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Thick brane in 7D and 8D spacetimes

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Abstract We consider a thick brane model using two interacting scalar fields in 7D and 8D gravity. Using a special choice of potential energy, we obtain numerically regular asymptotically flat vacuum solutions. The possibility of obtaining the similar solutions for an arbitrary number of extra spatial dimensions is being estimated.

Keywords Branes, Scalar fields

1 Introduction

The idea about a multidimensional universe was suggested in the 1920s for unification of gravitational and electromagnetic interactions. But just in the 1980s, a great deal of interest in multidimensional models has been renewed within the framework of superstring theory. This theory pretends on unification all of the fundamental interactions in one theory. Perhaps the most important problem of all multidimensional theories consists in the unobservability of extra dimensions. To solve this problem, one can get an idea that extra dimensions are compactified and unobservable up to the Planck scales. Despite the numerous attempts for solving the problem of compactification of extra dimensions, a satisfactory solution is not found so far.

At the end of the 1990s, the idea about noncompact (infinite) extra dimensions had been proposed [1; 2] (see also the earlier works [3; 4; 5; 6]). This is the so-called brane model. In this model, matter is somehow confined (trapped) on a 4D surface (brane). On the basis of this idea, new ways for solution of some

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old problems of high-energy physics (the problem of mass hierarchy, stability of proton, etc.) were proposed. At the present time, the models of 4D-branes embedded in 5D and 6D spaces [7], and also the crossing 5D-branes in 6D [8] are under consideration.

It is assumed in the brane models that various types of matter (bosons, fermions, gauge fields and so on) are localized on a hypersurface (brane) embedded into external multidimensional space-time. However, in contrast to the multidimensional Kaluza-Klein theory, the extra dimensions can be macroscopic ones and, generally speaking, non-compact ones. Another important difference consists in different behavior of gravitational and matter fields: if the former exists and propagates in the bulk (i.e., in the external space), the matter fields are localized on the brane, and they are 4D objects. However, at such an approach it is possible realization of a situation when a multidimensional gravitational field can be also localized on the brane [9; 10] (see also [11]), i.e., it becomes effectively four dimensional, despite the fact that the extra dimensions can be macroscopic ones. The arising effective 4D gravitational constant is defined not only by physics on the brane but also as a result of presence of the extra dimensions. These effects can be directly checked in experiments and observations both on small and cosmological scales (see, e.g., Ref. [11]).

A special interest consists in consideration of various scalar fields within the framework of brane world scenarios. The point is that scalar fields are widely used in particle physics, astrophysics and cosmology [12]. Scalar fields were considered in different aspects within the framework of brane theory in Refs. [13; 14; 15; 16; 17; 18; 19] (for a review, see [20]). This interest to the scalar fields is also being explained by relative simplicity of equations and solutions being obtained with use of them. It allows to make qualitative analysis of equations and find sufficiently clear physical interpretation of results by analogy to other fields of physics.

One of the main point at a consideration of the brane models is that the infinitely thin brane models are under consideration in most cases. Certainly, it is absolutely unsatisfactory from the physical point of view. The point is that obtaining of self-consistent solutions of gravitational equations interacting with matter and describing a thick brane is not a simple mathematical problem. At the present time, the following thick-brane solutions exist: in Ref. [21] such solutions were obtained as monopole-like solutions. The set of gravitating scalar fields with non-trivial topological configuration far from the brane was considered. It was shown that such solutions exist for co-dimension of the brane more or equal to 2. In Ref. [23] a thick-brane solution in non-Riemannian generalization of the 5D Kaluza-Klein theory was obtained. In that case the affine connection, except the Christoffel symbols, has also the Weyl term which was being described by a scalar field. In Refs. [24; 25] thick branes were obtained for a special form of scalar field potential. It has allowed to integrate once the field equations for the scalar field. It is important for existence of these solutions that the potential of the scalar field is unbounded from below, i.e., $V(\phi) \rightarrow -\infty$ as $\phi \rightarrow \pm\infty$.

In the light of the above, it is interesting to consider various types of the multidimensional brane models with scalar fields, and to search for localized solutions with finite energy for an arbitrary number of extra dimensions. One of possibilities consists in a consideration of two nonlinear gravitating scalar fields which can create a 4D brane in multidimensional space-times. From the physical point

of view, there is the following situation: an interaction potential of these fields has two local and two global minima. It means that there are two different vacuums. The multidimensional space is filled by these scalar fields which are located in that vacuum in which the scalar fields are in the local minimum. Therefore, there is a defect in the form of the 4D brane on the background of this vacuum.

We have already investigated similar problems earlier: for the 5D case in Ref. [26], for the 6D one in Ref. [27]. The obtained solutions show a possibility of localization of probe scalar fields on the brane.

A search of thick brane solutions is very complicated mathematical problem. The authors know only three classes of such solutions: (a) the thick branes with scalar field potentials unbounded from below [24; 25]; (b) the thick branes with topologically non-trivial scalar fields [21; 22]; and (c) the thick branes with two nonlinear scalar fields bounded from below [26; 27]. Thus the goal of our paper is to extend the solutions from Refs. [26; 27] to higher dimensions. We do not consider trapping of any matter on such branes. This problem should be considered elsewhere.

2 General equations

In general case of $D = 4 + n$ dimensional gravity the action can be written as follows [28]:

$$S = \int d^D x \sqrt{g} \left[-\frac{M^{n+2}}{2} R + L_m \right], \quad (1)$$

where M is the fundamental mass scale and n is a number of extra dimensions. As a source of matter fields L_m we chose two interacting scalar fields ϕ, χ with the Lagrangian

$$L_m = \frac{1}{2} \partial_A \phi \partial^A \phi + \frac{1}{2} \partial_A \chi \partial^A \chi - V(\phi, \chi), \quad (2)$$

where the potential energy

$$V(\phi, \chi) = \frac{\Lambda_1}{4} (\phi^2 - m_1^2)^2 + \frac{\Lambda_2}{4} (\chi^2 - m_2^2)^2 + \phi^2 \chi^2 - V_0. \quad (3)$$

(This potential energy was obtained in [29] at approximate modeling of a condensate of gauge field in the SU(3) Yang-Mills theory.) Here and further capital Latin indices run over $A, B = 0, 1, 2, 3, \dots, D$ and small Greek indices $\alpha, \beta = 0, 1, 2, 3$ refer to four dimensions; Λ_1, Λ_2 are the self-coupling constants, m_1, m_2 are the masses of the scalar fields ϕ, χ , respectively; V_0 is an arbitrary constant which will be chosen below from physical reasons.

Use of two fields ensures presence of two global minima of the potential (3) at $\phi = 0, \chi = \pm m_2$ and two local minima at $\chi = 0, \phi = \pm m_1$ for values of the parameters Λ_1, Λ_2 used in the paper. The conditions for existence of the local minima are: $\Lambda_1 > 0, m_1^2 > \Lambda_2 m_2^2 / 2$, and for the global minima: $\Lambda_2 > 0, m_2^2 > \Lambda_1 m_1^2 / 2$. The

presence of these minima has allowed to find solutions localized on the brane for 5D and 6D cases in Refs. [26; 27] when the solutions have tended to one of the local minima asymptotically.¹

Variation of the action (1) with respect to the D -dimensional metric tensor g_{AB} gives the Einstein equations:

$$R_B^A - \frac{1}{2} \delta_B^A R = \frac{1}{M^{n+2}} T_B^A, \quad (4)$$

where R_B^A and T_B^A are the D -dimensional Ricci and the energy-momentum tensors, respectively. The corresponding scalar field equations can be obtained from (1) by its variation with respect to the field variables φ, χ . Then these equations will be

$$\frac{1}{\sqrt{D}g} \frac{\partial}{\partial x^A} \left[\sqrt{D}g g^{AB} \frac{\partial(\varphi, \chi)}{\partial x^B} \right] = - \frac{\partial V}{\partial(\varphi, \chi)}. \quad (5)$$

3 7D case

In this case $n = 3$, and the metric can be chosen in the form [28]

$$ds^2 = \phi^2(r) \eta_{\alpha\beta} dx^\alpha dx^\beta - \lambda(r) (dr^2 + r^2 d\Omega_2^2), \quad (6)$$

where $\eta_{\alpha\beta}$ is a flat 4D spacetime metric, and the metric functions ϕ and λ depend only on the extra coordinate r . $d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\psi^2$ is an angular part of the metric depending only on 6th and 7th coordinates.

Then, using (2), (4) and (6), one can get the system of gravitational equations in the following form

$$3 \left(2 \frac{\phi''}{\phi} - \frac{\phi' \lambda'}{\phi \lambda} \right) + 6 \left(\frac{\phi'}{\phi} \right)^2 + 2 \left\{ 3 \frac{\phi'}{\phi} \frac{(r^2 \lambda)'}{r^2 \lambda} + \frac{(r^2 \lambda)''}{r^2 \lambda} - \frac{1}{4} \left[\frac{(r^2 \lambda)'}{r^2 \lambda} \right]^2 - \frac{1}{2} \frac{\lambda'}{\lambda} \frac{(r^2 \lambda)'}{r^2 \lambda} - \frac{1}{r^2} \right\} = - \frac{2\lambda}{M^5} T_\alpha^\alpha, \quad (7)$$

$$12 \left(\frac{\phi'}{\phi} \right)^2 + 2 \left\{ 4 \frac{\phi'}{\phi} \frac{(r^2 \lambda)'}{r^2 \lambda} + \frac{1}{4} \left[\frac{(r^2 \lambda)'}{r^2 \lambda} \right]^2 - \frac{1}{r^2} \right\} = - \frac{2\lambda}{M^5} T_r^r, \quad (8)$$

$$4 \left(2 \frac{\phi''}{\phi} - \frac{\phi' \lambda'}{\phi \lambda} \right) + 12 \left(\frac{\phi'}{\phi} \right)^2 + 4 \frac{\phi'}{\phi} \frac{(r^2 \lambda)'}{r^2 \lambda} + \frac{(r^2 \lambda)''}{r^2 \lambda} - \frac{1}{2} \left[\frac{(r^2 \lambda)'}{r^2 \lambda} \right]^2$$

¹ Note here that there are the following vacuums for the potential (3): two true vacuums in the global minima, two false vacuums in the local minima, one metastable vacuum in the local maximum, and four metastable vacuums in the saddle points.

$$-\frac{1}{2} \frac{\lambda'}{\lambda} \frac{(r^2 \lambda)'}{r^2 \lambda} = -\frac{2\lambda}{M^5} T_\theta^\theta, \quad (9)$$

where a prime denotes differentiation with respect to r , and the corresponding components of the energy–momentum tensor are:

$$T_\alpha^\alpha = T_\theta^\theta = \frac{1}{2\lambda} (\phi'^2 + \chi'^2) + V(\phi, \chi), \quad (10)$$

$$T_r^r = -\frac{1}{2\lambda} (\phi'^2 + \chi'^2) + V(\phi, \chi). \quad (11)$$

Multiplying equation (9) by $3/4$ and subtracting its from (7), and also subtracting equation (8) from (9), we have

$$\begin{aligned} \frac{\lambda''}{\lambda} - \frac{3}{5} \left(\frac{\lambda'}{\lambda} \right)^2 - \frac{12}{5} \left(\frac{\phi'}{\phi} \right)^2 + \frac{4}{5r} \left(6 \frac{\phi'}{\phi} + \frac{13}{4} \frac{\lambda'}{\lambda} \right) + \frac{12}{5} \frac{\phi' \lambda'}{\phi \lambda} \\ = -\frac{2}{5} \lambda \left[\frac{1}{2\lambda} (\phi'^2 + \chi'^2) + V(\phi, \chi) \right], \end{aligned} \quad (12)$$

$$8 \left(\frac{\phi''}{\phi} - \frac{\phi' \lambda'}{\phi \lambda} - \frac{\phi'}{r\phi} \right) + \frac{\lambda''}{\lambda} - \frac{3}{2} \left(\frac{\lambda'}{\lambda} \right)^2 - \frac{\lambda'}{r\lambda} = -2 (\phi'^2 + \chi'^2), \quad (13)$$

where the following rescalings have been used $r \rightarrow r/M^{5/2}$, $\phi \rightarrow M^{5/2}\phi$, $\chi \rightarrow M^{5/2}\chi$, $m_{1,2} \rightarrow M^{5/2}m_{1,2}$. For the 7D metric given by (6) the ϕ, χ scalar field equations become

$$\phi'' + \left(\frac{2}{r} + 4 \frac{\phi'}{\phi} + \frac{1}{2} \frac{\lambda'}{\lambda} \right) \phi' = \lambda \phi [2\chi^2 + \Lambda_1 (\phi^2 - m_1^2)], \quad (14)$$

$$\chi'' + \left(\frac{2}{r} + 4 \frac{\phi'}{\phi} + \frac{1}{2} \frac{\lambda'}{\lambda} \right) \chi' = \lambda \chi [2\phi^2 + \Lambda_2 (\chi^2 - m_2^2)]. \quad (15)$$

The system of equations (12)–(15) is a system of nonlinear equations with solutions whose behavior depends essentially from values of the parameters m_1, m_2 and Λ_1, Λ_2 . As it was shown in Refs. [26; 27], a problem of a search of *regular* solutions for systems similar to (12)–(15) reduced to evaluation of eigenvalues of the parameters m_1, m_2 at some values of the self-coupling constants Λ_1, Λ_2 . Only for these values of the parameters *regular* solutions with *finite* energy exist.

We will search for such regular solutions in a whole space. Because of the fact that it is not possible to find analytical solutions, we search for a solution numerically. In order to make a solution regular everywhere, let us consider a solution near the brane, i.e., at $r \approx 0$. We represent the solutions by a power series in r :

$$\phi(r) \approx \phi_0 + \phi_2 \frac{r^2}{2}, \quad (16)$$

$$\lambda(r) \approx \lambda_0 + \lambda_2 \frac{r^2}{2}, \quad (17)$$

$$\varphi(r) \approx \varphi_0 + \varphi_2 \frac{r^2}{2}, \quad (18)$$

$$\chi(r) \approx \chi_0 + \chi_2 \frac{r^2}{2}, \quad (19)$$

where the subscript 0 denotes a value of the function at $r = 0$, and the subscript 2 denotes a second derivative of the corresponding function at $r = 0$. Substitution of these series into equations (12)–(15) gives the following values for $\phi_2, \lambda_2, \varphi_2, \chi_2$

$$\lambda_2 = \frac{1}{15} \lambda_0^2 V(0), \quad \phi_2 = -\frac{2}{15} \phi_0 \lambda_0 V(0), \quad (20)$$

$$\varphi_2 = \frac{1}{3} \lambda_0 \phi_0 [2\chi_0^2 + \Lambda_1 (\phi_0^2 - m_1^2)], \quad (21)$$

$$\chi_2 = \frac{1}{3} \lambda_0 \chi_0 [2\varphi_0^2 + \Lambda_2 (\chi_0^2 - m_2^2)]. \quad (22)$$

Let us note here that, as it was shown by previous investigations [26; 27], existence of regular solutions is possible when the field ϕ has a minimum at $r = 0$, and the field χ has a maximum, i.e., it is necessary to ensure the fulfilment of inequalities $\varphi_2 > 0$ and $\chi_2 < 0$. As follows from (21)–(22), these conditions impose restrictions on the boundary conditions ϕ_0, χ_0 and values of the parameters Λ_1, Λ_2 .

Equations (12)–(15) cannot be solved analytically. But numerical analysis also faces some difficulties. As it was shown by preliminary numerical analysis, regular solutions exist not for all $m_{1,2}$ but for some special values of these parameters. This means that we have deal with a nonlinear eigenvalue problem. The functions $\phi(r), \chi(r)$ are eigenfunctions, and the parameters $m_{1,2}$ are eigenvalues, and the condition $\Lambda_1 \neq \Lambda_2$ should be satisfied. In particular, we chose $\Lambda_1 = 0.1, \Lambda_2 = 1.0$.

3.1 Numerical solution

In this section we describe a method of numerical solution of equations (12)–(15) in details. As it was mentioned above, the specific feature of the system of equations is that determining of the functions ϕ and χ is a nonlinear eigenvalue problem. Usually, numerical solution of one ordinary differential equation for a search of eigenfunctions and eigenvalues is carrying out by the shooting method. The essence of the method is to try to find (using a method of step-by-step approximation) some eigenvalue at which an eigenfunction is regular. For example, in such a

way one can find a discrete energy spectrum of a particle in a 1D potential well of an arbitrary shape. Unfortunately, this method does not work in a case when one has a eigenvalues and eigenfunctions problem with two variables. It takes place in our case: we have two eigenfunctions ϕ, χ and two eigenvalues $m_{1,2}$.

We therefore will search for a numerical solution in the following way. Searching for eigenvalues of equations (14)–(15), we only solve on each step either equation (14) or (15). Since there is also another function in this equation, its value is being taken from a previous step. Having found (with some accuracy) some values of the functions ϕ, χ and eigenvalues $m_{1,2}$, one should insert them into the Einstein equations (12)–(13) for determining the metric functions ϕ, λ . Then these functions should be inserted in one of field equations (14)–(15) which are being solved as an eigenfunctions problem for the functions ϕ, χ and eigenvalues $m_{1,2}$ once again. These iterations are being repeated necessary number of times for obtaining of a solution in the range $r \in [\Delta, r_f]$, where r_f is some final point on the axis r .

Note the most essential features of the method:

- One cannot start a numerical solution at $r = 0$ because of terms like $y(r)/r$ in equations (12)–(15), where $y(r)$ is one of the functions $\phi, \varphi, \chi, \lambda$. One therefore should start a numerical solution at $r = \Delta \neq 0$ using the following boundary conditions

$$\phi(\Delta) \approx \phi_0 + \phi_2 \frac{\Delta^2}{2}, \quad (23)$$

$$\lambda(\Delta) \approx \lambda_0 + \lambda_2 \frac{\Delta^2}{2}, \quad (24)$$

$$\varphi(\Delta) \approx \varphi_0 + \varphi_2 \frac{\Delta^2}{2}, \quad (25)$$

$$\chi(\Delta) \approx \chi_0 + \chi_2 \frac{\Delta^2}{2}. \quad (26)$$

- One should not try to find a numerical solution for large r_f because any numerical method has some inaccuracy. This inaccuracy defines (by some unknown way) the interval $[\Delta, r_f]$ in which an exact solution differs insignificantly from numerical one.
- An eigenvalues problem is very sensible to calculated eigenvalues. For example, let us consider the equation

$$\frac{d^2 y(x)}{dx^2} = 2y(x) [y(x)^2 - e]. \quad (27)$$

This equation, being considered as an eigenvalues problem with the eigenfunction $y(x)$ and the eigenvalue e , has an exact solution $y = \tanh(x), e = 1$. If one solves this equation numerically with initial conditions which follow from the exact solution

Fig. 1 The scalar fields $\varphi(r), \chi(r)$ for the boundary conditions given in (30). The *solid lines* correspond to the 7D case, the *dashed lines* correspond to the 8D case

Fig. 2 The metric functions $\phi(r), \lambda(r)$ for the boundary conditions given in (30). The *solid lines* correspond to the 7D case, the *dashed lines* correspond to the 8D case

$$y(0) = 0, \quad y'(0) = 1, \quad (28)$$

then one can see that for $x \approx 6$ the numerical solution significantly differs from the exact one.

Thus, we chose the following algorithm for numerical solution of equations (12)–(15):

1. The choice of a zeroth approximation for the function $\varphi(r)$ or $\chi(r)$.
2. Numerical solution of equation (14) or (15) by the shooting method with a calculation of the functions $\varphi(r)$ or $\chi(r)$, and m_1 or m_2 .
3. Numerical solution of equation (15) or (14) by the shooting method with calculation of the functions $\chi(r)$ or $\varphi(r)$, and m_2 or m_1 .
4. Reiteration of the steps 2 and 3 a necessary number of times.
5. Substitution of the functions $\varphi(r)$ and $\chi(r)$ into the Einstein equations (12) and (13) and their solution by usual numerical method.
6. Substitution of the functions $\phi(r)$ and $\lambda(r)$ into equations (14) or (15) and reiteration of the steps 2–5 a necessary number of times.
7. Check of the obtained solutions by solution of initial equations (12)–(15) by using the parameters $m_{1,2}$ which have been calculated on the steps 2–3.

For these calculations we use the program package *Mathematica* for numerical solution of differential equations.

3.2 Results

Using the procedure described in the previous section, let us find a self-consistent numerical solution of the system (12)–(15). We chose the following boundary conditions at $r = 0$

$$\begin{aligned} \varphi_0 &= \sqrt{3}, & \varphi'_0 &= 0, \\ \chi_0 &= \sqrt{0.6}, & \chi'_0 &= 0, \\ \phi_0 &= 1.0, & \phi'_0 &= 0, \\ \lambda_0 &= 1.0, & \lambda'_0 &= 0, \end{aligned} \quad (29)$$

and $\Lambda_1 = 0.1, \Lambda_2 = 1.0$. The arbitrary constant V_0 is being chosen as $V_0 = (\Lambda_2/4)m_2^4$ for zeroing of the energy density as $r \rightarrow \infty$.

In this case regular solutions exist at $m_1 \approx 2.31505626$ and $m_2 \approx 3.08288116$. The obtained solutions are presented in Figs. 1 and 2. The corresponding plot for the energy density is shown in Fig. 3. As one can see from the last figure, the

Fig. 3 The energy density $T_0^0(r)$ from (10). The *solid line* corresponds to the 7D case, the *dashed line* corresponds to the 8D case

energy density tends asymptotically to zero as $r \rightarrow \infty$, i.e., the scalar fields are trapped on the 4D-brane.

It is difficult to see an asymptotic behavior of the metric functions from Fig. 2. To clarify this question, let us search for asymptotic solutions of equations (14)–(15) in the form

$$\varphi = m_1 - \delta\varphi, \quad \chi = \delta\chi, \quad (30)$$

where $\delta\varphi, \delta\chi \ll 1$ as $r \rightarrow \infty$. Such an asymptotic behavior of the fields means that the solutions tend to the local minimum at $\varphi = m_1, \chi = 0$. Then the right hand sides of equations (12)–(13) tend to zero, i.e., one has the following asymptotic equations for the metric functions

$$\frac{\lambda''}{\lambda} - \frac{3}{5} \left(\frac{\lambda'}{\lambda} \right)^2 - \frac{12}{5} \left(\frac{\phi'}{\phi} \right)^2 + \frac{4}{5r} \left(6 \frac{\phi'}{\phi} + \frac{13}{4} \frac{\lambda'}{\lambda} \right) + \frac{12}{5} \frac{\phi'}{\phi} \frac{\lambda'}{\lambda} = 0, \quad (31)$$

$$8 \left(\frac{\phi''}{\phi} - \frac{\phi'}{\phi} \frac{\lambda'}{\lambda} - \frac{\phi'}{r\phi} \right) + \frac{\lambda''}{\lambda} - \frac{3}{2} \left(\frac{\lambda'}{\lambda} \right)^2 - \frac{\lambda'}{r\lambda} = 0. \quad (32)$$

Let us search for solutions of these equations in the form

$$\phi \approx \phi_\infty - \frac{C_1}{r^\alpha}, \quad (33)$$

$$\lambda \approx \lambda_\infty + \frac{C_2}{r^\beta}. \quad (34)$$

(The subscript “ ∞ ” indicates the asymptotic value of the variable, $C_1 > 0, C_2 > 0$ and α, β are some arbitrary constants.) One can check that in the case under consideration α and β are equal to each other. Then we have from (31)–(32)

$$\phi \approx \phi_\infty - \frac{C_1}{r}, \quad (35)$$

$$\lambda \approx \lambda_\infty + \frac{C_2}{r}. \quad (36)$$

Inserting these solutions in (14)–(15), and taking into account (29) we have

$$\delta\phi'' + \frac{2}{r}\delta\phi' = 2\lambda_\infty\Lambda_1 m_1^2 \delta\phi, \quad (37)$$

$$\delta\chi'' + \frac{2}{r}\delta\chi' = \lambda_\infty (2m_1^2 - \Lambda_2 m_2^2) \delta\chi \quad (38)$$

with regular solutions

$$\delta\phi \approx C_\phi \frac{\exp\left(-\sqrt{2\lambda_\infty\Lambda_1 m_1^2} r\right)}{r}, \quad (39)$$

$$\delta\chi \approx C_\chi \frac{\exp\left(-\sqrt{\lambda_\infty(2m_1^2 - \Lambda_2 m_2^2)} r\right)}{r}, \quad (40)$$

where C_ϕ, C_χ are integration constants. Thus one can see from (29), (32), (33), (38), and (39) that the asymptotic behavior corresponds to a 7D Minkowski space-time with the scalar fields energy density equal to zero.

Now let us calculate the dimensionless brane tension. For this purpose we take an integral over the extra space:

$$\sigma = 8\pi \int_0^\infty T_0^0(r) r^2 \lambda(r)^{3/2} dr \approx 5.$$

One can see that, despite the negative energy density near the brane, the tension is positive because the energy density is positive as $r \rightarrow \infty$.

4 8D case

We now consider the 8D problem. For this case $n = 4$, and we chose the metric as follows:

$$ds^2 = \phi^2(r) \eta_{\mu\nu} dx^\mu dx^\nu - \lambda(r) \{ dr^2 + r^2 [d\theta^2 + \sin^2 \theta (d\psi^2 + \sin^2 \psi d\xi^2)] \}, \quad (41)$$

here $\mu, \nu = 0, 1, 2, 3$ refer to four dimensions; r, θ, ψ, ξ are extra coordinates; θ, ψ, ξ are polar angles on a 3D sphere; $\eta_{\mu\nu} = \{+1, -1, -1, -1\}$ is the 4D Minkowski metric. Inserting the metric (41) in (4)–(5), one can obtain the following equations

$$4 \left(\frac{\phi''}{\phi} - \frac{\lambda' \phi}{\lambda \phi} - \frac{\phi'}{r \phi} \right) + \frac{\lambda''}{\lambda} - \frac{3}{2} \left(\frac{\lambda'}{\lambda} \right)^2 - \frac{\lambda'}{r \lambda} = - [\phi'^2 + \chi'^2], \quad (42)$$

$$\begin{aligned} \frac{\lambda''}{\lambda} - 2 \left(\frac{\phi'}{\phi} \right)^2 + \frac{4}{r} \left(\frac{\phi'}{\phi} + \frac{\lambda'}{\lambda} \right) + 2 \frac{\phi' \lambda'}{\phi \lambda} - \frac{1}{4} \left(\frac{\lambda'}{\lambda} \right)^2 \\ = -\frac{1}{3} \lambda \left[\frac{1}{2\lambda} (\phi'^2 + \chi'^2) + V \right], \end{aligned} \quad (43)$$

$$2 \frac{\phi'^2}{\phi^2} + 2 \frac{\phi' \lambda'}{\phi \lambda} + 4 \frac{\phi'}{r \phi} + \frac{3}{2} \frac{\lambda'^2}{\lambda^2} + 4 \frac{\lambda'}{r \lambda} = -\frac{1}{3} \lambda \left[-\frac{1}{2\lambda} (\phi'^2 + \chi'^2) + V \right], \quad (44)$$

$$\phi'' + \left(\frac{3}{r} + 4 \frac{\phi'}{\phi} + \frac{\lambda'}{\lambda} \right) \phi = \lambda \phi [2\chi^2 + \Lambda_1 (\phi^2 - m_1^2)], \quad (45)$$

$$\chi'' + \left(\frac{3}{r} + 4 \frac{\phi'}{\phi} + \frac{\lambda'}{\lambda} \right) \chi = \lambda \chi [2\phi^2 + \Lambda_2 (\chi^2 - m_2^2)], \quad (46)$$

where we have used the rescalings: $r \rightarrow r/M^3$, $\phi \rightarrow M^3 \phi$, $\chi \rightarrow M^3 \chi$, $m_{1,2} \rightarrow M^3 m_{1,2}$.

We will search for solutions of this system by analogy with the 7D case. Using the same boundary conditions (29), one can obtain the results presented in Figs. 1, 2 and 3.

It is interesting to estimate an asymptotic behavior of the solutions as $r \rightarrow \infty$. For this purpose let us search for a solution of equations (45)–(46) in the form (30). Then the right hand sides of equations (42)–(44) tend to zero, and we will search for solutions for the metric functions in the form (33)–(34). One can obtain from (42)–(44) that $\alpha = \beta = 2$, i.e., we have the following asymptotic behavior of the metric functions

$$\phi \approx \phi_\infty - \frac{C_1}{r^2}, \quad (47)$$

$$\lambda \approx \lambda_\infty + \frac{C_2}{r^2}. \quad (48)$$

Inserting these solutions in (45)–(46), and taking into account (29) we have

$$\delta\varphi'' + \frac{3}{r}\delta\varphi' = 2\lambda_\infty\Lambda_1 m_1^2 \delta\varphi, \quad (49)$$

$$\delta\chi'' + \frac{3}{r}\delta\chi' = \lambda_\infty (2m_1^2 - \Lambda_2 m_2^2) \delta\chi \quad (50)$$

with regular solutions

$$\delta\varphi \approx C_\varphi \frac{\exp\left(-\sqrt{2\lambda_\infty\Lambda_1 m_1^2} r\right)}{r^{3/2}}, \quad (51)$$

$$\delta\chi \approx C_\chi \frac{\exp\left(-\sqrt{\lambda_\infty(2m_1^2 - \Lambda_2 m_2^2)} r\right)}{r^{3/2}}, \quad (52)$$

where C_φ, C_χ are integration constants. Thus one can see from (29), (47), (48), (51) and (52) that, similar to the 7D case, the asymptotic behavior corresponds to a 8D Minkowski spacetime with the zero energy density of scalar fields.

Calculating the brane tension for this case, one can find

$$\sigma = 8\pi \int_0^\infty T_0^0(r) r^3 \lambda(r)^2 dr \approx 6.75,$$

i.e., one also has a positive brane tension as in the 7D case.

5 Arbitrary number of extra spatial dimensions

The numerical regular solutions obtained for the 7D and 8D cases allows us to hope that similar solutions can exist and for an arbitrary number of n extra spatial dimensions. Unfortunately, one cannot obtain numerical solutions for an arbitrary n because of the fact that it is not possible to eliminate n from the equations. However, we can estimate a possibility that such solutions exist. Let us use for this purpose the Einstein equations obtained in Ref. [28] for the generalized D -dimensional metric

$$ds^2 = \phi^2(r) \eta_{\alpha\beta} (x^\nu) dx^\alpha dx^\beta - \lambda(r) (dr^2 + r^2 d\Omega_{n-1}^2), \quad (53)$$

where $d\Omega_{n-1}^2$ is the solid angle for the $(n-1)$ sphere. It is convenient to rewrite the Einstein equations from [28] in the form

$$\begin{aligned} & 3 \left(2 \frac{\phi''}{\phi} - \frac{\phi' \lambda'}{\phi \lambda} \right) + 6 \frac{(\phi')^2}{\phi^2} + (n-1) \\ & \times \left[3 \frac{\phi'}{\phi} \left(\frac{\lambda'}{\lambda} + \frac{2}{r} \right) + \frac{\lambda''}{\lambda} - \frac{1}{2} \frac{\lambda'}{\lambda} \left(\frac{\lambda'}{\lambda} - \frac{6}{r} \right) + \frac{n-4}{4} \frac{\lambda'}{\lambda} \left(\frac{\lambda'}{\lambda} + \frac{4}{r} \right) \right] \\ & = - \frac{2\lambda}{M^{n+2}} T_\alpha^\alpha, \end{aligned} \quad (54)$$

$$12 \frac{(\phi')^2}{(\phi)^2} + (n-1) \left[4 \frac{\phi'}{\phi} \left(\frac{\lambda'}{\lambda} + \frac{2}{r} \right) + \frac{n-2}{4} \frac{\lambda'}{\lambda} \left(\frac{\lambda'}{\lambda} + \frac{4}{r} \right) \right] = - \frac{2\lambda}{M^{n+2}} T_r^r, \quad (55)$$

$$\begin{aligned} & 4 \left(2 \frac{\phi''}{\phi} - \frac{\phi' \lambda'}{\phi \lambda} \right) + 12 \frac{(\phi')^2}{\phi^2} + (n-2) \\ & \times \left[4 \frac{\phi'}{\phi} \left(\frac{\lambda'}{\lambda} + \frac{2}{r} \right) + \frac{\lambda''}{\lambda} - \frac{1}{2} \frac{\lambda'}{\lambda} \left(\frac{\lambda'}{\lambda} - \frac{6}{r} \right) + \frac{n-5}{4} \frac{\lambda'}{\lambda} \left(\frac{\lambda'}{\lambda} + \frac{4}{r} \right) \right] \\ & = - \frac{2\lambda}{M^{n+2}} T_\theta^\theta, \end{aligned} \quad (56)$$

and the scalar field equations are:

$$\phi'' + \left(\frac{n-1}{r} + 4 \frac{\phi'}{\phi} + \frac{n-2}{2} \frac{\lambda'}{\lambda} \right) \phi = \lambda \phi [2\chi^2 + \Lambda_1 (\phi^2 - m_1^2)], \quad (57)$$

$$\chi'' + \left(\frac{n-1}{r} + 4 \frac{\phi'}{\phi} + \frac{n-2}{2} \frac{\lambda'}{\lambda} \right) \chi = \lambda \chi [2\phi^2 + \Lambda_2 (\chi^2 - m_2^2)]. \quad (58)$$

As one can see from equations (54)–(56), the left hand sides could be regular at $r = 0$ if the boundary conditions $\lambda'(0) = 0, \phi'(0) = 0$ are hold true. Just these conditions for the metric functions were used by us at consideration of the 5D [26] and 6D [27] cases, and also the 7D and the 8D problems in this paper. Apparently, the situation remains the same and for a case of an arbitrary number of n extra spatial dimensions.

On the other hand, consideration of an asymptotic behavior of the metric functions ϕ, λ for the 7D and 8D cases shows that their behavior can be described for the arbitrary $n > 6$ in the following form

$$\phi \approx \phi_\infty - \frac{C_1}{r^{n-6}}, \quad (59)$$

$$\lambda \approx \lambda_\infty + \frac{C_2}{r^{n-6}}. \quad (60)$$

Such a behavior corresponds to fast transition of the solutions to a Minkowski spacetime. At the same time, an asymptotic behavior of the scalar fields can be described as follows:

$$\varphi \approx m_1 - C_\varphi \frac{\exp\left(-\sqrt{2\lambda_\infty \Lambda_1 m_1^2} r\right)}{r^{(n-1)/2}}, \quad (61)$$

$$\chi \approx C_\chi \frac{\exp\left(-\sqrt{\lambda_\infty (2m_1^2 - \Lambda_2 m_2^2)} r\right)}{r^{(n-1)/2}}. \quad (62)$$

So we have the regular solutions for an arbitrary n both near zero and as $r \rightarrow \infty$. Note that it is possible to expand the solutions near zero in a Taylor series (see (16)–(19)) and to obtain solutions with any necessary accuracy. One might expect that these solutions can be joined smoothly, and they would be regular in the entire range $0 \leq r < \infty$.

6 Conclusions

We have shown that two gravitating nonlinearly interacting scalar fields can form a 4-dimensional thick brane configuration in 7D and 8D spacetimes. Consideration of the problem has turned into investigation of the system of ordinary differential equations. It was shown that the regular solutions with the finite energy density exist only for some values of the masses of the scalar fields m_1, m_2 at some self-coupling constants Λ_1, Λ_2 which are given above. That is, the problem of a search of *eigenvalues* of the parameters m_1, m_2 for the system of nonlinear equations (4)–(5) was solved. The regular solutions with an asymptotically flat metric $\phi(\infty) = \phi_\infty, \lambda(\infty) = \lambda_\infty$ and with the zero energy density of the matter fields at $m_1 \approx 2.31505626$ and $m_2 \approx 3.08288116$ (for the 7D case) and $m_1 \approx 2.25005$ and $m_2 \approx 3.115$ (for the 8D case) were obtained. In both cases the solutions start with the boundary conditions (29) with subsequent transition to the local minimum of the potential (3).

A numerical analysis shows that the solutions exist because the potential $V(\varphi, \chi)$ has the local minimum. Our attempts to find a solution tending to the global minimum as $r \rightarrow \infty$ were unsuccessful. The possible reason is the Derrick's theorem [30] which forbids an existence of static regular solutions with finite energy for scalar fields in dimensions more than 2. At numerical solution (on the step 4) we obtain a solution which is regular both in 7D and 8D spacetimes. At first sight this

solution seems to be forbidden by the Derrick's theorem. But a more careful analysis shows that the obtained solution avoids the conditions of the given theorem because the scalar fields asymptotically tend to the local minimum but not to the global one.

Let us discuss a question about stability of the obtained solutions. If one consider these fields as classical ones, then it is possible to test the stability by the following way: (a) first of all, one can investigate small perturbations by a standard method; (b) then one should investigate large perturbations; (c) following which it is necessary to consider a behavior of this solution after quantizing of the scalar fields. In the last case the quantum fields can tunnel from a region of the local minimum (the false vacuum) to a region of the global minimum (the true vacuum). But in this case, apparently, the solution should decay into some waves because existence of a static solution with scalar fields tending asymptotically to a global minimum is forbidden by the Derrick's theorem.

But situation can turn out to be more interesting: the point is that in Ref. [29] there are some arguments in favour that the scalar fields, considered in this paper, are a quantum non-perturbative condensate of a $SU(3)$ gauge field. Briefly these arguments consist in the following: components of the $SU(3)$ gauge field can be divided in some natural way onto two parts. The first group contains those components which belong to a subgroup $SU(2) \in SU(3)$. The remaining components belong to a factor space $SU(3)/SU(2)$. Non-perturbative quantization is being carried out by the following approximate way: it is supposed that 2-point Green functions can be expressed via the scalar fields. The first field ϕ describes 2-point Green functions for $SU(2)$ components of a gauge potential, and the second scalar field χ describes 2-point Green functions for $SU(3)/SU(2)$ components of the gauge potential. It is supposed further that 4-point Green functions can be obtained as some bilinear combination of the 2-point Green functions. Consequently, the Lagrangian of the $SU(3)$ gauge field takes the form (2).

At such an interpretation, a question about stability of the obtained thick brane solution becomes rather nontrivial problem. In this case the obtained solution describes some defect in a spacetime filled by a condensate of the gauge field. A question about stability demands a non-perturbative quantized consideration of the $SU(3)$ gauge field with Green functions depending on time. The problem is that the suggested in Ref. [29] approximate method of description of Green functions can be used only for a static case.

Finally, in Sect. 5 we have discussed a possibility of obtaining of thick brane solutions with an arbitrary number of n extra spatial dimensions. It was shown for an arbitrary n that there are regular solutions near zero and as $r \rightarrow \infty$. It allows to hope for existence of solutions which smoothly join these regions.

All the results obtained in [26] for the 5D case and in [27] for the 6D case, and also in this paper, allow us to speak about principle possibility of localization of the scalar fields with the potential (3) on a brane in any dimensions. One has every reason to suppose that existence of similar regular solutions is possible for models with greater number of extra dimensions. In the future we suppose investigation of models with an arbitrary number of extra dimensions in studying this question.

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