1}x = 0 = xe_{n+1}$ which means x is a (possibly zero) multiple of f_{n+1}. But the trace of x is $P_{n+2}(\tau)$ so we are done by the previous theorem. □

Theorem 19.4.6. *Let $N \subseteq M$ be type II_1 factors. Then if $[M : N] < 4$ it is $4\cos^2 \pi/n$ for some $n = 3, 4, 5, \ldots$.*

Proof. Observe that $P_n(0) = 1$ for all n. If we put $q = e^{i\theta}$ in 19.4.2 we see that $\tau^{-1} = 4\cos^2\theta$ and

$$P_{n-1}(\tau) = \frac{\sin n\theta}{2^{n-1}\sin\theta(\cos\theta)^{n-1}}$$

This is zero for q a $2n$th. root of unity (except $q = 1$) and the one with largest cosine is $\theta = \pi/n$. Thus the smallest real zero of P_n is $\frac{1}{4\cos^2\pi/(n+1)}$. Moreover $\pi/(n+1) < \pi/n < 2\pi/(n+1)$. So $P_{n+1}(\tau) < 0$ between $\frac{1}{4\cos^2\pi/(n+1)}$ and $\frac{1}{4\cos^2\pi/n}$ while $P_k(\tau) > 0$ for $k \leq n$ and τ in the same interval. Thus if τ is strictly between $\frac{1}{4\cos^2\pi/(n+1)}$ and $\frac{1}{4\cos^2\pi/n}$ we conclude that $f_n > 0$ and $tr(f_{n+1}) < 0$ which is impossible.

□

19.5 Finite dimensions

It is nice to have these restrictions on the values of the index but at this stage the only values we know between 1 and 4 are 2 and 3. We will show that all the values of theorem 19.4.6 actually occur. We will use a kind of "bootstrap" method. If the value of the index exists then there are e_i's satsfying the relations of 19.3.2. But then we may consider the von Neumann algebra in the tower M_n generated by $\{e_1, e_2, e_3, \cdots\}$. We will show that this is a factor. Moreover we will see that the subfactor generated by $\{e_2, e_3, \cdots\}$ will be seen to have index τ^{-1}. But this presupposes the existence of the subfactor! For $\tau < 1/4$ we can get the e_i's from the tower obtained from example 19.2.9. For $\tau \geq 1/4$ we will be able to construct a tower coming from inclusions $A \subseteq B$ of *finite dimensional* von Neumann algebras which gets around the problem. For this we obviously need to know how the basic construction works in finite dimensions.

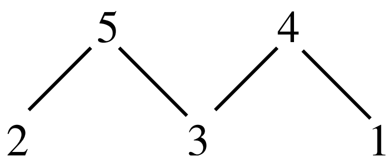
Recall from 4.4.3 that a unital inclusion $A \subseteq B$ of finite dimensional von Neumann algebras is given by a vector $\vec{v}$ whose entries are labelled by the minimal central projections of A and a matrix $\Lambda = \lambda_{p,q}$ where q runs over the minimal central projections in A an p over the minimal central projections in B. $\Lambda\vec{v}$ is then vector whose entries are the ranks of the simple components of B. If $e \leq p$ is minimal in A and $f \leq q$ is minimal in B then ef is a projection of rank $\lambda_{p,q}$in the factor qB.

Definition 19.5.1. *We call $\vec{v}$ as above the* dimension vector *of a finite dimensional von Neumann algebra and the matrix Λ the* inclusion matrix*. We will write $\vec{v}_A$ and Λ_A^B if we need to specify which algebras we are talking about. We will say the inclusion is* connected *if $Z(A) \cap Z(B) = \mathbb{C}id$, which can be recognised by connectedness of the obvious bipartite graph associated to the inclusion matrix.*

Thus in full:

$$\Lambda_A^B \vec{v}_A = \vec{v}_B$$

This information is conveniently recorded graphically:



Here $A = M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C}$ so $\vec{v}_A = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $B = M_5(\mathbb{C}) \oplus M_4(\mathbb{C})$ so $\vec{v}_B = \left(\begin{smallmatrix} 5 \\ 4 \end{smallmatrix}\right)$. There is no "multiplicity" so minimal projections in A are sums of minimal projections in different simple components of B and the inclusion matrix is $\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)$.

Exercise 19.5.2. *If $A \subseteq B$ and $B \subseteq C$ then $\Lambda_A^C = \Lambda_B^C \Lambda_A^B$.*

This can be done by pure thought observing that Λ_A^B is just the matrix of the inclusion map from $K_0(A)$ to $K_0(B)$.

The basic construction can be performed without recourse to a trace simply by defining it as the commutant on B of the right action of A which allows us to identify its centre with that of A. But we are after the e_i's so lets use (positive) traces.

Definition 19.5.3. *If A is a finite dimensional von Neumann algebra with trace tr define the* trace vector *$\vec{tr}$ to be the row vector whose entries are* *indexed by the central projections of A and whose pth. entry is $tr(e)$, e being a minimal projection in A, $e \leq p$.*

Remark 19.5.4.
(i)A trace is clearly normalised iff $\vec{tr} \cdot \vec{v}_A = 1$.
(ii) If $A \subseteq B$ are as above and Tr is a trace on B whose restriction to A is tr then:

$$\vec{Tr}\Lambda = \vec{tr}$$

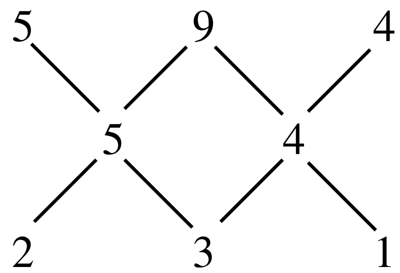
Given a (normalised) faithful trace Tr on B we may perform the basic construction $\langle B, e_A \rangle$ exactly as for type II_1 factors.

The centre of $\langle B, e_A \rangle$ can be identified with that of A by $x \mapsto JxJ$ so the inclusion matrix for $B \subseteq \langle B, e_A \rangle$ will have the same shape as the transpose of that of $A \subseteq B$.

Exercise 19.5.5. *Show that*

$$\Lambda_B^{\langle B, e_A \rangle} = (\Lambda_A^B)^t$$

Thus in the example above we would get the "Bratteli" diagram:



for the tower $A \subseteq B \subseteq \langle B, e_A \rangle$.

In the non-factor case there is no canonically defined trace on the basic construction. For obvious reasons we would like to have such a trace TR with the crucial property $TR(e_A x) = \tau Tr(x)$ for $x \in B$.

Theorem 19.5.6. *If $A \subseteq B$ is a connected inclusion with matrix Λ, there is a unique normalised trace Tr on B which extends to a trace TR on $\langle B, e_A \rangle$ such that $E_B(e_A) \in \mathbb{C}id$. $\vec{TR}\Lambda\Lambda^t = \tau^{-1}\vec{TR}$ for τ satisfying $E_B(e_A) = \tau e_A$.*

Proof. Observe that if f is a minimal projection in A then $e_A f$ is a minimal projection in $\langle B, e_A \rangle$ by (v) of 19.2.17. If p is a minimal central projection in A with $pf = f$, we want to show that $e_A f$ is under JpJ. To do this it is enough to show that $JpJe_A f \neq 0$. But applying it to the identity in B we get fp. So if Tr has an extension TR satisfying $TR(e_A x) = \tau Tr(x)$ for $x \in B$,

$TR(e_A f) = \tau Tr(f)$. This means that the trace vector $\vec{TR}$ is $\tau \vec{tr}$ where tr is the restriction of Tr to A. On the other hand by exercises 19.5.2 and 19.5.5 we have $\vec{tr} = \vec{TR}\Lambda\Lambda^t$. So $\vec{TR}$ is the suitably normalised Perron-Frobenius eigenvector for the irreducible matrix $\Lambda\Lambda^t$ with eigenvalue τ^{-1}. Hence TR is unique and so is Tr. □

Corollary 19.5.7. *If τ^{-1} is the Perron Frobenius eigenvalue for an irreducible matrix $\Lambda^t\Lambda$ for an $\mathbb{N}$-valued matrix Λ, there exists a von Neumann algebra M with faithful trace tr containing an infinite sequence of projections e_i satisfying the relations of 19.3.2.*

Proof. Choose a connected inclusion $A \subseteq B$ with matrix Λ and trace TR on $\langle B, e_A\rangle$ as above. Then if we consider the inclusion $B \subseteq \langle B, e_A\rangle$, we see that $\vec{Tr} = \vec{TR}\Lambda$ is the Perron-Frobenius eigenvector for $\Lambda\Lambda^t$ so the trace on $\langle B, e_A\rangle$ guaranteed by the previous theorem has the same value of τ and is equal to TR. We may thus iterate the basic construction always using the trace given by the theorem. To get M just use GNS on the union of the (C*-) algebras in the tower. □

Remark 19.5.8. *In fact the M constructed above is a type II_1 factor (provided $\tau \neq 1$ ....). This follows from the fact that the* only *trace on the tower is in fact the one used. See exercise 6.2.1.*

19.6 Existence of the $4\cos^2\pi/n$ subfactors.

Definition 19.6.1. *GIven a finite von Neumann algebra M with faithful normal normalised trace tr containing a sequence e_i of projections satisfying 19.3.2 we define the algebra $P = \{e_1, e_2, e_3 \cdots\}''$ and the subalgebra $Q = \{e_2, e_3, \cdots\}$.*

We will have shown the existence of subfactors of index $4\cos^2\pi/n$ for each $n = 3, 4, 5, \cdots$ if we can show:

(i) For each n there exists an $\mathbb{N}$-valued matrix Λ whose norm is $2\cos\pi/n$.

(ii) P and Q are type II_1 factors and $[P : Q] = 4\cos^2\pi/n$.

Let us begin with (i) since it is easy. Just consider the matrix which is the adjacency matrix Λ_n in the bipartite sense for the graph A_n with n vertices:

Thus for $n = 2m$ even, Λ_n is $m \times m$ and for $n = 2m+1$ it is $m \times (m+1)$. In both cases

$$\lambda_{i,j} = \begin{cases} 1 & \text{if } i = j \text{ or } j+1 \\ 0 & \text{otherwise} \end{cases}$$

Exercise 19.6.2. *Show that* $||\Lambda_n|| = 2\cos\pi/(n+1)$.

Note that these are not the only choices for Λ. If Λ is the bipartite adjacency matrix for a Coxeter-Dynkin graph of type A, D or E one has:

$$||\Lambda|| = 2\cos\pi/n \text{ where } n = \begin{cases} n+1 & \text{for } A_n \\ 2n-2 & \text{for } D_n \\ 12 & \text{for } E_6 \\ 18 & \text{for } E_7 \\ 30 & \text{for } E_8 \end{cases}$$

These are the only possibilities for $||\Lambda|| < 2$ (see []).

Now let us show that P (and hence obviously Q) is a factor.

We will need a simple lemma.

Lemma 19.6.3. *With notation as in 19.6.1, any normal trace on P is determined by its restriction to all commutative subalgebras of the form $A_{\mathcal{I}} = \{e_i | i \in \mathcal{I}\}''$ where $\mathcal{I}$ is a subset of $\mathbb{N}$ with the property that*

$$i, j \in \mathcal{I} \implies |i-j| \geq 2$$

.

Proof. If ϕ is a normal trace on P it is determined by its value on words on the e_i's. But it is a simple matter to deduce from exercise 19.3.3 that any word can be reduced after cyclic permuations to a multiple of a word in which all the indices of e_i's differ by at least two. □

Theorem 19.6.4. *Let P, M and tr be as in definition 19.6.1. Then P is a type* II_1 *factor (provided $\tau \neq 1$).*

Proof. By 19.6.3 it suffices to show that any normal normalised trace on P is equal to tr. But let ϕ be such a trace. Let $\mathcal{I}$ be as in 19.6.3. Embed $\mathcal{I}$ into an infinite set $\mathcal{J}$ with the same property. Let $i < j$ be elements of $\mathcal{J}$ with nothing in between i and j in $\mathcal{J}$. We claim the the normaliser of $A_{\mathcal{J}}$ contains a self-adjoint unitary u such that $ue_iu = e_j$ and $ue_ku = e_k$ for $k \neq i, j$. For this just consider the algebra generated by $1, e_i, e_{i+1}, e_{i+2} \cdots e_j$. The projections e_i and e_j are equivalent in this finite dimensional von Neumann algebra and it is a simple exercise to see that two equivalent projections in a matrix algebra are always conjugate under a self-adjoint unitary.

But $A_{\mathcal{J}}$ is the infinite tensor product of copies of $\mathbb{C}^2$ with product state given by $tr(e_i) = \tau$. And the normaliser contains the group S_∞ acting by permuting the tensor product components. So just as in 19.2.11, the action of S_∞ is ergodic and there is only one invariant probability measure absolutely continuous with respect to tr. Thus $tr = \phi$ on $A_{\mathcal{J}}$ and we are done. □

The last detail is to show that $Q \subseteq P$ has the right index.

Theorem 19.6.5. $[P : Q] = \tau^{-1}$.

Proof. Perform the basic construction $\langle P, e_Q\rangle$. P is spanned by words of the form ae_1b with a and b in Q. Let $R = \{e_3, e_4, \cdots\}''$. Using 19.3.2 we have $e_1(ae_1b) = E_R(a)e_1b$ and $e_1e_Qe_1(ae_1b) = \tau E_R(a)e_1b$. And easily $e_Qe_1e_Q = \tau e_Q$. Thus e_Q and e_1 are equivalent in $\langle P, e_Q\rangle$.

We conclude first that $\langle P, e_Q\rangle$ is a type II_1 factor since e_Q is a finite projection ($e_Q\langle P, e_Q\rangle e_Q = Qe_Q$), and a finite projection in a II_∞ factor cannot be in a II_1 subfactor. So $tr(e_Q) = \tau^{-1}$ since $tr(e_1) = \tau^{-1}$. □

19.7 The structure of the algebras $\mathcal{E}_n = \{e_1, e_2, \cdots e_n\}''$.

We have that $\mathcal{E}_n$ is finite dimensional but we will see that its dimension depends on τ. Clearly $\mathcal{E}_n \subseteq \mathcal{E}_{n+1}$ so there is a Bratteli diagram to compute.

Theorem 19.7.1. *"Generically", that is for* $0 < \tau \leq 1/4$*, the Bratteli diagram for the tower* $\mathcal{E}_n$ *is below:*

Where $\mathcal{E}_0 = \mathbb{C}id$ *and we have recorded the traces of minimal projections in each simple summand.*

Proof. The calculation of $\mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2$ is trivial. So proceed inductively. Then $\mathcal{E}_{n+1}$ is obtained from $\mathcal{E}_n$ by adding e_{n+1} which satisfies

$$e_{n+1}xe_{n+1} = E_{\mathcal{E}_{n-1}}(x)e_{n+1} \text{ for } x \in \mathcal{E}_n$$

□

Appendix A

Kauffman's diagrammatics for the Temperley-LIeb algebra.

Appendix B

Proof of the KMS condition.

Theorem B.0.2. *Let ϕ be a faithful normal state on a von Neumann algebra M. Then the modular group σ_t^ϕ is the unique one parameter automorphism group of M which satisfies the KMS condition for ϕ.*

Proof. Perform the GNS construction with canonical cyclic and separating vector Ω and modular operators $S = J\Delta^{1/2}$. Recall that $f(\Delta)\Omega = \Omega$ for any function of Δ with $f(1) = 1$. In particular $\phi(\sigma_t^\phi(x) = \langle(\Delta^{it}x\Delta^{-it}\Omega, \Omega\rangle$ so σ_t^ϕ preserves ϕ.

Now let us check the rest of the KMS condition. We have

$$\phi(\sigma_t^\phi(x)y) = \langle\Delta^{-it}y\Omega, x^*\Omega\rangle$$

and

$$\begin{aligned} \phi(y\sigma_t^\phi(x)) &= \langle y\sigma_t^\phi(x)\Omega, \Omega\rangle \\ &= \langle J\Delta^{1/2}\sigma_t^\phi(x^*)\Omega, J\Delta^{1/2}y\Omega\rangle \\ &= \langle\Delta^{1/2}y\Omega, \Delta^{1/2}\Delta^{it}x^*\Omega\rangle \\ &= \langle\Delta^{1/2-it}y\Omega, \Delta^{1/2}x^*\Omega\rangle \end{aligned}$$

So let $\xi = y\Omega$, $\eta = x^*\Omega$ and let p_n be the spectral projection for Δ for the interval $[1/n, n]$ so that p_n tends strongly to 1 and $\Delta^{\pm 1}$ are bounded on $p_n\mathcal{H}_\phi$. The functions

$$F_n(z) = \langle\Delta^{-iz}p_n\xi, \eta\rangle$$

are then entire and

$$\begin{aligned} |F_n(t) - \phi(\sigma_t^\phi(x)y)| &= |\langle\Delta^{-it}(1-p_n)\xi, \eta\rangle| \leq ||(1-p_n)\xi||\ ||\eta|| \\ |F_n(t+i) - \phi(y\sigma_t^\phi(x))| &= |\langle\Delta^{1/2-it}(1-p_n)\xi, \Delta^{1/2}\eta\rangle| \leq ||(1-p_n)\Delta^{1/2}\xi||\ ||\Delta^{1/2}\eta||. \end{aligned}$$

Hence the F_n are bounded and continuous on the strip $\{z : 0 < \Im m z < 1\}$ and converge uniformly on its boundary. By the Phragmen-Lindelof theorem we are done.

Now let us prove uniqueness of the modular group with the KMS condition.

Let α_t be another continous one-parameter automorphism group satisfying KMS for ϕ. The fact that α_t preserves ϕ means we can define a strongly continous one-parameter unitary group $t \mapsto u_t$ by $u_t x\Omega = \alpha_t(x)\Omega$. By Stone's theorem it is of the form $t \mapsto D^{it}$ for some non-singular positive self-adjoint operator A. The goal is to prove that $D = \Delta$. As a first step we construct a dense set of analytic vectors in $M\Omega$ by Fourier transform. Let A be the set of all operators of the form

$$\int_{-\infty}^{\infty} \hat{f}(t)\alpha_t(x)dx$$

for all C^∞ functions f of compact support on $\mathbb{R}$. The integral converges strongly so

$$f(\log(D))x\Omega = \int_{-\infty}^{\infty} \hat{f}(t)D^{it}(x\Omega)dx$$

is in $A\Omega$. Thus the spectral projections of D are in the strong closure of A and $A\Omega$ is dense. Moreover $z \mapsto D^z x\Omega$ is analytic for $x \in A$ since $x\Omega$ is in the spectral subspace of A for a bounded interval. Also $A\Omega$ is invariant under D^z by the functional calculus. To compare with ϕ define, for x and y in A, the entire function

$$F_1(z) = \langle D^{-iz}y\Omega, x^*\Omega\rangle.$$

Let F be the function, analytic inside the strip and continuous and bounded on it, guaranteed for x and y by the KMS condition. Then if we define $G(z)$ for $-1 \leq \Im m z \leq 1$ by

$$G(z) = \begin{cases} F(z) - F_1(z) & \text{if } \Im m z \geq 0; \\ \\ \overline{F(\overline{z}) - F_1(\overline{z})} & \text{if } \Im m z \leq 0. \end{cases}$$

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Since F and F_1 agree on the real line G is analytic for $-1 < \Im m z < 1$, hence equal to 0, and since both F and F_1 are continous on the strip, $\phi(y\sigma_t(x)) = F(t+i) = F_1(t+i) = \langle D^{1-it}y\Omega, x^*\Omega\rangle$. In particular putting $t = 0$ we get

$$\begin{aligned} \langle Dy\Omega, x^*\Omega\rangle &= \phi(yx) \\ &= \langle x\Omega, y^*\Omega\rangle \\ &= \langle J\Delta^{1/2}x^*\Omega, J\Delta^{1/2}y\Omega\rangle \\ &= \langle \Delta^{1/2}y\Omega, \Delta^{1/2}x^*\Omega\rangle \end{aligned}$$

So $\Delta^{1/2}y\Omega$ is in the domain of $\Delta^{1/2}$ and $\Delta y\Omega = Dy\Omega$.

Thus D and Δ agree on $A\Omega$. But multiplication by the function $e^z + 1$ is a linear isomorphism of C_c^∞ so by functional calculus $(D+1)A\Omega = A\Omega$ which is thus dense. Since $D + 1$ is invertible by spectral theory, any $(\xi, (D + 1)\xi)$ in the graph of $D+1$ can be approximated by $(A_n\Omega, (D+1)A_n\Omega)$. Thus D is essentially self-adjoint on $A\Omega$, and both Δ and D are self-adjoint extensions of the restriction of D to this domain. Thus $D = \Delta$. $\square$

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