A Higgs Mechanism for Gravity

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Abstract

We start with a review of gauge theory and all its aspects in general. The concept of unitarity of a QFT will be explained at length. Afterwards, we turn our focus to symmetry breaking. First, the breaking of a symmetry is defined, which leads us to a derivation of the Goldstone Theorem and the accompanying massless Goldstone particles. Then, by adding gauge vector bosons we explain how the massless Goldstone and vector particles can be avoided, which is called the Higgs mechanism. In the last part of the paper we try to find a similar working Higgs mechanism model for General Relativity, which makes the graviton massive. The problem of unitarity of the Higgs mechanism for GR is addressed. In order to tackle the persisting problem of unitarity, an effective field theory approach to the massive GR model is taken. Finally, we briefly look at recent studies of infrared-modified cosmological models and discuss the similarly encountered problems.

Contents

Chapter 1

Introduction

The Higgs mechanism for spin-1 particles was discovered in the 1960's. It has since been successfully implemented into many theories, most notably into the Salam-Weinberg model of the electroweak theory and the Standard Model. Then, in the 1970's the question was raised how one could apply such a mechanism to spin-2 gravitons. The symmetry that should be broken in such a mechanism is the diffeomorphism symmetry. Such a Higgs mechanism was discussed in a 1975 article by Duff [Duf75]. The author did not explicitly specify the lagrangian but derived the mass of the spin-2 particle through scattering elements. One year later, an explicit massless spin-2 lagrangian was given which led to a massive spin-2 lagrangian through the Higgs mechanism [SS76]. However, this explicit lagrangian was not diffeomorphism invariant. Therefore, one may not speak of a specific Higgs mechanism for spin-2 gravitons which transform as tensors in a general covariant lagrangian.
At the same time as the Higgs mechanism revolution in the Standard model, articles began to appear on the topic of massive gravity. In 1970 [vDV70] it was shown that the theory of general relativity does not have any neighbours. In other words, the theory of massive gravity does not smoothly approach general relativity when the mass of the graviton goes to zero. Also, in 1972 [BD72a] it was argued that a massive graviton lagrangian with local interactions leads to a theory which has negative metric particles called *ghosts*. These problems for gravity are now summarized into the so-called no-go theorem of massive gravity. Since then, many articles have appeared where massive theories of gravity are discussed which avoid the no-go theorem. Most notably the DGP model in 2000 and Lorentz breaking models in 2003.
All these articles obviously had cosmology as their main motivation. Secondly, most articles did not discuss the problem of massive gravity in relation to the Higgs mechanism. In this paper we will discuss such a Higgs mechanism. However, our main motivation lies in a *string theoretic approach* to the gluonic sector of QCD. A string-like approach to this sector of QCD would probably make use of bosonic string theory, which has a massless spin-2 graviton in its spectrum. As is well known, there are no spin-2 massless particles in QCD, hence one should derive a mechanism to get rid of the massless graviton. The question which we will ask ourselves thus becomes: how would one give the spin-2 graviton a mass? The Higgs mechanism for spin-1 massless vector bosons will come here into play. We will be interested to know if there is a similar mechanism for spin-2 particles. In this way one also would get a nice relation between gravity and QCD[1].

[1] Strong gravity [ISS71] was a somewhat different approach to linking the strong and gravitational forces.

This paper is subdivided as follows. In chapter 2 we will look at gauge theories and explain their purpose in physics. Most importantly, we will discuss the concept of unitarity. In chapter 3, symmetry breaking in general will be discussed. The Goldstone Theorem will be derived in a special case. Afterwards, the Higgs Mechanism for spin-1 particles will be derived. In chapter 4 we apply the Higgs mechanism for spin-1 particles to gravity. We discuss two lagrangians which both lead to a massive graviton with the help of the usual spin-1 Higgs mechanism. The first model will be shown not to satisfy unitarity. The second model will satisfy unitarity up to a certain extent. We will thus focus on the second model and explain its unitarity problems. This will then motivate us to use the effective field theory approach, in order to analyze the problem of unitarity of the second model. In the last chapter 5 we will discuss the root of the unitarity problem in massive gravity with the help of the no-go theorem. At the end of chapter 5 we will look at other models of massive gravity which do not have the problem of unitarity. In appendix A and B we have gathered a few important results which are used in chapter 2 and chapter 4.

Throughout the paper we will use the following notations and conventions:

1. The metric of gravity has the sign $(- + \cdots +)$.
2. If the space-time dimension $D = 4$, we take $k_4 := ik_0$.
3. Greek letters are raised/lowered by the metric $\eta_{\mu\nu} = diag(-1, 1, \ldots, 1)$ and the latin letters by the invariant delta metric δ^a_b, unless mentioned otherwise or if it is clear from the context.
4. Repeated indices are to be summed over, unless mentioned otherwise.

Chapter 2

Symmetries in physics

In the 1920's, physicists began applying the theory of groups and their representations to symmetries in quantum mechanics. In general, by symmetry we mean any transformation which leaves the laws of physics unchanged. The transformations which leave the laws of a physical system invariant form a group, called the *symmetry group* of the theory describing the system. From the symmetry group of a theory one can extract information and restrictions on the physical states of the system. If G is a symmetry group of a theory describing a physical system, the physical states of the system transform into each other according to some representation of the group G. In other words, the group transformations are mathematically represented in the state space by operations relating the states to each other. Quantum mechanics thus offers a particularly favourable framework for the application of symmetry principles.

The symmetry group G may either be *discrete* or *continuous*. A group is called discrete if the elements of the group form a countable set. On the other hand, the elements of a continuous group can be parameterized by continuous parameters and thus form in particular a non-countable set. In most examples, these parameters will either be real or complex numbers. An advantage of continuous symmetries is that they will be characterized by their infinitesimal transformations. A symmetry is called *global* if the transformation does not depend on the space-time coordinates, otherwise it is called *local*. Furthermore, one can characterize the symmetry groups in so-called *internal* and *space-time* symmetries. Internal symmetries are transformations which act on the internal degrees of freedom of the physical system, which in Quantum Field Theory (**QFT**) are represented by the fields in the action of the theory. Space-time symmetries are transformations acting on the space-time coordinates, with Poincaré symmetry being the prime example in QFT.

In theories with general coordinate invariance, such as general relativity, the space-time symmetries may be interpreted as internal symmetries via the tensor transformations. Hence, in the following chapters we will focus our attention on internal symmetries. We will mostly be interested in continuous internal symmetries, these are called *gauge symmetries*. The term *gauge* is sometimes used for all (global as well as local) continuous internal symmetries, and is sometimes reserved for the local versions (these being at the core of the Standard Model for elementary particles). In this chapter we will closely follow the global ideas set out in [tH07], [Ryd96], [Zee03] and [tHV73].

2.1 Gauge Symmetry

The starting point for the idea of continuous internal symmetries was the interpretation of the presence of particles with (approximately) the same mass as the components (states) of a single physical system. These components were believed to be connected to each other by the transformations of an internal symmetry group. This idea was in fact due to Heisenberg, who in a 1932 paper introduced the $SU(2)$ symmetry connecting the proton and the neutron (interpreted as the two states of a single system). This symmetry was further studied by Wigner, who in 1937 introduced the term isotopic spin (later contracted to isospin). The various internal symmetries considered then were *global* phase transformations of the quantum states and were described in terms of the unitary groups $SU(N)$. The global $SU(N)$ symmetry was then apparent in the *action*, or equivalently in the *lagrangian* up to total derivatives. It was then interesting to ask how one could extend this global $SU(N)$ symmetry to a local one. In this section we are going to start with a theory having a general global gauge symmetry G, and then show how to extend it to a local gauge symmetry G.

2.1.1 Global Gauge Symmetries

Consider a QFT for complex scalar fields $\phi(x) = (\phi_1(x), \ldots, \phi_n(x))$ with a global gauge symmetry lie group G. The scalar fields should transform as an irreducible n-dimensional unitary representation of the gauge group G:

$$\begin{aligned} &\phi'(x) = U\phi(x) = (\mathbb{1} + i\sum_a \theta^a T_a + \mathcal{O}(\theta^2))\phi(x) \ ; \\ &UU^\dagger = \mathbb{1} \qquad ; \qquad T_a = (T_a)^\dagger, \quad a = 1, \ldots, |G| \ ; \end{aligned} \tag{2.1}$$

with the $n \times n$ matrices T_a[1] the infinitesimal generators of the gauge group in the corresponding representation. The matrices T_a satisfy the following commutation relations:

$$[T_a, T_b] = i f_{abc} T_c,$$

with f_{abc} the characteristic structure constants of the gauge group. Assume the following form for the lagrangian:

$$\mathcal{L}^{\text{global inv}} = -(\partial_\mu\phi)^\dagger(\partial^\mu\phi) - V(\phi^\dagger\phi), \tag{2.2}$$

which is obviously invariant under the global unitary transformation (2.1).The conserved currents (which follow from the equations of motion) corresponding to the above symmetry transformations (2.1) are:

$$\begin{aligned} J_a^\mu &= i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_b)}(T_a)_{bc}\phi_c - i\phi_c^\dagger(T_a)_{cb}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_b)^\dagger} \\ &= -i(\phi^\dagger T_a(\partial^\mu\phi) - (\partial^\mu\phi)^\dagger T_a\phi) \ ; \\ \partial_\mu J_a^\mu &= 0. \end{aligned} \tag{2.3}$$

The corresponding conserved quantities are:

$$Q_a = \int J_a^0 d^3x \ ; \ \frac{dQ_a}{dt} = 0. \tag{2.4}$$

[1]For real scalar fields one drops the imaginary i's and one takes the matrices T_a real and antisymmetric.

2.1.2 From Global to Local

If the above parameters θ^a are allowed to become space-time dependent, $\theta^a(x)$, we have a local symmetry transformation. In this case, the field energy of the transformed system is changed and the original symmetry is lost. If however there is yet another field present that "reads off" the local changes and compensates for them in such a way that the system behaves as under the global symmetry transformation, the symmetry is nevertheless maintained. In the case of an internal local symmetry, such a "compensating" field is called a *gauge field*. Let us see how this works in this case. Under the transformation (2.1):

$$\partial_\mu \phi \longrightarrow (\partial_\mu \phi)' = U(\partial_\mu \phi) + (\partial_\mu U)\phi; \tag{2.5}$$

plugging this in the lagrangian (2.2) one notices that the derivative term is what spoils the invariance ($V(\phi^\dagger \phi)$ stays invariant). One would like to form a covariant derivative $D_\mu \phi$ which transforms in the same way as the fields ϕ:

$$D_\mu \phi \longrightarrow (D_\mu \phi)' = U D_\mu \phi,$$

since this will leave the kinetic term $(D_\mu \phi)^\dagger (D_\mu \phi)$ invariant, in the same way as the kinetic term $(\partial_\mu \phi)^\dagger (\partial_\mu \phi)$ was left invariant under the global transformation. In general relativity one encounters the similar problem of the usual derivative of a tensor not being a tensor, which is then fixed by adding to the original derivative a term linear in the tensor, which precisely compensates for the terms that make the normal derivative not a tensor. The "coefficient" of this linear term is the Cristoffel symbol which depends on the metric. Similarly, in this case one should define a covariant derivative $D_\mu \phi$ by adding a term linear in the fields ϕ which compensates the second term in (2.5):

$$D_\mu \phi := \partial_\mu \phi - i g A^a_\mu T_a \phi, \tag{2.6}$$

where g is a coupling constant. One easily finds that the new *gauge fields* $A^a_\mu(x)$ should transform as:

$$A^a_\mu(x) T_a \longrightarrow A'^a_\mu(x) T_a = U(A^a_\mu(x) T_a + \frac{i}{g}\partial_\mu) U^\dagger,$$

in order to cancel the second term in (2.5). The gauge fields do not transform as a representation of the gauge group:

$$A'^a_\mu(x) = A^a_\mu(x) + \frac{1}{g}\partial_\mu \theta^a(x) + f_{abc} A^b_\mu(x) \theta^c(x) + \mathcal{O}(\theta^2), \tag{2.7}$$

but the following quantity does:

$$F^a_{\mu\nu}(x) := \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + g f_{abc} A^b_\mu A^c_\nu.$$

From this quantity one can form the invariant kinetic term $-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$ for the gauge fields. The new lagrangian then becomes:

$$\mathcal{L}^{\text{local inv}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi^\dagger \phi). \tag{2.8}$$

The vector currents corresponding to the local symmetry transformations are:

$$\begin{aligned} J^\mu_a &= i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_b)} (T_a)_{bc} \phi_c - i \phi^\dagger_c (T_a)_{cb} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_b)^\dagger} \\ &= -i(\phi^\dagger T_a (D^\mu \phi) - (D^\mu \phi)^\dagger T_a \phi). \end{aligned} \tag{2.9}$$

In the local case, these vector currents are not conserved (except in the abelian case):

$$D_\mu J^\mu_a := \partial_\mu J^\mu_a - g A^b_\mu f_{bac} J^\mu_c = 0 \tag{2.10}$$

Only scalar fields were considered, but one could have easily included spinor fields. Comparing the globally gauge invariant lagrangian (2.2) and the locally gauge invariant lagrangian (2.8), one notices that the only difference lies in changing the derivative $\partial_\mu \longrightarrow D_\mu$ and adding a kinetic term for the gauge fields. Consider for example the global spinor lagrangian:

$$\mathcal{L}^{\text{global inv}} = -\overline{\psi}\gamma^\mu \partial_\mu \psi - V(\overline{\psi}\psi),$$

with γ^μ the usual Dirac matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta_{\mu\nu}\mathbb{1}$. The local spinor lagrangian then becomes:

$$\mathcal{L}^{\text{local inv}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \overline{\psi}\gamma^\mu D_\mu \psi - V(\overline{\psi}\psi).$$

Next we are going to study the propagators which follow from the above local lagrangian.

2.1.3 Path-integral Quantization of Gauge Fields

Let us focus on finding the propagator of the gauge fields. The quadratic part in the gauge fields in $S = \int \mathcal{L}^{\text{local inv}} d^4x$ is:

$$\int -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a) d^4x.$$

After partial integration, and discarding surface terms, this may be written:

$$\int \frac{1}{2} A^\mu_a (\eta_{\mu\nu}\Box - \partial_\mu \partial_\nu)\delta^{ab} A^\nu_b d^4x.$$

The propagator $D^{\nu\lambda}_{bc}(x-y)$ is defined as the inverse of the quadratic part in the action above:

$$\delta^{ab}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu) D^{\nu\lambda}_{bc}(x-y) = \delta^a_c \delta^\lambda_\mu \delta^4(x-y).$$

Applying now the operator $\delta^{ab}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)$ to $\partial^\nu\Lambda$ gives zero:

$$\delta^{ab}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\partial^\nu\Lambda = \delta^{ab}(\partial_\mu\Box - \Box\partial_\mu)\Lambda = 0.$$

The operator has a zero eigenvalue and therefore no inverse. Why did our straightforward attempt to find the photon propagator fail? One can try to explain this by going back and studying the generating functional:

$$Z = \int \prod_{a,\mu} \mathcal{D}A^a_\mu e^{iS}.$$

The action S is invariant under the gauge transformation (2.7), but the integration is taken over all the A^a_μ's, including those that are related only by a gauge transformation. This clearly gives an infinite contribution to Z and therefore to the Green's functions, obtained by functional differentiation of Z. In the following section we will see how to solve this problem.

2.2 Faddeev-Popov method

Notice that since the parameters $\theta(x)$ can be varied freely one can gauge away degrees of freedom by taking the appropriate $\theta(x)$ functions. Local symmetries are thus not actual symmetries but redundancies of the description of the fields. Let us take the shorthand notation ψ for all the fields that participate in the gauge theory or lagrangian. Also, write ψ^U for the gauge transformed fields. Hence, in the above scalar example we had:

$$\begin{aligned} \psi &= \{A_\mu, \phi\} \overset{\text{gauge transf.}}{\longrightarrow} \psi^U = \{A^U_\mu, \phi^U\} \ ; \\ A^U_\mu &= U A_\mu U^\dagger + \frac{i}{g} U(\partial_\mu U^\dagger) \ ; \\ \phi^U &= U\phi. \end{aligned} \tag{2.11}$$

The degrees of freedom are gauged away by imposing constraints:

$$F^a[\psi] = 0, \quad a = 1, \ldots, |G|. \tag{2.12}$$

These gauge constraints should impose constraints on the $\theta^a(x)$ functions, therefore in particular the constraints should *not* be gauge invariant. The functions $F[\psi]$ should be taken such that (2.12) can *always* be satisfied by taking the correct gauge transform. Also, we assume that there is *precisely one* gauge transform which achieves this. In short, given an arbitrary ψ, there should be precisely one gauge transform U such that (2.12) is satisfied for ψ^U.

Let us consider the following generating functional:

$$Z = \int \mathcal{D}\psi \ e^{iS}, \tag{2.13}$$

which gives an infinite value as was mentioned above. One may get a finite value for Z if the integration is taken over *different* values of the gauge fields that are not simply related by a gauge transformation. In appendix A it is shown how to implement the above (first class) constraints (2.12) in the generating functional Z. The result is the following constrained and finite generating functional Z:

$$Z = \int \mathcal{D}\psi \ \delta[F^a[\psi] - C^a(x)] |Det(\frac{\delta F^a(x)}{\delta\theta^b(y)})| e^{iS}, \tag{2.14}$$

where $\frac{\delta F^a(x)}{\delta\theta^b(y)}$ is short for $\frac{\delta F^a[\psi^U(x)]}{\delta\theta^b(y)}|_{\theta^b=0}$. The functions $C^a(x)$ were introduced in the appendix and are arbitrary. The determinant in the above equation (2.14) is called the Faddeev-Popov (**FP**) determinant. Since equation (2.14) is independent of $C^a(x)$, one can perform a weighted average over them, with the weight functions $e^{-\frac{i}{2\xi^a}\int (C^a)^2 d^4x}$, ξ^a being a set of parameters, which takes care of the δ functions in (2.14). The FP determinant may be expressed through so-called ghost fields $\eta_a, \overline{\eta}_a$ using:

$$|Det(\frac{\delta F^a(x)}{\delta\theta^b(y)})| \propto \int \mathcal{D}\eta_a \mathcal{D}\overline{\eta}_a \ e^{-i\int\int \overline{\eta}_a(x) \frac{\delta F^a(x)}{\delta\theta^b(y)} \eta_b(y) d^4x d^4y}.$$

As a result (2.14) can be written in terms of a path integral:

$$Z = \int \mathcal{D}\psi \mathcal{D}\eta_b \mathcal{D}\overline{\eta}_b \ e^{i\int \mathcal{L}_{\text{eff}} d^4x},$$

with an effective lagrangian $\mathcal{L}_{\text{eff}}$ given by:

$$\begin{aligned}\mathcal{L}_{\text{eff}}(x) &= \mathcal{L}(x) - \frac{1}{2\xi^a}(F^a(x))^2 - \int \overline{\eta}_a(x)\frac{\delta F^a(x)}{\delta\theta^b(y)}\eta_b(y)d^4y \\ &=: \mathcal{L} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}}.\end{aligned} \tag{2.15}$$

In most situations one has $\frac{\delta F^a(x)}{\delta\theta^b(y)} = \delta(x-y)\frac{\delta F^a(x)}{\delta\theta^b(y)}$, which thus leads to:

$$\mathcal{L}_{\text{eff}}(x) = \mathcal{L}(x) - \frac{1}{2\xi^a}(F^a(x))^2 - \overline{\eta}_a(x)\frac{\delta F^a(x)}{\delta\theta^b(x)}\eta_b(x). \tag{2.16}$$

The $\eta_a, \overline{\eta}_a$ are called the FP ghost fields. These fields should be treated as fermion fields, i.e. an extra minus sign for every ghost loops. Furthermore the ghost fields do not represent physical fields, since they were artificially introduced in order to get an exponential expression for the path integral. Note that the extra $\mathcal{L}_{\text{GF}}$ term breaks the gauge invariance, the problem of an infinite path integral Z has thus been fixed by constraining the integral to only go over different (not gauge related) fields ψ. With this new lagrangian $\mathcal{L}_{\text{eff}}$, it should be possible to calculate the propagator. For this, a constraint should be taken such that $\mathcal{L}_{\text{GF}}$ breaks the gauge invariance of the operator in the quadratic part of $\mathcal{L}_{\text{eff}}$. The operator in the quadratic part will then have an inverse, which will obviously depend on the constraints set on the parameters.

Sometimes one can integrate over the fields ψ directly, instead of introducing the weight functions (which lead to the term $\mathcal{L}_{\text{GF}}$). This can be done if the constraints are easy to read off for the fields ψ. For example in the so-called unitary gauge the non-physical fields are set to zero:

$$F[\psi] := \phi^a = 0, \quad a = 1, \ldots, |G|, \tag{2.17}$$

with ϕ^a being the non-physical fields and ψ' the physical fields, $\psi = \{\psi', \phi^a\}$. The generating function then becomes:

$$\begin{aligned}Z &= \int \mathcal{D}\psi'\mathcal{D}\phi^a\delta[\phi^a]|Det(\frac{\delta F^a(x)}{\delta\theta^b(y)})|e^{iS} \\ &= \int \mathcal{D}\psi' e^{i\int(\mathcal{L}|_{\phi^a=0}+\mathcal{L}_{\text{FP}}|_{\phi^a=0})d^4x}.\end{aligned}$$

Thus in the unitary gauge one has the following effective lagrangian:

$$\mathcal{L}^{\text{eff}} = \mathcal{L}|_{\phi^a=0} + \mathcal{L}_{\text{FP}}|_{\phi^a=0}. \tag{2.18}$$

2.2.1 Residual BRST Symmetry

The above effective lagrangian (2.15) still possesses a residual gauge symmetry, even after gauge fixing. One can show (Appendix A) that the effective lagrangian is invariant under the following *global* gauge transformation:

$$\begin{aligned}A'^a_\mu(x) &= A^a_\mu(x) + \lambda\frac{\delta A^b_\mu(x)}{\delta\theta^c(x)}\eta^a(x) = A^a_\mu(x) - \frac{1}{g}\partial_\mu\eta^a(x)\lambda - f_{abc}A^b_\mu(x)\eta^a(x)\lambda\ ; \\ \phi'(x) &= e^{iT_a\lambda\eta^a(x)}\phi(x) = \phi(x) + i\phi(x)\lambda T_a\eta^a(x)\ ; \\ \eta'^a(x) &= \eta^a(x) - \frac{1}{2}f_{abc}\eta^b(x)\eta^c(x)\lambda\ ; \\ \overline{\eta}'^a(x) &= \overline{\eta}^a(x) - \frac{1}{\xi^a}\lambda F^a(x),\end{aligned} \tag{2.19}$$

where the *global* parameter λ is taken to be anticommuting (Grassmann quantity). This symmetry is called the BRST symmetry, a tribute to its discoverers Becchi, Rouet, Stora and Tyutin. Let us check all of the above with an explicit example. For simplicity we will only focus on the gauge fields and the ghost fields, and leave the scalar fields out. Take for instance the Lorentz gauge:

$$\partial^\mu A^a_\mu =: F^a[\psi]. \tag{2.20}$$

Together with (2.7):

$$\delta A^a_\mu = \frac{1}{g}\partial_\mu(\delta\theta^a) + f_{abc}A^b_\mu \delta\theta^c,$$

gives for the variation of F under a gauge transformation:

$$\delta F^a = \frac{1}{g}\Box(\delta\theta^a) + f_{abc}\partial^\mu(A^b_\mu \delta\theta^c),$$

and hence:

$$\frac{\delta F^a(x)}{\delta\theta^b(x)} = \frac{1}{g}\delta^{ab}\Box - f_{abc}\partial^\mu A^c_\mu - f_{abc}A^c_\mu\partial^\mu.$$

The effective lagrangian (2.15) then becomes:

$$\begin{aligned}\mathcal{L}^{\text{eff}}(x) &= \mathcal{L}(x) - \frac{1}{2\xi^a}(\partial^\mu A^a_\mu)^2 - \overline{\eta}_a(x)(\frac{1}{g}\delta^{ab}\Box - f_{abc}\partial^\mu A^c_\mu - f_{abc}A^c_\mu\partial^\mu)\eta_b(x)\\ &=: \mathcal{L}(x) - \frac{1}{2\xi^a}(\partial^\mu A^a_\mu)^2 - \frac{1}{g}\overline{\eta}_a(x)\partial^\mu D_\mu \eta^a(x)\\ &=: \mathcal{L}^{\text{quadratic}}(x) + \mathcal{L}^{\text{interaction}}(x),\end{aligned} \tag{2.21}$$

with:

$$\begin{aligned}\mathcal{L}^{\text{quadratic}}(x) &= \tfrac{1}{2}A^\mu_a[\eta_{\mu\nu}\Box + (\tfrac{1}{\xi^a}-1)\partial_\mu\partial_\nu]\delta^{ab}A^\nu_b - \overline{\eta}_a(x)\tfrac{1}{g}\Box\eta_b(x)\ ;\\ \mathcal{L}^{\text{interaction}}(x) &= -\tfrac{1}{2}gf_{abc}A^b_\mu A^c_\nu(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a) - \tfrac{1}{4}g^2 f_{abc}f_{amn}A^b_\mu A^c_\nu A^\mu_m A^\nu_n\\ &\quad + f_{abc}(\partial^\mu A^c_\mu)\overline{\eta}_a(x)\eta_b(x) + f_{abc}A^c_\mu\overline{\eta}_a(x)\partial^\mu\eta_b(x).\end{aligned}$$

We introduced the covariant derivative corresponding to the adjoint representation $D_\mu\eta^a := \partial_\mu\eta^a - gf_{bac}A^b_\mu\eta^c$ in (2.21) (compare with 2.10). Notice that there is a ghost-gauge field interaction. The propagator of the ghost and gauge fields in momentum representation are:

$$\begin{aligned}(D^{\text{gauge field}})^{ab}_{\mu\nu}(k) &= -\frac{1}{k^2}(\eta_{\mu\nu} + (\xi^a - 1)\frac{k_\mu k_\nu}{k^2})\delta^{ab}\ ;\\ (D^{\text{ghost field}})^{ab}(k) &= \frac{g}{k^2}\delta^{ab}.\end{aligned} \tag{2.22}$$

As one finds in this explicit gauge, the problem of the ill-defined propagator in section 2.1.3 is solved. In Appendix A the invariance of $\mathcal{L}^{\text{eff}}$ under the BRST transformation is further shown:

$$\begin{aligned}\delta A^a_\mu &= -\frac{1}{g}(D_\mu\eta^a)\lambda\ ;\\ \delta\eta^a &= -\frac{1}{2}f_{abc}\eta^b\eta^c\lambda\ ;\\ \delta\overline{\eta}^a &= -\frac{1}{\xi^a}(\partial^\mu A^a_\mu)\lambda.\end{aligned} \tag{2.23}$$

Note that for abelian (commutative) groups one has $f_{abc} = 0$ and thus $\mathcal{L}^{\text{interaction}} = 0$, i.e. the ghost decouples. In the non-abelian case this does not happen and the ghost does not decouple. Another complication with non-abelian groups is that the gauge field couples to itself because there are cubic and quartic terms in $\mathcal{L}^{\text{interaction}}$.

2.2.2 The Slavnov-Taylor Identities

We are now going to derive the so-called Slavnov-Taylor identities. These follow from the fact that the full generating functional Z including sources for all the fields (including ghost fields) should be independent of the gauge constraints. One can show that these identities are equivalent to requiring that the full generating functional Z be invariant under the BRST transformation (2.19). They are thus the quantum version of the classical Noether's theorem, which similarly follows from the invariance under a gauge (differentiable) symmetry. We will again work in the Lorentz gauge (2.20) and leave the scalar fields out. First, introduce sources for the various fields in the lagrangian:

$$Z[J, J_\eta, J_{\overline{\eta}}; u, v] = \int \mathcal{D}\eta \mathcal{D}\overline{\eta} \mathcal{D}A_\mu e^{i\int \mathcal{L}_{\rm tot} dx}, \tag{2.24}$$

with:

$$\mathcal{L}_{\rm tot} = \mathcal{L}_{\rm eff} + J^a_\mu A^\mu_a + J^a_\eta \eta^a + J^a_{\overline{\eta}} \overline{\eta}^a + u^a_\mu (\frac{1}{g} D^\mu \eta)^a + v^a(-\frac{1}{2} f_{abc}\eta^b\eta^c). \tag{2.25}$$

The sources $J_\eta, J_{\overline{\eta}}$ and u are anticommuting. Next, subject the above generating functional Z to a BRST transformation. In Appendix A it is proven that the coefficients of the sources u and v are invariant under the BRST transformation and that the Jacobian of the BRST transformation is unity. Also, $\mathcal{L}^{\rm eff}$ is invariant as was shown in the previous section. The only changes in Z are thus caused by the changes in the coefficients of the sources J^a_μ, J_η and $J_{\overline{\eta}}$:

$$\begin{aligned} Z' &= \int \mathcal{D}\eta \mathcal{D}\overline{\eta} \mathcal{D}A_\mu e^{i(S + \int dx (J^a_\mu \delta A^\mu_a + J^a_\eta \delta\eta^a + J^a_{\overline{\eta}} \delta\overline{\eta}^a))} \\ &\simeq \int \mathcal{D}\eta \mathcal{D}\overline{\eta} \mathcal{D}A_\mu e^{iS}(1 + i\int dx (J^a_\mu \delta A^\mu_a + J^a_\eta \delta\eta^a + J^a_{\overline{\eta}} \delta\overline{\eta}^a)). \end{aligned}$$

The Slavnov-Taylor identity is the requirement that the generating functional (2.24) be invariant under the BRST transformation, hence $Z' = Z$ gives:

$$\int \mathcal{D}\eta \mathcal{D}\overline{\eta} \mathcal{D}A_\mu e^{iS} \int dx (J^a_\mu \delta A^\mu_a + J^a_\eta \delta\eta^a + J^a_{\overline{\eta}} \delta\overline{\eta}^a) = 0, \tag{2.26}$$

where δA^μ_a, $\delta\eta^a$ and $\delta\overline{\eta}^a$ are given by (2.23). It will be noticed that these quantities are precisely the coefficients of u and v, and proportional to $\partial^\mu A^a_\mu$, so (2.26) implies:

$$\lambda \int dx (J^a_\mu(x) \frac{\delta Z}{\delta u^a_\mu(x)} + J^a_\eta(x) \frac{\delta Z}{\delta v^a(x)} - \frac{1}{\xi^a} J^a_{\overline{\eta}}(x) \partial_\mu \frac{\delta Z}{\delta J^a_\mu(x)}) = 0. \tag{2.27}$$

This equation contains only first order derivatives, which is a consequence of introducing the sources u and v for the non-linear terms δA and $\delta\eta$. Putting $Z = e^{iW}$, a similar equation (in fact the same one) holds for W:

$$\int dx (J^a_\mu \frac{\delta W}{\delta u^a_\mu} + J^a_\eta \frac{\delta W}{\delta v^a} - \frac{1}{\xi^a} J^a_{\overline{\eta}} \partial_\mu \frac{\delta W}{\delta J^a_\mu}) = 0. \tag{2.28}$$

One usually converts this into a condition on the generating functional Γ, which is defined as follows:

$$W[J, J_\eta, J_{\overline{\eta}}; u, v] = \Gamma[A, \eta, \overline{\eta}; u, v] + \int dx (J^a_\mu A^\mu_a + J^a_\eta \eta^a + J^a_{\overline{\eta}} \overline{\eta}^a). \tag{2.29}$$

Finally, defining the functional Γ' by:

$$\Gamma' = \Gamma + \frac{1}{2\xi^a}\int dx(\partial^\nu A^a_\nu)^2,$$

one may show [Ryd96] that (2.28) gives:

$$\int dx(\frac{\delta\Gamma'}{\delta u^a_\mu}\frac{\delta\Gamma'}{\delta A^a_\mu} + \frac{\delta\Gamma'}{\delta v^a}\frac{\delta\Gamma'}{\delta\eta^a}) = 0. \tag{2.30}$$

This expresses the content of the Slavnov-Taylor identities, though in a different form from that of the original authors Slavnov and Taylor. These identities are used when considering renormalization of gauge theories. Namely, it can be shown that the Slavnov-Taylor identities (or equivalently the BRST invariance) imply the gauge symmetry of a theory. These identities will thus impose conditions on the possible counterterms in the renormalized lagrangian, which should have the same gauge symmetry as the original lagrangian. Written as they are here (2.30), they are in a form which may most easily be shown [Ryd96] to imply the renormalizability of Yang-Mills (gauge) theories. More importantly, the Slavnov-Taylor identities are used to show the gauge-invariance and unitarity of the so-called S-matrix which will be the subject of the following section.

2.3 Unitarity of the S-matrix

Every sensible QFT needs to satisfy unitarity. Unitarity is the expression that probability amplitudes should sum up to one. This implies that the operator which relates the end states to the initial states must be a unitary operator. The matrix representation of this operator is called the *S-matrix*. Unitarity of the S-matrix is mostly shown with the previously derived Slavnov-Taylor identities. In the next section it will be shown what is implied quantitatively with unitarity, we will also see what roles the previously introduced FP ghost fields play in this aspect.

2.3.1 S-matrix

Let us start by defining the so-called S-matrix. For this one needs to introduce sources J in the original lagrangian:

$$Z[J] = \int \mathcal{D}\psi e^{i(S+\int(J^\dagger_\psi\psi+\psi^\dagger J_\psi))},$$

where again $\psi = (\phi, A_\mu, \dots)$ stands for all the fields in the lagrangian and $J_\psi = (J_\phi, J_{A_\mu}, \dots)$ the corresponding sources. Real fields only need one extra real source term $\int \psi J d^4x$ in the lagrangian. $Z[J]$ is then the generating function for the n-point Green's functions:

$$\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z[J]|_{J=0} =: G(x_1,\dots,x_n). \tag{2.31}$$

Consider now the Feynman diagrams connecting two sources:

$$J^*_i(k')G_{ij}(k,k')J_j(k). \tag{2.32}$$

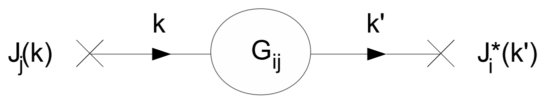


Figure 2.1: Feynman diagrams for the two-point Green's function.

The two-point Green's function will generally have a pole at some value $k^2 = -M^2$ of the squared four-momentum. If there is no pole there will be no corresponding S-matrix element. At the pole the Green's function will be of the form:

$$G_{ij}(k,k') = (2\pi)^4 i\delta^4(k-k')\frac{K_{ij}(k)}{k^2+M^2} \ , \tag{2.33}$$

where the matrix residue K_{ij} can be a function of the components k_μ, with the restriction that $k^2 = -M^2$.

In order to actually calculate (2.32) we should first define the incoming/emission and outgoing/absorbing sources of the (anti-)particles. We begin with treating the sources corresponding to *incoming* particles (anti-particles) of the S-matrix. For every non-zero eigenvalue of $K(k)$ (use the transpose $K^t(-k)$ for anti-particles), define a new set of currents $J_i^{(a)}$ ($J_i^{(\mathrm{anti},a)}$) which are mutually orthogonal and eigenstates of the matrix $K(k)$ ($K^t(-k)$):

$$\begin{aligned}
\text{particle source} \ &= \begin{cases} J_i^{(a)*}J_i^{(b)} & = 0 \qquad a \neq b \\ K_{ij}(k)J_j^{(a)}(k) & = f^a(k)J_i^{(a)}(k) \end{cases} \\
\text{anti-particle source} \ &= \begin{cases} J_i^{(\mathrm{anti},a)*}J_i^{(\mathrm{anti},b)} & = 0 \qquad a \neq b \\ (K^t)_{ij}(-k)J_j^{(\mathrm{anti},a)}(k) & = f^{\mathrm{anti},a}(k)J_i^{(\mathrm{anti},a)}(k) \end{cases}
\end{aligned} \tag{2.34}$$

and normalized for particles as well as anti-particles as follows (no summation over a is implied):

$$J_i^{(a)*}(k)K_{ij}(k)J_j^{(a)}(k) \ = \ \begin{cases} 1 & \text{integer spin} \\ \frac{k_0}{m} & \text{half-integer spin} \end{cases} \tag{2.35}$$

The above normalization is only possible if all eigenvalues f^a ($f^{\mathrm{anti,a}}$) of $K(k)$ ($K^t(-k)$) are positive. In the case of negative eigenvalues normalization is done with minus the right-hand side of (2.35).

The sources defined above are the properly normalized sources for emission of (anti-)particles. The properly defined sources for outgoing/absorption of (anti-)particles follow by considering the above equations (2.34) and (2.35), but with K replaced by its transpose K^t.

The above procedures defines the sources up to a phase factor. The phase factor for the emission of a certain (anti-)particle must agree with that for absorption of that same (anti-)particle. This phase factor is fixed by requiring that the two-point Green's function provided with such sources has precisely the residue 1. Thus the third identity which the sources should satisfy is the following (no summation over a is implied):

$$(J^{\mathrm{out}})_i^a K_{ij}(J^{\mathrm{in}})_j^a = 1. \tag{2.36}$$

The matrix elements of the matrix S' for n ingoing particles and m outgoing particles are defined by:

$$\begin{aligned} &< p_1 b_1, \ldots, p_m b_m | S' | k_1 a_1, \ldots k_n a_n >= \prod_{r=1} \lim_{k_r^2 = -M_r^2} J_{i_r}^{(a_r)}(k_r)(k_r^2 + M_r^2) \times \\ &\prod_{s=1} \lim_{k_s^2 = -M_s^2} J_{j_s}^{(b_s)}(k_s)(k_s^2 + M_s^2) \times G_{i_1,\ldots,i_n,j_1,\ldots,j_m}(k_1, \ldots, k_n, p_1, \ldots, p_m), \end{aligned} \tag{2.37}$$

where the sources $J^{(a_r)}$ are the appropriate incoming sources corresponding to the incoming (anti-)particles and similarly $J^{(b_s)}$ the outgoing sources corresponding to the outgoing (anti-)particles. The entries $k_{10}, \ldots, k_{n0}$ and $p_{10}, \ldots, p_{m0}$ are all positive. The prescription given above results in zero when applied to the two-point Green's function. Thus evidently one does not get exactly the S-matrix that one has in such cases. The S-matrix is defined as the above S'-matrix, but also including lines where particles go through without any interaction. By also counting these diagrams, we get a non-zero result when applied to the two-point function, namely $< p\ b | S | k\ a >= \delta^4(p - k)\delta_{ab}$. This completes the definition of the S-matrix.

The S-matrix may be expressed in other equivalent ways. The S-matrix elements as defined above are equal to the transition amplitudes:

$$< \beta | S | \alpha > = < \beta, t \to \infty | \alpha, t \to -\infty > .$$

This implies that the S operator can also be defined as:

$$\phi_{\text{out}}(x) \ = \ S^\dagger \phi_{\text{in}}(x) S, \tag{2.38}$$

where the operators $\phi_{\text{out,in}}$ are the free field operators satisfying:

$$\lim_{t \longrightarrow \pm\infty} < \alpha | \phi(x) | \beta > = < \alpha | \phi_{\text{in}}^{\text{out}}(x) | \beta > .$$

One [Ryd96] may then prove that the operator which satisfies (2.38) is the following:

$$S = \, : e^{\int \phi_{in} \Delta^{-1} \frac{\delta}{\delta J(z)} dz} : Z[J] \ , \tag{2.39}$$

where the exponential of the operators is normal ordered and $Z[J]$ is the generating function. Note that ϕ_{in} here is an operator satisfying the free field equations and Δ^{-1} is the inverse propagator, i.e. the differential operator corresponding to the free field equations. This equation for the S operator is called the LSZ-reduction formula. This prescription gives the same S-matrix elements as defined earlier in (2.37).

In order to even speak about unitarity one should first define the operator $S^\dagger$. The matrix elements of $S^\dagger$ are defined as usual by:

$$< \alpha | S^\dagger | \beta > = < \beta | S | \alpha >^* . \tag{2.40}$$

The matrix elements of $S^\dagger$ can also be found in another way. In addition to the lagrangian $\mathcal{L}$ defining S, consider the complex conjugated lagrangian $\mathcal{L}^\dagger$. This $\mathcal{L}^\dagger$ may be used to define another S matrix obtained in the usual way from $\mathcal{L}^\dagger$, however with the opposite sign for the $i\epsilon$

in the propagators and also the replacement $i \to -i$ in the factors $(2\pi)^4 i$. Then one [tHV73] can easily show that the following is true:

$$< \alpha|S(\mathcal{L}, i)^\dagger|\beta > = < \alpha|S(\mathcal{L}^\dagger, -i)|\beta > . \tag{2.41}$$

To summarize, the matrix elements of $S^\dagger$ can be obtained either directly from their definition (2.40), or by use of different Feynman rules. These new rules follow from using the complex conjugate of the lagrangian, in particular one reverses all arrows in vertices and propagators (including the arrows of the momenta k through the propagators) and replaces all vertex functions and propagators by their complex conjugate (for the propagator this means using the Hermitian conjugate propagators). The in- and out-state source functions are defined by the usual procedure, involving now $K^\dagger(-k)$ instead of K.

The S-matrix elements are the transition amplitudes of the theory. In the usual way, lifetimes and cross-sections can be deduced from the transition amplitudes. The probability amplitudes are defined by the absolute value squared of these amplitudes. Conservation of probability then requires that the S-matrix be unitary:

$$\sum_\beta | < \beta|S|\alpha > |^2 = \sum_\beta < \alpha|S^\dagger|\beta >< \beta|S|\alpha > = \ 1. \tag{2.42}$$

This property will only be true if the QFT satisfies certain conditions, which will be investigated in the following section.

2.3.2 Unitarity

Unitarity of a QFT, (2.42), is the requirement that the S-matrix be unitary, i.e. $SS^\dagger = \mathbb{1}$. If one writes $S = \mathbb{1} + iT$, this implies the following for the imaginary part of the reaction matrix iT:

$$\begin{aligned} < out|(iT)|in > + < out|(iT)^\dagger|in > &= - < out|(iT)^\dagger(iT)|in > \\ &= - \sum_{\mathrm{k}} < k|(iT)|in >< out|(iT)^\dagger|k > . \end{aligned} \tag{2.43}$$

This unitarity equation (2.43) should always be checked for QFT's, in order to get physical results. The above identity puts a few requirements on the lagrangian of a QFT, which will now be derived.

One usually starts with deriving a similar identity which is *always* true for *every* Feynman diagram in every QFT. The starting point is the decomposition of the bare propagator into negative and positive energy parts:

$$\begin{aligned} \Delta_{ij}(x) &= \theta(x_0)\Delta^+_{ij} + \theta(-x_0)\Delta^-_{ij} \ ; \\ \Delta^\pm_{ij}(x) &= \frac{1}{(2\pi)^3} \int d^4k e^{ikx} \theta(\pm k_0) \rho(k^2), \end{aligned} \tag{2.44}$$

with $x = x_i - x_j$, and here $\theta(x)$ is the Heaviside step function. The functions ρ are called the spectral functions and we assume them to be real, hence one then has $\Delta^\pm_{ij} = (\Delta^\mp_{ij})^*$. Also $\Delta^\pm_{ij} = \Delta^\mp_{ji}$, consequently:

$$\Delta^*_{ij} = \theta(x_0)\Delta^-_{ij} + \theta(-x_0)\Delta^+_{ij}.$$

This is called the Källen-Lehmann representation of the propagators. The above functions can also be viewed in momentum representation:

$$\begin{aligned} \Delta^{\pm}(k) &= \frac{1}{(2\pi)^3}\theta(\pm k_0)\int_0^{\infty} ds'\rho(s')\delta(k^2+s')\ ; \\ \Delta(k) &= \frac{1}{(2\pi)^4 i}\int_0^{\infty} ds'\rho(s')\frac{1}{k^2+s'-i\epsilon}. \end{aligned} \tag{2.45}$$

Let us now consider a (Feynman) diagram in momentum representation, denoted by $F(k_1,\ldots,k_n)$. Here, only the *internal* momenta are denoted, by $k_1,\ldots,k_n$. To get the corresponding expression contributing to a scattering matrix element one should multiply each external line by the matrix residue K and the appropriate incoming and outgoing source functions. We define a new diagram called a *cut diagram* of the original diagram, which is diagrammatically represented as the original diagram together with a cut line dividing the diagram into two connected parts, with one part shaded. The analytical expression $F_{cut}(k_1,\ldots,k_n)$ for this cut diagram is the same as the original diagram but with the *internal* propagators changed as follows:

1. A propagator $\Delta_{ki}(k)$ connecting two vertices is unchanged if both vertices lie in the unshaded part.
2. A propagator $\Delta_{ki}(k)$ is replaced by $\Delta^+_{ki}(k)$ if one vertex lies in the shaded part and the momentum k is directed towards the shaded part.
3. A propagator $\Delta_{ki}(k)$ is replaced by $\Delta^-_{ki}(k)$ if one vertex lies in the shaded part and the momentum k is directed towards the unshaded part.
4. A propagator $\Delta_{ki}(k)$ is replaced by Δ^*_{ki} if both vertices lie in the shaded part.
5. Any vertex lying in the shaded part carries an extra factor -1.

For example, in momentum space one represents:

$$F_{cut}(k_1,k_2,k_3,k_4) = (-1)^2\Delta^+(k_1)\Delta^*(k_2)\Delta^-(k_3)\Delta(k_4)$$

as the following cut diagram:

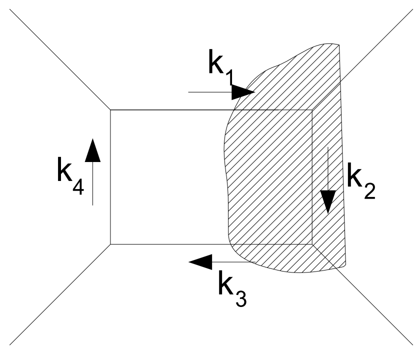


Figure 2.2: A four-point cut diagram. The external lines on the left are always taken to be the incoming sources, while the external lines on the right are the outgoing sources. Energy should always flow from the left incoming sources to the right outgoing sources.

The external propagators and source functions play no role in this prescription and are left unchanged, only the internal propagators and vertices are possibly changed. The above prescriptions (2) and (3) make sure that energy in the cut diagram only flows from the unshaded part towards the shaded part. This restriction in energy flow will cause that many cut diagrams are equal to zero, either because of energy conservation or the fact that energy in the diagrams should always flow from the left to the right. For example, the two following cut diagrams are zero:

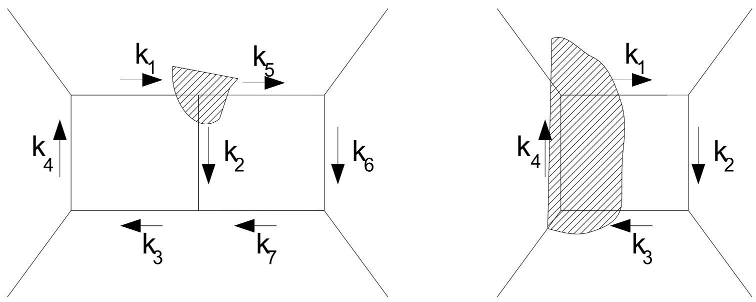


Figure 2.3: Two cut diagrams that are equal to zero. The left cut diagram is zero because of energy conservation in the shaded vertex. The right cut diagram is zero because the restriction in energy flow from the cut lines is in conflict with the restriction that one takes energy to flow from the left incoming sources to the right outgoing sources.

One [tHV73] can now show that the following identity is true for every given (Feynman) diagram:

$$F(k_1, \ldots, k_n) + \widehat{F}(k_1, \ldots, k_n) = -\sum_{\text{cuttings}} F_{cut}(k_1, \ldots, k_n). \tag{2.46}$$

Here F is the original (not shaded) diagram, $\widehat{F}$ is the diagram with all vertices in the shaded region. The sum goes over all possible non-zero cut diagrams F_{cut} where minimally one internal line is cut, in other words the original (not cut) diagram and the $\widehat{F}$ diagram are not included in the sum. Equation (2.46) is called Cutkosky's rule. Cutkosky's rule can be diagrammatically represented as follows:

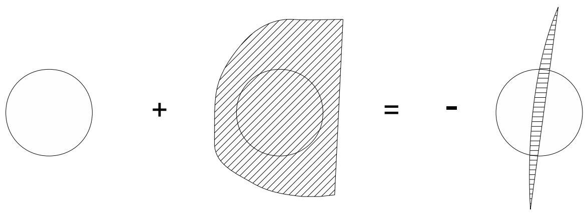


Figure 2.4: Diagrammatic representation of Cutkosky's rule.

Again, the right hand side equals minus the sum of the non-zero cut diagrams corresponding to all possible internal cuttings of the original diagram, with the prescriptions (1)-(5) given on the previous page for the analytical expressions. We only took the internal lines into consideration. The matrix residues K will have to be multiplied on both sides of the equation by hand together with the external sources in order to get an equation for a diagram contributing to the scattering element.

Example Consider a scalar theory with ϕ^3 interaction. For the loop diagram with three vertices connected by three internal propagators, one finds the following Cutkosky equation:

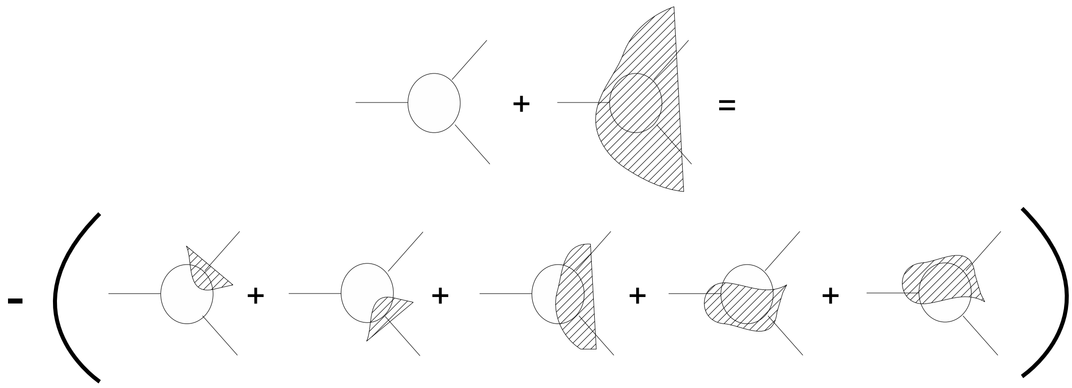
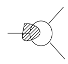


Figure 2.5: Cutkosky equation for the three vertex loop diagram.

Note that the cut diagram  with the left vertex in the shaded region is zero because energy is always taken to flow out of incoming sources towards the right outgoing sources.

The Cutkosky rule (2.46) is of the same structure as the unitarity requirement (2.43), except that the unitarity identity is a property for a scattering element, that is, for a sum of *different* Feynman diagrams, while Cutkosky's rule only holds for *one* Feynman diagram. Also Cutkosky's rule is true for every theory described by a Lagrangian, whether the theory is unitary or not.

Comparing the two equations (2.43) and (2.46) we see that unitarity basically requires the following:

1. The diagrams in the shaded part are those that occur in $S^\dagger$.
2. The $\Delta^\pm$ functions must be equal to what is obtained when summing over intermediate states.

Actually, one should also take into account that multiplication of both sides of the cutting equation is done with the incoming $J^{in,S}$ and outgoing $J^{out,S}$ sources corresponding to the S-matrix. The sources $J^{in,S^\dagger}$ and $J^{out,S^\dagger}$ corresponding to the $S^\dagger$-matrix are generally different than $J^{in,S}$ respectively $J^{out,S}$, hence one must take caution when comparing the cutting equation and the unitarity equation. For example in the unitarity equation one has the second term $< out|(iT)^\dagger|in >\sim J^{out,S^\dagger} \cdot \widehat{F} \cdot J^{in,S^\dagger}$ and this will not be trivially equal to the second term $J^{out,S} \cdot \widehat{F} \cdot J^{in,S}$ in the cutting equation.

2.3.3 Examples and summary

The above point (1) makes it *necessary* that the interaction part of the lagrangian generating the S-matrix is real, this follows from our previous assertion (2.41). Point (2) makes it *necessary* that the quadratic part of the lagrangian generating the S-matrix is real. In other words the lagrangian should be a real function, which from now on forward will always be assumed for any considered QFT. There will also be other requirements on the lagrangian, shown below. Assume for simplicity $K(-k) = K(k)$, this will generally be true for integer spin particles. The lagrangian is also real, in other words one has $K^\dagger(-k) = K(k)$, and hence $K^\dagger(k) = K(k)$. The matrix K is hermitian and thus diagonalizable: $K = diag(\lambda_1(k), \ldots, \lambda_n(k))$ with real eigenvalues λ_i. The zero eigenvalues $\lambda_j = 0$ will have no corresponding sources. By only concentrating on the non-zero eigenvalues (i.e. on the non-zero part of the K-matrix) one can assume without loss of generality that all eigenvalues are non-zero. The normalized sources are:

$$\begin{aligned} \text{particle source} &= \begin{cases} (J^{(a)})_j^{in,S}(k) &= \frac{\delta_j^a}{\sqrt{|\lambda^a|}} \\ (J^{(a)})_j^{out,S}(k) &= sign(\lambda_a)\frac{\delta_j^a}{\sqrt{|\lambda^a|}} \end{cases} \\ \text{anti-particle source} &= \begin{cases} (J^{(a)})_j^{in,S}(k) &= sign(\lambda_a)\frac{\delta_j^a}{\sqrt{|\lambda^a|}} \\ (J^{(a)})_j^{out,S}(k) &= \frac{\delta_j^a}{\sqrt{|\lambda^a|}} \end{cases} \end{aligned} \tag{2.47}$$

If one denotes the incoming and outgoing sources corresponding to $S^\dagger$ with $(J^{(a)})^{in,S^\dagger}$ and $(J^{(a)})^{out,S^\dagger}$, from our previous observation (2.41) one then finds:

$$\begin{aligned} (J^{(a)})^{in,S^\dagger}(k) &= ((J^{(a)})^{out,S}(k))^\dagger \ ; \\ (J^{(a)})^{out,S^\dagger}(k) &= ((J^{(a)})^{in,S}(k))^\dagger, \end{aligned} \tag{2.48}$$

which is true for the particle as well as anti-particle sources. For integer spin particles, one condition which is oftentimes sufficient for unitarity is:

$$\sum_a (J^{(a)})_j^{in,S^\dagger}(J^{(a)})_l^{out,S} = K_{jl}^{-1}(k) = diag(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}), \tag{2.49}$$

since this will then imply point (2) for the intermediate states:

$$\begin{aligned} &\sum_a \delta(k^2+m^2)\theta(\pm k_0)((J^{(a)})_l^{out,S} K_{lm}(k))(K_{ij}^\dagger(-k)(J^{(a)})_j^{in,S^\dagger}) = \\ &K_{lm}(k)\ K_{ij}(k)\delta(k^2+m^2)\theta(\pm k_0)K_{jl}^{-1}(k) = K_{im}(k)\delta(k^2+m^2)\theta(\pm k_0) = \Delta^\pm(k). \end{aligned}$$

Using (2.47) and (2.48), this will only be true if *all* the non-zero eigenvalues are *positive*:

$$\sum_a (J^{(a)})_j^{in,S^\dagger}(J^{(a)})_l^{out,S} = diag(\frac{1}{|\lambda_1|}, \ldots, \frac{1}{|\lambda_n|}).$$

In the case of negative eigenvalues an extra mechanism or symmetry is required to prove unitarity. Also, one would still need to check that the left hand side of (2.43) and (2.46) are equal while taking into account the difference between the incoming (outgoing) sources of S and $S^\dagger$.

The residue matrices of spin 1/2 particles do not satisfy $K(-k) = K(k)$. The above statements are not applicable, which will make it even more difficult to check unitarity. In the following, we will give a thorough treatment of unitarity in the case of scalar particles since these will be needed in the later chapters. Afterwards, spin 1/2 and massless spin 1 particles will partly be mentioned in the end.

Spin 0

First, we take all the eigenvalues of K to be positive: $\lambda_i > 0$. The incoming and outgoing sources were defined above: $(J^{(a)})_j^{out} = (J^{(a)})_j^{in} = (J^{(a)})_j^{out,S^\dagger} = (J^{(a)})_j^{in,S^\dagger} = \frac{1}{\sqrt{\lambda_a}}\delta_j^a$. Multiplying both sides of the the cutting equation with the incoming and outgoing sources gives an equation for the scattering elements. Since the sources of S and $S^\dagger$ are equal we conclude that only (2.49) should be checked, which is indeed satisfied since all eigenvalues are positive.

If one of the eigenvalues λ_i is negative, let's say for $i = r$: $\lambda_r < 0$. The incoming sources can then be taken $(J^{(a)})_j^{in,S} = \frac{1}{\sqrt{|\lambda_a|}}\delta_j^a$ and for the outgoing sources (2.47):

$$(J^{(a)})_j^{out,S} = \begin{cases} \frac{1}{\sqrt{|\lambda_a|}}\delta_j^a & a \neq r \\ -\frac{1}{\sqrt{|\lambda_a|}}\delta_j^a & a = r \end{cases} \tag{2.50}$$

The incoming and outgoing sources for $S^\dagger$ are found from (2.48): $(J^{(a)})^{in,S^\dagger} = (J^{(a)})^{out,S}$ and $(J^{(a)})^{out,S^\dagger} = (J^{(a)})^{in,S}$. Note the minus sign when $a = r$ in (2.50). Multiplying both sides of the cutting equation (2.46) with the sources $(J^{(a)})_j^{in,S}$, $(J^{(a)})_j^{out,S}$ and then comparing the terms with the scattering elements in the unitarity equation gives:

$$\begin{aligned} &< out|(iT)|in > +(-1)^{n^r_{in}+n^r_{out}} < out|(iT)^\dagger|in > \\ &= -\sum_s (-1)^{n^r_s+n^r_{out}} < s|(iT)|in >< out|(iT)^\dagger|s > . \end{aligned}$$

The number n^r_{out} is the amount of outgoing "r-sources" $(J^{(r)})^{out}$ in $|out >$, n^r_{in} is the amount of incoming "r-sources" $(J^{(r)})^{in}$ in $|in >$ and n^r_s is the amount of incoming "r-sources" $(J^{(r)})^{in}$ in the intermediate state $|s >$. The above equation is equal to the unitarity equation (2.43) if and only if $(-1)^{n^r_{out}+n^r_{in}} = 1$ and $(-1)^{n^r_{out}+n^r_s} = 1$. This is equivalent with $n^r_{in} + n^r_{out}$ being an even number for any diagram, hence the amount of external "r-particles" should be even. This can only be true if the lagrangian has the internal global symmetry $\phi_r \longrightarrow -\phi_r$, in other words the lagrangian should only contain even powers of the field ϕ_r.

Conclusion 2.1 *The scalar sector of the theory will always be unitary if all the scalar propagators have positive eigenvalues. If the scalar propagator has negative eigenvalues, then the scalar sector of the theory will only be unitary if the lagrangian is* **even** *in the fields corresponding to the negative eigenvalues.*

This also easily follows from the fact that if the lagrangian is even in those fields with negative propagators, we can always substitute the fields $\phi \longrightarrow i\phi$. Such a substitution leads to a lagrangian with the correct signs in front of the kinetic (quadratic) terms without changing the physics of the theory (the lagrangian is kept real).

Spin 1/2

For spin 1/2 fields the matrix residue equals:

$$K_{ij}(k) = (-i\gamma k + M)_{ij}.$$

We no longer have $K(k) = K(-k)$, thus the above equation (2.48) is not applicable in the spin 1/2 case. The corresponding normalized sources in this case are:

$$\begin{aligned}
\text{incoming particle source}: \quad J_a^{in,S}(k) &= \frac{\sqrt{2k_0}}{2M} u^a(k), \quad a = 1,2 \\
\text{outgoing particle source}: \quad J_a^{out,S}(k) &= \frac{\sqrt{2k_0}}{2M} \overline{u}^a(k), \quad a = 1,2 \\
\text{incoming anti-particle source}: \quad J_a^{in,S}(k) &= -\frac{\sqrt{2k_0}}{2M} \overline{v}^a(k), \quad a = 1,2 \\
\text{outgoing anti-particle source}: \quad J_a^{out,S}(k) &= \frac{\sqrt{2k_0}}{2M} v^a(k), \quad a = 1,2
\end{aligned} \tag{2.51}$$

The outgoing and incoming particles for $S^\dagger$ are:

$$\begin{aligned}
\text{incoming particle source}: \quad J_a^{in,S^\dagger}(k) &= J_a^{in,S}(k), \quad a = 1,2 \\
\text{outgoing particle source}: \quad J_a^{out,S^\dagger}(k) &= J_a^{out,S}(k), \quad a = 1,2 \\
\text{incoming anti-particle source}: \quad J_a^{in,S^\dagger}(k) &= -J_a^{in,S}(k), \quad a = 1,2 \\
\text{outgoing anti-particle source}: \quad J_a^{out,S^\dagger}(k) &= -J_a^{out,S}(k), \quad a = 1,2
\end{aligned} \tag{2.52}$$

Note the minus sign for the anti-particle sources in the above equation (2.52). This can easily be checked by using the fact that the lagrangian is real and the following:

$$K^\dagger(k)\gamma^4 = \gamma^4 K(k).$$

Because of these minus signs, unitarity will be more difficult to verify. The above equations (2.51) and (2.52), together with Cutkosky's rule gives the following equation for the S-matrix elements:

$$\begin{aligned}
&< out|iT|in > + (-1)^{n_{anti}^{out} + n_{anti}^{in}} < out|(iT)^\dagger|in >= \\
&-\sum_s (-1)^{n_{loops}^s + n_{anti}^s + n_{anti}^{out}} < s|(iT)|in >< out|(iT)^\dagger|s >,
\end{aligned}$$

where $n_{anti}^{out}, n_{anti}^{in}$ and n_{anti}^s are the amount of anti-particles in the incoming $|in >$, outgoing $< out|$ and intermediate $|s >$ state and n_{loops}^s refers to the amount of loops cut in the Cutkosky diagrams. For unitarity to be true one would have to either check that the minus signs in the above equation cancel, or that those diagrams with a minus sign that do not cancel, do not contribute to the above equation. Using the fact that an extra minus sign is given for changing two fermionic lines and for loops, unitarity is then proved.

Spin 1

Consider the spin 1 photons in Quantum Electrodynamics (**QED**). The matrix K for the photon is a tensor, which in the Lorentz gauge was found above (2.22) to be equal to $K(k) = \eta_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}$. Taking $\xi = 1$, the matrix residue becomes $K(k) = diag(-1, 1, 1, 1)$, with one negative eigenvalue. Noting that the sources are four-vectors, we might as well calculate in the rest system $k = (k_0, 0, 0, k_0)$. The eigenvectors of $K(k) = diag(-1, 1, 1, 1)$ are obviously $J^a_\mu = \delta^a_\mu$; for arbitrary k one should perform a lorentz transformation. An extra requirement for the sources is that they satisfy $k_\mu J^\mu = 0$, these are called gauge-invariant sources since they apply for all ξ. The gauge invariant sources are $J^1_\mu(k)$ and $J^2_\mu(k)$, which we suspect to be the physical sources. Using (2.47), (2.48) gives for the physical sources: $J^a = (J^{in,S})^a = (J^{out,S})^a = (J^{in,S^\dagger})^a = (J^{out,S^\dagger})^a$, with $a = 1, 2$. The sources of S and $S^\dagger$ are equal, in other words only (2.49) should be checked:

$$\sum_{a=1,2} J^a_\mu J^a_\nu = \eta_{\mu\nu} - \frac{k_\mu z_\nu + z_\mu k_\nu}{k.z}. \tag{2.53}$$

Unitarity implies that the right hand side of (2.53) is equal to $K^{-1}_{\mu\nu} = \eta_{\mu\nu}$, which means that the terms linear in k_μ should cancel. This cancellation follows from the Slavnov-Taylor identities which are: $k_\mu \mathcal{M}^\mu = 0$ for every scattering amplitude $\mathcal{M}$. This proves unitarity in a two-dimensional physical Hilbert state space.

Summary

In this chapter it was shown how one can derive a local gauge theory starting from a global gauge theory. Local gauge theories have been crucial to our understanding of nature, for example almost all QFT's nowadays are gauge theories. After gauge fixing, the lagrangian was left with a BRST symmetry, which led to the Slavnov-Taylor identities. Lastly, we studied the implications of unitarity, which is crucial for a theory to make sense. To prove unitarity one normally uses the Slavnov-Taylor identities, where also ghost particles play a role. Hence, one correctly expects that the sum over the intermediate states in the unitarity equation, also should include the ghost particles. These ghost particles decoupled in QED and consequently they were not summed over in the above spin 1 example. In the next chapter we are going to take a look at the so-called Higgs mechanism, which will be the main subject for the rest of this paper.

Chapter 3

Symmetry Breaking

A symmetry can be *exact*, *approximate*, or *broken.* Exact means unconditionally valid; approximate means valid under certain conditions; broken can mean different things, depending on the object considered and its context. The latter is going to be the focus of this chapter. Generally, the breaking of a certain symmetry does not imply that no symmetry is present, but rather that the situation where this symmetry is broken is characterized by a lower symmetry than the situation where this symmetry is not broken. In group-theoretic terms, this means that the initial symmetry group is broken to one of its subgroups. It is therefore possible to describe symmetry breaking in terms of relations between transformation groups, in particular between a group (the unbroken symmetry group) and its subgroup(s). Thus when considering symmetry breaking one encounters such questions as "which subgroups can occur" and "when does a given subgroup occur". Symmetry breaking was first explicitly studied in physics with respect to physical objects and phenomena. However, it is symmetry breaking of the laws that has come to have greater significance in physics. There are two different types of symmetry breaking of the laws: explicit and spontaneous.

Explicit symmetry breaking indicates a situation where the dynamical equations are not manifestly invariant under the symmetry group considered. This means, that the dynamical equations contain one or more terms explicitly breaking the symmetry. Such terms in the equations can have different origins, among others:
(a) From terms introduced in the lagrangian of the theory by hand on the basis of theoretical/experimental results, as in the case of the QFT of the weak interactions, which is expressly constructed in a way that manifestly violates parity.
(b) The terms may appear in the dynamical equations because of quantum-mechanical effects, this is then called an *anomaly.* An anomalous symmetry in a quantum theory is a symmetry of the action, but not of the measure.

Spontaneous symmetry breaking (**SSB**) occurs in a situation where, given a symmetry of the equations of motion, *solutions* exist which are not invariant under the action of this symmetry without any explicit asymmetric input. Hence, one speaks of SSB when the fundamental laws of nature are symmetric while at the same time the physical world (a solution of the dynamical equations) appears to us to be asymmetric. From a physical point of view SSB is thus very interesting. In this chapter we will closely follow the global ideas set out in [Ryd96] and [Zee03].

3.1 Spontaneous Symmetry Breaking

Historically, the concept of SSB first emerged in condensed matter physics. The prototype case is the 1928 Heisenberg theory of the ferromagnet as an infinite array of interacting spin 1/2 magnetic dipoles, such that neighbouring dipoles tend to align. Although the theory *is* rotationally invariant, below the critical Curie temperature T_c the actual ground state of the ferromagnet has the spin all aligned in some particular direction, thus *not* respecting the rotational symmetry. What happens is that below T_c there exists an infinitely degenerate set of ground states, in each of which the spins are all aligned in a given direction. A complete set of quantum states can be built upon each ground state. We thus have many different "possible worlds" (sets of solutions to the same equations), each one built on one of the possible ground states. A little man living inside one of these possible asymmetric worlds would have a hard time detecting the rotational symmetry of the laws of nature (all his experiments being under the effect of the background magnetic field). The symmetry is still there - the Hamiltonian being rotationally invariant - but *hidden* to the little man. The same picture can be generalized to QFT, the lowest energy state being the vacuum state, and the role of the little man being played by ourselves. This means that there may exist symmetries of the laws of nature which are not manifest to us because the physical world in which we live is built on a vacuum state which is not invariant under them. In substance SSB is the following: when some parameter (e.g. temperature) reaches a critical value, the lowest energy solution respecting the symmetry of the theory ceases to be stable under small perturbations and new asymmetric (but stable) lower energy solutions appear. The consequences of SSB on the physical states will be addressed in the rest of this chapter.

3.1.1 Goldstone Theorem

Consider the global gauge invariant lagrangian for *real* scalars from chapter 2.1:

$$\mathcal{L}^{\text{global inv}} = -(\partial_\mu \phi)^t (\partial^\mu \phi) - V(\phi) \; ;$$
$$\phi(x) = (\phi_1(x), \ldots, \phi_n(x)) \in \mathbb{R}^n.$$

The value assumed by an operator O in the ground state is known as the *vacuum expectation value* (**VEV**) $\langle O \rangle$ of O. It may be shown that the VEV[1] $\langle \phi \rangle$ is approximately the lowest energy solution of the potential $V(\phi)$:

$$\left(\frac{\partial V}{\partial \phi_a} \right)_{\phi = \langle \phi \rangle} = 0, \quad a = 1, \ldots, n. \tag{3.1}$$

Since V is gauge invariant, the lowest energy solutions will form a set $\{U\langle\phi\rangle : U \in G\}$. The lowest energy solution is normally equal to zero, hence $\{U\langle\phi\rangle : U \in G\} = \{0\}$. On the other hand, if $\langle \phi \rangle \neq 0$ and non-invariant (under G), there will be a degeneracy (possible worlds) of distinct asymmetric solutions of identical (lowest) energy. Nature chooses one of these solutions for $\langle \phi \rangle$, consequently we say that the symmetry group G is *spontaneously broken* by the non-invariant VEV[2].

[1] The precise definition of the VEV of a field will be defined in section 3.2.1. For now (3.1) suffices.

[2] Only scalars can get a non-zero VEV in a non-broken Lorentz invariant theory. If the Lagrangian has an internal symmetry which is precisely the inverse Lorentz transformation of a specific field (any tensor quantity), the Lorentz invariance will not broken if that tensor field gets a non-zero VEV. The apparent Lorentz invariance of the vacuum after SSB manifests itself as the combined internal and Lorentz symmetry.

To be precise, we will take the following definition of SSB:

Spontaneous Symmetry Breaking: The vacuum state is not invariant under the gauge group, apart from a possible remaining invariance under a subgroup. In other words the charge corresponding to the gauge symmetry does not satisfy $Q^a|0\rangle = 0, a = 1, \ldots, |G|$ anymore.

Let us assume (by for example tuning the parameters of the potential) that there is a non-zero lowest energy solution $\langle\phi\rangle \neq 0$ which is not invariant under the whole gauge group G, this implies in particular that $Q^a|0\rangle = 0$ cannot be true for all[3] $a = 1, \ldots, |G|$. The VEV will thus be invariant under a *smaller* subgroup $H \lhd G$, generated by (a subset of linear combinations of the original generators) T_a^H. The generators which do not leave $\langle\phi\rangle$ invariant will be denoted by $T_a^{G/H}$:

$$\begin{aligned} T_a^H\langle\phi\rangle &= 0, \quad a = 1, \ldots, |H| && (3.2) \\ T_a^{G/H}\langle\phi\rangle &\neq 0, \quad a = 1, \ldots, |G/H|. && (3.3) \end{aligned}$$

Since the vectors $T_a^{G/H}\langle\phi\rangle$ are linearly independent[4], one may form a basis by adding vectors v^a if necessary: $\mathbb{C}^n = \text{span}\{T_1^{G/H}\langle\phi\rangle, \ldots, T_{|G/H|}^{G/H}\langle\phi\rangle, v^1, \ldots, v^{n-|G/H|}\}$. The actual quantum fields $\tilde{\phi}(x)$ are the fluctuations around the VEV, which can be written in the previously mentioned basis: $\phi(x) = \langle\phi\rangle + \tilde{\phi}(x) =: \langle\phi\rangle + (T_a^{G/H}\langle\phi\rangle)\phi_a^{G/H}(x) + v^a\phi_a^H(x)$. Expanding $V(\phi)$ about its minimum $\langle\phi\rangle$ gives:

$$V(\phi) = V(\langle\phi\rangle) + \frac{1}{2}M_{ij}\tilde{\phi}_i(x)\tilde{\phi}_j(x) + O(\tilde{\phi}^3). \tag{3.4}$$

The mass matrix equals:

$$M_{ij} := \left(\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right)_{\phi=\langle\phi\rangle}.$$

Since $V(\langle\phi\rangle)$ is the minimum, after diagonalizing M_{ij} it must have either positive or zero mass entries. To find out for which fields it is zero, we do a group transformation. The invariance of V under the gauge transformation (2.1), gives:

$$V(\langle\phi\rangle) = V(U\langle\phi\rangle) = V(\langle\phi\rangle) + \frac{1}{2}M_{ij}\delta\langle\phi\rangle_i\delta\langle\phi\rangle_j + O((\delta\langle\phi\rangle)^3), \tag{3.5}$$

where $\delta\langle\phi\rangle_a = \sum_b \theta^b(T_b\langle\phi\rangle)_a$ is the variation in $\langle\phi\rangle_a$ under the gauge transformation. From (3.5) one then finds:

$$(T_a\langle\phi\rangle)_i M_{ij}(T_b\langle\phi\rangle)_j = 0, \quad a, b = 1, \ldots, |G|. \tag{3.6}$$

If the group transformation U belongs to the subgroup H generated by (3.2), then $\delta\langle\phi\rangle_a = 0$, so (3.6) is trivially satisfied. If however U does not belong to H but the broken transformations generated by (3.3), then (3.6) is an actual restriction on the mass terms:

$$(T_a^{G/H}\langle\phi\rangle)_i M_{ij}(T_b^{G/H}\langle\phi\rangle)_j = 0, \quad a, b = 1, \ldots, |G/H|. \tag{3.7}$$

[3]Otherwise $\langle\phi\rangle = \langle 0|e^{Q^a\theta_a}\phi(x)e^{-Q^a\theta_a}|0\rangle$ is invariant under the whole group G. The VEV is thus what one calls an off-shell (not physical) quantity/observable.

[4]A linear dependence would imply that a linear combination of those generators is an element in the Lie algebra of H.

Since M_{ij} is a symmetric semi-positive definite matrix, (3.7) leads to $M_{ij}(T_b^{G/H}\langle\phi\rangle)_j = 0$, $b = 1, \ldots, |G/H|$. Hence (3.4) becomes:

$$V(\phi) = V(\langle\phi\rangle) + \frac{1}{2}(v_i^a M_{ij} v_j^b)\phi_a^H(x)\phi_b^H(x) + O(\tilde{\phi}^3). \tag{3.8}$$

The scalar fields $\phi_a^{G/H}(x), a = 1, \ldots, |G/H|$ have missing mass terms in the above potential (3.8). These massless scalar particles are known as *Goldstone bosons* and appear because of SSB. This is a special case of the following theorem where the scalar fields do not have to be fundamental, first stated by J. Goldstone [Gol61], [GSW62]:

Theorem 3.1 (Goldstone Theorem) *New massless scalar particles appear in the spectrum of possible excitations if a gauge symmetry is spontaneously broken. There is one massless scalar particle for each generator of the symmetry that is broken.*

For the Goldstone theorem one only assumes that the vacuum is not invariant under the conserved charge Q, which generates the gauge transformation in the operator formalism: $[Q_a, \phi_b] = (T_a\phi)_b$. The above invariant scalar lagrangian was an example of this, since a VEV $\langle\phi\rangle \neq 0$ which is not G-invariant implies $Q|0> \neq 0$; in this case the charge Q was given in (2.4). In other words, SSB and the Goldstone particles are in practice a result of a non-singlet scalar field (not necessarily fundamental) having a non-zero VEV[5]. However, after SSB one still has a conserved current and time-independent charge.

Example Consider the example of a $U(1)$ gauge group of a one-component complex scalar field $\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$. Take the following so-called Mexican hat potential:

$$V(\phi) = m^2(T)\phi^*\phi + \frac{1}{2}\lambda(\phi^*\phi)^2. \tag{3.9}$$

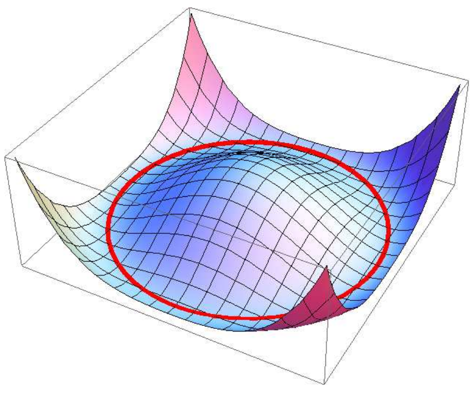


Figure 3.1: The Mexican Hat potential

The parameter $m^2(T)$ is dependent on the temperature T. If $m^2(T) \geq 0$ one finds the solution $\langle\phi\rangle = 0$. On the other hand , if we let $m^2(T)$ become negative (by changing the temperature

[5]The Goldstone is a scalar particle in most examples, except in SSB of supersymmetry where you get a fermion called the goldstino.

T) we get non-zero lowest energy solutions $\langle\phi\rangle = \sqrt{\frac{-m^2(T)}{\lambda}} e^{i\alpha}, \alpha \in [0, 2\pi[$. By choosing one vacuum solution, let's say $\alpha = 0$, we break the $U(1)$ symmetry spontaneously. The potential written in terms of the quantum fluctuations $\tilde{\phi}(x) = \frac{1}{\sqrt{2}}(\tilde{\phi}_1(x) + i\tilde{\phi}_2(x)) = \phi(x) - \sqrt{\frac{-m^2(T)}{\lambda}}$ is:

$$V(\phi) = \lambda F \tilde{\phi}_1^2 + \frac{\lambda F}{\sqrt{2}} \tilde{\phi}_1 (\tilde{\phi}_1^2 + \tilde{\phi}_2^2) + \frac{\lambda}{8} (\tilde{\phi}_1^2 + \tilde{\phi}_2^2)^2.$$

The field $\tilde{\phi}_2$ is the massless Goldstone boson in this case.

3.1.2 Higgs Mechanism

In the "proof" of the Goldstone theorem given above we did not explicitly use the fact that the gauge was global, thus one may think that the Goldstone theorem is also applicable to *local* gauge theories, leading to massless Goldstone particles. But actually, in the local gauge case the Goldstone particles are not physical, since the Goldstone particles may be locally varied without changing the energy of the state.
Let us now consider the above arguments for the following locally gauge invariant lagrangian:

$$\begin{aligned} \mathcal{L}^{\text{local inv}} &= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - (D_\mu \phi)^t (D^\mu \phi) - V(\phi) \ ; \\ D_\mu \phi :&= \partial_\mu \phi - g A_\mu \phi := \partial_\mu \phi - g A^a_\mu T_a \phi \ ; \\ \phi(x) &= (\phi_1(x), \ldots, \phi_n(x)) \in \mathbb{R}^n \ ; \\ (T_a)^t &= -T_a \in \mathrm{Mat}(n, \mathbb{R}). \end{aligned} \tag{3.10}$$

Again, we assume that the potential has a non-invariant minimum solution (3.1). The VEV will thus be non-invariant and according to the above definition we say that SSB has occurred. Since the Goldstone particles are not physical one may then take the unitary gauge (2.17), and thus gauge the Goldstone particles to zero[6].

But where did the degrees of freedom of the Goldstone particles go? Let us examine the gauge fields more closely. We again assume that the VEV is left invariant by generators $T^H_a, a = 1, \ldots, |H|$ (3.2) of the gauge group G and not invariant under the generators $T^{G/H}_a, a = 1, \ldots, |G/H|$ (3.3). Writing $\phi(x) = \langle\phi\rangle + \tilde{\phi}(x)$ and $A_\mu = T^H_a (A^H)^a_\mu + T^{G/H}_a (A^{G/H})^a_\mu$, one gets for the kinetic term of the scalar fields:

$$\begin{aligned} -(D_\mu \phi)^t (D^\mu \phi) &= -(D_\mu \langle\phi\rangle)^t (D^\mu \langle\phi\rangle) + \text{“}(A_\mu \tilde{\phi}) \text{ interactions”} \\ &= g^2 (A^{G/H})^{a\mu} (\langle\phi\rangle^t T^{G/H}_a T^{G/H}_b \langle\phi\rangle)(A^{G/H})^b_\mu + \text{interactions} \\ &= -g^2 (A^{G/H})^{a\mu} (\mu)_{ab} (A^{G/H})^b_\mu + \text{interactions}. \end{aligned}$$

The mass matrix is a semi-positive definite symmetric matrix:

$$(\mu)_{ab} := -\langle\phi\rangle^t T^{G/H}_a T^{G/H}_b \langle\phi\rangle = (T^{G/H}_a \langle\phi\rangle)^t (T^{G/H}_b \langle\phi\rangle), \tag{3.11}$$

hence after diagonalizing $(\mu)_{ab}$, it must have either positive or zero mass entries. The vectors $T^{G/H}_a \langle\phi\rangle$ are independent, in other words $det((\mu)_{ab}) = det((T^{G/H}_i \langle\phi\rangle)_j)^2 \neq 0$ and hence one

[6] There still might be a residual invariance present after gauging the Goldstones to zero, such an example will be encountered in the next chapter.

finds that the masses are all positive. The $|H|$ gauge fields A_μ^H remain massless, while all the $|G/H|$ gauge fields $A_\mu^{G/H}$ get a non-zero mass. The G/H Goldstone bosons are "eaten up" to give mass to the gauge bosons, and this happens without explicitly breaking the gauge invariance of the theory. The G/H degrees of freedom of the Goldstones thus become the G/H extra longitudinal degrees of freedom of the G/H massive gauge fields.

The above is a special case of the so-called *Higgs mechanism.* According to the Higgs mechanism, established in a general way in 1964 ([EB64], [Hig64], [GHK64], [Hig66], [Kib67]), in the case that the currents associated with the broken generators of the symmetry are coupled to gauge bosons, the Goldstone bosons "disappear" and the gauge bosons acquire a mass. The main reason for the failure of the Goldstone Theorem in the local case, lies in the fact that there are no time-independent charges that satisfy $[Q^c, \phi_a] = T^c_{ab}\phi_b$. Actually, in the case of local gauge breaking one should not even speak of SSB, since the physical states are *always* invariant under the local gauge symmetry. In spite of this we will still speak of SSB if a non-singlet field has a non-invariant VEV, even in the case of a local gauge.

Example Consider the example of a $U(1)$ local gauge group of a one-component complex scalar field $\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$. Take again the Mexican Hat potential:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\phi)^\dagger(D^\mu\phi) - m^2\phi^*\phi - \frac{1}{2}\lambda(\phi^*\phi)^2. \tag{3.12}$$

In the case $m^2 < 0$, the vacuum is at $|\langle\phi\rangle| = \sqrt{\frac{-m^2}{\lambda}} =: a$. By choosing one vacuum solution, let's say $\langle\phi\rangle = a$, we break the $U(1)$ symmetry spontaneously. The lagrangian (3.10) written in terms of the quantum fluctuations $\tilde{\phi}(x) = \frac{1}{\sqrt{2}}(\tilde{\phi_1}(x) + i\tilde{\phi_2}(x)) = \phi(x) - a$ is:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - g^2a^2A_\mu A^\mu - \frac{1}{2}(\partial_\mu\tilde{\phi_1})^2 - \frac{1}{2}(\partial_\mu\tilde{\phi_2})^2 - \lambda a^2\tilde{\phi}_1^2 - \sqrt{2}gaA^\mu\partial_\mu\tilde{\phi_2} + \text{cubic+quartic terms.}$$

Hence, the gauge particle gets a mass ga. The $\tilde{\phi_2}$ field is the unphysical field and may be gauged to zero. The left over massive real scalar field $\tilde{\phi_1}$ is called the Higgs boson.

To conclude, the Higgs mechanism follows from a non-singlet scalar field (not necessarily fundamental) having a non-invariant VEV[7]. In theories such as the unified model of electroweak interactions, the SSB follows from the symmetry-violating VEV of scalar fields (the so-called Higgs fields) that are introduced ad hoc in the theory. Note that this mechanism for the mass generation for the gauge fields is also what ensures the renormalizability of theories involving massive gauge fields, such as the Glashow-Weinberg-Salam electroweak theory. The ad hoc character of these scalar fields, for which there is no experimental evidence (no "Higgs particle" has been observed up to now), has drawn increasing attention to the possibility that the Higgs fields could be phenomenological rather than fundamental, that is bound states resulting from a specified dynamical mechanism. SSB realized in this way has been called Dynamical Symmetry Breaking (**DSB**).

[7] If the the gauge particle associated to the symmetry itself has a non-invariant VEV, the Higgs mechanism will not follow. For example in the case of gravity the VEV of the metric is Minkowski but there are no massive particles.

3.2 Effective Action

The phenomenon of SSB was based on a scalar field having a non-invariant VEV, which minimized the classical potential energy $V(\phi)$ (3.1). Consider again the example of the Mexican hat potential (3.9) with $m^2 = 0$, i.e. $V(\phi) = \frac{1}{2}\lambda(\phi^*\phi)^2$. The potential is minimized at $\phi = 0$. According to (3.1) $\langle\phi\rangle = 0$, which leads to the conclusion that there is no SSB. However the $m^2 = 0$ theory is posed on the edge of symmetry breaking, and one would, correctly, guess that quantum fluctuations would break the symmetry. In other words, the VEV will be non-zero (and hence non-invariant) due to quantum fluctuations. To show this we should give the precise definition of the VEV.

3.2.1 Vacuum Expectation Value

Consider a QFT defined by:

$$Z[J] = e^{iW[J]} = \int \mathcal{D}\psi e^{i(S+\int J\psi)},$$

where $\psi = (\phi, A_\mu, \dots)$ stands for all the fields in the lagrangian and $J = (J_\phi, J_{A_\mu}, \dots)$ the corresponding sources. By differentiating W with respect to the sources J, one can obtain any connected Green's function and hence any scattering amplitude. In particular:

$$\psi_a^c(x) := \frac{\delta W}{\delta J_a(x)} = \frac{\langle 0^+|\psi_a(x)|0^-\rangle_J}{\langle 0^+|0^-\rangle_J} = \frac{1}{Z}\int \mathcal{D}\psi e^{i(S+\int J\psi)}\psi_a(x). \tag{3.13}$$

The *vacuum expectation value* (**VEV**) of the field ψ_a is now defined as:

$$\langle\psi_a(x)\rangle := \lim_{J\to 0}\psi_a^c. \tag{3.14}$$

Given a functional W of J, one can perform a Legendre transformation to obtain a functional Γ of ψ^c:

$$\Gamma[\psi^c] := W[J] - \int d^4x J_a(x)\psi_a^c(x). \tag{3.15}$$

Γ is called the *effective action* and equals the generating functional of the vertex functions. The vertex functions are calculated by considering the so-called one-particle-irreducible (**1PI**) Feynman diagrams. The sources J in (3.15) are to be expressed in ψ^c by solving (3.13) for J. The functional derivative of Γ is:

$$\frac{\delta\Gamma[\psi^c]}{\delta\psi_a^c(x)} = -J_a(x). \tag{3.16}$$

Letting $J \to 0$ in (3.16) and using the definition (3.14) leads to:

$$\left.\frac{\delta\Gamma[\psi^c]}{\delta\psi_a^c(x)}\right|_{\langle\psi(x)\rangle} = 0. \tag{3.17}$$

The VEV is thus the value of the fields which actually minimizes the *effective* action $\Gamma[\psi^c]$, instead of the *classical* action $S[\psi]$ (3.1). The effective action is thus the actual action which one needs to work with when considering SSB. For example, the Green's functions can be calculated using the effective action, but then only tree diagrams should be taken into account. In the next section it will be shown that the $m^2 = 0$ theory discussed in the introduction of this section generally leads to an effective action which induces a non-zero VEV, with SSB (and it's consequences such as Goldstone particles or mass-generation) as a result.

3.2.2 Effective Potential

The notion of effective action enables us to view SSB in the same way as theories with unbroken symmetry. First, notice that only scalar fields can have a non-zero VEV in a non-broken Lorentz invariant theory, thus we expand the effective action $\Gamma[\psi^c]$ in terms of the scalar fields and set the other (spinor, vector, tensor etc.) fields to zero:

$$\Gamma[\psi^c = (\phi^c(x), 0, 0, \dots)] =: \int d^4x[-V^{\text{eff}}(\phi^c(x)) + \frac{1}{2}Z(\phi^c(x))(\partial_\mu \phi^c(x))^2 + \dots], \tag{3.18}$$

where the dots refer to higher derivatives in the scalar fields. The VEV is not necessarily space-time independent in theories which include gravity. However, QFT's without gravity generally have $\langle\phi\rangle$ =constant[8], hence we find:

$$\lim_{J\to 0} \Gamma[\psi^c = (\phi^c, 0, 0, \dots)] = -(\int d^4x) V^{\text{eff}}(\langle\phi\rangle). \tag{3.19}$$

The function V^{eff} is called the *effective potential.* The condition (3.17) on the VEV becomes:

$$\left.\frac{\partial V^{\text{eff}}}{\partial \phi^c_a}\right|_{\langle\phi\rangle} = 0. \tag{3.20}$$

To investigate SSB, one should calculate the effective potential and then derive the minimal points (compare with (3.1)). The ground state VEV is then the value which the scalar fields take at the global minimum of the effective potential[9]. The effective potential is the generating function for the vertex functions with all external momenta set to zero. It may be calculated in a loop expansion $V^{\text{eff}} = V + \hbar V_1 + \hbar^2 V_2 + \dots$, where V_n are the 1PI Feynman diagrams made up of n loops with zero external momenta. In addition, when calculating V_n one only considers the 1PI Feynman diagrams with scalar particles as external sources, which are set equal to ϕ^c.

Example Take again the $U(1)$ local gauge $m^2 = 0$ theory from (3.12) written out in terms of the real fields $\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{2}(\partial_\mu\phi_1 - eA_\mu\phi_2)^2 + \frac{1}{2}(\partial_\mu\phi_2 + eA_\mu\phi_1)^2 - \frac{1}{4!}\lambda(\phi_1^2 + \phi_2^2)^2,$$

where we took the Lorentz gauge (2.20). In this gauge the effective action will depend on $(\phi^c)^2 = (\phi^c_1)^2 + (\phi^c_2)^2$. Renormalization conditions for the vertex functions may be stated in terms of the derivatives of the effective potential:

$$\left.\frac{d^2V^{\text{eff}}(\phi^c)}{d(\phi^c)^2}\right|_{\phi^c=0} = 0 \;;$$
$$\left.\frac{d^4V^{\text{eff}}(\phi^c)}{d(\phi^c)^4}\right|_{\phi^c=M} = \lambda,$$

[8] In gauge theories, the effective action and thus also the VEV are gauge dependent ([Jac73]). Physical observables, such as the mass, scattering elements and identities (cross sections, lifetimes etc.) derived from them are gauge independent. To work with the effective potential one should then choose a wise gauge fixing condition, such that $\langle\phi\rangle$ =constant. The unitary, R_ξ and Lorentz gauges are a few of many good gauges with $\langle\phi\rangle$ =constant [Nie75], [AF84], [FK76].

[9] The effective potential is gauge dependent, however the value of the effective action *at* the stationary points are gauge independent, hence the procedure for finding the stationary points are gauge independent [FN75], [FB75].

where M is an arbitrary mass parameter. These conditions put constraints on the counterterms $\frac{1}{2}B(\phi^c)^2 + \frac{1}{4!}C(\phi^c)^4$ that are added to the effective potential. The form of the effective action in the Landau gauge (Lorentz gauge (2.20) with gauge parameter $\xi = 0$) can then be shown [CW73] to be as follows:

$$V(\phi^c) = \frac{\lambda}{4!}(\phi^c)^4 + (\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2})(\phi^c)^4(\log\frac{(\phi^c)^2}{M^2} - \frac{25}{6}) + \mathcal{O}(\lambda^3).$$

This potential clearly has a non-zero global minimum, hence we say that SSB has taken place caused by *radiative corrections*. These are the quantum effects that were mentioned at the beginning of this section.

Summary

In this chapter Spontaneous Symmetry Breaking was discussed. SSB is achieved in practice by a scalar field having a non-invariant VEV. The first section treated the consequence of SSB: the Goldstone theorem, which states that there is one massless particle for each broken generator of a gauge group. The presence of these massless scalar particles was first seen as a serious problem since no particles of the sort had been observed in the context considered. The answer to this problem was the so-called Higgs mechanism, which stated that if the broken generators correspond to a local gauge group, then the massless Goldstone bosons are "eaten" by the corresponding massless gauge bosons, which in turn become massive. This was also the solution of another similar problem, that is the fact that the 1954 Yang-Mills theory for the (Electroweak) $SU(2) \times U(1)$ gauge group predicted unobservable massless gauge bosons. Lastly it was shown that SSB can be a purely quantum effect, since the VEV of a scalar field is the extremal point of the quantum effective action, instead of the classical action. In the next chapter we are going to take a look at the Higgs mechanism applied to the theory of gravity.

Chapter 4

From Gravity to a Massive Spin-2 Theory

In recent years there has been renewed interest in the possibility of giving a mass to the graviton. This idea belongs to a broader class of proposals for modifying gravity at large distances. These models could be phenomenologically relevant as possible alternatives to dark matter and dark energy. In this chapter our focus will be on another interesting motivation for giving a mass to the graviton, namely QCD. If QCD is to be described by a string theory which contains a massless graviton particle in its spectrum, then the graviton would somehow have to acquire mass (for example to describe massive spin 2 glueball states in QCD). Gravitational Higgs mechanism is one plausible way of achieving this. The gravitational Higgs mechanism is the spin-2 analog of the spin-1 Higgs mechanism studied in the previous chapter.

The idea of using a string-like description for QCD, a picture in which massive string-like hadrons interact in Minkowski space-time, has been known for a long time. In [GT91] the question was raised if the methods of string theory may be used for describing QCD. It was suggested that such a theory might be viewed as a gravitational theory in which the graviton becomes massive, through a Higgs mechanism. In that case the spin-2 glueball states may be considered as the massive gravitons in the string theory approach. Unfortunately, the question of how this would specifically happen was left open. The suggestions of [GT91] were then further elaborated in an article by W. Siegel [Sie94]. In the article by Siegel a closed form was found for the unbroken lagrangian, which included a graviton field and scalars. These scalars developed space-time dependent VEV's, which broke the space-time diffeomorphism symmetry. As a result, the broken lagrangian had a mass term for the graviton after fixing the scalar fluctuations to zero. However, Siegel did not explicitly study the unitarity of the resulting lagrangian.

Recently, the question of finding a unitary lagrangian describing massive gravity, through SSB, was brought up by G. 't Hooft [tH08]. In 't Hooft's article, the QFT theoretic analogy of a string-like approach to QCD was studied. In particular, it was asked how the string-like description would basically work, as seen from a normal QFT (point-like) description. In the first section of this chapter we will follow the thought and reasoning from 't Hooft. Afterwards, we will study a revised model by Zurab [Kak08b]. In the last section, we will look at the problem of a string-like description of QCD from an effective field theory perspective.

4.1 Higgs Mechanism for Gravity in a Minkowski Background

Before starting, let us establish our notation. At the beginning, the space-time dimension D will be kept arbitrary. The Einstein-Hilbert action in D space-time dimensions is:

$$S = M_P^{D-2} \int (\mathcal{L}_{EH} + \mathcal{L}_\Lambda + \mathcal{L}_M) d^D x \ ;$$
$$\mathcal{L}_{EH} + \mathcal{L}_\Lambda := \sqrt{-g}(R - \Lambda).$$

R is the Ricci scalar curvature, g is the determinant of the metric tensor $g_{\mu\nu}$ and $M_P = \frac{1}{\sqrt{16\pi G}}$. Λ is a possible cosmological constant and $\mathcal{L}_M$ denotes the lagrangian of the matter fields. We have the usual definitions:

$$R = g^{\alpha\nu} R^\mu_{\alpha\mu\nu} \ ;$$
$$R^\lambda_{\alpha\mu\nu} = \partial_\mu \Gamma^\lambda_{\alpha\nu} - \partial_\nu \Gamma^\lambda_{\alpha\nu} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\alpha\nu} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\alpha\mu} \ ;$$
$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}).$$

For the metric one writes:

$$g_{\mu\nu} = g_{0\mu\nu} + h_{\mu\nu}, \tag{4.1}$$

where $g_{0\mu\nu}$ is taken to be the background metric. In the following two sections all calculations will be around flat Minkowski space-time, i.e. $g_{0\mu\nu} = \eta_{\mu\nu}$. The inverse of the metric is:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^\mu_\alpha h^{\nu\alpha} + \mathcal{O}(h^3).$$

Note that the terms consisting of $h_{\mu\nu}$ are always raised and lowered with $g_{0\mu\nu} = \eta_{\mu\nu}$. Expanding the Einstein-Hilbert action around flat space-time gives:

$$\begin{aligned} \mathcal{L}_{EH} &= \frac{1}{8}(\partial_\mu h^\alpha_\alpha)^2 - \frac{1}{4}(\partial_\mu h_{\alpha\beta})^2 + \frac{1}{2}(\partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h^\lambda_\lambda)^2 + \mathcal{O}(h^3) \ ; \\ \mathcal{L}_\Lambda &= -\Lambda\sqrt{-g} \\ &= -\Lambda(1 + \frac{1}{2}h^\alpha_\alpha - \frac{1}{4}(h_{\mu\alpha})^2 + \frac{1}{8}(h^\alpha_\alpha)^2 + \mathcal{O}(h^3)). \end{aligned} \tag{4.2}$$

The Einstein-Hilbert lagrangian has a general space-time diffeomorphism symmetry $x \to x(\tilde{x})$:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x). \tag{4.3}$$

In particular, under the following coordinate transformation:

$$x^\mu = \tilde{x}^\mu + \xi^\mu(\tilde{x}), \tag{4.4}$$

the metric tensor in the new coordinate system becomes:

$$\tilde{h}_{\mu\nu}(\tilde{x}) = h_{\mu\nu}(\tilde{x}) + \partial_\nu\xi_\mu(\tilde{x}) + \partial_\mu\xi_\nu(\tilde{x}) + (\xi^\alpha\partial_\alpha h_{\mu\nu} + h_{\alpha\nu}\partial_\mu\xi^\alpha + h_{\mu\alpha}\partial_\nu\xi^\alpha)(\tilde{x}) + \mathcal{O}(\xi^2). \tag{4.5}$$

In other words the space-time symmetry can, and will be interpreted as a gauge symmetry. Similarly, scalars will transform as follows under the gauge transformation:

$$\tilde{\phi}(\tilde{x}) = \phi(\tilde{x}) + \xi^\lambda\partial_\lambda\phi(\tilde{x}) + \mathcal{O}(\xi^2). \tag{4.6}$$

As became clear in chapter 2, one should always impose gauge conditions. The D-dimensional gauge freedom in choosing $h_{\mu\nu}$ in (4.5) allows us to impose for example the so-called harmonic gauge condition:

$$F_\nu[h] = \partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h^\lambda_\lambda.$$

The function F changes under the gauge transformation (4.5) as follows:

$$\delta F_\nu[h] = \partial^2 \xi_\nu + \text{(ghost-interactions)}.$$

Together with (2.15), the effective lagrangian becomes:

$$\begin{aligned}\mathcal{L}^{\text{eff}} &= \frac{1}{8}(\partial_\mu h^\alpha_\alpha)^2 - \frac{1}{4}(\partial_\mu h_{\alpha\beta})^2 + \mathcal{O}(h^3) + \mathcal{L}_{\text{FP}} \ ; \\ \mathcal{L}_{\text{FP}} &= -\overline{\eta}_\mu \partial^2 \eta^\mu + \text{(ghost-interactions)}.\end{aligned}$$

In conclusion, the graviton propagator in the harmonic gauge around flat space-time is:

$$D_{\mu\nu,\lambda\sigma}(k) = \frac{1}{2}\frac{\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\sigma}}{k^2 + i\epsilon}. \tag{4.7}$$

4.1.1 Gravitational Higgs Model

Now the matter lagrangian $\mathcal{L}_M$ will be added into the mix. Our goal is to find a matter lagrangian, which after SSB leaves a mass term for the graviton. The SSB may be achieved in a variety of ways, including via a tensor VEV or a scalar VEV. Also, since we are looking for a *Poincaré-invariant* model for QCD with a massive spin-2 graviton, the lagrangian should have the Poincaré symmetry as a residual symmetry after SSB. This implies in particular that the background metric[1] should be Minkowski, hence the linear term in the expansion of the lagrangian around the Minkowski metric should vanish. In other words, the term $-\Lambda\frac{1}{2}h^\alpha_\alpha$ from (4.2), should cancel the linear (in the metric) term coming from the matter lagrangian. These are all conditions on the matter lagrangian. The easiest way of achieving an SSB is via the simplest massless scalar matter lagrangian:

$$\begin{aligned}\mathcal{L}_M &= \mathcal{L}_\phi + \mathcal{L}_m \ ; \\ \mathcal{L}_\phi &:= -\frac{1}{2}\sqrt{-g}g^{\mu\nu}\eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b,\end{aligned} \tag{4.8}$$

where $a, b = 0, \ldots, D-1$ is an internal index and $\mathcal{L}_m$ denotes other matter fields. The term $\mathcal{L}_m$ will be left out for now.

One immediately finds a possible problem with unitarity when referring to the Conclusion (2.1). The residue matrix $K_{\mu\nu} = \eta_{\mu\nu} = diag(-1, 1, 1, \ldots, 1)$ of the scalars has one negative eigenvalue, namely the one corresponding to the scalar ϕ_0. But luckily the lagrangian is even in the scalar fields, hence in the above model this should pose no unitarity problems as long as the symmetry $\phi_0 \to -\phi_0$ remains unbroken. This implies that the VEV of the scalar ϕ_0 must remain zero while some (or all others) should get a space-time dependent VEV (in order to break the gauge symmetry). Unfortunately, in that case, the *Lorentz* symmetry (a subgroup of the residual Poincaré symmetry) does not remain unbroken, since the VEV of the scalar particles would not be treated in a Lorentz invariant way. Realizing that having the residual Poincaré symmetry

[1] Recall that the background metric is used for raising and lowering of indices.

seems more important than possibly breaking $\phi_0 \to -\phi_0$, we will close our eyes for the moment and look at the case where the VEV equals:

$$\begin{aligned} \langle \phi^a(x) \rangle &= m x^a, \quad a = 0, \ldots, D-1 \ ; \\ \langle g_{\mu\nu} \rangle &= \eta_{\mu\nu}. \end{aligned} \tag{4.9}$$

The above should be a minimum solution of the *classical* action: the linear terms in h^α_α and ϕ^a should vanish. Note that the VEV *must* depend on the coordinates, in order to break the coordinate symmetry. This in turn, implies that the linear term $-\frac{1}{2}\Lambda h^\alpha_\alpha$, and hence also Λ, should be non-zero[2], in order to cancel the linear term coming from the matter lagrangian. Also, because of the internal "Poincaré" global symmetry of the scalars in (4.8), one is assured of the residual Poincaré symmetry[3]. Plugging $\phi^a = \langle \phi^a \rangle(x) + \varphi^a(x)$ in (4.8) and using:

$$\begin{aligned} \sqrt{-g} &= 1 + \frac{1}{2} h^\alpha_\alpha - \frac{1}{4}(h_{\mu\alpha})^2 + \frac{1}{8}(h^\alpha_\alpha)^2 + \mathcal{O}(h^3) \ ; \\ g^{\mu\nu}\eta_{\mu\nu} &= D - h^\alpha_\alpha + (h_{\mu\nu})^2 + \mathcal{O}(h^3), \end{aligned} \tag{4.10}$$

gives:

$$\mathcal{L}_\phi = -\frac{1}{2} m^2 \sqrt{-g} g^{\mu\nu}\eta_{\mu\nu} + (\varphi h) - \text{interactions} =$$

$$-\frac{1}{2} m^2 (D + (\frac{1}{2}D - 1) h^\alpha_\alpha + (-\frac{1}{4}D + 1)(h_{\mu\alpha})^2 + (\frac{1}{8}D - \frac{1}{2})(h^\alpha_\alpha)^2) + \mathcal{O}(h^3) + (\varphi h) - \text{interactions}.$$

We will impose the unitary gauge $F^a[\phi] := \varphi^a = 0$, via the gauge transformations (4.6):

$$\tilde{\varphi}^a(\tilde{x}) = \varphi^a(\tilde{x}) + \xi^\lambda \partial_\lambda \phi^a(\tilde{x}) + \mathcal{O}(\xi^2) = \varphi^a(\tilde{x}) + m\xi^a(\tilde{x}) + \xi^\lambda \partial_\lambda \varphi(\tilde{x}) + \mathcal{O}(\xi^2) \ ; \tag{4.11}$$

$$\delta F^a[h] = \tilde{\varphi}^a(\tilde{x}) - \varphi^a(\tilde{x}) = m\xi^a(\tilde{x}) + \xi^\lambda \partial_\lambda \varphi(\tilde{x}) + \mathcal{O}(\xi^2). \tag{4.12}$$

This gives:

$$\mathcal{L}_{\text{FP}}|_{\varphi^a=0} = m\overline{\eta}_a \eta_a,$$

and thus the ghosts decouple (they also do not propagate at all). In the rest of this section we will take $D = 4$ for the QCD model. Referring to (2.18) gives the following effective lagrangian:

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{O}(h^3) \\ \mathcal{L}^{(0)} &= -(2m^2 + \Lambda) \ ; \\ \mathcal{L}^{(1)} &= -\frac{1}{2}(m^2 + \Lambda) h^\alpha_\alpha \ ; \\ \mathcal{L}^{(2)} &= -\frac{1}{4}\Lambda (h_{\mu\alpha})^2 + \frac{1}{8}\Lambda (h^\alpha_\alpha)^2 + \frac{1}{8}(\partial_\mu h^\alpha_\alpha)^2 - \frac{1}{4}(\partial_\mu h_{\alpha\beta})^2 + \frac{1}{2}(\partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h^\lambda_\lambda)^2. \end{aligned} \tag{4.13}$$

One concludes that the following should be taken for Λ, in order to cancel the linear term:

$$\Lambda = -m^2. \tag{4.14}$$

Let us now go to Fourier space. We want to read off the masses of the dynamical fields, for this it suffices to look at the effective lagrangian in the rotated direction $k_\mu = (k, 0, 0, 0)$. The lagrangian in the rest frame is:

$$\mathcal{L}_{\text{eff}} = -m^2 + (k^2 + m^2)(-\frac{1}{4}(h_{\alpha\beta})^2 + \frac{1}{8}(h^\alpha_\alpha)^2) + \frac{1}{2}k^2(h_{0\mu} - \frac{1}{2}h^\alpha_\alpha \eta_{0\mu})^2 + \mathcal{O}(h^3_{\mu\nu}).$$

[2] One can also reason the other way around, as is done in 't Hooft's article: a non-zero Λ implies a space-time dependent VEV.

[3] 4.9 is invariant under the combined Poincaré *space-time* and (inverse) *internal* transformation.

We split the field $h_{\mu\nu}$ into irreducible representations of the rotation group:

$$
\begin{aligned}
h_{ii} = h, \quad h_{ij} = \tilde{h}_{ij} + \frac{1}{3} h\delta_{ij}, \quad \tilde{h}_{ii} = 0, \\
h_{i0} = h_{0i} = h_i.
\end{aligned}
\tag{4.15}
$$

In terms of these new variables one finds:

$$
\mathcal{L}_{\text{eff}} = -m^2 + (k^2 + m^2)(-\frac{1}{4}(\tilde{h}_{ij})^2 + \frac{1}{6}h^2) + \frac{1}{2}m^2 h_i^2 - \frac{1}{8}m^2(h_{00} + h)^2 + \mathcal{O}(h_{\mu\nu}^3). \tag{4.16}
$$

The fields $\tilde{h}_{ij}$ are the 5 components of a massive spin 2 field. The field h is also a dynamical field. On the other hand, h_i and $h_{00} + h$ do not propagate. Note that the dynamical field h has a wrong sign in the lagrangian, and thus in the propagator:

$$
D^{(h)}(k) = \frac{-3}{k^2 + m^2 - i\epsilon}.
$$

This was expected, since originally the field ϕ_0 also had the wrong sign. Unitarity is still satisfied if the lagrangian only has even powers of h (Conclusion (2.1)). Let us now examine this.

The other matter fields in $\mathcal{L}_m$ couple to the gravitational fields:

$$
\mathcal{L}_m[g_{\mu\nu}] = \mathcal{L}_m[\eta_{\mu\nu}] - \frac{1}{2}T^{\mu\nu}h_{\mu\nu} + \mathcal{O}(h^2).
$$

In momentum space we again rotate into the direction $k_\mu = (k, 0, 0, 0)$. Up to linear order one has $\partial_\mu T^{\mu\nu} = 0$:

$$
\begin{aligned}
T^{0\mu} = 0 \longrightarrow T := \eta_{\mu\nu}T^{\mu\nu} = T^{ii} \ ; \\
T^{ij} = \tilde{T}^{ij} + \frac{1}{3}T\delta^{ij}, \quad \tilde{T}^{ii} = 0.
\end{aligned}
$$

In the rotated frame one finds the following coupling term:

$$
\frac{1}{2}T^{\mu\nu}h_{\mu\nu} = \frac{1}{2}T^{ij}h_{ij} = \frac{1}{2}\tilde{T}^{ij}\tilde{h}_{ij} + \frac{1}{6}Th.
$$

In QCD, $T \neq 0$, thus there will be an uneven term in u which will spoil unitarity. This is expected, since we ignored the fact that we broke the $\phi_0 \to -\phi_0$ symmetry. This linear term may be cancelled by coupling matter not to the metric $g_{\mu\nu}$, but to the following:

$$
\begin{aligned}
g_{\mu\nu}^{\text{matter}} = g_{\mu\nu}\left(\frac{-g^\phi}{-g}\right)^\alpha \ ; \\
g_{\mu\nu}^\phi = \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b, \quad g^\phi = det(g_{\mu\nu}^\phi) \ ; \\
S_m[g_{\mu\nu}^{\text{matter}}] = \int d^4x \mathcal{L}_m[g_{\mu\nu}^{\text{matter}}].
\end{aligned}
$$

In the unitary gauge (4.9): $-g^\phi = 1$, thus $T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m[g]}{\delta g_{\mu\nu}} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m[g^{\text{matter}}]}{\delta g_{\mu\nu}^{\text{matter}}}$ couples as follows:

$$
\begin{aligned}
\mathcal{L}_m[g_{\mu\nu}^{\text{matter}}] &= \mathcal{L}_m(\eta_{\mu\nu}) - \frac{1}{2}T^{\mu\nu}h_{\mu\nu}^{\text{matter}} + \mathcal{O}((h^{\text{matter}})^2) \ ; \\
h_{\mu\nu}^{\text{matter}} &:= g_{\mu\nu}^{\text{matter}} - \eta_{\mu\nu} = g_{\mu\nu}(-g)^{-\alpha} - \eta_{\mu\nu} = (\eta_{\mu\nu} + h_{\mu\nu})(1 - \alpha h_\lambda^\lambda + \mathcal{O}(h^2)) - \eta_{\mu\nu} \\
&= h_{\mu\nu} - \alpha h_\lambda^\lambda \eta_{\mu\nu} + \mathcal{O}(h^2).
\end{aligned}
$$

The eventual coupling is:

$$\begin{aligned}\frac{1}{2}T^{\mu\nu}h^{\text{matter}}_{\mu\nu} &= \frac{1}{2}T^{\mu\nu}(h_{\mu\nu} - \alpha h^{\lambda}_{\lambda}\eta_{\mu\nu}) + \mathcal{O}(h^2)\\ &= \frac{1}{2}\tilde{T}^{ij}\tilde{h}_{ij} + ((\frac{1}{6} - \frac{1}{2}\alpha)h + \frac{1}{2}\alpha h_{00})T + \mathcal{O}(h^2)\\ &= \frac{1}{2}\tilde{T}^{ij}\tilde{h}_{ij} + (\frac{1}{6} - \alpha)hT + \mathcal{O}(h^2),\end{aligned}$$

where $h_{00} = -h$ was used, according to the dynamical equations following from the lagrangian (4.16). The linear term is cancelled if one takes $\alpha = \frac{1}{6}$. Of course there will be higher order uneven coupling terms in h and the matter section, thus unitarity is still not satisfied. In order to satisfy unitarity, one must find a matter lagrangian which either gives a positive propagator for the field h, or one where the field h completely decouples. In the following subsection, we are going to look at what conditions this puts on the effective lagrangian in the unitary gauge.

4.1.2 Pauli-Fierz Lagrangian

Let us consider the effective gauge fixed broken lagrangian to second order. The kinetic (derivative) terms come from the Einstein-Hilbert lagrangian $\mathcal{L}_{EH} = \sqrt{-g}R$. In the rest frame $(k, 0, 0, 0)$ they will always be of the following form after SSB:

$$\mathcal{L}_{\text{kin}} = k^2(-\frac{1}{4}\tilde{h}^2_{ij} + \frac{1}{6}h^2).$$

The only dynamical fields are again h and $\tilde{h}_{ij}$. Note that the field h will always have a wrong sign in front of the momentum k^2 coming from the derivatives. Because of this, the field h is called a *ghost*. Our only hope for having a bounded from below (and thus unitary) hamiltonian, is if the ghost field h decouples.

The second order non-derivative mass terms come from the matter lagrangian after SSB and are required to be of the following form because of Lorentz invariance:

$$\mathcal{L}_{\text{mass}} = -\frac{1}{4}m^2(h_{\mu\nu}h^{\mu\nu} - a(h^{\alpha}_{\alpha})^2), \tag{4.17}$$

where a is an arbitrary real number. Decomposing $\mathcal{L}_{\text{mass}}$ again in terms of (4.15):

$$\mathcal{L}_{\text{mass}} = -\frac{1}{4}m^2(\tilde{h}^2_{ij} + (\frac{1}{3} - a)h^2 - 2h^2_i + (1 - a)h^2_{00} + 2ahh_{00}). \tag{4.18}$$

The non-dynamical fields h_{00} and h_i may be integrated out. For general $a \neq 1$ the dynamical equations of the fields h_{00} and h_i are:

$$\begin{aligned}h_{00} &= \frac{a}{a-1}h\ ;\\ h_i &= 0.\end{aligned}$$

One is then left with 6 d.o.f.: 5 correspond to the massive spin 2 field $\tilde{h}_{ij}$ and 1 d.o.f. corresponds to the ghost field h which does not decouple. The special case $a = 1$ must be separately considered. In that case the field h_{00} only has the linear term $2ahh_{00}$ in the lagrangian and thus acts as a lagrange multiplier, giving the constraint $h = 0$. The h field decouples, reducing the d.o.f. to 5. This is thus a healthy, unitary lagrangian describing a massive spin 2 field without ghosts, first described by Pauli and Fierz [FP39]. In the following section we are going to see how such a form for the lagrangian can result from SSB.

4.2 Revised Higgs Mechanism for Gravity in Minkowski Background

To avoid the non-unitary mode, one needs to effectively eliminate one of the scalars. In this section we will find that this can be achieved by adding higher derivative terms in the scalar sector of the matter lagrangian. Decoupling of the ghost then requires appropriate tuning of the cosmological constant.

4.2.1 Revised Gravitational Higgs Model

There are various ways to achieve massive gravity via SSB without the ghost. One is for example by only breaking the $D-1$ spatial coordinates with $D-1$ scalars. This leaves a unitary theory with a massive graviton in an expanding background [Kak08a], [tH08], which will not be considered in this section, since we want a flat background for QCD. One other way is to include higher order terms in the scalar [Kak08b]. It will now be shown that this way has Minkowski background as a possible groundstate. We define the induced metric for the scalar sector:

$$Y_{mn} := \eta_{ab}\partial_m\phi^a\partial_n\phi^b, \quad a,b = 0,\dots,D-1. \tag{4.19}$$

In the following we will raise and lower *every* index with the Minkowski metric. It is natural to generalize the action (4.1) as follows:

$$\begin{aligned} S_Y &= M_P^{D-2}\int(\mathcal{L}_{EH}+\mathcal{L}_Y)d^Dx\ ; \\ \mathcal{L}_Y &= -\sqrt{-g}V(Y)\ ; \\ Y :&= Y_{mn}g^{mn}, \end{aligned} \tag{4.20}$$

where the potential $V(Y)$ is a function of Y. For now the space-time dimension D will be arbitrary. In particular, note that the case considered in the previous section is the special case $V(Y) = \Lambda + \frac{1}{2}Y$ with $D = 4$. We are interested in the resulting lagrangian after SSB, with the same VEV as before (4.9). Again the scalar fluctuations $\varphi^a = \phi^a - \langle\phi^a\rangle$ can be gauged away with (4.11). After taking the unitary gauge:

$$\begin{gathered} Y_{mn} = m^2\eta_{mn}\ ; \\ Y = m^2\eta_{mn}g^{mn} = m^2D + (-m^2h^\alpha_\alpha + m^2(h_{\mu\nu})^2) + \mathcal{O}(h^3). \end{gathered}$$

Expand for a general potential term $V(Y)$ around m^2D:

$$\begin{aligned} V(Y) &= V(m^2D + (-m^2h^\alpha_\alpha + m^2(h_{\mu\nu})^2) + \mathcal{O}(h^3)) \\ &= V(m^2D) + (-m^2h^\alpha_\alpha + m^2(h_{\mu\nu})^2)V'(m^2D) + \frac{1}{2}m^4(h^\alpha_\alpha)^2V''(m^2D) + \mathcal{O}(h^3)), \end{aligned}$$

where the prime denotes differentiation w.r.t. Y. Using the expansion for $\sqrt{-g}$ given in (4.10) gives:

$$\begin{aligned} -\sqrt{-g}V(Y) = &- V(m^2D) + (\frac{1}{2}V(m^2D) - m^2V'(m^2D))h^\alpha_\alpha \\ &- \frac{1}{4}(-V(m^2D) + 4m^2V'(m^2D))(h_{\mu\nu})^2 \\ &+ \frac{1}{4}(-\frac{1}{2}V(m^2D) + 2m^2V'(m^2D) - 2m^4V''(m^2D))(h^\alpha_\alpha)^2 + \mathcal{O}(h^3). \end{aligned} \tag{4.21}$$

Note that for the linear potential $V(Y) = \Lambda + \frac{1}{2}Y$ with $D = 4$ we reproduce the result (4.13) of the previous section. The linear term should again vanish, giving the following constraint on the function $V(Y)$:

$$V(m^2 D) = 2m^2 V'(m^2 D). \tag{4.22}$$

In particular, for $V(Y) = \Lambda + \frac{1}{2}Y$ the constraint (4.22) is precisely (4.14) for $D = 4$. The equation (4.21), taking into account (4.22), can be put into the form:

$$\mathcal{L}_Y = -\sqrt{-g}V(Y) = -V(m^2 D) - \frac{1}{4}(m_1^2 (h_{\mu\nu})^2 - m_2^2 (h^\alpha_\alpha)^2) + \mathcal{O}(h^3), \tag{4.23}$$

where:

$$\begin{aligned} m_1^2 &= -V(m^2 D) + 4m^2 V'(m^2 D) = V(m^2 D) \ ; \\ m_2^2 &= \frac{1}{2}m_1^2 - 2m^4 V''(m^2 D). \end{aligned} \tag{4.24}$$

In particular one gets the Pauli-Fierz lagrangian, i.e. vanishing linear term and $m_1^2 = m_2^2$ if:

$$\begin{aligned} V(m^2 D) &= -4m^4 V''(m^2 D) \ ; \\ V(m^2 D) &= 2m^2 V'(m^2 D). \end{aligned} \tag{4.25}$$

These are two sufficient constraints on the general potential $V(Y)$ in order for the lagrangian to reproduce the unitary Pauli-Fierz lagrangian. The mass of the graviton becomes $m_h^2 := m_1^2 = V(m^2 D)$. Note that the second constraint implies that one should look for potentials that have higher order, non-linear terms in Y.

Example Consider the following example:

$$V(Y) = \Lambda + \frac{1}{2}Y + \lambda Y^2.$$

The first term is a cosmological constant, the second term is the kinetic term for the scalars, and the third term is a four-derivative term. The constraints (4.25) in this case become:

$$\begin{aligned} m^2 &= \frac{-1}{4\lambda(D+2)} \ ; \\ \Lambda &= \frac{D^2 + 4D - 8}{16\lambda(D+2)^2}. \end{aligned}$$

The parameter λ can be taken arbitrary, as long as $\lambda < 0$ (also needed for a bounded hamiltonian from below). Note that $\lambda < 0$ implies a negative cosmological constant.

In conclusion, a lagrangian (4.20) with appropriate constraints (4.25) leads to the Pauli-Fierz form. The mass of the spin-2 glueball state is $V(m^2 D)$. In the following subsection we are going to study why the ghost actually decoupled.

4.2.2 The reason for the decoupling

Why did we get a unitary lagrangian in the previous subsection? After all, we did break the symmetry $\phi_0 \to -\phi_0$, which implies uneven terms in the lagrangian for the scalar fluctuation φ_0. How come do we still find a unitary lagrangian? It seems that one d.o.f. was lost, since we started with massless gravity, whose number of d.o.f. is given by $D(D-3)/2$, plus D scalars, one of which is time-like. One therefore expects $D(D-1)/2$ propagating d.o.f., $(D+1)(D-2)/2$ of which correspond to a massive graviton, and one to a ghost.
This is similar to what happens with a vector particle: the vector particle only has D components, however it is well known that the vector field only has a maximum of $D-1$ actual physical fields, where at least one decouples. Thus, we suspect that the scalar fields reorganize themselves in the form of a vector boson, where one scalar component (the ghost) is then unphysical to begin with.

Another way of seeing this is as follows: after SSB the broken lagrangian will have terms of second power in the scalar fluctuations φ:

$$\mathcal{L}_Y = \text{constant} + \text{linear terms} - \frac{1}{2}a(\partial_\mu\varphi_\nu)^2 + \frac{1}{2}b(\partial_\mu\varphi^\mu)^2 + \text{higher order terms.} \tag{4.26}$$

Again, we raise and lower the indices with the Minkowski metric $\eta_{\mu\nu}$. It can be shown [tH07] that the form of the above lagrangian *must* become that of a vector lagrangian, i.e. $a = b$, in order to have a bounded hamiltonian. It is then well known that effectively only $D-1$ components are physically propagating, whereas the time-like component is not physical to begin with and does not propagate. Let us see how this happens in detail in our case.

Let us not take the unitary gauge yet; instead, expand Y in terms of the scalar fluctuations φ and the metric $h_{\mu\nu}$:

$$Y = m^2 D + 2m\partial_\mu\varphi^\mu + \partial^\mu\varphi_a\partial_\mu\varphi^b - 2mh^{\mu\nu}\partial_\mu\varphi_\nu - m^2h^\alpha_\alpha + m^2(h_{\mu\nu})^2 + \text{third order terms.}$$

One may then plug in the above expansion of Y into the potential $V(Y)$, which gives for $\mathcal{L}_Y$:

$$\begin{aligned}\mathcal{L}_Y = -\sqrt{-g}V(Y) = &-\frac{1}{4}m_h^2((h_{\mu\nu})^2 - (h^\alpha_\alpha)^2) - \frac{1}{m}\partial_\mu\varphi^\mu - \frac{1}{m}(h^\alpha_\alpha\partial_\mu\varphi^\mu - h^{\mu\nu}\partial_\mu\varphi_\nu)\\ &- \frac{1}{2m^2}((\partial^\mu\varphi_\nu)^2 - (\partial_\mu\varphi^\mu)^2) + \text{higher order terms,}\end{aligned}$$

where we assumed that the constraints (4.25) are satisfied. This is precisely of the form (4.26) as expected. Up to surface terms and quadratic order, one has the following:

$$\mathcal{L}_Y = -\sqrt{-g}V(Y) = -\frac{1}{4}m_h^2((h_{\mu\nu})^2 - (h^\alpha_\alpha)^2) - \frac{m_h^2}{m}A^\mu Q_\mu - \frac{m_h^2}{4m^2}F_{\mu\nu}F^{\mu\nu}, \tag{4.27}$$

where again $m_h^2 := V(m^2D)$, and:

$$\begin{aligned} A_\mu &:= \varphi_\mu \;; \\ Q_\mu &:= \partial^\nu h_{\mu\nu} - \partial_\mu h^\alpha_\alpha \;; \\ F_{\mu\nu} &:= \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \tag{4.28}$$

Hence, at the tuned values of the parameters of the potential $V(Y)$, the kinetic term of the scalar fluctuations reorganizes into that of a vector boson. Note that the above action is invariant under the full diffeomorphisms:

$$\begin{aligned} \delta A_\mu &= m\xi_\mu \ ; \\ \delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu . \end{aligned} \tag{4.29}$$

It is now well known that the modes with a pole in the propagator are those satisfying:

$$\partial_\mu A^\mu = 0.$$

In the appropriate coordinate frame, the time-like component of A_μ is not propagating to begin with and thus need not be gauged to zero. One can then use the spatial diffeomorphisms to set the space-like components of A_μ to zero and the remaining time-like diffeomorphism can be used to gauge away the ghost, i.e. the trace component $h = h_{ii}$ (instead of gauging the already unphysical time-like component of A_μ to zero). In appendix B it will be shown that the fields satisfy:

$$\begin{aligned} h^\alpha_\alpha = -h_{00} + h &= 0 \ ; \\ \partial^\nu h_{\mu\nu} &= 0 \ ; \\ \partial^\alpha \partial_\alpha h_{\mu\nu} &= m_h^2 h_{\mu\nu} . \end{aligned} \tag{4.30}$$

Going to Fourier space again and using the decomposition in (4.15), in the rest frame $k_\mu = (k, 0, 0, 0)$ the first two equations from (4.30) give:

$$\left. \begin{aligned} h^\alpha_\alpha &= 0 \\ k h_{\mu 0} &= 0 \end{aligned} \right\} \quad h_{00} = 0, \quad h_i = 0, \quad h = 0. \tag{4.31}$$

One is thus left with the 5 d.o.f. satisfying:

$$\partial^\alpha \partial_\alpha \tilde{h}_{\mu\nu} = m_h^2 \tilde{h}_{\mu\nu} . \tag{4.32}$$

The troublesome ghost h decouples, and we are left with a massive spin-2 glueball having a mass m_h^2. This is a unitary theory, as found in the previous subsection.

To summarize, the reason why the ghost decoupled is that, at the tuned value of the parameters of $V(Y)$, the propagating scalar d.o.f. is not D, but $D-1$. The reduction in scalar d.o.f. is due to the fact that the kinetic term for scalar fluctuations reorganizes into that of a vector boson, which has one d.o.f. less. The total physical d.o.f. is then $D(D-3)/2 + (D-1) = (D+1)(D-2)/2$, which is the correct d.o.f. of a massive graviton. Note that in this section we only kept terms up to quadratic order. In the next section we are going to look more closely at the Pauli-Fierz lagrangian, but now including higher orders.

4.3 Effective Field Theory Approach

In the previous section we looked at how SSB can lead to the Pauli-Fierz lagrangian. This was achieved by considering a matter lagrangian with higher order scalar terms. These were all non-renormalizable interactions, but this was not a problem since the Einstein Hilbert lagrangian was already non-renormalizable to begin with. We were able to gauge away D d.o.f. using the gauge freedom, while the troubling ghost d.o.f. was decoupled by the Pauli-Fierz choice of the mass terms. It is reasonable to assume that radiative *quantum* corrections do not preserve the Pauli-Fierz form of the mass term, implying that the ghost d.o.f. will reemerge. Even worse, it will be shown in this section that the ghost will reemerge already in the *classical* regime by also taking into account higher order terms of the metric [BD72b], [BD72a]. In other words, the ghost decoupling is only possible at the linearized level in the dynamical equations, or equivalently by only considering up to second order terms of the metric in the classical lagrangian. Let us study the Pauli-Fierz lagrangian further, but now also including higher orders.

Start with the Einstein-Hilbert lagrangian $\mathcal{L}_{EH} = \sqrt{-g}R$. For our purpose here it will be useful to take the so-called ADM variables [ADM60]:

$$\begin{aligned} \gamma_{ij} &:= g_{ij}, \quad \pi^{ij} := \frac{\delta \mathcal{L}_{EH}}{\delta \dot{\gamma}_{ij}}, \\ N_i &:= g_{0i}, \quad N := (-g^{00})^{-1/2}. \end{aligned} \tag{4.33}$$

In these variables, the Einstein-Hilbert lagrangian becomes:

$$\mathcal{L}_{EH} = \pi^{ij}\dot{\gamma}_{ij} - \sqrt{\gamma}(NR^0 + N_iR^i). \tag{4.34}$$

Here γ is the determinant of γ_{ij}. The specific form of the functions R^0 and R^i will not be needed in this calculation and is thus not given. We only need that R^0 and R^i *solely* depend on γ_{ij} and its conjugate momenta π^{ij}. The variables N and N_i are thus linear in $\mathcal{L}_{EH}$. In ordinary general relativity they thus serve as lagrange multipliers, giving four constraints, and thereby reducing the d.o.f. to $10 - 2 \times 4 = 2$. Let us now add the massive terms to $\mathcal{L}_{EH}$:

$$\begin{aligned} \mathcal{L}_{\text{mass}} &= -\frac{1}{4}m_h^2(h_{\mu\nu}h^{\mu\nu} - a(h^\alpha_\alpha)^2) = -\frac{1}{4}m_h^2(\tilde{h}_{ij}^2 + (\frac{1}{3} - a)h^2 - 2N_i^2 + (1-a)h_{00}^2 + 2ahh_{00}) \\ &= -\frac{1}{4}m_h^2(\tilde{h}_{ij}^2 + (\frac{1}{3} - a)h^2 - 2N_i^2 + (1-a)(1 - N^2 + N_iN_j\gamma^{ij})^2 + 2ah(1 - N^2 + N_iN_j\gamma^{ij})). \end{aligned}$$

γ^{ij} is the inverse of γ_{ij}. Also, we used $h_{00} = 1 - N^2 + N_iN_j\gamma^{ij}$ at the second equality. As a result of taking cubic and higher orders into consideration, the variables N and N_i are no longer lagrange multipliers. The added mass term is quadratic in the variable N_i, irrespective of a, hence the dynamical equation of N_i is no longer a constraint on the other variables, but instead is an equation for N_i itself. This raises the d.o.f. from 2 to 5. If $a = 1$, the mass term only has linear terms of the variable N when calculating up to quadratic terms in the ADM variables, thus the variable N is in that case a lagrange multiplier and its dynamical equation is a constraint. So for the best case scenario we might as well take $a = 1$. However, the mass term always has a quadratic term in N when one takes cubic and higher orders into account. The dynamical equation of N is no longer a constraint on the other variables, but instead is an equation for N itself. This further raises the d.o.f. from 5 to 6 and hence the ghost returns.

Let us try to see the effect of the ghost on the Hamiltonian when $a = 1$. The dynamical equations of N and N_i are:

$$\sqrt{\gamma}R^0 = m_h^2 h N \; ;$$
$$\sqrt{\gamma}R^i = m_h^2(\eta^{ij} - h\gamma^{ij})N_j.$$

Substituting this into the full hamiltonian density one can show:

$$\mathcal{H} = \pi^{ij}\dot{\gamma}_{ij} - (\mathcal{L}_{EH} + \mathcal{L}_{\text{mass}}) = \frac{1}{2m_h^2}(\frac{(\sqrt{\gamma}R^0)^2}{h} + \gamma R^i(\eta^{ij} - h\gamma^{ij})^{-1}R^j) + \frac{1}{4}m^2(\tilde{h}_{ij}^2 - \frac{2}{3}h^2 + 2h).$$

This is a hamiltonian density of an ill-defined theory. The first term on the r.h.s. is not bounded from below and singular in m and h. For instance consider $\sqrt{\gamma}R^0$ fixed and $R^i = 0$; when $h \uparrow 0$ the term $(\sqrt{\gamma}R^0)^2/(m_h^2 h)$ is unbounded from below. This demonstrates the presence of a ghost-like instability in the theory. Thus unitarity is not satisfied and the theory is hence hard to make sense of.

One may think that it may be possible to still have a unitary theory, by having higher order terms in the metric $h_{\mu\nu}$ which supplies terms of N^n such that they all get cancelled, except the linear term. If this would be possible, the variable N would again be a Lagrange multiplier and thus as before one would get 5 massive degrees of freedom instead of 6. The ghost would decouple and one would be left with a unitary theory. Unfortunately this is not possible, as will be shown in the Appendix B. The ghost will thus never decouple by just adding higher order lorentz invariant, non-derivative terms in the metric $h_{\mu\nu}$. It thus seems that one might as well just stop searching for a way to a unitary massive lagrangian via SSB. Indeed, the SSB only adds higher non-derivative metric terms[4], which always results in a non-unitary lagrangian, irrespective of the coefficients in front of the higher non-derivative metric terms (see Appendix B).

However, from an effective field theory point of view this is not a problem, until one specifies the scale at which the ghost shows up, i.e. it mass. If this scale is much higher than the other particles (such as the massive glueball), the ghost can be consistently disregarded. For this effective field theory approach, we are going to use the so-called Stueckelberg formalism, which was first invented for spin 1 [Stu38] massive gauge fields.

4.3.1 Stueckelberg Formalism

In the previous sections we *started* with a lagrangian that included scalars and a massless graviton. Then a coordinate dependent VEV was assumed for the scalars which broke the gauge symmetry, leading to a mass term for the graviton. Finally, the matter lagrangian of the scalars was tuned such that we got the Pauli-Fierz form for the mass terms, which decoupled the ghost up to quadratic order. One can also work the other way around: *start* with the Pauli-Fierz lagrangian, and restore local coordinate invariance. This is called the Stueckelberg trick for spin 2 [SD75]: one introduces a set of Goldstone fields and requires that they transform under a coordinate transformation precisely such that local coordinate invariance is restored.

[4]In the following chapter, a brief account of recent studies will be given, in particular one study [DPR08] in which the mass terms are non-local.

In the rest of this section we will work in $D = 4$. Consider a non-covariant action that can be split as follows:

$$S = \int \mathcal{L} d^4x := M_P^2 \int d^4x (\mathcal{L}_{EH} + \mathcal{L}_{non-inv}). \tag{4.35}$$

The term $\mathcal{L}_{non-inv}$ is assumed to be *not* invariant under general coordinate transformations. One can study this lagrangian as a non-linear Sigma model[5], by introducing component functions $Y^\mu(x)$ as follows:

$$\tilde{g}_{\mu\nu}(x) = \frac{\partial Y^\alpha(x)}{\partial x^\mu} \frac{\partial Y^\beta(x)}{\partial x^\nu} g_{\alpha\beta}(Y(x)). \tag{4.36}$$

In the action (4.35) one then replaces the metric $g_{\mu\nu}$ with the r.h.s. of (4.36). Referring to (4.3), we note that $\tilde{g}_{\mu\nu}$ is just the image of $g_{\mu\nu}$ under the coordinate transformation $x^\mu \to Y^\mu(x)$. Analogously, in the presence of matter fields ψ one replaces them everywhere with their images under such a coordinate transformation. Clearly, because of the invariance of $\mathcal{L}_{EH}$ under general coordinate transformations, the functions Y^μ will only appear in the non-invariant part of the action. In particular, no direct coupling between these functions and matter arises, provided the action for matter is covariant.

It now follows easily that the action (4.35), considered as a functional for $g_{\mu\nu}(x)$ and functions $Y^\mu(x)$ is invariant under the general gauge transformations (4.4), accompanied by the following gauge transformations:

$$\begin{aligned} g'_{\mu\nu}(x') &= \frac{\partial x^\alpha}{\partial (x')^\mu} \frac{\partial x^\beta}{\partial (x')^\nu} g_{\alpha\beta}(x) \ ; \\ (Y')^\mu(x') &= \lambda^\mu(Y(x')). \end{aligned} \tag{4.37}$$

where $(x')^\mu = \lambda^\mu(x)$ is a change of coordinates. One can then easily show that the metric and matter fields transform as:

$$\begin{aligned} \tilde{g}'_{\mu\nu}(x') &= \tilde{g}_{\mu\nu}(x') \ ; \\ \psi'(x') &= \psi(x'), \end{aligned}$$

hence the action (4.35) is left invariant under general coordinate transformations:

$$S' = \int d^4x' \mathcal{L}'(x') = \int d^4x' \mathcal{L}(x') = \int d^4x \mathcal{L}(x) = S.$$

Transformation (4.37) implies that the functions Y^μ do not transform as scalars under changes of coordinates. However, the inverse functions Y^{-1} do transform as scalars:

$$(Y^{-1})'(x') = (\lambda \circ Y)^{-1}(x') = Y^{-1} \circ \lambda^{-1}(x') = Y^{-1}(x).$$

These inverse functions are called the Stueckelberg scalar fields:

$$\phi(x) := Y^{-1}(x). \tag{4.38}$$

Now, the physics described by the lagrangian with the Stueckelberg scalars included is identical to the original lagrangian without the Stueckelberg fields. This follows from the gauge invariance (4.37). Indeed taking the unitary gauge, which can be seen as a constraint [GKK93]:

$$\phi(x) = x,$$

[5]Non-linear Sigma models and gravity theories go along way back [OP80], [GMZ84].

implies:

$$Y(x) = x \longrightarrow \tilde{g}_{\mu\nu} = g_{\mu\nu},$$

and one returns to the old unitary lagrangian. Off course, the above action should be expressed in the fields ϕ and the metric g, using (4.36). This is obviously difficult in general, thus one should expand ϕ around its background value x. Equivalently, expanding $Y(x)$ around its background value x gives:

$$Y(x) = x + \xi(x). \tag{4.39}$$

Using (4.36) and (4.5) to quadratic order in the field $\xi(x)$ we then have the following:

$$\tilde{h}_{\mu\nu}(x) = h_{\mu\nu}(x) + \partial_\nu \xi_\mu(x) + \partial_\mu \xi_\nu(x) + \partial_\mu \xi^\gamma \partial_\nu \xi_\gamma + \text{interactions}. \tag{4.40}$$

This should then be plugged into the action for $h_{\mu\nu}$.

The main advantage when working with the Stueckelberg fields ξ, is that one can easily derive the mass modes of the ghost. This follows from the fact that the ghost can be made visible by studying the *longitudinal* component of the Stueckelberg fields. Also, with the Stueckelberg trick, one can easily derive the scale of strong coupling [AHGS03]. This is especially handy since the theory (of gravity) is non-renormalizable. Lastly the terms in the effective field lagrangian are most easily derived with the help of the Stueckelberg fields [AHGS03]. In the next subsection we will see how the above concept of Stueckelberg is used to derive the mass of the ghost.

4.3.2 Higher orders

One may take an effective field theory approach to the problem of studying higher orders. This is made easier by using the Stueckelberg trick. We will be particularly interested in finding the mass of the ghost due to the higher order terms. Consider again the action of the form (4.35). In this subsection we are going to start by splitting the fluctuating field ξ (4.39) in a transverse mode and a longitudinal mode:

$$\xi_\mu := A_\mu + \partial_\mu \pi. \tag{4.41}$$

First consider the non-invariant Pauli-Fierz mass term:

$$\mathcal{L}_{non-inv} = \mathcal{L}_{\text{mass}} = -\frac{1}{4} m_h^2 ((h_{\mu\nu})^2 - (h_\alpha^\alpha)^2). \tag{4.42}$$

The troublesome ghost lies in the excitations of the longitudinal component π. The reason for this is that the longitudinal component has higher derivative terms, which usually implies ghosts [CNPT05]. Plugging (4.40) into $\mathcal{L}_{\text{mass}}$ gives:

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2} m_h^2 (\partial_\mu \xi_\nu - \partial_\nu \xi_\mu)(\partial_\nu \xi_\mu - \partial_\mu \xi_\nu) - m_h^2 (\partial_\nu \xi^\mu h_\mu^\nu - \partial_\mu \xi^\mu h_\nu^\nu),$$

up to quadratic terms. Note again the vector boson form for the lagrangian. The vector A_μ has 2 d.o.f. with healthy kinetic terms. For convenience, we will set A_μ to zero and only focus on the longitudinal component. The longitudinal mode only has the kinetic term due to mixing with the field $h_{\mu\nu}$. The kinetic terms for $h_{\mu\nu}$ and π thus become:

$$M_P^2 \mathcal{L}_{\text{kin}} = M_P^2 L_{EH}(h_{\mu\nu}) - m_h^2 M_P^2 (h_\mu^\nu \partial_\nu \partial^\mu \pi - h_\nu^\nu \partial_\mu \partial^\mu \pi) \ ;$$
$$L_{EH}(h_{\mu\nu}) = \frac{1}{8}(\partial_\mu h_\alpha^\alpha)^2 - \frac{1}{4}(\partial_\mu h_{\alpha\beta})^2 + \frac{1}{2}(\partial_\mu h_\nu^\mu - \frac{1}{2}\partial_\nu h_\lambda^\lambda)^2.$$

It can be diagonalized by the conformal transformation:

$$h_{\mu\nu} =: \hat{h}_{\mu\nu} - m_h^2 \eta_{\mu\nu} \pi. \tag{4.43}$$

Then the kinetic term becomes:

$$\mathcal{L}_{\text{kin}} = M_P^2 L_{EH}(\hat{h}_{\mu\nu}) - 3m_h^4 M_P^2 (\partial_\mu \pi)^2 = M_P^2 L_{EH}(\hat{h}_{\mu\nu}) - \frac{1}{2}(\partial_\mu \pi^c)^2 \ ;$$
$$\pi^c := \sqrt{6} M_P m_h^2 \pi.$$

The canonically normalized scalar π^c is 1 d.o.f. with a healthy kinetic term. Because of the covariance that we have in our Stueckelberg formalism we may then gauge fix the $h_{\mu\nu}$, resulting in a tensor with 2 d.o.f. The total d.o.f. is thus $2 + 2 + 1 = 5$, with all healthy kinetic terms. However, the ghost comes in when we include higher order terms as we saw earlier. Here the strength of the Stueckelberg will be seen. The first higher order term in π becomes:

$$m_h^2 M_P^2 (\partial^2 \pi)^3 = \frac{1}{6\sqrt{6} m_h^4 M_P} (\partial^2 \pi^c)^3 =: \frac{1}{\Lambda_5^5} (\partial^2 \pi^c)^3. \tag{4.44}$$

It can be shown that the above cubic interaction (4.44) is the strongest interaction [AHGS03] at small enough energies. Strong coupling sets in at:

$$\Lambda_5 = (6\sqrt{6} m_h^4 M_P)^{1/5}. \tag{4.45}$$

If we now add a matter tensor coupled to the original metric $h_{\mu\nu}$, the field π^c will also couple to the matter tensor via the conformal rescaling (4.43). This thus leads to the eventual action for the longitudinal mode:

$$S = \int d^4x \left(-\frac{1}{2}(\partial_\mu \pi^c)^2 + \frac{1}{\Lambda_5^5}(\partial^2 \pi^c)^3 + \frac{1}{2M_P} \pi^c T\right). \tag{4.46}$$

Note again that the cubic term is the largest interaction term at small enough energies. This action induces a non-zero classical background $\langle \pi^c(x) \rangle$ for π. Expanding the above lagrangian around the background, $\pi^c = \langle \pi^c(x) \rangle + \varphi$, gives up to quadratic order:

$$\mathcal{L}_\varphi = -\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{3\partial^2 \langle \pi^c(x) \rangle}{\Lambda_5^5} (\partial^2 \varphi)^2. \tag{4.47}$$

The second term has two derivatives for each term φ, this implies a ghost. To see this we introduce an auxiliary scalar field χ. The lagrangian (4.47) is equivalent with:

$$\mathcal{L}'_\varphi = -\frac{1}{2}(\partial_\mu \varphi)^2 - 6\partial^2 \langle \pi^c(x) \rangle \partial_\mu \chi \partial^\mu \varphi - (3\partial^2 \langle \pi^c(x) \rangle) \Lambda_5^5 \chi^2.$$

$\mathcal{L}$' is diagonalized by the substitution $\varphi = \varphi' - (6\partial^2 \langle \pi^c(x) \rangle)\chi$:

$$\mathcal{L}'_{\varphi'} = -\frac{1}{2}(\partial_\mu \varphi')^2 + \frac{1}{2}(6\partial^2 \langle \pi^c(x) \rangle)^2 (\partial_\mu \chi)^2 - \frac{1}{2}(6\partial^2 \langle \pi^c(x) \rangle) \Lambda_5^5 \chi^2. \tag{4.48}$$

One has $\partial^2 \langle \pi^c(x) \rangle = \frac{-\langle T \rangle}{2M_P} = \textit{constant}$ as the solution to the equations of motion when coupled to a source $\langle T \rangle = \textit{constant}$. This clearly signals the presence of a ghost with mass:

$$m_{ghost}^2 = \frac{\Lambda_5^5}{6\partial^2 \langle \pi^c(x) \rangle} = \frac{-\Lambda_5^5 M_P}{3\langle T \rangle}. \tag{4.49}$$

Remember that we are dealing with an effective theory with a cutoff Λ_5, therefore we should not worry if the mass of the ghost is much higher than the other particles in play, i.e. $m_{ghost} \gg m_h$. Unfortunately this is not the case for particles in QCD. Indeed, one has for the QCD gluonic condensate $\langle T \rangle \sim \langle -\frac{9\alpha_s}{8\pi} G^a_{\mu\nu} G^{a\mu\nu} \rangle \sim -(0.4 \text{ GeV})^4$ [CFG92]. Plugging in $m_h \sim 1$ GeV [Col87] for the squared mass of the massive spin 2 (glueball) state and $M_P = M_{QCD} \sim 0.2$ GeV, gives for the ghost mass:

$$m^2_{ghost} = \left(\frac{2\sqrt{6} m_h^2 M_P^2}{T} \right) m_h^2 \sim 7m_h^2 \longrightarrow m_{ghost} \sim 3m_h = 3 \text{ GeV} \sim m_h. \tag{4.50}$$

The ghost does not decouple in the effective field theory. Therefore, the model is not even useful as a low energy effective field theory. It thus seems hopeless to get an effective QCD model for the glueballs from an SSB of a gravity model, since the ghost enters already at very low energies.

Summary

In this chapter we looked at how the Higgs mechanism could be applied to gravity. The general coordinate invariance was broken by scalar fields, leading to the Higgs mechanism. The main motivation for giving a mass to the graviton came from cosmology and QCD. We were interested in the application to QCD in $D = 4$. This implied that the broken lagrangian should have a residual Poincaré invariance. The first model that we discussed was a model from 't Hooft [tH08]. The constraint of a residual Poincaré invariance led to the model not being unitary. Afterwards, a modified lagrangian [Kak08b], having higher derivatives was discussed. This led to a unitary lagrangian, up to quadratic order. In the last section we focussed on the higher orders. It was first shown that these higher orders *always* spoil the unitarity, leading to a ghost. To study the ghost interactions, we took an effective field theory approach. By using the Stueckelberg formalism it was shown that the ghost mass was about the same order as the massive spin-2 glueball states, in other words the ghost does not decouple. It therefore seems that one should search for other means to get to a QCD model from an SSB of a gravity theory. One interesting possibility is to add scale dependent or non-local mass terms [DPR08]. In the last chapter, we will briefly look at this and a few other recent studies in massive gravity. The other motivation for massive gravity, namely for cosmological applications, will be explained. The problems that are encountered when applying massive gravity models to problems in cosmology will briefly be addressed. These include, among others, the same problems that we encountered in this chapter.

Chapter 5

Discussion & Recent Work

In the previous chapter we focussed on giving mass to a graviton through the Higgs mechanism. The hope was that the mechanism would give a unitary, or at best an effective field theory where the ghost decouples. The general covariance symmetry was broken to a residual Poincaré symmetry. The main motivation was to understand in basic QFT language how a (General covariant) string-like approach to (Lorentz invariant) QCD would work. Unfortunately, in the eventual broken lagrangian there was a ghost in the spectrum which coupled at the same energies as the mass of the graviton. The main reason for the failure to find such a mechanism lies in a so-called *no-go theorem of massive gravity.* The no-go theorem of massive gravity excludes theories of *massive* spin-2 particles which have *all* the following properties:

1. Lorentz invariance.
2. Local equations of motion, or equivalently a local lagrangian.
3. Massive spin-2 interactions.
4. Finite amount of fields.
5. No ghosts in the spectrum.

The Pauli-Fierz model (to quadratic order) does not have interactions (3), hence it is possible to satisfy (1), (2), (4), (5) without contradicting the no-go theorem. The Pauli-Fierz model (to quadratic order) did not have ghosts (5) and was thus unitary. On the other hand, by adopting the normal spin-1 Higgs mechanism for the Einstein-Hilbert lagrangian in 4 dimensions, the eventual broken lagrangian satisfied (1)-(4). With the no-go theorem one then concludes that there *must* be ghosts in the spectrum, i.e. (5) *cannot* be satisfied. To possibly avoid ghosts, a Higgs mechanism for the Einstein-Hilbert lagrangian should *not* satisfy (2) or (3)[1]. The property (2) will always be satisfied if one starts with a local matter lagrangian, which in our case is assumed. Hence, our only possibility is to exclude property (3). In order for the third property not to be satisfied, the matter lagrangian (after taking the unitary gauge) would need to cancel all the cubic and higher terms coming from the Einstein-Hilbert lagrangian. This seems highly unlikely, even if one takes other forms for the VEV of the (fundamental or composite) scalars. Hence, excluding exotic matter lagrangians, we conclude the following:

[1]The properties (1) and (4) are always assumed to be satisfied in our 4-dimensional case.

Conclusion 5.1 *It is not possible to have a unitary Higgs mechanism which breaks the diffeomorphism symmetry of a local lagrangian $\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_m$, down to the Poincaré symmetry.*

At the beginning of chapter 4 we noted another important motivation for studying massive gravity models. The motivation for studying massive gravity models comes from cosmology. In these models the graviton particle has a mass, which thus makes gravity into a finite range force. By assuming that gravity has a finite range one can explain interesting measurements taken of the universe in other ways normal GR cannot. For example, the accelerating expansion of the universe can then be seen as a *weakening* of gravity at large distance scales, without resorting to a vacuum energy. Usual problems encountered with massive theories of gravity are:

1. The vDVZ discontinuity [vDV70]: A massive graviton has 5 degrees of freedom. The scalar longitudinal d.o.f. couples to the trace of the energy tensor. For the Pauli-Fierz model one can show that this scalar does not decouple if the mass of the graviton goes to zero. Thus the PF lagrangian gives rise to unacceptable predictions either for light bending or for Newtonian interactions, no matter how small the graviton mass is. In other words, when the zero mass model is different than the non-zero (no matter how small) mass model, one speaks of the vDVZ discontinuity.
2. Ghosts in the spectrum.
3. Having an unacceptably low strong coupling scale.

The vDVZ discontinuity (1) has been shown to be an artifact of non-linearity [Vai72] and can normally be avoided by taking into account the full non-linear theory. The infrared modifications of gravity avoids the ghost that accompanies the usual massive graviton (2), by dropping one or more assumptions of the no-go theorem; most mainstream massive gravity theories either break Lorentz invariance or have non-local equations of motion. The low strong coupling scale (3) cannot be fully avoided. Two massive gravity models will now briefly be explained.

DGP Models

The DGP model was proposed in 2000 [DGP00] and assumes that a 3-brane (3 space dimensions) is embedded in a $(4+n)$-dimensional space-time. A 4-dimensional Einstein-Hilbert term concentrated near the brane is added to the usual $(4+n)$-dimensional EH lagrangian. This assumption makes gravity four-dimensional at short distances and $(4+n)$-dimensional at long distances. The model avoids the vDVZ discontinuity because of the Vainshtein effect [DDGV02]. The effective action on the brane is [DPR08]:

$$L^{eff} = h_{\mu\nu}(x) \int d^4x' G_{\Box}^{-1}(x-x')(h_{\mu\nu}(x') + \frac{1}{n-2}\eta_{\mu\nu}h(x')) \; ; \tag{5.1}$$

$$G_{\Box}^{-1}(x-x') = \int d^4p \frac{1}{G(-p^2)} e^{ip(x-x')}, \quad G(-p^2) \sim \int \frac{d^nq}{p^2+q^2}. \tag{5.2}$$

Hence, the no-go theorem is avoided by non-local interactions in the lagrangian. In the $n = 1$ model one can show the absence of ghosts to be stable against quantum corrections [NR04]. The strong coupling scale is approximately $(\Lambda_3)^{-1} \sim 1000$ km [LPR03]. However, by adding appropriate counterterms one can raise this cutoff. These models may serve as a solution to the Cosmological Constant Problem, since the vacuum energy will couple extremely weak to gravity [DGS03]. The weakening effect of the vacuum energy on the universe follows from:

$$H \sim \Lambda^{\frac{1}{2-n}}. \tag{5.3}$$

For $n > 2$, a *naturally* big cosmological constant can still lead to a small measured acceleration rate $H \sim 10^{-33}$ eV. Also, the model itself can serve as a self-accelerating de Sitter model without the vacuum energy.

Lorentz Breaking Models

First published in [AHCLM04], these models avoid the no-go theorem by breaking Lorentz invariance. The vDVZ discontinuity is also avoided. The lorentz breaking terms come from a lagrangian where the diffeomorphism symmetry is partly broken by scalars VEV's[2]. The terms considered in the mass lagrangian are:

$$\mathcal{L}_m = \frac{1}{2} M_P^2 (m_0^2 h_{00}^2 + 2m_1^2 h_{0i}^2 - m_2^2 h_{ij}^2 + m_3^2 h^2 - 2m_4^2 h_{00} h). \tag{5.4}$$

The above mass lagrangian has more parameters, thus it may be easier to avoid ghosts by fine-tuning the coefficients [Rub04]. This may be compared with the PF lagrangian where we only have one parameter, the mass of the graviton, to our disposal in order to cancel the ghost (which is not possible). The fine-tunings of the coefficients may be achieved by residual *local* symmetries that are not broken [Dub04] and are thus stable against quantum corrections. Also, the cutoff may be driven up to:

$$\Lambda_2 \sim \sqrt{m M_P}, \quad m_0 = 0,\ m_1 = m_2 = m_3 = m_4 = m. \tag{5.5}$$

One big advantage of these models is that they have a much higher strong coupling scale than the DGP or Pauli-Fierz model, thus being much more UV insensitive. Also, a de Sitter or inflation phase in the universe may be driven by the scalars, called the *ghost condensate* [AHCLM04]. The ghost condensate can even serve as a dark matter candidate [DTT05]. The main disadvantage is of course the breaking of Lorentz invariance. Because of this, among other problems, experiments confirming Lorentz invariance up to small measurement errors put many constraints on the model. For a review applied to cosmological models, we refer to [RT08].

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[2]The SSB may also be achieved by a vector field VEV [Gri04].

Appendix A

BRST Transformation

Faddeev-Popov gauge fixing

Here we prove (2.14) by following [Ryd96]. Define the new quantity $\triangle_F[\psi]$ as:

$$1 = \triangle_F[\psi] \int \mathcal{D}U \delta[F^a[\psi^U] - C^a], \tag{A.1}$$

with ψ^U given by (2.11) and $C^a(x)$ an arbitrary function. Note that $\delta[F^a[\psi^U] - C^a]$ is a delta functional:

$$\delta[F^a[\psi^U] - C^a] := \prod_{x^\mu, a} \delta[F^a[\psi(x^\mu)] - C^a(x^\mu)];$$

a product of Dirac delta functions at each point of space-time. Observe that $\triangle_F[\psi]$ is gauge invariant. We have from (A.1):

$$(\triangle_F[\psi^{U'}])^{-1} = \int \mathcal{D}U \delta[F^a[\psi_\mu^{U'U}] - C^a].$$

Now putting $U'' = U'U$ and using the result that for compact groups the volume element in group space defines an invariant measure:

$$\mathcal{D}U = \mathcal{D}U'',$$

we then find:

$$(\triangle_F[\psi_\mu^{U'}])^{-1} = \int \mathcal{D}U'' \delta[F^a[\psi_\mu^{U''}] - C^a] = (\triangle_F[\psi])^{-1}, \tag{A.2}$$

so $\triangle_F[\psi]$ is indeed gauge invariant. Inserting (A.1) into (2.13) gives:

$$Z = \int \mathcal{D}\psi \triangle_F[\psi] \int \mathcal{D}U \delta[F^a[\psi^U] - C^a] e^{iS}.$$

Now perform a gauge transformation taking ψ^U to ψ and use the fact that $\mathcal{D}\psi^U$ is the same as $\mathcal{D}\psi$, the action S is gauge invariant and so is $\triangle_F[\psi]$ (A.2). This gives:

$$\begin{aligned} Z &= \int \mathcal{D}\psi \triangle_F[\psi] \int \mathcal{D}U \delta[F^a[\psi^U] - C^a] e^{iS} \\ &= \int \mathcal{D}U \int \mathcal{D}\psi \triangle_F[\psi] \delta[F^a[\psi] - C^a] e^{iS}, \end{aligned}$$

where we have been able to take out the factor $\mathcal{D}U$ since the integrand is independent of U. The isolation of this factor is exactly what we wanted to achieve. It contributes only an overall multiplicative constant to Z and may therefore be ignored. The correct *finite* expression for Z is therefore:

$$Z = \int \mathcal{D}\psi \triangle_F[\psi]\delta[F^a[\psi] - C^a]e^{iS}. \tag{A.3}$$

We now want an expression for $\triangle_F[\psi]$. First, define the functional derivative $\frac{\delta F^a[\psi^U(x)]}{\delta\theta^b(y)}|_{\theta^b=0}$ as follows:

$$F^a[\psi(x)] \stackrel{\text{gauge transf.}}{\longrightarrow} F^a[\psi^U(x)] = F^a[\psi(x)] + \int \frac{\delta F^a[\psi^U(x)]}{\delta\theta^b(y)}|_{\theta^b=0}\theta^b(y)d^4y + \mathcal{O}(\theta^2),$$

where again $U(x) = (\mathbb{1} + i\sum_a \theta^a(x)T_a + \mathcal{O}(\theta^2))$. We will write $\frac{\delta F^a(x)}{\delta\theta^b(y)}$ for $\frac{\delta F^a[\psi^U(x)]}{\delta\theta^b(y)}|_{\theta^b=0}$ in order to save notation. Next, note the following for the Dirac delta function:

$$\delta(g(x)) = \sum_i (\frac{dg}{dx}|_{x=x^i})^{-1}\delta(x - x^i), \tag{A.4}$$

where x^i are the roots of the function $g(x)$. Applying the above equation (A.4) we similarly find in this case:

$$\begin{aligned}
(\triangle_F[\psi])^{-1}\delta[F^a[\psi] - C^a] &= \int \mathcal{D}U\delta[F^a[\psi^U] - C^a]\delta[F^a[\psi] - C^a] \\
&= \int \mathcal{D}U\delta[F^a[\psi^U] - F^a[\psi]]\delta[F^a[\psi] - C^a] \\
&= \int \mathcal{D}U|Det(\frac{\delta F^a[\psi^U] - F^a[\psi]}{\delta\theta^b(y)}|_{\theta^b=0})|^{-1}\delta[\theta^b(x)]\delta[F^a[\psi] - C^a] \\
&= |Det(\frac{\delta F^a(x)}{\delta\theta^b(y)})|^{-1}\delta[F^a[\psi] - C^a].
\end{aligned} \tag{A.5}$$

At the third equality in (A.5) we used the similar expression (A.4) to inspire a change of variables, we also assumed:

$$F^a[\psi^U] - F^a[\psi] = 0 \Longleftrightarrow \theta^b = 0, \tag{A.6}$$

which is equivalent with the function $g(x) = F^a[\psi^U] - F^a[\psi]$ only having one root, namely the root $U = \mathbb{1}$ (or equivalently $\theta^b = 0$). The above equation (A.6) is precisely what we previously assumed for the constraints, that is, for every ψ there should be *precisely one* gauge transform U which achieves $F^a[\psi^U] = 0$. We thus get:

$$(\triangle_F[\psi])\delta[F^a[\psi] - C^a] = |Det(\frac{\delta F^a(x)}{\delta\theta^b(y)})|\delta[F^a[\psi] - C^a]. \tag{A.7}$$

Plugging the above equation (A.7) in (A.3) gives (2.14). The above FP gauge fixing method is actually a special case of a more general formalism which is known as the *BRST formalism.*

BRST formalism

In this section we again follow [Ryd96]. The BRST formalism is a method of implementing first class constraints, which the FP gauge fixing method is a special case of. We will first show the invariance of $\mathcal{L}^{\text{eff}}$ under the BRST transformation (2.19). Notice that the original lagrangian $\mathcal{L}$ is invariant under the BRST transformation since it is just a gauge transformation for the gauge fields with parameters $\theta^a = \lambda\eta^a$. Thus the change in $\mathcal{L}^{\text{eff}}$ comes from the changes in $\mathcal{L}^{\text{GF}}$ and $\mathcal{L}^{\text{FP}}$:

$$\delta\mathcal{L}^{\text{eff}} = \delta\mathcal{L}^{\text{GF}} + \delta\mathcal{L}^{\text{FP}},$$

with:

$$\begin{aligned}
\delta\mathcal{L}^{\text{GF}} &= -\frac{1}{\xi^a}F^a\delta F^a = -\frac{1}{\xi^a}F^a\frac{\delta F^a}{\delta\theta^b}\theta^b = -\frac{1}{\xi^a}F^a\frac{\delta F^a}{\delta\theta^b}\lambda\eta^b \ ;\\
\delta\mathcal{L}^{\text{FP}} &= -\delta\overline{\eta}^a\frac{\delta F^a}{\delta\theta^b}\eta^b - \overline{\eta}^a\delta(\frac{\delta F^a}{\delta\theta^b}\eta^b) = \frac{1}{\xi^a}F^a\lambda\frac{\delta F^a}{\delta\theta^b}\eta^b - \overline{\eta}^a\delta(\frac{\delta F^a}{\delta\theta^b}\eta^b).
\end{aligned}$$

The first terms of $\mathcal{L}^{\text{GF}}$ and $\mathcal{L}^{\text{FP}}$ cancel against each other to give:

$$\begin{aligned}
\delta\mathcal{L}^{\text{eff}} &= -\overline{\eta}^a\delta(\frac{\delta F^a}{\delta\theta^b})\eta^b - \overline{\eta}^a\frac{\delta F^a}{\delta\theta^b}\delta\eta^b\\
&= -\overline{\eta}^a\frac{\delta}{\delta\theta^d}\frac{\delta F^a}{\delta\theta^b}\theta^d\eta^b + \frac{1}{2}f_{bcd}\overline{\eta}^a\frac{\delta F^a}{\delta\theta^b}\lambda\eta^c\eta^d\\
&= \frac{1}{2}f_{bcd}\overline{\eta}^a\frac{\delta F^a}{\delta\theta^b}\lambda\eta^c\eta^d - \overline{\eta}^a\frac{\delta}{\delta\theta^c}\frac{\delta F^a}{\delta\theta^d}\lambda\eta^c\eta^d.
\end{aligned} \tag{A.8}$$

The remainder can be checked to be equal to zero for the most usual constraints. For example, for the lorentz gauge, the remainder becomes together with (2.21):

$$\begin{aligned}
\delta\mathcal{L}^{\text{eff}} &= -\overline{\eta}^a\partial^\mu(\delta D_\mu\eta^a) \ ;\\
\delta(D_\mu\eta^a) &= \delta(\partial_\mu\eta^a + gf_{abc}A_\mu^b\eta^c) = \partial_\mu(\delta\eta^a) + gf_{abc}(\delta A_\mu^b)\eta^c + gf_{abc}A_\mu^b(\delta\eta^c)\\
&= \frac{1}{2}f_{abc}\partial_\mu(\eta^b\eta^c)\lambda - f_{abc}(\partial_\mu\eta^b + gf_{bmn}A_\mu^m\eta^n)\lambda\eta^c - \frac{1}{2}gf_{abc}f_{cmn}A_\mu^b\eta^m\eta^n\lambda.
\end{aligned}$$

Using the fact that η, λ are anticommuting and the Jacobi identity for the structure constants:

$$f_{abc}f_{cmn} + f_{amc}f_{cnb} + f_{anc}f_{cbm} = 0, \tag{A.9}$$

one can easily prove that $\delta(D_\mu\eta^a) = 0$ and hence $\delta\mathcal{L}^{\text{eff}} = 0$. This proves the BRST invariance of the effective lagrangian.

Now we prove the invariance of the coefficients of the sources u and v (in the total lagrangian (2.25)) under the BRST transformation. The coefficient of u has already be shown to be invariant: $\delta(D^\mu\eta^a) = 0$. For the coefficient of v we have:

$$\begin{aligned}
\delta(f_{abc}\eta^b\eta^c) &= f_{abc}((\delta\eta^b)\eta^c + \eta^b(\delta\eta^c))\\
&= -\frac{1}{2}f_{abc}(f_{bmn}\eta^m\eta^n\lambda\eta^c + f_{cmn}\eta^b\eta^m\eta^n\lambda)\\
&= -f_{abc}f_{cmn}\eta^m\eta^n\eta^b\lambda\\
&= -\frac{1}{3}(f_{abc}f_{cmn} + f_{amc}f_{cnb} + f_{anc}f_{cbm})\eta^m\eta^n\eta^b\lambda\\
&= 0,
\end{aligned}$$

where we again used the Jacobi identity. Note, in passing, that the changes in η^a and A_μ^a are nilpotent, i.e. $\delta^2\eta^a = 0, \delta^2 A_\mu^a = 0$. This is the reason why we speak of the *BRST cohomology*, which is generally used in the BRST formalism. Our last task is to show that the Jacobian of the BRST transformation is unity. The Jacobian is:

$$J = J(\frac{A_\mu^a(x) + \delta A_\mu^a(x), \eta^a(x) + \delta\eta^a(x), \overline{\eta}^a(x) + \delta\overline{\eta}^a(x)}{A_\nu^b(y), \eta^b(y), \overline{\eta}^b(y)}).$$

The only non-vanishing elements of this determinant are:

$$\begin{aligned}
\frac{\partial(A_\mu^a(x) + \delta A_\mu^a(x))}{A_\nu^b(y)} &= \delta_\mu^\nu \delta^4(x-y)(\delta_{ab} - f_{abc}\eta^c\lambda) \ ; \\
\frac{\partial(\eta^a(x) + \delta\eta^a(x))}{\partial\eta^b(y)} &= \delta^4(x-y)(\delta_{ab} - \frac{1}{2}\frac{\partial}{\partial\eta^b}(f_{amn}\eta^m\eta^n)\lambda) \\
&= \delta^4(x-y)(\delta_{ab} + f_{abc}\eta^c\lambda) \ ; \\
\frac{\partial(A_\mu^a(x) + \delta A_\mu^a(x))}{\eta^b(y)} &= \delta^4(x-y) f_{abc} A_\mu^b \lambda \ ; \\
\frac{\partial(\overline{\eta}^a(x) + \delta\overline{\eta}^a(x))}{\partial A_\nu^b(y)} &= -\frac{\lambda}{\xi^a}\delta_\mu^\nu \partial^\mu \delta^4(x-y),
\end{aligned}$$

where we have taken the η-differentiation as ‘right differentiation’. So in schematic form, the Jacobian is (where n is the number of generators of the group):

$$J = (\delta_\mu^\nu)^n (\delta^4(x-y))^{3n} \begin{vmatrix} 1 - f\eta\lambda & fA\lambda & 0 \\ 0 & 1 + f\eta\lambda & 0 \\ -\frac{\lambda}{\xi^a}\partial^\mu & 0 & 1 \end{vmatrix} = (\delta_\mu^\nu)^n (\delta^4(x-y))^{3n},$$

since $\lambda^2 = 0$. In other words, the Jacobian is unity. This completes the proofs that we needed in section 2.2.2 on gauge theory.

Appendix B

Dynamical Equations of Massive Gravity

Linear Equations

Here we follow the ideas set out in [Kak08b]. In this section the linearized field equations (4.30) will be derived. These follow from variation of the lagrangian:

$$\begin{aligned}\mathcal{L} &= \sqrt{-g}R - \sqrt{-g}V(Y) \\ &= \frac{1}{8}(\partial_\mu h^\alpha_\alpha)^2 - \frac{1}{4}(\partial_\mu h_{\alpha\beta})^2 + \frac{1}{2}(\partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h^\lambda_\lambda)^2 - \frac{1}{4}m_h^2((h_{\mu\nu})^2 - (h^\alpha_\alpha)^2) \\ &- \frac{m_h^2}{m}A^\mu Q_\mu - \frac{m_h^2}{4m^2}F_{\mu\nu}F^{\mu\nu} + \text{higher order terms} \ ,\end{aligned} \tag{B.1}$$

where we used (4.27). The "vector" field A_μ can be decomposed as:

$$A_\mu = A^{tr}_\mu + \partial_\mu \chi, \quad \partial^\mu A^{tr}_\mu = 0.$$

Similarly for the diffeomorphisms:

$$\xi_\mu = \xi^{tr}_\mu + \partial_\mu \psi, \quad \partial^\mu \chi^{tr}_\mu = 0.$$

The diffeomorphisms ξ^{tr}_μ can be used to gauge away the transverse components A^{tr}_μ through the transformation (4.29), such that A_μ becomes:

$$A_\mu = \partial_\mu \chi.$$

The lagrangian then reads after partial integration, to quadratic order:

$$\mathcal{L} = \frac{1}{8}(\partial_\mu h^\alpha_\alpha)^2 - \frac{1}{4}(\partial_\mu h_{\alpha\beta})^2 + \frac{1}{2}(\partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h^\lambda_\lambda)^2 - \frac{1}{4}m_h^2((h_{\mu\nu})^2 - (h^\alpha_\alpha)^2) - \frac{m_h^2}{m}\chi\partial^\mu Q_\mu. \tag{B.2}$$

As we see χ is not a propagating d.o.f., but a Lagrange multiplier leading to the constraint $\partial_\mu Q^\mu = 0$. We still have one remaining diffeomorphism, which can be written as:

$$\xi_\mu = \partial_\mu \psi.$$

Under this diffeomorphism we have:

$$\begin{aligned} \delta\chi &= m\psi \ ; \\ \delta h_{\mu\nu} &= 2\partial_\mu\partial_\nu\psi. \end{aligned} \tag{B.3}$$

Variation of (B.1) gives the equations:

$$\begin{aligned} &\partial^\mu\partial^\nu h_{\mu\nu} - \partial^\mu\partial_\mu h^\alpha_\alpha = 0 \ ; \\ &\partial^\alpha\partial_\alpha h_{\mu\nu} + \partial_\mu\partial_\nu h^\alpha_\alpha - \partial_\mu\partial^\alpha h_{\alpha\nu} - \partial_\nu\partial^\alpha h_{\alpha\mu} + \eta_{\mu\nu}(\partial^\alpha\partial^\beta h_{\alpha\beta} - \partial^\beta\partial_\beta h^\alpha_\alpha) = \\ &m_h^2(h_{\mu\nu} - \eta_{\mu\nu}h^\alpha_\alpha) - \frac{2m_h^2}{m}(\partial_\mu\partial_\nu\chi - \eta_{\mu\nu}\partial^\alpha\partial_\alpha\chi). \end{aligned} \tag{B.4}$$

Differentiating the second equation of (B.4) w.r.t. ∂^μ and contracting indices, we obtain:

$$\partial^\nu h_{\mu\nu} = \partial_\mu h^\alpha_\alpha. \tag{B.5}$$

Substituting this into the second equation of motion gives:

$$\partial^\alpha\partial_\alpha h_{\mu\nu} = m_h^2 h_{\mu\nu} + \partial_\mu\partial_\nu\tilde{h}, \tag{B.6}$$

where we define:

$$\tilde{h} := h^\alpha_\alpha - \frac{2m_h^2}{m}\chi.$$

Under the remaining diffeomorphism ψ, the variable $\tilde{h}$ will transform as:

$$\delta\tilde{h} = 2(\partial^\mu\partial_\mu - m_h^2)\psi.$$

With the help of the Green's function D_{KG} of the massive Klein-Gordon equation, satisfying $(\partial^\mu\partial_\mu - m_h^2)D_{\mathrm{KG}}(x) = \delta(x)$, we may take $\psi = -\frac{1}{2}\int D_{\mathrm{KG}}(x-y)\tilde{h}(y)d^Dy$, giving $\delta\tilde{h} = -\tilde{h}$ and hence gauge:

$$\tilde{h}' = 0. \tag{B.7}$$

However even after the gauge fixing (B.7), there is residual gauge symmetry in the system. The gauge fixing condition (B.7) is preserved by diffeomorphisms satisfying the following massive Klein-Gordon equation:

$$(\partial^\mu\partial_\mu - m_h^2)\psi = 0, \tag{B.8}$$

which is the same as that for the graviton modes:

$$\partial^\alpha\partial_\alpha h_{\mu\nu} = m_h^2 h_{\mu\nu}. \tag{B.9}$$

Because of (B.9), we have that $\psi = \frac{-h^\alpha_\alpha}{2m_h^2}$ satisfies (B.8). Using (B.8) and (B.3), it follows that this diffeomorphism can be used to gauge h^α_α to zero:

$$\delta h^\alpha_\alpha = -h^\alpha_\alpha \longrightarrow (h^\alpha_\alpha)' = 0. \tag{B.10}$$

Dropping the primes, equations (B.7) and (B.10) give $h^\alpha_\alpha = \chi = 0$. In other words, the longitudinal component of the field A_μ can be set to zero. Taking $h^\alpha_\alpha = \chi = 0$ into account, then (B.5) and (B.6) give the linearized equations of motion (4.30).

Non-linear Equations

In this section we will show that one cannot end up with the normal 5 d.o.f. of massive gravity, even with the freedom of adding non-derivative higher order terms of the metric fluctuation with *arbitrary* coefficients. Instead, we will see that one *always* ends up with 6 degrees of freedom: 5 d.o.f. coming from the massive graviton and 1 d.o.f. from a scalar. We will not actually show that the scalar is a ghost. This is more work and we refer to [CNPT05], [BD72a].

Consider a general completion of the EH lagrangian, where *non-derivative* terms of the metric fluctuation are added:

$$\mathcal{L} = \mathcal{L}_{EH} + \sum_{n\geq 2} \mathcal{L}_n. \tag{B.11}$$

All terms of power n in the metric fluctuation h are grouped in $\mathcal{L}_n$. We will again express everything in the ADM variables (4.33) and show that $\delta N := N - 1$ is no longer a lagrange multiplier as in normal GR, thus the dynamical equation of δN (or equivalently N) is no longer a constraint but is an equation which expresses δN in the other variables. This then raises the d.o.f. from 2 to 3. The d.o.f. is further raised from 3 to 6 since the N_j also are no longer Lagrange multipliers[1]. We will show that one cannot cancel all higher (than linear) order terms of δN simultaneously, thus implying that δN (or equivalently N) is not a Lagrange multiplier. Notice that, given the non-linear relation between $h_{\mu\nu}$ and the ADM variables, a generic n-th order expression in $h_{\mu\nu}$ also contributes to orders higher than n when expressed in ADM variables.

Quadratic terms $\mathcal{L}_2$: The most general lagrangian at quadratic order is:

$$\mathcal{L}_2 = a_2[h^2] + b_2[h]^2. \tag{B.12}$$

In this expression[2] we find the term proportional to δN^2:

$$\mathcal{L}_2 \supset 4(a_2 + b_2)\delta N^2. \tag{B.13}$$

Hence, again the Pauli-Fierz combination $b_2 = -a_2$ should be taken in order to cancel the second order term in δN. The coefficient a_2 fixes the mass of the graviton, and for our purposes we can take it to be 1. At quadratic level the lagrangian is thus unitary, but $\mathcal{L}_2$ also contributes to third and fourth order terms in δN; in particular it contains a term $-2h\delta N^2$. We are therefore forced to introduce cubic terms in $h_{\mu\nu}$.

Cubic terms $\mathcal{L}_3$: The most general lagrangian at cubic order is:

$$\mathcal{L}_3 = a_3[h^3] + b_3[h][h^2] + c_3[h]^3. \tag{B.14}$$

Cubic terms in the ADM variables come both from $\mathcal{L}_3$ and $\mathcal{L}_2$. In particular those non-linear terms involving δN are:

$$\mathcal{L}_2 + \mathcal{L}_3 \supset (12c_3 + 4b_3 - 2)h\delta N^2 + 8(a_3 + b_3 + c_3)\delta N^3. \tag{B.15}$$

[1] This will not be showed. However, because of the symmetry of the spatial indices, either all of the variables N_j are Lagrange multipliers, or none of them are. Thus the d.o.f. is either 3 or 6. Either way we do not get massive gravity.

[2] For convenience we use the notation $[h] = \eta^{\mu\nu}h_{\mu\nu}$, $[h^2] = \eta^{\mu\nu}\eta^{\alpha\beta}h_{\mu\alpha}h_{\nu\beta}$, and its straightforward generalization to higher orders.

Note that $h = h_{ii}$ on the right hand side of (B.15). We can set both terms in (B.15) to zero by choosing:

$$a_3 = 2c_3 - \frac{1}{2}, \quad b_3 = \frac{1}{2} - 3c_3. \tag{B.16}$$

The coefficient c_3 is still undetermined. Now we are forced to introduce quartic terms in $h_{\mu\nu}$ to cancel undesired quartic terms containing δN coming both from $\mathcal{L}_2$ and $\mathcal{L}_3$.

Quartic terms $\mathcal{L}_4$: The most general lagrangian at quartic order is:

$$\mathcal{L}_4 = a_4[h^4] + b_4[h][h^3] + c_4[h^2]^2 + d_4[h]^2[h^2] + e_4[h]^4. \tag{B.17}$$

Quartic terms in the ADM variables come from $\mathcal{L}_2, \mathcal{L}_3$ and $\mathcal{L}_4$. In particular those non-linear terms involving δN are:

$$\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 \supset (Ah^2 + Bh_{ij}h^{ij} + CN_jN^j)\delta N^2 + Dh\delta N^3 + E\delta N^4. \tag{B.18}$$

After a lengthy calculation the following relationship between the coefficients $(A, ..., E)$ and $(c_3, a_4, ..., e_4)$ can be shown:

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 & 4 & 24 \\ -3 & 0 & 0 & 8 & 4 & 0 \\ 0 & -16 & -12 & -16 & -8 & 0 \\ 0 & 0 & 8 & 0 & 16 & 32 \\ 0 & 16 & 16 & 16 & 16 & 16 \end{pmatrix} \cdot \begin{pmatrix} c_3 \\ a_4 \\ b_4 \\ c_4 \\ d_4 \\ e_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ 2 \\ 0 \end{pmatrix}. \tag{B.19}$$

We would like to set the vector $(A, ..., E)$ to zero. One naively thinks this is possible since one has 5 conditions and 6 free parameters. On the contrary, this is impossible. This is because the matrix above has rank 4 and the space spanned by it thus not contain the inhomogeneous term. Thus, there is now way of cancelling all the quartic non-linear terms containing δN. In summary, we tried to tune all interactions $h^n_{\mu\nu}$ in order to keep the hamiltonian linear in N (or equivalently δN), this to ensure the presence of a constraint equation that eliminates the troublesome sixth degree of freedom. We found that when fourth order terms are taken into account, this tuning is impossible. We conclude that the ghost is unavoidable when one adds non-derivative terms to the EH lagrangian.

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