Topics in Supersymmetric Quantum Field Theory

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Abstract

This thesis describes several new tools for analyzing supersymmetric quantum field theories, focusing on theories with four supercharges in three and four dimensions. In chapter two, we discuss supercurrents, supersymmetry multiplets that include the energy-momentum tensor. Physically, different supercurrents give rise to different brane charges in the supersymmetry algebra. They also encode different ways of placing supersymmetric field theories on a curved manifold. Under certain conditions this procedure preserves some of the supersymmetry. In chapter three, we explore these conditions for the case of four-dimensional $\mathcal{N} = 1$ theories with a $U(1)_R$ symmetry. In particular, we find that a manifold admits a single supercharge if and only if it is Hermitian.

In chapter four, we shift the focus to three-dimensional field theories. We study Chern-Simons contact terms – contact terms of conserved currents and the energy-momentum tensor, which are associated with Chern-Simons terms for background fields. While the integer parts of these contact terms are ambiguous, their fractional parts constitute new meaningful observables. In $\mathcal{N} = 2$ supersymmetric theories with a $U(1)_R$ symmetry certain Chern-Simons contact terms can lead to a novel superconformal anomaly. In chapter five, we use this understanding to elucidate the structure of the free energy F of these theories on a three sphere. In particular, we prove the F-maximization principle for $\mathcal{N} = 2$ superconformal theories. We also explain why computing F via localization leads to a complex answer, even though we expect it to be real in unitary theories.

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The material presented in this thesis is based on previously published work. Chapter 2 is based on [1], which was written in collaboration with Nathan Seiberg. Chapter 3 is based on [2], which was written together with Guido Festuccia and Nathan Seiberg. Chapters 4 and 5 are based on [3] and [4], which were written in collaboration with Cyril Closset, Guido Festuccia, Zohar Komargodski, and Nathan Seiberg. I would like to thank my co-authors on these papers, as well as those not included here [5–8]. I am particularly grateful to Guido Festuccia and Zohar Komargodski for many intense and enjoyable collaborations. I would also like to thank Zohar and the Weizmann Institute of Science for hosting me in January 2012.

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Chapter 1

Introduction

1.1 A Brief Tour of Supersymmetry

1.1.1 Supersymmetry in Particle Physics

The concept of symmetry plays a fundamental role in theoretical physics. The standard model (SM) of particle physics furnishes a prime example for the central role of symmetry in defining a physical theory and exploring its consequences. The formulation of the theory in terms of spontaneously broken gauge symmetry is crucial for its theoretical consistency, and it enables the SM particles to acquire their masses. The particle that is most closely associated with this effect is the formerly elusive Higgs boson. Recently, the ATLAS and CMS experiments at the Large Hadron Collider (LHC) have reported the discovery of a Higgs-like particle with mass $M_H \simeq 125$ GeV [9, 10]. Symmetry also plays an important role in many other aspects of the SM. For instance, our understanding of low-energy QCD is based on the paradigm of spontaneously broken chiral symmetry.

With the discovery of the Higgs boson, the SM is theoretically and experimentally complete: it is a consistent theory that explains all known phenomena in particle physics with impressive precision. Nevertheless, much work in theoretical particle physics has been dedicated to the exploration of possible SM extensions – physics beyond the standard model (BSM). One of the motivations for these efforts is a basic guiding principle known as naturalness. Originally formalized by 't Hooft [11], it can be stated as follows: a physical parameter is naturally small if the theory possesses an approximate symmetry that becomes exact when the parameter is set to zero. In the SM, the Higgs mass $M_H \simeq 125\,\text{GeV}$, which defines the weak scale, is much smaller than the Planck mass $M_P \simeq 10^{19}$ GeV, a natural short-distance scale associated with quantum gravity. This is known as the SM hierarchy problem. However, there is no associated approximate symmetry. A closely related fact is that the SM Higgs mass is subject to large radiative corrections, which render it highly sensitive to short-distance physics. This constitutes the SM fine-tuning problem. The hierarchy and fine-tuning problems are manifestations of the fact that the SM Higgs mass is not natural in the sense defined above. Much of the effort in BSM physics has been dedicated to constructing extensions of the SM that render the Higgs mass natural.

Supersymmetry (SUSY) is a leading candidate for BSM physics. For each SM particle, it posits the existence of a superpartner particle of opposite statistics. SUSY relates particles and their superpartners, i.e. bosons and fermions. The introduction of the superpartners stabilizes the Higgs mass by eliminating the leading short-distance contributions, and hence it solves the SM fine-tuning problem. Since the superpartners have not yet been observed, SUSY must be spontaneously broken. This should happen near the weak scale if we would like to retain SUSY as a solution to the fine-tuning problem. As pointed out by Witten [12], SUSY also allows for an appealing solution to the SM hierarchy problem: if SUSY is broken dynamically, i.e. by small, non-perturbative effects, the scale of SUSY breaking is naturally much lower than the Planck mass. Explicit models of this type were first realized in the work of Affleck, Dine, and Seiberg [13]. Dynamical SUSY breaking near the weak scale thus solves the SM fine-tuning and hierarchy problems, and it renders the theory natural. As such, it has become a powerful paradigm for BSM physics. For a review, see [14] and references therein.

The paradigm of weak-scale SUSY leads to the exciting possibility that some of the superpartners could be detected at current or future colliders. However, to date there is no experimental evidence for the existence of superpartners (or other BSM physics) near the weak scale. In the years ahead, the LHC is expected to continue its successful program to explore the energy frontier. In particular, it will probe the extent to which the principle of naturalness applies to the weak scale. Either outcome – the presence of additional particles, such as superpartners, to restore naturalness, or the breakdown of naturalness as a physical principle – will have a profound impact on our understanding of the fundamental laws of nature.

1.1.2 Supersymmetry in Field Theory and String Theory

In addition to its application in BSM particle physics, supersymmetry has played an important role in elucidating the dynamics of quantum field theories and string theories. Supersymmetry dramatically constrains the dynamics of these theories, and certain quantities that are protected by supersymmetry can often be analyzed exactly. A typical example is the holomorphic superpotential W in four-dimensional $\mathcal{N} = 1$ theories, which is closely associated with the moduli space of supersymmetric vacua. Supersymmetry implies that W is not renormalized at any order in perturbation theory [15, 16]. It can receive non-perturbative corrections, but they too are severely constrained by supersymmetry and can sometimes be determined exactly [13, 16, 17].

The holomorphy of W has proven to be a powerful handle on the dynamics of supersymmetric field theories, leading to many exact results. In particular, it has lead to a detailed understanding of the phase diagram of four-dimensional $\mathcal{N} = 1$ SUSY QCD, which displays a veritable cornucopia of interesting dynamics: confinement, chiral symmetry breaking, non-perturbative topology change of the moduli space, the emergence of a free magnetic phase at long distances, and non-Abelian electric-magnetic duality [18–20]. In four-dimensional $\mathcal{N} = 2$ gauge theories the analogue of the superpotential is the holomorphic prepotential $\mathcal{F}$, which encodes the entire low-energy effective action on the moduli space. In this case the holomoprhy of $\mathcal{F}$ is sufficiently powerful to determine it, and hence the low-energy action, exactly [21, 22].

Similarly, supersymmetry has been a powerful tool in exploring the dynamics of string theory, to date the only known example of a consistent theory that successfully incorporates quantum gravity. For instance, the first calculation of black hole entropy in string theory relied on supersymmetry to reduce the problem to a weak coupling computation [23]. As in field theory, supersymmetry has played an important role in establishing proposed dualities, since quantities that are protected by supersymmetry can often be computed on both sides of the duality. The most prominent example is the holographic duality between type IIB string theory on $AdS_5 \times S^5$ and maximally supersymmetric Yang-Mills theory in four dimensions, which is a conformal field theory (CFT) [24–26]. Many detailed tests of the duality rely heavily on quantities protected by supersymmetry. Such tests played a crucial role in establishing and generalizing the AdS/CFT correspondence, which by now has come to be viewed as a general principle of quantum gravity, including string theory as a particular example.

Supersymmetric quantum field theories display a rich set of phenomena that emulate many aspects of more realistic, non-supersymmetric theories. We have mentioned confinement, chiral symmetry breaking, and electric-magnetic duality in four dimensions. Similarly, supersymmetric theories provide a unique theoretical laboratory for deepening our understanding of quantum field theory in two and three dimensions, where many qualitatively new phenomena arise. These theories, and their non-supersymmetric cousins, describe a large variety of phases and phase transitions in two and three-dimensional materials. They also furnish holographic descriptions of three- and four-dimensional theories of quantum gravity. Finally, there is a fruitful connection between supersymmetric field theories and certain problems in mathematics. These are good reasons to continue studying such theories, and to develop new and general tools for analyzing their dynamics.

1.2 Overview and Summary

In the spirit of the preceding discussion, this thesis explores several complementary tools that have recently emerged in the study of supersymmetric quantum field theories in three and four dimensions.

Chapter 2 is dedicated to a systematic analysis of supercurrents, SUSY multiplets that include the supersymmetry current $S_{\alpha\mu}$ and the energy-momentum tensor $T_{\mu\nu}$. In this overview, we will focus on four-dimensional $\mathcal{N} = 1$ theories, but in chapter 2 we also discuss theories in two and three dimensions. We find the most general consistent supercurrent multiplet that satisfies certain basic physical requirements. It is given by a real superfield $\mathcal{S}_{\alpha\dot{\alpha}}$, such that

$$\begin{aligned}
&\overline{D}^{\dot{\alpha}}\mathcal{S}_{\alpha\dot{\alpha}} = \chi_\alpha + \mathcal{Y}_\alpha \ , \\
&\overline{D}_{\dot{\alpha}}\chi_\alpha = 0 \ , \qquad D^\alpha\chi_\alpha = \overline{D}_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}} \ , \\
&D_\alpha\mathcal{Y}_\beta + D_\beta\mathcal{Y}_\alpha = 0 \ , \qquad \overline{D}^2\mathcal{Y}_\alpha = 0 \ .
\end{aligned} \tag{1.2.1}$$

The component expressions of the superfields $\mathcal{S}_{\alpha\dot{\alpha}}, \chi_\alpha$, and $\mathcal{Y}_\alpha$ are given in chapter 2. In addition to the supersymmetry current and the energy-momentum tensor, the $\mathcal{S}$-multiplet contains several other operators. We interpret some of these as brane currents – conserved, totally antisymmetric tensor currents associated with charged branes, such as particles, strings, or domain walls. Upon integration, they give rise to brane charges that appear in the supersymmetry algebra. The absence of certain brane charges implies that the corresponding currents are total derivatives and can be removed by an improvement transformation. In this case the supercurrent multiplet decomposes into a shorter supercurrent and another decoupled multiplet.

The structure of the $\mathcal{S}$-multiplet leads to non-trivial restrictions on supersymmetric field theories. For instance, we show that $U(1)$ gauge theories with Fayet-Iliopoulos (FI) terms do not contain magnetic charges. Similarly, the absence of certain brane charges, i.e. the ability to decompose the $\mathcal{S}$-multiplet into shorter supercurrent multiplets, constrains the IR behavior of supersymmetric theories. For instance, the absence of a certain string charge implies that the theory cannot develop a compact moduli space of vacua. Similarly, in theories with a $U(1)_R$ symmetry, a certain two-brane charge is absent, so that these theories do not admit BPS domain walls.

The fact that supercurrent multiplets contain the energy-momentum tensor also means that they encode the necessary information to place a supersymmetric field theory on a rigid curved background. Following the work of [27–30], which considered supersymmetric theories on round spheres, it has become clear that studying these theories on curved manifolds can shed new light on the original flat-space theory. A systematic approach to this subject was developed in [31] using background supergravity. In ordinary supergravity, the metric $g_{\mu\nu}$ is dynamical and belongs to a supermultiplet that also includes the gravitino $\psi_{\mu\alpha}$ and various auxiliary fields. Instead, we can view these fields as classical backgrounds and allow arbitrary field configurations. Rigid supersymmetry corresponds to the subalgebra of supergravity transformations that leaves a given background invariant.

In chapter 3, we explore this procedure for the case of four-dimensional $\mathcal{N} = 1$ theories with a $U(1)_R$ symmetry. The auxiliary fields in the corresponding background supergravity multiplet consist of an Abelian gauge field A_μ, which couples to the R-symmetry, and a two-form gauge field $B_{\mu\nu}$. The dual field strength V^μ of $B_{\mu\nu}$ is a well-defined, conserved vector field,

$$V^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda} , \qquad \nabla_\mu V^\mu = 0 . \tag{1.2.2}$$

A given configuration of the background fields $g_{\mu\nu}$, A_μ, and V_μ preserves rigid supersymmetry if and only if we can solve

$$\begin{aligned} (\nabla_\mu - iA_\mu)\, \zeta &= -iV_\mu\zeta - iV^\nu\sigma_{\mu\nu}\zeta , \\ (\nabla_\mu + iA_\mu)\, \widetilde{\zeta} &= iV_\mu\widetilde{\zeta} + iV^\nu\widetilde{\sigma}_{\mu\nu}\widetilde{\zeta} , \end{aligned} \tag{1.2.3}$$

for some choice of spinors ζ and $\widetilde{\zeta}$. Note that the presence of rigid supersymmetry does not depend on the details of the field theory, since these equations only involve supergravity background fields.

We analyze these equations and classify Riemannian four-manifolds $\mathcal{M}$ that admit rigid supersymmetry. (The corresponding problem for three-dimensional $\mathcal{N} = 2$

theories with a $U(1)_R$ symmetry was analyzed in [8].) We find that $\mathcal{M}$ admits a single supercharge if and only if it is a Hermitian manifold. The supercharge transforms as a scalar on $\mathcal{M}$. We then consider the restrictions imposed by the presence of additional supercharges. Two supercharges of opposite R-charge exist on certain fibrations of a two-torus over a Riemann surface. Upon dimensional reduction, these give rise to an interesting class of supersymmetric geometries in three dimensions. We further show that compact manifolds admitting two supercharges of equal R-charge must be hyperhermitian. Finally, four supercharges imply that $\mathcal{M}$ is locally isometric to $\mathcal{M}_3 \times \mathbb{R}$, where $\mathcal{M}_3$ is a maximally symmetric space.

In the second part of this thesis, we study quantum field theories in three dimensions. One of the main goals is to achieve a detailed understanding of the partition function of three-dimensional $\mathcal{N} = 2$ theories on a round three-sphere, but along the way we uncover several general phenomena in three-dimensional field theory.

In chapter 4, we study contact terms of conserved currents and the energy-momentum tensor. They are associated with Chern-Simons terms for background fields. For concreteness, consider a global, compact $U(1)$ symmetry and couple the associated current j_μ to a background gauge field a_μ. A contact term in the two-point function

$$\langle j_\mu(x) j_\nu(0)\rangle = \cdots + \frac{i\kappa}{2\pi}\varepsilon_{\mu\nu\rho}\partial^\rho \delta^{(3)}(x) \tag{1.2.4}$$

corresponds to a Chern-Simons term for a_μ,

$$\frac{i\kappa}{4\pi}\int d^3x\, \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \ . \tag{1.2.5}$$

Typically, contact terms are scheme-dependent, and hence not well-defined observables. However, the Chern-Simons term (1.2.5) is not the integral of a gauge-invariant local density and this restricts the scheme-dependence of κ. One way to see this is to place the theory on a curved manifold that allows non-trivial bundles for the background gauge field a_μ. Demanding that well-defined counterterms be invariant under large gauge transformations implies that such counterterms can only shift κ by an integer. Therefore, the fractional part of κ is physical and does not depend on the short-distance cutoff. It is an observable of the field theory.

A careful analysis of Chern-Simons contact terms in $\mathcal{N}=2$ supersymmetric theories with a $U(1)_R$ symmetry reveals the presence of a new superconformal anomaly.

In chapter 5, we consider three-dimensional $\mathcal{N}=2$ superconformal field theories on a three-sphere and analyze their free energy F as a function of background gauge and supergravity fields. A crucial role is played by certain local terms in these background fields, including the Chern-Simons terms discussed in chapter 4. The presence of these terms clarifies a number of subtle properties of F.

This understanding allows us to prove that the real part $\operatorname{Re} F(t)$ satisfies

$$\frac{\partial}{\partial t^a} \operatorname{Re} F\bigg|_{t=t_*} = 0 \ , \qquad \frac{\partial^2}{\partial t^a \partial t^b} \operatorname{Re} F\bigg|_{t=t_*} = -\frac{\pi^2}{2} \tau_{ab} \ . \tag{1.2.6}$$

Here, the t^a are real parameters that encode the mixing of the R-symmetry with Abelian flavor symmetries and $t = t_*$ is the superconformal point, at which the R-symmetry resides in the $\mathcal{N}=2$ superconformal algebra. The matrix τ_{ab} is determined by the flat-space two-point functions of the Abelian flavor currents j_a^μ corresponding to the parameters t^a at separated points,

$$\langle j_a^\mu(x) j_b^\nu(0)\rangle = \frac{\tau_{ab}}{16\pi^2}\left(\delta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)\frac{1}{x^2} \ . \tag{1.2.7}$$

In a unitary theory τ_{ab} is a positive definite matrix.

These conditions can be stated as a maximization principle: *the superconformal R-symmetry $R(t_*)$ locally maximizes* $\operatorname{Re} F(t)$ *over the space of trial R-symmetries $R(t)$.* The local maximum $\operatorname{Re} F(t_*)$ determines the SCFT partition function on S^3. This F-maximization principle is similar to a-maximization in four dimensions [32]. The first condition in (1.2.6) is the extremization condition proposed in [29]. The fact that the extremum should be a maximum was conjectured in [33].

We also explain why computing F via localization leads to a complex answer, even though we expect it to be real in unitary theories. We discuss several corollaries of our results and comment on the relation to the F-theorem.

Chapter 2

Supercurrents and Brane Currents in Diverse Dimensions

2.1 Introduction

The goal of this chapter is to present a systematic analysis of *supercurrents* – supersymmetry (SUSY) multiplets that include the supersymmetry current and the energy-momentum tensor. We find the most general consistent supercurrent and we show under what conditions it can be decomposed into smaller multiplets. Furthermore, we give a physical interpretation of the various supercurrents.

For concreteness, we initially focus on $\mathcal{N} = 1$ theories in four dimensions. Later we extend our discussion to $\mathcal{N} = 2$ theories in three dimensions, as well as $\mathcal{N} = (0, 2)$ and $\mathcal{N} = (2, 2)$ theories in two dimensions.

In its simplest form, the $\mathcal{N}=1$ algebra in four dimensions is[1]

$$\begin{aligned} \{Q_\alpha, \overline{Q}_{\dot\alpha}\} &= 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \ , \\ \{Q_\alpha, Q_\beta\} &= 0 \ . \end{aligned} \tag{2.1.2}$$

Following [35–37], we can add additional charges Z_μ and $Z_{\mu\nu}$ to this algebra,

$$\begin{aligned} \{Q_\alpha, \overline{Q}_{\dot\alpha}\} &= 2\sigma^\mu_{\alpha\dot\alpha} \left(P_\mu + Z_\mu\right) \ , \\ \{Q_\alpha, Q_\beta\} &= \sigma^{\mu\nu}_{\alpha\beta} Z_{\mu\nu} \ . \end{aligned} \tag{2.1.3}$$

The charges Z_μ and $Z_{\mu\nu}$ are *brane charges.* They are nonzero for one-branes (strings) and two-branes (domain walls) respectively. These brane charges commute with the supercharges, but they are not central charges of the super-Poincaré algebra, because they do not commute with the Lorentz generators. Other known modifications of the supersymmetry algebra (2.1.2) include terms that do not commute with the supercharges; we do not discuss them here.

The brane charges Z_μ and $Z_{\mu\nu}$ are generally infinite – only the charge per unit volume is meaningful. This motivates us to replace the algebra (2.1.3) by its local version,

$$\begin{aligned} \{\overline{Q}_{\dot\alpha}, S_{\alpha\mu}\} &= 2\sigma^\nu_{\alpha\dot\alpha} \left(T_{\nu\mu} + C_{\nu\mu}\right) + \cdots \ , \\ \{Q_\beta, S_{\alpha\mu}\} &= \sigma^{\nu\rho}_{\alpha\beta} C_{\nu\rho\mu} + \cdots \ . \end{aligned} \tag{2.1.4}$$

Here $C_{\mu\nu}$ and $C_{\mu\nu\rho}$ are *brane currents.* They are the conserved currents corresponding to the brane charges Z_μ and $Z_{\mu\nu}$. The SUSY current algebra (2.1.4) implies that these brane currents are embedded in a supercurrent multiplet, along with the supersymmetry current $S_{\alpha\mu}$ and the energy-momentum tensor $T_{\mu\nu}$. The ellipses in (2.1.4) represent Schwinger terms; they will be discussed below.

[1]We follow the conventions of Wess and Bagger [34], except that our convention for switching between vectors and bi-spinors is

$$\ell_{\alpha\dot\alpha} = -2\sigma^\mu_{\alpha\dot\alpha}\ell_\mu \ , \qquad \ell_\mu = \frac{1}{4}\overline{\sigma}^{\dot\alpha\alpha}_\mu \ell_{\alpha\dot\alpha} \ . \tag{2.1.1}$$

In general, both brane charges in (2.1.3) are present and we must study the current algebra (2.1.4). However, under certain conditions some of these charges are absent. This means that the corresponding current is a total derivative. In that case we should be able to set it to zero by an improvement transformation, which modifies the various operators in (2.1.4) without affecting the associated charges. If this can be done, then the supercurrent multiplet contains fewer operators.

Supercurrent multiplets have been studied by many authors [38–49]; see also section 7.10 of [50]. Our discussion differs from earlier approaches in two crucial respects: First, some authors view rigid supersymmetric field theory as a limit of a supergravity theory. Supergravity has several known presentations, which differ in the choice of auxiliary as well as propagating fields. These different supergravity theories are closely related to various supercurrents. We will pursue a complementary approach, focusing on the different supercurrent multiplets in the rigid theory. We then have the option of gauging these supercurrents to obtain a supergravity theory. One advantage of this approach is that it can be used to derive constraints on consistent supergravity theories [43–49, 51].

Second, we insist on discussing only well-defined operators. These must be gauge invariant and globally well-defined, even when the target space of the theory has nontrivial topology. It is sometimes useful to describe such well-defined operators in terms of other operators, which are not themselves well-defined. A commonly known example arises in electrodynamics, where the field strength $F_{\mu\nu}$ is gauge invariant and well-defined, but it is useful to express it in terms of the gauge non-invariant vector potential A_μ. We will see that physically distinct supercurrent multiplets appear to be identical, if we are careless about allowing operators that are not well-defined.

Throughout our discussion of the various supercurrent multiplets, we impose the following basic requirements:

(a) *The multiplet includes the energy-momentum tensor* $T_{\mu\nu}$. Every local quantum field theory possesses a real, conserved, symmetric energy-momentum tensor (see appendix A):

$$\partial^\nu T_{\mu\nu} = 0 \ , \qquad\qquad P_\mu = \int d^{D-1}x\, T_\mu{}^0 \ . \tag{2.1.5}$$

The energy-momentum tensor is not unique. It can be modified by an improvement transformation

$$T_{\mu\nu} \to T_{\mu\nu} + \partial_\mu U_\nu - \eta_{\mu\nu}\partial^\rho U_\rho \ , \qquad\qquad \partial_{[\mu}U_{\nu]} = 0 \ . \tag{2.1.6}$$

More general improvement transformations include operators of higher spin; they will not be important for us. The improvement term is automatically conserved, and it does not contribute to the total momentum P_μ. The fact that U_μ is closed ensures that $T_{\mu\nu}$ remains symmetric. If there is a well-defined real scalar u such that $U_\mu = \partial_\mu u$, then the improvement (2.1.6) takes the more familiar form

$$T_{\mu\nu} \to T_{\mu\nu} + \left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\right) u \ . \tag{2.1.7}$$

(b) *The multiplet includes the supersymmetry current $S_{\alpha\mu}$.* Every supersymmetric quantum field theory possesses a conserved supersymmetry current:

$$\partial^\mu S_{\alpha\mu} = 0 \ , \qquad\qquad Q_\alpha = \int d^3x\, S_\alpha{}^0 \ . \tag{2.1.8}$$

Like the energy-momentum tensor, the supersymmetry current is not unique. It can be modified by an improvement transformation

$$S_{\alpha\mu} \to S_{\alpha\mu} + \left(\sigma_{\mu\nu}\right)_\alpha{}^\beta \partial^\nu \omega_\beta \ . \tag{2.1.9}$$

As before, more general improvements include operators of higher spin; we do not discuss them. The improvement term is automatically conserved and it does not affect the supercharges Q_α.

(c) *The energy-momentum tensor and the supersymmetry current are the only operators with spin larger than one.* This can be motivated by noting that when a rigid supersymmetric field theory is weakly coupled to supergravity, the supercurrent is the source of the metric superfield. Since the graviton and the gravitino are the only fields of spin larger than one in the supergravity multiplet, we demand that $T_{\mu\nu}$ and $S_{\alpha\mu}$ be the only operators of spin larger than one in the supercurrent.

(d) *The multiplet is indecomposable.* In other words, it cannot be separated into two decoupled supersymmetry multiplets. This does not mean that the multiplet is irreducible. As we will see below, most supercurrents are reducible – they include a non-trivial sub-multiplet, which is closed under supersymmetry transformations. However, if the complement of that sub-multiplet is not a separate supersymmetry multiplet, then the multiplet is indecomposable.

In section 2.2 we show that the most general supercurrent that satisfies the four basic requirements (a)–(d) is a real superfield $\mathcal{S}_{\alpha\dot{\alpha}}$ obeying the constraints

$$\begin{aligned}
&\overline{D}^{\dot{\alpha}}\mathcal{S}_{\alpha\dot{\alpha}} = \chi_\alpha + \mathcal{Y}_\alpha \ , \\
&\overline{D}_{\dot{\alpha}}\chi_\alpha = 0 \ , \qquad D^\alpha\chi_\alpha = \overline{D}_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}} \ , \\
&D_\alpha\mathcal{Y}_\beta + D_\beta\mathcal{Y}_\alpha = 0 \ , \qquad \overline{D}^2\mathcal{Y}_\alpha = 0 \ .
\end{aligned} \tag{2.1.10}$$

This multiplet must exist in every supersymmetric field theory. If there is a well-defined chiral superfield X such that $\mathcal{Y}_\alpha = D_\alpha X$, the multiplet (2.1.10) reduces to the $\mathcal{S}$-multiplet of [46]:

$$\begin{aligned}
&\overline{D}^{\dot{\alpha}}\mathcal{S}_{\alpha\dot{\alpha}} = \chi_\alpha + D_\alpha X \ , \\
&\overline{D}_{\dot{\alpha}}\chi_\alpha = 0 \ , \qquad D^\alpha\chi_\alpha = \overline{D}_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}} \ , \\
&\overline{D}_{\dot{\alpha}}X = 0 \ .
\end{aligned} \tag{2.1.11}$$

However, the superfield X does not always exist. Throughout this chapter, we will refer to (2.1.10) as the $\mathcal{S}$-multiplet, and distinguish (2.1.11) as a special case.

The $\mathcal{S}$-multiplet is reducible, because the superfields χ_α and $\mathcal{Y}_\alpha$ are non-trivial sub-multiplets, but in general it is indecomposable. There are, however, special cases in which the $\mathcal{S}$-multiplet is decomposable, so that we can set either χ_α, or $\mathcal{Y}_\alpha$, or both to zero by an improvement transformation. This gives rise to smaller supercurrent multiplets: the Ferrara-Zumino (FZ) multiplet [38] with $\chi_\alpha = 0$, the $\mathcal{R}$-multiplet [40, 44–46, 50] with $\mathcal{Y}_\alpha = 0$, and the superconformal multiplet with $\chi_\alpha = \mathcal{Y}_\alpha = 0$.

As an example, we discuss how the different supercurrent multiplets arise in general Wess-Zumino models.

In section 2.3 we analyze the supersymmetry current algebra (2.1.4) that follows from the different supercurrents discussed in section 2.2. We find that the difference between these multiplets is reflected in the brane currents they contain.

In section 2.4 we repeat the analysis of sections 2.2 and 2.3 for $\mathcal{N} = 2$ theories in three dimensions. We present the analogue of the $\mathcal{S}$-multiplet, and we explore the resulting current algebra to identify the different brane currents that can arise. As we will see, these theories admit space-filling brane currents, which are not present in four-dimensional theories with $\mathcal{N} = 1$ supersymmetry.

In section 2.5 we discuss the $\mathcal{S}$-multiplet and the resulting current algebra in two-dimensional $\mathcal{N} = (0, 2)$ theories.

In section 2.6 we present additional examples. In particular, we show that there are no magnetic charges in $U(1)$ gauge theories with a Fayet-Iliopoulos (FI) term.

In section 2.7 we discuss partial supersymmetry breaking and its connection with space-filling brane currents. We show that these brane currents deform the supersymmetry current algebra by constants [52]. This highlights the fundamental qualitative difference between partial supersymmetry breaking and ordinary spontaneous SUSY-breaking, where the current algebra is not modified.

In section 2.8 we consider the behavior of the supercurrent multiplet under renormalization group flow. This allows us to constrain the IR behavior of supersymmetric field theories. For instance, we can establish whether a given theory admits certain charged branes. We also comment on the fact that quantum corrections can modify the supercurrent multiplet and show how these corrections are constrained by the structure of the multiplet.

Appendix A summarizes some facts about the energy-momentum tensor and its improvements. Our conventions for two- and three-dimensional theories are summarized in appendix B. In appendix C, we describe the $\mathcal{S}$-multiplet in two-dimensional theories with $\mathcal{N} = (2,2)$ supersymmetry. Appendix D explains the relation between some additional supercurrents, which were discussed in [40,42,47–50], and our general framework.

2.2 Supercurrents in Four Dimensions

In this section we show that the $\mathcal{S}$-multiplet (2.1.10) is the most general supercurrent satisfying the general requirements (a)–(d) laid out in the introduction. This multiplet must exist in any four-dimensional field theory with $\mathcal{N} = 1$ supersymmetry. We then discuss the allowed improvements of the $\mathcal{S}$-multiplet and we use them to establish when the multiplet is decomposable. We will illustrate this using general Wess-Zumino models.

2.2.1 Deriving the $\mathcal{S}$-Multiplet

The most general supercurrent multiplet satisfying (a)–(c) must contain a conserved supersymmetry current $S_{\alpha\mu}$, a real, conserved, symmetric energy-momentum tensor $T_{\mu\nu}$, and possibly other operators of lower spin. Since $T_{\mu\nu}$ is the highest-spin operator, such a multiplet can be represented by a real superfield $\mathcal{T}_\mu$ with

$$\mathcal{T}_\mu\big|_{\theta\sigma^\nu\bar{\theta}} \sim T_{\nu\mu} + \cdots \ , \tag{2.2.1}$$

where the ellipsis denotes lower-spin operators and their derivatives. The component structure of $\mathcal{T}_\mu$ must be consistent with the supersymmetry current algebra (2.1.4). A detailed analysis shows that this completely fixes $\mathcal{T}_\mu$ and the Schwinger terms in (2.1.4). (We do not describe this arduous computation here.) Furthermore, the resulting expression for $\mathcal{T}_\mu$ is always decomposable. It can be separated into a sub-multiplet $\mathcal{Z}_\alpha$ and a smaller supercurrent[2]

$$\mathcal{S}_{\alpha\dot\alpha} = \mathcal{T}_{\alpha\dot\alpha} + i\left(D_\alpha \overline{\mathcal{Z}}_{\dot\alpha} + \overline{D}_{\dot\alpha}\mathcal{Z}_\alpha\right) \ . \tag{2.2.3}$$

This is the $\mathcal{S}$-multiplet (2.1.10), which we repeat here for convenience:

$$\begin{aligned}
&\overline{D}^{\dot\alpha}\mathcal{S}_{\alpha\dot\alpha} = \chi_\alpha + \mathcal{Y}_\alpha \ , \\
&\overline{D}_{\dot\alpha}\chi_\alpha = 0 \ , \qquad D^\alpha\chi_\alpha = \overline{D}_{\dot\alpha}\overline{\chi}^{\dot\alpha} \ , \\
&D_\alpha\mathcal{Y}_\beta + D_\beta\mathcal{Y}_\alpha = 0 \ , \qquad \overline{D}^2\mathcal{Y}_\alpha = 0 \ .
\end{aligned} \tag{2.2.4}$$

Thus, every supersymmetric field theory admits an $\mathcal{S}$-multiplet.

It is straightforward to solve the constraints (2.2.4) in components:

$$\begin{aligned}
\mathcal{S}_\mu = \ & j_\mu - i\theta\left(S_\mu - \frac{i}{\sqrt{2}}\sigma_\mu\overline{\psi}\right) + i\overline{\theta}\left(\overline{S}_\mu - \frac{i}{\sqrt{2}}\overline{\sigma}_\mu\psi\right) + \frac{i}{2}\theta^2\overline{Y}_\mu - \frac{i}{2}\overline{\theta}^2 Y_\mu \\
&+ \left(\theta\sigma^\nu\overline{\theta}\right)\left(2T_{\nu\mu} - \eta_{\nu\mu}A - \frac{1}{8}\varepsilon_{\nu\mu\rho\sigma}F^{\rho\sigma} - \frac{1}{2}\varepsilon_{\nu\mu\rho\sigma}\partial^\rho j^\sigma\right) \\
&- \frac{1}{2}\theta^2\overline{\theta}\left(\overline{\sigma}^\nu\partial_\nu S_\mu + \frac{i}{\sqrt{2}}\overline{\sigma}_\mu\sigma^\nu\partial_\nu\overline{\psi}\right) + \frac{1}{2}\overline{\theta}^2\theta\left(\sigma^\nu\partial_\nu\overline{S}_\mu + \frac{i}{\sqrt{2}}\sigma_\mu\overline{\sigma}^\nu\partial_\nu\psi\right) \\
&+ \frac{1}{2}\theta^2\overline{\theta}^2\left(\partial_\mu\partial^\nu j_\nu - \frac{1}{2}\partial^2 j_\mu\right) \ .
\end{aligned} \tag{2.2.5}$$

The chiral superfield χ_α is given by

$$\begin{aligned}
&\chi_\alpha = -i\lambda_\alpha(y) + \theta_\beta\left(\delta_\alpha{}^\beta D(y) - i(\sigma^{\mu\nu})_\alpha{}^\beta F_{\mu\nu}(y)\right) + \theta^2\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\overline{\lambda}^{\dot\alpha}(y) \ , \\
&\lambda_\alpha = 2\sigma^\mu_{\alpha\dot\alpha}\overline{S}^{\dot\alpha}{}_\mu + 3\sqrt{2}i\psi_\alpha \ , \\
&D = -4T^\mu{}_\mu + 6A \ , \\
&F_{\mu\nu} = -F_{\nu\mu} \ , \qquad \partial_{[\mu}F_{\nu\rho]} = 0 \ ,
\end{aligned} \tag{2.2.6}$$

---

[2]The superfield $\mathcal{Z}_\alpha$ satisfies the defining relations

$$\begin{aligned}
&D_\alpha\mathcal{Z}_\beta + D_\beta\mathcal{Z}_\alpha = 0 \ , \\
&\overline{D}^2\mathcal{Z}_\alpha + 2\overline{D}_{\dot\alpha}D_\alpha\overline{\mathcal{Z}}^{\dot\alpha} + D_\alpha\overline{D}_{\dot\alpha}\overline{\mathcal{Z}}^{\dot\alpha} = 0 \ .
\end{aligned} \tag{2.2.2}$$

See appendix D for a related discussion.

and the superfield $\mathcal{Y}_\alpha$ is given by

$$\begin{aligned}\mathcal{Y}_\alpha &= \sqrt{2}\psi_\alpha + 2\theta_\alpha F + 2i\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha} Y_\mu - 2\sqrt{2}i\left(\theta\sigma^\mu\overline{\theta}\right)\left(\sigma_{\mu\nu}\right)_\alpha{}^\beta \partial^\nu\psi_\beta \\ &\quad + i\theta^2\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_\mu F + \overline{\theta}^2\theta_\alpha\partial^\mu Y_\mu - \frac{1}{2\sqrt{2}}\theta^2\overline{\theta}^2\partial^2\psi_\alpha \ , \\ \partial_{[\mu}Y_{\nu]} &= 0 \ , \\ F &= A + i\partial^\mu j_\mu \ .\end{aligned} \tag{2.2.7}$$

The supersymmetry current $S_{\alpha\mu}$ is conserved, and the energy-momentum tensor $T_{\mu\nu}$ is real, conserved, and symmetric. The $\mathcal{S}$-multiplet contains $16+16$ independent real operators.[3]

If there is a well-defined complex scalar x such that the complex closed one-form Y_μ in (2.2.7) can be written as $Y_\mu = \partial_\mu x$, then we can express

$$\mathcal{Y}_\alpha = D_\alpha X \ , \qquad \overline{D}_{\dot\alpha} X = 0 \ , \tag{2.2.8}$$

where the chiral superfield X is given by

$$X = x(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \ . \tag{2.2.9}$$

In this case the $\mathcal{S}$-multiplet takes the form (2.1.11) discussed in [46]. However, there are situations in which X does not exist and we must use $\mathcal{Y}_\alpha$ (for an example, see subsection 2.2.3).

2.2.2 Improvements and Decomposability

The $\mathcal{S}$-multiplet is not unique. It can be modified by an improvement transformation,

$$\begin{aligned}\mathcal{S}_{\alpha\dot\alpha} &\to \mathcal{S}_{\alpha\dot\alpha} + [D_\alpha, \overline{D}_{\dot\alpha}]U \ , \\ \chi_\alpha &\to \chi_\alpha + \frac{3}{2}\overline{D}^2 D_\alpha U \ , \\ \mathcal{Y}_\alpha &\to \mathcal{Y}_\alpha + \frac{1}{2}D_\alpha\overline{D}^2 U \ ,\end{aligned} \tag{2.2.10}$$

[3] We define the number of independent operators as the number of components minus the number of conservation laws. For example, the $4 \times 5/2 = 10$ components of the energy-momentum tensor lead to 6 independent operators, because there are 4 conservation laws.

where the real superfield U takes the form

$$U = u + \theta\eta + \overline{\theta}\overline{\eta} + \theta^2 N + \overline{\theta}^2 \overline{N} - \left(\theta\sigma^\mu\overline{\theta}\right) V_\mu + \cdots \ . \tag{2.2.11}$$

The transformation (2.2.10) preserves the constraints (2.2.4). It modifies the supersymmetry current and the energy-momentum tensor by improvement terms as in (2.1.9) and (2.1.6),

$$\begin{aligned} S_{\alpha\mu} &\to S_{\alpha\mu} + 2(\sigma_{\mu\nu})_\alpha{}^\beta \partial^\nu \eta_\beta \ , \\ T_{\mu\nu} &\to T_{\mu\nu} + \frac{1}{2}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\right) u \ , \end{aligned} \tag{2.2.12}$$

and it also shifts

$$\begin{aligned} F_{\mu\nu} &\to F_{\mu\nu} - 6\left(\partial_\mu V_\nu - \partial_\nu V_\mu\right) \ , \\ Y_\mu &\to Y_\mu - 2\partial_\mu \overline{N} \ . \end{aligned} \tag{2.2.13}$$

In order for the improvement transformation (2.2.10) to be well-defined, the superfield U must be well-defined up to shifts by a real constant. It is possible to express this transformation entirely in terms of the well-defined superfield $\zeta_\alpha = D_\alpha U$.[4] With this understanding and for ease of notation, we continue to work in terms of U.

As we explained in the introduction, the $\mathcal{S}$-multiplet is reducible, since χ_α and $\mathcal{Y}_\alpha$ are non-trivial sub-multiplets, but it is generally indecomposable. However, there are

[4]The superfield ζ_α satisfies the constraints

$$\begin{aligned} & D_\alpha\zeta_\beta + D_\beta\zeta_\alpha = 0 \ , \\ & \overline{D}^2\zeta_\alpha + 2\overline{D}_{\dot\alpha} D_\alpha \overline{\zeta}^{\dot\alpha} + D_\alpha \overline{D}_{\dot\alpha}\overline{\zeta}^{\dot\alpha} = 0 \ . \end{aligned} \tag{2.2.14}$$

In terms of ζ_α, the improvement transformation (2.2.10) takes the form

$$\begin{aligned} \mathcal{S}_{\alpha\dot\alpha} &\to \mathcal{S}_{\alpha\dot\alpha} + D_\alpha\overline{\zeta}_{\dot\alpha} - \overline{D}_{\dot\alpha}\zeta_\alpha \ , \\ \chi_\alpha &\to \chi_\alpha + \frac{3}{2}\overline{D}^2\zeta_\alpha \ , \\ \mathcal{Y}_\alpha &\to \mathcal{Y}_\alpha + \frac{1}{2}D_\alpha\overline{D}_{\dot\alpha}\overline{\zeta}^{\dot\alpha} \ . \end{aligned} \tag{2.2.15}$$

This is similar, but not identical, to the transformation (2.2.3), which involves the superfield $\mathcal{Z}_\alpha$ defined in (2.2.2).

special cases in which we can use improvements (2.2.10) to decompose the $\mathcal{S}$-multiplet. This gives rise to smaller supercurrent multiplets:

1.) If there is a well-defined real U such that $\chi_\alpha = -\frac{3}{2}\overline{D}^2 D_\alpha U$, then χ_α can be improved to zero. In this case the $\mathcal{S}$-multiplet decomposes into χ_α and a supercurrent $\mathcal{J}_{\alpha\dot\alpha}$ satisfying

$$\begin{aligned} &\overline{D}^{\dot\alpha}\mathcal{J}_{\alpha\dot\alpha} = \mathcal{Y}_\alpha \ , \\ &D_\alpha \mathcal{Y}_\beta + D_\beta \mathcal{Y}_\alpha = 0 \ , \qquad \overline{D}^2 \mathcal{Y}_\alpha = 0 \ . \end{aligned} \tag{2.2.16}$$

This is the FZ-multiplet [38]. It contains $12+12$ independent real operators. If it is possible to write $\mathcal{Y}_\alpha = D_\alpha X$ as in (2.2.8), then we recover the more familiar form of the FZ-multiplet,

$$\begin{aligned} &\overline{D}^{\dot\alpha}\mathcal{J}_{\alpha\dot\alpha} = D_\alpha X \ , \\ &\overline{D}_{\dot\alpha} X = 0 \ . \end{aligned} \tag{2.2.17}$$

2.) If there is a well-defined real U such that $X = -\frac{1}{2}\overline{D}^2 U$, then $\mathcal{Y}_\alpha = D_\alpha X$ can be improved to zero. In this case the $\mathcal{S}$-multiplet decomposes into $\mathcal{Y}_\alpha$ and a supercurrent $\mathcal{R}_{\alpha\dot\alpha}$ satisfying

$$\begin{aligned} &\overline{D}^{\dot\alpha}\mathcal{R}_{\alpha\dot\alpha} = \chi_\alpha \ , \\ &\overline{D}_{\dot\alpha}\chi_\alpha = 0 \ , \quad D^\alpha \chi_\alpha = \overline{D}_{\dot\alpha}\overline{\chi}^{\dot\alpha} \ . \end{aligned} \tag{2.2.18}$$

This is the $\mathcal{R}$-multiplet [40, 44–46, 50]. Like the FZ-multiplet, the $\mathcal{R}$-multiplet contains $12+12$ independent real operators. The constraints (2.2.18) imply that $\partial^\mu \mathcal{R}_\mu = 0$, so that the bottom component of $\mathcal{R}_\mu$ is a conserved R-current. Conversely, any theory with a continuous R-symmetry admits an $\mathcal{R}$-multiplet.

3.) If we can set both χ_α and $\mathcal{Y}_\alpha$ to zero by a single improvement transformation, then the theory is superconformal and the $\mathcal{S}$-multiplet decomposes into χ_α, $\mathcal{Y}_\alpha$, and an $8+8$ supercurrent $\mathcal{J}_{\alpha\dot\alpha}$ satisfying

$$\overline{D}^{\dot\alpha}\mathcal{J}_{\alpha\dot\alpha} = 0 \ . \tag{2.2.19}$$

The FZ-multiplet and the $\mathcal{R}$-multiplet allow residual improvement transformations, which preserve the conditions $\chi_\alpha = 0$ and $\mathcal{Y}_\alpha = 0$ respectively.

With the exception of the special cases discussed above, the $\mathcal{S}$-multiplet is indecomposable. This is because we insist on discussing only well-defined operators. Other authors have decomposed the $\mathcal{S}$-multiplet even when it is indecomposable, because they were willing to consider operators that are either not gauge invariant or not globally well-defined.

2.2.3 The $\mathcal{S}$-Multiplet in Wess-Zumino Models

As an example, we consider a general Wess-Zumino model with Kähler potential $K(\Phi^i, \overline{\Phi}^{\bar{i}})$ and superpotential $W(\Phi^i)$. The Kähler potential and the superpotential need not be well-defined: K may be shifted by Kähler transformations,

$$K(\Phi^i, \overline{\Phi}^{\bar{i}}) \to K(\Phi^i, \overline{\Phi}^{\bar{i}}) + \Lambda(\Phi^i) + \overline{\Lambda}(\overline{\Phi}^{\bar{i}}) \ , \tag{2.2.20}$$

and W may be shifted by constants. This is because the component Lagrangian of the theory only depends on the Kähler metric $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ of the target space, and on the derivatives $\partial_i W$ of the superpotential. Thus, only $g_{i\bar{j}}$ and $\partial_i W$ must be well-defined. We can use the metric to construct the Kähler form

$$\Omega = i g_{i\bar{j}} d\Phi^i \wedge d\overline{\Phi}^{\bar{j}} \ , \tag{2.2.21}$$

which is real and closed, $d\Omega = 0$. Locally, it can be expressed as

$$\Omega = d\mathcal{A} \ , \qquad \mathcal{A} = -\frac{i}{2}\partial_i K d\Phi^i + \frac{i}{2}\partial_{\bar{i}} K d\overline{\Phi}^{\bar{i}} \ . \tag{2.2.22}$$

In general, the Kähler connection $\mathcal{A}$ is not globally well-defined.

Using the equations of motion $\overline{D}^2 \partial_i K = 4\partial_i W$, we can check that the superfields

$$\begin{aligned} &\mathcal{S}_{\alpha\dot{\alpha}} = 2 g_{i\bar{j}} D_\alpha \Phi^i \overline{D}_{\dot{\alpha}} \overline{\Phi}^{\bar{j}} \ , \\ &\chi_\alpha = \overline{D}^2 D_\alpha K \ , \\ &\mathcal{Y}_\alpha = 4 D_\alpha W \ , \end{aligned} \tag{2.2.23}$$

satisfy the constraints (2.1.10). These operators are well-defined under Kähler transformations (2.2.20) and shifts of W by a constant. Thus, the Wess-Zumino model has a well-defined $\mathcal{S}$-multiplet, as must be the case in any supersymmetric field theory. We would like to know under what conditions this multiplet is decomposable, so that it can be improved to an FZ-multiplet or an $\mathcal{R}$-multiplet.

If we take $U = -\frac{2}{3}K$ in (2.2.10), then χ_α is improved to zero and we obtain an FZ-multiplet. This is allowed only if $U \sim K$ is well-defined up to shifts by a real constant. In other words, the Kähler connection $\mathcal{A}$ must be globally well-defined [46]. Note that this never happens on a compact manifold, where some power of Ω is proportional to the volume form, which cannot be exact. As an example, consider a single chiral superfield Φ with Kähler potential

$$K = f^2 \log\left(1 + |\Phi|^2\right) \ , \tag{2.2.24}$$

where f is a real constant of dimension one and Φ is dimensionless. This Kähler potential gives rise to the Fubini-Study metric on $\mathbb{CP}^1$, which is compact. In this theory, the $\mathcal{S}$-multiplet cannot be improved to an FZ-multiplet.

If W is not well-defined, then it is not possible to express $\mathcal{Y}_\alpha = D_\alpha X$ as in (2.2.8). Therefore, such a model cannot have an $\mathcal{R}$-multiplet. A simple example is a cylinder-valued chiral superfield $\Phi \sim \Phi + 1$, with canonical Kähler potential and superpotential $W \sim \Phi$. Going around the cylinder shifts W by a constant, and hence it is not well-defined.

If W is well-defined, then so is $X = 4W$. We can improve X to zero and obtain an $\mathcal{R}$-multiplet if and only if the theory has a continuous R-symmetry. This requires a basis in which the fields Φ^i can be assigned R-charges R_i such that the superpotential has R-charge 2,

$$2W = \sum_i R_i \Phi^i \partial_i W \ , \tag{2.2.25}$$

and the Kähler potential is R-invariant up to a Kähler transformation,

$$\sum_i \left(R_i \Phi^i \partial_i K - R_i \overline{\Phi}^{\bar{i}} \partial_{\bar{i}} K \right) = \Xi(\Phi^j) + \overline{\Xi}(\overline{\Phi}^{\bar{j}}) \ . \tag{2.2.26}$$

Using (2.2.25) and the equations of motion, we can write

$$X = -\frac{1}{2}\overline{D}^2 U \ , \qquad U = -\sum_i R_i \Phi^i \partial_i K \ . \tag{2.2.27}$$

This U is real as long as the chiral superfield Ξ in (2.2.26) is a constant. In other words, K must be R-invariant up to shifts by a real constant. Furthermore, U is well-defined as long as we only perform Kähler transformations that preserve this R-invariance of K. If both of these conditions are satisfied, then we can use U in (2.2.10) to obtain an $\mathcal{R}$-multiplet.

For example, the $\mathbb{CP}^1$ model (2.2.24) has an $\mathcal{R}$-multiplet. However, the cylinder-valued superfield $\Phi \sim \Phi + 1$ with canonical K and $W \sim \Phi$ does not have an $\mathcal{R}$-multiplet. This follows from the fact that the theory does not have a well-defined X. More explicitly, the superpotential $W \sim \Phi$ forces us to assign $R_\Phi = 2$, so that the R-transformation multiplies the bottom component of Φ by a phase, but this is incompatible with the cylindrical field space $\Phi \sim \Phi + 1$.

2.3 Physical Interpretation in Terms of Brane Currents

We have seen that the $\mathcal{S}$-multiplet, though generally indecomposable, can sometimes be improved to a smaller supercurrent multiplet. This is possible whenever χ_α or $\mathcal{Y}_\alpha$ can be expressed in terms of a real superfield U. The non-existence of such a U is an obstruction to the decomposability of the $\mathcal{S}$-multiplet. In this section we interpret this obstruction physically.

Let us consider the current algebra that follows from the $\mathcal{S}$-multiplet,[5]

$$\begin{aligned} \{\overline{Q}_{\dot\alpha}, S_{\alpha\mu}\} &= \sigma^{\nu}_{\alpha\dot\alpha}\left(2T_{\nu\mu} - \frac{1}{8}\varepsilon_{\nu\mu\rho\sigma}F^{\rho\sigma} + i\partial_\nu j_\mu - i\eta_{\nu\mu}\partial^\rho j_\rho - \frac{1}{2}\varepsilon_{\nu\mu\rho\sigma}\partial^\rho j^\sigma\right) \ , \\ \{Q_\beta, S_{\alpha\mu}\} &= 2i\left(\sigma_{\mu\nu}\right)_{\alpha\beta}\overline{Y}^\nu \ . \end{aligned} \tag{2.3.1}$$

Recall from (2.2.6) and (2.2.7) that the real closed two-form $F_{\mu\nu}$ is embedded in χ_α and that the complex closed one-form Y_μ is embedded in $\mathcal{Y}_\alpha$. To elucidate the role of these operators, we define

$$\begin{aligned} C_{\mu\nu} &= -\frac{1}{16}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \ , \qquad \partial^\nu C_{\mu\nu} = 0 \ , \\ C_{\mu\nu\rho} &= -\varepsilon_{\mu\nu\rho\sigma}\overline{Y}^\sigma \ , \qquad \partial^\rho C_{\mu\nu\rho} = 0 \ . \end{aligned} \tag{2.3.2}$$

The current algebra (2.3.1) then takes the form (2.1.4). We see that the Schwinger terms depend only on j_μ and that there are no such terms in $\{Q_\beta, S_{\alpha\mu}\}$. As we mentioned in the introduction, the two-form current $C_{\mu\nu}$ is associated with strings and the three-form current $C_{\mu\nu\rho}$ is associated with domain walls. The appearance of such currents in the four-dimensional $\mathcal{N} = 1$ current algebra was pointed out in [37, 53].

We can formally define the string and domain wall charges

$$Z_\mu = \int d^3x\, C_\mu{}^0 \ , \qquad Z_{\mu\nu} = \int d^3x\, C_{\mu\nu}{}^0 \ , \tag{2.3.3}$$

and integrate the current algebra (2.3.1) to obtain the modified supersymmetry algebra (2.1.3), which we repeat here for convenience:

$$\begin{aligned} \{Q_\alpha, \overline{Q}_{\dot\alpha}\} &= 2\sigma^\mu_{\alpha\dot\alpha}\left(P_\mu + Z_\mu\right) \ , \\ \{Q_\alpha, Q_\beta\} &= \sigma^{\mu\nu}_{\alpha\beta}Z_{\mu\nu} \ . \end{aligned} \tag{2.3.4}$$

In obtaining this algebra, we have dropped the contributions from the j_μ-dependent Schwinger terms in (2.3.1), which can contribute a boundary term to $\{Q_\alpha, \overline{Q}_{\dot\alpha}\}$. The

[5]We use the fact that $[\xi^\alpha Q_\alpha + \overline{\xi}_{\dot\alpha}\overline{Q}^{\dot\alpha}, S] = i(\xi^\alpha \mathcal{Q}_\alpha + \overline{\xi}_{\dot\alpha}\overline{\mathcal{Q}}^{\dot\alpha})S$, for any superfield S. Here Q_α is the supercharge and $\mathcal{Q}_\alpha$ is the corresponding superspace differential operator. The additional factor of i is needed for consistency with Hermitian conjugation.

imaginary part of this boundary term must vanish by unitarity. The real part is due to the term $\sim \varepsilon_{\mu\nu\rho\sigma}\partial^\rho j^\sigma$ in (2.3.1), and we assume that it vanishes as well. (We will revisit this point below.) Note that the string charge Z_μ is algebraically indistinguishable from the momentum P_μ. However, they are distinguished at the level of the current algebra (2.3.1).

As we mentioned in the introduction, the brane charges Z_μ and $Z_{\mu\nu}$ are not central charges of the super-Poincaré algebra. Moreover, they are generally infinite, and only the charge per unit volume is meaningful. For instance, it determines the tension of BPS branes. Many authors have studied such BPS configurations (see e.g. [54,55] and references therein). Our new point here is the relation between the brane currents and the different supercurrent multiplets.

Under improvements (2.2.10) of the $\mathcal{S}$-multiplet, the shifts of $F_{\mu\nu}$ and Y_μ in (2.2.13) imply that the brane currents also change by improvement terms,

$$\begin{aligned} C_{\mu\nu} &\to C_{\mu\nu} + \frac{3}{4}\varepsilon_{\mu\nu\rho\sigma}\partial^\rho V^\sigma \ , \\ C_{\mu\nu\rho} &\to C_{\mu\nu\rho} + 2\varepsilon_{\mu\nu\rho\sigma}\partial^\sigma N \ , \end{aligned} \tag{2.3.5}$$

where V_μ and N belong to the superfield U in (2.2.11). Upon integration, these improvement terms can contribute boundary terms to the brane charges Z_μ and $Z_{\mu\nu}$. Whether or not such boundary terms arise depends on the behavior of V_μ and N at spatial infinity. Note that the Schwinger term $\sim \varepsilon_{\mu\nu\rho\sigma}\partial^\rho j^\sigma$ in (2.3.1) looks like an improvement term for $C_{\mu\nu}$ with $V_\mu \sim j_\mu$. As long as j_μ, V_μ, and N are sufficiently well-behaved at spatial infinity, all boundary terms vanish and the brane charges are not affected by the improvements (2.3.5). This is the case for isolated branes, as long as the fields approach a supersymmetric vacuum far away from the brane.[6] The fact

[6] In the presence of more complicated configurations, such as certain brane bound states, this is no longer true. For instance, the Schwinger term $\sim \varepsilon_{\mu\nu\rho\sigma}\partial^\rho j^\sigma$ in (2.3.1) gives rise to a boundary contribution in the presence of domain wall junctions [37,56]. However, the string-like defect on which the domain walls end does not exist in isolation, and hence this boundary term is not a conventional string charge.

that improvements of the supersymmetry current do not affect the brane charges was pointed out in [37, 57].

With these assumptions, we conclude that the string charge Z_μ must vanish in theories in which $F_{\mu\nu}$ can be set to zero by an improvement transformation. This is the case if and only if the $\mathcal{S}$-multiplet can be improved to an FZ-multiplet. Likewise, the domain wall charge $Z_{\mu\nu}$ must vanish in theories in which Y_μ can be set to zero by an improvement transformation, and this happens if and only if the $\mathcal{S}$-multiplet can be improved to an $\mathcal{R}$-multiplet. Conversely, the existence of strings that carry charge Z_μ is a physical obstruction to improving the $\mathcal{S}$-multiplet to an FZ-multiplet, and the existence of domain walls that carry charge $Z_{\mu\nu}$ is a physical obstruction to improving the $\mathcal{S}$-multiplet to an $\mathcal{R}$-multiplet.[7]

This point of view emphasizes the fact that the $\mathcal{S}$-multiplet always exists, but that it may be decomposable. The existence of brane charges in the supersymmetry algebra is an obstruction to decomposability, and it forces us to consider different supercurrents containing the corresponding brane currents: charged domain walls lead to the FZ-multiplet, and charged strings give rise to the $\mathcal{R}$-multiplet. Theories that support both domain walls and strings with charges in the supersymmetry algebra require the $\mathcal{S}$-multiplet.

To illustrate this, we return to the Wess-Zumino models of subsection 2.2.3. If such a model admits strings with charge Z_μ, then $\chi_\alpha = \overline{D}^2 D_\alpha K$ in (2.2.23) cannot be improved to zero. Therefore, the Kähler form Ω in (2.2.20) is not exact. In this case, the operator $F_{\mu\nu}$ in the $\mathcal{S}$-multiplet is proportional to the pull-back to spacetime of Ω, and the string current $C_{\mu\nu} \sim i\varepsilon_{\mu\nu\rho\sigma} g_{i\bar{j}} \partial^\rho \phi^i \partial^\sigma \overline{\phi}^{\bar{j}}$ is topological. (This is familiar in the context of two-dimensional sigma models, where the analogues of four-dimensional strings are instantons.) If the string is oriented along the z-axis in

[7]This does not apply to branes whose charges do not appear in the supersymmetry algebra. For instance, there can be strings in theories with FZ-multiplets, provided the string charge does not appear in the supersymmetry algebra [58]. Clearly, such strings cannot be BPS.

its rest frame, then the string charge is given by $Z_\mu = \pm T_{\rm BPS} L \delta_{\mu 3}$, where $T_{\rm BPS} > 0$ is a constant, $L \to \infty$ is the length of the string, and the sign is determined by the chirality of the string. From (2.3.4), we see that the mass M of the string satisfies the BPS bound $M \geq T_{\rm BPS} L$. If this bound is saturated, then the string has tension $T_{\rm BPS}$, and it preserves two real supercharges. A typical example is the $\mathbb{CP}^1$ model (2.2.24), which supports BPS strings with $T_{\rm BPS} \sim f^2$.

If the Wess-Zumino model admits domain walls that carry charge $Z_{\mu\nu}$, then $\mathcal{Y}_\alpha$ cannot be improved to zero. Hence, the theory does not have a continuous R-symmetry. In this case, the operator Y_μ in the $\mathcal{S}$-multiplet is proportional to the pull-back to spacetime of the holomorphic one-form $\partial_i W d\Phi^i$. If the domain wall is at rest and lies in the xy-plane, then the non-vanishing components of the domain wall charge are $Z_{12} = -Z_{21} = 2 z_{\rm BPS} A$, where $z_{\rm BPS}$ is a complex constant and $A \to \infty$ is the area of the wall. From (2.3.4), we see that the mass M of the wall satisfies the BPS bound $M \geq |z_{\rm BPS}| A$. If this bound is saturated, then the wall has tension $|z_{\rm BPS}|$, and it preserves two real supercharges. A simple example is a single chiral superfield Φ with canonical Kähler potential and superpotential $W = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3$. This model has two degenerate supersymmetric vacua, and it supports a BPS domain wall, which interpolates between them. In this case $z_{\rm BPS} = -2\Delta\overline{W}$, where $\Delta W = \pm\frac{m^3}{6\lambda^2}$ is the difference of the superpotential evaluated in the two vacua; the sign is determined by the choice of vacuum on either side of the wall.

2.4 Supercurrents in Three Dimensions

In this section, we discuss the analogue of the $\mathcal{S}$-multiplet in three-dimensional theories with $\mathcal{N} = 2$ supersymmetry. (Our conventions are summarized in appendix B.) Just as in four-dimensions, this multiplet is the most general supercurrent satisfying the requirements (a)–(d) laid out in the introduction. Consequently, it exists in every supersymmetric field theory.

2.4.1 The $\mathcal{S}$-Multiplet

In three-dimensional $\mathcal{N} = 2$ theories, the $\mathcal{S}$-multiplet is a real superfield $\mathcal{S}_\mu$, which satisfies the constraints

$$\begin{aligned}
&\overline{D}^\beta \mathcal{S}_{\alpha\beta} = \chi_\alpha + \mathcal{Y}_\alpha \ , \\
&\overline{D}_\alpha \chi_\beta = \frac{1}{2} C \varepsilon_{\alpha\beta} \ , \qquad D^\alpha \chi_\alpha = -\overline{D}^\alpha \overline{\chi}_\alpha \ , \\
&D_\alpha \mathcal{Y}_\beta + D_\beta \mathcal{Y}_\alpha = 0 \ , \qquad \overline{D}^\alpha \mathcal{Y}_\alpha = -C \ ,
\end{aligned} \tag{2.4.1}$$

where $\mathcal{S}_{\alpha\beta} = \mathcal{S}_{\beta\alpha}$ is the symmetric bi-spinor corresponding to $\mathcal{S}_\mu$, and C is a complex constant. We will see that C gives rise to a new kind of brane current, which is qualitatively different from the brane currents we encountered in four dimensions.

It is straightforward to solve the constraints (2.4.1) in components:

$$\begin{aligned}
\mathcal{S}_\mu = \ & j_\mu - i\theta \left(S_\mu + \frac{i}{\sqrt{2}} \gamma_\mu \overline{\psi} \right) - i\overline{\theta} \left(\overline{S}_\mu - \frac{i}{\sqrt{2}} \gamma_\mu \psi \right) + \frac{i}{2} \theta^2 \overline{Y}_\mu + \frac{i}{2} \overline{\theta}^2 Y_\mu \\
& - \left(\theta \gamma^\nu \overline{\theta} \right) \left(2T_{\nu\mu} - \eta_{\mu\nu} A + \frac{1}{4} \varepsilon_{\nu\mu\rho} H^\rho \right) - i\theta\overline{\theta} \left(\frac{1}{4} \varepsilon_{\mu\nu\rho} F^{\nu\rho} + \varepsilon_{\mu\nu\rho} \partial^\nu j^\rho \right) \\
& + \frac{1}{2} \theta^2 \overline{\theta} \left(\gamma^\nu \partial_\nu S_\mu - \frac{i}{\sqrt{2}} \gamma_\nu \gamma_\mu \partial^\nu \overline{\psi} \right) + \frac{1}{2} \overline{\theta}^2 \theta \left(\gamma^\nu \partial_\nu \overline{S}_\mu + \frac{i}{\sqrt{2}} \gamma_\nu \gamma_\mu \partial^\nu \psi \right) \\
& - \frac{1}{2} \theta^2 \overline{\theta}^2 \left(\partial_\mu \partial^\nu j_\nu - \frac{1}{2} \partial^2 j_\mu \right) \ .
\end{aligned} \tag{2.4.2}$$

The chiral superfield χ_α is given by

$$\begin{aligned}
&\chi_\alpha = -i\lambda_\alpha(y) + \theta_\beta \Big(\delta_\alpha{}^\beta D(y) - \gamma^\mu{}_\alpha{}^\beta \big(H_\mu(y) - \frac{i}{2} \varepsilon_{\mu\nu\rho} F^{\nu\rho}(y) \big) \Big) \\
&\qquad + \frac{1}{2} \overline{\theta}_\alpha C - \theta^2 \gamma^\mu{}_\alpha{}^\beta \partial_\mu \overline{\lambda}_\beta(y) \ , \\
&\lambda_\alpha = -2\gamma^\mu{}_\alpha{}^\beta \overline{S}_{\beta\mu} + 3\sqrt{2} i \psi_\alpha \ , \\
&D = -4T^\mu{}_\mu + 4A \ , \\
&\partial_{[\mu} H_{\nu]} = 0 \ , \\
&F_{\mu\nu} = -F_{\nu\mu} \ , \qquad \partial_{[\mu} F_{\nu\rho]} = 0 \ , \\
&y^\mu = x^\mu - i\theta\gamma^\mu\overline{\theta} \ ,
\end{aligned} \tag{2.4.3}$$

and the superfield $\mathcal{Y}_\alpha$ is given by

$$\begin{aligned}
\mathcal{Y}_\alpha &= \sqrt{2}\psi_\alpha + 2\theta_\alpha F - \frac{1}{2}\overline{\theta}_\alpha C + 2i\gamma^\mu_{\alpha\beta}\overline{\theta}^\beta Y_\mu + \sqrt{2}i\left(\theta\gamma^\mu\overline{\theta}\right)\varepsilon_{\mu\nu\rho}\gamma^\nu{}_\alpha{}^\beta\partial^\rho\psi_\beta \\
&\quad + \sqrt{2}i\theta\overline{\theta}\gamma^\mu{}_\alpha{}^\beta\partial_\mu\psi_\beta + i\theta^2\gamma^\mu_{\alpha\beta}\overline{\theta}^\beta\partial_\mu F - \overline{\theta}^2\theta_\alpha\partial^\mu Y_\mu + \frac{1}{2\sqrt{2}}\theta^2\overline{\theta}^2\partial^2\psi_\alpha \ , \\
&\partial_{[\mu}Y_{\nu]} = 0 \ , \\
&F = A + i\partial^\mu j_\mu \ .
\end{aligned} \tag{2.4.4}$$

The supersymmetry current $S_{\alpha\mu}$ is conserved, and the energy-momentum tensor $T_{\mu\nu}$ is real, conserved, and symmetric. The $\mathcal{S}$-multiplet now contains $12+12$ independent real operators, and the complex constant C.

If there is a well-defined complex scalar x such that the complex closed one-form Y_μ in (2.4.4) can be written as $Y_\mu = \partial_\mu x$, then we can express

$$\mathcal{Y}_\alpha = D_\alpha X \ , \qquad D_\alpha\overline{D}_\beta X = -\frac{1}{2}C\varepsilon_{\alpha\beta} \ , \qquad \overline{D}^2 X = 0 \ , \tag{2.4.5}$$

where $X| = x$. If the constant C vanishes, then X is chiral, just as in four dimensions.

If there is a well-defined real scalar J such that the real closed one-form H_μ in (2.4.3) can be written as $H_\mu = \partial_\mu J$, then we can express

$$\chi_\alpha = i\overline{D}_\alpha\mathcal{J} \ , \qquad \overline{D}^2\mathcal{J} = -iC \ , \tag{2.4.6}$$

where $\mathcal{J}| = J$. If the constant C vanishes, then $\mathcal{J}$ is a real linear multiplet.[8]

2.4.2 Improvements and Decomposability

The $\mathcal{S}$-multiplet (2.4.1) can be modified by an improvement transformation

$$\begin{aligned}
&\mathcal{S}_{\alpha\beta} \to \mathcal{S}_{\alpha\beta} + \frac{1}{2}\left([D_\alpha, \overline{D}_\beta] + [D_\beta, \overline{D}_\alpha]\right)U \ , \\
&\chi_\alpha \to \chi_\alpha - \overline{D}^2 D_\alpha U \ , \\
&\mathcal{Y}_\alpha \to \mathcal{Y}_\alpha - \frac{1}{2}D_\alpha\overline{D}^2 U \ ,
\end{aligned} \tag{2.4.7}$$

where the real superfield U takes the form

$$U = \cdots + \theta^2 N - \overline{\theta}^2\overline{N} + \left(\theta\gamma^\mu\overline{\theta}\right)V_\mu - i\theta\overline{\theta}K + \cdots \ . \tag{2.4.8}$$

[8]A real linear multiplet $\mathcal{O}$ satisfies $\overline{D}^2\mathcal{O} = 0$ and hence also $D^2\mathcal{O} = 0$.

The transformation (2.4.7) preserves the constraints (2.4.1), and it changes the supersymmetry current and the energy-momentum tensor by improvement terms. It also shifts

$$\begin{aligned} &H_\mu \to H_\mu - 4\partial_\mu K \ , \\ &F_{\mu\nu} \to F_{\mu\nu} - 4\left(\partial_\mu V_\nu - \partial_\nu V_\mu\right) \ , \\ &Y_\mu \to Y_\mu - 2\partial_\mu \overline{N} \ . \end{aligned} \tag{2.4.9}$$

The constant C is not affected. As in four dimensions, the superfield U in (2.4.7) is only well-defined up to shifts by a real constant, and we could instead work with the well-defined superfield $\zeta_\alpha = D_\alpha U$.[9]

Again, we distinguish cases in which the $\mathcal{S}$-multiplet can be improved to a smaller supercurrent:

1.) If $C = 0$ and there is a well-defined real U such that $\mathcal{J} = 2i\overline{D}DU$, then $\chi_\alpha = i\overline{D}_\alpha \mathcal{J}$ can be improved to zero and we obtain an FZ-multiplet

$$\begin{aligned} &\overline{D}^\beta \mathcal{J}_{\alpha\beta} = \mathcal{Y}_\alpha \ , \\ &D_\alpha \mathcal{Y}_\beta + D_\beta \mathcal{Y}_\alpha = 0 \ , \qquad \overline{D}^\alpha \mathcal{Y}_\alpha = 0 \ . \end{aligned} \tag{2.4.11}$$

This multiplet contains $8+8$ independent real operators.

2.) If $C = 0$ and there is a well-defined real U such that $X = \frac{1}{2}\overline{D}^2 U$, then $\mathcal{Y}_\alpha = D_\alpha X$ can be improved to zero and we obtain an $\mathcal{R}$-multiplet

$$\begin{aligned} &\overline{D}^\beta \mathcal{R}_{\alpha\beta} = \chi_\alpha \ , \\ &\overline{D}_\alpha \chi_\beta = 0 \ , \qquad D^\alpha \chi_\alpha = -\overline{D}^\alpha \overline{\chi}_\alpha \ . \end{aligned} \tag{2.4.12}$$

[9] In three dimensions, the superfield ζ_α satisfies the constraints (compare with (2.2.14))

$$\begin{aligned} &\overline{D}^\alpha \zeta_\alpha = D^\alpha \overline{\zeta}_\alpha \ , \\ &D_\alpha \zeta_\beta + D_\beta \zeta_\alpha = 0 \ , \\ &\overline{D}^2 \zeta_\alpha + 2\overline{D}^\beta D_\alpha \overline{\zeta}_\beta + D_\alpha \overline{D}^\beta \overline{\zeta}_\beta = 0 \ . \end{aligned} \tag{2.4.10}$$

Like the FZ-multiplet, it contains $8+8$ independent real operators. As in four dimensions, the bottom component of the $\mathcal{R}$-multiplet is a conserved R-current, and the $\mathcal{R}$-multiplet exists in every theory with a continuous R-symmetry.

3.) If $C=0$ and we can set both χ_α and $\mathcal{Y}_\alpha$ to zero by a single improvement transformation, then the theory is superconformal, and it has a $4+4$ multiplet satisfying

$$\overline{D}^\beta \mathcal{J}_{\alpha\beta} = 0 \ . \tag{2.4.13}$$

Note that the $\mathcal{S}$-multiplet is decomposable only if the constant C vanishes.

2.4.3 Brane Currents

The current algebra that follows from the $\mathcal{S}$-multiplet takes the form

$$\begin{aligned}\{\overline{Q}_\alpha, S_{\beta\mu}\} &= \gamma^\nu_{\alpha\beta}\left(2T_{\nu\mu} + \frac{1}{4}\varepsilon_{\nu\mu\rho}H^\rho + i\partial_\nu j_\mu - i\eta_{\mu\nu}\partial^\rho j_\rho\right)\\ &\quad + i\varepsilon_{\alpha\beta}\left(\frac{1}{4}\varepsilon_{\mu\nu\rho}F^{\nu\rho} + \varepsilon_{\mu\nu\rho}\partial^\nu j^\rho\right) \ ,\\ \{Q_\alpha, S_{\beta\mu}\} &= \frac{1}{4}\overline{C}\gamma_{\mu\alpha\beta} + i\varepsilon_{\mu\nu\rho}\gamma^\nu_{\alpha\beta}\overline{Y}^\rho \ .\end{aligned} \tag{2.4.14}$$

This allows us to identify the conserved brane currents

$$C_\mu \sim \varepsilon_{\mu\nu\rho}F^{\nu\rho} \ , \qquad C_{\mu\nu} \sim \varepsilon_{\mu\nu\rho}H^\rho \ , \qquad C'_{\mu\nu} \sim \varepsilon_{\mu\nu\rho}\overline{Y}^\rho \ , \qquad C_{\mu\nu\rho} \sim \overline{C}\varepsilon_{\mu\nu\rho} \ . \tag{2.4.15}$$

The current C_μ is associated with zero-branes (it gives rise to a well-defined central charge Z; see appendix B), while $C_{\mu\nu}$ and $C'_{\mu\nu}$ are associated with one-branes. The current $C_{\mu\nu\rho}$ is associated with space-filling two-branes.

As in four-dimensions, improvement transformations (2.4.7) of the $\mathcal{S}$-multiplet shift the brane-currents (2.4.15) by improvement terms, so that the corresponding brane charges are unchanged. (The space-filling brane current $C_{\mu\nu\rho}$ is not affected.) Thus, the zero-brane charge corresponding to C_μ and the one-brane charge corresponding to $C_{\mu\nu}$ must vanish, if the $\mathcal{S}$-multiplet can be improved to an FZ-multiplet. The one-brane charge corresponding to $C'_{\mu\nu}$ must vanish, if the $\mathcal{S}$-multiplet can be improved to an $\mathcal{R}$-multiplet. Conversely, the existence of branes carrying these charges is a physical obstruction to improving the $\mathcal{S}$-multiplet to one of the smaller supercurrents.

2.4.4 Relation to Four-Dimensional Supercurrents

It is instructive to reduce the four-dimensional $\mathcal{S}$-multiplet (2.1.10) to three dimensions. Upon reduction, the four-dimensional superfield $\mathcal{S}_{\alpha\dot\beta}$ decomposes into a symmetric bi-spinor $\widehat{\mathcal{S}}_{\alpha\beta}$ and a real scalar $\widehat{\mathcal{J}}$, which arises as the component of the four-dimensional $\mathcal{S}_\mu$ in the reduced direction. Thus $\widehat{\mathcal{J}}$ contains a conserved current corresponding to translations in the reduced direction. The four-dimensional superfields $\chi_\alpha, \mathcal{Y}_\alpha$ reduce to $\widehat{\chi}_\alpha, \widehat{\mathcal{Y}}_\alpha$. The constraints (2.1.10) then take the form

$$\begin{aligned} &\overline{D}^\beta \widehat{\mathcal{S}}_{\alpha\beta} = 2i\overline{D}_\alpha \widehat{\mathcal{J}} + \widehat{\chi}_\alpha + \widehat{\mathcal{Y}}_\alpha \ , \\ &\overline{D}_\alpha \widehat{\chi}_\beta = 0 \ , \qquad D^\alpha \widehat{\chi}_\alpha = -\overline{D}^\alpha \overline{\widehat{\chi}}_\alpha \ , \\ &D_\alpha \widehat{\mathcal{Y}}_\beta + D_\beta \widehat{\mathcal{Y}}_\alpha = 0 \ , \qquad \overline{D}^2 \widehat{\mathcal{Y}}_\alpha = 0 \ . \end{aligned} \tag{2.4.16}$$

These constraints imply that $\overline{D}^\alpha \widehat{\mathcal{Y}}_\alpha = -C$, where C is a complex constant, and thus

$$\overline{D}^2 \widehat{\mathcal{J}} = -\frac{iC}{2} \ . \tag{2.4.17}$$

The constant C arises from the four-dimensional domain wall current $C_{\mu\nu\rho}$ in (2.3.2), but in three dimensions it represents a space-filling brane current. We identify (2.4.16) as a three-dimensional $\mathcal{S}$-multiplet (2.4.1) with

$$\chi_\alpha = \widehat{\chi}_\alpha + 2i\overline{D}_\alpha \widehat{\mathcal{J}} \ . \tag{2.4.18}$$

In general, $\widehat{\mathcal{J}}$ is non-trivial, so that it cannot be set to zero by a three-dimensional improvement transformation (2.4.7).

We see that the four-dimensional $\mathcal{S}$-multiplet, which has $16 + 16$ independent operators, becomes decomposable upon reduction to three dimensions. It decomposes into a three-dimensional $\mathcal{S}$-multiplet, which has $12+12$ independent operators, and another $4 + 4$ multiplet. Likewise, the reduction of the four-dimensional $\mathcal{R}$-multiplet (2.2.18) decomposes into a three-dimensional $\mathcal{R}$-multiplet (2.4.12), and another 4+4 multiplet. This is expected, because a continuous R-symmetry is preserved by dimensional reduction.

However, the four-dimensional FZ-multiplet (2.2.16), which has $12+12$ independent operators, reduces to a three-dimensional $\mathcal{S}$-multiplet (2.4.1), which is generally indecomposable. This is because $\widehat{\mathcal{J}}$ gives rise to a non-trivial χ_α in (2.4.18), even when $\widehat{\chi}_\alpha = 0$.

2.5 Supercurrents in Two Dimensions

In this section, we present the analogue of the $\mathcal{S}$-multiplet in two-dimensional theories with $\mathcal{N} = (0,2)$ supersymmetry. (Our conventions are summarized in appendix B.) In appendix C we extend our results to theories with $\mathcal{N} = (2,2)$ supersymmetry.

In two-dimensional $\mathcal{N} = (0,2)$ theories, the $\mathcal{S}$-multiplet consists of two real superfields $\mathcal{S}_{++}$, $\mathcal{T}_{----}$ and a complex superfield $\mathcal{W}_-$, which satisfy the constraints

$$\begin{aligned} &\partial_{--}\mathcal{S}_{++} = D_+\mathcal{W}_- - \overline{D}_+\overline{\mathcal{W}}_- \ , \\ &\overline{D}_+\mathcal{T}_{----} = \frac{1}{2}\partial_{--}\mathcal{W}_- \ , \\ &\overline{D}_+\mathcal{W}_- = C \ . \end{aligned} \tag{2.5.1}$$

Here C is a complex constant. As in three dimensions, it is associated with a space-filling brane current.

It is straightforward to solve the constraints (2.5.1) in components:

$$\begin{aligned} &\mathcal{S}_{++} = j_{++} - i\theta^+ S_{+++} - i\overline{\theta}^+\overline{S}_{+++} - \theta^+\overline{\theta}^+ T_{++++} \ , \\ &\mathcal{W}_- = -\overline{S}_{+--} - i\theta^+\left(T_{++--} + \frac{i}{2}\partial_{--}j_{++}\right) - \overline{\theta}^+ C + \frac{i}{2}\theta^+\overline{\theta}^+\partial_{++}\overline{S}_{+--} \ , \\ &\mathcal{T}_{----} = T_{----} - \frac{1}{2}\theta^+\partial_{--}S_{+--} + \frac{1}{2}\overline{\theta}^+\partial_{--}\overline{S}_{+--} + \frac{1}{4}\theta^+\overline{\theta}^+\partial_{--}^2 j_{++} \ . \end{aligned} \tag{2.5.2}$$

The supersymmetry current is conserved, and the energy-momentum tensor is real, conserved, and symmetric,

$$\begin{aligned} &\partial_{++}S_{+--} + \partial_{--}S_{+++} = 0 \ , \\ &\partial_{++}T_{\pm\pm--} + \partial_{--}T_{\pm\pm++} = 0 \ , \\ &T_{++--} = T_{--++} \ . \end{aligned} \tag{2.5.3}$$

Thus, the $\mathcal{S}$-multiplet contains 2+2 independent real operators,[10] and the constant C. Note that j_{++} is not in general accompanied by another real operator j_{--}.

The improvements of the $\mathcal{S}$-multiplet (2.5.1) take the form

$$\begin{aligned}
&\mathcal{S}_{++} \to \mathcal{S}_{++} + [D_+, \overline{D}_+]U \ , \\
&\mathcal{W}_- \to \mathcal{W}_- + \partial_{--}\overline{D}_+ U \ , \\
&\mathcal{T}_{----} \to \mathcal{T}_{----} + \frac{1}{2}\partial_{--}^2 U \ ,
\end{aligned} \tag{2.5.4}$$

where U is a real superfield, whose bottom component is well-defined up to shifts by a real constant. The transformation (2.5.4) preserves the constraints (2.5.1) and it changes the supersymmetry current and the energy-momentum tensor by improvement terms. The constant C is not affected.

As before, we distinguish special cases:

1.) If $C = 0$ and there is a well-defined real superfield $\mathcal{R}_{--}$ such that $\mathcal{W}_- = i\overline{D}_+\mathcal{R}_{--}$, we obtain an $\mathcal{R}$-multiplet

$$\begin{aligned}
&\partial_{--}\mathcal{R}_{++} + \partial_{++}\mathcal{R}_{--} = 0 \ , \\
&\overline{D}_+\left(\mathcal{T}_{----} - \frac{1}{2}\partial_{--}\mathcal{R}_{--}\right) = 0 \ .
\end{aligned} \tag{2.5.5}$$

Here we have relabeled $\mathcal{S}_{++} \to \mathcal{R}_{++}$. The bottom components of $\mathcal{R}_{\pm\pm}$ form a conserved R-current with $R = -\frac{1}{4}\int dx\,(j_{++} + j_{--})$. Unlike in higher dimensions, the $\mathcal{R}$-multiplet now includes the same number $(2+2)$ of independent real operators as the $\mathcal{S}$-multiplet: the conserved, symmetric energy-momentum tensor, the conserved R-current, and two conserved supersymmetry currents.

2.) If $C = 0$ and we can set $\mathcal{W}_-$ to zero by an improvement transformation, the theory is superconformal and the $\mathcal{S}$-multiplet decomposes into the right-moving supercurrent

$$\partial_{--}\mathcal{S}_{++} = 0 \ , \tag{2.5.6}$$

[10]We count independent operators according to the rules explained in footnote 3. This can obscure the counting in two dimensions. For instance, we do not count left-moving operators, which satisfy $\partial_{++}\mathcal{O} = 0$.

and the left-moving component T_{----} of the energy-momentum tensor.

The current algebra that follows from the $\mathcal{S}$-multiplet takes the from

$$\begin{aligned}
&\{\overline{Q}_+, S_{+++}\} = -T_{++++} - \frac{i}{2}\partial_{++}j_{++} \ , \\
&\{\overline{Q}_+, S_{+--}\} = -T_{++--} + \frac{i}{2}\partial_{--}j_{++} \ , \\
&\{Q_+, S_{+++}\} = 0 \ , \\
&\{Q_+, S_{+--}\} = i\overline{C} \ .
\end{aligned} \tag{2.5.7}$$

As in three dimensions, we interpret the constant C as a space-filling brane current. This brane current is not affected by improvement transformations (2.5.4), and it must vanish whenever the theory admits an $\mathcal{R}$-multiplet (2.5.5).

2.6 Examples

2.6.1 Fayet-Iliopoulos Terms

Consider a free $U(1)$ gauge theory with an FI-term in four dimensions:

$$\mathscr{L} = \frac{1}{4e^2}\int d^2\theta\, W^\alpha W_\alpha + \text{c.c.} + \xi \int d^4\theta\, V \ . \tag{2.6.1}$$

Here $W_\alpha = -\frac{1}{4}\overline{D}^2 D_\alpha V$ is the usual field-strength superfield. Using the equations of motion $D^\alpha W_\alpha = e^2\xi$, we find that this theory has an $\mathcal{R}$-multiplet

$$\begin{aligned}
&\mathcal{R}_{\alpha\dot{\alpha}} = -\frac{4}{e^2}W_\alpha \overline{W}_{\dot{\alpha}} \ , \\
&\chi_\alpha = -4\xi W_\alpha \ .
\end{aligned} \tag{2.6.2}$$

It cannot be improved to an FZ-multiplet. Such an improvement would require $U \sim \xi V$ in (2.2.10), and this is not gauge invariant [43, 46]. If we couple (2.6.1) to matter with a generic superpotential, there will no longer be a continuous R-symmetry. In this case the theory has an indecomposable $\mathcal{S}$-multiplet; it admits neither an $\mathcal{R}$-multiplet nor an FZ-multiplet.

We see that $\chi_\alpha \sim \xi W_\alpha$ cannot be improved to zero in theories with an FI-term, and therefore they do not have an FZ-multiplet. From our discussion in section 2.3

we expect these theories to admit strings carrying charge Z_μ. This is indeed the case: even the simplest nontrivial example, supersymmetric QED with an FI-term, supports such strings [59]. In this theory they turn out to be BPS, with tension $T_{\rm BPS} \sim \xi$.

Note that the real two-form $F_{\mu\nu}$ in χ_α is proportional to the $U(1)$ field strength in W_α. Since $F_{\mu\nu}$ must be closed, we conclude that there are no magnetic charges in $U(1)$ gauge theories with an FI-term.

2.6.2 Chern-Simons Terms

Consider a free $U(1)$ gauge theory with a Chern-Simons term and an FI-term in three dimensions:

$$\mathscr{L} = -\frac{1}{4e^2}\int d^4\theta\, \Sigma^2 + k\int d^4\theta\, \Sigma V + \xi\int d^4\theta\, V\ . \tag{2.6.3}$$

Here $\Sigma = i\overline{D}DV$ is the three-dimensional field strength; it is a real linear superfield. Using the equations of motion $i\overline{D}D\Sigma = 2e^2\xi + 4e^2k\Sigma$, we find that the theory has an $\mathcal{R}$-multiplet

$$\begin{aligned} &\mathcal{R}_{\alpha\beta} = \frac{1}{2e^2}\left(D_\alpha\Sigma\,\overline{D}_\beta\Sigma + D_\beta\Sigma\,\overline{D}_\alpha\Sigma\right)\ ,\\ &\chi_\alpha = i\overline{D}_\alpha\mathcal{J}\ , \qquad \mathcal{J} = -\xi\Sigma - \frac{i}{4e^2}\overline{D}D\left(\Sigma^2\right)\ . \end{aligned} \tag{2.6.4}$$

If $\xi = 0$, we can perform an improvement transformation (2.4.7) with $U \sim \frac{1}{e^2}\Sigma^2$ to obtain an FZ-multiplet

$$\begin{aligned} &\mathcal{J}_{\alpha\beta} = \frac{1}{2e^2}\left(D_\alpha\Sigma\,\overline{D}_\beta\Sigma + D_\beta\Sigma\,\overline{D}_\alpha\Sigma\right) - \frac{1}{16e^2}\left([D_\alpha,\overline{D}_\beta] + [D_\beta,\overline{D}_\alpha]\right)\left(\Sigma^2\right)\ ,\\ &\mathcal{Y}_\alpha = D_\alpha X, \qquad X = \frac{1}{16e^2}\overline{D}^2\left(\Sigma^2\right)\ . \end{aligned} \tag{2.6.5}$$

Note that the Chern-Simons level k does not appear explicitly.

2.6.3 Real Mass Terms

Three-dimensional $\mathcal{N} = 2$ theories allow real mass terms. Each real mass parameter m is associated with a $U(1)$ flavor symmetry. The flavor current is usually embedded in a real linear multiplet $\mathcal{J}_m$, which contributes to the operator χ_α in the $\mathcal{S}$-multiplet,

$$\chi_\alpha \sim im\overline{D}_\alpha\mathcal{J}_m\ . \tag{2.6.6}$$

Thus χ_α cannot be improved to zero in theories with real mass terms.

2.6.4 Two-Dimensional $\mathcal{N} = (0,2)$ Kähler Sigma Models

Consider a two-dimensional $\mathcal{N} = (0,2)$ sigma model, whose target space is a Kähler manifold, such as $\mathbb{CP}^1$. The Lagrangian is

$$\mathscr{L} = \frac{i}{8}\int d\theta^+ d\overline{\theta}^+ \, \partial_i K \partial_{--}\Phi^i + \text{c.c.} \,, \tag{2.6.7}$$

where K is the Kähler potential and the Φ^i are chiral, $\overline{D}_+\Phi^i = 0$. The classical theory is superconformal, and it admits an $\mathcal{S}$-multiplet (2.5.1) with $\mathcal{S}_{++} \sim g_{i\bar{j}} D_+\Phi^i \overline{D}_+ \overline{\Phi}^{\bar{j}}$ and $\mathcal{W}_- = 0$. Quantum corrections lead to a breakdown of conformal invariance, and a non-zero $\mathcal{W}_-$ is generated at one-loop,

$$\mathcal{W}_- \sim R_{i\bar{j}}\, \partial_{--}\Phi^i \overline{D}_+ \overline{\Phi}^{\bar{j}} \ . \tag{2.6.8}$$

Here $R_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log\det g_{k\bar{l}}$ is the Ricci tensor of the target space. This also shows that the R-symmetry of the classical theory is anomalous. Note that we can write $\mathcal{W}_- = i\overline{D}_+\mathcal{R}_{--}$ with $\mathcal{R}_{--} \sim -i\partial_i \log\det g_{k\bar{l}}\, \partial_{--}\Phi^i$, which is not globally well-defined. Therefore, the R-symmetry is not violated in perturbation theory, even though the theory does not admit a well-defined $\mathcal{R}$-multiplet. In particular, the constant C in (2.5.1) cannot be generated perturbatively.

Nonperturbatively, instantons activate the anomaly and explicitly break the R-symmetry. To see this in more detail, let us consider the Euclidean two-point function $\langle S_{+--}(0) S_{+++}(z,\overline{z})\rangle$,[11] where $S_{+--} \sim R_{i\bar{j}} \psi^i_+ \partial_{--}\overline{\phi}^{\bar{j}}$ is generated by the one-loop anomaly (2.6.8) and $S_{+++} \sim g_{i\bar{j}} \psi^i_+ \partial_{++}\overline{\phi}^{\bar{j}}$. Since this correlation function violates the R-symmetry by two units, it vanishes in perturbation theory. However, instantons that violate the R-symmetry by the same amount can lead to a nonzero answer. For instance, this happens in the $\mathbb{CP}^1$ model, where the (anti-) instanton of degree -1 has two fermion zero modes, and thus gives rise to a contribution

$$\langle S_{+--}(0) S_{+++}(z,\overline{z})\rangle_{\text{instanton}} \sim \frac{\Lambda^2}{\overline{z}} \ . \tag{2.6.9}$$

[11] Here $z, \overline{z}$ are the Euclidean continuations of x^{--}, x^{++}.

Here Λ is the strong coupling scale of the theory. Upon integration, the residue at $\bar{z} = 0$ gives rise to a contribution $C \sim \Lambda^2$ in (2.5.7). We conclude that instantons generate the constant C in (2.5.1). This was pointed out in [60] and explicitly verified in [61].

2.6.5 A Quantum Mechanical Example

An interesting class of examples in which the superfield $\mathcal{Y}_\alpha$ in the $\mathcal{S}$-multiplet cannot be expressed in terms of a chiral superfield X consists of Wess-Zumino models whose superpotential W is not well-defined (see subsection 2.2.3). To briefly illustrate the interesting quantum effects that can arise in such models, we consider the $\mathcal{N} = 2$ quantum mechanics of a real superfield Φ:

$$L = \int d\theta d\bar{\theta} \left(\overline{D}\Phi D\Phi + W(\Phi)\right) \ . \tag{2.6.10}$$

Here the superpotential W is real. Since we are interested in the case where W is not well-defined, we identify

$$\Phi \sim \Phi + 2\pi \ , \tag{2.6.11}$$

and we choose

$$W = f\Phi + \cos\Phi \ , \tag{2.6.12}$$

where f is a real constant. The classical vacua are determined by the equation

$$\sin\Phi = f \ . \tag{2.6.13}$$

When $|f| > 1$, there is no solution to (2.6.13) and SUSY is spontaneously broken at tree level. When $0 < |f| < 1$, there are two classical supersymmetric vacua satisfying (2.6.13). In this case the system has two different instantons, which interpolate between these vacua – one for each arc of the circle (2.6.11). These instantons mix the two vacuum states and lead to spontaneous SUSY-breaking. Thus, the model (2.6.10) spontaneously breaks SUSY for all non-zero values of f. When $f = 0$, there are supersymmetric vacua at $\Phi = 0, \pi$. Now the two instantons are still present and each one mixes the two vacua, but their contributions exactly cancel and supersymmetry is unbroken.

Similar effects can arise in two-dimensional $\mathcal{N} = (2, 2)$ theories when W is not well-defined. These models often admit BPS solitons that preserve some of the supercharges to all orders in perturbation theory. However, just as in the quantum mechanical example above, nonperturbative effects can break the remaining supersymmetries, so that the BPS property is not maintained in the full quantum theory [62, 63].

2.7 Partial Supersymmetry Breaking and Space-Filling Branes

The goal of this section is to clarify some issues about the phenomenon known as partial supersymmetry breaking, and to relate it to our previous discussion about supercurrent multiplets and brane currents.

2.7.1 A Quantum Mechanical Example

Following [52], we consider a quantum mechanical system with $\mathcal{N} = 2$ supersymmetry

$$\begin{aligned} \{Q, \overline{Q}\} &= 2H \ , \\ \{Q, Q\} &= 2Z \ , \\ \{\overline{Q}, \overline{Q}\} &= 2\overline{Z} \ . \end{aligned} \tag{2.7.1}$$

Here H is the Hamiltonian and the complex constant Z is a central charge. Note that the energy E satisfies the BPS bound $E \geq |Z|$. Let us study the representations of the algebra (2.7.1) as a function of Z.

If $Z = 0$, the algebra has two-dimensional representations with generic energy $E > 0$, and a one-dimensional representation with $E = 0$. The one-dimensional representation is supersymmetric; it is annihilated by both supercharges. If the Hilbert space includes a state in this representation, $\mathcal{N} = 2$ supersymmetry is unbroken. If there is no such state in the Hilbert space, supersymmetry is completely broken.

For $Z \neq 0$, the situation is more interesting. The representations with generic energy $E > |Z|$ are two-dimensional, and they are similar to the two-dimensional representations of the $Z = 0$ algebra. In particular, both supercharges act non-trivially. There is also a one-dimensional representation with $E = |Z|$, which saturates the BPS bound. It is annihilated by one linear combination of the supercharges, while the other linear combination acts as a constant. We say that such a state breaks $\mathcal{N} = 2$ to $\mathcal{N} = 1$. In other words, it partially breaks supersymmetry.

Virtually all models have a $\mathbb{Z}_2$ symmetry, implemented by $(-1)^F$, under which all fermions are odd. Let us add this operator to the algebra (2.7.1). Most of the representations discussed above easily accommodate this operator. The only exception is the one-dimensional representation with $E = |Z| \neq 0$, which must be extended to a two-dimensional representation.[12]

There is a fundamental difference between the partial supersymmetry breaking that can happen when $Z \neq 0$ and the spontaneous supersymmetry breaking that can happen when $Z = 0$.

If $Z = 0$, the algebra (2.7.1) admits supersymmetric representations. It is a dynamical question whether or not the Hilbert space of the system includes such supersymmetric states. Thus, whether or not supersymmetry is spontaneously broken is a property of the ground state. The high-energy behavior of the system is supersymmetric.

Turning on a non-zero Z does not spontaneously break $\mathcal{N} = 2$ to $\mathcal{N} = 1$. Instead, the original $\mathcal{N} = 2$ supersymmetry algebra with $Z = 0$ is deformed. This deformation of the algebra is a property of the high-energy theory rather than a property of the ground state. The ground state is determined by the dynamics. If it saturates the BPS bound, $E = |Z|$, then $\mathcal{N} = 1$ is preserved. If all states have $E > |Z|$, supersymmetry is completely broken.

[12]More generally, adding $(-1)^F$ to the SUSY algebra doubles the size of a representation, whenever the number of supercharges that do not annihilate that representation is odd.

From this point of view, Witten's argument ruling out spontaneous partial supersymmetry breaking [12] is correct. It applies to the algebra with $Z = 0$. The observation of [52] is that the algebra can be deformed to admit states that partially break supersymmetry.

This quantum mechanical discussion also emphasizes the fact that partial supersymmetry breaking has nothing to do with infinite volume or with the non-existence of the supercharges. It is simply a consequence of deforming the supersymmetry algebra.

2.7.2 Relation to Space-Filling Branes

In higher-dimensional systems, a central charge like Z in (2.7.1) is proportional to the volume of space. For example, integrating the three-dimensional $\mathcal{N} = 2$ current algebra (2.4.14) gives

$$\{\overline{Q}_\alpha, Q_\beta\} = -2EA\gamma^0_{\alpha\beta} \ , \qquad \{Q_\alpha, Q_\beta\} = \frac{\overline{C}A}{4}\gamma^0_{\alpha\beta} \ . \tag{2.7.2}$$

Here E is the vacuum energy density, and $A \to \infty$ is the spatial volume. This leads to the BPS bound $E \geq |C|/8$, so that the vacuum has positive energy whenever $C \neq 0$. If the BPS bound is saturated, $E = |C|/8$, then the vacuum breaks half of the supercharges, preserving only $\mathcal{N} = 1$ supersymmetry. If the BPS bound is not saturated, then SUSY is completely broken.

The same phenomenon occurs in two-dimensional $\mathcal{N} = (0, 2)$ theories. Integrating the current algebra (2.5.7) leads to

$$\{\overline{Q}_\pm, Q_\pm\} = 2EL \ , \qquad \{Q_+, Q_+\} = -\frac{\overline{C}L}{4} \ , \tag{2.7.3}$$

where $L \to \infty$ is the spatial volume. Just as in three dimensions, we obtain a BPS bound $E \geq |C|/8$. If this bound is saturated, then the vacuum preserves only one real supercharge and SUSY is partially broken from $\mathcal{N} = (0, 2)$ to $\mathcal{N} = (0, 1)$. Otherwise, supersymmetry is completely broken.

It should now be clear that partial supersymmetry breaking can be interpreted in terms of space-filling brane currents, which give rise to constants in the SUSY current algebra [52]. The deformation of the ordinary current algebra by these constants implies that some of the supersymmetries are always realized non-linearly.

2.7.3 Examples

There are copious known examples of theories with partial supersymmetry breaking. Many of them arise as effective field theories on various BPS branes in field theory and string theory.

Perhaps the simplest examples occur in quantum mechanics. The theory of D0-branes exhibits partial supersymmetry breaking. This was used in the BFSS matrix model [64] and further explored in [65]. Two-dimensional examples arise on the world sheet of strings. The standard Green-Schwarz light-cone string exhibits $\mathcal{N} = 16$ supersymmetry broken to $\mathcal{N} = 8$. Other examples in two dimensions were studied in [52, 66].

An interesting phenomenon arises in the two-dimensional $\mathcal{N} = (0, 2)$ sigma model (2.6.7). As we discussed in subsection 2.6.4, the constant C in (2.7.3) cannot be generated perturbatively. Thus, the theory and its vacuum preserve $\mathcal{N} = (0, 2)$ supersymmetry to all orders in perturbation theory. Nonperturbatively, the constant C is generated by instantons, and the supersymmetry algebra is deformed [60, 61]. Therefore, the vacuum preserves at most one real supercharge. As was pointed out in section 2.7.1, BPS vacua that preserve one real supercharge must come in pairs in order to represent $(-1)^F$. Such pairs of BPS vacua do not constitute short representations, and consequently it is not easy to establish their existence.

Three-dimensional theories with partial supersymmetry breaking can be found on the world-volume of BPS domain walls embedded in four dimensions. These theories admit a three-dimensional $\mathcal{S}$-multiplet (2.4.1) with $C \sim z_{\text{BPS}}$, which leads to partial SUSY-breaking [67]. In this class of models, the constants in the SUSY current algebra arise due to the presence of physical space-filling branes embedded in a higher-dimensional theory.

Another three-dimensional example is a variant of the two-dimensional model studied in [66]. It has a space-filling brane current at tree-level. We start with a Wess-Zumino model with a single chiral superfield Φ, canonical Kähler potential, and superpotential

$$W = \omega \log \Phi \ . \tag{2.7.4}$$

Here ω is a complex constant. Note that this W is not globally well-defined. The model has a $U(1)$ flavor symmetry under which Φ has charge 1 and W is shifted by a constant,

$$\begin{aligned} &\delta_{U(1)}\Phi = -i\Phi \ , \\ &\delta_{U(1)}W = -i\omega \ . \end{aligned} \tag{2.7.5}$$

The scalar potential leads to runaway behavior and the theory does not have a ground state. In order to avoid the runaway, we turn on a real mass m for the $U(1)$ flavor symmetry. This stabilizes the runaway potential and it deforms the supercovariant derivatives by the action of the $U(1)$ symmetry,

$$\begin{aligned} &D_\alpha \to D_\alpha + m\overline{\theta}_\alpha \delta_{U(1)} \ , \\ &\overline{D}_\alpha \to \overline{D}_\alpha + m\theta_\alpha \delta_{U(1)} \ . \end{aligned} \tag{2.7.6}$$

The chiral superfield Φ still satisfies $\overline{D}_\alpha \Phi = 0$. Using the equations of motion $\overline{D}^2\overline{\Phi} = -\frac{4\omega}{\Phi}$, we can check that this model has an $\mathcal{S}$-multiplet (2.4.1) with

$$\begin{aligned} &\mathcal{S}_{\alpha\beta} = D_\alpha\Phi\overline{D}_\beta\overline{\Phi} + D_\beta\Phi\overline{D}_\alpha\overline{\Phi} \ , \\ &\chi_\alpha = -\frac{1}{2}\overline{D}^2 D_\alpha\left(\overline{\Phi}\Phi\right) - 4im\overline{D}_\alpha\left(\overline{\Phi}\Phi\right) \ , \\ &\mathcal{Y}_\alpha = 4\omega\frac{D_\alpha\Phi}{\Phi} \ , \\ &C = 16im\omega \ . \end{aligned} \tag{2.7.7}$$

The vacuum saturates the BPS bound and supersymmetry is partially broken from $\mathcal{N}=2$ to $\mathcal{N}=1$.

The interpretation of partial supersymmetry breaking in terms of space-filling brane currents also applies to four-dimensional $\mathcal{N}=2$ theories. Examples of such theories are world-volume theories of BPS three-branes embedded in six dimensions [68], and gauge theories with magnetic FI-terms [69,70]. At low energies, these models are described by four-dimensional Born-Infeld actions with $\mathcal{N}=1$ supersymmetry [71,72].

2.8 Constraints on Renormalization Group Flow

Consider a supersymmetric quantum field theory with a UV cutoff. This theory must have a well-defined supercurrent multiplet. In this section we discuss the behavior of this multiplet under renormalization group flow. This allows us to constrain the IR behavior of the theory.

All supercurrents furnish short representations of the supersymmetry algebra. (Equivalently, they satisfy certain constraints in superspace.) As is typical in supersymmetric theories, short multiplets are protected: they must remain short under renormalization group flow. Therefore, the structure of the supercurrent multiplet is determined in the UV. This structure is then preserved at all energy scales along the renormalization group flow to the IR.

Before presenting specific applications of this reasoning, we would like to emphasize three important subtleties:

1.) In the extreme UV the theory is superconformal and it has a superconformal multiplet. As we start flowing toward the IR, the superconformal multiplet mixes with another multiplet and becomes larger – it turns into one of the multiplets discussed above. In this section, we would like to discuss the renormalization group flow starting at a high, but finite UV cutoff.

2.) The opposite phenomenon happens in the extreme IR, where the theory is again superconformal and the multiplet becomes shorter. This happens because some non-trivial operators flow to zero at the IR fixed point. (If the low-energy theory is completely massive, the entire multiplet flows to zero in the extreme IR.) Therefore, our conclusions about the low-energy theory will be most interesting when we consider the theory at long, but finite distances.

3.) The supercurrent multiplet must retain its form under renormalization group flow. In particular, constants that appear in the multiplet cannot change along the flow. This does not mean that these constants, or other operators in the multiplet, are not corrected in perturbation theory, or even nonperturbatively. However, these corrections are completely determined by the UV theory.

Consider a four-dimensional theory that admits an FZ-multiplet in the UV. This FZ-multiplet must exist at all energy scales. Therefore, the theory cannot have strings carrying charge Z_μ. If the low-energy theory is a weakly coupled Wess-Zumino model, perhaps with some IR-free gauge fields, the existence of the FZ-multiplet in the IR implies that the target space of the Wess-Zumino model has an exact Kähler form (in particular, it cannot be compact), and that there is no FI-term for any $U(1)$ gauge field [43, 46]. This statement is nonperturbatively exact. It holds even if the topology of the target space or the emergence of $U(1)$ gauge fields at low energies is the result of strong dynamics. (For earlier related results see [73] as referred to in [74], and [41,75,76].) This reasoning can also be applied to constrain the dynamics of SUSY-breaking [6].

Likewise, a four-dimensional theory with a non-anomalous continuous R-symmetry in the UV admits an $\mathcal{R}$-multiplet, and it must retain this multiplet at all energy scales. Consequently, a theory with a continuous R-symmetry cannot support domain walls that carry charge $Z_{\mu\nu}$. (Another application of tracking the $\mathcal{R}$-multiplet from the UV to the IR was recently found in [77].) A theory that admits both an FZ-multiplet and an $\mathcal{R}$-multiplet supports neither strings nor domain walls with charges in the supersymmetry algebra.

Let us demonstrate this in specific examples. Pure $SU(N_c)$ SUSY Yang-Mills theory admits an FZ-multiplet, but no $\mathcal{R}$-multiplet.[13] It has N_c isolated vacua, and it supports domain walls carrying charge $Z_{\mu\nu}$ that interpolate between these

[13]The situation in this theory is similar to the discussion in subsection 2.6.4. The superconformal invariance of the classical theory is broken by quantum corrections. At one-loop we find an FZ-multiplet with $X \sim \operatorname{Tr} W^\alpha W_\alpha$, so that the R-symmetry is anomalous. Even though the theory does not admit a well-defined $\mathcal{R}$-multiplet, the R-symmetry is not violated in perturbation theory. Nonperturbatively, instantons activate the anomaly and explicitly break the R-symmetry.

vacua [53]. However, it does not support charged strings. On the other hand, SUSY QCD with $N_f \geq N_c$ massless flavors has an FZ-multiplet and an $\mathcal{R}$-multiplet, and thus it supports neither strings carrying charge Z_μ nor domain walls carrying charge $Z_{\mu\nu}$.[14] For $N_c \leq N_f \leq \frac{3}{2}N_c$, the IR theory is a weakly coupled Wess-Zumino model, in some cases with IR-free non-Abelian gauge fields [18–20]. The target spaces of these Wess-Zumino models all have exact Kähler forms [46]. This is particularly interesting in the case $N_f = N_c$, when the topology of the IR target space is deformed [18].

Just as in four dimensions, we can use supercurrents to constrain the IR behavior of supersymmetric field theories in two and three dimensions. In particular, we can establish whether a given theory admits branes, whose charges appear in the supersymmetry algebra. This is especially interesting for space-filling branes, which manifest themselves as constants in the various supercurrent multiplets. As such, they are not affected by renormalization group flow. If they are not present in the UV theory, they do not arise at low energies.

When comparing the UV and the IR theories, we must use the supercurrents of the full quantum theories. These may differ from the classical multiplets by perturbative or nonperturbative corrections. For example, we saw in subsection 2.6.4 that anomalies can modify the multiplet at one-loop. Likewise, the constant C in the two-dimensional $\mathcal{S}$-multiplet (2.5.1) can be be generated by instantons. However, we emphasize again that this change in the value of C can be seen by performing an instanton computation in the UV theory.

One way to constrain the form of these quantum corrections is to follow [16] and promote all coupling constants to background superfields. For instance, we can introduce a coupling constant τ in the sigma model (2.6.7) by letting $\partial_i K \rightarrow \tau \partial_i K$. We then promote τ to a background superfield. It is clear from (2.5.1) that the constant C in the $\mathcal{S}$-multiplet can be modified by quantum corrections only if τ is a chiral superfield, $\overline{D}_+ \tau = 0$. This is the case for the $\mathbb{CP}^1$ model, since Kähler

[14]When $N_f < N_c$, the theory does not have a stable vacuum and we do not discuss it [13].

transformations in an instanton background force τ to be chiral, and in this theory C is indeed generated by instantons [60,61]. In sigma models whose target space has an exact Kähler form, τ can be promoted to an arbitrary complex superfield, and in this case C is not generated.[15] (For a recent discussion of nonrenormalization theorems in two-dimensional $\mathcal{N} = (0, 2)$ theories see [78].)

2.9 Appendix A: The Energy-Momentum Tensor

In this appendix we review some facts about the energy-momentum tensor. Noether's theorem guarantees that any translation invariant local field theory possesses a real, conserved energy-momentum tensor $\widehat{T}_{\mu\nu}$,

$$\partial^\nu \widehat{T}_{\mu\nu} = 0 \ , \tag{2.9.1}$$

which integrates to the total momentum

$$P_\mu = \int d^{D-1}x \ \widehat{T}_\mu^{\ 0} \ . \tag{2.9.2}$$

The energy-momentum tensor is not unique. It can be modified by an improvement transformation,

$$\widehat{T}_{\mu\nu} \to \widehat{T}_{\mu\nu} + \partial^\rho B_{\mu\nu\rho} \ , \qquad B_{\mu\nu\rho} = -B_{\mu\rho\nu} \ . \tag{2.9.3}$$

The improvement term $\partial^\rho B_{\mu\nu\rho}$ is automatically conserved and it does not contribute to the total momentum (2.9.2). For some choices of $B_{\mu\nu\rho}$, the energy-momentum tensor $\widehat{T}_{\mu\nu}$ is not symmetric. (This is emphasized by the hat.) For instance, the canonical energy-momentum tensor in Lagrangian field theories is not symmetric, if the theory contains fields with non-zero spin.

[15] C violates the R-symmetry by two units and therefore it can only be generated by instantons with two fermionic zero modes. Such instantons must be BPS, and they only exist in sigma models, whose Kähler form is not exact. (See the related discussion around (2.6.9).)

Lorentz invariance guarantees that there is a choice for $B_{\mu\nu\rho}$ that leads to a symmetric energy-momentum tensor $T_{\mu\nu} = T_{\nu\mu}$. This is well-known for Lagrangian field theories [79], but it holds more generally. Lorentz invariance implies the existence of a real conserved current $j_{\mu\nu\rho}$,

$$\partial^\rho j_{\mu\nu\rho} = 0 \ , \qquad j_{\mu\nu\rho} = -j_{\nu\mu\rho} \ , \tag{2.9.4}$$

which integrates to the Lorentz generators

$$J_{\mu\nu} = \int d^{D-1}x \; j_{\mu\nu}{}^0 \ . \tag{2.9.5}$$

The generators $J_{\mu\nu}$ are time-independent and they satisfy $i[P_\mu, J_{\nu\rho}] = \eta_{\mu\nu} P_\rho - \eta_{\mu\rho} P_\nu$, so that the current $j_{\mu\nu\rho}$ must take the form

$$j_{\mu\nu\rho} = x_\mu \widehat{T}_{\nu\rho} - x_\nu \widehat{T}_{\mu\rho} + s_{\mu\nu\rho} \ , \qquad s_{\mu\nu\rho} = -s_{\nu\mu\rho} \ . \tag{2.9.6}$$

Here $s_{\mu\nu\rho}$ is a local operator without explicit x-dependence. We can obtain a symmetric energy-momentum tensor $T_{\mu\nu}$ by performing an improvement transformation (2.9.3) with

$$B_{\mu\nu\rho} = \frac{1}{2} \left(s_{\nu\rho\mu} + s_{\nu\mu\rho} + s_{\mu\rho\nu} \right) \ . \tag{2.9.7}$$

In terms of $T_{\mu\nu}$, the currents (2.9.6) can be written as $j_{\mu\nu\rho} = x_\mu T_{\nu\rho} - x_\nu T_{\mu\rho}$, up to an overall improvement term.

The symmetric energy-momentum tensor $T_{\mu\nu}$ is also not unique. It can be modified by further improvement transformations (2.9.3), as long as $B_{\mu\nu\rho}$ satisfies

$$\partial^\rho B_{\mu\nu\rho} = \partial^\rho B_{\nu\mu\rho} \ , \tag{2.9.8}$$

so that $T_{\mu\nu}$ remains symmetric.[16] In general $B_{\mu\nu\rho}$ has spin-1 and spin-2 components. If we restrict ourselves to the spin-1 component, we can write

$$B_{\mu\nu\rho} = \eta_{\mu\rho} U_\nu - \eta_{\mu\nu} U_\rho \ , \qquad \partial_{[\mu} U_{\nu]} = 0 \ , \tag{2.9.9}$$

[16]Locally, we can express $B_{\mu\nu\rho} = \partial^\sigma Y_{\mu\sigma\nu\rho}$, where $Y_{\mu\sigma\nu\rho}$ has the symmetries of the Riemann curvature tensor. (It is antisymmetric in each pair $\mu\sigma$ and $\nu\rho$, but symmetric under the exchange of these pairs). However, $Y_{\mu\sigma\nu\rho}$ may not be well-defined.

so that the remaining allowed improvements for $T_{\mu\nu}$ are given by

$$T_{\mu\nu} \to T_{\mu\nu} + \partial_\mu U_\nu - \eta_{\mu\nu}\partial^\rho U_\rho \ . \tag{2.9.10}$$

If there is a well-defined real scalar u such that $U_\mu = \partial_\mu u$, these improvements take the more familiar form

$$T_{\mu\nu} \to T_{\mu\nu} + \left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\right) u \ . \tag{2.9.11}$$

2.10 Appendix B: Conventions in Two and Three Dimensions

In this appendix we summarize our conventions for spinors and supersymmetry in two and three dimensions. Whenever possible, we use the dimensionally reduced conventions of Wess and Bagger [34]. In any number D of dimensions, we take the Minkowski metric to be $\eta_{\mu\nu} = - + \cdots +$, where the Lorentz indices μ, ν run from 0 to $D-1$. We normalize the totally antisymmetric Levi-Civita symbol as $\varepsilon_{01\cdots(D-1)} = -1$.

2.10.1 Conventions in Three Dimensions

In $D = 3$, the Lorentz group is $SL(2,\mathbb{R})$ and the fundamental representation is a real two-component spinor $\psi_\alpha = \overline{\psi}_\alpha$ $(\alpha = 1,2)$. There are only undotted indices and as in $D = 4$, they are raised and lowered by acting from the left with the antisymmetric symbols $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$,

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \ , \qquad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta \ . \tag{2.10.1}$$

There is now only one way to suppress contracted spinor indices,

$$\psi\chi = \psi^\alpha\chi_\alpha \ , \tag{2.10.2}$$

and this leads to some unfamiliar signs, which are absent in $D = 4$. For instance, under Hermitian conjugation we have

$$\overline{(\psi\chi)} = -\overline{\chi}\overline{\psi} \ . \tag{2.10.3}$$

We work in a basis in which the three-dimensional gamma matrices are given by[17]

$$\gamma^\mu_{\alpha\beta} = \left(-\mathbb{1}, \sigma^1, \sigma^3\right) \ . \tag{2.10.4}$$

Here $\mathbb{1}$ is the 2×2 unit matrix, and σ^1, σ^3 are Pauli matrices. The gamma matrices (2.10.4) are real, and they satisfy the following identities:

$$\begin{aligned}
&\gamma^\mu_{\alpha\beta} = \gamma^\mu_{\beta\alpha} \ , \\
&(\gamma^\mu)_\alpha{}^\beta (\gamma^\nu)_\beta{}^\lambda = \eta^{\mu\nu}\delta_\alpha{}^\lambda + \varepsilon^{\mu\nu\rho}(\gamma_\rho)_\alpha{}^\lambda \ , \\
&(\gamma^\mu)_{\alpha\beta}\,(\gamma_\mu)_{\lambda\kappa} = \varepsilon_{\alpha\lambda}\varepsilon_{\kappa\beta} + \varepsilon_{\alpha\kappa}\varepsilon_{\lambda\beta} \ .
\end{aligned} \tag{2.10.5}$$

We can use these to switch between vectors and symmetric bi-spinors,

$$\ell_{\alpha\beta} = -2\gamma^\mu_{\alpha\beta}\ell_\mu \ , \qquad \ell_\mu = \frac{1}{4}\gamma^{\alpha\beta}_\mu \ell_{\alpha\beta} \ , \qquad \ell_{\alpha\beta} = \ell_{\beta\alpha} \ . \tag{2.10.6}$$

The conventional $\mathcal{N}=2$ supersymmetry algebra in $D=3$ takes the form

$$\begin{aligned}
&\{Q_\alpha, \overline{Q}_\beta\} = 2\gamma^\mu_{\alpha\beta}P_\mu + 2i\varepsilon_{\alpha\beta}Z \ , \\
&\{Q_\alpha, Q_\beta\} = 0 \ .
\end{aligned} \tag{2.10.7}$$

The real scalar Z is a central charge. (As in four dimensions, we can extend (2.10.7) by adding additional brane charges [36].) This algebra admits a $U(1)_R$ automorphism under which Q_α has charge -1,

$$[R, Q_\alpha] = -Q_\alpha \ . \tag{2.10.8}$$

If the central charge Z in (2.10.7) vanishes, then $\mathcal{N}=2$ superspace in $D=3$ is the naive dimensional reduction of $\mathcal{N}=1$ superspace in $D=4$. The supercharges Q_α are represented on superfields $S(x,\theta,\overline{\theta})$ by differential operators $\mathcal{Q}_\alpha$,

$$[\xi^\alpha Q_\alpha - \overline{\xi}^\alpha \overline{Q}_\alpha, S] = i\left(\xi^\alpha \mathcal{Q}_\alpha - \overline{\xi}^\alpha\overline{\mathcal{Q}}_\alpha\right) S \ , \tag{2.10.9}$$

with

$$\begin{aligned}
&\mathcal{Q}_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\left(\gamma^\mu\overline{\theta}\right)_\alpha \partial_\mu \ , \\
&\overline{\mathcal{Q}}_\alpha = -\frac{\partial}{\partial\overline{\theta}^\alpha} - i\left(\gamma^\mu\theta\right)_\alpha \partial_\mu \ .
\end{aligned} \tag{2.10.10}$$

[17]These gamma-matrices are obtained by reducing $\sigma^\mu_{\alpha\dot\beta}$ along the four-dimensional 2-direction.

The corresponding supercovariant derivatives are given by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\left(\gamma^\mu\overline{\theta}\right)_\alpha \partial_\mu \ ,$$
$$\overline{D}_\alpha = -\frac{\partial}{\partial\overline{\theta}^\alpha} + i\left(\gamma^\mu\theta\right)_\alpha \partial_\mu \ . \tag{2.10.11}$$

They satisfy the identities

$$\{D_\alpha, \overline{D}_\beta\} = i\partial_{\alpha\beta} \ ,$$
$$D^\alpha\overline{D}_\alpha = \overline{D}^\alpha D_\alpha \ , \tag{2.10.12}$$
$$\{D_\alpha, D_\beta\} = 0 \ .$$

These formulas can be used to derive other useful identities, such as $\overline{D}_\alpha\overline{D}^\beta D_\beta = -\frac{1}{2}\overline{D}^\beta\overline{D}_\beta D_\alpha$. To write supersymmetric actions, we also need the superspace integrals

$$\int d^2\theta\,\theta^2 = 1 \ , \qquad \int d^2\overline{\theta}\,\overline{\theta}^2 = -1 \ , \qquad \int d^4\theta\,\theta^2\overline{\theta}^2 = -1 \ . \tag{2.10.13}$$

2.10.2 Conventions in Two Dimensions

In $D = 2$, the irreducible representations of the Lorentz group are real and one-dimensional. There are two inequivalent real spinors $\psi_\pm$. They can be obtained by reducing from $D = 3$ and identifying[18]

$$\psi_{\alpha=1} \to \psi_- \ , \qquad \psi_{\alpha=2} \to \psi_+ \ . \tag{2.10.14}$$

As in (2.10.1), we raise and lower indices according to

$$\psi^+ = -\psi_- \ , \qquad \psi^- = \psi_+ \ . \tag{2.10.15}$$

We will only use spinor indices $\pm$, so that every vector ℓ_μ is written as a bi-spinor

$$\ell_{\pm\pm} = \ell^{\mp\mp} = 2\left(\ell_0 \pm \ell_1\right) \ . \tag{2.10.16}$$

This leads to some unfamiliar numerical factors. For instance,

$$\ell^2 = -\frac{1}{8}\left(\ell^{++}\ell_{++} + \ell^{--}\ell_{--}\right) = -\frac{1}{4}\ell_{++}\ell_{--} \ . \tag{2.10.17}$$

[18] In our conventions, this corresponds to reducing along the three-dimensional 1-direction.

The conventional $\mathcal{N}=(2,2)$ supersymmetry algebra in $D=2$ takes the form

$$\begin{aligned} \{Q_\pm, \overline{Q}_\pm\} &= -P_{\pm\pm} \ , \\ \{Q_+, Q_-\} &= Z \ , \\ \{Q_+, \overline{Q}_-\} &= \widetilde{Z} \ . \end{aligned} \tag{2.10.18}$$

The complex scalars Z and $\widetilde{Z}$ are central charges. This algebra admits a continuous $U(1)_{R_V} \times U(1)_{R_A}$ automorphism

$$\begin{aligned} [R_V, Q_\pm] &= -Q_\pm \ , \qquad [R_V, Z] = -2Z \ , \\ [R_A, Q_\pm] &= \mp Q_\pm \ , \qquad [R_A, \widetilde{Z}] = -2\widetilde{Z} \ , \end{aligned} \tag{2.10.19}$$

as well as a $\mathbb{Z}_2$ mirror automorphism

$$Q_- \leftrightarrow \overline{Q}_- \ , \qquad Z \leftrightarrow \widetilde{Z} \ , \qquad R_V \leftrightarrow R_A \ . \tag{2.10.20}$$

If the central charges in (2.10.18) vanish, then $\mathcal{N}=(2,2)$ superspace in $D=2$ is the naive dimensional reduction of $\mathcal{N}=2$ superspace in $D=3$. The supercharges $Q_\pm$ are represented on superfields $S(x,\theta,\overline{\theta})$ by differential operators $\mathcal{Q}_\pm$,

$$[\xi^+ Q_+ + \xi^- Q_- - \overline{\xi}^+ \overline{Q}_+ - \overline{\xi}^- \overline{Q}_-, S] = i\left(\xi^+ \mathcal{Q}_+ + \xi^- \mathcal{Q}_- - \overline{\xi}^+ \overline{\mathcal{Q}}_+ - \overline{\xi}^- \overline{\mathcal{Q}}_-\right) S \ , \tag{2.10.21}$$

with

$$\begin{aligned} \mathcal{Q}_\pm &= \frac{\partial}{\partial \theta^\pm} + \frac{i}{2}\overline{\theta}^\pm \partial_{\pm\pm} \ , \\ \overline{\mathcal{Q}}_\pm &= -\frac{\partial}{\partial \overline{\theta}^\pm} - \frac{i}{2}\theta^\pm \partial_{\pm\pm} \ . \end{aligned} \tag{2.10.22}$$

The corresponding supercovariant derivatives are given by

$$\begin{aligned} D_\pm &= \frac{\partial}{\partial \theta^\pm} - \frac{i}{2}\overline{\theta}^\pm \partial_{\pm\pm} \ , \\ \overline{D}_\pm &= -\frac{\partial}{\partial \overline{\theta}^\pm} + \frac{i}{2}\theta^\pm \partial_{\pm\pm} \ . \end{aligned} \tag{2.10.23}$$

They satisfy the identities

$$\begin{aligned} \{D_\pm, \overline{D}_\pm\} &= i\partial_{\pm\pm} \ , \\ D_\pm^2 = \overline{D}_\pm^2 &= \{D_\pm, D_\mp\} = 0 \ . \end{aligned} \tag{2.10.24}$$

The $\mathcal{N} = (0,2)$ subalgebra of (2.10.18) takes the from

$$\begin{aligned} &\{Q_+, \overline{Q}_+\} = -P_{++} \ , \\ &Q_+^2 = 0 \ . \end{aligned} \tag{2.10.25}$$

It admits a $U(1)_R$ automorphism under which Q_+ has charge -1. To obtain $\mathcal{N} = (0,2)$ superspace, we simply set $\theta^- = 0$ in $\mathcal{N} = (2,2)$ superspace.

2.11 Appendix C: The $\mathcal{S}$-Multiplet in Two-Dimensional $\mathcal{N} = (2,2)$ Theories

The $\mathcal{S}$-multiplet in two-dimensional theories with $\mathcal{N} = (2,2)$ supersymmetry consists of two real superfields $\mathcal{S}_{\pm\pm}$, which satisfy the constraints

$$\overline{D}_\pm \mathcal{S}_{\mp\mp} = \pm \left(\chi_\mp + \mathcal{Y}_\mp\right) \ , \tag{2.11.1}$$

where

$$\begin{aligned} &\overline{D}_\pm \chi_\pm = 0 \ , \\ &\overline{D}_\pm \chi_\mp = \pm C^{(\pm)} \ , \\ &D_+ \chi_- - \overline{D}_- \overline{\chi}_+ = k \ , \end{aligned} \tag{2.11.2}$$

and

$$\begin{aligned} &D_\pm \mathcal{Y}_\pm = 0 \ , \\ &\overline{D}_\pm \mathcal{Y}_\mp = \mp C^{(\pm)} \ , \\ &D_+ \mathcal{Y}_- + D_- \mathcal{Y}_+ = k' \ . \end{aligned} \tag{2.11.3}$$

Here k, k' and $C^{(\pm)}$ are real and complex constants respectively.

It is straightforward to solve the constraints (2.11.1) in components:

$$\begin{aligned}\mathcal{S}_{\pm\pm} = \; & j_{\pm\pm} - i\theta^{\pm} S_{\pm\pm\pm} - i\theta^{\mp}\left(S_{\mp\pm\pm} \mp 2\sqrt{2}i\overline{\psi}_{\pm}\right) - i\overline{\theta}^{\pm}\overline{S}_{\pm\pm\pm} \\ & - i\overline{\theta}^{\mp}\left(\overline{S}_{\mp\pm\pm} \pm 2\sqrt{2}i\psi_{\pm}\right) - \theta^{\pm}\overline{\theta}^{\pm} T_{\pm\pm\pm\pm} + \theta^{\mp}\overline{\theta}^{\mp}\left(A \mp \frac{k+k'}{2}\right) \\ & + i\theta^{+}\theta^{-}\overline{Y}_{\pm\pm} + i\overline{\theta}^{+}\overline{\theta}^{-} Y_{\pm\pm} \pm i\theta^{+}\overline{\theta}^{-}\overline{G}_{\pm\pm} \mp i\theta^{-}\overline{\theta}^{+} G_{\pm\pm} \\ & \mp \frac{1}{2}\theta^{+}\theta^{-}\overline{\theta}^{\pm}\partial_{\pm\pm}S_{\mp\pm\pm} \mp \frac{1}{2}\theta^{+}\theta^{-}\overline{\theta}^{\mp}\partial_{\pm\pm}\left(S_{\pm\mp\mp} \pm 2\sqrt{2}i\overline{\psi}_{\mp}\right) \\ & \mp \frac{1}{2}\overline{\theta}^{+}\overline{\theta}^{-}\theta^{\pm}\partial_{\pm\pm}\overline{S}_{\mp\pm\pm} \mp \frac{1}{2}\overline{\theta}^{+}\overline{\theta}^{-}\theta^{\mp}\partial_{\pm\pm}\left(\overline{S}_{\pm\mp\mp} \mp 2\sqrt{2}i\psi_{\mp}\right) \\ & + \frac{1}{4}\theta^{+}\theta^{-}\overline{\theta}^{+}\overline{\theta}^{-}\partial_{\pm\pm}^{2} j_{\mp\mp} \; .\end{aligned} \tag{2.11.4}$$

The chiral superfields $\chi_{\pm}$ are given by

$$\begin{aligned}& \chi_{+} = -i\lambda_{+}(y) - i\theta^{+}\overline{G}_{++}(y) + \theta^{-}\left(E(y) + \frac{k}{2}\right) + \overline{\theta}^{-}C^{(-)} + \theta^{+}\theta^{-}\partial_{++}\overline{\lambda}_{-}(y) \; , \\ & \chi_{-} = -i\lambda_{-}(y) - \theta^{+}\left(\overline{E}(y) - \frac{k}{2}\right) + i\theta^{-}G_{--}(y) - \overline{\theta}^{+}C^{(+)} - \theta^{+}\theta^{-}\partial_{--}\overline{\lambda}_{+}(y) \; , \\ & \lambda_{\pm} = \pm\overline{S}_{\mp\pm\pm} + \sqrt{2}i\psi_{\pm} \; , \\ & E = \frac{1}{2}\left(T_{++--} - A\right) + \frac{i}{4}\left(\partial_{++}j_{--} - \partial_{--}j_{++}\right) \; , \\ & \partial_{++}G_{--} - \partial_{--}G_{++} = 0 \; , \\ & y^{\pm\pm} = x^{\pm\pm} + 4i\theta^{\pm}\overline{\theta}^{\pm} \; ,\end{aligned} \tag{2.11.5}$$

and the twisted (anti-) chiral superfields $\mathcal{Y}_{\pm}$ are given by

$$\begin{aligned}& \mathcal{Y}_{+} = \sqrt{2}\psi_{+}(\overline{\widetilde{y}}) + \theta^{-}\left(F(\overline{\widetilde{y}}) + \frac{k'}{2}\right) - i\overline{\theta}^{+}Y_{++}(\overline{\widetilde{y}}) - \overline{\theta}^{-}C^{(-)} + \sqrt{2}i\theta^{-}\overline{\theta}^{+}\partial_{++}\psi_{-}(\overline{\widetilde{y}}) \; , \\ & \mathcal{Y}_{-} = \sqrt{2}\psi_{-}(\widetilde{y}) - \theta^{+}\left(F(\widetilde{y}) - \frac{k'}{2}\right) + \overline{\theta}^{+}C^{(+)} - i\overline{\theta}^{-}Y_{--}(\widetilde{y}) + \sqrt{2}i\theta^{+}\overline{\theta}^{-}\partial_{--}\psi_{+}(\widetilde{y}) \; , \\ & F = -\frac{1}{2}\left(T_{++--} + A\right) - \frac{i}{4}\left(\partial_{++}j_{--} + \partial_{--}j_{++}\right) \; , \\ & \partial_{++}Y_{--} - \partial_{--}Y_{++} = 0 \; , \\ & \widetilde{y}^{\pm\pm} = x^{\pm\pm} \pm 4i\theta^{\pm}\overline{\theta}^{\pm} \; .\end{aligned} \tag{2.11.6}$$

The supersymmetry current is conserved, and the energy-momentum tensor is real, conserved, and symmetric. The $\mathcal{S}$-multiplet contains $8+8$ independent real operators and the constants $k, k', C^{(\pm)}$.

The mirror automorphism (2.10.20) acts on superspace by exchanging $\theta^- \leftrightarrow -\overline{\theta}^-$ and $D_- \leftrightarrow \overline{D}_-$. The constraints (2.11.1), (2.11.2), and (2.11.3) are invariant, if we accompany this action on superspace by

$$\begin{aligned} &\mathcal{S}_{\pm\pm} \leftrightarrow \pm\mathcal{S}_{\pm\pm} , \qquad \chi_+ \leftrightarrow \overline{\mathcal{Y}}_+ , \qquad \chi_- \leftrightarrow -\mathcal{Y}_- , \\ &k \leftrightarrow -k' , \qquad C^{(+)} \leftrightarrow C^{(+)} , \qquad C^{(-)} \leftrightarrow \overline{C}^{(-)} . \end{aligned} \tag{2.11.7}$$

This implies that $Q_- \leftrightarrow \overline{Q}_-$, $Z \leftrightarrow \widetilde{Z}$, and $R_V \leftrightarrow R_A$.

The $\mathcal{S}$-multiplet of $\mathcal{N} = (2,2)$ decomposes into multiplets of the $\mathcal{N} = (0,2)$ subalgebra. This decomposition includes the $\mathcal{S}$-multiplet of $\mathcal{N} = (0,2)$, which is given by

$$\begin{aligned} &\mathcal{S}_{++}\big|_{\theta^-=0} , \\ &\mathcal{W}_- = i\left(\chi_- - \mathcal{Y}_-\right)\big|_{\theta^-=0} , \\ &\mathcal{T}_{----} = \frac{1}{2}[\overline{D}_-, D_-]S_{--}\big|_{\theta^-=0} . \end{aligned} \tag{2.11.8}$$

These superfields satisfy the constraints (2.5.1) with $C = 2iC^{(+)}$. After the $\mathcal{N} = (0,2)$ projection, the constants k, k' can be eliminated by a shift of T_{++--}, which amounts to an unobservable shift in the total energy.

The $\mathcal{S}$-multiplet (2.11.1) can be modified by an improvement transformation

$$\begin{aligned} &\mathcal{S}_{\pm\pm} \to \mathcal{S}_{\pm\pm} + [D_\pm, \overline{D}_\pm]U , \\ &\chi_\pm \to \chi_\pm - \overline{D}_+\overline{D}_-D_\pm U , \\ &\mathcal{Y}_\pm \to \mathcal{Y}_\pm - D_\pm\overline{D}_+\overline{D}_-U . \end{aligned} \tag{2.11.9}$$

Here U is a real superfield, which is well-defined up to shifts by a real constant.

In some cases, the $\mathcal{S}$-multiplet can be improved to a smaller supercurrent:

1.) If $k = C^{(\pm)} = 0$ and there is a well-defined real U such that $\chi_\pm = \overline{D}_+\overline{D}_-D_\pm U$, then $\chi_\pm$ can be improved to zero and we obtain an FZ-multiplet

$$\begin{aligned} &\overline{D}_\pm\mathcal{J}_{\mp\mp} = \pm\mathcal{Y}_\mp , \\ &D_\pm\mathcal{Y}_\pm = 0 , \qquad \overline{D}_\pm\mathcal{Y}_\mp = 0 , \\ &D_+\mathcal{Y}_- + D_-\mathcal{Y}_+ = k' . \end{aligned} \tag{2.11.10}$$

This multiplet contains 4+4 independent real operators and the real constant k'. It follows from (2.11.10) that

$$\partial_{++}\mathcal{J}_{--} - \partial_{--}\mathcal{J}_{++} = 0 \ , \tag{2.11.11}$$

so that the bottom component of the FZ-multiplet gives rise to a conserved R_A-current with $R_A = -\frac{1}{4}\int dx\,(j_{++} - j_{--})$.

2.) If $k' = C^{(\pm)} = 0$ and there is a well-defined real U such that $\mathcal{Y}_\pm = D_\pm \overline{D}_+ \overline{D}_- U$, then $\mathcal{Y}_\pm$ can be improved to zero and we obtain an $\mathcal{R}$-multiplet

$$\begin{aligned}
&\overline{D}_\pm \mathcal{R}_{\mp\mp} = \pm\chi_\mp \ , \\
&\overline{D}_\pm \chi_+ = 0 \ , \qquad \overline{D}_\pm \chi_- = 0 \ , \\
&D_+\chi_- - \overline{D}_-\overline{\chi}_+ = k \ .
\end{aligned} \tag{2.11.12}$$

Like the FZ-multiplet, it contains $4+4$ real operators, as well as the real constant k. It follows from (2.11.12) that

$$\partial_{++}\mathcal{R}_{--} + \partial_{--}\mathcal{R}_{++} = 0 \ , \tag{2.11.13}$$

so that the bottom component of the $\mathcal{R}$-multiplet is a conserved R_V-current with $R_V = -\frac{1}{4}\int dx\,(j_{++} + j_{--})$. Note that the mirror automorphism (2.10.20) exchanges the $\mathcal{R}$-multiplet and the FZ-multiplet.

3.) If $k = k' = C^{(\pm)} = 0$ and we can set both $\chi_\pm$ and $\mathcal{Y}_\pm$ to zero by a single improvement transformation, then the theory is superconformal and the supercurrent satisfies

$$\overline{D}_\pm \mathcal{J}_{++} = 0 \ , \qquad \overline{D}_\pm \mathcal{J}_{--} = 0 \ . \tag{2.11.14}$$

The current algebra that follows from the $\mathcal{S}$-multiplet takes the form

$$\begin{aligned}
&\{\overline{Q}_\pm, S_{\pm\pm\pm}\} = -T_{\pm\pm\pm\pm} - \frac{i}{2}\partial_{\pm\pm}j_{\pm\pm} \ , \\
&\{\overline{Q}_\pm, S_{\pm\mp\mp}\} = -T_{++--} \pm \frac{1}{2}(k - k') + \frac{i}{2}\partial_{\mp\mp}j_{\pm\pm} \ , \\
&\{Q_\pm, S_{\pm\mp\mp}\} = 2\overline{C}^{(\pm)} \ , \\
&\{Q_+, S_{-\pm\pm}\} = \mp i\overline{Y}_{\pm\pm} \ , \\
&\{\overline{Q}_+, S_{-\pm\pm}\} = \mp iG_{\pm\pm} \ .
\end{aligned} \tag{2.11.15}$$

This allows us to identify the conserved brane currents. The zero-brane currents $\mp i\overline{Y}_{\pm\pm}$ and $\pm i\overline{G}_{\pm\pm}$ give rise to well-defined central charges Z and $\widetilde{Z}$. These currents are exchanged by the mirror automorphism (2.10.20). The constants $C^{(\pm)}$ and $k-k'$ are interpreted as space-filling brane currents, which can lead to partial SUSY-breaking.

2.12 Appendix D: Additional Supercurrent Multiplets?

In this appendix we consider certain multiplets that are more general than the $\mathcal{S}$-multiplet. We show that they are acceptable supercurrents only if they differ from the $\mathcal{S}$-multiplet by an improvement transformation.

One such multiplet was proposed in [42, 47]; see also [48, 49]. It is a real superfield $\mathcal{K}_\mu$ that satisfies the constraints

$$\begin{aligned} &\overline{D}^{\dot\alpha}\mathcal{K}_{\alpha\dot\alpha} = \chi_\alpha + i\chi'_\alpha + \mathcal{Y}_\alpha \ , \\ &\overline{D}_{\dot\alpha}\chi_\alpha = 0 \ , \qquad D^\alpha\chi_\alpha = \overline{D}_{\dot\alpha}\overline{\chi}^{\dot\alpha} \ , \\ &\overline{D}_{\dot\alpha}\chi'_\alpha = 0 \ , \qquad D^\alpha\chi'_\alpha = \overline{D}_{\dot\alpha}\overline{\chi}'^{\dot\alpha} \ , \\ &D_\alpha\mathcal{Y}_\beta + D_\beta\mathcal{Y}_\alpha = 0 \ , \qquad \overline{D}^2\mathcal{Y}_\alpha = 0 \ . \end{aligned} \tag{2.12.1}$$

If $\chi'_\alpha = 0$, we recover the $\mathcal{S}$-multiplet (2.1.10).

It is straightforward to solve the constraints (2.12.1) in components. In particular, we find that

$$\mathcal{K}_\mu\big|_{\theta\sigma^\nu\overline\theta} = 2\widehat{T}_{\nu\mu} - \eta_{\mu\nu}A - \frac{1}{8}\varepsilon_{\nu\mu\rho\sigma}\left(F^{\rho\sigma} + 4\partial^\rho j^\sigma\right) - \frac{1}{2}F'_{\nu\mu} \ . \tag{2.12.2}$$

The operators $A, F_{\mu\nu}, j_\mu$ are familiar from the $\mathcal{S}$-multiplet, while the real closed two-form $F'_{\mu\nu}$ comes from the superfield χ'_α. The energy-momentum tensor $\widehat{T}_{\mu\nu}$ is real and conserved,

$$\partial^\nu\widehat{T}_{\mu\nu} = 0 \ , \tag{2.12.3}$$

but it is *not* symmetric,

$$\widehat{T}_{\mu\nu} - \widehat{T}_{\nu\mu} = \frac{1}{4}F'_{\mu\nu} \ . \tag{2.12.4}$$

However, Lorentz invariance guarantees that $\widehat{T}_{\mu\nu}$ can be improved to a symmetric energy-momentum tensor $T_{\mu\nu}$ (see appendix A).

The allowed improvements of (2.12.1) take the form

$$\begin{aligned}
&\mathcal{K}_{\alpha\dot\alpha} \to \mathcal{K}_{\alpha\dot\alpha} + D_\alpha \overline{\Sigma}_{\dot\alpha} - \overline{D}_{\dot\alpha}\Sigma_\alpha \ , \\
&\chi_\alpha \to \chi_\alpha + \frac{3}{4}\left(\overline{D}^2\Sigma_\alpha - 2\overline{D}_{\dot\alpha}D_\alpha\overline{\Sigma}^{\dot\alpha} - D_\alpha\overline{D}_{\dot\alpha}\overline{\Sigma}^{\dot\alpha}\right) \ , \\
&\chi'_\alpha \to \chi'_\alpha - \frac{i}{4}\left(\overline{D}^2\Sigma_\alpha + 2\overline{D}_{\dot\alpha}D_\alpha\overline{\Sigma}^{\dot\alpha} + D_\alpha\overline{D}_{\dot\alpha}\overline{\Sigma}^{\dot\alpha}\right) \ , \\
&\mathcal{Y}_\alpha \to \mathcal{Y}_\alpha + \frac{1}{2}D_\alpha\overline{D}_{\dot\alpha}\overline{\Sigma}^{\dot\alpha} \ ,
\end{aligned} \tag{2.12.5}$$

where Σ_α satisfies the constraint

$$D_\alpha\Sigma_\beta + D_\beta\Sigma_\alpha = 0 \ . \tag{2.12.6}$$

The transformation (2.12.5) shifts the energy-momentum tensor by an improvement term of the form (2.1.6),

$$\widehat{T}_{\nu\mu} \to \widehat{T}_{\nu\mu} + \partial_\nu U_\mu - \eta_{\mu\nu}\partial^\rho U_\rho \ , \qquad \partial_{[\mu}U_{\nu]} = 0 \ . \tag{2.12.7}$$

If this improvement makes $\widehat{T}_{\mu\nu}$ symmetric, then (2.12.4) shows that it also sets the two-form $F'_{\mu\nu}$ to zero, and hence the entire superfield χ'_α vanishes. Thus, the multiplet (2.12.1) is an acceptable supercurrent only if it is decomposable and can be improved to an $\mathcal{S}$-multiplet.

As an example, we consider a supercurrent that arises in conjunction with non-minimal supergravity theories [40, 50]. In our conventions it takes the form

$$\begin{aligned}
&\overline{D}^{\dot\alpha}\mathcal{G}_{\alpha\dot\alpha} = i\chi'_\alpha + D_\alpha X \ , \\
&\chi'_\alpha = -\frac{i}{6}\left(\overline{D}^2\lambda_\alpha + 2\overline{D}_{\dot\alpha}D_\alpha\overline{\lambda}^{\dot\alpha} + D_\alpha\overline{D}_{\dot\alpha}\overline{\lambda}^{\dot\alpha}\right) \ , \\
&X = \frac{1}{3(3n+1)}\overline{D}_{\dot\alpha}\overline{\lambda}^{\dot\alpha} \ ,
\end{aligned} \tag{2.12.8}$$

where

$$D_\alpha\lambda_\beta + D_\beta\lambda_\alpha = 0 \ , \tag{2.12.9}$$

and n is a complex parameter. We immediately see that it is possible to set χ'_α to zero by an improvement transformation (2.12.5) with $\Sigma_\alpha = -\frac{2}{3}\lambda_\alpha$. This gives rise to an $\mathcal{S}$-multiplet

$$\mathcal{S}_{\alpha\dot{\alpha}} = \mathcal{G}_{\alpha\dot{\alpha}} - \frac{2}{3}\left(D_\alpha \overline{\lambda}_{\dot{\alpha}} - \overline{D}_{\dot{\alpha}} \lambda_\alpha\right) \ , \tag{2.12.10}$$

with

$$\begin{aligned} \chi_\alpha &= -\frac{1}{2}\left(\overline{D}^2 \lambda_\alpha - 2\overline{D}_{\dot{\alpha}} D_\alpha \overline{\lambda}^{\dot{\alpha}} - D_\alpha \overline{D}_{\dot{\alpha}} \overline{\lambda}^{\dot{\alpha}}\right) \ , \\ X &= -\frac{n}{3n+1} \overline{D}_{\dot{\alpha}} \overline{\lambda}^{\dot{\alpha}} \ . \end{aligned} \tag{2.12.11}$$

This form of the multiplet makes manifest two special cases: if $n = 0$, we obtain an $\mathcal{R}$-multiplet, and if $n \to -\frac{1}{3}$, we obtain an FZ-multiplet. (These values of n correspond to the new-minimal and the old-minimal limits of non-minimal supergravity.) The $\mathcal{S}$-multiplet (2.12.10) is decomposable when $\lambda_\alpha = D_\alpha U$, where the real superfield U is well-defined up to shifts by a real constant. Then, it can be improved to either an FZ-multiplet or an $\mathcal{R}$-multiplet. In particular, in this case the theory has a continuous R-symmetry.

Chapter 3

Exploring Curved Superspace

3.1 Introduction

In this chapter, we present a systematic analysis of Riemannian manifolds that admit rigid supersymmetry, focusing on four-dimensional $\mathcal{N} = 1$ theories with a $U(1)_R$ symmetry. We can place any such theory on a Riemannian manifold $\mathcal{M}$ by minimally coupling it to the metric. The resulting theory is invariant under supersymmetry variations with spinor parameter ζ,[1] as long as ζ is covariantly constant,

$$\nabla_\mu \zeta = 0 \ . \tag{3.1.1}$$

The presence of a covariantly constant spinor dramatically restricts the geometry of $\mathcal{M}$, and it is not necessary in order to preserve supersymmetry. In many cases it is possible to place the theory on $\mathcal{M}$ in a certain non-minimal way, such that it is invariant under some appropriately modified supersymmetry variations. In this case the differential equation satisfied by the spinor ζ is a generalization of (3.1.1).

Several such generalizations have been considered in the literature. For instance,

[1]The spinor ζ is left-handed and carries un-dotted indices, ζ_α. Right-handed spinors are distinguished by a tilde and carry dotted indices, $\widetilde{\zeta}^{\dot{\alpha}}$. Our conventions are summarized in appendix A.

we can twist by a line bundle L. Given a connection A_μ on L, this leads to

$$(\nabla_\mu - iA_\mu)\,\zeta = 0 \ . \tag{3.1.2}$$

This equation admits a solution if and only if $\mathcal{M}$ is Kähler [80]; see also [81]. The relation between twisting and rigid supersymmetry on Kähler manifolds is discussed in [82]. A different generalization of (3.1.1) arises if we set only the spin-$\frac{3}{2}$ component of $\nabla_\mu\zeta$ to zero,

$$\nabla_\mu\zeta = \sigma_\mu\widetilde{\eta} \ . \tag{3.1.3}$$

The spinor $\widetilde{\eta}$ is not independent. Rather, it captures the spin-$\frac{1}{2}$ component of $\nabla_\mu\zeta$,

$$\widetilde{\eta} = -\frac{1}{4}\widetilde{\sigma}^\mu\nabla_\mu\zeta \ . \tag{3.1.4}$$

Equation (3.1.3) is known as the twistor equation. It has been studied extensively in the mathematical literature; see for instance [83, 84] and references therein. Finally, we can consider the twistor equation (3.1.3) in conjunction with the twist by L,

$$(\nabla_\mu - iA_\mu)\,\zeta = \sigma_\mu\widetilde{\eta} \ . \tag{3.1.5}$$

This equation clearly includes (3.1.1), (3.1.2), and (3.1.3) as special cases. It was recently studied in the context of conformal supergravity [85].

As we will see below, a systematic approach to supersymmetric field theory on curved manifolds leads to a different generalization of (3.1.1) and (3.1.2),

$$(\nabla_\mu - iA_\mu)\,\zeta = -iV_\mu\zeta - iV^\nu\sigma_{\mu\nu}\zeta \ . \tag{3.1.6}$$

Here V^μ is a smooth, conserved vector field, $\nabla_\mu V^\mu = 0$. This equation is closely related to (3.1.5), although there are important differences. We can express (3.1.6) as

$$\big(\nabla_\mu - i\widehat{A}_\mu\big)\zeta = -\frac{i}{2}\,\sigma_\mu\big(V^\nu\widetilde{\sigma}_\nu\zeta\big) \ , \tag{3.1.7}$$

where $\widehat{A}_\mu = A_\mu - \frac{3}{2}V_\mu$. Therefore, every solution ζ of (3.1.6) is a solution of (3.1.5). However, given a solution ζ of (3.1.5), we see from (3.1.7) that it satisfies (3.1.6) only if $\widetilde{\eta}$ in (3.1.5) can be expressed in terms of a smooth conserved V^μ,

$$\widetilde{\eta} = -\frac{i}{2}V^\nu\widetilde{\sigma}_\nu\zeta \ , \qquad \nabla_\mu V^\mu = 0 \ . \tag{3.1.8}$$

This is always possible in a neighborhood where ζ does not vanish. By counting degrees of freedom, we see that V^μ is determined up to two functions, which must satisfy a differential constraint to ensure $\nabla_\mu V^\mu = 0$. Locally, any solution of (3.1.5) is therefore a solution of (3.1.6), as long as ζ does not vanish. This is no longer true if ζ has zeros, since we cannot satisfy (3.1.8) for any smooth V^μ. It is known that (3.1.5) admits nontrivial solutions with zeros; see for instance [83, 84]. By contrast, it is easy to show that every nontrivial solution of (3.1.6) is nowhere vanishing.

We will now explain how (3.1.6) arises in the study of supersymmetric field theories on Riemannian manifolds. Following [27–30], much work has focused on supersymmetric theories on round spheres. (See [86, 87] for some earlier work.) Recently, it was shown that rigid supersymmetry also exists on certain squashed spheres [88–94]. A systematic approach to this subject was developed in [31] using background supergravity. In ordinary supergravity, the metric $g_{\mu\nu}$ is dynamical and belongs to a supermultiplet that also includes the gravitino $\psi_{\mu\alpha}$ and various auxiliary fields. Here, we would like to view these fields as classical backgrounds and allow arbitrary field configurations. This can be achieved by starting with supergravity and appropriately scaling the Planck mass to infinity. Rigid supersymmetry corresponds to the subalgebra of supergravity transformations that leaves a given background invariant. This procedure captures all deformations of the theory that approach the original flat-space theory at short distances. (There are known modifications of flat-space supersymmetry, but we will not discuss them here.) See appendix B, which also contains a brief review of [31].

In this chapter, we will discuss $\mathcal{N} = 1$ theories in four dimensions. The corresponding supergravity has several presentations, which differ in the choice of propagating and auxiliary fields. Since we do not integrate out the auxiliary fields, these formulations are not equivalent and can lead to different backgrounds with rigid supersymmetry. We will focus on theories with a $U(1)_R$ symmetry, which can be coupled to the new minimal formulation of supergravity [95, 96].[2] In this formulation,

[2]The corresponding analysis for old minimal supergravity [97,98] is described in [7].

the auxiliary fields in the supergravity multiplet consist of an Abelian gauge field A_μ and a two-form gauge field $B_{\mu\nu}$. The dual field strength V^μ of $B_{\mu\nu}$ is a well-defined, conserved vector field,

$$V^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda} \ , \qquad \nabla_\mu V^\mu = 0 \ . \tag{3.1.9}$$

The gauge field A_μ couples to the $U(1)_R$ current of the field theory, which leads to invariance under local R-transformations.

In new minimal supergravity, the variation of the gravitino takes the form

$$\begin{aligned} \delta\psi_\mu &= -2\left(\nabla_\mu - iA_\mu\right)\zeta - 2iV_\mu\zeta - 2iV^\nu\sigma_{\mu\nu}\zeta \ , \\ \delta\widetilde{\psi}_\mu &= -2\left(\nabla_\mu + iA_\mu\right)\widetilde{\zeta} + 2iV_\mu\widetilde{\zeta} + 2iV^\nu\widetilde{\sigma}_{\mu\nu}\widetilde{\zeta} \ . \end{aligned} \tag{3.1.10}$$

The spinor parameters ζ and $\widetilde{\zeta}$ have R-charge $+1$ and -1 respectively. In Lorentzian signature, left-handed and right-handed spinors are exchanged by complex conjugation and the background fields A_μ and V_μ are real. This is not the case in Euclidean signature, where ζ and $\widetilde{\zeta}$ are independent and the background fields A_μ and V_μ may be complex. However, we will always take the metric $g_{\mu\nu}$ to be real.

A given configuration of the background fields $g_{\mu\nu}, A_\mu$, and V_μ on $\mathcal{M}$ preserves rigid supersymmetry, if and only if both variations in (3.1.10) vanish for some choice of ζ and $\widetilde{\zeta}$. Moreover, we can always consider variations of definite R-charge. A supercharge δ_ζ of R-charge -1 corresponds to a solution ζ of

$$\left(\nabla_\mu - iA_\mu\right)\zeta = -iV_\mu\zeta - iV^\nu\sigma_{\mu\nu}\zeta \ , \tag{3.1.11}$$

while a supercharge $\delta_{\widetilde{\zeta}}$ of R-charge $+1$ corresponds to a solution $\widetilde{\zeta}$ of

$$\left(\nabla_\mu + iA_\mu\right)\widetilde{\zeta} = iV_\mu\widetilde{\zeta} + iV^\nu\widetilde{\sigma}_{\mu\nu}\widetilde{\zeta} \ . \tag{3.1.12}$$

Note that the presence of rigid supersymmetry does not depend on the details of the field theory, since (3.1.11) and (3.1.12) only involve supergravity background fields. From the algebra of local supergravity transformations [95, 96], we find that

See also [99, 100].

the commutation relations satisfied by the supercharges corresponding to ζ and $\widetilde{\zeta}$ take the form

$$\begin{aligned}
&\{\delta_\zeta, \delta_{\widetilde{\zeta}}\} = 2i\delta_K \ , \\
&\{\delta_\zeta, \delta_\zeta\} = \{\delta_{\widetilde{\zeta}}, \delta_{\widetilde{\zeta}}\} = 0 \ , \\
&[\delta_K, \delta_\zeta] = [\delta_K, \delta_{\widetilde{\zeta}}] = 0 \ .
\end{aligned} \tag{3.1.13}$$

The fact that $\delta_\zeta^2 = 0$ follows from the R-symmetry. If $\widetilde{\zeta}$ is absent, this comprises the entire superalgebra. In the presence of $\widetilde{\zeta}$, we can form a complex vector $K = K^\mu \partial_\mu$ with $K^\mu = \zeta\sigma^\mu\widetilde{\zeta}$ and δ_K is the variation generated by the R-covariant Lie derivative along K. When acting on objects of R-charge q, it is given by

$$\delta_K = \mathcal{L}_K^A = \mathcal{L}_K - iqK^\mu A_\mu \ , \tag{3.1.14}$$

where $\mathcal{L}_K$ is the conventional Lie derivative.[3] As we will see below, K is a Killing vector. The fact that δ_K commutes with δ_ζ and $\delta_{\widetilde{\zeta}}$ is required for the consistency of (3.1.13).[4]

In this chapter, we will analyze Riemannian four-manifolds $\mathcal{M}$ that admit one or several solutions of (3.1.11) and (3.1.12). In section 3.2, we will discuss the various objects that appear in these equations, and comment on some of their general properties that will be used subsequently. The equations (3.1.11) and (3.1.12) do not admit solutions for arbitrary values of $g_{\mu\nu}$, A_μ, and V_μ. This is due to the fact that they are partial differential equations, which are only consistent if the background fields satisfy

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[3] The Lie derivative of ζ along a vector $X = X^\mu\partial_\mu$ is given by

$$\mathcal{L}_X\zeta = X^\mu\nabla_\mu\zeta - \frac{1}{2}\nabla_\mu X_\nu \sigma^{\mu\nu}\zeta \ , \tag{3.1.15}$$

and similarly for $\widetilde{\zeta}$. See appendix A.

[4] If there are other supercharges, which correspond to additional solutions η or $\widetilde{\eta}$ of (3.1.11) or (3.1.12), the Killing vector K need not commute with them,

$$[\delta_K, \delta_\eta] = -\delta_{\mathcal{L}_K^A \eta} \ , \qquad [\delta_K, \delta_{\tilde{\eta}}] = -\delta_{\mathcal{L}_K^A \tilde{\eta}} \ . \tag{3.1.16}$$

certain integrability conditions. Additionally, there may be global obstructions. We would like to understand the restrictions imposed by the presence of one or several solutions, and formulate sufficient conditions for their existence.

In section 3.3, we show that $\mathcal{M}$ admits a single scalar supercharge, if and only if it is Hermitian. In this case, we can rewrite (3.1.11) as

$$\left(\nabla^c_\mu - A^c_\mu\right)\zeta = 0 \ , \tag{3.1.17}$$

where ∇^c_μ is the Chern connection adapted to the complex structure and A^c_μ is simply related to A_μ. The ability to cast (3.1.11) in this form crucially relies on the presence of V_μ, which is related to the torsion of the Chern connection. On a Kähler manifold (3.1.17) reduces to (3.1.2). More generally, it allows us to adapt the twisting procedure of [82] to Hermitian manifolds that are not Kähler. As we will see, the auxiliary fields A_μ and V_μ are not completely determined by the geometry. This freedom, which resides in the non-minimal couplings parametrized by A_μ and V_μ, reflects the fact that we can place a given field theory on $\mathcal{M}$ in several different ways, while preserving one supercharge (see appendix B).

In section 3.4, we consider manifolds admitting two solutions ζ and $\widetilde{\zeta}$ of opposite R-charge. As was mentioned above, we can use them to construct a complex Killing vector $K^\mu = \zeta\sigma^\mu\widetilde{\zeta}$. This situation is realized on any Hermitian manifold with metric

$$ds^2 = \Omega(z,\overline{z})^2\left((dw + h(z,\overline{z})dz)(d\overline{w} + \overline{h}(z,\overline{z})d\overline{z}) + c(z,\overline{z})^2 dz d\overline{z}\right) \ , \tag{3.1.18}$$

where w, z are holomorphic coordinates. The metric (3.1.18) describes a two-torus fibered over a Riemann surface Σ with metric $ds^2_\Sigma = \Omega^2 c^2 dz d\overline{z}$. As in the case of a single supercharge, ζ and $\widetilde{\zeta}$ turn out to be scalars on $\mathcal{M}$. Upon dimensional reduction, they give rise to two supercharges on Seifert manifolds that are circle bundles over Σ. Rigid supersymmetry on such manifolds was recently discussed in [101,102]. Reducing once more, we make contact with the A-twist on Σ [103, 104].

Section 3.5 describes manifolds admitting two supercharges of equal R-charge. This case turns out to be very restrictive. When $\mathcal{M}$ is compact, we will show that it must be hyperhermitian. Using the classification of [105], this allows us to constrain $\mathcal{M}$ to be one of the following: a flat torus T^4, a $K3$ surface with Ricci-flat Kähler metric, or $S^3 \times S^1$ with the standard metric $ds^2 = d\tau^2 + r^2 d\Omega_3$ and certain quotients thereof. We also comment on the non-compact case, which is less constrained.

In section 3.6 we describe manifolds admitting four supercharges. They are locally isometric to $\mathcal{M}_3 \times \mathbb{R}$, where $\mathcal{M}_3$ is one of the maximally symmetric spaces S^3, T^3, or H^3. (The size of $\mathcal{M}_3$ does not vary along $\mathbb{R}$.) In this case, the auxiliary fields A_μ and V_μ are tightly constrained.

We conclude in section 3.7 by considering several explicit geometries that illustrate our general analysis. Our conventions are summarized in appendix A. In appendix B we review the procedure of [31] to place a four-dimensional $\mathcal{N} = 1$ theory on a Riemannian manifold $\mathcal{M}$ in a supersymmetric way, focusing on theories with a $U(1)_R$ symmetry. Appendix C contains some supplementary material related to section 3.4.

3.2 General Properties of the Equations

In this section we will lay the groundwork for our discussion of the equations (3.1.11) and (3.1.12),

$$\begin{aligned} (\nabla_\mu - iA_\mu)\,\zeta &= -iV_\mu\zeta - iV^\nu\sigma_{\mu\nu}\zeta\ , \\ (\nabla_\mu + iA_\mu)\,\widetilde{\zeta} &= iV_\mu\widetilde{\zeta} + iV^\nu\widetilde{\sigma}_{\mu\nu}\widetilde{\zeta}\ . \end{aligned} \tag{3.2.1}$$

We will study them on a smooth, oriented, connected four-manifold $\mathcal{M}$, endowed with a Riemannian metric $g_{\mu\nu}$. The Levi-Civita connection is denoted by ∇_μ. As we have explained in the introduction, the background fields A_μ and V_μ are generally complex, and V^μ is conserved, $\nabla_\mu V^\mu = 0$. Note that the equations (3.2.1) are invariant under

$$\zeta \to \zeta^\dagger\ , \qquad \widetilde{\zeta} \to \widetilde{\zeta}^\dagger\ , \qquad A_\mu \to -\overline{A}_\mu\ , \qquad V_\mu \to -\overline{V}_\mu\ . \tag{3.2.2}$$

Under local frame rotations $SU(2)_+ \times SU(2)_-$, the spinors ζ and $\widetilde{\zeta}$ transform as $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Additionally, they carry charge $+1$ and -1 under conventional R-

transformations. Locally, the equations (3.2.1) are also invariant under complexified R-transformations, and this is reflected in various formulas below. However, we will not make use of such transformations. (One reason is that they could lead to pathologies in the field theory.) Therefore, the real part of A_μ transforms as a gauge field for the local $U(1)_R$ symmetry, while the imaginary part is a well-defined one-form. In summary, ζ is a section of $L \otimes S_+$, where L is a unitary line bundle and S_+ is the bundle of left-handed spinors, and $\widetilde{\zeta}$ is a section of $L^{-1} \otimes S_-$ with S_- the bundle of right-handed spinors. The transition functions of L consist of local $U(1)_R$ transformations, and the connection on L is given by the real part of A_μ.

Let us briefly comment on some global properties of the various objects introduced above. (For a more thorough discussion, see for instance [80].) If $\mathcal{M}$ is a spin manifold, we can choose well-defined bundles $S_\pm$. In this case the line bundle L is also well defined. In general, an oriented Riemannian four manifold does not possess a spin structure. It does, however, admit a spin^c structure. In this case it is possible to define well-behaved product bundles $L \otimes S_+$ and $L^{-1} \otimes S_-$, even though $S_\pm$ and L do not exist. However, even powers of L are well defined.

Since the equations (3.2.1) are linear, the solutions have the structure of a complex vector space, which decomposes into solutions ζ with R-charge $+1$ and solutions $\widetilde{\zeta}$ with R-charge -1. The fact that the equations are also first-order, with smooth coefficients, implies that any solution is determined by its value at a single point. Therefore, any nontrivial solution is nowhere vanishing, and this will be crucial below. Moreover, there are at most two solutions of R-charge $+1$, and likewise for R-charge -1.

The equations (3.2.1) do not admit solutions for arbitrary values of $g_{\mu\nu}$, A_μ, and V_μ. This is due to the fact that they are partial differential equations, which are only consistent if the background fields satisfy certain integrability conditions. Additionally, there may be global obstructions. Before attempting to solve the equations in general, we will analyze the restrictions on the background fields due to the presence of one or several solutions. Given one or several spinors satisfying (3.2.1), it is useful to construct spinor bilinears, and these will feature prominently in our analysis. Here we will introduce various interesting bilinears and list some of their properties. These follow only from Fierz identities and do not make use of the equations (3.2.1). We will only need the fact that the spinors are non-vanishing.

Given a spinor $\zeta \in L \otimes S_+$, its norm $|\zeta|^2$ is a scalar. More interestingly, we can define a real, self-dual two-form,

$$J_{\mu\nu} = \frac{2i}{|\zeta|^2}\zeta^\dagger \sigma_{\mu\nu}\zeta \ , \tag{3.2.3}$$

which satisfies

$$J^\mu{}_\nu J^\nu{}_\rho = -\delta^\mu{}_\rho \ . \tag{3.2.4}$$

Therefore, $J^\mu{}_\nu$ is an almost complex structure, which splits the complexified tangent space at every point into holomorphic and anti-holomorphic subspaces. The holomorphic tangent space has the following useful characterization [80]: a vector X^μ is holomorphic with respect to $J^\mu{}_\nu$ if and only if $X^\mu \widetilde{\sigma}_\mu \zeta = 0$.[5]

We can also define another complex bilinear,

$$P_{\mu\nu} = \zeta\sigma_{\mu\nu}\zeta \ , \tag{3.2.5}$$

which is a section of $L^2 \otimes \Lambda^2_+$, where Λ^2_+ denotes the bundle of self-dual two-forms. We find that

$$J_\mu{}^\rho P_{\rho\nu} = iP_{\mu\nu} \ , \tag{3.2.6}$$

and hence $P_{\mu\nu}$ is anti-holomorphic with respect to the almost complex structure $J^\mu{}_\nu$.

Suppose we are given another spinor $\widetilde{\zeta} \in L^{-1} \otimes S_-$. Then we can define an anti-self-dual two-form,

$$\widetilde{J}_{\mu\nu} = \frac{2i}{|\widetilde{\zeta}|^2}\widetilde{\zeta}^\dagger \widetilde{\sigma}_{\mu\nu}\widetilde{\zeta} \ . \tag{3.2.7}$$

Again, we find that $\widetilde{J}^\mu{}_\nu \widetilde{J}^\nu{}_\rho = -\delta^\mu{}_\rho$, so that $\widetilde{J}^\mu{}_\nu$ is another almost complex structure. The two almost complex structures $J^\mu{}_\nu$ and $\widetilde{J}^\mu{}_\nu$ commute,

$$J^\mu{}_\nu \widetilde{J}^\nu{}_\rho - \widetilde{J}^\mu{}_\nu J^\nu{}_\rho = 0 \ . \tag{3.2.8}$$

[5]To see this, we can multiply $X^\nu \widetilde{\sigma}_\nu \zeta = 0$ by $\zeta^\dagger \sigma^\mu$ and use (3.2.3) to obtain $J^\mu{}_\nu X^\nu = iX^\mu$. Conversely, if X^μ is holomorphic then $\zeta^\dagger \sigma^\mu \widetilde{\sigma}_\nu \zeta X^\nu = 0$. Multiplying by $\overline{X}_\mu$ we find $|X^\mu \widetilde{\sigma}_\mu \zeta|^2 = 0$, and hence $X^\mu \widetilde{\sigma}_\mu \zeta = 0$.

Combining ζ and $\widetilde{\zeta}$, we can also construct a complex vector $K = K^\mu \partial_\mu$ with

$$K^\mu = \zeta \sigma^\mu \widetilde{\zeta} \ . \tag{3.2.9}$$

It squares to zero, $K^\mu K_\mu = 0$, and it is holomorphic with respect to both $J^\mu{}_\nu$ and $\widetilde{J}^\mu{}_\nu$,

$$J^\mu{}_\nu K^\nu = \widetilde{J}^\mu{}_\nu K^\nu = i K^\mu \ . \tag{3.2.10}$$

The norm of K is determined by the norms of ζ and $\widetilde{\zeta}$,

$$\overline{K}^\mu K_\mu = 2|\zeta|^2 |\widetilde{\zeta}|^2 \ . \tag{3.2.11}$$

It will be useful to express $J_{\mu\nu}$ and $\widetilde{J}_{\mu\nu}$ directly in terms of K_μ,

$$\begin{aligned} J_{\mu\nu} &= Q_{\mu\nu} + \frac{1}{2} \varepsilon_{\mu\nu\rho\lambda} Q^{\rho\lambda} \ , \\ \widetilde{J}_{\mu\nu} &= Q_{\mu\nu} - \frac{1}{2} \varepsilon_{\mu\nu\rho\lambda} Q^{\rho\lambda} \ , \\ Q_{\mu\nu} &= \frac{i}{\overline{K}^\lambda K_\lambda} \left(K_\mu \overline{K}_\nu - K_\nu \overline{K}_\mu \right) \ . \end{aligned} \tag{3.2.12}$$

Finally, we consider two spinors $\zeta, \eta \in L \otimes S_+$. As above, they give rise to almost complex structures,

$$J^\mu{}_\nu = \frac{2i}{|\zeta|^2} \zeta^\dagger \sigma^\mu{}_\nu \zeta \ , \qquad I^\mu{}_\nu = \frac{2i}{|\eta|^2} \eta^\dagger \sigma^\mu{}_\nu \eta \ . \tag{3.2.13}$$

Their anticommutator is given by

$$\begin{aligned} & J^\mu{}_\nu I^\nu{}_\rho + I^\mu{}_\nu J^\nu{}_\rho = -2 f \delta^\mu{}_\rho \ , \\ & f = 2 \, \frac{|\zeta^\dagger \eta|^2}{|\zeta|^2 |\eta|^2} - 1 \ . \end{aligned} \tag{3.2.14}$$

It follows from the Cauchy-Schwarz inequality that $-1 \leq f \leq 1$, so that $f = 1$ if and only if ζ is proportional to η. In this case $J^\mu{}_\nu = I^\mu{}_\nu$. Similarly, $f = -1$ if and only if ζ is proportional to $\eta^\dagger$, so that $J^\mu{}_\nu = -I^\mu{}_\nu$. By appropriately choosing independent solutions ζ and η of (3.2.1) we can always arrange for $f \neq \pm 1$ at a given point. This fact will be used in section 3.5.

3.3 Manifolds Admitting One Supercharge

In this section we will analyze manifolds $\mathcal{M}$ that admit a solution ζ of (3.1.11),

$$(\nabla_\mu - iA_\mu)\,\zeta = -iV_\mu\zeta - iV^\nu\sigma_{\mu\nu}\zeta \ . \tag{3.3.1}$$

The presence of such a solution implies that $\mathcal{M}$ is Hermitian. Conversely, we will show that a solution exists on any Hermitian manifold.

3.3.1 Restrictions Imposed by ζ

In section 3.2 we used the fact that solutions of (3.3.1) are nowhere vanishing to construct various bilinears out of ζ, and we established some of their properties at a fixed point on $\mathcal{M}$. Here we will use the fact that ζ satisfies (3.3.1) to study their derivatives. We begin by proving that the almost complex structure $J^\mu{}_\nu$ defined in (3.2.3) is integrable, so that $\mathcal{M}$ is a complex manifold with Hermitian metric $g_{\mu\nu}$. It suffices to show that the commutator of two holomorphic vector fields is also holomorphic. Recall from section 3.2 that a vector X^μ is holomorphic with respect to $J^\mu{}_\nu$ if and only if $X^\mu\widetilde{\sigma}_\mu\zeta = 0$. By differentiating this formula, contracting with another holomorphic vector Y^μ, and antisymmmetrizing, one finds that $[X, Y]$ is holomorphic if and only if [80]

$$X^{[\mu}Y^{\nu]}\widetilde{\sigma}_\mu\nabla_\nu\zeta = 0 \ . \tag{3.3.2}$$

Using (3.3.1) and the fact that X^μ, Y^μ are holomorphic, we find that this is indeed the case, and hence $J^\mu{}_\nu$ is integrable.

Alternatively, we can use (3.3.1) to compute $\nabla_\mu J^\nu{}_\rho$ directly (this is straightforward but tedious), and show that the Nijenhuis tensor of $J^\mu{}_\nu$ vanishes,

$$N^\mu{}_{\nu\rho} = J^\lambda{}_\nu\nabla_\lambda J^\mu{}_\rho - J^\lambda{}_\rho\nabla_\lambda J^\mu{}_\nu - J^\mu{}_\lambda\nabla_\nu J^\lambda{}_\rho + J^\mu{}_\lambda\nabla_\rho J^\lambda{}_\nu = 0 \ . \tag{3.3.3}$$

Again, it follows that the almost complex structure $J^\mu{}_\nu$ is integrable.

Using the complex structure, we can introduce local holomorphic coordinates z^i $(i = 1, 2)$. We will denote holomorphic and anti-holomorphic indices by un-barred and barred lowercase Latin letters respectively. In these coordinates, the complex structure takes the form,

$$J^i{}_j = i\delta^i{}_j \ , \qquad J^{\bar{i}}{}_{\bar{j}} = -i\delta^{\bar{i}}{}_{\bar{j}} \ . \tag{3.3.4}$$

Lowering both indices, we obtain the Kähler form of the Hermitian manifold,

$$J_{i\bar{j}} = -ig_{i\bar{j}} \ . \tag{3.3.5}$$

It is a real $(1,1)$ form. The Kähler form $J_{\mu\nu}$ is not covariantly constant with respect to the Levi-Civita connection, unless the manifold is Kähler. Instead, we compute using (3.3.1),

$$\nabla_\mu J^\mu{}_\nu = -(V_\nu + \overline{V}_\nu) + i(V_\mu - \overline{V}_\mu)J^\mu{}_\nu \ . \tag{3.3.6}$$

This implies that V_μ takes the form

$$V_\mu = -\frac{1}{2}\nabla_\nu J^\nu{}_\mu + U_\mu \ , \qquad J_\mu{}^\nu U_\nu = iU_\mu \ . \tag{3.3.7}$$

Since U_μ only has anti-holomorphic components $U_{\bar{i}}$, we see that $V_{\bar{i}}$ is not determined by $J^\mu{}_\nu$. This freedom in V_μ was already mentioned in the introduction, where it reflected an ambiguity in passing from (3.1.5) to (3.1.6); see also the discussion in appendix B. Imposing conservation of V_μ leads to

$$\nabla^\mu U_\mu = 0 \ . \tag{3.3.8}$$

Recall from (3.1.9) that V_μ is the dual field strength of a two-form gauge field $B_{\mu\nu}$. We can then express (3.3.7) as $B_{\mu\nu} = \frac{1}{2}J_{\mu\nu} + \cdots$, where the ellipsis denotes additional terms that reflect the freedom in $V_{\bar{i}}$.

Since $\mathcal{M}$ is Hermitian, it is natural to adopt a connection that is compatible with both the metric $g_{\mu\nu}$ and the complex structure $J^\mu{}_\nu$. As we remarked above, this is not the case for the Levi-Civita connection ∇_μ, unless the manifold is Kähler. We will instead use the Chern connection ∇^c_μ, which has the property that $\nabla^c_\mu g_{\nu\rho} = 0$ and $\nabla^c_\mu J^\nu{}_\rho = 0$. This corresponds to replacing the ordinary spin connection $\omega_{\mu\nu\rho}$ by

$$\omega^c_{\mu\nu\rho} = \omega_{\mu\nu\rho} - \frac{1}{2}J_\mu{}^\lambda\left(\nabla_\lambda J_{\nu\rho} + \nabla_\nu J_{\rho\lambda} + \nabla_\rho J_{\lambda\nu}\right) \ . \tag{3.3.9}$$

Rewriting the spinor equation (3.3.1) in terms of the Chern connection, we obtain

$$\left(\nabla_\mu^c - iA_\mu^c\right)\zeta = 0 \;, \tag{3.3.10}$$

where we have defined

$$A_\mu^c = A_\mu + \frac{1}{4}\left(\delta_\mu{}^\nu - iJ_\mu{}^\nu\right)\nabla_\rho J^\rho{}_\nu - \frac{3}{2}U_\mu \;. \tag{3.3.11}$$

Note that A_μ^c and A_μ only differ by a well-defined one-form, and hence they shift in the same way under R-transformations.

To summarize, a solution ζ of (3.3.1) defines an integrable complex structure $J^\mu{}_\nu$ and an associated Chern connection. In turn, the spinor ζ is covariantly constant with respect to the Chern connection twisted by A_μ^c in (3.3.11).

When $\mathcal{M}$ is Kähler, the Chern connection coincides with the Levi-Civita connection, and $A_\mu^c = A_\mu$ if we choose $U_\mu = 0$. In this case (3.3.10) reduces to (3.1.2),

$$\left(\nabla_\mu - iA_\mu\right)\zeta = 0 \;. \tag{3.3.12}$$

Conversely, it is well-known that this equation admits a solution on any Kähler manifold [80]. Intuitively, this follows from the $\mathcal{N} = 1$ twisting procedure described in [82]. On a Kähler manifold, the holonomy of the Levi-Civita connection is given by $U(2) = U(1)_+ \times SU(2)_-$ with $U(1)_+ \subset SU(2)_+$. For an appropriate choice of $U(1)_R$ connection A_μ, we can cancel the $U(1)_+$ component of the spin connection to obtain a scalar supercharge on $\mathcal{M}$. Similarly, it was shown in [106] that the $\mathcal{N} = 2$ twisting procedure of [107] can be interpreted in terms of a certain generalization of (3.3.12).

Equation (3.3.10) allows us to generalize this argument to an arbitrary Hermitian manifold. Given a complex structure $J^\mu{}_\nu$, the holonomy of the Chern connection is contained in $U(2)$. As above, we can twist by A_μ^c to obtain a solution ζ, which transforms as a scalar. This solution is related to the complex structure as in (3.2.3). Choosing V_μ as in (3.3.7) and A_μ as in (3.3.11), we see that ζ also satisfies (3.3.1). Therefore, we can solve (3.3.1) on any Hermitian manifold to obtain a scalar supercharge. We will describe the explicit solution in the next subsection. Here we will explore some of its properties, assuming that it exists.

Consider $P_{\mu\nu} = \zeta\sigma_{\mu\nu}\zeta$, which was defined in (3.2.5). Note that $P_{\mu\nu}$ locally determines ζ up to a sign. It follows from (3.2.6) that $P_{\mu\nu}$ is a nowhere vanishing section of $L^2 \otimes \overline{\mathcal{K}}$, where $\overline{\mathcal{K}} = \Lambda^{0,2}$ is the anti-canonical bundle of $(0,2)$ forms. This implies that the line bundle $L^2 \otimes \overline{\mathcal{K}}$ is trivial, and hence we can identify $L = (\overline{\mathcal{K}})^{-\frac{1}{2}}$, up to a trivial line bundle. If $\mathcal{M}$ is not spin, the line bundle $(\overline{\mathcal{K}})^{-\frac{1}{2}}$ is not globally well defined. However, it does correspond to a good spin^c structure on $\mathcal{M}$.

More explicitly, we work in a patch with coordinates z^i and define $p = P_{\overline{12}}$. Since the induced metric on $\overline{\mathcal{K}}$ is given by $\frac{1}{\sqrt{g}}$ with $g = \det(g_{\mu\nu})$, it follows that $\frac{1}{\sqrt{g}}|p|^2$ is a positive scalar on $\mathcal{M}$. We are therefore led to consider

$$s = p\, g^{-\frac{1}{4}} \ , \tag{3.3.13}$$

which is nowhere vanishing and has R-charge 2. Under holomorphic coordinate changes s transforms by a phase,

$$z'^i = z'^i(z) \ , \qquad s'(z') = s(z)\left(\det\left(\frac{\partial z'^i}{\partial z^j}\right)\right)^{\frac{1}{2}}\left(\det\left(\overline{\frac{\partial z'^i}{\partial z^j}}\right)\right)^{-\frac{1}{2}} \ . \tag{3.3.14}$$

We can locally compensate these phase rotations by appropriate R-transformations. Under these combined transformations s transforms as a scalar. Starting from a section p of the trivial line bundle $L^2 \otimes \overline{\mathcal{K}}$ and dividing by a power of the trivial determinant bundle, we have thus produced a scalar s. As we will see in the next subsection, the scalar s determines the scalar supercharge corresponding to ζ.

We will now solve for A_μ in terms of s. It follows from (3.3.10) that

$$\left(\nabla^c_\mu - 2iA^c_\mu\right) p = 0 \ . \tag{3.3.15}$$

The Chern connection acts on sections of the anti-canonical bundle in a simple way,

$$\nabla^c_i p = \partial_i p \ , \qquad \nabla^c_{\bar{i}} p = \partial_{\bar{i}} p - \frac{p}{2}\partial_{\bar{i}} \log g \ . \tag{3.3.16}$$

Substituting into (3.3.15) and using (3.3.13), we obtain A^c_μ and hence A_μ,

$$\begin{aligned} A_\mu &= A^c_\mu - \frac{1}{4}\left(\delta_\mu{}^\nu - iJ_\mu{}^\nu\right)\nabla_\rho J^\rho{}_\nu + \frac{3}{2}U_\mu \ , \\ A^c_i &= -\frac{i}{8}\partial_i \log g - \frac{i}{2}\partial_i \log s \ , \\ A^c_{\bar{i}} &= \frac{i}{8}\partial_{\bar{i}} \log g - \frac{i}{2}\partial_{\bar{i}} \log s \ . \end{aligned} \tag{3.3.17}$$

Note that s appears in (3.3.17) as the parameter of complexified local R-transformations.

3.3.2 Solving for ζ on a Hermitian Manifold

We will now show that it is possible to solve the equation (3.3.1) on a general Hermitian manifold $\mathcal{M}$, given its metric $g_{\mu\nu}$ and complex structure $J^\mu{}_\nu$. The solution is not completely determined by these geometric structures. It also depends on a choice of conserved, anti-holomorphic U_μ and a complex, nowhere vanishing scalar s on $\mathcal{M}$. In terms of this additional data, the background fields V_μ and A_μ are given by (3.3.7) and (3.3.17).

We will work in a local frame that is adapted to the Hermitian metric on $\mathcal{M}$. This corresponds to a choice of vielbein $e^1, e^2 \in \Lambda^{(1,0)}$ and $e^{\bar{1}}, e^{\bar{2}} \in \Lambda^{(0,1)}$, which satisfies

$$ds^2 = e^1 e^{\bar{1}} + e^2 e^{\bar{2}} \ . \tag{3.3.18}$$

Any two such frames are related by a transformation in $U(2) \subset SU(2)_+ \times SU(2)_-$. Since (3.3.18) is preserved by parallel transport with the Chern connection, we see that its holonomy is contained in $U(2)$. More explicitly, we choose

$$\frac{1}{\sqrt{2}} e^1 = \sqrt{g_{1\bar{1}}}\, dz^1 + \frac{g_{2\bar{1}}}{\sqrt{g_{1\bar{1}}}}\, dz^2, \qquad \frac{1}{\sqrt{2}} e^2 = \frac{g^{\frac{1}{4}}}{\sqrt{g_{1\bar{1}}}}\, dz^2 \ . \tag{3.3.19}$$

In this frame, the solution of (3.3.1) with our choice of background fields is given by

$$\zeta_\alpha = \frac{\sqrt{s}}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \tag{3.3.20}$$

The complex structure is then indeed given by ζ as in (3.2.3).

We have specified that s is a scalar on $\mathcal{M}$, yet ζ in (3.3.20) only depends on s. We will now discuss the transformation properties of ζ, and explain to what extent it can be considered a scalar as well. Under a holomorphic coordinate change $z'^i = z'^i(z)$, the metric and the vielbein transform in the usual way. In the z'-coordinates, we can also define another frame f'^1, f'^2, which is related to $g'_{i\bar{j}}$ as in (3.3.19). In this frame, the spinor ζ' takes the same form as in (3.3.20). The frames f' and e' are related by a matrix $\mathcal{U} \in U(2)$ via $f' = \mathcal{U}e'$. To relate the spinors ζ' and ζ, we will only need the determinant of $\mathcal{U}$,[6]

$$\begin{aligned} \zeta' &= \sqrt{\det \mathcal{U}}\, \zeta \ , \\ \det \mathcal{U} &= \left(\det \left(\frac{\partial z'^i}{\partial z^j} \right) \right)^{\frac{1}{2}} \left(\det \left(\overline{\frac{\partial z'^i}{\partial z^j}} \right) \right)^{-\frac{1}{2}} \ . \end{aligned} \tag{3.3.21}$$

Hence, ζ' and ζ only differ by a phase, and this can be undone by an appropriate R-transformation. Under this combined transformation ζ transforms as a scalar, which is related to the scalar s via (3.3.20). Note that the phase of s can be removed by a globally well-defined R-transformation. (If we were to allow complexified R-transformations, we could set $s = 1$ everywhere on $\mathcal{M}$.)

3.3.3 Restrictions Imposed by $\widetilde{\zeta}$

It is straightforward to repeat the analysis above in the presence of a solution $\widetilde{\zeta}$ of (3.1.12). As before, the complex structure $\widetilde{J}^\mu{}_\nu$ in (3.2.7) is integrable and determines the holomorphic part of V_μ,

$$V_\mu = \frac{1}{2} \nabla_\nu \widetilde{J}^\nu{}_\mu + \widetilde{U}_\mu \ , \qquad \widetilde{J}_\mu{}^\nu \widetilde{U}_\nu = i \widetilde{U}_\mu \ , \qquad \nabla^\mu \widetilde{U}_\mu = 0 \ . \tag{3.3.22}$$

The gauge field A_μ then takes the form

$$\begin{aligned} A_\mu &= A^c_\mu + \frac{1}{4} \left(\delta_\mu{}^\nu - i \, \widetilde{J}_\mu{}^\nu \right) \nabla_\rho \widetilde{J}^\rho{}_\nu + \frac{3}{2} \widetilde{U}_\mu \ , \\ A^c_i &= \frac{i}{8} \partial_i \log g + \frac{i}{2} \partial_i \log \widetilde{s} \ , \\ A^c_{\bar{i}} &= -\frac{i}{8} \partial_{\bar{i}} \log g + \frac{i}{2} \partial_{\bar{i}} \log \widetilde{s} \ . \end{aligned} \tag{3.3.23}$$

As above, $\widetilde{s}$ is a complex scalar that determines $\widetilde{\zeta}$.

[6]This follows from the fact that the complex structure $J^\mu{}_\nu$ can be written in terms of ζ as in (3.2.3), which implies that local $U(2)$ frame rotations are identified with $U(1)_+ \times SU(2)_- \subset SU(2)_+ \times SU(2)_-$.

3.4 Manifolds Admitting Two Supercharges of Opposite R-Charge

In this section we will consider manifolds $\mathcal{M}$ on which it is possible to find a pair ζ and $\widetilde{\zeta}$ that solves the equations in (3.1.11) and (3.1.12),

$$\begin{aligned}(\nabla_\mu - iA_\mu)\,\zeta &= -iV_\mu\zeta - iV^\nu\sigma_{\mu\nu}\zeta\ ,\\ (\nabla_\mu + iA_\mu)\,\widetilde{\zeta} &= iV_\mu\widetilde{\zeta} + iV^\nu\widetilde{\sigma}_{\mu\nu}\widetilde{\zeta}\ .\end{aligned} \tag{3.4.1}$$

Again, we begin by analyzing the restrictions imposed by the presence of ζ and $\widetilde{\zeta}$, before establishing sufficient conditions for their existence. As discussed in the introduction, the solutions ζ and $\widetilde{\zeta}$ give rise to a Killing vector field $K = K^\mu\partial_\mu$ with $K^\mu = \zeta\sigma^\mu\widetilde{\zeta}$. Together with its complex conjugate $\overline{K}$, it generates part of the isometry group of $\mathcal{M}$. There are two qualitatively different cases depending on whether K and $\overline{K}$ commute. In this section we will discuss the case when they do commute, and we will show that $\mathcal{M}$ can be described as a fibration of a torus T^2 over an arbitrary Riemann surface Σ. The non-commuting case turns out to be very restrictive. It is discussed in sections 3.6 and 3.7, as well as appendix C.

3.4.1 Restrictions Imposed by ζ and $\widetilde{\zeta}$

We begin by assuming the existence of two spinors ζ and $\widetilde{\zeta}$ that solve the equations (3.4.1). From the analysis of the previous section we know that they give rise to two complex structures $J^\mu{}_\nu$ and $\widetilde{J}^\mu{}_\nu$, both of which are compatible with the metric. Recall from section 3.2 that the nowhere vanishing complex vector field $K^\mu = \zeta\sigma^\mu\widetilde{\zeta}$ is holomorphic with respect to both complex structures. We can now use the fact that ζ and $\widetilde{\zeta}$ satisfy the equations (3.4.1) to show that K is a Killing vector,

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0\ . \tag{3.4.2}$$

The fact that $K^\mu K_\mu = 0$ allows us to constrain the algebra satisfied by K and its complex conjugate $\overline{K}$ (see appendix C). When they do not commute, there are additional Killing vectors and the equations (3.4.1) imply that the manifold is locally isometric to $S^3 \times \mathbb{R}$ with metric

$$ds^2 = d\tau^2 + r^2 d\Omega_3 \ . \tag{3.4.3}$$

Here $d\Omega_3$ is the round metric on the unit three-sphere. This case will be discussed in sections 3.6 and 3.7.

In the remainder of this section we will analyze the case in which the Killing vector K commutes with its complex conjugate $\overline{K}$,

$$\overline{K}^\nu \nabla_\nu K^\mu - K^\nu \nabla_\nu \overline{K}^\mu = 0 \ . \tag{3.4.4}$$

Using the complex structure $J^\mu{}_\nu$, we can introduce holomorphic coordinates w, z. Since K is holomorphic with respect to $J^\mu{}_\nu$ and satisfies (3.4.4), we can choose these coordinates so that $K = \partial_w$. The metric then takes the form

$$ds^2 = \Omega(z,\overline{z})^2 \left((dw + h(z,\overline{z})dz)(d\overline{w} + \overline{h}(z,\overline{z})d\overline{z}) + c(z,\overline{z})^2 dz d\overline{z} \right) \ . \tag{3.4.5}$$

The conformal factor Ω^2 is determined by the norm of K, which in turn depends on the norms of ζ and $\widetilde{\zeta}$ as in (3.2.11),

$$\Omega^2 = 2\overline{K}^\mu K_\mu = 4|\zeta|^2|\widetilde{\zeta}|^2 \ . \tag{3.4.6}$$

The metric (3.4.5) describes a two-torus T^2 fibered over a Riemann surface Σ with metric $ds^2_\Sigma = \Omega^2 c^2 dz d\overline{z}$. As we will see below, the metric (3.4.5) admits a second compatible complex structure, which can be identified with $\widetilde{J}^\mu{}_\nu$.

We will now constrain the form of the background field V_μ. First note that $\nabla_\nu J^\nu{}_\mu = -\nabla_\nu \widetilde{J}^\nu{}_\mu$, which follows from (3.4.2), (3.4.4), and the expressions (3.2.12) for $J^\mu{}_\nu$ and $\widetilde{J}^\mu{}_\nu$ in terms of K. Since ζ is a solution of (3.4.1), it must be that V_μ satisfies (3.3.7). Similarly (3.3.22) must hold because $\widetilde{\zeta}$ is also a solution. Consistency of these expressions requires the two conserved vectors U_μ and $\widetilde{U}_\mu$ to satisfy

$$U_\mu = \widetilde{U}_\mu \ , \quad J_\mu{}^\nu U_\nu = i U_\mu \ , \quad \widetilde{J}_\mu{}^\nu \widetilde{U}_\nu = i \widetilde{U}_\mu \ . \tag{3.4.7}$$

Recall from (3.2.8) that the two complex structures $J^\mu{}_\nu$ and $\widetilde{J}^\mu{}_\nu$ commute. Moreover, they have opposite self-duality, so that the space of vectors that are holomorphic under both is one dimensional. Hence, $U_\mu = \widetilde{U}_\mu$ must be proportional to K_μ everywhere. In summary,

$$V_\mu = -\frac{1}{2}\nabla_\nu J^\nu{}_\mu + \kappa K_\mu = \frac{1}{2}\nabla_\nu \widetilde{J}^\nu{}_\mu + \kappa K_\mu \ , \qquad K^\mu \partial_\mu \kappa = 0 \ . \tag{3.4.8}$$

Here κ is a complex scalar function on $\mathcal{M}$, which is constrained by the conservation of V_μ.

Given the form of V_μ in (3.4.8) and the spinors ζ and $\widetilde{\zeta}$, the gauge field A_μ is completely determined. It is given by (3.3.17), or alternatively (3.3.23). It can be checked that the consistency of these two equations does not impose additional restrictions on the metric or the background fields. This also follows from the explicit solution presented in the next subsection.

3.4.2 Solving for ζ and $\widetilde{\zeta}$

Here we will establish a converse to the results of the previous subsection: we can find a pair ζ and $\widetilde{\zeta}$ that solves the equations (3.4.1) whenever the metric $g_{\mu\nu}$ admits a complex Killing vector K that squares to zero, $K^\mu K_\mu = 0$, and commutes with its complex conjugate as in (3.4.4). Note that we do not assume that $\mathcal{M}$ is Hermitian. Instead, we can use K to define $J^\mu{}_\nu$ and $\widetilde{J}^\mu{}_\nu$ through the formula (3.2.12), without making reference to ζ and $\widetilde{\zeta}$. Since $K^\mu K_\mu = 0$ these are indeed almost complex structures, and K is holomorphic with respect to both. Using (3.4.2) and (3.4.4), we can show that they are integrable, i.e. their Nijenhuis tensor (3.3.3) vanishes. Choosing complex coordinates adapted to $J^\mu{}_\nu$, the metric takes the same form as in (3.4.5),

$$ds^2 = \Omega(z,\bar{z})^2 \left((dw + h(z,\bar{z})dz)(d\bar{w} + \bar{h}(z,\bar{z})d\bar{z}) + c(z,\bar{z})^2 dz d\bar{z} \right) \ . \tag{3.4.9}$$

In order to exhibit the explicit solution for ζ and $\widetilde{\zeta}$, we introduce a local frame adapted to the Hermitian metric (3.4.9) as in (3.3.19),

$$e^1 = \Omega(dw + hdz) \ , \qquad e^2 = \Omega c dz \ . \tag{3.4.10}$$

Choosing the background fields V_μ and A_μ as in (3.4.8) and (3.3.17), we solve for ζ and $\widetilde{\zeta}$,

$$\zeta_\alpha = \frac{\sqrt{s}}{2}\begin{pmatrix}0\\1\end{pmatrix} , \qquad \widetilde{\zeta}^{\dot{\alpha}} = \frac{\Omega}{\sqrt{s}}\begin{pmatrix}0\\1\end{pmatrix} . \tag{3.4.11}$$

As before, we have the freedom of choosing a nowhere vanishing complex s, which transforms as a scalar under holomorphic coordinate changes followed by appropriate R-transformations. Hence, ζ can be regarded as a scalar, and the same is true for $\widetilde{\zeta}$, since Ω is a scalar as well. (Recall from (3.4.6) that it is proportional to the norm of K.) The freedom in choosing s reflects the underlying invariance of the equations (3.4.1) under complexified R-transformations, and as above we could use this freedom to set $s = 1$.

We would like to comment on the isometries generated by K and $\overline{K}$. Recall that K appeared on the right-hand-side of the supersymmetry algebra (3.1.13). This is not the case for $\overline{K}$. Nevertheless, both K and $\overline{K}$ are Killing vectors, because the metric is real. However, $\overline{K}$ need not be a symmetry of the auxiliary fields V_μ and A_μ. For instance, to ensure that V_μ in (3.4.8) commutes with $\overline{K}$ we must impose an additional restriction on κ,

$$\overline{K}^\mu \partial_\mu \kappa = 0 . \tag{3.4.12}$$

Similarly, to ensure that A_μ is invariant under K and $\overline{K}$ up to ordinary gauge transformations, we must impose

$$K^\mu \partial_\mu |s| = \overline{K}^\mu \partial_\mu |s| = 0 . \tag{3.4.13}$$

Note that A_μ is always invariant under K up to complexified gauge transformations. The conditions (3.4.12) and (3.4.13) ensure that K and $\overline{K}$ are good symmetries of all background fields. Although this choice is natural, we are free to consider auxiliary fields that are not invariant under $\overline{K}$.

If we choose to impose (3.4.12) and (3.4.13), we would like to add $\overline{K}$ to the supersymmetry algebra (3.1.13). When acting on objects of R-charge q, we define

$$\delta_{\overline{K}} = \mathcal{L}_{\overline{K}}^{\overline{A}} = \mathcal{L}_{\overline{K}} - iq\overline{K}^\mu \overline{A}_\mu , \tag{3.4.14}$$

which is similar to (3.1.14), except that we use $\overline{A}_\mu$ instead of A_μ. This is covariant under ordinary gauge transformations. With this definition, we find that

$$
\begin{aligned}
&[\delta_{\overline{K}}, \delta_\zeta] = [\delta_{\overline{K}}, \delta_{\widetilde{\zeta}}] = 0 \ , \\
&[\delta_K, \delta_{\overline{K}}] = 0 \ .
\end{aligned}
\tag{3.4.15}
$$

Together with (3.1.13), these commutation relations comprise the two-dimensional $(2,0)$ supersymmetry algebra. Here it acts on the T^2 fibers in (3.4.5).

3.4.3 Trivial Fibrations and Dimensional Reduction

Here we will comment on the case when one or both cycles of the torus are trivially fibered over the base Σ. By reducing along these cycles, we obtain manifolds admitting two supercharges in three and two dimensions.

Let us consider the case when one of the cycles is trivially fibered, so that the manifold is of the form $\mathcal{M}_3 \times S^1$. The three-manifold $\mathcal{M}_3$ is itself a circle bundle over the Riemann surface Σ. Let us choose $K = \partial_\tau + i\partial_\psi$, where the real coordinates τ and ψ parametrize the trivial S^1 and the circle fiber of $\mathcal{M}_3$ respectively. In this case, the metric takes the form

$$
\begin{aligned}
&ds^2 = \Omega^2(z,\overline{z})d\tau^2 + ds^2_{\mathcal{M}_3} \ , \\
&ds^2_{\mathcal{M}_3} = \Omega^2(z,\overline{z}) \Big(\big(d\psi + a(z,\overline{z})dz + \overline{a}(z,\overline{z})d\overline{z}\big)^2 + c^2(z,\overline{z})dzd\overline{z} \Big) \ .
\end{aligned}
\tag{3.4.16}
$$

Since $K = \partial_\tau + i\partial_\psi$ squares to zero and commutes with its complex conjugate, this metric is in the class considered in the previous subsection. Hence, we can find two solutions ζ and $\widetilde{\zeta}$. Imposing the additional conditions (3.4.12) and (3.4.13) ensures that the spinors and the background fields do not vary along the two circles parametrized by τ and ψ.

We can now reduce along τ to obtain two scalar supercharges on Seifert manifolds that are circle bundles over a Riemann surface Σ, as long as the metric is invariant under translations along the fiber. Rigid supersymmetry on such manifolds was recently discussed in [101,102]. The supercharges we find exist in any three-dimensional $\mathcal{N} = 2$ theory with a $U(1)_R$ symmetry.

If we choose the circle bundle to be trivial, we can reduce once more and obtain two scalar supercharges on any Riemann surface Σ. They are analogous to the ones obtained by the A-twist of a two-dimensional $\mathcal{N} = (2,2)$ theory on Σ [103, 104].

3.5 Manifolds Admitting Two Supercharges of Equal R-Charge

In this section we analyze manifolds that admit two independent solutions of (3.1.11). The presence of these two solutions turns out to be very restrictive. If $\mathcal{M}$ is compact, we will prove that it must be one of the following:

- A torus T^4 with flat metric.
- A $K3$ surface with Ricci-flat Kähler metric.
- Certain discrete quotients of $S^3 \times S^1$ with the standard metric $ds^2 = d\tau^2 + r^2 d\Omega_3$.

Given two independent solutions ζ and η of (3.1.11), we derive a set of consistency conditions for the metric and the auxiliary fields. Using $\frac{1}{2} R_{\mu\nu\kappa\lambda}\sigma^{\kappa\lambda}\zeta = [\nabla_\mu, \nabla_\nu]\zeta$ and the fact that ζ satisfies (3.1.11), we obtain

$$\begin{aligned} \frac{1}{2} R_{\mu\nu\kappa\lambda}\sigma^{\kappa\lambda}\zeta = \ & V^\rho V_\rho \sigma_{\mu\nu}\zeta + i(\partial_\mu(A_\nu - V_\nu) - \partial_\nu(A_\mu - V_\mu))\zeta \\ & - i(\nabla_\mu + iV_\mu)V^\rho\sigma_{\nu\rho}\zeta + i(\nabla_\nu + iV_\nu)V^\rho\sigma_{\mu\rho}\zeta \ , \end{aligned} \tag{3.5.1}$$

and similarly for η. Since ζ and η are independent at every point, we arrive at the following integrability conditions:

1.) The Weyl tensor is anti-self-dual, $W_{\mu\nu\rho\lambda} = -\frac{1}{2}\varepsilon_{\mu\nu\kappa\sigma}W^{\kappa\sigma}{}_{\rho\lambda}$.

2.) The curl of V_μ is anti-self-dual, $\partial_\mu V_\nu - \partial_\nu V_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}(\partial^\rho V^\lambda - \partial^\lambda V^\rho)$.

3.) The difference $A_\mu - V_\mu$ is closed, $\partial_\mu(A_\nu - V_\nu) - \partial_\nu(A_\mu - V_\mu) = 0$.

4.) The Ricci tensor is given by

$$R_{\mu\nu} = i(\nabla_\mu V_\nu + \nabla_\nu V_\mu) - 2(V_\mu V_\nu - g_{\mu\nu}V_\rho V^\rho) \ . \tag{3.5.2}$$

If we instead consider two independent solutions $\widetilde{\zeta}$ and $\widetilde{\eta}$ of (3.1.12), the Weyl tensor and the curl of V_μ in 1.) and 2.) are self-dual rather than anti-self-dual, and the Ricci tensor is given by (3.5.2) with $V_\mu \to -V_\mu$.

These conditions locally constrain the geometry of the manifold. They take a particularly simple form on manifolds of $SU(2)$ holonomy, which are Ricci-flat and have anti-self-dual Weyl tensor. In this case, we can satisfy the integrability conditions by choosing $V_\mu = A_\mu = 0$. Indeed, such manifolds admit two independent covariantly constant spinors. Further examples are discussed in sections 3.6 and 3.7.

Here we will not attempt to classify all manifolds that satisfy the conditions above. Instead, we will focus on the case when $\mathcal{M}$ is compact, and prove the following global result: the existence of two spinors ζ and η that satisfy (3.1.11) everywhere on a compact manifold $\mathcal{M}$ implies that $\mathcal{M}$ is hyperhermitian. Compact hyperhermitian four-manifolds have been classified in [105]. Up to a global conformal transformation, they are given by the manifolds listed at the beginning of this section.[7] Using the fact that V_μ is conserved, we will find that the conformal factor must be a constant.

A hyperhermitian structure on $\mathcal{M}$ arises whenever there are two anti-commuting hermitian structures $J^{(1)}$ and $J^{(2)}$. Together with their commutator $J^{(3)}$ they satisfy the quaternion algebra,

$$\{J^{(a)}, J^{(b)}\} = -2\delta^{ab} , \qquad (a, b = 1, 2, 3) . \tag{3.5.3}$$

This implies that there is an entire S^2 of Hermitian structures parametrized by

$$J(\vec{n}) = \sum_a n^a J^{(a)} , \qquad |\vec{n}| = 1 . \tag{3.5.4}$$

Since ζ and η satisfy (3.1.11), the almost complex structures $J^\mu{}_\nu$ and $I^\mu{}_\nu$ constructed in (3.2.13) are integrable. Recall from (3.2.14) that the anticommutator of $J^\mu{}_\nu$ and $I^\mu{}_\nu$ gives rise to a real function f, which is determined in terms of ζ

[7] See also the discussion in [108], where these manifolds are identified with compact Hermitian surfaces for which the restricted holonomy of the Bismut connection is contained in $SU(2)$.

and η. Moreover, we are free to choose ζ and η such that $f \neq \pm 1$ at a given point; at this point $J^\mu{}_\nu \neq \pm I^\mu{}_\nu$. We will now prove that $J^\mu{}_\nu$ and $I^\mu{}_\nu$ are elements of a hyperhermitian structure on $\mathcal{M}$.

In order to establish this result, we will consider the Lee forms associated with $J^\mu{}_\nu$ and $I^\mu{}_\nu$,

$$\begin{aligned} \theta^J_\mu &= J_\mu{}^\rho \nabla_\nu J^\nu{}_\rho \ , \\ \theta^I_\mu &= I_\mu{}^\rho \nabla_\nu I^\nu{}_\rho \ . \end{aligned} \tag{3.5.5}$$

Using (3.3.6) they can be expressed as follows:

$$\begin{aligned} \theta^J_\mu &= i(V_\mu - \overline{V}_\mu) - J_\mu{}^\rho (V_\rho + \overline{V}_\rho) \ , \\ \theta^I_\mu &= i(V_\mu - \overline{V}_\mu) - I_\mu{}^\rho (V_\rho + \overline{V}_\rho) \ . \end{aligned} \tag{3.5.6}$$

We will also need the following formula, which follows from (3.1.11) by direct computation:

$$\partial_\mu \log \frac{|\zeta|^2}{|\eta|^2} = -\frac{1}{2} \left(J_{\mu\nu} - I_{\mu\nu}\right) \left(V^\nu + \overline{V}^\nu\right) \ . \tag{3.5.7}$$

Subtracting the two equations in (3.5.6) and using (3.5.7), we find that the Lee forms differ by an exact one-form,

$$\theta^J_\mu - \theta^I_\mu = \partial_\mu h \ , \qquad h = 2 \log \frac{|\zeta|^2}{|\eta|^2} \ . \tag{3.5.8}$$

Recall from the first integrability condition listed above that the Weyl tensor must be anti-self-dual. We thus have a compact four-manifold $\mathcal{M}$ with anti-self-dual Weyl tensor that admits two Hermitian structures J and I such that $J \neq \pm I$ somewhere on $\mathcal{M}$. Applying proposition (3.7) in [109], it follows that

a) The function h in (3.5.8) is constant.

b) The manifold admits a hyperhermitian structure as in (3.5.3). Moreover, J and I belong to it, and hence they can be expressed in terms of the $J^{(a)}$ as in (3.5.4).

We conclude that $\mathcal{M}$ is one of the manifolds listed at the beginning of this section, up to a global conformal rescaling of the metric. In order to fix the conformal factor, we use the fact that V_μ is conserved. Observe that since $J^\mu{}_\nu$ and $I^\mu{}_\nu$ are independent elements of the hyperhermitian structure, $J^\mu{}_\nu - I^\mu{}_\nu$ is invertible. It therefore follows from (3.5.7) and (3.5.8) with $\partial_\mu h = 0$ that V_μ is purely imaginary. Hence, we see from (3.5.6) that

$$\theta^J_\mu = \theta^I_\mu = i(V_\mu - \overline{V}_\mu) \ . \tag{3.5.9}$$

Since V_μ is conserved, θ^I_μ and θ^J_μ are conserved as well. To see that this fixes the conformal factor, consider a conformal rescaling of the metric, $\widehat{g}_{\mu\nu} = e^\phi g_{\mu\nu}$. It follows from (3.5.5) that the Lee forms shift by an exact one-form,

$$\widehat{\theta}^J_\mu = \theta^J_\mu + \partial_\mu \phi \ , \tag{3.5.10}$$

and similarly for θ^I_μ. Since θ^J_μ is conserved, $\widehat{\theta}^J_\mu$ can only be conserved if ϕ is harmonic. On a compact manifold, this is only possible for constant ϕ. It can be checked that the manifolds listed at the beginning of this section all have conserved Lee forms, and hence they are the correct hyperhermitian representatives within each conformal class. This is trivial for a flat T^4 or a $K3$ surface with Ricci-flat metric, since both are hyperkähler manifolds and hence the Lee forms (3.5.5) vanish. Even though $S^3 \times S^1$ is not Kähler, it can be checked that the Lee forms are conserved if we choose the standard metric $ds^2 = d\tau^2 + r^2 d\Omega_3$. This case will be discussed in section 3.7.

3.6 Manifolds Admitting Four Supercharges

In this section we will formulate necessary conditions for the existence of four supercharges. These follow straightforwardly from the integrability conditions discussed in section 3.5. Assuming the existence of two independent solutions of (3.1.11), we found that the Weyl tensor and the curl of V_μ must be anti-self-dual. Similarly, two solutions of (3.1.12) imply that they are also self-dual, and hence they must vanish. It follows that $\mathcal{M}$ is locally conformally flat and that V_μ is closed. Since $A_\mu - V_\mu$ is also closed, it follows that the gauge field A_μ is flat. Finally, the Ricci tensor must satisfy (3.5.2) and the same relation with $V_\mu \to -V_\mu$. This implies that $\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$, and since V_μ is also closed, it must be covariantly constant,

$$\nabla_\mu V_\nu = 0 \ . \tag{3.6.1}$$

The Ricci tensor is then given by

$$R_{\mu\nu} = -2(V_\mu V_\nu - g_{\mu\nu} V_\rho V^\rho) \ . \tag{3.6.2}$$

Since V_μ is covariantly constant, $\mathcal{M}$ is locally isometric to $\mathcal{M}_3 \times \mathbb{R}$. It follows from (3.6.2) that $\mathcal{M}_3$ is a space of constant curvature. Let r be a positive constant. There are three possible cases:

1.) If $V^\mu V_\mu = -\frac{1}{r^2}$ then $\mathcal{M}_3$ is locally isometric to a round S^3 of radius r. In this case V^μ is purely imaginary and points along $\mathbb{R}$.

2.) If $V_\mu = 0$ then $\mathcal{M}$ is locally isometric to flat $\mathbb{R}^4$. This is the case of ordinary $\mathcal{N} = 1$ supersymmetry in flat space.

3.) If $V^\mu V_\mu = \frac{1}{r^2}$ then $\mathcal{M}_3$ is locally isometric to H^3, the three-dimensional hyperbolic space of radius r and constant negative curvature. In this case V^μ is real and points along $\mathbb{R}$.

We will discuss cases 1.) and 3.) below.

3.7 Examples

3.7.1 $S^3 \times \mathbb{R}$

Consider $S^3 \times \mathbb{R}$ with metric

$$ds^2 = d\tau^2 + r^2 d\Omega_3 \ , \tag{3.7.1}$$

where $d\Omega_3$ is the round metric on a unit three sphere. As we saw in section 3.6, this manifold admits four supercharges. Supersymmetric field theories on this space have been studied in [110, 111] and more recently in [31]. Here we will examine this example from the point of view of the preceding discussion.

Since this manifold admits rigid supersymmetry, it must be Hermitian. We can introduce holomorphic coordinates w, z, so that the metric (3.7.1) takes the form

$$ds^2 = \Big(dw - \frac{r\overline{z}}{r^2+|z|^2}\,dz\Big)\Big(d\overline{w} - \frac{rz}{r^2+|z|^2}\,d\overline{z}\Big) + \frac{r^4}{(r^2+|z|^2)^2}\,dzd\overline{z}\ . \qquad (3.7.2)$$

Here the imaginary part of w is periodic, $w \sim w + 2\pi i r$.[8] The vector $\partial_w + \partial_{\overline{w}}$ is covariantly constant and points along $\mathbb{R}$, while $i(\partial_w - \partial_{\overline{w}})$ generates translations along a Hopf fiber of S^3. Since the metric (3.7.2) is of the form (3.4.5), it allows for two supercharges ζ and $\widetilde{\zeta}$ such that $K = \zeta\sigma^\mu\widetilde{\zeta}\partial_\mu$ is equal to the holomorphic Killing vector ∂_w. With the choice of frame in (3.4.10), these two solutions are given by (3.4.11).

As discussed in section 3.6, it is possible to choose the auxiliary fields A_μ and V_μ to obtain four supercharges on $S^3 \times \mathbb{R}$. Up to a sign, this fixes

$$V = -\frac{i}{r}(\partial_w + \partial_{\overline{w}})\ . \qquad (3.7.4)$$

We will comment on the other choice of sign below. The gauge field A_μ must be flat, but it is otherwise undetermined, and hence we can add an arbitrary complex Wilson line for A_μ along $\mathbb{R}$. In general, the resulting supercharges vary along $\mathbb{R}$. This is not the case if we choose $A_\mu = V_\mu$, so that we can compactify to $S^3 \times S^1$. In the frame (3.4.10) the supercharges take the form

$$\zeta_\alpha = \begin{pmatrix} a_1 e^{-(w-\overline{w})/2r} \\ a_2 e^{(w-\overline{w})/2r} \end{pmatrix}\ , \quad \widetilde{\zeta}^{\dot\alpha} = \begin{pmatrix} a_3 e^{(w-\overline{w})/2r} \\ a_4 e^{-(w-\overline{w})/2r} \end{pmatrix}\ , \qquad (3.7.5)$$

[8] The point $z = \infty$ is covered by different coordinates w', z',

$$z' = \frac{r^2}{z}\ , \qquad w' = w - r\log\frac{z}{r}\ , \qquad (3.7.3)$$

as long as $z \neq 0$. In these coordinates, the metric takes the same form as in (3.7.2). Due to the periodicity of w, we do not need to choose a specific branch for the logarithm in (3.7.3).

where the a_i are arbitrary complex constants. Setting $a_1 = a_3 = 0$, we obtain the two supercharges ζ and $\widetilde{\zeta}$ discussed above, which are of the form (3.4.11). Since A_μ and V_μ are purely imaginary, we can use (3.2.2) to obtain two other supercharges $\zeta^\dagger$ and $\widetilde{\zeta}^\dagger$. They correspond to setting $a_2 = a_4 = 0$ in (3.7.5).

Setting $a_3 = a_4 = 0$, we obtain two supercharges of equal R-charge on the compact manifold $S^3 \times S^1$. This manifold is hyperhermitian but not Kähler. If V is given by (3.7.4), the Lee forms in (3.5.9) are non-vanishing but conserved, in accord with the general discussion in section 3.5.

Using the spinors ζ and $\widetilde{\zeta}$ in (3.7.5), we can construct four independent complex Killing vectors of the form $K^\mu = \zeta\sigma^\mu\widetilde{\zeta}$. Since the supercharges are related by $\zeta \leftrightarrow \zeta^\dagger$ and $\widetilde{\zeta} \leftrightarrow \widetilde{\zeta}^\dagger$, these vectors are linear combinations of four real, orthogonal Killing vectors L_a $(a = 1, 2, 3)$ and T, which satisfy the algebra

$$[L_a, L_b] = \epsilon_{abc} L_c \ , \qquad [L_a, T] = 0 \ . \tag{3.7.6}$$

The L_a generate the $SU(2)_l$ inside the $SU(2)_l \times SU(2)_r$ isometry group of S^3, while T generates translations along $\mathbb{R}$. The supercharges form two $SU(2)_l$ doublets that carry opposite R-charge and are invariant under $SU(2)_r$. (If we choose the opposite sign for V in (3.7.4), the spinors are invariant under $SU(2)_l$ and transform as doublets under $SU(2)_r$.) Using (3.1.13) and (3.1.16) we find that the supersymmetry algebra is $SU(2|1)$.

If we remain on $S^3 \times S^1$ but only require two supercharges ζ and $\widetilde{\zeta}$, the auxiliary fields V_μ and A_μ are are less constrained. For instance, we can choose ζ and $\widetilde{\zeta}$ corresponding to $a_1 = a_3 = 0$ in (3.7.5). As discussed above, they give rise to the holomorphic Killing vector $K = \partial_w$, which commutes with its complex conjugate $\overline{K}$. According to the discussion in section 3.4, we can preserve ζ and $\widetilde{\zeta}$ for any choice

$$V = \frac{i}{r}(\partial_w + \partial_{\overline{w}}) + \kappa K \ , \qquad A = \frac{i}{r}(\partial_w + \partial_{\overline{w}}) + \frac{3}{2}\kappa K \ , \tag{3.7.7}$$

where κ is a complex function that satisfies $K^\mu\partial_\mu\kappa = 0$, so that V^μ is conserved. If we instead choose ζ and $\widetilde{\zeta}$ corresponding to $a_1 = a_4 = 0$ in (3.7.5), the resulting

Killing vector K points along the S^3. Together with its complex conjugate $\overline{K}$, it generates the $SU(2)_l$ isometry subgroup. For generic choices of a_i we find that K and $\overline{K}$ generate the $SU(2)_l \times U(1)$ isometry subgroup that also includes translations along S^1. These possibilities for the algebra of Killing vectors are precisely the ones discussed in appendix C.

3.7.2 $H^3 \times \mathbb{R}$

Consider $H^3 \times \mathbb{R}$, where H^3 is the three-dimensional hyperbolic space of constant negative curvature. This manifold also admits four supercharges, and in some respects it is similar to the previous example $S^3 \times \mathbb{R}$, but there are also qualitative differences.

As before, we can introduce holomorphic coordinates w, z and write the metric in the form,

$$ds^2 = e^{(z+\overline{z})/r}\, dw d\overline{w} + dz d\overline{z} \ , \tag{3.7.8}$$

where r is the radius of H^3. Now the coordinates w, z cover the entire space. The covariantly constant vector $i(\partial_z - \partial_{\overline{z}})$ points along $\mathbb{R}$. In order to preserve four supercharges, we must choose

$$V = \frac{i}{r}(\partial_z - \partial_{\overline{z}}) \ , \tag{3.7.9}$$

up to a sign (see below). As in the previous example, we are free to add an arbitrary complex Wilson line for A_μ along $\mathbb{R}$. Setting $A_\mu = V_\mu$ and choosing the frame (3.4.10), the four supercharges are given by

$$\zeta_\alpha = \begin{pmatrix} a_1 e^{-(z+\overline{z})/4r} \\ (a_2 - a_1 \frac{w}{r})\, e^{z+\overline{z}/4r} \end{pmatrix} \ , \qquad \widetilde{\zeta}^{\dot{\alpha}} = \begin{pmatrix} a_3 e^{-(z+\overline{z})/4r} \\ (a_4 - a_3 \frac{w}{r})\, e^{(z+\overline{z})/4r} \end{pmatrix} \ . \tag{3.7.10}$$

As before, the a_i are arbitrary complex constants. Since the metric (3.7.8) is of the form (3.4.5), two of the supercharges are given by (3.4.11). They correspond to $a_1 = a_3 = 0$ in (3.7.10). Note that the spinors in (3.7.10) do not depend on $z - \overline{z}$, so that we can compactify to $H^3 \times S^1$.

In contrast to $S^3 \times S^1$, the supercharges $\zeta^\dagger$ and $\widetilde{\zeta}^\dagger$ correspond to choosing the opposite sign in (3.7.9). Now the four independent complex Killing vectors $K^\mu = \zeta \sigma^\mu \widetilde{\zeta}$

constructed from ζ and $\widetilde{\zeta}$ in (3.7.10) give rise to seven independent real Killing vectors, which comprise the full $SL(2,\mathbb{C})\times U(1)$ isometry group of $H^3\times S^1$. The four complex Killing vectors K commute with all four complex conjugates $\overline{K}$.

3.7.3 Squashed $S^3\times\mathbb{R}$

We now consider $S^3\times\mathbb{R}$, where the S^3 is one of the squashed three-spheres discussed in [88]. They are defined by their isometric embedding in flat $\mathbb{R}^4$, where they satisfy the constraint

$$\frac{1}{a^2}\left(x_1^2+x_2^2\right)+\frac{1}{b^2}\left(x_3^2+x_4^2\right)=1~. \tag{3.7.11}$$

Rotations in the x_1x_2 and x_3x_4 planes generate a $U(1)\times U(1)$ isometry. Introducing angular coordinates $\theta\in[0,\pi/2]$, $\alpha\sim\alpha+2\pi$, and $\beta\sim\beta+2\pi$ for the squashed S^3 and a coordinate τ along $\mathbb{R}$, we can write the metric in the form

$$\begin{aligned}&ds^2=d\tau^2+F(\theta)^2d\theta^2+a^2\cos^2\theta d\alpha^2+b^2\sin^2\theta d\beta^2~,\\&F(\theta)=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}~.\end{aligned} \tag{3.7.12}$$

In these coordinates, the $U(1)\times U(1)$ isometry of the squashed sphere is generated by the real Killing vectors ∂_α and ∂_β. By combining them with translations ∂_τ along $\mathbb{R}$, we obtain a complex Killing vector K, which squares to zero and commutes with its complex conjugate,

$$K=\partial_\tau-\frac{i}{a}\partial_\alpha-\frac{i}{b}\partial_\beta~. \tag{3.7.13}$$

According to the discussion in section 3.4, this guarantees the existence of two supercharges ζ and $\widetilde{\zeta}$. Introducing the frame

$$e^1=d\tau~,\quad e^2=F(\theta)\,d\theta~,\quad e^3=a\cos\theta\,d\alpha~,\quad e^4=b\sin\theta\,d\beta~, \tag{3.7.14}$$

they are given by

$$\zeta_\alpha=-\frac{i}{\sqrt{2}}\begin{pmatrix}e^{\frac{i}{2}(\alpha+\beta-\theta)}\\ ie^{\frac{i}{2}(\alpha+\beta+\theta)}\end{pmatrix}~,\qquad\widetilde{\zeta}^{\dot\alpha}=-\frac{i}{\sqrt{2}}\begin{pmatrix}e^{-\frac{i}{2}(\alpha+\beta-\theta)}\\ ie^{-\frac{i}{2}(\alpha+\beta+\theta)}\end{pmatrix}~. \tag{3.7.15}$$

The auxiliary fields take the form

$$\begin{aligned} V_\mu dx^\mu &= -\frac{i}{F(\theta)}\, d\tau + \kappa K_\mu dx^\mu \ , \qquad K^\mu \partial_\mu \kappa = 0 \ , \\ A_\mu dx^\mu &= -\frac{1}{2F(\theta)}\left(2id\tau + ad\alpha + bd\beta\right) + \frac{1}{2}(d\alpha + d\beta) + \frac{3}{2}\kappa K_\mu dx^\mu \ . \end{aligned} \tag{3.7.16}$$

As in section 3.4, we can use K to define a complex structure compatible with the metric (3.7.12). In the corresponding holomorphic coordinates, the metric is of the form (3.4.5) and the supercharges are given by (3.4.11).

We can again fix the metric (3.7.12) and obtain two supercharges for different choices of the auxiliary fields A_μ and V_μ. For instance, we can obtain a second solution by replacing $\tau \to -\tau$ and $\beta \to \pi - \beta$ in (3.7.13) and repeating the previous construction. Note that this does not change the orientation of the manifold. From these two solutions, we can obtain two more by using (3.2.2).

3.8 Appendix A: Conventions

We follow the conventions of [34], adapted to Euclidean signature. This leads to some differences in notation, which are summarized here, together with various relevant formulas.

3.8.1 Flat Euclidean Space

The metric is given by $\delta_{\mu\nu}$, where $\mu, \nu = 1, \ldots, 4$. The totally antisymmetric Levi-Civita symbol is normalized so that $\varepsilon_{1234} = 1$. The rotation group is given by $SO(4) = SU(2)_+ \times SU(2)_-$. A left-handed spinor ζ is an $SU(2)_+$ doublet and carries undotted indices, ζ_α. Right-handed spinors $\widetilde\zeta$ are doublets under $SU(2)_-$. They are distinguished by a tilde and carry dotted indices, $\widetilde\zeta^{\dot\alpha}$. In Euclidean signature, $SU(2)_+$ and $SU(2)_-$ are not related by complex conjugation, and hence ζ and $\widetilde\zeta$ are independent spinors.

The Hermitian conjugate spinors $\zeta^\dagger$ and $\widetilde\zeta^\dagger$ transform as doublets under $SU(2)_+$

and $SU(2)_-$ respectively. They are defined with the following index structure,

$$(\zeta^\dagger)^\alpha = \overline{(\zeta_\alpha)} \ , \qquad (\widetilde{\zeta}^\dagger)_{\dot\alpha} = \overline{(\widetilde{\zeta}^{\dot\alpha})} \ , \tag{3.8.1}$$

where the bars denote complex conjugation. Changing the index placement on both sides of these equations leads to a relative minus sign,

$$(\zeta^\dagger)_\alpha = -\overline{(\zeta^\alpha)} \ , \qquad (\widetilde{\zeta}^\dagger)^{\dot\alpha} = -\overline{(\widetilde{\zeta}_{\dot\alpha})} \ . \tag{3.8.2}$$

We can therefore write the $SU(2)_+$ invariant inner product of ζ and η as $\zeta^\dagger\eta$. Similarly, the $SU(2)_-$ invariant inner product of $\widetilde{\zeta}$ and $\widetilde{\eta}$ is given by $\widetilde{\zeta}^\dagger\widetilde{\eta}$. The corresponding norms are denoted by $|\zeta|^2 = \zeta^\dagger\zeta$ and $|\widetilde{\zeta}|^2 = \widetilde{\zeta}^\dagger\widetilde{\zeta}$.

The sigma matrices take the form

$$\sigma^\mu_{\alpha\dot\alpha} = (\vec{\sigma}, -i) \ , \qquad \widetilde{\sigma}^{\mu\dot\alpha\alpha} = (-\vec{\sigma}, -i) \ , \tag{3.8.3}$$

where $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices. We use a tilde (rather than a bar) to emphasize that σ^μ and $\widetilde{\sigma}^\mu$ are not related by complex conjugation in Euclidean signature. The sigma matrices (3.8.3) satisfy the identities

$$\sigma_\mu\widetilde{\sigma}_\nu + \sigma_\nu\widetilde{\sigma}_\mu = -2\delta_{\mu\nu} \ , \qquad \widetilde{\sigma}_\mu\sigma_\nu + \widetilde{\sigma}_\nu\sigma_\mu = -2\delta_{\mu\nu} \ . \tag{3.8.4}$$

The generators of $SU(2)_+$ and $SU(2)_-$ are given by the antisymmetric matrices

$$\sigma_{\mu\nu} = \frac{1}{4}(\sigma_\mu\widetilde{\sigma}_\nu - \sigma_\nu\widetilde{\sigma}_\mu) \ , \qquad \widetilde{\sigma}_{\mu\nu} = \frac{1}{4}(\widetilde{\sigma}_\mu\sigma_\nu - \widetilde{\sigma}_\nu\sigma_\mu) \ . \tag{3.8.5}$$

They are self-dual and anti-self-dual respectively,

$$\frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\sigma^{\rho\lambda} = \sigma_{\mu\nu} \ , \qquad \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\widetilde{\sigma}^{\rho\lambda} = -\widetilde{\sigma}_{\mu\nu} \ . \tag{3.8.6}$$

3.8.2 Differential Geometry

We will use lowercase Greek letters $\mu, \nu, \ldots$ to denote curved indices and lowercase Latin letters $a, b, \ldots$ to denote frame indices. Given a Riemannian metric $g_{\mu\nu}$, we can define an orthonormal tetrad $e^a{}_\mu$. The Levi-Civita connection is denoted ∇_μ and the corresponding spin connection is given by

$$\omega_{\mu a}{}^b = e^b{}_\nu \nabla_\mu e_a{}^\nu \ . \tag{3.8.7}$$

The Riemann tensor takes the form

$$R_{\mu\nu a}{}^{b} = \partial_\mu \omega_{\nu a}{}^{b} - \partial_\nu \omega_{\mu a}{}^{b} + \omega_{\nu a}{}^{c}\omega_{\mu c}{}^{b} - \omega_{\mu a}{}^{c}\omega_{\nu c}{}^{b} \ . \tag{3.8.8}$$

The Ricci tensor is defined by $R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho}$, and $R = R_\mu{}^\mu$ is the Ricci scalar. Note that in these conventions, the Ricci scalar is negative on a round sphere.

The covariant derivatives of the spinors ζ and $\widetilde{\zeta}$ are given by

$$\nabla_\mu \zeta = \partial_\mu \zeta + \frac{1}{2}\omega_{\mu ab}\sigma^{ab}\zeta \ , \qquad \nabla_\mu \widetilde{\zeta} = \partial_\mu \widetilde{\zeta} + \frac{1}{2}\omega_{\mu ab}\widetilde{\sigma}^{ab}\widetilde{\zeta} \ . \tag{3.8.9}$$

We will also need the commutator of two covariant derivatives,

$$[\nabla_\mu, \nabla_\nu]\zeta = \frac{1}{2}R_{\mu\nu ab}\sigma^{ab}\zeta \ , \qquad [\nabla_\mu, \nabla_\nu]\widetilde{\zeta} = \frac{1}{2}R_{\mu\nu ab}\widetilde{\sigma}^{ab}\widetilde{\zeta} \ . \tag{3.8.10}$$

Finally, the Lie derivatives of ζ and $\widetilde{\zeta}$ along a vector field $X = X^\mu \partial_\mu$ are given by [112],

$$\begin{aligned} \mathcal{L}_X \zeta &= X^\mu \nabla_\mu \zeta - \frac{1}{2}\nabla_\mu X_\nu \sigma^{\mu\nu}\zeta \ , \\ \mathcal{L}_X \widetilde{\zeta} &= X^\mu \nabla_\mu \widetilde{\zeta} - \frac{1}{2}\nabla_\mu X_\nu \widetilde{\sigma}^{\mu\nu}\widetilde{\zeta} \ . \end{aligned} \tag{3.8.11}$$

3.9 Appendix B: Review of Curved Superspace

In this appendix we explain how to place a four-dimensional $\mathcal{N} = 1$ theory on a Riemannian manifold $\mathcal{M}$ in a supersymmetric way. We review the procedure of [31] and comment on several points that play an important role in our analysis.

3.9.1 Supercurrents

Given a flat-space field theory, we can place it on $\mathcal{M}$ by coupling its energy-momentum tensor $T_{\mu\nu}$ to the background metric $g_{\mu\nu}$ on $\mathcal{M}$. In a supersymmetric theory, the energy-momentum tensor resides in a supercurrent multiplet $\mathcal{S}_\mu$, which also contains the supersymmetry current $S_{\mu\alpha}$ and various other operators. In the spirit of [16] we can promote the background metric $g_{\mu\nu}$ to a background supergravity multiplet, which also contains the gravitino $\psi_{\mu\alpha}$ and several auxiliary fields. They couple to the operators in $\mathcal{S}_\mu$.

In general, the supercurrent multiplet $\mathcal{S}_\mu$ contains 16 + 16 independent operators [1, 46]. In many theories, it can be reduced to a smaller multiplet, which only contains 12 + 12 operators. There are two such 12 + 12 supercurrents: the Ferrara-Zumino (FZ) multiplet [38] and the $\mathcal{R}$-multiplet (see for instance [50]). The $\mathcal{R}$-multiplet exists whenever the field theory possesses a $U(1)_R$ symmetry, and this is the case we will focus on here.

The $\mathcal{R}$-multiplet satisfies the defining relations[9]

$$\widetilde{D}^{\dot{\alpha}}\mathcal{R}_{\alpha\dot{\alpha}} = \chi_\alpha \ , \qquad \widetilde{D}_{\dot{\alpha}}\chi_\alpha = 0 \ , \qquad D^\alpha\chi_\alpha = \widetilde{D}_{\dot{\alpha}}\widetilde{\chi}^{\dot{\alpha}} \ . \tag{3.9.2}$$

Here $\mathcal{R}_{\alpha\dot{\alpha}} = -2\sigma^\mu_{\alpha\dot{\alpha}}\mathcal{R}_\mu$ is the bi-spinor corresponding to $\mathcal{R}_\mu$. In components,

$$\begin{aligned}
\mathcal{R}_\mu =\ & j^{(R)}_\mu - i\theta S_\mu + i\widetilde{\theta}\widetilde{S}_\mu + \theta\sigma^\nu\widetilde{\theta}\left(2T_{\mu\nu} + \frac{i}{2}\varepsilon_{\mu\nu\rho\lambda}\mathcal{F}^{\rho\lambda} - \frac{i}{2}\varepsilon_{\mu\nu\rho\lambda}\partial^\rho j^{(R)\lambda}\right) \\
& - \frac{1}{2}\theta^2\widetilde{\theta}\widetilde{\sigma}^\nu\partial_\nu S_\mu + \frac{1}{2}\widetilde{\theta}^2\theta\sigma^\nu\partial_\nu\widetilde{S}_\mu - \frac{1}{4}\theta^2\widetilde{\theta}^2\partial^2 j^{(R)}_\mu \ , \\
\chi_\alpha =\ & -2i(\sigma^\mu\widetilde{S}_\mu)_\alpha - 4\theta_\beta\left(\delta_\alpha{}^\beta T_\mu{}^\mu - i(\sigma^{\mu\nu})_\alpha{}^\beta\mathcal{F}_{\mu\nu}\right) - 4\theta^2(\sigma^{\mu\nu}\partial_\mu S_\nu)_\alpha + \cdots \ .
\end{aligned} \tag{3.9.3}$$

Here $j^{(R)}_\mu$ is R-current, $S_{\mu\alpha}$ is the supersymmetry current, $T_{\mu\nu}$ is the energy-momentum tensor, and $\mathcal{F}_{\mu\nu}$ is a closed two-form, which gives rise to a string current $\varepsilon_{\mu\nu\rho\lambda}\mathcal{F}^{\rho\lambda}$. All of these currents are conserved. Note that (3.9.3) contains several unfamiliar factors of i, because we are working in Euclidean signature. In Lorentzian signature, the superfield $\mathcal{R}_\mu$ is real.

It is convenient to express the closed two-form $\mathcal{F}_{\mu\nu}$ in terms of a one-form $\mathcal{A}_\mu$,

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu \ . \tag{3.9.4}$$

In general $\mathcal{A}_\mu$ is not well defined, because it can shift by an exact one-form, $\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu\alpha$. An exception occurs if the theory is superconformal, in which case $\mathcal{A}_\mu$ is

[9] The supercovariant derivatives D_α and $\widetilde{D}_{\dot{\alpha}}$ are given by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\widetilde{\theta}^{\dot{\alpha}}\partial_\mu \ , \qquad \widetilde{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\widetilde{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \ . \tag{3.9.1}$$

a well-defined conserved current. The superfield χ_α can then be set to zero by an improvement transformation. Below, we will need the variation of the bosonic fields in the $\mathcal{R}$-multiplet under ordinary flat-space supersymmetry transformations,

$$\begin{aligned}
\delta j_\mu^{(R)} &= -i\zeta S_\mu + i\widetilde{\zeta}\widetilde{S}_\mu \ , \\
\delta T_{\mu\nu} &= \frac{1}{2}\zeta\sigma_{\mu\rho}\partial^\rho S_\nu + \frac{1}{2}\widetilde{\zeta}\,\widetilde{\sigma}_{\mu\rho}\partial^\rho\widetilde{S}_\nu + (\mu\leftrightarrow\nu) \ , \\
\delta\mathcal{A}_\mu &= -\frac{i}{2}\left(\zeta S_\mu - \widetilde{\zeta}\widetilde{S}_\mu - 2\zeta\sigma_{\mu\rho}S^\rho + 2\widetilde{\zeta}\,\widetilde{\sigma}_{\mu\rho}\widetilde{S}^\rho\right) + \partial_\mu\left(\cdots\right) \ .
\end{aligned} \tag{3.9.5}$$

The ellipsis denotes a possible ambiguity in the variation of $\mathcal{A}_\mu$ due to shifts by an exact one-form.

3.9.2 Background Supergravity and the Rigid Limit

We would like to place a supersymmetric flat-space theory on a curved manifold $\mathcal{M}$ by coupling it to background supergravity fields. A straightforward but tedious approach is to follow the Noether procedure. This can be avoided if an off-shell formulation of dynamical supergravity is available. As explained in [31], we can couple this supergravity to the field theory of interest and freeze the supergravity fields in arbitrary background configurations by rescaling them appropriately and sending the Planck mass to infinity. This was termed the rigid limit in [31]. In this limit, the fluctuations of the supergravity fields decouple and they become classical backgrounds, which can be chosen arbitrarily. In particular, we do not eliminate the auxiliary fields via their equations of motion.

We will apply this procedure to $\mathcal{N}=1$ theories in four dimensions, which admit different supercurrent multiplets. These give rise to different off-shell formulations of supergravity, which differ in the choice of propagating and auxiliary fields. For instance, the FZ-multiplet couples to the old minimal formulation of supergravity [97, 98], while the $\mathcal{R}$-multiplet couples to new minimal supergravity [95, 96]. We will focus on the latter. In addition to the metric $g_{\mu\nu}$ and the gravitino $\psi_{\mu\alpha}$, new minimal supergravity contains two auxiliary fields: an Abelian gauge field A_μ and a two-form gauge field $B_{\mu\nu}$. The dual field strength V^μ of $B_{\mu\nu}$ is a conserved vector field,

$$V^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda} \ , \qquad \nabla_\mu V^\mu = 0 \ . \tag{3.9.6}$$

In an expansion around flat space, $g_{\mu\nu} = \delta_{\mu\nu} + 2h_{\mu\nu}$, the linearized couplings to new minimal supergravity are determined by the operators in the $\mathcal{R}$-multiplet (3.9.3),

$$\mathscr{L} = T_{\mu\nu}h^{\mu\nu} - j_\mu^{(R)}\big(A^\mu - \frac{3}{2}V^\mu\big) - \mathcal{A}_\mu V^\mu + (\text{fermions}) \ . \tag{3.9.7}$$

Here A^μ and V^μ both have dimension 1, while $g_{\mu\nu}$ and $B_{\mu\nu}$ are dimensionless. The fermion terms contain the couplings of the gravitino to the supersymmetry current, which will not be important for us. We see from (3.9.7) that A_μ is the gauge field associated with local $U(1)_R$ transformations. Under these transformations the gravitino $\psi_{\mu\alpha}$ has R-charge $+1$. Note that the couplings in (3.9.7) are well defined under shifts of $\mathcal{A}_\mu$ by an exact one form, because V^μ is conserved. The fact that A^μ and V^μ couple to $j_\mu^{(R)}$ and $\mathcal{A}_\mu$ is a general feature of the rigid limit that persists beyond the linearized approximation around flat space. At higher order there are also terms quadratic in the auxiliary fields, as well as curvature terms, which are described in [31].

Note that the couplings (3.9.7) do not modify the short-distance structure of the field theory, which is the same as in flat space. To see this, we can choose a point on $\mathcal{M}$ and examine the theory in Riemann normal coordinates around this point. If the curvature scale is given by r, the metric is flat up to terms of order $\frac{1}{r^2}$. In these coordinates, the deformation (3.9.7) of the flat-space Lagrangian reduces to operators of dimension 3 or less, and hence the short-distance structure is not affected.

We are interested in configurations of the bosonic supergravity background fields that preserve some amount of rigid supersymmetry. The gravitino is set to zero. Such backgrounds must be invariant under a subalgebra of the underlying supergravity transformations. This subalgebra must leave the gravitino invariant,

$$\begin{aligned} \delta\psi_\mu &= -2\left(\nabla_\mu - iA_\mu\right)\zeta - 2iV_\mu\zeta - 2iV^\nu\sigma_{\mu\nu}\zeta \ , \\ \delta\widetilde{\psi}_\mu &= -2\left(\nabla_\mu + iA_\mu\right)\widetilde{\zeta} + 2iV_\mu\widetilde{\zeta} + 2iV^\nu\widetilde{\sigma}_{\mu\nu}\widetilde{\zeta} \ . \end{aligned} \tag{3.9.8}$$

Since the variations of the bosonic supergravity fields are proportional to the gravitino, they vanish automatically. Therefore, any nontrivial choice of ζ and $\widetilde{\zeta}$ that satisfies (3.9.8) gives rise to a rigid supercharge. In general, the algebra satisfied by these supercharges differs from the ordinary supersymmetry algebra in flat space. Rather, it is a particular subalgebra of the local supergravity transformations that is determined by the background fields.

3.9.3 Freedom in the Auxiliary Fields

In section 3.3 we found that the auxiliary fields A_μ and V_μ are not completely determined by the geometry of the underlying Hermitian manifold. For instance, it follows from (3.3.7) that we have the freedom of shifting V^μ by a conserved holomorphic vector U^μ. Here we would like to elucidate the origin of this freedom by linearizing the metric around flat space, so that the deformation of the Lagrangian is given by (3.9.7), and using our knowledge of the $\mathcal{R}$-multiplet.

We can choose

$$\zeta_\alpha = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{3.9.9}$$

and use holomorphic coordinates w, z adapted to the complex structure defined by ζ as in (3.2.3). In these coordinates, the linearized Hermitian metric only has components $h_{i\bar{j}}$. We would like to determine the values of the auxiliary fields A_μ and V_μ for which the bosonic terms in (3.9.7) are invariant under the supercharge δ_ζ corresponding to ζ in (3.9.9). This amounts to finding combinations of $T_{i\bar{j}}$ and the other bosonic operators $j_\mu^{(R)}$, $\mathcal{A}_\mu$ in the $\mathcal{R}$-multiplet that are invariant under δ_ζ. Since we are working to linear order, we can use the flat-space transformations (3.9.5) to find

$$\begin{aligned} &\delta_\zeta\Big(T_{w\overline{w}} - \frac{i}{2}\mathcal{F}_{z\overline{z}} - \frac{i}{4}\big(\partial_w j_{\overline{w}}^{(R)} - \partial_{\overline{w}} j_w^{(R)}\big) + \frac{i}{2}\partial_z j_{\overline{z}}^{(R)}\Big) = 0 \ , \\ &\delta_\zeta\Big(T_{w\overline{z}} + \frac{i}{2}\mathcal{F}_{w\overline{z}} - \frac{3i}{4}\partial_w j_{\overline{z}}^{(R)} + \frac{i}{4}\partial_{\overline{z}} j_w^{(R)}\Big) = 0 \ , \end{aligned} \tag{3.9.10}$$

and two more with $w \leftrightarrow z$, $\overline{w} \leftrightarrow \overline{z}$. Moreover, up to shifts of $\mathcal{A}_\mu$ by an exact one-form,

$$\delta_\zeta \mathcal{A}_w = \delta_\zeta \mathcal{A}_z = 0 \ . \tag{3.9.11}$$

We can therefore add $\mathcal{A}_w$ and $\mathcal{A}_z$ with coefficients that are arbitrary functions of the coordinates, as long as we ensure that the Lagrangian is invariant under shifts of $\mathcal{A}_\mu$. Hence, the following Lagrangian is invariant under δ_ζ,

$$\begin{aligned}\mathscr{L} = \ &\Big(2h^{w\overline{w}}\big(T_{w\overline{w}} - \frac{i}{2}\mathcal{F}_{z\overline{z}} - \frac{i}{4}(\partial_w j^{(R)}_{\overline{w}} - \partial_{\overline{w}} j^{(R)}_w) + \frac{i}{2}\partial_z j^{(R)}_{\overline{z}}\big) \\ &+ 2h^{w\overline{z}}\big(T_{w\overline{z}} + \frac{i}{2}\mathcal{F}_{w\overline{z}} - \frac{3i}{4}\partial_w j^{(R)}_{\overline{z}} + \frac{i}{4}\partial_{\overline{z}} j^{(R)}_w\big)\Big) + (w \leftrightarrow z\ , \overline{w} \leftrightarrow \overline{z}) \\ &- U^w \mathcal{A}_w - U^z \mathcal{A}_z\ .\end{aligned} \tag{3.9.12}$$

Here $U^{w,z}$ is a holomorphic vector field. Invariance under shifts of $\mathcal{A}_\mu$ by an exact one-form implies that it must be conserved,

$$\partial_w U^w + \partial_z U^z = 0\ . \tag{3.9.13}$$

We can now determine the auxiliary fields V_μ and A_μ by comparing (3.9.12) to (3.9.7),

$$\begin{aligned}&V_w = -2i\left(\partial_w h_{z\overline{z}} - \partial_z h_{w\overline{z}}\right)\ , \qquad V_{\overline{w}} = 2i\left(\partial_{\overline{w}} h_{z\overline{z}} - \partial_{\overline{z}} h_{z\overline{w}}\right) + U_{\overline{w}}\ , \\ &A_w = -i\partial_w\left(h_{w\overline{w}} + h_{z\overline{z}}\right)\ , \qquad A_{\overline{w}} = i\partial_{\overline{w}}\left(h_{w\overline{w}} + h_{z\overline{z}}\right) + V_{\overline{w}} + \frac{1}{2}U_{\overline{w}}\ ,\end{aligned} \tag{3.9.14}$$

and four more with $w \leftrightarrow z$, $\overline{w} \leftrightarrow \overline{z}$. This exactly agrees with (3.3.7) and (3.3.17). We see that the freedom in $U_{\overline{i}}$ is the result of (3.9.11), which allows us to add $\mathcal{A}_w$ and $\mathcal{A}_z$ with coefficients that are arbitrary functions of the coordinates, as long as we ensure invariance under shifts of $\mathcal{A}_\mu$ by an exact one-form.

3.10 Appendix C: Solutions with $[K, \overline{K}] \neq 0$

In this appendix we analyze the constraints due to a complex Killing vector K that squares to zero, $K^\mu K_\mu = 0$, and does not commute with its complex conjugate, $[K, \overline{K}] \neq 0$. In this case, we will show that $\mathcal{M}$ is locally isometric to $S^3 \times \mathbb{R}$ with warped metric

$$ds^2 = d\tau^2 + r(\tau)^2 d\Omega_3\ . \tag{3.10.1}$$

Here $d\Omega_3$ is the round metric on a unit three-sphere. If K is constructed from solutions ζ and $\widetilde{\zeta}$ of (3.4.1), we further prove that $r(\tau)$ must be a constant.

3.10.1 Algebra of Killing Vectors

For ease of notation, we will write $\langle X, Y \rangle = X^\mu Y_\mu$ for any two complex vectors X and Y, and refer to it as their inner product even though the vectors are complex. Since the complex conjugate $\overline{K}$ of the Killing vector K is also a Killing vector, their commutator gives rise to a third Killing vector L, which must be real,

$$[K, \overline{K}] = -iL \ . \tag{3.10.2}$$

The case $L = 0$ was analyzed in section 3.4. Here we assume that $L \neq 0$. In order to constrain the algebra generated by $K, \overline{K}$, and L, we will differentiate their inner products along the vectors themselves. For instance, differentiating $\langle K, K \rangle = 0$ along $\overline{K}$ gives

$$0 = \mathcal{L}_{\overline{K}} \langle K, K \rangle = 2i \langle L, K \rangle \ . \tag{3.10.3}$$

Since L is real, this implies that the three real Killing vectors $K + \overline{K}$, $i(K - \overline{K})$, and L are orthogonal.

We will now consider two distinct cases. If $K, \overline{K}$, and L form a closed algebra, it follows from constraints similar to (3.10.3) that this algebra must be $SU(2)$ in its usual compact form. If the algebra does not close, we find a fourth real Killing vector, which is orthogonal to the first three. In this case the algebra is $SU(2) \times U(1)$.

In the first case, we can introduce $SU(2)$-invariant one-forms ω^a and write the metric as $ds^2 = d\tau^2 + h_{ab}(\tau)\omega^a\omega^b$. The fact that the three Killing vectors are orthogonal implies that $h_{ab}(\tau) = r(\tau)^2\delta_{ab}$, see for instance [113]. The metric is therefore given by (3.10.1) and the isometry group is enhanced to $SU(2) \times SU(2)$. So far, $r(\tau)$ is an arbitrary positive function.

In the second case, we can similarly show that the metric must take the form (3.10.1), but in this case the presence of the additional $U(1)$ isometry corresponds to translations along τ, and hence $r(\tau)$ is a constant.

3.10.2 Proof that $r(\tau)$ is a Constant

As we saw above, the algebra of Killing vectors is not always sufficient to prove that $r(\tau)$ is constant. We will now show that this must be the case if $K^\mu = \zeta\sigma^\mu\widetilde{\zeta}$, where ζ and $\widetilde{\zeta}$ are solutions of (3.4.1). If we demand that the auxiliary fields A_μ and V_μ respect the $SU(2)\times SU(2)$ isometry of (3.10.1), the equations (3.4.1) can be analyzed explicitly and a solution exists if and only if $r(\tau)$ is a constant. We will now give a proof that does not rely on this additional assumption.

Recall that ζ and $\widetilde{\zeta}$ give rise to integrable complex structures $J^\mu{}_\nu$ and $\widetilde{J}^\mu{}_\nu$, which can be expressed in terms of K and $\overline{K}$ as in (3.2.12). Using (3.3.7) and (3.3.22), we find that

$$V_\mu = -\frac{1}{2}\nabla_\nu J^\nu{}_\mu + U_\mu = \frac{1}{2}\nabla_\nu \widetilde{J}^\nu{}_\mu + \widetilde{U}_\mu \ . \tag{3.10.4}$$

The vectors U^μ and $\widetilde{U}^\mu$ are conserved and holomorphic with respect to $J^\mu{}_\nu$ and $\widetilde{J}^\mu{}_\nu$ respectively. We can use the Killing vectors to parametrize them at every point:

$$\begin{aligned} U_\mu &= \kappa K_\mu + \sigma(L_\mu - iT_\mu) \ , \\ \widetilde{U}_\mu &= \widetilde{\kappa} K_\mu - \widetilde{\sigma}(L_\mu + iT_\mu) \ , \end{aligned} \tag{3.10.5}$$

where κ, σ and $\widetilde{\kappa}, \widetilde{\sigma}$ are complex functions on $\mathcal{M}$. Here we have defined an additional vector,[10]

$$T = \frac{i}{\langle K, \overline{K}\rangle}\varepsilon^{\mu\nu\rho\lambda} L_\nu K_\rho \overline{K}_\lambda \partial_\mu \sim r(t)\partial_\tau \ . \tag{3.10.6}$$

The fact that $K, L - iT$ and $K, L + iT$ are holomorphic with respect to $J^\mu{}_\nu$ and $\widetilde{J}^\mu{}_\nu$ respectively follows from (3.2.12).

Substituting (3.10.5) into (3.10.4) and demanding consistency leads to

$$\sigma = \widetilde{\sigma} \sim \frac{1}{r(\tau)^2} \ , \qquad \kappa = \widetilde{\kappa} \ . \tag{3.10.7}$$

Here we have used the fact that

$$\nabla_\mu (J^\mu{}_\nu + \widetilde{J}^\mu{}_\nu) = \frac{2}{\langle K, \overline{K}\rangle} L_\nu \sim \frac{1}{r(\tau)^2} L_\nu \ , \tag{3.10.8}$$

[10] It follows from the form of the metric (3.10.1) that $\langle K, \overline{K}\rangle$ is proportional to $r^2(\tau)$. For ease of notation, we will omit an overall real constant in some of the formulas below. This will be indicated by a tilde.

which follows from (3.2.12) and the commutation relation (3.10.2). Substituting (3.10.7) into (3.10.5) and using the fact that U_μ is conserved, we find that

$$\nabla_\mu(\kappa K^\mu) \sim i\frac{r'(\tau)}{r(\tau)^2} \ . \tag{3.10.9}$$

The orbits of $K, \overline{K}, L$ are given by surfaces of constant τ. Since the isometry is $SU(2)$, they must be compact. Integrating (3.10.9) over such an orbit, we find that

$$\int d^3x \sqrt{g}\,\nabla_\mu(\kappa K^\mu) \sim ir'(\tau)r(\tau) = 0 \ . \tag{3.10.10}$$

Therefore $r(\tau)$ is a constant.

Chapter 4

Comments on Chern-Simons Contact Terms in Three Dimensions

4.1 Introduction

In quantum field theory, correlation functions of local operators may contain δ-function singularities at coincident points. Such contributions are referred to as contact terms. Typically, they are not universal. They depend on how the operators and coupling constants of the theory are defined at short distances, i.e. they depend on the regularization scheme. This is intuitively obvious, since contact terms probe the theory at very short distances, near the UV cutoff Λ. If Λ is large but finite, correlation functions have features at distances of order Λ^{-1}. In the limit $\Lambda \to \infty$ some of these features can collapse into δ-function contact terms.

In this chapter, we will discuss contact terms in two-point functions of conserved currents in three-dimensional quantum field theory. As we will see, they do not suffer from the scheme dependence of conventional contact terms, and hence they lead to interesting observables.

It is convenient to promote all coupling constants to classical background fields and specify a combined Lagrangian for the dynamical fields and the classical backgrounds. As an example, consider a scalar operator $\mathcal{O}(x)$, which couples to a classical background field $\lambda(x)$,

$$\mathscr{L} = \mathscr{L}_0 + \lambda(x)\mathcal{O}(x) + c\lambda^2(x) + c'\lambda(x)\partial^2\lambda(x) + \cdots \ . \tag{4.1.1}$$

Here $\mathscr{L}_0$ only depends on the dynamical fields and c, c' are constants. The ellipsis denotes other allowed local terms in $\lambda(x)$. If the theory has a gap, we can construct a well-defined effective action $F[\lambda]$ for the background field $\lambda(x)$,

$$e^{-F[\lambda]} = \left\langle e^{-\int d^3x\, \mathscr{L}} \right\rangle \ , \tag{4.1.2}$$

which captures correlation functions of $\mathcal{O}(x)$. (Since we are working in Euclidean signature, $F[\lambda]$ is nothing but the free energy.) At separated points, the connected two-point function $\langle \mathcal{O}(x)\mathcal{O}(y)\rangle$ arises from the term in (4.1.1) that is linear in $\lambda(x)$. Terms quadratic in $\lambda(x)$ give rise to contact terms: $c\delta^{(3)}(x-y) + c'\partial^2\delta^{(3)}(x-y) + \cdots$.

A change in the short-distance physics corresponds to modifying the Lagrangian (4.1.1) by local counterterms in the dynamical and the background fields. For instance, we can change the constants c, c' by modifying the theory near the UV cutoff, and hence the corresponding contact terms are scheme dependent. Equivalently, a scheme change corresponds to a field redefinition of the coupling $\lambda(x)$. This does not affect correlation functions at separated points, but it shifts the contact terms [114]. A related statement concerns redundant operators, i.e. operators that vanish by the equations of motion, which have vanishing correlation functions at separated points but may give rise to non-trivial contact terms.

Nevertheless, contact terms are meaningful in several circumstances. For example, this is the case for contact terms associated with irrelevant operators, such as the magnetic moment operator. Dimensionless contact terms are also meaningful whenever some physical principle, such as a symmetry, restricts the allowed local counterterms. A well-known example is the seagull term in scalar electrodynamics, which is fixed by gauge invariance. Another example is the trace anomaly of the energy-momentum tensor $T_{\mu\nu}$ in two-dimensional conformal field theories. Conformal invariance implies that T^μ_μ is a redundant operator. However, imposing the conservation law $\partial^\mu T_{\mu\nu} = 0$ implies that T^μ_μ has non-trivial contact terms. These contact terms are determined by the correlation functions of $T_{\mu\nu}$ at separated points, and hence they are unambiguous and meaningful. This is typical of local anomalies [115–117].

If we couple $T_{\mu\nu}$ to a background metric $g_{\mu\nu}$, the requirement that $T_{\mu\nu}$ be conserved corresponds to diffeomorphism invariance, which restricts the set of allowed counterterms. In two dimensions, the contact terms of T^μ_μ are summarized by the formula $\langle T^\mu_\mu \rangle = \frac{c}{24\pi} R$, where c is the Virasoro central charge and R is the scalar curvature of the background metric.[1] This result cannot be changed by the addition of diffeomorphism-invariant local counterterms.

The contact terms discussed above are either completely arbitrary or completely meaningful. In this chapter we will discuss a third kind of contact term. Its integer part is scheme dependent and can be changed by adding local counterterms. However, its fractional part is an intrinsic physical observable.

Consider a three-dimensional quantum field theory with a global $U(1)$ symmetry and its associated current j_μ. We will assume that the symmetry group is compact, i.e. only integer charges are allowed. The two-point function of j_μ can include a contact term,

$$\langle j_\mu(x) j_\nu(0) \rangle = \cdots + \frac{i\kappa}{2\pi} \, \varepsilon_{\mu\nu\rho} \partial^\rho \delta^{(3)}(x) \ . \qquad (4.1.3)$$

Here κ is a real constant. Note that this term is consistent with current conservation. We can couple j_μ to a background gauge field a_μ. The contact term in (4.1.3) corresponds to a Chern-Simons term for a_μ in the effective action $F[a]$,

$$F[a] = \cdots + \frac{i\kappa}{4\pi} \int d^3x \, \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \ . \qquad (4.1.4)$$

We might attempt to shift $\kappa \to \kappa + \delta\kappa$ by adding a Chern-Simons counterterm to the

[1] In our conventions, a d-dimensional sphere of radius r has scalar curvature $R = -\frac{d(d-1)}{r^2}$.

UV Lagrangian,

$$\delta\mathscr{L} = \frac{i\delta\kappa}{4\pi}\,\varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \ . \tag{4.1.5}$$

However, this term is not gauge invariant, and hence it is not a standard local counterterm.

We will now argue that (4.1.5) is only a valid counterterm for certain quantized values of $\delta\kappa$. Since counterterms summarize local physics near the cutoff scale, they are insensitive to global issues. Their contribution to the partition function (4.1.2) must be a well-defined, smooth functional for arbitrary configurations of the background fields and on arbitrary curved three-manifolds $\mathcal{M}_3$. Since we are interested in theories with fermions, we require $\mathcal{M}_3$ to be a spin manifold. Therefore (4.1.5) is an admissible counterterm if its integral is a well-defined, smooth functional up to integer multiples of $2\pi i$. This restricts $\delta\kappa$ to be an integer.

Usually, the quantization of $\delta\kappa$ is said to follow from gauge invariance, but this is slightly imprecise. If the $U(1)$ bundle corresponding to a_μ is topologically trivial, then a_μ is a good one-form. Since (4.1.5) shifts by a total derivative under small gauge transformations, its integral is well defined. This is no longer the case for non-trivial bundles. In order to make sense of the integral, we extend a_μ to a connection on a suitable $U(1)$ bundle over a spin four-manifold $\mathcal{M}_4$ with boundary $\mathcal{M}_3$, and we define

$$\frac{i}{4\pi}\int_{\mathcal{M}_3} d^3x\,\varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho = \frac{i}{16\pi}\int_{\mathcal{M}_4} d^4x\,\varepsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda} \ , \tag{4.1.6}$$

where $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the field strength. The right-hand side is a well-defined, smooth functional of a_μ, but it depends on the choice of $\mathcal{M}_4$. The difference between two choices $\mathcal{M}_4$ and $\mathcal{M}_4'$ is given by the integral over the closed four-manifold X_4, which is obtained by properly gluing $\mathcal{M}_4$ and $\mathcal{M}_4'$ along their common boundary $\mathcal{M}_3$. Since X_4 is also spin, we have

$$\frac{i}{16\pi}\int_{X_4} d^4x\,\varepsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda} = 2\pi i n \ , \qquad n \in \mathbb{Z} \ . \tag{4.1.7}$$

Thus, if $\delta\kappa$ is an integer, the integral of (4.1.5) is well defined up to integer multiples of $2\pi i$.[2]

We conclude that a counterterm of the from (4.1.5) can only shift the contact term κ in (4.1.3) by an integer. Therefore, the fractional part $\kappa \bmod 1$ does not depend on short-distance physics. It is scheme independent and gives rise to a new meaningful observable in three-dimensional field theories. This observable is discussed in section 4.2.

In section 4.2, we will also discuss the corresponding observable for the energy-momentum tensor $T_{\mu\nu}$. It is related to a contact term in the two-point function of $T_{\mu\nu}$,

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle = \cdots - \frac{i\kappa_g}{192\pi}\Big(\big(\varepsilon_{\mu\rho\lambda}\partial^\lambda(\partial_\nu\partial_\sigma - \partial^2\delta_{\nu\sigma}) + (\mu\leftrightarrow\nu)\big) + (\rho\leftrightarrow\sigma)\Big)\,\delta^{(3)}(x)\ . \tag{4.1.8}$$

This contact term is associated with the gravitational Chern-Simons term, which is properly defined by extending the metric $g_{\mu\nu}$ to a four-manifold,

$$\frac{i}{192\pi}\int_{\mathcal{M}_3}\sqrt{g}\,d^3x\,\varepsilon^{\mu\nu\rho}\ \mathrm{Tr}\left(\omega_\mu\partial_\nu\omega_\rho + \frac{2}{3}\omega_\mu\omega_\nu\omega_\rho\right) = \frac{i}{768\pi}\int_{\mathcal{M}_4}\sqrt{g}\,d^3x\,\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\kappa\lambda}R_{\rho\sigma}{}^{\kappa\lambda}\ . \tag{4.1.9}$$

Here ω_μ is the spin connection and $R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor. Note that we do not interpret the left-hand side of (4.1.9) as a Chern-Simons term for the $SO(3)$ frame bundle. (See for instance the discussion in [118].) As above, two different extensions of $\mathcal{M}_3$ differ by the integral over a closed spin four-manifold X_4,

$$\frac{i}{768\pi}\int_{X_4}\sqrt{g}\,d^3x\,\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\kappa\lambda}R_{\rho\sigma}{}^{\kappa\lambda} = 2\pi i n\ , \qquad n\in\mathbb{Z}\ . \tag{4.1.10}$$

Therefore, the gravitational Chern-Simons term (4.1.9) is a valid counterterm, as long as its coefficient is an integer.[3] Consequently, the integer part of the contact term κ_g in (4.1.8) is scheme dependent, while the fractional part $\kappa_g \bmod 1$ gives rise to a meaningful observable.

[2] In a purely bosonic theory we do not require $\mathcal{M}_3$ to be spin. In this case $\delta\kappa$ must be an even integer.

[3] If $\mathcal{M}_3$ is not spin, then the coefficient of (4.1.9) should be an integer multiple of 16.

We would briefly like to comment on another possible definition of Chern-Simons counterterms, which results in the same quantization conditions for their coefficients. It involves the Atiyah-Patodi-Singer η-invariant [119–121], which is defined in terms of the eigenvalues of a certain Dirac operator on $\mathcal{M}_3$ that couples to a_μ and $g_{\mu\nu}$. (Loosely speaking, it counts the number of eigenvalues, weighted by their sign.) Therefore, $\eta[a, g]$ is intrinsically three-dimensional and gauge invariant. The Atiyah-Patodi-Singer theorem states that $i\pi\eta[a, g]$ differs from the four-dimensional integrals in (4.1.6) and (4.1.9) by an integer multiple of $2\pi i$. Hence, its variation gives rise to contact terms of the form (4.1.3) and (4.1.8). Although $\eta[a, g]$ is well defined, it jumps discontinuously by 2 when an eigenvalue of its associated Dirac operator crosses zero. Since short-distance counterterms should not be sensitive to zero-modes, we only allow $i\pi\eta[a, g]$ with an integer coefficient.

In section 4.3, we discuss the observables $\kappa \bmod 1$ and $\kappa_g \bmod 1$ in several examples. We use our understanding of these contact terms to give an intuitive proof of a non-renormalization theorem due to Coleman and Hill [122].

In section 4.4 we extend our discussion to three-dimensional theories with $\mathcal{N} = 2$ supersymmetry. Here we must distinguish between $U(1)$ flavor symmetries and $U(1)_R$ symmetries. Some of the contact terms associated with the R-current are not consistent with conformal invariance. As we will see in section 4.5, this leads to a new anomaly in $\mathcal{N} = 2$ superconformal theories, which is similar to the framing anomaly of [123]. The anomaly can lead to violations of conformal invariance and unitarity when the theory is placed on curved manifolds.

In section 4.6, we explore these phenomena in $\mathcal{N} = 2$ supersymmetric QED (SQED) with a dynamical Chern-Simons term. For some range of parameters, this model is accessible in perturbation theory.

In supersymmetric theories, the observables defined in section 4.4 can be computed exactly using localization [4]. In section 4.7, we compute them in several theories that were conjectured to be dual, subjecting these dualities to a new test.

Appendix A contains simple free-field examples. In appendix B we summarize relevant aspects of $\mathcal{N}=2$ supergravity.

4.2 Two-Point Functions of Conserved Currents in Three Dimensions

In this section we will discuss two-point functions of flavor currents and the energy-momentum tensor in three-dimensional quantum field theory, and we will explain in detail how the contact terms in these correlators give rise to a meaningful observable.

4.2.1 Flavor Currents

We will consider a $U(1)$ flavor current j_μ. The extension to multiple $U(1)$'s or to non-Abelian symmetries is straightforward. Current conservation restricts the two-point function of j_μ. In momentum space,[4]

$$\langle j_\mu(p) j_\nu(-p)\rangle = \tau\left(\frac{p^2}{\mu^2}\right)\frac{p_\mu p_\nu - p^2\delta_{\mu\nu}}{16|p|} + \kappa\left(\frac{p^2}{\mu^2}\right)\frac{\varepsilon_{\mu\nu\rho}p^\rho}{2\pi} \ . \tag{4.2.2}$$

Here $\tau\left(p^2/\mu^2\right)$ and $\kappa\left(p^2/\mu^2\right)$ are real, dimensionless structure functions and μ is an arbitrary mass scale.

In a conformal field theory (CFT), $\tau=\tau_{\rm CFT}$ and $\kappa=\kappa_{\rm CFT}$ are independent of p^2. (We assume throughout that the symmetry is not spontaneously broken.) In this case (4.2.2) leads to the following formula in position space:[5]

$$\langle j_\mu(x) j_\nu(0)\rangle = \left(\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)\frac{\tau_{\rm CFT}}{32\pi^2 x^2} + \frac{i\kappa_{\rm CFT}}{2\pi}\varepsilon_{\mu\nu\rho}\partial^\rho\delta^{(3)}(x) \ . \tag{4.2.3}$$

---

[4] Given two operators $\mathcal{A}(x)$ and $\mathcal{B}(x)$, we define

$$\langle \mathcal{A}(p)\mathcal{B}(-p)\rangle = \int d^3x\, e^{ip\cdot x}\,\langle \mathcal{A}(x)\mathcal{B}(0)\rangle \ . \tag{4.2.1}$$

[5] A term proportional to $\varepsilon_{\mu\nu\rho}\partial^\rho|x|^{-3}$, which is conserved and does not vanish at separated points, is not consistent with conformal invariance.

This makes it clear that $\tau_{\rm CFT}$ controls the behavior at separated points, while the term proportional to $\kappa_{\rm CFT}$ is a pure contact term of the form (4.1.3). Unitarity implies that $\tau_{\rm CFT} \geq 0$. If $\tau_{\rm CFT} = 0$ then j_μ is a redundant operator.

If the theory is not conformal, then $\kappa\left(p^2/\mu^2\right)$ may be a non-trivial function of p^2. In this case the second term in (4.2.2) contributes to the two-point function at separated points, and hence it is manifestly physical. Shifting $\kappa\left(p^2/\mu^2\right)$ by a constant $\delta\kappa$ only affects the contact term (4.1.3). It corresponds to shifting the Lagrangian by the Chern-Simons counterterm (4.1.5). As explained in the introduction, shifts with arbitrary $\delta\kappa$ may not always be allowed. We will return to this issue below.

It is natural to define the UV and IR values

$$\kappa_{\rm UV} = \lim_{p^2\to\infty} \kappa\left(\frac{p^2}{\mu^2}\right) , \qquad \kappa_{\rm IR} = \lim_{p^2\to 0} \kappa\left(\frac{p^2}{\mu^2}\right) . \tag{4.2.4}$$

Adding the counterterm (4.1.5) shifts $\kappa_{\rm UV}$ and $\kappa_{\rm IR}$ by $\delta\kappa$. Therefore $\kappa_{\rm UV} - \kappa_{\rm IR}$ is not modified, and hence it is a physical observable.

We will now assume that the $U(1)$ symmetry is compact, i.e. only integer charges are allowed. (This is always the case for theories with a Lagrangian description, as long as we pick a suitable basis for the Abelian flavor symmetries.) In this case, the coefficient $\delta\kappa$ of the Chern-Simons counterterm (4.1.5) must be an integer. Therefore, the entire fractional part $\kappa(p^2/\mu^2) \bmod 1$ is scheme independent. It is a physical observable for every value of p^2. In particular, the constant $\kappa_{\rm CFT} \bmod 1$ is an intrinsic physical observable in any CFT.

The fractional part of $\kappa_{\rm CFT}$ has a natural bulk interpretation for CFTs with an AdS_4 dual. While the constant $\tau_{\rm CFT}$ is related to the coupling of the bulk gauge field corresponding to j_μ, the fractional part of $\kappa_{\rm CFT}$ is related to the bulk θ-angle. The freedom to shift $\kappa_{\rm CFT}$ by an integer reflects the periodicity of θ, see for instance [124].

In order to calculate the observable $\kappa_{\rm CFT} \bmod 1$ for a given CFT, we can embed the CFT into an RG flow from a theory whose κ is known – for instance a free theory. We can then unambiguously calculate $\kappa(p^2/\mu^2)$ to find the value of $\kappa_{\rm CFT}$ in the IR. This procedure is carried out for free massive theories in appendix A. More generally, if the RG flow is short, we can calculate the change in κ using (conformal) perturbation theory. In certain supersymmetric theories it is possible to calculate $\kappa_{\rm CFT} \bmod 1$ exactly using localization [4]. This will be discussed in section 4.7.

We would like to offer another perspective on the observable related to $\kappa(p^2)$. Using (4.2.2), we can write the difference $\kappa_{\rm UV} - \kappa_{\rm IR}$ as follows:

$$\kappa_{\rm UV} - \kappa_{\rm IR} = \frac{i\pi}{6} \int_{\mathbb{R}^3 - \{0\}} d^3x \, x^2 \, \varepsilon^{\mu\nu\rho} \, \partial_\mu \langle j_\nu(x) j_\rho(0) \rangle \ . \tag{4.2.5}$$

The integral over $\mathbb{R}^3 - \{0\}$ excludes a small ball around $x = 0$, and hence it is not sensitive to contact terms. The integral converges because the two-point function $\varepsilon^{\mu\nu\rho}\partial_\mu \langle j_\nu(x) j_\rho(0)\rangle$ vanishes at separated points in a conformal field theory, so that it decays faster than $\frac{1}{x^3}$ in the IR and diverges more slowly than $\frac{1}{x^3}$ in the UV. Alternatively, we can use Cauchy's theorem to obtain the dispersion relation

$$\kappa_{\rm UV} - \kappa_{\rm IR} = \frac{1}{\pi} \int_0^\infty \frac{ds}{s} \, {\rm Im}\, \kappa \left(-\frac{s}{\mu^2} \right) \ . \tag{4.2.6}$$

This integral converges for the same reasons as (4.2.5). Since it only depends on the imaginary part of $\kappa(p^2/\mu^2)$, it is physical.

The formulas (4.2.5) and (4.2.6) show that the difference between $\kappa_{\rm UV}$ and $\kappa_{\rm IR}$ can be understood by integrating out massive degrees of freedom as we flow from the UV theory to the IR theory. Nevertheless, they capture the difference between two quantities that are intrinsic to these theories. Although there are generally many different RG flows that connect a pair of UV and IR theories, the integrals in (4.2.5) and (4.2.6) are invariant under continuous deformations of the flow. This is very similar to well-known statements about the Virasoro central charge c in two dimensions. In particular, the sum rules (4.2.5) and (4.2.6) are analogous to the sum rules in [125, 126] for the change in c along an RG flow.

4.2.2 Energy-Momentum Tensor

We can repeat the analysis of the previous subsection for the two-point function of the energy-momentum tensor $T_{\mu\nu}$, which depends on three dimensionless structure functions $\tau_g(p^2/\mu^2)$, $\tau'_g(p^2/\mu^2)$, and $\kappa_g(p^2/\mu^2)$,

$$
\begin{aligned}
\langle T_{\mu\nu}(p)T_{\rho\sigma}(-p)\rangle = & -(p_\mu p_\nu - p^2\delta_{\mu\nu})(p_\rho p_\sigma - p^2\delta_{\rho\sigma})\frac{\tau_g\left(p^2/\mu^2\right)}{|p|} \\
& - \big((p_\mu p_\rho - p^2\delta_{\mu\rho})(p_\nu p_\sigma - p^2\delta_{\nu\sigma}) + (\mu \leftrightarrow \nu)\big)\,\frac{\tau'_g\left(p^2/\mu^2\right)}{|p|} \\
& + \frac{\kappa_g\left(p^2/\mu^2\right)}{192\pi}\left(\big(\varepsilon_{\mu\rho\lambda}p^\lambda(p_\nu p_\sigma - p^2\delta_{\nu\sigma}) + (\mu \leftrightarrow \nu)\big) + (\rho \leftrightarrow \sigma)\right) \ .
\end{aligned}
\tag{4.2.7}
$$

Unitarity implies that $\tau_g(p^2/\mu^2) + \tau'_g(p^2/\mu^2) \geq 0$. If the equality is saturated, the trace T^μ_μ becomes a redundant operator. This is the case in a CFT, where $\tau_g = -\tau'_g$ and κ_g are constants. The terms proportional to τ_g determine the correlation function at separated points. The term proportional to κ_g gives rise to a conformally invariant contact term (4.1.8). It is associated with the gravitational Chern-Simons term (4.1.9), which is invariant under a conformal rescaling of the metric. Unlike the Abelian case discussed above, the contact term κ_g is also present in higher-point functions of $T_{\mu\nu}$. (This is also true for non-Abelian flavor currents.)

Repeating the logic of the previous subsection, we conclude that $\kappa_{g,\mathrm{UV}} - \kappa_{g,\mathrm{IR}}$ is physical and can in principle be computed along any RG flow. Moreover, the quantization condition on the coefficient of the gravitational Chern-Simons term (4.1.9) implies that the fractional part $\kappa_g(p^2/\mu^2) \bmod 1$ is a physical observable for any value of p^2. In particular $\kappa_{g,\mathrm{CFT}} \bmod 1$ is an intrinsic observable in any CFT.

4.3 Examples

In this section we discuss a number of examples that illustrate our general discussion above. An important example with $\mathcal{N} = 2$ supersymmetry will be discussed in section 4.6. Other examples with $\mathcal{N} = 4$ supersymmetry appear in [127].

4.3.1 Free Fermions

We begin by considering a theory of N free Dirac fermions of charge $+1$ with real masses m_i. Here we make contact with the parity anomaly of [117, 128, 129]. As is reviewed in appendix A, integrating out a Dirac fermion of mass m and charge $+1$ shifts κ by $-\frac{1}{2}\,\mathrm{sign}(m)$, and hence we find that

$$\kappa_{\mathrm{UV}} - \kappa_{\mathrm{IR}} = \frac{1}{2}\sum_{i=1}^{N} \mathrm{sign}\,(m_i) \ . \tag{4.3.1}$$

If N is odd, this difference is a half-integer. Setting $\kappa_{\mathrm{UV}} = 0$ implies that κ_{IR} is a half-integer, even though the IR theory is empty. In the introduction, we argued that short-distance physics can only shift κ by an integer. The same argument implies that κ_{IR} must be an integer if the IR theory is fully gapped.[6] We conclude that it is inconsistent to set κ_{UV} to zero; it must be a half-integer. Therefore,

$$\begin{aligned} \kappa_{\mathrm{UV}} &= \frac{1}{2} + n \ , \qquad n \in \mathbb{Z} \ , \\ \kappa_{\mathrm{IR}} &= \kappa_{\mathrm{UV}} - \frac{1}{2}\sum_{i=1}^{N} \mathrm{sign}(m_i) \in \mathbb{Z} \ . \end{aligned} \tag{4.3.2}$$

The half-integer value of κ_{UV} implies that the UV theory is not parity invariant, even though it does not contain any parity-violating mass terms. This is known as the parity anomaly [117, 128, 129].

We can use (4.3.2) to find the observable $\kappa_{\mathrm{CFT}} \bmod 1$ for the CFT that consists of N free massless Dirac fermions of unit charge:

$$\kappa_{\mathrm{CFT}} \bmod 1 = \begin{cases} 0 & N \text{ even} \\ \frac{1}{2} & N \text{ odd} \end{cases} \tag{4.3.3}$$

This illustrates the fact that we can calculate κ_{CFT}, if we can connect the CFT of interest to a theory with a known value of κ. Here we used the fact that the fully gapped IR theory has integer κ_{IR}.

We can repeat the above discussion for the contact term κ_g that appears in the two-point function of the energy-momentum tensor. Integrating out a Dirac fermion of mass m shifts κ_g by $-\,\mathrm{sign}(m)$, so that

$$\kappa_{g,\mathrm{UV}} - \kappa_{g,\mathrm{IR}} = \sum_{i} \mathrm{sign}(m_i) \ . \tag{4.3.4}$$

[6] We refer to a theory as fully gapped when it does not contain any massless or topological degrees of freedom.

If we instead consider N Majorana fermions with masses m_i, then $\kappa_{g,\mathrm{UV}} - \kappa_{g,\mathrm{IR}}$ would be half the answer in (4.3.4). Since $\kappa_{g,\mathrm{IR}}$ must be an integer in a fully gapped theory, we conclude that $\kappa_{g,\mathrm{UV}}$ is a half-integer if the UV theory consists of an odd number of massless Majorana fermions. This is the gravitational analogue of the parity anomaly.

4.3.2 Topological Currents and Fractional Values of κ

Consider a dynamical $U(1)$ gauge field A_μ, and the associated topological current

$$j_\mu = \frac{ip}{2\pi}\,\varepsilon_{\mu\nu\rho}\partial^\nu A^\rho \ , \qquad p \in \mathbb{Z} \ . \tag{4.3.5}$$

Note that the corresponding charges are integer multiples of p. We study the free topological theory consisting of two $U(1)$ gauge fields – the dynamical gauge field A_μ and a classical background gauge field a_μ – with Lagrangian [51, 124, 127, 130–132]

$$\mathscr{L} = \frac{i}{4\pi}\left(k\,\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + 2\,p\,\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu A_\rho + q\,\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho\right) \ , \qquad k, p, q \in \mathbb{Z} \ . \tag{4.3.6}$$

The background field a_μ couples to the topological current j_μ in (4.3.5). In order to compute the contact term κ corresponding to j_μ, we naively integrate out the dynamical field A_μ to obtain an effective Lagrangian for a_μ,

$$\mathscr{L}_{\mathrm{eff}} = \frac{i\kappa}{4\pi}\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \ , \qquad \kappa = q - \frac{p^2}{k} \ . \tag{4.3.7}$$

Let us examine the derivation of (4.3.7) more carefully. The equation of motion for A_μ is

$$k\varepsilon^{\mu\nu\rho}\partial_\nu A_\rho = -p\varepsilon^{\mu\nu\rho}\partial_\nu a_\rho \ . \tag{4.3.8}$$

Assuming, for simplicity, that k and p are relatively prime, this equation can be solved only if the flux of a_μ through every two-cycle is an integer multiple of k. When this is not the case the functional integral vanishes. If the fluxes of a_μ are multiples of k, the derivation of (4.3.7) is valid. For these configurations the fractional value of κ is harmless.

This example shows that κ is not necessarily an integer, even if the theory contains only topological degrees of freedom. Equivalently, the observable $\kappa \bmod 1$ is sensitive to topological degrees of freedom. We would like to make a few additional comments:

1.) The freedom in shifting the Lagrangian by a Chern-Simons counterterm (4.1.5) with integer $\delta\kappa$ amounts to changing the integer q in (4.3.6).

2.) The value $\kappa = q - \frac{p^2}{k}$ can be measured by making the background field a_μ dynamical and studying correlation functions of Wilson loops for a_μ in flat Euclidean space $\mathbb{R}^3$. These correlation functions can be determined using either the original theory (4.3.6) or the effective Lagrangian (4.3.7).

3.) Consider a CFT that consists of two decoupled sectors: a nontrivial CFT$_0$ with a global $U(1)$ current $j_\mu^{(0)}$ and a $U(1)$ Chern-Simons theory with level k and topological current $\frac{ip}{2\pi}\varepsilon_{\mu\nu\rho}\partial^\nu A^\rho$. We will study the linear combination $j_\mu = j_\mu^{(0)} + \frac{ip}{2\pi}\varepsilon_{\mu\nu\rho}\partial^\nu A^\rho$. Denoting the contact term in the two-point function of $j_\mu^{(0)}$ by κ_0, the contact term κ corresponding to j_μ is given by

$$\kappa = \kappa_0 - \frac{p^2}{k} + (\text{integer}) \ . \tag{4.3.9}$$

Since the topological current is a redundant operator, it is not possible to extract κ by studying correlation functions of local operators at separated points. Nevertheless, the fractional part of κ is an intrinsic physical observable. This is an example of a general point that was recently emphasized in [133]: a quantum field theory is not uniquely characterized by its local operators and their correlation functions at separated points. The presence of topological degrees of freedom makes it necessary to also study various extended objects, such as line or surface operators.

4.3.3 A Non-Renormalization Theorem

Consider an RG flow from a free theory in the UV to a fully gapped theory in the IR. (Recall that a theory is fully gapped when it does not contain massless or topological degrees of freedom.) In this case, we can identify κ_{IR} with the coefficient of the Chern-Simons term for the background field a_μ in the Wilsonian effective action. Since the IR theory is fully gapped, κ_{IR} must be an integer. Depending on the number of fermions in the free UV theory, $\kappa_{\rm UV}$ is either an integer or a half-integer. Therefore, the difference $\kappa_{\rm UV} - \kappa_{\rm IR}$ is either an integer or a half-integer, and hence it cannot change under smooth deformations of the coupling constants. It follows that this difference is only generated at one-loop. This is closely related to a non-renormalization theorem due to Coleman and Hill [122], which was proved through a detailed analysis of Feynman diagrams. Note that our argument applies to Abelian and non-Abelian flavor currents, as well as the energy-momentum tensor.

When the IR theory has a gap, but contains some topological degrees of freedom, κ need not be captured by the Wilsonian effective action. As in the previous subsection, it can receive contributions from the topological sector. If the flow is perturbative, we can distinguish 1PI diagrams. The results of [122] imply that 1PI diagrams only contribute to κ associated with a flavor current at one-loop. (The fractional contribution discussed in the previous subsection arises from diagrams that are not 1PI.) However, this is no longer true for κ_g, which is associated with the energy-momentum tensor. For instance, κ_g receives higher loop contributions from 1PI diagrams in pure non-Abelian Chern-Simons theory [123].

4.3.4 Flowing Close to a Fixed Point

Consider an RG flow with two crossover scales $M \gg m$. The UV consists of a free theory that is deformed by a relevant operator. Below the scale M, the theory flows very close to a CFT. This CFT is further deformed by a relevant operator, so that it flows to a gapped theory below a scale $m \ll M$.

If the theory has a $U(1)$ flavor current j_μ, the structure functions in (4.2.2) interpolate between their values in the UV, through the CFT values, down to the IR:

$$
\tau \approx \begin{cases} \tau_{\rm UV} & p^2 \gg M^2 \\ \tau_{\rm CFT} & m^2 \ll p^2 \ll M^2 \\ \tau_{\rm IR} & p^2 \ll m^2 \end{cases}
\qquad
\kappa \approx \begin{cases} \kappa_{\rm UV} & p^2 \gg M^2 \\ \kappa_{\rm CFT} & m^2 \ll p^2 \ll M^2 \\ \kappa_{\rm IR} & p^2 \ll m^2 \end{cases}
\tag{4.3.10}
$$

Since the UV theory is free, $\tau_{\rm UV}$ is easily computed (see appendix A). In a free theory we can always take the global symmetry group to be compact. This implies that $\kappa_{\rm UV}$ is either integer or half-integer, depending on the number of fermions that are charged under j_μ. If j_μ does not mix with a topological current in the IR, then $\tau_{\rm IR}$ vanishes and $\kappa_{\rm IR}$ must be an integer. This follows from the fact that the theory is gapped.

Since we know $\kappa_{\rm UV}$ and $\kappa_{\rm IR}$, we can use the flow to give two complementary arguments that $\kappa_{\rm CFT} \bmod 1$ is an intrinsic observable of the CFT:

1.) *The flow from the UV to the CFT:* Here we start with a well- defined $\kappa_{\rm UV}$, which can only be shifted by an integer. Since $\kappa_{\rm UV} - \kappa_{\rm CFT}$ is physical, it follows that $\kappa_{\rm CFT}$ is well defined modulo an integer.

2.) *The flow from the CFT to the IR:* We can discuss the CFT without flowing into it from a free UV theory. If the CFT can be deformed by a relevant operator such that it flows to a fully gapped theory, then $\kappa_{\rm IR}$ must be an integer. Since $\kappa_{\rm CFT} - \kappa_{\rm IR}$ is physical and only depends on information intrinsic to the CFT, i.e. the relevant deformation that we used to flow out, we conclude that the fractional part of $\kappa_{\rm CFT}$ is an intrinsic observable of the CFT.

Below, we will see examples of such flows, and we will use them to compute $\kappa_{\rm CFT} \bmod 1$. For the theory discussed in section 4.6, we will check explicitly that flowing into or out of the CFT gives the same answer for this observable.

4.4 Theories with $\mathcal{N} = 2$ Supersymmetry

In this section we extend the previous discussion to three-dimensional theories with $\mathcal{N} = 2$ supersymmetry. Here we must distinguish between $U(1)$ flavor symmetries and $U(1)_R$ symmetries.

4.4.1 Flavor Symmetries

A $U(1)$ flavor current j_μ is embedded in a real linear superfield $\mathcal{J}$, which satisfies $D^2\mathcal{J} = \overline{D}^2\mathcal{J} = 0$. In components,

$$\mathcal{J} = J + i\theta j + i\overline{\theta}\overline{j} + i\theta\overline{\theta}K - \left(\theta\gamma^\mu\overline{\theta}\right) j_\mu - \frac{1}{2}\theta^2\overline{\theta}\gamma^\mu\partial_\mu j - \frac{1}{2}\overline{\theta^2}\theta\gamma^\mu\partial_\mu\overline{j} + \frac{1}{4}\theta^2\overline{\theta^2}\partial^2 J \ . \quad (4.4.1)$$

The supersymmetry Ward identities imply the following extension of (4.2.2):[7]

$$\begin{aligned}
\langle j_\mu(p) j_\nu(-p)\rangle &= (p_\mu p_\nu - p^2\delta_{\mu\nu})\frac{\widehat{\tau}_{ff}}{8|p|} + \varepsilon_{\mu\nu\rho}p^\rho\frac{\kappa_{ff}}{2\pi} \ , \\
\langle J(p)J(-p)\rangle &= \frac{\widehat{\tau}_{ff}}{8|p|} \ , \\
\langle K(p)K(-p)\rangle &= -\frac{|p|}{8}\widehat{\tau}_{ff} \ , \\
\langle J(p)K(-p)\rangle &= \frac{\kappa_{ff}}{2\pi} \ .
\end{aligned} \quad (4.4.2)$$

Here we have defined $\widehat{\tau}_{ff} = \frac{1}{2}\tau$, so that $\widehat{\tau}_{ff} = 1$ for a free massless chiral superfield of charge $+1$, and we have also renamed $\kappa_{ff} = \kappa$. The subscript ff emphasizes the fact that we are discussing two-point functions of flavor currents.

As in the non-supersymmetric case, we can couple the flavor current to a background gauge field. Following [16, 134], we should couple $\mathcal{J}$ to a background vector superfield,

$$\mathcal{V} = \cdots + \left(\theta\gamma^\mu\overline{\theta}\right) a_\mu - i\theta\overline{\theta}\sigma - i\theta^2\overline{\theta}\overline{\lambda} + i\overline{\theta^2}\theta\lambda - \frac{1}{2}\theta^2\overline{\theta^2}D \ . \quad (4.4.3)$$

[7]Supersymmetry also fixes the two-point function of the fermionic operators j_α and $\overline{j}_\alpha$ in terms of $\widehat{\tau}_{ff}$ and κ_{ff}, but in order to simplify the presentation, we will restrict our discussion to bosonic operators.

Background gauge transformations shift $\mathcal{V} \to \mathcal{V} + \Lambda + \overline{\Lambda}$ with chiral Λ, so that σ and D are gauge invariant, while a_μ transforms like an ordinary gauge field. (The ellipsis denotes fields that are pure gauge modes and do not appear in gauge-invariant functionals of $\mathcal{V}$.) The coupling of $\mathcal{J}$ to $\mathcal{V}$ takes the form

$$\delta\mathscr{L} = -2\int d^4\theta\, \mathcal{J}\mathcal{V} = -j_\mu a^\mu - K\sigma - JD + (\text{fermions}) \ . \qquad (4.4.4)$$

As before, it may be necessary to also add higher-order terms in $\mathcal{V}$ to maintain gauge invariance.

We can now adapt our previous discussion to κ_{ff}. According to (4.4.2), a constant value of κ_{ff} gives rise to contact terms in both $\langle j_\mu(p) j_\nu(-p)\rangle$ and $\langle J(p)K(-p)\rangle$. These contact terms correspond to a supersymmetric Chern-Simons term for the background field $\mathcal{V}$,

$$\mathscr{L}_{ff} = -\frac{\kappa_{ff}}{2\pi}\int d^4\theta\, \Sigma\, \mathcal{V} = \frac{\kappa_{ff}}{4\pi}\left(i\varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho - 2\sigma D + (\text{fermions})\right) \ . \qquad (4.4.5)$$

Here the real linear superfield $\Sigma = \frac{i}{2}\overline{D}D\mathcal{V}$ is the gauge-invariant field strength corresponding to $\mathcal{V}$. If the $U(1)$ flavor symmetry is compact, then the same arguments as above imply that short-distance counterterms can only shift κ_{ff} by an integer, and hence the analysis of section 4.2 applies. In particular, the fractional part $\kappa_{ff} \bmod 1$ is a good observable in any superconformal theory with a $U(1)$ flavor symmetry.

4.4.2 R-Symmetries

Every three-dimensional $\mathcal{N} = 2$ theory admits a supercurrent multiplet $\mathcal{S}_\mu$ that contains the supersymmetry current and the energy-momentum tensor, as well as other operators. A thorough discussion of supercurrents in three dimensions can be found in [1]. If the theory has a $U(1)_R$ symmetry, the $\mathcal{S}$-multiplet can be improved to a multiplet $\mathcal{R}_\mu$, which satisfies

$$\overline{D}^\beta \mathcal{R}_{\alpha\beta} = -4i\overline{D}_\alpha \mathcal{J}^{(Z)} \ , \qquad D^2\mathcal{J}^{(Z)} = \overline{D}^2\mathcal{J}^{(Z)} = 0 \ . \qquad (4.4.6)$$

Here $\mathcal{R}_{\alpha\beta} = -2\gamma^\mu_{\alpha\beta}\mathcal{R}_\mu$ is the symmetric bi-spinor corresponding to $\mathcal{R}_\mu$. Note that $\mathcal{J}^{(Z)}$ is a real linear multiplet, and hence $\mathcal{R}_\mu$ is also annihilated by D^2 and $\overline{D}^2$. In components,

$$\begin{aligned}
\mathcal{R}_\mu &= j^{(R)}_\mu - i\theta S_\mu - i\overline{\theta}\overline{S}_\mu - (\theta\gamma^\nu\overline{\theta})\big(2T_{\mu\nu} + i\varepsilon_{\mu\nu\rho}\partial^\rho J^{(Z)}\big) \\
&\quad - i\theta\overline{\theta}\big(2j^{(Z)}_\mu + i\varepsilon_{\mu\nu\rho}\partial^\nu j^{(R)\rho}\big) + \cdots \ , \\
\mathcal{J}^{(Z)} &= J^{(Z)} - \frac{1}{2}\theta\gamma^\mu S_\mu + \frac{1}{2}\overline{\theta}\gamma^\mu\overline{S}_\mu + i\theta\overline{\theta}T^\mu_\mu - (\theta\gamma^\mu\overline{\theta})j^{(Z)}_\mu + \cdots \ ,
\end{aligned} \tag{4.4.7}$$

where the ellipses denote terms that are determined by the lower components as in (4.4.1). Here $j^{(R)}_\mu$ is the R-current, $S_{\alpha\mu}$ is the supersymmetry current, $T_{\mu\nu}$ is the energy-momentum tensor, and $j^{(Z)}_\mu$ is the current associated with the central charge in the supersymmetry algebra. The scalar $J^{(Z)}$ gives rise to a string current $i\varepsilon_{\mu\nu\rho}\partial^\rho J^{(Z)}$. All of these currents are conserved. Note that there are additional factors of i in (4.4.7) compared to the formulas in [1], because we are working in Euclidean signature. (In Lorentzian signature the superfield $\mathcal{R}_\mu$ is real.)

The $\mathcal{R}$-multiplet is not unique. It can be changed by an improvement transformation,

$$\begin{aligned}
\mathcal{R}'_{\alpha\beta} &= \mathcal{R}_{\alpha\beta} - \frac{t}{2}\left([D_\alpha, \overline{D}_\beta] + [D_\beta, \overline{D}_\alpha]\right)\mathcal{J} \ , \\
\mathcal{J}'^{(Z)} &= \mathcal{J}^{(Z)} - \frac{it}{2}\overline{D}D\mathcal{J} \ ,
\end{aligned} \tag{4.4.8}$$

where $\mathcal{J}$ is a flavor current and t is a real parameter. In components,

$$\begin{aligned}
j'^{(R)}_\mu &= j^{(R)}_\mu + tj_\mu \ , \\
T'_{\mu\nu} &= T_{\mu\nu} - \frac{t}{2}(\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2)J \ , \\
J'^{(Z)} &= J^{(Z)} + tK \ , \\
j'^{(Z)}_\mu &= j^{(Z)}_\mu - it\varepsilon_{\mu\nu\rho}\partial^\nu j^\rho \ .
\end{aligned} \tag{4.4.9}$$

Note that the R-current $j^{(R)}_\mu$ is shifted by the flavor current j_μ. If the theory is superconformal, it is possible to set $\mathcal{J}^{(Z)}$ to zero by an improvement transformation, so that $J^{(Z)}, T^\mu_\mu$, and $j^{(Z)}_\mu$ are redundant operators.

We first consider the two-point functions of operators in the flavor current multiplet $\mathcal{J}$ with operators in the $\mathcal{R}$-multiplet. They are parameterized by two dimensionless structure functions $\widehat{\tau}_{fr}$ and κ_{fr}, where the subscript fr emphasizes the fact that we are considering mixed flavor-R two-point functions:

$$\begin{aligned}
\langle j_\mu(p) j_\nu^{(R)}(-p)\rangle &= (p_\mu p_\nu - p^2\delta_{\mu\nu})\frac{\widehat{\tau}_{fr}}{8|p|} + \varepsilon_{\mu\nu\rho}p^\rho \frac{\kappa_{fr}}{2\pi} \ , \\
\langle j_\mu(p) j_\nu^{(Z)}(-p)\rangle &= (p_\mu p_\nu - p^2\delta_{\mu\nu})\frac{\kappa_{fr}}{2\pi} - \varepsilon_{\mu\nu\rho}p^\rho \frac{|p|\widehat{\tau}_{fr}}{8} \ , \\
\langle J(p) J^{(Z)}(-p)\rangle &= \frac{\kappa_{fr}}{2\pi} \ , \\
\langle K(p) J^{(Z)}(-p)\rangle &= -\frac{|p|\widehat{\tau}_{fr}}{8} \ , \\
\langle J(p) T_{\mu\nu}(-p)\rangle &= (p_\mu p_\nu - p^2\delta_{\mu\nu})\frac{\widehat{\tau}_{fr}}{16|p|} \ , \\
\langle K(p) T_{\mu\nu}(-p)\rangle &= (p_\mu p_\nu - p^2\delta_{\mu\nu})\frac{\kappa_{fr}}{4\pi} \ .
\end{aligned} \tag{4.4.10}$$

Under an improvement transformation (4.4.9), the structure functions shift as follows:

$$\begin{aligned}
\widehat{\tau}'_{fr} &= \widehat{\tau}_{fr} + t\,\widehat{\tau}_{ff} \ , \\
\kappa'_{fr} &= \kappa_{fr} + t\,\kappa_{ff} \ .
\end{aligned} \tag{4.4.11}$$

As explained above, in a superconformal theory there is a preferred $\mathcal{R}'_{\alpha\beta}$, whose corresponding $\mathcal{J}'^{(Z)}$ is a redundant operator. Typically, it differs from a natural choice $\mathcal{R}_{\alpha\beta}$ in the UV by an improvement transformation (4.4.8). In order to find the value of t that characterizes this improvement, we can use (4.4.10) and the fact that the operators in $\mathcal{J}'^{(Z)}$ are redundant to conclude that $\widehat{\tau}'_{fr}$ must vanish [135]. Alternatively, we can determine t by applying the F-maximization principle, which was conjectured in [29, 33] and proved in [4].

We will now discuss two-point functions of operators in the $\mathcal{R}$-multiplet. They are parameterized by four dimensionless structure functions $\widehat{\tau}_{rr}$, $\widehat{\tau}_{zz}$, κ_{rr}, and κ_{zz},

$$\begin{aligned}
\langle j_\mu^{(R)}(p) j_\nu^{(R)}(-p)\rangle &= (p_\mu p_\nu - p^2\delta_{\mu\nu})\frac{\widehat{\tau}_{rr}}{8|p|} + \varepsilon_{\mu\nu\rho}p^\rho \frac{\kappa_{rr}}{2\pi} \ , \\
\langle j_\mu^{(Z)}(p) j_\nu^{(Z)}(-p)\rangle &= (p_\mu p_\nu - p^2\delta_{\mu\nu})\frac{|p|\widehat{\tau}_{zz}}{8} + \varepsilon_{\mu\nu\rho}p^\rho p^2 \frac{\kappa_{zz}}{2\pi} \ , \\
\langle j_\mu^{(Z)}(p) j_\nu^{(R)}(-p)\rangle &= -(p_\mu p_\nu - p^2\delta_{\mu\nu})\frac{\kappa_{zz}}{2\pi} + \varepsilon_{\mu\nu\rho}p^\rho \frac{|p|\widehat{\tau}_{zz}}{8} \ , \\
\langle J^{(Z)}(p) J^{(Z)}(-p)\rangle &= \frac{|p|\widehat{\tau}_{zz}}{8} \ , \\
\langle J^{(Z)}(p) T_{\mu\nu}(-p)\rangle &= -\frac{\kappa_{zz}}{4\pi}(p_\mu p_\nu - p^2\delta_{\mu\nu}) \ .
\end{aligned} \tag{4.4.12}$$

The two-point function $\langle T_{\mu\nu}(p)T_{\rho\lambda}(-p)\rangle$ is given by (4.2.7) with

$$\tau_g = \frac{\widehat{\tau}_{rr} + 2\widehat{\tau}_{zz}}{32} \ , \qquad \tau'_g = -\frac{\widehat{\tau}_{rr} + \widehat{\tau}_{zz}}{32} \ , \qquad \kappa_g = 12\left(\kappa_{rr} + \kappa_{zz}\right) \ . \tag{4.4.13}$$

The subscripts rr and zz are associated with two-point functions of the currents $j_\mu^{(R)}$ and $j_\mu^{(Z)}$. Note that $\tau_g + \tau'_g = \frac{\widehat{\tau}_{zz}}{32}$, which is non-negative and vanishes in a superconformal theory. As before, an improvement transformation (4.4.8) shifts the structure functions,

$$\begin{aligned}
\widehat{\tau}'_{rr} &= \widehat{\tau}_{rr} + 2t\,\widehat{\tau}_{fr} + t^2\,\widehat{\tau}_{ff} \ , \\
\widehat{\tau}'_{zz} &= \widehat{\tau}_{zz} - 2t\,\widehat{\tau}_{fr} - t^2\,\widehat{\tau}_{ff} \ , \\
\kappa'_{rr} &= \kappa_{rr} + 2t\,\kappa_{fr} + t^2\,\kappa_{ff} \ , \\
\kappa'_{zz} &= \kappa_{zz} - 2t\,\kappa_{fr} - t^2\,\kappa_{ff} \ .
\end{aligned} \tag{4.4.14}$$

Note that τ'_g and κ_g in (4.4.13) are invariant under these shifts.

In a superconformal theory, the operators $J^{(Z)}$, T^μ_μ, and $j_\mu^{(Z)}$ are redundant. However, we see from (4.4.10) and (4.4.12) that they give rise to contact terms, which are parameterized by κ_{fr} and κ_{zz}. These contact terms violate conformal invariance. Unless κ_{fr} and κ_{zz} are properly quantized, they cannot be set to zero by a local counterterm without violating the quantization conditions for Chern-Simons counterterms explained in the introduction. This leads to a new anomaly, which will be discussed in section 4.5.

4.4.3 Background Supergravity Fields

In order to get a better understanding of the contact terms discussed in the previous subsection, we couple the $\mathcal{R}$-multiplet to background supergravity fields. (See appendix B for relevant aspects of $\mathcal{N} = 2$ supergravity.) To linear order, the $\mathcal{R}$-multiplet couples to the linearized metric superfield $\mathcal{H}_\mu$. In Wess-Zumino gauge,

$$\mathcal{H}_\mu = \frac{1}{2}\left(\theta\gamma^\nu\overline{\theta}\right)(h_{\mu\nu} - iB_{\mu\nu}) - \frac{1}{2}\theta\overline{\theta}C_\mu - \frac{i}{2}\theta^2\overline{\theta}\overline{\psi}_\mu + \frac{i}{2}\overline{\theta^2}\theta\psi_\mu + \frac{1}{2}\theta^2\overline{\theta^2}\left(A_\mu - V_\mu\right) \ . \tag{4.4.15}$$

Here $h_{\mu\nu}$ is the linearized metric, so that $g_{\mu\nu} = \delta_{\mu\nu} + 2h_{\mu\nu}$. The vectors C_μ and A_μ are Abelian gauge fields, and $B_{\mu\nu}$ is a two-form gauge field. It will be convenient to define the following field strengths,

$$
\begin{aligned}
&V_\mu = -\varepsilon_{\mu\nu\rho}\partial^\nu C^\rho \ , \qquad \partial^\mu V_\mu = 0 \ , \\
&H = \frac{1}{2}\varepsilon_{\mu\nu\rho}\partial^\mu B^{\nu\rho} \ .
\end{aligned}
\tag{4.4.16}
$$

Despite several unfamiliar factors of i in (4.4.15) that arise in Euclidean signature, the fields V_μ and H are naturally real. Below, we will encounter situations with imaginary H, see also [4,31].

If the theory is superconformal, we can reduce the $\mathcal{R}$-multiplet to a smaller supercurrent. Consequently, the linearized metric superfield $\mathcal{H}_\mu$ enjoys more gauge freedom, which allows us to set $B_{\mu\nu}$ and $A_\mu - \frac{1}{2}V_\mu$ to zero. The combination $A_\mu - \frac{3}{2}V_\mu$ remains and transforms like an Abelian gauge field.

Using $\mathcal{H}_\mu$, we can construct three Chern-Simons terms (see appendix B), which capture the contact terms described in the previous subsection. As we saw there, not all of them are conformally invariant.

- *Gravitational Chern-Simons Term:*

$$
\begin{aligned}
\mathscr{L}_g =& \frac{\kappa_g}{192\pi}\Big(i\varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\omega_\mu\partial_\nu\omega_\rho + \frac{2}{3}\omega_\mu\omega_\nu\omega_\rho\right) \\
&+ 4i\varepsilon^{\mu\nu\rho}\big(A_\mu - \frac{3}{2}V_\mu\big)\partial_\nu\big(A_\rho - \frac{3}{2}V_\rho\big) + (\text{fermions})\Big) \ .
\end{aligned}
\tag{4.4.17}
$$

  We see that the $\mathcal{N} = 2$ completion of the gravitational Chern-Simons term (4.1.9) also involves a Chern-Simons term for $A_\mu - \frac{3}{2}V_\mu$. Like the flavor-flavor term (4.4.5), the gravitational Chern-Simons term (4.4.17) is conformally invariant. It was previously studied in the context of conformal $\mathcal{N} = 2$ supergravity [136], see also [137,138].

- *Z-Z Chern-Simons Term:*

$$
\mathscr{L}_{zz} = -\frac{\kappa_{zz}}{4\pi}\left(i\varepsilon^{\mu\nu\rho}\big(A_\mu - \frac{1}{2}V_\mu\big)\partial_\nu\big(A_\rho - \frac{1}{2}V_\rho\big) + \frac{1}{2}HR + \cdots + (\text{fermions})\right) \ .
\tag{4.4.18}
$$

  Here the ellipsis denotes higher-order terms in the bosonic fields, which go beyond linearized supergravity. The presence of the Ricci scalar R and the fields H, $A_\mu - \frac{1}{2}V_\mu$ implies that (4.4.18) is not conformally invariant.

- *Flavor-R Chern-Simons Term:*

$$\mathscr{L}_{fr} = -\frac{\kappa_{fr}}{2\pi}\left(i\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu\big(A_\rho - \frac{1}{2}V_\rho\big) + \frac{1}{4}\sigma R - DH + \cdots + (\text{fermions})\right) . \tag{4.4.19}$$

  The meaning of the ellipsis is as in (4.4.18). Again, the presence of R, H, and $A_\mu - \frac{1}{2}V_\mu$ shows that this term is not conformally invariant. The relative sign between the Chern-Simons terms (4.4.5) and (4.4.19) is due to the different couplings of flavor and R-currents to their respective background gauge fields.

Note that both (4.4.17) and (4.4.18) give rise to a Chern-Simons term for A_μ. Its overall coefficient is $\kappa_{rr} = \frac{\kappa_g}{12} - \kappa_{zz}$, in accord with (4.4.13).

It is straightforward to adapt the discussion of section 4.2 to these Chern-Simons terms. Their coefficients can be modified by shifting the Lagrangian by appropriate counterterms, whose coefficients are quantized according to the periodicity of the global symmetries. Instead of stating the precise quantization conditions, we will abuse the language and say that the fractional parts of these coefficients are physical, while their integer parts are scheme dependent.

4.5 A New Anomaly

In the previous section, we have discussed four Chern-Simons terms in the background fields: the flavor-flavor term (4.4.5), the gravitational term (4.4.17), the Z-Z term (4.4.18), and the flavor-R term (4.4.19). They correspond to certain contact terms in two-point functions of operators in the flavor current $\mathcal{J}$ and the $\mathcal{R}$-multiplet. As we saw above, the flavor-flavor and the gravitational Chern-Simons terms are superconformal, while the Z-Z term and the flavor-R term are not. The latter give rise to non-conformal contact terms proportional to κ_{zz} and κ_{fr}.

The integer parts of κ_{zz} and κ_{fr} can be changed by adding appropriate Chern-Simons counterterms, but the fractional parts are physical and cannot be removed. This leads to an interesting puzzle: if κ_{zz} or κ_{fr} have non-vanishing fractional parts in a superconformal theory, they give rise to non-conformal contact terms. This is similar to the conformal anomaly in two dimensions, where the redundant operator T^μ_μ has nonzero contact terms. However, in two-dimensions the non-conformal contact terms arise from correlation functions of the conserved energy-momentum tensor at separated points, and hence they cannot be removed by a local counterterm. In our case, the anomaly is a bit more subtle.

An anomaly arises whenever we are unable to impose several physical requirements at the same time. Although the anomaly implies that we must sacrifice one of these requirements, we can often choose which one to give up. In our situation we would like to impose supersymmetry, conformal invariance, and compactness of the global symmetries, including the R-symmetry. Moreover, we would like to couple the global symmetries to arbitrary background gauge fields in a fully gauge-invariant way. As we saw above, this implies that the corresponding Chern-Simons counterterms must have integer coefficients.[8] If the fractional part of κ_{zz} or κ_{fr} is nonzero, we cannot satisfy all of these requirements, and hence there is an anomaly. In this case we have the following options:

1.) *We can sacrifice supersymmetry.* Then we can shift the Lagrangian by non-supersymmetric counterterms that remove the non-conformal terms in (4.4.18) and (4.4.19) and restore conformal invariance. Note that these counterterms are gauge invariant.

2.) *We can sacrifice conformal invariance.* Then there is no need to add any counterterm. The correlation functions at separated points are superconformal, while the contact terms are supersymmetric but not conformal.

3.) *We can sacrifice invariance under large gauge transformations.* Now we can

[8]Here we will abuse the language and attribute the quantization of these coefficients to invariance under large gauge transformations. As we reviewed in the introduction, a more careful construction requires a choice of auxiliary four-manifold. The quantization follows by demanding that our answers do not depend on that choice.

shift the Lagrangian by supersymmetric Chern-Simons counterterms with fractional coefficients to restore conformal invariance. These counterterms are not invariant under large gauge transformations, if the background gauge fields are topologically non-trivial.

The third option is the most conservative, since we retain both supersymmetry and conformal invariance. If the background gauge fields are topologically non-trivial, the partition function is multiplied by a phase under large background gauge transformations. In order to obtain a well-defined answer, we need to specify additional geometric data.[9] By measuring the change in the phase of the partition function as we vary this data, we can extract the fractional parts of κ_{zz} and κ_{fr}. Therefore, these observables are not lost, even if we set the corresponding contact terms to zero by a counterterm.

This discussion is similar to the framing anomaly of [123]. There, a Lorentz Chern-Simons term for the frame bundle is added with fractional coefficient, in order to make the theory topologically invariant. This introduces a dependence on the trivialization of the frame bundle. In our case the requirement of topological invariance is replaced with superconformal invariance and we sacrifice invariance under large gauge transformations rather than invariance under a change of framing.

Finally, we would like to point out that the anomaly described above has important consequences if the theory is placed on a curved manifold [4]. For some configurations of the background fields, the partition function is not consistent with conformal invariance and even unitarity.

[9]More precisely, the phase of the partition function depends on the choice of auxiliary four-manifold, which is the additional data needed to obtain a well-defined answer.

4.6 A Perturbative Example: SQED with a Chern-Simons Term

Consider $\mathcal{N} = 2$ SQED with a level k Chern-Simons term for the dynamical $U(1)_v$ gauge field and N_f flavor pairs $Q_i, \widetilde{Q}_{\tilde{i}}$ that carry charge ± 1 under $U(1)_v$. The theory also has a global $U(1)_a$ flavor symmetry under which $Q_i, \widetilde{Q}_{\tilde{i}}$ all carry charge $+1$. Here v and a stand for 'vector' and 'axial' respectively. The Euclidean flat-space Lagrangian takes the form

$$\mathscr{L} = -\int d^4\theta \left(\overline{Q}_i e^{2\widehat{\mathcal{V}}_v} Q_i + \overline{\widetilde{Q}}_{\tilde{i}} e^{-2\widehat{\mathcal{V}}_v} \widetilde{Q}_{\tilde{i}} - \frac{1}{e^2}\widehat{\Sigma}_v^2 + \frac{k}{2\pi}\widehat{\mathcal{V}}_v \widehat{\Sigma}_v \right) , \tag{4.6.1}$$

where e is the gauge coupling and $\widehat{\mathcal{V}}_v$ denotes the dynamical $U(1)_v$ gauge field. (The hat emphasizes the fact that it is dynamical.) Note that the theory is invariant under charge conjugation, which maps $\widehat{\mathcal{V}}_v \to -\widehat{\mathcal{V}}_v$ and $Q_i \leftrightarrow \widetilde{Q}_{\tilde{i}}$. This symmetry prevents mixing of the axial current with the topological current, so that some of the subtleties discussed in section 4.3 are absent in this theory.

The Chern-Simons term leads to a mass for the dynamical gauge multiplet,

$$M = \frac{ke^2}{2\pi} . \tag{4.6.2}$$

This mass is the crossover scale from the free UV theory to a non-trivial CFT labeled by k and N_f in the IR. We will analyze this theory in perturbation theory for $k \gg 1$. In particular, we will study the contact terms of the axial current,

$$\mathcal{J} = |Q_i|^2 + |\widetilde{Q}_{\tilde{i}}|^2 , \tag{4.6.3}$$

and the $\mathcal{R}$-multiplet,

$$\begin{aligned} \mathcal{R}_{\alpha\beta} &= \frac{2}{e^2}\left(D_\alpha \widehat{\Sigma}_v \overline{D}_\beta \widehat{\Sigma}_v + D_\beta \widehat{\Sigma}_v \overline{D}_\alpha \widehat{\Sigma}_v\right) + \mathcal{R}^m_{\alpha\beta} , \\ \mathcal{J}^{(Z)} &= \frac{i}{4e^2}\overline{D}D\left(\widehat{\Sigma}_v^2\right) . \end{aligned} \tag{4.6.4}$$

Here $\mathcal{R}^m_{\alpha\beta}$ is associated with the matter fields and assigns canonical dimensions to $Q_i, \widetilde{Q}_{\tilde{i}}$. In the IR, the $\mathcal{R}$-multiplet flows to a superconformal multiplet, up to an

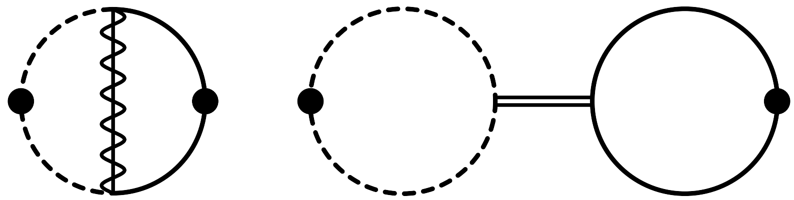

Figure 4.1: Feynman diagrams for flavor-flavor.

improvement by the axial current $\mathscr{J}$. Therefore, at long distances $\mathscr{J}^{(Z)}$ is proportional to $i\overline{D}D\mathscr{J}$.

We begin by computing the flavor-flavor contact term $\kappa_{ff,\mathrm{CFT}}$ in the two-point function of the axial current (4.6.3), by flowing from the free UV theory to the CFT in the IR. Using (4.4.2), we see that it suffices to compute the correlation function $\langle J(p)K(-p)\rangle$ at small momentum $p^2 \to 0$. In a conformal field theory, the correlator $\langle J(x)K(0)\rangle$ vanishes at separated points, and hence we must obtain a pure contact term. More explicitly, we have

$$J = |q_i|^2 + |\widetilde{q}_{\widetilde{i}}|^2 \; , \qquad K = -i\overline{\psi}_i\psi_i - i\overline{\widetilde{\psi}}_{\widetilde{i}}\widetilde{\psi}_{\widetilde{i}} \; . \tag{4.6.5}$$

There are two diagrams at leading order in $\frac{1}{k}$, displayed in figure 4.1.[10] The first diagram, with the intermediate gaugino, is paired with a seagull diagram, which ensures that we obtain a pure contact term. The second diagram vanishes by charge conjugation. Evaluating these diagrams, we find

$$\lim_{p^2\to 0} \langle J(p)K(-p)\rangle = \frac{\pi N_f}{8k} + \mathcal{O}\left(\frac{1}{k^3}\right) \; , \tag{4.6.6}$$

and hence

$$\kappa_{ff,\mathrm{CFT}} = \frac{\pi^2 N_f}{4k} + \mathcal{O}\left(\frac{1}{k^3}\right) \; . \tag{4.6.7}$$

[10]The solid dots denote the appropriate operator insertions. The dashed and solid lines represent scalar and fermion matter. The double line denotes the scalar and the auxiliary field in the vector multiplet, while the zigzag line represents the gaugino.

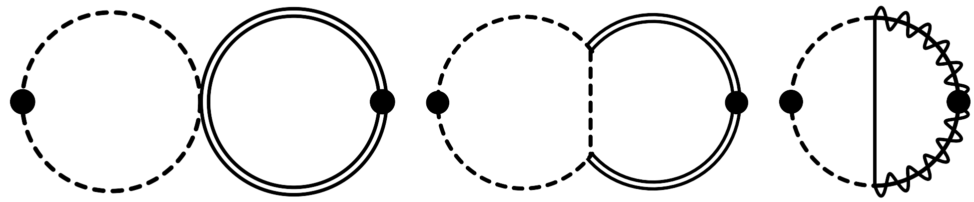

Figure 4.2: Feynman diagrams for flavor-gravity.

We similarly compute the flavor-R contact term $\kappa_{fr,\mathrm{CFT}}$ by flowing into the CFT from the free UV theory. It follows from (4.4.10) that it can be determined by computing the two-point function $\langle J(p)J^{(Z)}(-p)\rangle$ at small momentum $p^2 \to 0$. Using (4.6.4), we find

$$J^{(Z)} = -\frac{1}{e^2}\left(\widehat{\sigma}_v \widehat{D}_v - \frac{i}{2}\overline{\widehat{\lambda}}_v \widehat{\lambda}_v\right) . \tag{4.6.8}$$

Since $\mathcal{J}^{(Z)}$ is proportional to $i\overline{D}D\mathcal{J}$ at low energies, the operator $J^{(Z)}$ flows to an operator proportional to K. The coefficient is determined by the mixing of the R-symmetry with the axial current $\mathcal{J}$, which occurs at order $\frac{1}{k^2}$. Since $\langle J(x)K(0)\rangle$ vanishes at separated points, the two-point function of J and $J^{(Z)}$ must be a pure contact term. Unlike the flavor-flavor case, several diagrams contribute to this correlator at order $\frac{1}{k}$ (see figure 4.2). Each diagram gives rise to a term proportional to $\frac{1}{|p|}$. However, these contributions cancel, and we find a pure contact term,

$$\lim_{p^2\to 0}\langle J(p)J^{(Z)}(-p)\rangle = -\frac{N_f}{4\pi k} + \mathcal{O}\left(\frac{1}{k^3}\right) , \tag{4.6.9}$$

so that

$$\kappa_{fr,\mathrm{CFT}} = -\frac{N_f}{2k} + \mathcal{O}\left(\frac{1}{k^3}\right) . \tag{4.6.10}$$

Since this value is fractional, it implies the presence of the anomaly discussed in the previous section.

We have computed $\kappa_{ff,\mathrm{CFT}}$ and $\kappa_{fr,\mathrm{CFT}}$ by flowing into the CFT from the free UV theory. It is instructive to follow the discussion in subsection 4.3.4 and further deform the theory by a real mass $m \ll M$. In order to preserve charge conjugation, we assign the same real mass m to all flavors $Q_i, \widetilde{Q}_{\tilde{i}}$. This deformation leads to a gap in the IR. Even though a topological theory with Lagrangian proportional to $i\varepsilon^{\mu\nu\rho}\widehat{v}_\mu \partial_\nu \widehat{v}_\rho$ can remain, it does not mix with $\mathcal{J}$ or $\mathcal{R}_{\alpha\beta}$ because of charge conjugation. Therefore, the contact terms κ_{ff} and κ_{fr} must be properly quantized in the IR. (Since the matter fields in this example have half-integer R-charges, this means that κ_{fr} should be a half-integer.)

For the axial current, we have

$$\widehat{\tau}_{ff} \approx \begin{cases} 2N_f & p^2 \gg M^2 \\ \widehat{\tau}_{ff,\text{CFT}} = 2N_f - \mathcal{O}\left(\frac{1}{k^2}\right) & m^2 \ll p^2 \ll M^2 \\ 0 & p^2 \ll m^2 \end{cases} \tag{4.6.11}$$

The fact that $\widehat{\tau}_{ff} = 0$ in the IR follows from the fact that the theory is gapped. Similarly,

$$\kappa_{ff} \approx \begin{cases} 0 & p^2 \gg M^2 \\ \kappa_{ff,\text{CFT}} = \frac{\pi^2 N_f}{4k} + \mathcal{O}\left(\frac{1}{k^3}\right) & m^2 \ll p^2 \ll M^2 \\ -N_f \,\text{sign}(m) & p^2 \ll m^2 \end{cases} \tag{4.6.12}$$

Note that parity, which acts as $k \to -k$, $m \to -m$, $\kappa_{ff} \to -\kappa_{ff}$, with $\widehat{\tau}_{ff}$ invariant, is a symmetry of (4.6.11) and (4.6.12).

For the two-point function of the axial current and the $\mathcal{R}$-multiplet, we find

$$\widehat{\tau}_{fr} \approx \begin{cases} 0 & p^2 \gg M^2 \\ \widehat{\tau}_{fr,\text{CFT}} = \mathcal{O}\left(\frac{1}{k^2}\right) & m^2 \ll p^2 \ll M^2 \\ 0 & p^2 \ll m^2 \end{cases} \tag{4.6.13}$$

Here $\widehat{\tau}_{fr,\text{CFT}}$ measures the mixing of the axial current with the UV $\mathcal{R}$-multiplet (4.6.4). For the superconformal $\mathcal{R}$-multiplet of the CFT, we would have obtained $\widehat{\tau}_{fr,\text{CFT}} = 0$, as explained after (4.4.11). Similarly,

$$\kappa_{fr} \approx \begin{cases} 0 & p^2 \gg M^2 \\ \kappa_{fr,\text{CFT}} = -\frac{N_f}{2k} + \mathcal{O}\left(\frac{1}{k^3}\right) & m^2 \ll p^2 \ll M^2 \\ \frac{N_f}{2} \,\text{sign}(m) & p^2 \ll m^2 \end{cases} \tag{4.6.14}$$

As before, (4.6.13) and (4.6.14) transform appropriately under parity.

Let us examine the flow from the CFT to the IR in more detail, taking the UV crossover scale $M \to \infty$. In the CFT, the operator $J^{(Z)}$ is redundant, up to $\mathcal{O}\left(\frac{1}{k^2}\right)$ corrections due to the mixing with the axial current. Once the CFT is deformed by the real mass m, we find that

$$J^{(Z)} = mJ + \mathcal{O}\left(\frac{1}{e^2}, \frac{1}{k^2}\right) , \tag{4.6.15}$$

where J is the bottom component of the axial current (4.6.3), which is given by (4.6.5). (As always, the operator equation (4.6.15) holds at separated points.) Substituting into (4.4.10), we find that

$$\frac{\kappa_{fr}}{2\pi} = \frac{\kappa_{fr,\mathrm{CFT}}}{2\pi} + m\langle J(p)J(-p)\rangle + \mathcal{O}\left(\frac{1}{e^2}, \frac{1}{k^2}\right) = \frac{\kappa_{fr,\mathrm{CFT}}}{2\pi} + \frac{m}{8|p|}\widehat{\tau}_{ff} + \mathcal{O}\left(\frac{1}{e^2}, \frac{1}{k^2}\right) . \tag{4.6.16}$$

Here it is important that the two-point function of J does not have a contact term in the CFT. Explicitly computing $\widehat{\tau}_{ff}$, we find that

$$\widehat{\tau}_{ff} = \begin{cases} 2N_f - \mathcal{O}\left(\frac{1}{k^2}\right) & p^2 \gg m^2 \\ \frac{|p|}{|m|}\frac{2N_f}{\pi}\left(1 + \frac{1}{k}\,\mathrm{sign}(m)\right) + \mathcal{O}\left(\frac{1}{k^2}\right) & p^2 \ll m^2 \end{cases} \tag{4.6.17}$$

This is consistent with (4.6.14) and (4.6.16).

4.7 Checks of Dualities

In this section we examine dual pairs of three-dimensional $\mathcal{N} = 2$ theories, which are conjectured to flow to the same IR fixed point. In this case, the various contact terms discussed above, computed on either side of the duality, should match.

First, as in [139–143], the three-sphere partition functions of the two theories should match, up to the contribution of Chern-Simons counterterms in the background fields. Denote their coefficients by $\delta\kappa$.

Second, as in the parity anomaly matching condition discussed in [134], the fractional parts of these contact terms are intrinsic to the theories. Therefore, the Chern-

Simons counterterms that are needed for the duality must be properly quantized. This provides a new non-trivial test of the duality.

Finally, these counterterms can often be determined independently. Whenever different pairs of dual theories are related by renormalization group flows, the counterterms for these pairs are similarly related. In particular, given the properly quantized Chern-Simons counterterms that are needed for one dual pair, we can determine them for other related pairs by a one-loop computation in flat space. This constitutes an additional check of the duality.

In this section we demonstrate this matching for $\mathcal{N} = 2$ supersymmetric level-rank duality and Giveon-Kutasov duality [144]. We compute some of the relative Chern-Simons counterterms, both in flat space and using the three-sphere partition function, and verify that they are properly quantized.

4.7.1 Level-Rank Duality

Consider an $\mathcal{N} = 2$ supersymmetric $U(n)$ gauge theory with a level k Chern-Simons term. We will call this the 'electric' theory and denote it by $U(n)_k$. In terms of the $SU(n)$ and $U(1)$ subgroups, this theory is equivalent to $(SU(n)_k \times U(1)_{nk})/\mathbb{Z}_n$, where we have used the conventional normalization for Abelian gauge fields. This theory flows to a purely topological $U(n)$ Chern-Simons theory with shifted levels, denoted by $U(n)^{top}_{\mathrm{sign}(k)(|k|-n),\, kn}$. The first subscript specifies the level of the $SU(n)$ subgroup, which is shifted by integrating out the charged, massive gauginos (recall that their mass has the same sign as the level k), and the second subscript denotes the level of the $U(1)$ subgroup, which is not shifted.

The dual 'magnetic' theory is a supersymmetric $U(|k| - n)_{-k}$ Yang-Mills Chern-Simons theory. It flows to the purely topological theory $U(|k| - n)^{top}_{-\mathrm{sign}(k)n, -k(|k|-n)}$. This theory is related to the other topological theory described above by conventional level-rank duality for unitary gauge groups [145].[11]

[11]The authors of [145] restricted n to be odd and k to be even. This restriction is unnecessary on spin manifolds. Furthermore, we reversed the orientation on the

These theories have two Abelian symmetries: a $U(1)_R$ symmetry under which all gauginos have charge $+1$, and a topological symmetry $U(1)_J$. The topological symmetry corresponds to the current $j_\mu = \frac{i}{2\pi}\varepsilon_{\mu\nu\rho}\,\mathrm{Tr}\,F^{\nu\rho}$ on the electric side, and to $j_\mu = -\frac{i}{2\pi}\varepsilon_{\mu\nu\rho}\,\mathrm{Tr}\,F^{\nu\rho}$ on the magnetic side.

We can integrate out the gauginos to obtain the contact term κ_{rr} in the two-point function (4.4.12) of the R-current. On the electric side, we find $\kappa_{rr,e} = -\frac{1}{2}\,\mathrm{sign}(k)n^2$, and on the magnetic side we have $\kappa_{rr,m} = \frac{1}{2}\,\mathrm{sign}(k)(|k|-n)^2$. We must therefore add a counterterm

$$\delta\kappa_{rr} = -\frac{1}{2}\,\mathrm{sign}(k)\left((|k|-n)^2 + n^2\right) \ , \tag{4.7.1}$$

to the magnetic theory. Taking into account possible half-integer counterterms that must be added on either side of the duality because of the parity anomaly, what remains of the relative counterterm (4.7.1) is always an integer.

In order to compute the contact term associated with $U(1)_J$, we follow the discussion in subsection 4.3.2 and integrate out the dynamical gauge fields to find the effective theory for the corresponding background gauge field. In the electric theory, this leads to $\kappa_{JJ,e} = -\frac{n}{k}$, and in the magnetic theory we find $\kappa_{JJ,m} = \frac{|k|-n}{k}$. Hence we need to add an integer Chern-Simons counterterm to the magnetic theory,

$$\delta\kappa_{JJ} = -\,\mathrm{sign}(k) \ . \tag{4.7.2}$$

4.7.2 Giveon-Kutasov Duality

Consider the duality of Giveon and Kutasov [144]. The electric theory consists of a $U(n)_k$ Chern-Simons theory with N_f pairs $Q_i, \widetilde{Q}_{\widetilde{i}}$ of quarks in the fundamental and the anti-fundamental representation of $U(n)$. The global symmetry group is $SU(N_f)\times SU(N_f)\times U(1)_A\times U(1)_J\times U(1)_R$. The quantum numbers of the fundamental fields are given by

The magnetic dual is given by a $U(\widetilde{n} = N_f + |k| - n)_{-k}$ Chern-Simons theory. It contains N_f pairs $q^i, \widetilde{q}^{\widetilde{i}}$ of dual quarks and N_f^2 singlets $M_i^{\widetilde{i}}$, which interact through a

magnetic side.

Fields	$U(n)_k$	$SU(N_f)$	$SU(N_f)$	$U(1)_A$	$U(1)_J$	$U(1)_R$
Q	$\square$	$\square$	$\mathbf{1}$	1	0	$\frac{1}{2}$
$\widetilde{Q}$	$\square$	$\mathbf{1}$	$\overline{\square}$	1	0	$\frac{1}{2}$

Table 4.1: Quantum numbers of the electric fundamental fields

superpotential $W = q^i M_i^{\tilde{i}} \widetilde{q}_{\tilde{i}}$. The quantum numbers in the magnetic theory are given by

Fields	$U(\widetilde{n})_{-k}$	$SU(N_f)$	$SU(N_f)$	$U(1)_A$	$U(1)_J$	$U(1)_R$
q	$\square$	$\overline{\square}$	$\mathbf{1}$	-1	0	$\frac{1}{2}$
$\widetilde{q}$	$\overline{\square}$	$\mathbf{1}$	$\square$	-1	0	$\frac{1}{2}$
M	$\mathbf{1}$	$\square$	$\overline{\square}$	2	0	1

Table 4.2: Quantum numbers of the magnetic fundamental fields

As before, the topological symmetry $U(1)_J$ corresponds to $j_\mu = \frac{i}{2\pi}\varepsilon_{\mu\nu\rho} \operatorname{Tr} F^{\nu\rho}$ on the electric side, and to $j_\mu = -\frac{i}{2\pi}\varepsilon_{\mu\nu\rho} \operatorname{Tr} F^{\nu\rho}$ on the magnetic side. Note that none of the fundamental fields are charged under $U(1)_J$.

This duality requires the following Chern-Simons counterterms for the Abelian symmetries, which must be added to the magnetic theory:[12]

$$\begin{aligned}
&\delta\kappa_{AA} = -\operatorname{sign}(k) N_f (N_f - |k|) \ , \\
&\delta\kappa_{JJ} = -\operatorname{sign}(k) \ , \\
&\delta\kappa_{Ar} = \frac{1}{2}\operatorname{sign}(k) N_f (N_f + |k| - 2n) \ , \\
&\delta\kappa_{rr} = -\frac{1}{4}\operatorname{sign}(k)\left(2k^2 - 4|k|n + 3|k|N_f + 4n^2 - 4nN_f + N_f^2\right) \ .
\end{aligned} \tag{4.7.3}$$

This was derived in [143] by flowing into Giveon-Kutasov duality from Aharony duality [146] via a real mass deformation.[13] Note that these Chern-Simons counterterms

[12]Similar counterterms are required for the $SU(N_f) \times SU(N_f)$ flavor symmetry [140–143].

[13]The R-symmetry used in [143] assigns R-charge 0 to the electric quarks $Q_i, \widetilde{Q}_{\tilde{i}}$. Therefore, our results for $\delta\kappa_{Ar}$ and $\delta\kappa_{rr}$ differ from those of [143] by improve-

are properly quantized: $\delta\kappa_{AA}$ and $\delta\kappa_{JJ}$ are integers, while $\delta\kappa_{Ar}$ is half-integer and $\delta\kappa_{rr}$ is quantized in units of $\frac{1}{4}$. This is due to the presence of fields with R-charge $\frac{1}{2}$.

We can also understand (4.7.3) by flowing out of Giveon-Kutasov duality to a pair of purely topological theories. If we give a real mass to all electric quarks, with its sign opposite to that of the Chern-Simons level k, we flow to a $U(n)_{k+\text{sign}(k)N_f}$ theory without matter. The corresponding deformation of the magnetic theory flows to $U(|k|-n)_{-(k+\text{sign}(k)N_f)}$. Level-rank duality between these two theories without matter was discussed in the previous subsection. Given the counterterms (4.7.1) and (4.7.2) that are needed for this duality and accounting for the Chern-Simons terms generated by the mass deformation, we reproduce (4.7.3).

4.7.3 Matching the Three-Sphere Partition Function

As explained in [4], we can read off the contact terms κ_{ff} and κ_{fr} from the dependence of the free energy F_{S^3} on a unit three-sphere on the real mass parameter m associated with the flavor symmetry:

$$\kappa_{ff} = -\frac{1}{2\pi}\frac{\partial^2}{\partial m^2}\operatorname{Im} F_{S^3}\bigg|_{m=0}, \qquad \kappa_{fr} = \frac{1}{2\pi}\frac{\partial}{\partial m}\operatorname{Re} F_{S^3}\bigg|_{m=0}. \tag{4.7.4}$$

We can use this to rederive some of the relative Chern-Simons counterterms in (4.7.3). Let us denote by m and ξ the real mass parameters corresponding to $U(1)_A$ and $U(1)_J$. (Equivalently, ξ is a Fayet-Iliopoulos term for the dynamical gauge fields.) Using the results of [147], it was shown in [142] that the difference between the three-sphere partition functions of the electric and the magnetic theories requires a counterterm

$$\delta F_{S^3} = \text{sign}(k)\left(\pi i N_f(N_f-|k|)m^2 + \pi i\xi^2 + \pi N_f(N_f+|k|-2n)m\right) + \cdots . \tag{4.7.5}$$

where the ellipsis represents terms that are independent of m and ξ. (Our conventions for the Chern-Simons level k differ from those of [142] by a sign.) An analogous result was obtained in [143] for a different choice of R-symmetry. Using (4.7.4), we find the

ments (4.4.11) and (4.4.14).

same values for $\delta\kappa_{AA}$, $\delta\kappa_{JJ}$, and $\delta\kappa_{Ar}$ as in (4.7.3). Note that the counterterm (4.7.5) does not just affect the phase of the partition function, because the term linear in m is real.

Many other dualities have been shown to require relative Chern-Simons counterterms [139–143, 148, 149]. It would be interesting to repeat the preceding analysis in these examples.

4.8 Appendix A: Free Massive Theories

Consider a complex scalar field ϕ of mass m,

$$\mathscr{L} = |\partial_\mu \phi|^2 + m^2 |\phi|^2 \ . \tag{4.8.1}$$

This theory is invariant under parity and has a $U(1)$ flavor symmetry under which ϕ has charge $+1$. The corresponding current is given by

$$j_\mu = i\left(\overline{\phi}\partial_\mu \phi - \phi \partial_\mu \overline{\phi}\right) \ . \tag{4.8.2}$$

In momentum space, the two-point function of j_μ is given by (4.2.2) with

$$\begin{aligned} &\tau\left(\frac{p^2}{m^2}\right) = \frac{2}{\pi}\left[\left(1+\frac{4m^2}{p^2}\right)\operatorname{arccot}\left(\frac{2|m|}{|p|}\right) - \frac{2|m|}{|p|}\right] \ , \\ &\kappa = 0 \ . \end{aligned} \tag{4.8.3}$$

The fact that $\kappa = 0$ follows from parity. The function $\tau(p^2/m^2)$ interpolates between $\tau = 1$ in the UV and the empty theory with $\tau = 0$ in the IR,

$$\tau\left(\frac{p^2}{m^2}\right) = \begin{cases} 1 + \mathcal{O}\left(\frac{|m|}{|p|}\right) & p^2 \gg m^2 \\ \frac{2|p|}{3\pi|m|} + \mathcal{O}\left(\frac{|p|^3}{|m|^3}\right) & p^2 \ll m^2 \end{cases} \tag{4.8.4}$$

Now consider a Dirac fermion ψ with real mass m,

$$\mathscr{L} = -i\overline{\psi}\gamma^\mu \partial_\mu \psi + im\overline{\psi}\psi \ . \tag{4.8.5}$$

The mass term explicitly breaks parity. The $U(1)$ flavor symmetry that assigns charge $+1$ to ψ gives rise to the current

$$j_\mu = -\overline{\psi}\gamma_\mu \psi \ , \tag{4.8.6}$$

whose two-point function is given by (4.2.2) with

$$\begin{aligned}
\tau\left(\frac{p^2}{m^2}\right) &= \frac{2}{\pi}\left[\left(1-\frac{4m^2}{p^2}\right)\text{arccot}\left(\frac{2|m|}{|p|}\right)+\frac{2|m|}{|p|}\right] , \\
\kappa\left(\frac{p^2}{m^2}\right) &= -\frac{m}{|p|}\,\text{arccot}\left(\frac{2|m|}{|p|}\right) .
\end{aligned} \tag{4.8.7}$$

Note that $m\to -m$ under parity, so that τ is invariant and $\kappa\to-\kappa$. Again, the function $\tau\left(p^2/m^2\right)$ interpolates between $\tau=1$ in the UV and $\tau=0$ in the IR,

$$\tau\left(\frac{p^2}{m^2}\right) = \begin{cases} 1+\mathcal{O}\left(\frac{m^2}{p^2}\right) & p^2\gg m^2 \\ \frac{4|p|}{3\pi|m|}+\mathcal{O}\left(\frac{|p|^3}{|m|^3}\right) & p^2\ll m^2 \end{cases} \tag{4.8.8}$$

The function $\kappa\left(p^2/m^2\right)$ interpolates from $\kappa=0$ in the UV, where the theory is massless and parity invariant, to $\kappa=-\frac{1}{2}\,\text{sign}(m)$ in the empty IR theory,

$$\kappa\left(\frac{p^2}{m^2}\right) = \text{sign}(m)\begin{cases} -\frac{\pi|m|}{2|p|}+\mathcal{O}\left(\frac{m^2}{p^2}\right) & p^2\gg m^2 \\ -\frac{1}{2}+\mathcal{O}\left(\frac{p^2}{m^2}\right) & p^2\ll m^2 \end{cases} \tag{4.8.9}$$

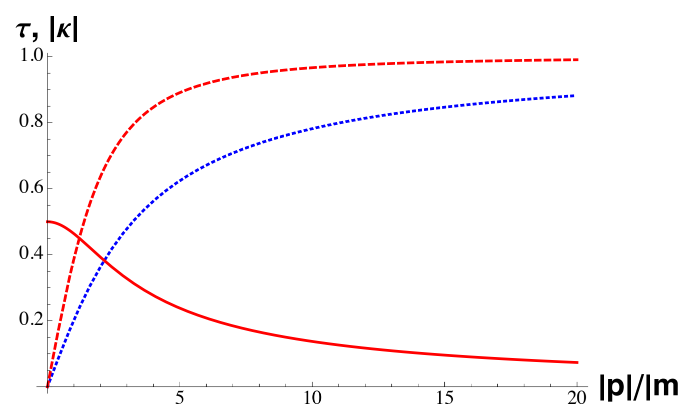

Figure 4.3: Functions $\tau, |\kappa|$ for the free scalar and fermion.

The function $\tau\left(p^2/m^2\right)$ in (4.8.3) for a free scalar (blue, dotted) and the functions $\tau\left(p^2/m^2\right)$ and $\kappa\left(p^2/m^2\right)$ in (4.8.7) for a free fermion (red, dashed and solid) are shown in figure 4.3. At the scale $p^2\approx m^2$ these functions display a rapid crossover from the UV to the IR.

In theories with $\mathcal{N}=2$ supersymmetry, we can consider a single chiral superfield Φ with real mass m. This theory has a global $U(1)$ flavor symmetry, and the associated conserved current j_μ, which resides in the real linear multiplet $\mathcal{J}=\overline{\Phi}e^{2im\theta\overline{\theta}}\Phi$, is the sum of the currents in (4.8.2) and (4.8.6). Therefore, the function $\tau\left(\frac{p^2}{m^2}\right)$ is the sum of the corresponding functions in (4.8.3) and (4.8.7). Since $\kappa\left(\frac{p^2}{m^2}\right)$ only receives contributions from the fermion ψ, it is the same as in (4.8.7). From (4.8.4) and (4.8.8), we see that the total $\tau\left(\frac{p^2}{m^2}\right)\approx 2$ when $p^2\gg m^2$. In supersymmetric theories it is thus convenient to define $\widehat{\tau}=\frac{\tau}{2}$, so that that $\widehat{\tau}\left(\frac{p^2}{m^2}\right)\approx 1$ when $p^2\gg m^2$ for a chiral superfield of charge $+1$ and real mass m.

4.9 Appendix B: Supergravity in Three Dimensions

In this appendix we review some facts about three-dimensional $\mathcal{N}=2$ supergravity, focusing on the supergravity theory associated with the $\mathcal{R}$-multiplet. It closely resembles $\mathcal{N}=1$ new minimal supergravity in four dimensions [95]. For a recent discussion, see [150, 151].

4.9.1 Linearized Supergravity

We can construct a linearized supergravity theory by coupling the $\mathcal{R}$-multiplet to the metric superfield $\mathcal{H}_\mu$,

$$\delta\mathscr{L}=-2\int d^4\theta\,\mathcal{R}_\mu\mathcal{H}^\mu\ . \tag{4.9.1}$$

The supergravity gauge transformations are embedded in a superfield L_α,

$$\delta\mathcal{H}_{\alpha\beta}=\frac{1}{2}\left(D_\alpha\overline{L}_\beta-\overline{D}_\beta L_\alpha\right)+(\alpha\leftrightarrow\beta)\ . \tag{4.9.2}$$

Demanding gauge invariance of (4.9.1) leads to the following constraints:

$$D^\alpha\overline{D}^2L_\alpha+\overline{D}^\alpha D^2\overline{L}_\alpha=0\ . \tag{4.9.3}$$

In Wess-Zumino gauge, the metric superfield takes the form[14]

$$\mathcal{H}_\mu = \frac{1}{2}\left(\theta\gamma^\nu\overline{\theta}\right)\left(h_{\mu\nu} - iB_{\mu\nu}\right) - \frac{1}{2}\theta\overline{\theta}C_\mu - \frac{i}{2}\theta^2\overline{\theta}\overline{\psi}_\mu + \frac{i}{2}\overline{\theta^2}\theta\psi_\mu + \frac{1}{2}\theta^2\overline{\theta^2}\left(A_\mu - V_\mu\right) \ . \quad (4.9.4)$$

Here $h_{\mu\nu}$ is the linearized metric, so that $g_{\mu\nu} = \delta_{\mu\nu} + 2h_{\mu\nu}$. The vectors C_μ and A_μ are Abelian gauge fields, and $B_{\mu\nu}$ is a two-form gauge field. The gravitino ψ_μ will not be important for us. We will also need the following field strengths,

$$\begin{aligned} V_\mu &= -\varepsilon_{\mu\nu\rho}\partial^\nu C^\rho \ , \qquad \partial^\mu V_\mu = 0 \\ H &= \frac{1}{2}\varepsilon_{\mu\nu\rho}\partial^\mu B^{\nu\rho} \ . \end{aligned} \quad (4.9.5)$$

We can now express the coupling (4.9.1) in components,

$$\delta\mathscr{L} = -T_{\mu\nu}h^{\mu\nu} + j_\mu^{(R)}\left(A^\mu - \frac{3}{2}V^\mu\right) - ij_\mu^{(Z)}C^\mu + J^{(Z)}H + (\text{fermions}) \ . \quad (4.9.6)$$

Since the gauge field A^μ couples to the R-current, we see that the gauge transformations (4.9.2) include local R-transformations. This supergravity theory is the three-dimensional analog of $\mathcal{N} = 1$ new minimal supergravity in four dimensions [95].

It will be convenient to introduce an additional superfield,

$$\mathcal{V}_\mathcal{H} = \frac{1}{4}\gamma_\mu^{\alpha\beta}[D_\alpha, \overline{D}_\beta]\mathcal{H}^\mu \ , \quad (4.9.7)$$

which transforms like an ordinary vector superfield under (4.9.2). Up to a gauge transformation, it takes the form

$$\mathcal{V}_\mathcal{H} = \left(\theta\gamma^\mu\overline{\theta}\right)\left(A_\mu - \frac{1}{2}V_\mu\right) - i\theta\overline{\theta}H + \frac{1}{4}\theta^2\overline{\theta^2}\left(\partial^2 h^\mu{}_\mu - \partial^\mu\partial^\nu h_{\mu\nu}\right) + (\text{fermions}) \ . \quad (4.9.8)$$

The corresponding field strength $\Sigma_\mathcal{H} = \frac{i}{2}D\overline{D}\mathcal{V}_\mathcal{H}$ is a gauge-invariant real linear superfield. The top component of $\mathcal{V}_\mathcal{H}$ is proportional to the linearized Ricci scalar,

$$R = 2\left(\partial^2 h^\mu{}_\mu - \partial^\mu\partial^\nu h_{\mu\nu}\right) + \mathcal{O}\left(h^2\right) \ . \quad (4.9.9)$$

With this definition, a d-dimensional sphere of radius r has scalar curvature $R = -\frac{d(d-1)}{r^2}$.

[14]Like the $\mathcal{R}$-multiplet in (4.4.7), the metric superfield contains factors of i that are absent in Lorentzian signature.

In a superconformal theory, the $\mathcal{R}$-multiplet can be improved to a superconformal multiplet with $\mathcal{J}^{(Z)} = 0$, as discussed in subsection 4.4.2. In this case the superfield L_α is no longer constrained by (4.9.3), and hence $\mathcal{H}_\mu$ enjoys more gauge freedom. In particular, this allows us to set H and $A_\mu - \frac{1}{2}V_\mu$ to zero. The combination $A_\mu - \frac{3}{2}V_\mu$ remains and transforms like an Abelian gauge field.

4.9.2 Supergravity Chern-Simons Terms

We will now derive the Chern-Simons terms (4.4.17), (4.4.18), and (4.4.19) in linearized supergravity. We begin by considering terms bilinear in the gravity fields,

$$\delta\mathscr{L} = -2\int d^4\theta\, \mathcal{H}^\mu \mathcal{W}_\mu(\mathcal{H}) \ . \tag{4.9.10}$$

Here $\mathcal{W}_\mu(\mathcal{H})$ is linear in $\mathcal{H}$. By dimensional analysis, it contains six supercovariant derivatives. Comparing to (4.9.1), we see that $\mathcal{W}_\mu(\mathcal{H})$ should be invariant under (4.9.2) and satisfy the defining equation (4.4.6) of the $\mathcal{R}$-multiplet. It follows that the bottom component of $\mathcal{W}_\mu(\mathcal{H})$ is a conserved current.

There are two possible choices for $\mathcal{W}_\mu(\mathcal{H})$,

$$\begin{aligned} \mathcal{W}_\mu^{(g)} &= i\left(\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)\overline{D}D\mathcal{H}^\nu + \frac{1}{4}\gamma_\mu^{\alpha\beta}[D_\alpha, \overline{D}_\beta]\Sigma_{\mathcal{H}} \ , \\ \mathcal{W}_\mu^{(zz)} &= \frac{1}{8}\gamma_\mu^{\alpha\beta}[D_\alpha, \overline{D}_\beta]\Sigma_{\mathcal{H}} \ . \end{aligned} \tag{4.9.11}$$

The first choice $\mathcal{W}_\mu^{(g)}$ leads to the $\mathcal{N} = 2$ completion of the gravitational Chern-Simons term (4.4.17),

$$\begin{aligned} \mathscr{L}^{(g)} = \ & \frac{i}{4}\varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\omega_\mu\partial_\nu\omega_\rho + \frac{2}{3}\omega_\mu\omega_\nu\omega_\rho\right) \\ & + i\varepsilon^{\mu\nu\rho}\big(A_\mu - \frac{3}{2}V_\mu\big)\partial_\nu\big(A_\rho - \frac{3}{2}V_\rho\big) + (\text{fermions}) \ . \end{aligned} \tag{4.9.12}$$

Here $(\omega_\mu)_{\nu\rho} = \partial_\nu h_{\rho\mu} - \partial_\rho h_{\nu\mu} + \mathcal{O}(h^2)$ is the spin connection. Note that we have included terms cubic in ω_μ, even though they go beyond second order in linearized supergravity, because we would like our final answer to be properly covariant. Both terms in (4.9.12) are conformally invariant and only the superconformal linear combination $A_\mu - \frac{3}{2}V_\mu$ appears. This is due to the fact that (4.9.12) is actually invariant under the superconformal gauge freedom (4.9.2) without the constraint (4.9.3).

Upon substituting the second choice $\mathcal{W}_\mu^{(zz)}$, we can integrate by parts in (4.9.10),

$$\mathscr{L}^{(zz)} = -\int d^4\theta \, \mathcal{V}_{\mathcal{H}} \Sigma_{\mathcal{H}} \ , \tag{4.9.13}$$

to obtain the Z-Z Chern-Simons term (4.4.18),

$$\mathscr{L}^{(zz)} = i\varepsilon^{\mu\nu\rho}\big(A_\mu - \frac{1}{2}V_\mu\big)\partial_\nu\big(A_\rho - \frac{1}{2}V_\rho\big) + \frac{1}{2}HR + \cdots + (\text{fermions}) \ . \tag{4.9.14}$$

Here the ellipsis denotes higher-order terms in the bosonic fields, which go beyond linearized supergravity. This term contains the Ricci scalar R, as well as H and $A_\mu - \frac{1}{2}V_\mu$, and thus it is not conformally invariant.

It is now straightforward to obtain the flavor-gravity Chern-Simons term (4.4.19) by replacing $\Sigma_{\mathcal{H}} \to \Sigma$ in (4.9.13). This amounts to shifting the $\mathcal{R}$-multiplet by an improvement term $\delta\mathcal{R}_\mu = \frac{1}{8}\gamma_\mu^{\alpha\beta}[D_\alpha, \overline{D}_\beta]\Sigma$. In components,

$$\mathscr{L}^{(fr)} = \frac{i}{2}\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu\big(A_\rho - \frac{1}{2}V_\rho\big) + \frac{1}{8}\sigma R - \frac{1}{2}DH + \cdots + (\text{fermions}) \ . \tag{4.9.15}$$

As above, the ellipsis denotes higher-order terms in the bosonic fields and the presence of R, H, and $A_\mu - \frac{1}{2}V_\mu$ shows that this term is also not conformally invariant.

Chapter 5

Contact Terms, Unitarity, and F-Maximization in Three-Dimensional Superconformal Theories

5.1 Introduction

Any conformal field theory (CFT) in d dimensions can be placed on the d-sphere S^d in a canonical, conformally invariant way, by using the stereographic map from flat Euclidean space. It is natural to study the partition function Z_{S^d} of the CFT compactified on S^d, or the associated free energy,

$$F_d = -\log Z_{S^d} \ . \tag{5.1.1}$$

Since the sphere is compact, F_d does not suffer from infrared (IR) ambiguities. However, it is generally divergent in the ultraviolet (UV). For instance, it may contain power divergences,

$$F_d \sim (\Lambda r)^d + \cdots \ , \tag{5.1.2}$$

where r is the radius of the sphere and Λ is a UV cutoff. (The ellipsis denotes less divergent terms.) These power divergences depend on r and are inconsistent with conformal invariance. They should be set to zero by a local counterterm. In the example (5.1.2) the divergence can be canceled by adjusting the cosmological constant counterterm $\int_{S^d} \sqrt{g}\, d^d x$.

What remains after all power divergences have been eliminated depends on whether the number of dimensions is even or odd. If d is even, the free energy contains a logarithmic term in the radius,

$$F_d \sim a \log(\Lambda r) + (\text{finite}) \ , \tag{5.1.3}$$

which cannot be canceled by a local, diffeomorphism invariant counterterm. It reflects the well-known trace anomaly. The coefficient a is an intrinsic observable of the CFT, while the finite part of F_d depends on the choice of UV cutoff.

If d is odd, there are no local trace anomalies and we remain with a pure number F_d. In unitary theories F_d is real.[1] There are no diffeomorphism invariant counterterms that can affect the value of F_d, and hence any UV cutoff that respects diffeomorphism invariance leads to the same answer. For this reason, F_d is an intrinsic observable of the CFT.

In two and four dimensions, it was shown [125, 152–154] that every unitary renormalization group (RG) flow connecting a CFT_{UV} at short distances to a CFT_{IR} at long distances must respect the inequality

$$a_{\text{UV}} > a_{\text{IR}} \ . \tag{5.1.4}$$

See [155] for a discussion of the six-dimensional case. (Another quantity conjectured to decrease under RG flow was recently discussed in [156].) It has been proposed [33, 157–160] that a similar inequality should hold in three dimensions,

$$F_{\text{UV}} > F_{\text{IR}} \ . \tag{5.1.5}$$

[1] Since our entire discussion is in Euclidean signature, we will not distinguish between unitarity and reflection positivity.

(Since we will remain in three dimensions for the remainder of this chapter, we have dropped the subscript $d = 3$.) This conjectured F-theorem has been checked for a variety of supersymmetric flows, and also for some non-supersymmetric ones; see for instance [143, 161–166]. Moreover, the free energy F on a three-sphere corresponds to a certain entanglement entropy [158]. This relation has been used recently [167] to argue for (5.1.5).

In practice, the free energy F is not easy to compute. Much recent work has focused on evaluating F in $\mathcal{N} = 2$ superconformal theories (SCFTs). (The flat-space dynamics of $\mathcal{N} = 2$ theories in three dimensions was first studied in [134, 168].) In such theories, it is possible to compute F exactly via localization [107], which reduces the entire functional integral to a finite-dimensional matrix model [28–30]. In this approach, one embeds the SCFT into the deep IR of a renormalization group flow from a free UV theory. The functional integral is then computed in this UV description and reduces to an integral over a finite number of zero modes. (A similar reduction of the functional integral occurs in certain four-dimensional field theories [27].)

Since this procedure breaks conformal invariance, the theory can no longer be placed on the sphere in a canonical way. Nevertheless, it is possible to place the theory on S^3 while preserving supersymmetry, and explicit Lagrangians were constructed in [28–30]. A systematic approach to this subject was developed in [31], where supersymmetric Lagrangians on curved manifolds were described in terms of background supergravity fields. This point of view will be important below. One finds that if the non-conformal theory has a $U(1)_R$ symmetry, it is possible to place it on S^3 while preserving an $SU(2|1) \times SU(2)$ symmetry. This superalgebra is a subalgebra of the superconformal algebra on the sphere, but as emphasized in [31], its presence is not related to superconformal invariance.

The choice of $SU(2|1) \times SU(2)$ symmetry is not unique. It depends on a continuous choice of R-symmetry in the UV, as well as a discrete choice of orientation on the sphere.[2] Given any reference R-symmetry R_0, the space of R-symmetries is

[2]The orientation determines whether the bosonic $SU(2) \subset SU(2|1)$ is the $SU(2)_l$

parameterized by the mixing with all Abelian flavor symmetries Q_a,

$$R(t) = R_0 + \sum_a t^a Q_a \ . \tag{5.1.6}$$

The free energy $F(t)$ explicitly depends on the real parameters t^a. Surprisingly, the function $F(t)$ is complex-valued [28–30], even though we expect it to be real in a unitary theory. This will be discussed extensively below. In order to make contact with the free energy of the SCFT, we must find the values $t^a = t^a_*$, such that $R(t_*)$ is the R-symmetry that appears in the $\mathcal{N} = 2$ superconformal algebra.

In this chapter, we will show that the real part $\operatorname{Re} F(t)$ satisfies

$$\frac{\partial}{\partial t^a} \operatorname{Re} F\bigg|_{t=t_*} = 0 \ , \qquad \frac{\partial^2}{\partial t^a \partial t^b} \operatorname{Re} F\bigg|_{t=t_*} = -\frac{\pi^2}{2} \tau_{ab} \ . \tag{5.1.7}$$

The matrix τ_{ab} is determined by the flat-space two-point functions of the Abelian flavor currents j^μ_a at separated points,

$$\langle j^\mu_a(x) j^\nu_b(0) \rangle = \frac{\tau_{ab}}{16\pi^2} \left(\delta^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu \right) \frac{1}{x^2} \ . \tag{5.1.8}$$

In a unitary theory τ_{ab} is a positive definite matrix.

These conditions can be stated as a maximization principle: *the superconformal R-symmetry $R(t_*)$ locally maximizes* $\operatorname{Re} F(t)$ *over the space of trial R-symmetries $R(t)$.* The local maximum $\operatorname{Re} F(t_*)$ is the SCFT partition function on S^3. This F-maximization principle is similar to a-maximization in four dimensions [32]. Analogously, it leads to (5.1.5) for a wide variety of renormalization group flows. The first condition in (5.1.7) is the extremization condition proposed in [29]. The fact that the extremum should be a maximum was conjectured in [33].

A corollary of (5.1.7) is that τ_{ab} is constant on conformal manifolds. It does not depend on deformations of the SCFT by exactly marginal operators, as long as these operators do not break the associated flavor symmetries. Another consequence of (5.1.7) is that τ_{ab} can be obtained from the same matrix integral that calculates

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or the $SU(2)_r$ subgroup of the $SU(2)_l \times SU(2)_r$ isometry group. Below, we will always assume the former.

the free energy, adding to the list of SCFT observables that can be computed exactly using localization. Below, we will discuss several new observables that can also be extracted from $F(t)$.

We will establish (5.1.7) by studying the free energy of the SCFT as a function of background gauge fields for the flavor currents j_a^μ, as well as various background supergravity fields. In theories with $\mathcal{N} = 2$ supersymmetry, every flavor current is embedded in a real linear superfield $\mathcal{J}_a$ and the corresponding background gauge field resides in a vector superfield $\mathcal{V}^a$. The supergravity fields are embedded in a multiplet $\mathcal{H}$. The free energy $F[\mathcal{V}^a, \mathcal{H}]$ of the SCFT now depends on these sources.

Localization allows us to compute $F[\mathcal{V}^a, \mathcal{H}]$ for certain special values of the background fields $\mathcal{V}^a, \mathcal{H}$. On a three-sphere, the answer turns out to violate several physical requirements: it is not real, in contradiction with unitarity, and it is not conformally invariant. The imaginary part arises because we must assign imaginary values to some of the background fields in order to preserve rigid supersymmetry on the sphere [31]. The lack of conformal invariance is more subtle. It reflects a new anomaly in three-dimensional $\mathcal{N} = 2$ superconformal theories [3].

As we will see below, $F[\mathcal{V}^a, \mathcal{H}]$ may contain Chern-Simons terms in the background fields, which capture contact terms in correlation functions of various currents. For instance, a contact term

$$\langle j_a^\mu(x) j_b^\nu(0)\rangle = \cdots + \frac{i\kappa_{ab}}{2\pi}\varepsilon^{\mu\nu\rho}\partial_\rho\delta^{(3)}(x) \ , \tag{5.1.9}$$

corresponds to a Chern-Simons term for the background gauge fields $\mathcal{V}^a$ and $\mathcal{V}^b$. Such contact terms are thoroughly discussed in [3], where it is shown that they lead to new observables in three-dimensional conformal field theories. Here we will use them to elucidate various properties of the three-sphere partition function in $\mathcal{N} = 2$ superconformal theories. In particular, we explain why some of these terms are responsible for the fact that $F[\mathcal{V}^a, \mathcal{H}]$ is not conformally invariant. Moreover, we show how the observables related to κ_{ab} in (5.1.9) can be computed exactly using localization.[3]

[3]In this chapter we explain how to compute the quantities τ_{ab} and κ_{ab}, which are

The outline of this chapter is as follows. Section 5.2 summarizes necessary material from [3]. We introduce the background fields $\mathcal{V}^a$, $\mathcal{H}$ and present various Chern-Simons terms in these fields. We explain why they give rise to new observables and how some of them lead to a violation of conformal invariance. In section 5.3 we place the theory on a three-sphere and review the relevant supergravity background that leads to rigid supersymmetry [31]. We then relate the linear and quadratic terms in the background gauge fields $\mathcal{V}^a$ to the flat-space quantities introduced in section 5.2. In section 5.4 we derive (5.1.7) and clarify the relation to (5.1.5). Section 5.5 contains some simple examples.

5.2 Background Fields and Contact Terms

In this section we discuss contact terms in two-point functions of conserved currents. In theories with $\mathcal{N} = 2$ supersymmetry, we distinguish between $U(1)$ flavor currents and $U(1)_R$ currents. These contact terms correspond to Chern-Simons terms in background gauge and supergravity fields. Their fractional parts are meaningful physical observables and some of them lead to a new anomaly in $\mathcal{N} = 2$ superconformal theories. This section is a summary of [3], which is the basis for chapter 4 of this thesis. We include it here to render the present chapter self-contained.

5.2.1 Non-Supersymmetric Theories

Consider a three-dimensional conformal field theory with a global, compact $U(1)$ symmetry, and the associated current j_μ. We can couple it to a background gauge field a_μ, and consider the free energy $F[a]$, which is defined by

$$e^{-F[a]} = \left\langle \exp\left(\int d^3x \, j_\mu a^\mu + \cdots \right) \right\rangle . \tag{5.2.1}$$

associated with global flavor symmetries, using localization. The corresponding observables for the R-symmetry, and other closely related objects, can also be computed exactly [8].

Here the ellipsis denotes higher-order terms in a_μ that may be required in order to ensure invariance of $F[a]$ under background gauge transformations of a_μ. A familiar example is the seagull term $a_\mu a^\mu |\phi|^2$, which is needed when a charged scalar field ϕ is coupled to a_μ.

We see from (5.2.1) that $F[a]$ is the generating functional for connected correlation functions of j_μ. The two-point function $\langle j_\mu(x) j_\nu(0)\rangle$ is constrained by current conservation and conformal symmetry, so that

$$\langle j_\mu(x) j_\nu(0)\rangle = \frac{\tau}{16\pi^2}\left(\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu\right)\frac{1}{x^2} + \frac{i\kappa}{2\pi}\varepsilon_{\mu\nu\rho}\partial^\rho\delta^{(3)}(x) \ . \tag{5.2.2}$$

Here τ and κ are dimensionless real constants. At separated points, only the first term contributes, and unitarity implies that $\tau \geq 0$. (If $\tau = 0$, then j_μ is a redundant operator.) The correlation function at separated points gives rise to a non-local term in $F[a]$. The term proportional to κ is a contact term, whose sign is not constrained by unitarity. It corresponds to a background Chern-Simons term in $F[a]$,

$$\frac{i\kappa}{4\pi}\int d^3x\, \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \ . \tag{5.2.3}$$

This term explicitly breaks parity.

Correlation functions at separated points are universal. They do not depend on short-distance physics. By contrast, contact terms depend on the choice of UV cutoff. They can be changed by adjusting local terms in the dynamical or background fields. Some contact terms are determined by imposing symmetries. For instance, the seagull term discussed above ensures current conservation. The contact term proportional to κ in (5.2.2) is not of this type. Nevertheless, it possesses certain universality properties, as we will now review.

The Chern-Simons term (5.2.3) is invariant under small background gauge transformations, as required by current conservation. However, it is not the integral of a gauge-invariant local density and this restricts the freedom in changing κ by adding a local counterterm in the exponent of (5.2.1). This restriction arises because we can place the theory on a curved manifold that allows non-trivial bundles for the background gauge field a_μ. Demanding invariance under large gauge transformations implies that κ can only be shifted by an integer.[4] Therefore, the fractional part of κ is universal and does not depend on the short-distance physics. It is an intrinsic observable of the CFT. If we choose to set κ to zero by a local counterterm, then $F[a]$ is no longer invariant under large background gauge transformations: its imaginary part shifts by an amount that is determined by the observable described above and the topology of the gauge bundle.

As described in [3], there are different ways to calculate this observable in flat space. Below, we will discuss its importance for supersymmetric theories on a three-sphere.

5.2.2 Supersymmetric Theories

In theories with $\mathcal{N}=2$ supersymmetry, we distinguish between $U(1)$ flavor symmetries and $U(1)_R$ symmetries. A global $U(1)$ flavor current j_μ is embedded in a real linear superfield $\mathcal{J}$, which satisfies $D^2\mathcal{J} = \overline{D}^2\mathcal{J} = 0$.[5] In components,

$$\mathcal{J} = J + i\theta j + i\overline{\theta}\overline{j} + i\theta\overline{\theta}K - \left(\theta\gamma^\mu\overline{\theta}\right) j_\mu + \cdots \ . \tag{5.2.4}$$

Superconformal invariance implies that J, K, and j_μ are conformal primaries of dimension $\Delta_J = 1$, $\Delta_K = 2$, and $\Delta_{j_\mu} = 2$. (Only J is a superconformal primary.) It

[4]Here we follow the common practice of attributing the quantization of Chern-Simons levels to invariance under large gauge transformations. A more careful treatment involves a definition of the Chern-Simons term (5.2.3) using an extension of the gauge field a_μ to an auxiliary four-manifold. Demanding that the answer be independent of how we choose this four-manifold leads to the same quantization condition as above.

[5]We follow the conventions of [1], continued to Euclidean signature. The gamma matrices are given by $(\gamma^\mu)_\alpha^{\ \beta} = (\sigma^3, -\sigma^1, -\sigma^2)$, where σ^i are the Pauli matrices. The totally antisymmetric Levi-Civita symbol is normalized so that $\varepsilon_{123} = 1$. Note the identity $\gamma_\mu\gamma_\nu = \delta_{\mu\nu} + i\varepsilon_{\mu\nu\rho}\gamma^\rho$.

follows that the one-point functions of J and K vanish, while their two-point functions are related to the two-point function (5.2.2) of j_μ with $\tau = \tau_{ff}$ and $\kappa = \kappa_{ff}$,

$$\begin{aligned} \langle J(x)J(0)\rangle &= \frac{\tau_{ff}}{16\pi^2}\frac{1}{x^2}\ , \\ \langle K(x)K(0)\rangle &= \frac{\tau_{ff}}{8\pi^2}\frac{1}{x^4}\ , \\ \langle J(x)K(0)\rangle &= \frac{\kappa_{ff}}{2\pi}\delta^{(3)}(x)\ . \end{aligned} \tag{5.2.5}$$

The subscript ff emphasizes the fact that we are considering the two-point function of a flavor current. The constant τ_{ff} is normalized so that $\tau_{ff} = 1$ for a free chiral superfield of charge $+1$.

We can couple $\mathcal{J}$ to a background vector superfield,

$$\mathcal{V} = \cdots + \left(\theta\gamma^\mu\overline{\theta}\right) a_\mu - i\theta\overline{\theta}\sigma - i\theta^2\overline{\theta\lambda} + i\overline{\theta^2}\theta\lambda - \frac{1}{2}\theta^2\overline{\theta^2}D\ . \tag{5.2.6}$$

Here a_μ, σ, and D are real. Background gauge transformations shift $\mathcal{V} \to \mathcal{V} + \Omega + \overline{\Omega}$ with chiral Ω, so that σ and D are gauge invariant, while a_μ transforms like an ordinary gauge field. (The ellipsis denotes fields that are pure gauge modes and do not appear in gauge-invariant functionals of $\mathcal{V}$.) The coupling of $\mathcal{J}$ to $\mathcal{V}$ takes the form

$$2\int d^4\theta\, \mathcal{J}\mathcal{V} = JD + j_\mu a^\mu + K\sigma + (\text{fermions})\ . \tag{5.2.7}$$

Now the free energy $F[\mathcal{V}]$ is a supersymmetric functional of the background gauge superfield $\mathcal{V}$. The supersymmetric generalization of the Chern-Simons term (5.2.3) takes the form

$$F_{ff} = -\frac{\kappa_{ff}}{2\pi}\int d^3x \int d^4\theta\, \Sigma\mathcal{V} = \frac{\kappa_{ff}}{4\pi}\int d^3x\ \left(i\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho - 2\sigma D\right) + (\text{fermions})\ . \tag{5.2.8}$$

Here the real linear superfield $\Sigma = \frac{i}{2}\overline{D}D\mathcal{V}$ is the gauge-invariant field strength corresponding to $\mathcal{V}$. This Chern-Simons term captures the contact terms in the two-point functions (5.2.2) and (5.2.5). It is conformally invariant.

A $U(1)_R$ current $j_\mu^{(R)}$ is embedded in a supercurrent multiplet $\mathcal{R}_\mu$, which also contains the supersymmetry current $S_{\mu\alpha}$, the energy-momentum tensor $T_{\mu\nu}$, a current $j_\mu^{(Z)}$ that corresponds to the central charge Z in the supersymmetry algebra, and a string current $\varepsilon_{\mu\nu\rho}\partial^\rho J^{(Z)}$. All of these currents are conserved. See [1] for a thorough discussion of supercurrents in three dimensions. In components,

$$\begin{aligned}\mathcal{R}_\mu &= j_\mu^{(R)} - i\theta S_\mu - i\overline{\theta}\overline{S}_\mu - \left(\theta\gamma^\nu\overline{\theta}\right)\left(2T_{\mu\nu} + i\varepsilon_{\mu\nu\rho}\partial^\rho J^{(Z)}\right) \\ &\quad - i\theta\overline{\theta}\left(2j_\mu^{(Z)} + i\varepsilon_{\mu\nu\rho}\partial^\nu j^{(R)\rho}\right) + \cdots \ .\end{aligned} \tag{5.2.9}$$

Note that there are additional factors of i in (5.2.9) compared to the formulas in [1], because we are working in Euclidean signature. (In Lorentzian signature the superfield $\mathcal{R}_\mu$ is real.)

The $\mathcal{R}$-multiplet couples to the linearized metric superfield $\mathcal{H}_\mu$. In Wess-Zumino gauge,

$$\mathcal{H}_\mu = \frac{1}{2}\left(\theta\gamma^\nu\overline{\theta}\right)\left(h_{\mu\nu} - iB_{\mu\nu}\right) - \frac{1}{2}\theta\overline{\theta}C_\mu - \frac{i}{2}\theta^2\overline{\theta}\overline{\psi}_\mu + \frac{i}{2}\overline{\theta}^2\theta\psi_\mu + \frac{1}{2}\theta^2\overline{\theta}^2\left(A_\mu - V_\mu\right) \ . \tag{5.2.10}$$

Here $h_{\mu\nu}$ is the linearized metric, so that $g_{\mu\nu} = \delta_{\mu\nu} + 2h_{\mu\nu}$, and $\psi_{\mu\alpha}$ is the gravitino. The vectors C_μ and A_μ are Abelian gauge fields, and $B_{\mu\nu}$ is a two-form gauge field. We will also need the following field strengths,

$$\begin{aligned}&V_\mu = -\varepsilon_{\mu\nu\rho}\partial^\nu C^\rho \ , \qquad \partial^\mu V_\mu = 0 \ , \\ &H = \frac{1}{2}\varepsilon_{\mu\nu\rho}\partial^\mu B^{\nu\rho} \ .\end{aligned} \tag{5.2.11}$$

As above, there are several unfamiliar factors of i in (5.2.10) that arise in Euclidean signature. The coupling of $\mathcal{R}_\mu$ to $\mathcal{H}_\mu$ takes the form

$$2\int d^4\theta\, \mathcal{R}_\mu\mathcal{H}^\mu = T_{\mu\nu}h^{\mu\nu} - j_\mu^{(R)}\left(A^\mu - \frac{3}{2}V^\mu\right) + ij_\mu^{(Z)}C^\mu - J^{(Z)}H + (\text{fermions}) \ . \tag{5.2.12}$$

Since the gauge field A_μ couples to the R-current, we see that the gauge freedom includes local R-transformations. This is analogous to $\mathcal{N} = 1$ new minimal supergravity in four dimensions [95, 96]. For a recent discussion, see [150, 151].

If the theory is superconformal, the $\mathcal{R}$-multiplet reduces to a smaller supercurrent. Consequently, the linearized metric superfield $\mathcal{H}_\mu$ enjoys more gauge freedom, which allows us to set $B_{\mu\nu}$ and $A_\mu - \frac{1}{2}V_\mu$ to zero. The combination $A_\mu - \frac{3}{2}V_\mu$ remains and transforms like an Abelian gauge field.

Using $\mathcal{H}_\mu$, we can construct three Chern-Simons terms. They are derived in [3]. Surprisingly, not all of them are conformally invariant.[6]

- *Gravitational Chern-Simons Term:*

$$F_g = \frac{\kappa_g}{192\pi}\int \sqrt{g}\, d^3x \left(i\varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\omega_\mu\partial_\nu\omega_\rho + \frac{2}{3}\omega_\mu\omega_\nu\omega_\rho\right) \right.$$
$$\left. + 4i\varepsilon^{\mu\nu\rho}\left(A_\mu - \frac{3}{2}V_\mu\right)\partial_\nu\left(A_\rho - \frac{3}{2}V_\rho\right)\right) + (\text{fermions}) \ . \tag{5.2.13}$$

Here ω_μ is the spin connection. We see that the $\mathcal{N} = 2$ completion of the usual gravitational Chern-Simons term also involves a Chern-Simons term for $A_\mu - \frac{3}{2}V_\mu$. Like the flavor-flavor term, the gravitational Chern-Simons term is conformally invariant. It was previously studied in the context of $\mathcal{N} = 2$ conformal supergravity [136], see also [137, 138].

- *Z-Z Chern-Simons Term:*

$$F_{zz} = -\frac{\kappa_{zz}}{4\pi}\int \sqrt{g}\, d^3x \left(i\varepsilon^{\mu\nu\rho}\left(A_\mu - \frac{1}{2}V_\mu\right)\partial_\nu\left(A_\rho - \frac{1}{2}V_\rho\right) + \frac{1}{2}HR + \cdots \right)$$
$$+ (\text{fermions}) \ . \tag{5.2.14}$$

Here R is the Ricci scalar.[7] The ellipsis denotes higher-order terms in the bosonic fields, which go beyond linearized supergravity. The *Z-Z* Chern-Simons term is not conformally invariant, as is clear from the presence of the Ricci scalar. This lack of conformal invariance is related to the following fact: in a superconformal theory, the $\mathcal{R}$-multiplet reduces to a smaller supercurrent and the operators conjugate to R, H and $A_\mu - \frac{1}{2}V_\mu$ are redundant.

[6] In order to write suitably covariant formulas, we will include some terms that go beyond linearized supergravity, such as the measure factor $\sqrt{g}$. We also endow $\varepsilon_{\mu\nu\rho}$ with a factor of $\sqrt{g}$, so that it transforms like a tensor. Consequently, the field strength $V_\mu = -\varepsilon_{\mu\nu\rho}\partial^\nu C^\rho$ is covariantly conserved, $\nabla_\mu V^\mu = 0$.

[7] In our conventions, a d-dimensional sphere of radius r has scalar curvature $R = -\frac{d(d-1)}{r^2}$.

- *Flavor-R Chern-Simons Term:*

$$F_{fr} = -\frac{\kappa_{fr}}{2\pi}\int \sqrt{g}\, d^3x \left(i\varepsilon^{\mu\nu\rho} a_\mu \partial_\nu \big(A_\rho - \frac{1}{2}V_\rho\big) + \frac{1}{4}\sigma R - DH + \cdots \right) \\ + (\text{fermions}) \ . \tag{5.2.15}$$

  The meaning of the ellipsis is as in (5.2.14) above. Again, the presence of R, H, and $A_\mu - \frac{1}{2}V_\mu$ shows that this term is not conformally invariant. The relative sign between the Chern-Simons terms (5.2.8) and (5.2.15) is due to the different couplings (5.2.7) and (5.2.12) of j_μ and $j_\mu^{(R)}$ to their respective background gauge fields. Unlike the conformal Chern Simons terms (5.2.8) and (5.2.13), the Z-Z term (5.2.14) and the flavor-R term (5.2.15) are novel. Their lack of conformal invariance will be important below.

The Chern-Simons terms (5.2.8), (5.2.13), (5.2.14), and (5.2.15) summarize contact terms in two-point functions of $\mathcal{J}$ and $\mathcal{R}_\mu$. As we stated above, the fractional parts of these contact terms are meaningful physical observables. This is thoroughly explained in [3]. Using the background fields $\mathcal{V}$ and $\mathcal{H}_\mu$, we can construct two additional local terms: the Fayet-Iliopoulos (FI) term,

$$F_{FI} = \Lambda \int \sqrt{g}\, d^3x\, (D + \cdots) + (\text{fermions}) \ , \tag{5.2.16}$$

and the Einstein-Hilbert term,

$$F_{EH} = \Lambda \int \sqrt{g}\, d^3x\, (R + \cdots) + (\text{fermions}) \ . \tag{5.2.17}$$

These terms are not conformally invariant, and they are multiplied by an explicit power of the UV cutoff Λ. They correspond to conventional contact terms, which can be adjusted at will. Below we will use them to remove certain linear divergences. A finite coefficient of (5.2.16) leads to a one-point function for J. In a scale-invariant theory it is natural to set such a dimensionful finite coefficient to zero. More generally, the dynamical generation of FI-terms is very constrained. For a recent discussion, see [1, 43, 46] and references therein. Note that a cosmological constant counterterm proportional to Λ^3 is not allowed by supersymmetry.

5.2.3 A Superconformal Anomaly

As we have seen above, the two Chern-Simons terms (5.2.14) and (5.2.15) are not conformally invariant. Moreover, we have argued that the fractional parts of their coefficients κ_{zz} and κ_{fr} are meaningful physical observables. If these fractional parts are non-vanishing, certain correlation functions have non-conformal contact terms. If we want to preserve supersymmetry, we have to choose between the following:

1.) Retain these Chern-Simons terms at the expense of conformal invariance. In this case, the free energy is invariant under large background gauge transformations.

2.) Restore conformal invariance by adding appropriate Chern-Simons counterterms with fractional coefficients. In this case the free energy in the presence of topologically nontrivial background fields is not invariant under large gauge transformations. Its imaginary part, which encodes the fractional parts of κ_{zz} and κ_{fr}, is only well defined if we specify additional geometric data. This is similar to the framing anomaly of [123].

This understanding is essential for our discussion below. A detailed explanation can be found in [3]. The second option above is the less radical of the two (the idea of adding Chern-Simons terms to a theory in order to ensure some physical requirements has already appeared long ago in several contexts [117, 123, 128, 129], but we will explore both alternatives.

5.3 The Free Energy on a Three-Sphere

Coupling the flat-space theory to the background supergravity multiplet $\mathcal{H}$ renders it invariant under all background supergravity transformations. For certain expectation values of the fields in $\mathcal{H}$, the theory also preserves some amount of rigid supersymmetry [31]. Here we are interested in round spheres [28–31].[8] In stereographic

[8]Recently, it was found that various squashed spheres also admit rigid supersymmetry [88–94]. Many of our results can be generalized to these backgrounds.

coordinates, the metric takes the form

$$g_{\mu\nu} = \frac{4r^4}{(r^2+x^2)^2}\,\delta_{\mu\nu}\ , \tag{5.3.1}$$

where r is the radius of the sphere. In order to preserve supersymmetry, we must also turn on a particular *imaginary* value for the background H-flux [31],

$$H = -\frac{i}{r}\ . \tag{5.3.2}$$

This expectation value explicitly violates unitarity, since H is real in a unitary theory. Given a generic $\mathcal{N}=2$ theory with a choice of R-symmetry, the background fields (5.3.1) and (5.3.2) preserve an $SU(2|1)\times SU(2)$ superalgebra. If the theory is superconformal, this is enhanced to the full superconformal algebra and the coupling to the background fields in $\mathcal{H}$ reduces to the one obtained by the stereographic map from flat space. In this case the imaginary value for H in (5.3.2) is harmless and does not lead to any violations of unitarity [31].

In this section, we will study an $\mathcal{N}=2$ SCFT on a three-sphere and consider its free energy $F[\mathcal{V}]$ in the presence of a background gauge field $\mathcal{V}$ for the current $\mathcal{J}$. For our purposes, it is sufficient to analyze $F[\mathcal{V}]$ for constant values of the background fields D and σ. The other fields in $\mathcal{V}$ are set to zero. We will study $F[\mathcal{V}]$ as a power series expansion in D and σ around zero, starting with the free energy $F[0]$ itself.

As we saw in the previous section, superconformal invariance may be violated by certain Chern-Simons contact terms. We can restore it by adding bare Chern-Simons counterterms with appropriate fractional coefficients, but this forces us to give up on invariance under large background gauge transformations. Here we will choose to retain the non-conformal terms and preserve invariance under large gauge transformations, since this setup is natural in calculations based on localization. Only the Z-Z Chern-Simons term (5.2.14) and the gravitational Chern-Simons term (5.2.13) can contribute to $F[0]$. On the sphere, the imaginary value of H in (5.3.2) implies that F_{zz} reduces to a purely imaginary constant, since the coefficient κ_{zz} in (5.2.14) is real. The value of this constant depends on non-linear terms in the gravity fields, which are not captured by the linearized formula (5.2.14). The gravitational Chern-Simons term is superconformal and it does not contribute on the round sphere. In general, we will therefore find a complex $F[0]$. Its real part is the conventional free energy of the SCFT, which must be real by unitarity. The imaginary part is due to a Chern-Simons term in the supergravity background fields.

The terms linear in D and σ reflect the one-point functions of J and K. If our theory were fully conformally invariant, these terms would be absent. However, in the presence of the non-conformal flavor-R Chern-Simons term (5.2.15) this is not the case. On the sphere, this term reduces to

$$F_{fr} = \frac{\kappa_{fr}}{2\pi} \int_{S^3} \sqrt{g}\, d^3x \left(\frac{\sigma}{r^2} - \frac{iD}{r} \right) \; . \tag{5.3.3}$$

The explicit factor of i, which violates unitarity, is due to the imaginary value of H in (5.3.2). The relative coefficient between σ and D depends on both the linearized terms that appear explicitly in (5.2.15) and on non-linear terms, denoted by an ellipsis. Instead of computing them, we can check that (5.3.3) is supersymmetric on the sphere. This term leads to non-trivial one-point functions for J and K. However, the fact that κ_{fr} is real implies that

$$\partial_\sigma \, \mathrm{Im}\, F\big|_{\mathcal{V}=0} = 0 \; , \qquad \partial_D \, \mathrm{Re}\, F\big|_{\mathcal{V}=0} = 0 \; . \tag{5.3.4}$$

In order to understand the terms quadratic in D and σ, we must determine the two-point functions of J and K on the sphere. At separated points, they are easily obtained from the flat-space correlators (5.2.5) using the stereographic map,

$$\begin{aligned} \langle J(x) J(y) \rangle_{S^3} &= \frac{\tau_{ff}}{16\pi^2} \frac{1}{s(x,y)^2} \; , \\ \langle K(x) K(y) \rangle_{S^3} &= \frac{\tau_{ff}}{8\pi^2} \frac{1}{s(x,y)^4} \; , \\ \langle J(x) K(y) \rangle_{S^3} &= 0 \; . \end{aligned} \tag{5.3.5}$$

Here $s(x,y)$ is the $SO(4)$ invariant distance function on the sphere. In stereographic coordinates,

$$s(x,y) = \frac{2r^2 |x-y|}{(r^2+x^2)^{1/2}(r^2+y^2)^{1/2}} \; . \tag{5.3.6}$$

Since we are discussing constant values of D and σ, we need to integrate the two-point functions in (5.3.5) over the sphere, and hence we will also need to understand possible contact terms at coincident points. Contact terms are short-distance contributions, which can be analyzed in flat space, and hence we can use results from section 5.2.

We begin by studying $\partial_D^2 F\big|_{\mathcal{V}=0}$. Since $\langle J(x)J(y)\rangle$ does not contain a contact term on dimensional grounds, we can calculate $\partial_D^2 F\big|_{\mathcal{V}=0}$ by integrating this two-point function over separated points on the sphere,

$$\frac{1}{r^4}\frac{\partial^2 F}{\partial D^2}\bigg|_{\mathcal{V}=0} = -\frac{\tau_{ff}}{16\pi^2 r^4}\int_{S^3}\sqrt{g}\,d^3x\int_{S^3}\sqrt{g}\,d^3y\,\frac{1}{s(x,y)^2} = -\frac{\pi^2}{4}\tau_{ff} < 0\ . \tag{5.3.7}$$

The answer is finite and only depends on the constant τ_{ff}. The sign follows from unitarity.

The second derivative $\partial_\sigma^2 F\big|_{\mathcal{V}=0}$ involves the two-point function $\langle K(x)K(y)\rangle_{S^3}$, which has a non-integrable singularity at coincident points. Since the resulting divergence is a short-distance effect, it can be understood in flat space. We can regulate the divergence by excising a small sphere of radius $\frac{1}{\Lambda}$ around $x=y$. Now the integral converges, but it leads to a contribution proportional to Λ. This contribution is canceled by a contact term $\langle K(x)K(0)\rangle \sim \Lambda\delta^{(3)}(x-y)$. The divergence and the associated contact term are related to the seagull term discussed in section 5.2. The removal of the divergence is unambiguously fixed by supersymmetry and current conservation, so that the answer is finite and well defined.[9] This leads to

$$\frac{1}{r^2}\frac{\partial^2 F}{\partial \sigma^2}\bigg|_{\mathcal{V}=0} = -\frac{\tau_{ff}}{8\pi^2 r^2}\int_{S^3}\sqrt{g}\,d^3x\int_{S^3}\sqrt{g}\,d^3y\,\frac{1}{s(x,y)^4} = \frac{\pi^2}{4}\tau_{ff} > 0\ . \tag{5.3.8}$$

[9] To see this, note that in momentum space $\langle J(p)J(-p)\rangle \sim \frac{1}{|p|}$. Supersymmetry implies that $\langle K(p)K(-p)\rangle \sim p^2\langle J(p)J(-p)\rangle \sim |p|$. Thus, a contact term proportional to Λ in $\langle K(p)K(-p)\rangle$ is incompatible with the two-point function of J at separated points. This shows that any UV cutoff that preserves supersymmetry does not allow a contact term, and hence it must lead to a finite and unambiguous answer for $\int d^3x\,\langle K(x)K(0)\rangle$. By contrast, excising a sphere of radius $\frac{1}{\Lambda}$ does not respect supersymmetry, and thus it requires a contact term.

Alternatively, we can evaluate the integral by analytic continuation of the exponent 4 in the denominator from a region in which the integral is convergent. Note that we have integrated a negative function to find a positive answer. This change of sign is not in conflict with unitarity, because we had to subtract the divergence.

Finally, the mixed derivative $\partial_D \partial_\sigma F\big|_{\mathcal{V}=0}$ is obtained by integrating the two-point function $\langle J(x) K(y) \rangle_{S^3}$, which vanishes at separated points. However, it may contain a non-vanishing contact term (5.2.5), and hence it need not integrate to zero on the sphere. Such a contact term gives rise to

$$\frac{1}{r^3} \frac{\partial^2 F}{\partial D\, \partial \sigma}\bigg|_{\mathcal{V}=0} = -\pi \kappa_{ff} \ . \tag{5.3.9}$$

As we explained in section 5.2, the fractional part of κ_{ff} is a well-defined observable in the SCFT.

5.4 Localization and F-Maximization

As we have explained in the introduction, localization embeds the SCFT of interest into the deep IR of an RG flow from a free theory in the UV. We can then compute $F[\mathcal{V}]$ on a three-sphere for certain supersymmetric choices of $\mathcal{V}$,

$$\sigma = m \ , \qquad D = \frac{im}{r} \ , \tag{5.4.1}$$

with all other fields in $\mathcal{V}$ vanishing. Here m is a real constant that can be thought of as a real mass associated with the flavor symmetry that couples to $\mathcal{V}$. Hence D is imaginary. In order to place the theory on the sphere, we must choose an R-symmetry. As explained in [29–31], the real parameter m can be extended to complex values,

$$m \to m + \frac{it}{r} \ , \tag{5.4.2}$$

where t parameterizes the choice of R-symmetry in the UV. The free energy computed via localization is then a holomorphic function of $m + \frac{it}{r}$.

In general, the UV R-symmetry parametrized by t does not coincide with the superconformal R-symmetry in the IR. This only happens for a special choice, $t = t_*$.

In this case $F[m+\frac{it_*}{r}]$ encodes the free-energy and various current correlation functions in the SCFT on the sphere, exactly as in section 5.3. Expanding around $m = 0$, we write

$$F\Big[m + \frac{it_*}{r}\Big] = F_0 + mrF_1 + \frac{1}{2}(mr)^2 F_2 + \cdots \ . \tag{5.4.3}$$

As we explained in section 5.3, the Chern-Simons term (5.2.14) in the background gravity fields leads to complex F_0, but it only affects the imaginary part. This explains the complex answers for F_0 found in the localization computations of [28–30]. Alternatively, we can remove the imaginary part by adding a Chern-Simons counterterm with appropriate fractional coefficient, at the expense of invariance under large background gauge transformations. The real part of F_0 is not affected. It appears in the F-theorem (5.1.5).

The first order term F_1 arises because of the flavor-R Chern-Simons term (5.2.15), which reduces to (5.3.3) on the three-sphere. Restricting to the supersymmetric subspace (5.4.1), we find that

$$F_1 = 2\pi\kappa_{fr} \ . \tag{5.4.4}$$

This accounts for the non-vanishing, real F_1 found in [28–30] and shows that κ_{fr} can be computed using localization. As we explained above, this term is not compatible with conformal symmetry. We can set it to zero and restore conformal invariance by adding an appropriate flavor-R Chern-Simons counterterm, at the expense of invariance under large background gauge transformations.

The imaginary part of F_1 always vanishes, in accord with conformal symmetry. Using holomorphy in $m + \frac{it}{r}$, we thus find

$$\frac{\partial}{\partial t} \operatorname{Re} F\Big|_{m=0,t=t_*} = -\frac{1}{r}\frac{\partial}{\partial m} \operatorname{Im} F\Big|_{m=0,t=t_*} = 0 \ . \tag{5.4.5}$$

This is the condition proposed in [29].

The real part of F_2 arises from (5.3.7) and (5.3.8),

$$\operatorname{Re} F_2 = \frac{1}{r^2}\frac{\partial^2}{\partial m^2} \operatorname{Re} F\Big|_{m=0,t=t_*} = \frac{\pi^2}{2}\,\tau_{ff} \ , \tag{5.4.6}$$

while the imaginary part is due to the flavor-flavor Chern-Simons term (5.2.8). Using (5.3.9), we obtain

$$\operatorname{Im} F_2 = \frac{1}{r^2}\frac{\partial^2}{\partial m^2}\operatorname{Im} F\bigg|_{m=0,t=t_*} = -2\pi\kappa_{ff}\ . \tag{5.4.7}$$

Combining the real and imaginary parts,

$$F_2 = \frac{\pi^2}{2}\tau_{ff} - 2\pi i\kappa_{ff}\ . \tag{5.4.8}$$

Thus, both τ_{ff} and κ_{ff} are computable using localization.

If we denote by $F(t) = F[0 + \frac{it}{r}]$ the free energy for $m = 0$, we can summarize (5.4.5) and (5.4.6) as follows,

$$\frac{\partial}{\partial t}\operatorname{Re} F\bigg|_{t=t_*} = 0\ , \qquad \frac{\partial^2}{\partial t^2}\operatorname{Re} F\bigg|_{t=t_*} = -\frac{\pi^2}{2}\,\tau_{ff} < 0\ . \tag{5.4.9}$$

The generalization to multiple Abelian flavor symmetries is straightforward and leads to (5.1.7),

$$\frac{\partial}{\partial t^a}\operatorname{Re} F\bigg|_{t=t_*} = 0\ , \qquad \frac{\partial^2}{\partial t^a \partial t^b}\operatorname{Re} F\bigg|_{t=t_*} = -\frac{\pi^2}{2}\,\tau_{ab}\ , \tag{5.4.10}$$

where the matrix τ_{ab} is determined by the flat-space two-point functions of the Abelian flavor currents j_a^μ at separated points,

$$\langle j_a^\mu(x) j_b^\nu(0)\rangle = \frac{\tau_{ab}}{16\pi^2}\left(\delta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)\frac{1}{x^2}\ . \tag{5.4.11}$$

Unitarity implies that τ_{ab} is a positive definite matrix. Note that our condition on the second derivatives is reminiscent of a similar condition in [135]. However, the precise relation of [135] to the three-sphere partition function is not understood.

As an immediate corollary, we obtain a non-renormalization theorem for the two-point function coefficients τ_{ab} and κ_{ab}. Since localization sets all chiral fields to zero, the free energy is independent of all superpotential couplings, and hence all exactly marginal deformations. Thus τ_{ab} and κ_{ab} are independent of exactly marginal deformations.

We would briefly like to mention the connection of (5.4.10) to the F-theorem (5.1.5). It is analogous to the relationship between a-maximization and the a-theorem in four dimensions [32]. Since relevant deformations in the UV generally break some flavor symmetries, there are more flavor symmetries in the UV than in the IR. Maximizing over this larger set in the UV should result in a larger value of F, thus establishing (5.1.5). This simple argument applies to a wide variety of RG flows, but there are several caveats similar to those discussed in [135]. An important restriction is that the argument only applies to flows induced by superpotential deformations. For such flows, the free energy is the same function in the UV and in the IR, since it is independent of all superpotential couplings. One can say less about RG flows triggered by real mass terms, since the free energy depends on them nontrivially.

One of the caveats emphasized in [135] is the existence of accidental symmetries in the IR of many RG flows. Similarly, to use localization at the point $t = t_*$, we need to find an RG flow with an R-symmetry that connects the SCFT in the IR to a free theory in the UV. This is generally impossible if there are accidental symmetries in the IR. Nevertheless, the maximization principle (5.4.10) holds. It would be interesting to find a three-dimensional analog of [169], which would enable exact computations in the presence of accidental symmetries. See [170] for recent work in this direction.

5.5 Examples

5.5.1 Free Chiral Superfield

Consider a free chiral superfield Φ of charge $+1$, coupled to σ and D in a background vector multiplet. The action on the sphere is given by

$$S = \int_{S^3} \sqrt{g}\, d^3x \left(|\nabla\phi|^2 - i\overline{\psi}\gamma^\mu \nabla_\mu \psi + \sigma^2 |\phi|^2 - D|\phi|^2 + i\sigma\overline{\psi}\psi + \frac{3}{4r^2}|\phi|^2 \right) . \quad (5.5.1)$$

For constant σ and D, we can compute the partition function by performing the Gaussian functional integral over ϕ and ψ,

$$F = \sum_{n=1}^{\infty} n^2 \log\left(n^2 - \frac{1}{4} + (\sigma^2 - D)r^2 \right) - \sum_{n=1}^{\infty} n(n+1) \log\left((n + \frac{1}{2})^2 + (\sigma r)^2 \right) . \quad (5.5.2)$$

The two sums arise from the bosonic and the fermionic modes respectively. (The eigenvalues of the relevant differential operators on S^3 can be found in [164].) As expected, the leading divergence cancels due to supersymmetry, but there are lower-order divergences.

Instead of evaluating (5.5.2), we will calculate its derivative,

$$\begin{aligned} \frac{1}{r^2}\frac{\partial F}{\partial D} &= \sum_{n=1}^{\infty} \frac{(\sigma^2 - D)r^2 - \frac{1}{4}}{n^2 - \frac{1}{4} + (\sigma^2 - D)r^2} - \sum_{n=1}^{\infty} 1 \\ &= \frac{\pi}{2}\sqrt{(\sigma^2 - D)r^2 - \frac{1}{4}} \coth\left[\pi\sqrt{(\sigma^2 - D)r^2 - \frac{1}{4}}\right] , \end{aligned} \tag{5.5.3}$$

where we set $\sum_n 1 \to -\frac{1}{2}$ by zeta function regularization.[10] Similarly, we find

$$\begin{aligned} \frac{1}{r}\frac{\partial F}{\partial \sigma} = & -\pi\sigma r\sqrt{(\sigma^2 - D)r^2 - \frac{1}{4}} \coth\left[\pi\sqrt{(\sigma^2 - D)r^2 - \frac{1}{4}}\right] \\ & + \pi\left((\sigma r)^2 + \frac{1}{4}\right)\tanh(\pi\sigma r) . \end{aligned} \tag{5.5.4}$$

Note that (5.5.3) and (5.5.4) both vanish when $\sigma = D = 0$, as required by conformal invariance. The derivative of the free energy on the supersymmetric subspace (5.4.1) is given by

$$\frac{1}{r}\frac{\partial F}{\partial m} = \pi\left(\frac{1}{2} + imr\right)\tanh\left(\pi m r\right) . \tag{5.5.5}$$

This exactly matches the result obtained via localization [29, 30].

We can also compute the second derivatives

$$\frac{1}{r^4}\frac{\partial^2 F}{\partial D^2}\bigg|_{\sigma=D=0} = -\frac{\pi^2}{4} , \qquad \frac{1}{r^2}\frac{\partial^2 F}{\partial \sigma^2}\bigg|_{\sigma=D=0} = \frac{\pi^2}{4} , \tag{5.5.6}$$

and therefore,

$$\frac{1}{r^2}\frac{\partial^2}{\partial m^2}\,\mathrm{Re}\,F\bigg|_{m=0} = \frac{\pi^2}{2} . \tag{5.5.7}$$

Since $\tau_{ff} = 1$ for a free chiral superfield of charge $+1$, these results are consistent with (5.3.7), (5.3.8), and (5.4.6).

[10] Equivalently, we can remove the divergence by an appropriate FI counterterm (5.2.16) for the background vector multiplet.

Finally, we discuss the mixed second derivatives,

$$\frac{1}{r^3}\frac{\partial^2 F}{\partial D\partial\sigma}\bigg|_{\sigma=D=0} = 0 \ , \qquad \lim_{\sigma r\to\pm\infty}\frac{1}{r^3}\frac{\partial^2 F}{\partial D\,\partial\sigma}\bigg|_{D=0} = \pm\frac{\pi}{2} \ . \tag{5.5.8}$$

Comparing with (5.3.9), we see that κ_{ff} vanishes in the UV theory. If we give the chiral superfield a real mass by turning on a non-zero value for σ, the RG flow to the IR will generate a contact term $\kappa_{ff} = -\frac{1}{2}\,\mathrm{sgn}(\sigma)$. This corresponds to the half-integer Chern-Simons term that arises when we integrate out a massive fermion [128, 129]. Therefore the free energy is not invariant under all large gauge transformations of the background vector multiplet on arbitrary manifolds. In order to preserve invariance under large gauge transformations, we must add a half-integer Chern-Simons term for the background gauge field to (5.5.1).

Note that the first derivative (5.5.5) has infinitely many zeros. By holomorphy, this means that $F(t)$ has infinitely many extrema, even for a free chiral superfield. However, only one physically acceptable extremum is a local maximum. The F-maximization principle may help resolve similar ambiguities in less trivial examples.

5.5.2 Pure Chern-Simons Theory

Consider a dynamical $\mathcal{N}=2$ Chern Simons theory with gauge group $U(1)$ and integer level k,

$$\frac{k}{4\pi}\left(i\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - 2\sigma D\right) + (\text{fermions}) \ . \tag{5.5.9}$$

Here A_μ denotes the dynamical gauge field rather than a background supergravity field. This theory has an Abelian flavor symmetry with topological current $j^\mu = \frac{i}{2\pi}\varepsilon^{\mu\nu\rho}\partial_\nu A_\rho$, whose correlation functions vanish at separated points. We can couple j^μ to a background gauge field a_μ, which resides in a vector multiplet that also contains the bosons σ_a, D_a, and we also add a background Chern-Simons term for a_μ,

$$\frac{1}{2\pi}\left(i\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu A_\rho - \sigma_a D - D_a\sigma\right) + \frac{q}{4\pi}\left(i\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho - 2\sigma_a D_a\right) + (\text{fermions}) \ . \tag{5.5.10}$$

Here q is an integer. This example is discussed at length in [3].

Naively integrating out A_μ generates a Chern-Simons term for a_μ with fractional coefficient

$$\kappa_{ff} = q - \frac{1}{k} \ . \tag{5.5.11}$$

On the supersymmetric subspace (5.4.1) appropriate to the three-sphere, this term evaluates to $F_{ff} = -i\pi\kappa_{ff}(mr)^2$. We can compare it to the answer obtained via localization. Following [28], we find that

$$e^{-F} = \int d(\sigma r)\, \exp(i\pi r^2(k\sigma^2 + 2\sigma m + qm^2)) = \frac{1}{\sqrt{|k|}} e^{i\,\mathrm{sgn}(k)\pi/4} \exp(i\pi\kappa_{ff}(mr)^2) \ . \tag{5.5.12}$$

We see that the term in F proportional to m^2 agrees with the flat-space calculation.

5.5.3 SQED with a Chern-Simons Term

Consider $\mathcal{N} = 2$ SQED with an integer level k Chern-Simons term for the dynamical $U(1)_v$ gauge field and N_f chiral flavor pairs $Q_i, \widetilde{Q}_{\tilde{i}}$ ($i, \tilde{i} = 1, \ldots, N_f$) that carry charge ± 1 under $U(1)_v$. The theory also has a global $U(1)_a$ flavor symmetry $\mathcal{J}$ under which $Q_i, \widetilde{Q}_{\tilde{i}}$ all carry charge $+1$. Here v and a stand for vector and axial respectively. The theory is invariant under charge conjugation, which flips the sign of the dynamical $U(1)_v$ gauge field and interchanges $Q_i \leftrightarrow \widetilde{Q}_{\tilde{i}}$. In the IR, the theory flows to an SCFT, which is labeled by the integers k and N_f.

In [3], this model is analyzed in perturbation theory for $k \gg 1$. Computing the appropriate two-point functions of the axial flavor current and the R-current leads to

$$\kappa_{ff} = \frac{\pi^2 N_f}{4k} + \mathcal{O}\left(\frac{1}{k^3}\right) \ , \qquad \kappa_{fr} = -\frac{N_f}{2k} + \mathcal{O}\left(\frac{1}{k^3}\right) \ . \tag{5.5.13}$$

We can now compare these flat-space calculations to the result obtained via localization [29, 30]. In the notation of (5.4.3), we find

$$\begin{aligned} F_0 &= N_f \log 2 + \frac{1}{2}\log|k| - \frac{i\pi}{4}\left(\mathrm{sgn}(k) - \frac{N_f}{k}\right) + \mathcal{O}\left(\frac{1}{k^2}\right) \ , \\ F_1 &= -\frac{\pi N_f}{k} + \mathcal{O}\left(\frac{1}{k^3}\right) \ , \\ F_2 &= \pi^2 N_f - \frac{i\pi^3 N_f}{2k} + \mathcal{O}\left(\frac{1}{k^2}\right) \ . \end{aligned} \tag{5.5.14}$$

The real part of F_0 is the conventional free energy for the SCFT in the IR. The imaginary part of F_0 corresponds to (5.2.14), whose coefficient we will not discuss here. The first order term F_1 exactly matches the contribution of the flavor-R term as in (5.4.4), while the imaginary part vanishes to this order in $\frac{1}{k}$. This is due to the fact that the mixing of the R-current and the axial current only arises at $\mathcal{O}\left(\frac{1}{k^2}\right)$. Likewise, the imaginary part of F_2 is captured by the flavor-flavor term as in (5.4.7). Finally, the real part of F_2 is in agreement with (5.4.6), since the two-point function coefficient of $\mathcal{J}$ is given by $\tau_{ff} = 2N_f + \mathcal{O}\left(\frac{1}{k^2}\right)$.

5.5.4 A Theory with a Gravity Dual

Equation (5.1.7) can be checked in $\mathcal{N} = 2$ SCFTs with AdS_4 supergravity duals. The AdS/CFT correspondence [24–26] relates global symmetries of the boundary theory to gauge symmetries in the bulk. The boundary values a_μ^a of the bulk gauge fields A_μ^a act as background gauge fields for the global symmetry currents j_a^μ on the boundary. The boundary free energy $F[a]$ in the presence of these background fields is equal to the on-shell supergravity action computed with the boundary conditions $A_\mu^a(x,z)|_{z=0} = a_\mu^a(x)$. The matrix τ_{ab} defined by the two-point functions (5.1.8) of global currents on the boundary is proportional to the matrix $\frac{1}{g_{ab}^2}$ of inverse gauge couplings that appears in the bulk Yang-Mills term [171].

Consider M-theory on $AdS_4 \times X_7$, where X_7 is a Sasaki-Einstein seven manifold. This background preserves $\mathcal{N} = 2$ supersymmetry on the three-dimensional boundary. The isometries of X_7 lead to AdS_4 gauge fields upon KK reduction from 11-dimensional supergravity. Hence, they correspond to global symmetries of the dual SCFT_3. Given a set of Killing vectors K_a on X_7 that are dual to the global symmetry currents j_a^μ, the matrix τ_{ab} is given by [172]

$$\tau_{ab} = \frac{32\pi N^{\frac{3}{2}}}{3\sqrt{6}(\text{Vol}(X_7))^{\frac{3}{2}}} \int G(K_a, K_b)\, vol(X_7)\ . \qquad (5.5.15)$$

Here G is the Sasaki-Einstein metric on X_7 and $vol(X_7)$ is the corresponding volume form. There are N units of flux threading X_7. We can use (5.5.15) to compute τ_{ab} in the gravity dual and compare to the answer obtained via localization on the boundary, providing a check of (5.1.7).

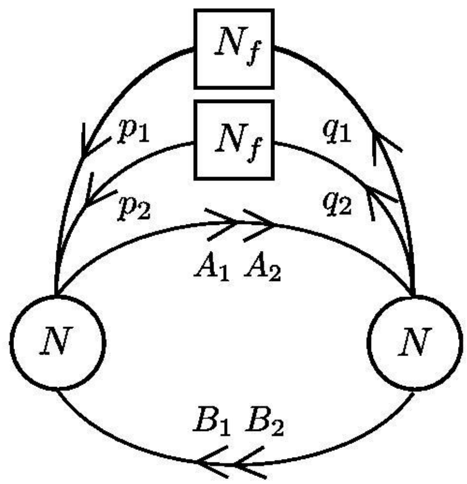

Figure 5.1: Flavored conifold quiver dual to M2-branes on the cone over $Q^{1,1,1}$.

Consider, for instance, the theory depicted in figure 5.1. It is the well-known conifold quiver with gauge group $U(N) \times U(N)$ and vanishing Chern-Simons levels, coupled to two $U(N_f)$ flavor groups. The superpotential is given by

$$W = A_1B_1A_2B_2 - A_1B_2A_2B_1 + \sum_{l=1}^{N_f} \left(p_{1l}A_1q_1^l + p_{2l}A_2q_2^l \right) \ . \qquad (5.5.16)$$

This theory describes N M2 branes on a $\mathbb{Z}_{N_f}$ orbifold of the cone over $Q^{1,1,1} \cong \frac{SU(2)\times SU(2)\times SU(2)}{U(1)\times U(1)}$. It is expected to flow to the SCFT dual to $AdS_4 \times Q^{1,1,1}/\mathbb{Z}_{N_f}$ in the infrared [173, 174].

The large-N partition function of this theory as a function of the trial R-charges was computed in [33]. For simplicity, we consider the free energy $F(t)$ as a function of a single mixing parameter t, which corresponds to the diagonal topological current[11] $j^\mu \sim \varepsilon^{\mu\nu\rho} \left(\operatorname{Tr} F^{(1)}_{\nu\rho} + \operatorname{Tr} F^{(2)}_{\nu\rho} \right)$. Here $F^{(1)}_{\mu\nu}$ and $F^{(2)}_{\mu\nu}$ are the field strengths of the two $U(N)$ gauge groups. The function $F(t)$ is maximized at $t = 0$ and its second

[11]The current is normalized so that certain diagonal monopole operators have charge ± 1.

derivative is given by

$$\left.\frac{\partial^2 F}{\partial t^2}\right|_{t=0} = -\frac{20\pi}{9\sqrt{3}}\left(\frac{N}{N_f}\right)^{\frac{3}{2}} . \tag{5.5.17}$$

We will now compute the two-point function coefficient τ_{ff} of j^μ via the AdS/CFT prescription (5.5.15). The Sasaki-Einstein metric on $Q^{1,1,1}$ takes the form

$$ds^2 = \frac{1}{16}(d\psi + \sum_{i=1}^{3}\cos\theta_i d\phi_i)^2 + \frac{1}{8}\sum_{i=1}^{3}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2) , \tag{5.5.18}$$

with $\psi \in [0, 4\pi)$, $\phi_i \in [0, 2\pi)$, $\theta_i \in [0, \pi]$. Using the results of [173], one can show that the Killing vector of $Q^{1,1,1}$ that corresponds to the current j^μ is given by[12]

$$K = \frac{1}{N_f}(-\partial_{\phi_1} + \partial_{\phi_2}) . \tag{5.5.19}$$

Substituting into (5.5.15) and using $\text{Vol}(Q^{1,1,1}/\mathbb{Z}_{N_f}) = \frac{\pi^4}{8N_f}$, we find

$$\tau_{ff} = \frac{40}{9\sqrt{3}\pi}\left(\frac{N}{N_f}\right)^{\frac{3}{2}} . \tag{5.5.20}$$

Comparing (5.5.17) and (5.5.20), we find perfect agreement with (5.1.7).

As was pointed out in [33, 175], the F-maximization principle is closely related to the volume minimization procedure of [176, 177]. It is natural to conjecture that the two procedures are in fact identical. In other words, the two functions that are being extremized should be related, even away from their critical points. (A similar relation between a-maximization in four dimensions and volume minimization was established in [178, 179].) The example discussed above is consistent with this conjecture: both the free energy at the critical point [33, 175] and its second derivative match.

[12]This identification relies on a certain chiral ring relation involving two diagonal monopole operators, which was conjectured in [173, 174]. Our final result below can be viewed as additional evidence for this conjecture.

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