FREE FERMION CONSTRUCTIONS OF SUPER VIRASORO AND SUPER KAC-MOODY ALGEBRAS

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Abstract. Free fermion constructions of the superconformal and Kac-Moody algebras are discussed. Coset representations provide examples for the $N = 1$, $c < \frac{3}{2}$ discrete series. They generalize the Kac-Todorov construction of the supercurrent which was valid for $N = 1$, $c \geq \frac{3}{2}$, and differ by the terms mixing the SuperVirasoro and Kac-Moody algebras. They thus provide a guide for searching for new forms for the lower and upper components of the superfields in one-to-one correspondence with the untwisted states in a twisted superconformal field theory, and may be useful in discussing the low energy phenomenology of superstring theory.

1. Introduction

Conformal, superconformal and extended superconformal algebras play a role in string theory. In this paper we investigate constructions of the supercurrent generator $F(z)$ of the two $N = 1$ world sheet supersymmetric extensions of the Virasoro algebra, i.e., the Ramond and Neveu-Schwarz sectors. The critical dimension of this system (determined by the absence of negative norm ghost states) is $D = 10$. Unitary representations of the $N = 1$ superconformal algebra with critical central charge $c = 15$ are constructed from the matter superfields. The superconformal BRST ghost system provides a non-unitary representation of the $N = 1$ superVirasoro algebra (SVA) with $c = -15$. Although the representation is non-unitary, the SVA generators still satisfy the hermiticity conditions $L_n^{\text{ghost}\dagger} = L_{-n}^{\text{ghost}}$, $F_n^{\text{ghost}\dagger} = F_{-n}^{\text{ghost}}$. In addition, the superconformal ghost system also carries a representation of the extended $N = 2$ world sheet algebra.[1]

The matter superconformal fields of conformal weight one-half close to form a super Kac-Moody algebra (SKMA). The mixing between the SVA and the SKMA differs depending on the particular construction of the supercurrent. The $N = 1$ algebra is given by operator products where the right hand side holds for $|z| > |\zeta|$ up to terms regular as $z \to \zeta$.

$$L(z)L(\zeta) = \frac{\frac{c}{2}}{(z-\zeta)^4} + \frac{2L(\zeta)}{(z-\zeta)^2} + \frac{\frac{dL(\zeta)}{d\zeta}}{(z-\zeta)}$$

$$L(z)F(\zeta) = \frac{\frac{3}{2}F(\zeta)}{(z-\zeta)^2} + \frac{\frac{dF(\zeta)}{d\zeta}}{(z-\zeta)}$$

$$F(z)F(\zeta) = \frac{\frac{2c}{3}}{(z-\zeta)^3} + \frac{2L(\zeta)}{(z-\zeta)} \tag{1}$$

In component form we have

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}$$

$$[L_n, F_m] = (\frac{n}{2} - m)F_{n+m}$$

$$[F_n, F_m] = 2L_{n+m} + \frac{c}{3}(n^2 - \frac{1}{4})\delta_{n,-m} \tag{2}$$

The super Kac-Moody algebra is

$$T^a(z)T^b(z) = \frac{k\delta_{ab}}{(z-\zeta)^2} + \frac{if_{abc}T^c(\zeta)}{(z-\zeta)}$$

$$T^a(z)d^b(z) = \frac{if_{abc}T^c(\zeta)}{(z-\zeta)}$$

$$d^a(z)d^b(z) = \frac{\delta_{ab}}{(z-\zeta)}. \tag{3}$$

Here $f_{abc}f_{abe} = c_\psi \delta_{ce}$; the level of the KMA is $x = \frac{2k}{\psi^2} = \frac{2k}{c_\psi}\tilde{h}$, where $\tilde{h}$ is the dual Coxeter number of the compact Lie algebra with structure constants f_{abc}.

Constructions of the matter supercurrent are given by the following.

1)The Kac-Todorov construction extends to a super Kac-Moody algebra and has a mixing between the SVA and SKMA which reflects the fact that the SKMA generators are conformal weight one-half superfields. The Virasoro generators form a Sugawara construction and $\frac{3}{2} \leq c < \frac{3dimg}{2}$.

2)The coset constructions have SVA generators which are seen to be a modification of the Kac-Todorov construction. This constructions also extends to a SKMA, but now the mixing between the SVA and the SKMA is different from the Kac-Todorov case. The central charge satisfies $0 \leq c < \frac{dimg}{2}$.

3)Complex free fermions provide a construction similar to Kac-Todorov, but now the supercurrent can carry automorphisms of groups other than $SU(2)^6$ in the presence of massless fermions. Here $c = \frac{dimg}{2}$.

2. The Kac-Todorov construction

This construction provides the general free real fermion representations of the internal space SVA and SKMA algebras with the following mixing characteristic of a weight one-half superfield.

$$L(z)T^a(\zeta) = \frac{T^a(\zeta)}{(z-\zeta)^2} + \frac{\frac{dT^a(\zeta)}{d\zeta}}{(z-\zeta)}$$

$$L(z)d^a(\zeta) = \frac{\frac{1}{2}d^a(\zeta)}{(z-\zeta)^2} + \frac{\frac{dd^a(\zeta)}{d\zeta}}{(z-\zeta)}$$

$$F(z)T^a(\zeta) = \sqrt{k}[\frac{d^a(\zeta)}{(z-\zeta)^2} + \frac{\frac{dd^a(\zeta)}{d\zeta}}{(z-\zeta)}]$$

$$F(z)d^a(\zeta) = \frac{1}{\sqrt{k}}\frac{T^a(\zeta)}{(z-\zeta)}. \tag{4}$$

A realization is given by

$$\check{L}(z) = \frac{1}{c_\psi}({}^x_x\check{T}^a(z)\check{T}^a(z)^x_x) = \frac{1}{2} : \frac{dd^a(z)}{dz}d^a(z) : + \frac{\epsilon}{16z^2}$$

$$\check{T}^a(z) = \frac{-i}{2} f_{abc}d^b(z)d^c(z)$$

$$\check{F}(z) = \frac{1}{3\sqrt{\frac{c_\psi}{2}}}d^a(z)\check{T}^a(z) = \frac{-i}{6\sqrt{\frac{c_\psi}{2}}}f_{abc}d^a(z)d^b(z)d^c(z). \tag{5}$$

Here $1 \leq a \leq \epsilon$. This representation has level $\check{x} = \frac{2k}{c_\psi}\tilde{h} = \tilde{h}$ and $\check{c} = \frac{dimg}{2}$, *i.e.* $\frac{3}{2} \leq \check{c}$. The most general realization is given by

$$L(z) = \check{L}(z) + \frac{1}{2k^q + c_\psi}({}^x_x q^a(z)q^a(z)^x_x) = \check{L}(z) + L^q_{Sug}(z)$$

$$T^a(z) = \check{T}^a(z) + q^a(z)$$

$$F(z) = \sqrt{\frac{\frac{c_\psi}{2}}{k^q + \frac{c_\psi}{2}}}\check{F}(z) + \frac{1}{\sqrt{k^q + \frac{c_\psi}{2}}}d^a(z)q^a(z) \tag{6}$$

with level $x = \tilde{h} + x^q$ and $c = \frac{dimg}{2} + \frac{x^q dimg}{x^q + \tilde{h}} = \frac{3\dim g}{2} - \frac{\tilde{h}dimg}{x}$, *i.e.* $\frac{3}{2} \leq \check{c} \leq c < \frac{3dimg}{2}$. The abelian SKMA is

$$L(z) = \frac{1}{2} : \frac{dd^a(z)}{dz}d^a(z) : + \frac{\epsilon}{16z^2} + \frac{1}{2k^q}({}^x_x q^a(z)q^a(z)^x_x)$$

$$T^a(z) = q^a(z) = \frac{-i}{2} f_{aIJ} b^I(z) b^J(z)$$

$$F(z) = \frac{1}{\sqrt{k^q}} d^a(z) q^a(z), \tag{7}$$

Here $f_{aIJ} f_{bIJ} = 2k^q \delta_{ab}$ and $c = \frac{1}{2}(\text{dim} d^a + \text{dim} b^I) \geq \frac{3}{2}$.

3. Coset construction

We now modify the Kac-Todorov construction of the supercurrent to be of the general form

$$F(z) = A\check{F}(z) + Bd^a(z)q^a(z). \tag{8}$$

It follows that

$$L(z) = C\check{L}(z) + DL^q_{Sug}(z) + E\check{T}^a(z)q^a(z)$$

$$T^a(z) = \check{T}^a(z) + q^a(z). \tag{9}$$

Case 1: for $E = 0$, we regain the Kac-Todorov forms: a) minimal $B = D = 0, A = C = 1$ and b) maximal $B = \frac{1}{\sqrt{\frac{c_\psi}{2}}} A = \frac{1}{\sqrt{k^q + \frac{c_\psi}{2}}}$.

Case 2: for $E = \frac{-2\tilde{h}}{c_\psi(x^q+2\tilde{h})}$; $C = \frac{x^q}{x^q+2\tilde{h}}$; $D = \frac{\tilde{h}}{x^q+2\tilde{h}}$; where $A = \frac{-x^q}{\sqrt{(x^q+\tilde{h})(x^q+2\tilde{h})}}$; $B = \frac{\tilde{h}\sqrt{\frac{2}{c_\psi}}}{\sqrt{(x^q+\tilde{h})(x^q+2\tilde{h})}}$, the mixing between the SVA and SKMA is given by

$$L(z)T^a(\zeta) = 0$$

$$L(z)d^a(\zeta) =$$

$$= (\frac{x^q}{x^q + 2\tilde{h}})(\frac{d^a(\zeta)}{2(z-\zeta)^2} + \frac{\frac{dd^a(\zeta)}{d\zeta}}{(z-\zeta)}) + (\frac{-2\tilde{h}}{(x^q + 2\tilde{h})c_\psi})\frac{if_{abc}q^b(z)d^c(z)}{(z-\zeta)}$$

$$F(z)T^a(\zeta) = 0$$

$$F(z)d^a(\zeta) = \frac{1}{(z-\zeta)}[\frac{\sqrt{\frac{2}{c_\psi}}}{\sqrt{(x^q+\tilde{h})(x^q+2\tilde{h})}}][-x^q T^a(\zeta) + \tilde{h}q^a(\zeta)]. \tag{10}$$

This representation has level $x = \tilde{h} + x^q$ and $c = \frac{dim g}{2}(1 - \frac{2\tilde{h}^2}{(x^q+\tilde{h})(x^q+2\tilde{h})})$, *i.e.* $0 \leq c < \frac{dim g}{2}$. For $g = SU(2)$ (so $\tilde{h} = 2$), *we see this is just the discrete series for unitary representations of the* $N = 1$ SVA (let $x^q \equiv m$):

$$c = \frac{3}{2}(1 - \frac{8}{(m+2)(m+4)}) = 0, \frac{7}{10}, 1, \ldots, \frac{3}{2}. \tag{11}$$

Case 2 is seen to be equivalent to the coset construction[3],

$$L(z) = L^G(z) - L^H(z) = \check{L}(z) + L^q_{Sug}(z) -$$

$$-\frac{1}{2k^q + 2c_\psi}[{}^x_x(\check{T}^a(z) + q^a(z))(\check{T}^a(z) + q^a(z)){}^x_x]T^a(z) =$$

$$= \check{T}^a(z) + q^a(z) \tag{12}$$

$$F(z) = \frac{-x^q}{\sqrt{(x^q + \tilde{h})(x^q + 2\tilde{h})}}\check{F}(z) + \frac{\tilde{h}\sqrt{\frac{2}{c_\psi}}}{\sqrt{(x^q + \tilde{h})(x^q + 2\tilde{h})}}d^a(z)q^a(z)$$

The coset here corresponds to $G = SU(2) \otimes SU(2)$ and $H = SU(2)$.

4. Complex fermions

For complex fermions satisfying twisted boundary conditions $f^a(e^{2\pi i}z) = e^{2\pi i\nu}f^a(z)$, $\tilde{f}^a(e^{2\pi i}z) = e^{-2\pi i\nu}\tilde{f}^a(z)$, the supercurrent construction generalizes the Kac-Todorov expression to be given by

$$F(z) = \frac{-i}{6\sqrt{\frac{c_\psi}{2}}}f_{abc}h^a(z)h^b(z)h^c(z). \tag{13}$$

The space-time fermi fields satisfy the periodicity condition $h^\mu(e^{2\pi i}z) = \delta_\alpha h^\mu(z)$ where $\delta_\alpha = \mp 1$ for R an NS fields respectively, so $F(e^{2\pi i}z) = \delta_\alpha F(z)$. In a given sector, all the fermionic boundary conditions can be specified by a matrix ω^a_b: so $h^a(e^{2\pi i}z) = \omega^a_b h^b(z)$ and

$$f_{def}\omega^d_a\omega^e_b\omega^f_c = \delta_\alpha f_{abc}, \tag{14}$$

i.e. $\mp\omega^a_b$ is an automorphism [4] of the Lie algebra g with structure constants f_{abc} used to define the supercurrent in a Ramond (Neveu-Schwarz) sector. The Virasoro generator is then given by

$$L(z) = \frac{1}{2} : \frac{d\tilde{f}(z)}{dz}f(z) : + \frac{1}{2} : \frac{df(z)}{dz}\tilde{f}(z) : + \frac{1}{16z^2}\mathrm{tr}([\frac{1}{i\pi}\log(-\omega)]^2) \tag{15}$$

where we are ultimately interested in the automorphisms ω of g for which the coboundary term $\frac{1}{16}\mathrm{tr}([\frac{1}{i\pi}\log(-\omega)]^2) = \frac{dim g}{16\cdot 3}$, *i.e.* the automorphisms for which the coboundary term takes its minimum value on the Ramond sector. In D space-time dimensions, the mass operator is

$$m^2 = -\frac{1}{2} + \frac{D-2}{16} + \frac{1}{16}\mathrm{tr}([\frac{1}{i\pi}\log(-\omega)]^2) \tag{16}$$

and the structure constants require $dim g = 3(10 - D)$. So for massless space-time fermions, $m^2 = 0$, the coboundary term realizes its minimum value, and in $D = 4$, *the dimension of g is 18, where g is the algebra of the structure constants occuring in the supercurrent (13).* This mass formula is to be contrasted with that of the Kac-Todorov construction where massless fermions require

$$m^2 = -\frac{1}{2} + \frac{D-2}{16} + \dim d^a = 0, \tag{17}$$

so for $D = 4$, *g is $U(1)^6$*, which is the gauge group of the states in string models using the Kac-Todorov supercurrent construction necessary for sectors with massless Ramond states.

In order to examine which gauge groups occur as the relevant internal gauge symmetry of the spectrum of states for the complex fermion form of the supercurrent given in (13), we investigate the SKMA (which mixes as a weight one-half superfield) with this representation of the internal superVirasoro algebra, $c = 9$. The modified Cartan Weyl basis for the SKMA diagonalizes inner automorphisms. The fermions in this basis are $h^i(z), h^\alpha(z)$, for $1 \leq i \leq \text{rank(g)}$ and $\alpha \in \text{roots(g)}$, thus h^i are real (R, NS) and $h^{\alpha *} = h^{-\alpha}$ are complex. The SKMA generators now form a twisted SKMA where $H^i(e^{2\pi i}z) = H^i(z)$ and the step operators $E^\alpha(z) = e^{2\pi i\alpha\cdot\lambda}E^\alpha(e^{-2\pi i}z)$. So for *inner* automorphisms, the zero mode subalgebra which is the gauge symmetry of the spectrum is $U(1)^{rank(g)}$. For outer automorphisms, one can check for the relevant groups $SU(3)$, $SU(4)$ and $SO(5)$ that the zero mode subalgebra is again a product of $U(1)$ factors.

5. Non-free fermion representations of the supercurrent

Not all known representations of the superconformal algebra can be expressed as free fermion constructions. In particular, the Waterson boson[5,6] provides a representation for $N = 2$, $c = 1$. The $N = 1$ subalgebra is generated by

$$L(z) = \frac{1}{2} : a(z) \cdot a(z) :$$

$$F(z) = \frac{1}{\sqrt{2}}(: e^{i\sqrt{3}X(z)} : + : e^{-i\sqrt{3}X(z)} :). \tag{18}$$

6. BRST superconformal ghost system

Non-unitary representations of the superVirasoro algebra are provided by the BRST superconformal ghost system[1,7]. The ghost superfields are $B(z) = \beta(z) + \theta b(z)$ and $C(z) = c(z) + \theta\gamma(z)$ with conformal spin $h_\beta = \frac{3}{2}$, $h_c = -1$, etc. The commutation relations on the Ramond sector are $\{b_n, c_m\} = \delta_{n,-m}$,

$[\beta_n, \gamma_m] = -\delta_{n,-m}$. The superVirasoro representation has $c = -15$, but the generators still satisfy the hermitian property $F_n^\dagger = F_{-n}$ and $L_n^\dagger = L_{-n}$:

$$L(z) = -2 : b(z)\frac{dc(z)}{dz} : - : \frac{db(z)}{dz}c(z) : -\frac{3}{2} : \beta(z)\frac{d\gamma(z)}{dz} : -$$

$$-\frac{1}{2} : \frac{d\beta(z)}{dz}\gamma(z) : -\frac{1}{2z^2}$$

$$F(z) =: b(z)\gamma(z) : -3 : \beta(z)\frac{dc(z)}{dz} : -2 : \frac{d\beta(z)}{dz}c(z) : \tag{19}$$

An alternative form for the supercurrent is given by Schwarz[8]:

$$F(z) = -2 : b(z)\gamma(z) : +\frac{3}{2} : \beta(z)\frac{dc(z)}{dz} : + : \frac{d\beta(z)}{dz}c(z) : \tag{20}$$

The ghost number current forms an abelian SKMA:

$$T(z) = - : b(z)c(z) : - : \beta(z)\gamma(z) : +\frac{1}{2z} \tag{21}$$

The supercurrent in (19) can be identified as $F^+ + F^-$ and can be used to construct a second $h = \frac{3}{2}$ supercurrent $-F^+ + F^-$ as the upper component of the superfield whose lower component is

$$H(z) = 2 : b(z)c(z) : +3 : \beta(z)\gamma(z) : \tag{22}$$

We find

$$-F^+(z) + F^-(z) =: b(z)\gamma(z) : +3 : \beta(z)\frac{dc(z)}{dz} : +2 : \frac{d\beta(z)}{dz}c(z) : \tag{23}$$

The set L, G^+, G^-, H form an $N = 2$ superconformal algebra with $c = -15$.

We include here for completeness, the unitarity restrictions on the central for the $N = 0, 1, 2$ superVirasoro algebras[9,10]. For $N = 0$, unitary representations occur for all values of $c \geq 1, h \geq 0$ and for discrete values below 1 given by $c = 1 - \frac{6}{(m+2)(m+3)} = 0, \frac{1}{2}, \frac{7}{10}, \frac{4}{5}, \ldots, 1$. The critical value of the central charge is $c = 26$. The $N = 1$ system provides representations of two supersymmetric extensions of the Virasoro algebra, i.e. the Ramond and the Neveu-Schwarz. The critical dimension is $D = 10$. The only possible unitary highest weight representations, i.e. representations generated from a state $|h\rangle$, satisfy $L_n|h\rangle = 0, n \geq 0$; $L_0|h\rangle = h|h\rangle; F_n|h\rangle = 0, n \geq 0$; are characterized by (c, h) where either $c \geq \frac{3}{2}$, $h \geq 0$; or for the discrete values $0 \leq c < \frac{3}{2}$ given by $c = \frac{3}{2}[1 - \frac{8}{(m+2)(m+4)}] = 0, \frac{7}{10}, 1, \ldots, \frac{3}{2}$. The critical value of the central charge is $c = 15$. For $N = 2$, the critical dimension is $D = 2$ complex or $D = 4$ real. Unitary representations occur for all values of $c \geq 3$ and for discrete values below 3 given by $c = \frac{3m}{m+2} = 0, 1, \ldots, 3$. The critical central charge is $c = 6$.

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