THE MANDELSTAM REPRESENTATION IN PERTURBATION THEORY

P. V. Landshoff, J. C. Polkinghorne, and J. C. Taylor

University of Cambridge, Cambridge, England

(presented by J. C. Polkinghorne)

1. METHODS

The aim of this work is to show that the Mandelstam representation holds to every finite order of perturbation theory under the condition that no anomalous vertex thresholds exist. We shall use induction procedure, considering at each stage the leading singularity of a Feynman graph [1]. Its two-dimensional curve of singularity Σ will be divided into a number of parts on each of which the singularity is either wholly present or wholly absent on the physical sheet. It will then be shown to be wholly absent. The only exception to this part of the real section of Σ lying in the overlap of two cuts which must be singular in the inappropriate sense (defined below [2]) to give the boundary of the relevant Mandelstam spectral function. A simple example in which one considers more than one component of Σ would be the case in which it consists of two irreducible algebraic curves. Another example comes later.

2. BEHAVIOUR OF SINGULARITIES

If a point of Σ is singular, the α-hypercontour discussed by Eden is pinched between two halves of a locally cone-shaped portion of the surface of singularities. As one moves around on Σ, the vertex of the cone moves, dragging the hypercontour with it. The only way that one can change from singularity to non-singularity is for the pinch to fall off an "edge" of the hypercontour. This corresponds to a point where some of the α's are zero and all α's have coincidences, that is, an *effective* (same values of the α's) intersection of Σ with a lower order singularity. Such points can only form a set of dimension zero on Σ so that at first sight they could always be avoided. However, we must not leave the physical sheet and so must not cross the normal threshold cuts. In going around the cuts we may be forced to go through a lower order singularity. This is what happens in Tarski's fourth order singular case (Fig. 1) with the anomalous threshold cuts. Avoiding the cut $A\,A'$ in going from I to II forces us to go through the anomalous threshold singularity at A'.

All this is just a simple illustration of the continuity theorem.

3. THE INDUCTION PROCEDURE

We impose the condition of no anomalous thresholds to start the induction off. The way it proceeds is not quite trivial. It is necessary to be sure that the behaviour of a singularity is not different from the case in which it is the leading singularity.

In considering this problem we have to think about the distortions of the α-hypercontour. The generalization of an end-point is an edge in which a number

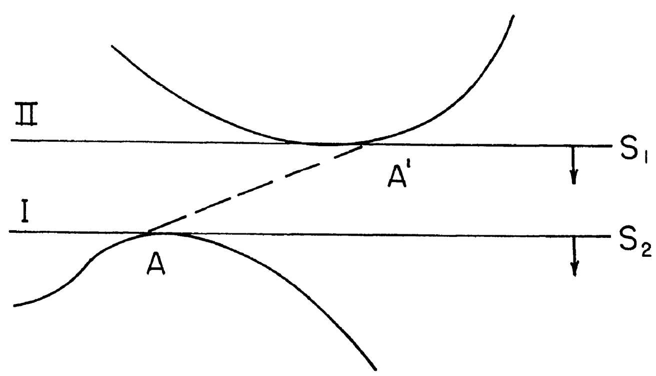

Fig. 1 Impossibility of avoiding threshold singularity because of pinched α-hypercontour.

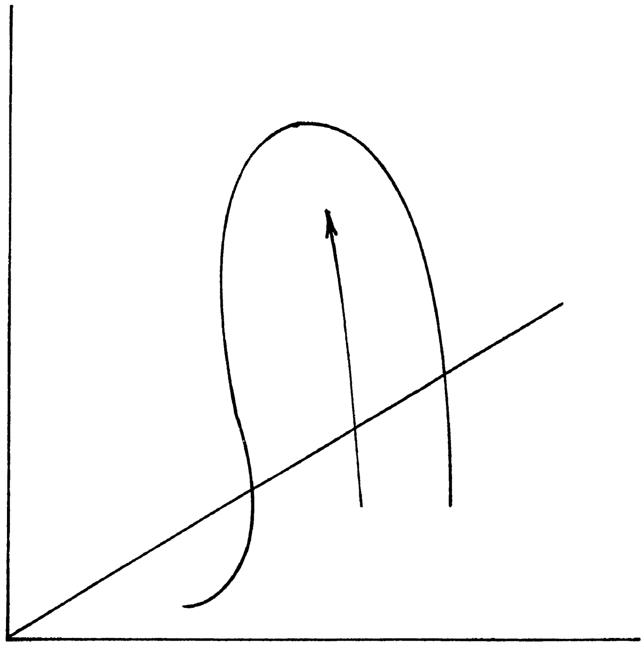

Fig. 2 Deformation of α-hypercontour and possible occurrence of irrelevant singularity.

of α's must be kept zero. The permissible distortions are then arbitrary in the sub-space of non-zero α's for that edge. Away from an edge it is completely arbitrary. Thus the edge gets distorted as it would when it is a leading singularity. But there may be intrusions into the relevant sub-space due to the contour being forced in from an interior point (Fig. 2). These could give extra singularities but, since they are not true edges where the pinch can fall off, they are irrelevant for us. Thus the induction procedure is indeed possible.

4. REALITY CONDITIONS

Symanzik has shown that there is a region in which the denominator of the Feynman function does not vanish and so the function is real. As we continue from this region the reality ceases to hold but there remains an important connection between behaviour at complex conjugate points. This implies that for the complex parts of Σ the two complex conjugate halves behave in a similar way. It implies also that for the real section there are only two distinct limits corresponding to the same or opposite signs for the imaginary parts of the two relevant variables z_1 and z_2 [2]. Which variables (out of s,t, and u) are relevant is determined by which two give the overlapping cuts at the point in question. If there are not two overlapping cuts, then all limits give the same behaviour.

5. THE CUTS

According to our induction hypothesis we have only the normal threshold cut. As far as possible we shall just go around any cuts we meet as we move around on Σ. A simple example of the difficulty that may arise is provided by Fig. 3. Our way out of Region I is blocked in each direction by a cut. To get to II we follow some such path as ACB (Fig. 3). We must cross two cuts to reach Region III, and at C we can cross the real section and so recross the two previous (dashed) cuts. Finally we circumnavigate S_1, and arrive at II. From A to C we have left the physical sheet but at the end of our journey we are back on it again. Our only worry is that between A and C we might have encountered dangerous singularities whose presence in an unphysical sheet is not covered immediately by our induction procedure.

Actually this point is all right. We need not go far into the unphysical sheet and will in fact skim along close to the real section. Thus only singularities which almost appear on the physical sheet will worry us, i.e., only lower order real curves Γ which are singular when a limit is taken in an inappropriate sense [3]. It is easy to see that these must have negative slope (see appendix) and so, since they must reach Σ

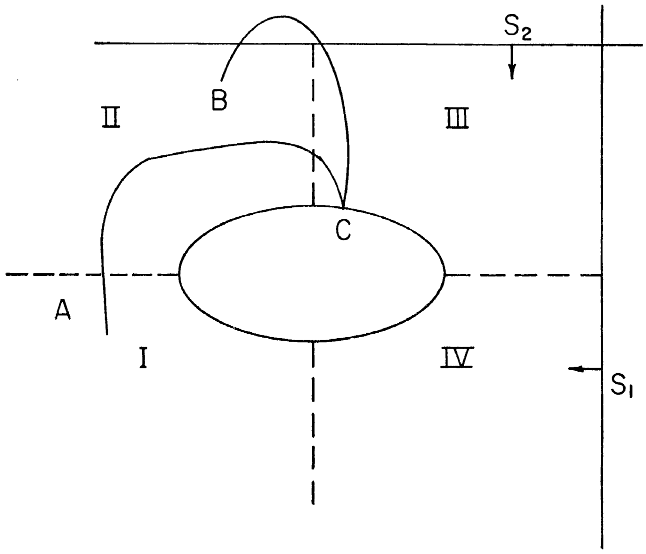

Fig. 3 Method of by-passing cuts which occur when moving along the curve of singularities Σ.

at an effective intersection, only the arc BC is really dangerous. However it may be made as short as we please to avoid trouble. Therefore all is well.

In this way we can cross cuts until we come to the normal threshold itself. It is easy to show that it cannot have a finite effective intersection and so the worst case is shown in Fig. 4, where a double point at infinity has been put in to make it as hard as possible. If Γ_1 and Γ_2 join up somewhere on the right then we can link up these complex surfaces there outside the cut. However if they do not, then there is no simple way of relating the parts of Σ connected to Γ_1 to those connected to Γ_2. In this case we must take these two parts of Σ and treat them separately.

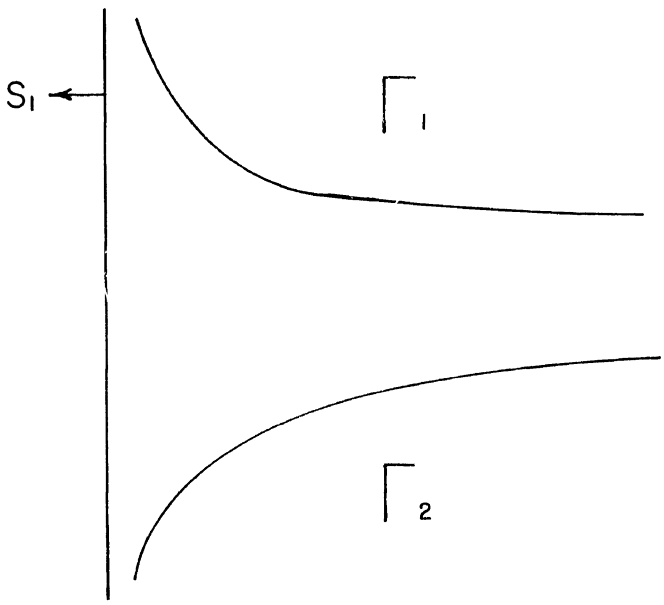

Fig. 4 The worst configurations.

6. NON-SINGULARITY

We now have a number of pieces of Σ and we wish to show that they are non-singular. Consider the line s = real constant ≤ 0. A point P_i of each piece lies on such a line. However, Symanzik and Eden have proven normal dispersion relations in t for such values. Therefore each point P_i cannot be complex or outside the normal cuts. However in the cuts it follows from unitarity that the only singularities are those given by normal thresholds. Therefore the P_i are non-singular and so each piece of Σ is also non-singular.

This completes the proof.

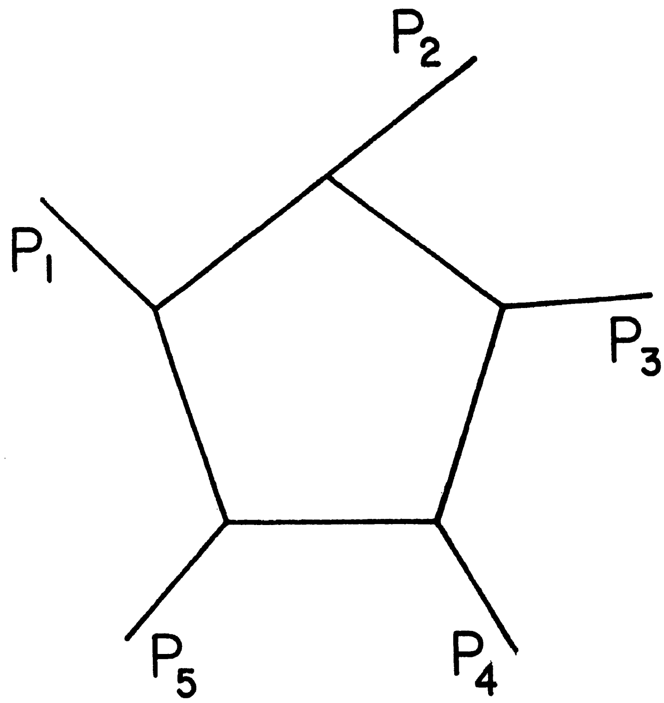

Fig. 5 A five-legged diagram.

7. HIGHER ORDER PROCESSES

The observation that the graph in Fig. 5 has singularities corresponding to the contraction shown in Fig. 6, and that the latter has singularities for $(p_1+p_2)^2$, $(p_3+p_4)^2$ complex, shows that the simplest generalizations of the Mandelstam representation to higher order processes are not true in perturbation theory.

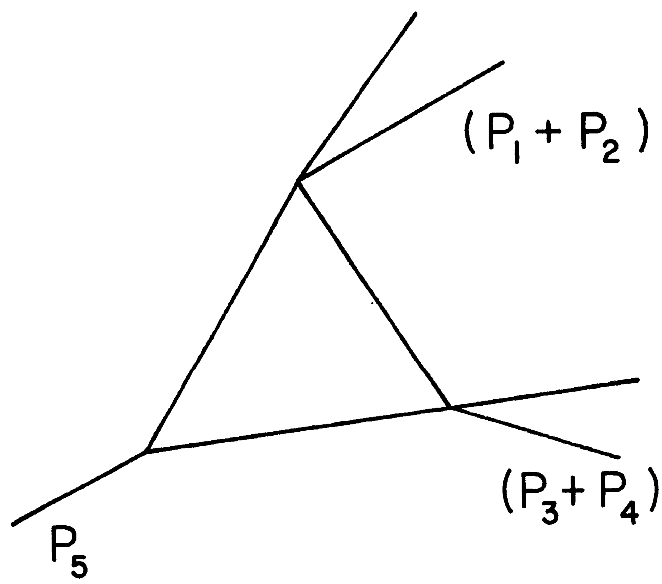

Fig. 6 Contracted diagram for the graph of Fig. 5.

8. UNITARITY

Mr. Lardner (St. John's College, Cambridge) has been able to show that the contributions to the scattering amplitude arising from three- and four-particle intermediate states in the unitarity condition are consistent with the Mandelstam representation under the condition that the corresponding production amplitudes have *single* variable analyticity in a cut plane.

APPENDIX

On the induction hypothesis the complex surface sprouting from Γ' is non-singular. Therefore Γ is non-singular in its appropriate limits. It must stay in the crossed cuts, otherwise it is non-singular in both limits. If it has a turning point as in Fig. 7 the inappropriate limit on AB is just obtained by crossing from II through the cut, and since II is non-singular this limit is non-singular. Thus Γ must be a convex curve and have negative slope.

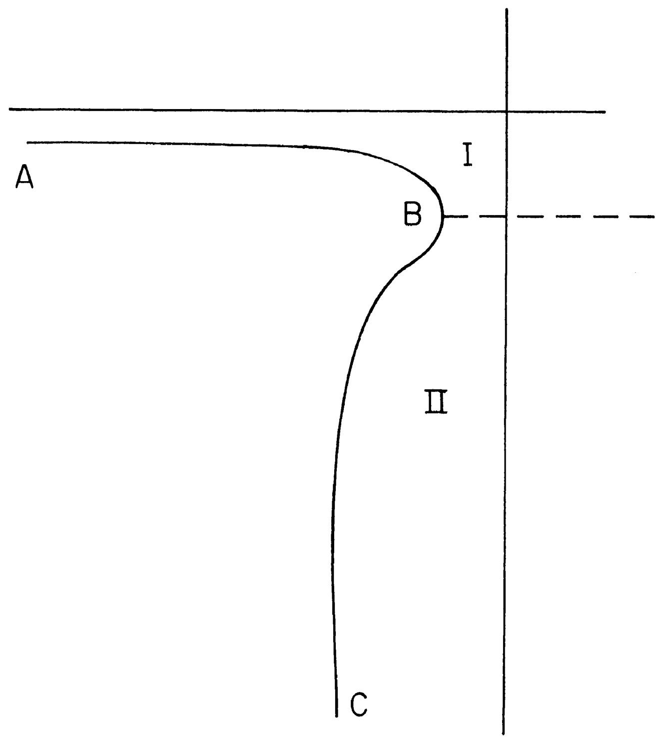

Fig. 7 The Γ-curve.

LIST OF REFERENCES AND NOTES

1. By leading singularity we mean the curve associated with coincidences in each integration over a Feynman variable. See, e.g., Polkinghorne. J. C., and Screaton, G. R., Nuovo Cimento **15**, (1960) or the preceding talk.
2. See in particular the remarks of Eden in the preceding talk in the material following Eq. (5.6). We call the behaviour of such a singularity when extended into the complex domain in this way "appropriate." The opposite behaviour would be "inappropriate."
3. The induction hypothesis forbids these to be singular in an appropriate sense.

DISCUSSION

J. G. Taylor: Where do the anomalous thresholds show up in this? Can one see that easily?

Polkinghorne: The main trouble is that an intersection with normal thresholds is going to be at infinity. Now, if you have anomalous thresholds, the effective intersection will be at a finite point. This just happens to spoil the argument because you do not have room to manoeuvre.

Oppenheimer: I would like to ask a question on what is probably a forbidden topic. Have you gained from this any insight into generalizations of the Mandelstam representation?

Polkinghorne: If we have, say, the five-point function and we think of a five-point loop, among its contractions is the graph shown in Fig. 6 where you short-circuit two of the lines. This is a vertex function in which one of the external masses is represented by $(P_1+P_2)^2$ and another one by $(P_3+P_4)^2$ and the last one by P_5^2. If we use these two variables in describing the five-point function, there exist complex singularities. So, certainly, the simplest generalization of the Mandelstam representation would not apply to the five-point function.

Eden: It is clear that the multiple integral for the five-point loop is going to be very complicated. The domain of integration is no longer real but is complex as has been indicated. The question one would really like to know is just what influence causality, unitarity, and Lorentz invariance have upon the general structure of an N-fold representation of a multiple Feynman integral. In the case of the four-point function, the condition of causality contained in the $m^2 - i\varepsilon$ implies that the spectral function is non-zero only in a real domain on the boundary of the physical sheet, and the interesting question is just what simple (or complicated) property of the higher functions contains the concept of causality.

A GENERALIZED UNITARITY RELATION

R. E. Cutkosky (*)

Carnegie Institute of Technology, Pittsburgh, Pennsylvania

Last year at the Ninth International Conference on High Energy Physics, Landau [1] presented some new theorems on singularities of perturbation theory amplitudes. He showed that when we discuss a particular singularity, we only need to look at a "skeleton" of the Feynman graph, a "reduced graph." (See Fig. 1). The circles, which are the vertices of the reduced graphs, stand for any arbitrarily complicated subgraphs. Landau showed that a singularity is obtained when all the lines of a reduced graph correspond to particles which are simultaneously on the mass shell, and in addition satisfy certain geometrical relations. Each of the reduced graphs shown could arise from the same Feynman graph, and correspond to different singularitics cf the same amplitude.

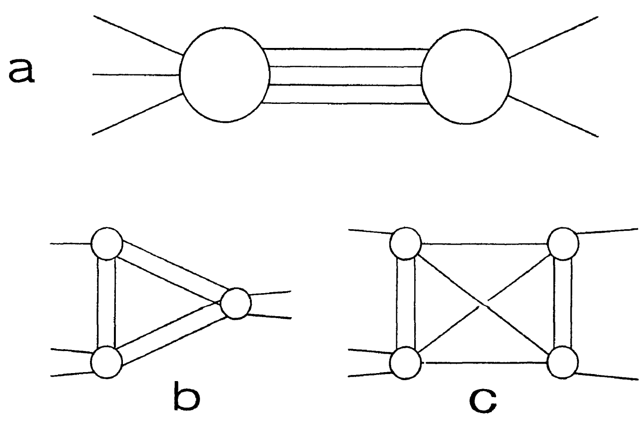

Fig. 1 Example of "reduced graph."

When we analyze these singularities further, we find that they are sometimes poles, and sometimes branch points. The residue of a pole is, of course, obtained by considering the subgraphs for the case that the lines leading into them represent particles which satisfy Landau's condition. When the singularity is a branch point, the discontinuity of the amplitude across the branch cut is obtained by an equally simple prescription. For each line of the Feynman graph which also appears explicitly in the reduced graph, the Feynman propagator is replaced by a delta function. In other words, the particles which correspond to the lines of the reduced graph are always taken to be on the mass shell. This prescription, when it is applied to a reduced graph like that on the top of Fig. 1, is equivalent to the familiar unitarity property of the S-matrix.

This theorem will perhaps be a little clearer after we outline a brief proof. The main idea of the proof is that we rewrite the Feynman integral in terms of the

(*) A. P. Sloan Foundation Fellow.