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On explicit formulae of LMOV invariants

Shengmao Zhu

Department of Mathematics, Zhejiang International Studies University,
No. 299 Liuhe Road, Hangzhou 310023, P.R. China
Center of Mathematical Sciences, Zhejiang University,
No. 38 Zheda Road, Hangzhou 310027, P.R. China

E-mail: shengmaozhu@126.com

Abstract: We started a program to study the open string integrality invariants (LMOV invariants) for toric Calabi-Yau 3-folds with Aganagic-Vafa brane (AV-brane) several years ago. This paper is devoted to the case of resolved conifold with one out AV-brane in any integer framing τ, which is the large N duality of Chern-Simons theory for a framed unknot with integer framing τ in S^3. By using the methods from string dualities, we compute several explicit formulae of the corresponding LMOV invariants for this special model, whose integrality properties have been proved in a separated paper [56].

Keywords: Topological Strings, String Duality, Chern-Simons Theories

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Contents

1 Introduction

We are interested in integrality structures of topological strings theory. Let X be a Calabi-Yau 3-fold with symplectic form ω, according to the work of Gopakumar and Vafa [26], the closed string free energy F^X, which is the generating function of Gromov-Witten invariants $K_{g,Q}$, has the following structure:

$$F^X = \sum_{g\geq 0} g_s^{2g-2} \sum_{Q\neq 0} K_{g,Q} e^{-Q\cdot\omega} = \sum_{g\geq 0, d\geq 1} \sum_{Q\neq 0} \frac{1}{d} N_{g,Q} \left(2\sin\frac{dg_s}{2}\right)^{2g-2} e^{-dQ\cdot\omega}$$

where $N_{g,Q}$ are integers and vanish for large g or Q. When X is a toric Calabi-Yau 3-fold, the above Gopakumar-Vafa conjecture was proved in [33, 67].

Then we are going to investigate the integrality structures of open topological strings. Let us consider a Calabi-Yau 3-fold X with a Lagrangian submanifold $\mathcal{D}$ in it. Based on Ooguri and Vafa's work [66], the generating function of open Gromov-Witten invariants can also be expressed in terms of a series of new integers which were later refined by Labastida, Mariño and Vafa in [45–47]:

$$\sum_{g\geq 0}\sum_{Q\neq 0} g_s^{2g-2+l(\mu)} K_{\mu,g,Q} e^{-Q\cdot\omega} \tag{1.1}$$
$$= \sum_{g\geq 0}\sum_{Q\neq 0}\sum_{d|\mu} \frac{(-1)^{l(\mu)+g}}{\prod_{i=1}^{l(\mu)}\mu_i} d^{l(\mu)-1} n_{\mu/d,g,Q} \prod_{j=1}^{l(\mu)}\left(2\sin\frac{\mu_j g_s}{2}\right)\left(2\sin\frac{d g_s}{2}\right)^{2g-2} e^{-dQ\cdot\omega}.$$

These new integers $n_{\mu,g,Q}$ (here μ denote a partition of a positive integer) are referred as LMOV invariants in this paper.

Although for any toric Calabi-Yau 3-fold with Agangica-Vafa brane (AV-brane for short) [1], we have an effective method to calculate the open string partition function by gluing topological vertices [4, 41], it is difficult to compute the corresponding LMOV invariants $n_{\mu,g,Q}$ and prove their integrality properties.

During the past several years, we started a program to study the LMOV invariants and their variations [10, 12, 55, 56, 82–84]. In this paper, we only focus on a special toric Calabi-Yau 3-fold, i.e. the resolved conifold $\hat{X}$ with one special Lagrangian submanifold (AV-brane D_τ in integer framing τ). More general cases will be discussed in a separated paper [84]. We also refer to [60, 61, 69] and the references therein for some recent developments.

Based on large N duality, the open string theory of $(\hat{X}, D_\tau)$ is the large N duality of the SU(N) Chern-Simons theory of (S^3, U_τ), where U_τ denotes a framed unknot (trivial knot) with integer framing τ. The large N duality of Chern-Simons and topological string theory was proposed by Witten [78], and developed further by [27, 47, 66]. Later, Mariño and Vafa [62] generalized it to the case of a knot with arbitrary framing. The large N duality of $(\hat{X}, D_\tau)$ and (S^3, U_τ) can be expressed in terms of the following identity:

$$Z_{\text{CS}}^{(S^3,U_\tau)}(q,a;\mathbf{x}) = Z_{\text{str}}^{(\hat{X},D_\tau)}(g_s,a;\mathbf{x}),\ q = e^{\sqrt{-1}g_s} \tag{1.2}$$

where the explicit expressions of the above two partition functions in identity (1.2) are given by the formulae (3.4) and (3.5) respectively. The identity (1.2) implies the Mariño-Vafa formula [42, 62, 65], a very powerful Hodge integral identity, which implies various important results in intersection theory of moduli spaces of curves, see [54, 80] for a review of the applications of Mariño-Vafa formula. Finally, the identity (1.2) was proved by J. Zhou [79] based on his previous joint works with C.-C. Liu and K. Liu [42, 44].

On the other hand side, through mirror symmetry, the partition function $Z_{\text{str}}^{(\hat{X},D_\tau)}(g_s,a;\mathbf{x})$ can also be computed by topological string B-model [7]. The mirror geometry information of $(\hat{X}, D_\tau)$ is encoded in a mirror curve $\mathcal{C}_{\hat{X}}$. Then the disc counting invariants of $(\hat{X}, D_\tau)$ were given by the coefficients of superpotential related to the mirror curve [1, 3], and this fact was proved in [19]. Furthermore, the open Gromov-Witten invariants of higher genus with more holes can be obtained by using Eynard-Orantin topological recursions [15]. This approach named BKMP conjecture, was proposed by Bouchard, Klemm, Mariño and Pasquetti in [9], and then fully proved in [16, 20] for any toric Calabi-Yau 3-fold with AV-brane, so one can also use the BKMP method to compute the LMOV invariants for $(\hat{X}, D_\tau)$.

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In conclusion, now we have three different approaches to compute the open string partition function $Z_{\rm str}^{(\hat{X},D_\tau)}(g_s,a;\mathbf{x})$: (i) topological vertex [4, 41]; (ii) Chern-Simons partition function (3.4); (iii) BKMP method [9].

In this paper, we provide several explicit formulae for LMOV invariants of the open string model $(\hat{X},D_\tau)$ by using above methods. Firstly, we illustrate the computations of the mirror curve of $(\hat{X},D_\tau)$ by the approach in [2]. It turns out that the mirror curve of this model is given by:

$$y-1-a^{-\frac{1}{2}}(-1)^\tau xy^\tau(ay-1)=0. \tag{1.3}$$

By using the mirror curve (1.3), we obtain an explicit formula for genus 0 and one-hole LMOV invariants (disc countings) $n_{m,0,l-\frac{m}{2}}(\tau)$ which is denoted by $n_{m,l}(\tau)$ for brevity:

$$n_{m,l}(\tau)=\sum_{d|m,d|l}\frac{\mu(d)}{d^2}c_{\frac{m}{d},\frac{l}{d}}(\tau),$$

where

$$c_{m,l}(\tau)=-\frac{(-1)^{m\tau+m+l}}{m^2}\binom{m}{l}\binom{m\tau+l-1}{m-1}$$

and $\mu(d)$ denotes the Möbius functions. In [56], we have proved that $n_{m,l}(\tau)\in\mathbb{Z}$ for any $\tau\in\mathbb{Z}$, $m\geq 1, l\geq 0$.

Remark 1.1. Recently, Panfil and Sulkowski [69] generalized the above disc counting formula to a class of toric Calabi-Yau manifolds without compact four-cycles which is also referred to as strip geometries. In our notations, their formula (cf. formula (4.19) in [69]) can be formulated as follow. Given two integers $r,s\geq 0$, set $\mathbf{l}=(l_1,\dots,l_r)$, $\mathbf{k}=(k_1,\dots,k_s)$, and $|\mathbf{l}|=\sum_{j=1}^r l_j$, $|\mathbf{k}|=\sum_{j=1}^s k_j$. We define

$$c_{m,\mathbf{l},\mathbf{k}}(\tau)=\frac{(-1)^{m(\tau+1)+|\mathbf{l}|}}{m^2}\binom{m\tau+|\mathbf{l}|+|\mathbf{k}|-1}{m-1}\prod_{j=1}^r\binom{m}{l_j}\prod_{j=1}^s\frac{m}{m+k_j}\binom{m+k_j}{k_j}.$$

Then, we have the following disc counting formula

$$n_{m,\mathbf{l},\mathbf{k}}(\tau)=\sum_{d|\gcd(m,\mathbf{l},\mathbf{k})}\frac{\mu(d)}{d^2}c_{m/d,\mathbf{l}/d,\mathbf{k}/d}(\tau)$$

In [84], we have generalized the number theory method used in [56] to show that $n_{m,\mathbf{l},\mathbf{k}}(\tau)\in\mathbb{Z}$.

For the LMOV invariants of genus 0 with two holes, we study the Bergmann kernel expansion in the BKMP construction, and find an explicit formula for the LMOV invariants $n_{(m_1,m_2),0,\frac{m_1+m_2}{2}}(\tau)$ which is denoted by $n_{(m_1,m_2)}(\tau)$ for short,

$$n_{(m_1,m_2)}(\tau)=\frac{1}{m_1+m_2}\sum_{d|m_1,d|m_2}\mu(d)(-1)^{(m_1+m_2)(\tau+1)/d}$$
$$\cdot\binom{(m_1\tau+m_1)/d-1}{m_1/d}\binom{(m_2\tau+m_2)/d}{m_2/d}.$$

In [56], we have also proved that $n_{(m_1,m_2)}(\tau)\in\mathbb{Z}$ for $m_1,m_2\geq 1$ and $\tau\in\mathbb{Z}$.

As to the genus 0 LMOV invariants with more than two holes, one can compute the LMOV invariant $n_{\mu,0,Q}(\tau)$ for general Q by using the BKMP construction. But it is hard to give an explicit formula for general Q, except $Q = \frac{|\mu|}{2}$ in which case

$$n_{\mu,0,\frac{|\mu|}{2}}(\tau) = (-1)^{l(\mu)} \sum_{d|\mu} \mu(d) d^{l(\mu)-1} K^{\tau}_{\frac{\mu}{d},0,\frac{|\mu|}{2d}}$$

where

$$K^{\tau}_{\mu,0,\frac{|\mu|}{2}} = (-1)^{|\mu|\tau}[\tau(\tau+1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \binom{\mu_i(\tau+1)-1}{\mu_i - 1} \left(\sum_{i=1}^{l(\mu)} \mu_i\right)^{l(\mu)-3}.$$

It is obvious that $K^{\tau}_{\mu,0,\frac{|\mu|}{2}} \in \mathbb{Z}$ for any $\tau \in \mathbb{Z}$, and since $l(\mu) \geq 3$, it immediately implies that $n_{\mu,0,\frac{|\mu|}{2}}(\tau) \in \mathbb{Z}$ for any partition μ with $l(\mu) \geq 3$ and $\tau \in \mathbb{Z}$.

Finally, we study the high genus LMOV invariants $n_{m,g,Q}(\tau)$ of framed unknot U_τ. We define

$$g_m(q,a) = \sum_{d|m} \mu(d) \mathcal{Z}_{m/d}(q^d, a^d),$$

where $\mathcal{Z}_m(q,a) = (-1)^{m\tau} \sum_{|\nu|=m} \frac{1}{\mathfrak{z}_\nu} \frac{\{m\nu\tau\}}{\{m\}\{m\tau\}} \frac{\{\nu\}_a}{\{\nu\}}$. In [56], we have proved that $g_m(q,a)$ is a polynomial of $n_{m,g,Q}(\tau)$. More precisely,

$$g_m(q,a) = \sum_{g\geq 0} \sum_{Q} n_{m,g,Q}(\tau) z^{2g-2} a^Q \in z^{-2}\mathbb{Z}[z^2, a^{\pm\frac{1}{2}}],$$

where $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. In other words, we have

$$n_{m,g,Q}(\tau) = \text{Coefficient of term } z^{2g-2}a^Q \text{ in the polynomial } g_m(q,a).$$

The rest of this paper is organized as follow: in section 2, we review the mathematical definitions of topological string partition functions, free energies, and the integrality structures appearing in topological strings. Then we introduce the LMOV invariants in open topological strings. In section 3, we first review Witten's Chern-Simons theory for three-manifolds and links, and the large N duality between the Chern-Simons theory and topological strings. Then, the basic case for framed unknot was illustrated explicitly. We also formulate the LMOV integrality conjecture for framed knot. In section 4, we study the LMOV invariants for framed unknot in detail. We first illustrate the computations of the mirror curve of $(\hat{X}, D_\tau)$ by using the approach of [2]. Then we compute the explicit formulae for genus 0 LMOV invariants by using this mirror curve, and high genus LMOV invariants with one hole as well. Finally, in section 5, we discuss several related questions and works.

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2 Topological strings

2.1 Closed topological strings and Gromov-Witten invariants

Topological strings on a Calabi-Yau 3-fold X have two types: A-models and B-models. The mathematical theory for A-model is Gromov-Witten theory [29, 31]. Let $\overline{\mathcal{M}}_{g,n}(X,Q)$ be the moduli space of stable maps $(f,\Sigma_g,p_1,\dots,p_n)$, where $f:\Sigma_g\to X$ is a holomorphic map from the nodal curve Σ_g to the Kähler manifold X with $f_*([\Sigma_g])=Q\in H_2(X,\mathbb{Z})$. In general, $\overline{\mathcal{M}}_{g,n}(X,Q)$ carries a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,Q)]^{\mathrm{vir}}$ in the sense of [8, 52]. The virtual dimension is given by:

$$\mathrm{vdim}[\overline{\mathcal{M}}_{g,n}(X,Q)]^{\mathrm{vir}}=\int_Q c_1(X)+(\dim X-3)(1-g)+n.$$

When X is a Calabi-Yau 3-fold, i.e. $c_1(X)=0$, then $\mathrm{vdim}[\overline{\mathcal{M}}_{g,0}(X,Q)]^{\mathrm{vir}}=0$. The genus g, degree Q Gromov-Witten invariants of X is defined by

$$K_{g,Q}=\int_{[\overline{\mathcal{M}}_{g,0}(X,Q)]^{\mathrm{vir}}}1$$

which is usually denoted by $K_{g,Q}$ for brevity without any confusions. In A-model, the genus g closed free energy F_g^X of X is the generating function of Gromov-Witten invariants $K_{g,Q}$, i.e.

$$F_g^X=\sum_{Q\neq 0}K_{g,Q}e^{-Q\cdot\omega},$$

where ω represents the Kähler class for X. We define the total free energy F^X and partition function Z^X as

$$F^X=\sum_{g\geq 0}g_s^{2g-2}F_g^X,\ \ Z^X=\exp(F^X),$$

where g_s denotes the string coupling constant. In mathematics, the free energy F^X are mainly computed by the method of localizations [24, 31]. Especially, when X is a toric Calabi-Yau 3-fold, we have a more effective approach to obtain the partition function Z^X by the method of gluing topological vertices [4, 41].

Usually, the Gromov-Witten invariants $K_{g,Q}$ are rational numbers, from the BPS counting in M-theory, Gopakumar and Vafa [26] expressed the total free energy F^X in terms of the generating function of a series of new integer numbers $N_{g,Q}$ as follow:

$$F^X=\sum_{g\geq 0}g_s^{2g-2}\sum_{Q\neq 0}K_{g,Q}e^{-Q\cdot\omega}=\sum_{g\geq 0,d\geq 1}\sum_{Q\neq 0}\frac{1}{d}N_{g,Q}\left(2\sin\frac{dg_s}{2}\right)^{2g-2}e^{-dQ\cdot\omega}$$

The integrality of Gopakumar-Vafa invariants $N_{g,Q}$ was first proved by P. Peng for the case of toric Del Pezzo surfaces [67]. The proof for general toric Calabi-Yau threefolds was then given by Konishi in [33].

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2.2 Open topological strings

Let us now consider the open sector of topological A-model of a Calabi-Yau 3-fold X with a Lagrangian submanifold $\mathcal{D}$ with dim $H_1(\mathcal{D},\mathbb{Z})=L$. The open sector topological A-model can be described by holomorphic maps ϕ from open Riemann surface of genus g and l-holes $\Sigma_{g,l}$ to X, with Dirichlet condition specified by $\mathcal{D}$. These holomorphic maps are referred as open string instantons. More precisely, an open string instanton is a holomorphic map $\phi:\Sigma_{g,l}\to X$ such that $\partial\Sigma_{g,l}=\cup_{i=1}^{l}\mathcal{C}_i\to\mathcal{D}\subset X$ where the boundary $\partial\Sigma_{g,l}$ of $\Sigma_{g,l}$ consists of l connected components $\mathcal{C}_i$ mapped to Lagrangian submanifold $\mathcal{D}$ of X. Therefore, the open string instanton ϕ is described by the following two different kinds of data: the first is the "bulk part" which is given by $\phi_*[\Sigma_{g,l}]=Q\in H_2(X,\mathcal{D})$, and the second is the "boundary part" which is given by $\phi_*[\mathcal{C}_i]=w_i^\alpha\gamma_\alpha$, for $i=1,\ldots l$, where γ_α, $\alpha=1,\ldots,L$ is a basis of $H_1(\mathcal{D},\mathbb{Z})$ and $w_i^\alpha\in\mathbb{Z}$. Let $\vec{w}=(w^1,\ldots,w^L)$, and where $w^\alpha=(w_1^\alpha,\ldots,w_l^\alpha)\in\mathbb{Z}^l$, for $\alpha=1,\ldots,L$. We expect there exist the corresponding open Gromov-Witten invariants $K_{\vec{w},g,Q}$ determined by the data $\vec{w},Q$ in the genus g. See [35, 51] for mathematical aspects of defining these invariants in special cases.

We take all $w_i\geq 1$ as in [62], and use the notations of partitions and symmetric functions [57]. We denote by $\mathcal{P}$ the set of all partitions including the empty partition 0, and by $\mathcal{P}_+$ the set of nonzero partitions. Let $\mathbf{x}=\{x_1,x_2,\ldots\}$ be the set of infinitely many independent variables. For $n\geq 0$, let $p_n(\mathbf{x})=\sum_{i\geq1}x_i^n$ be a power sum symmetric function. For a partition $\mu\in\mathcal{P}_+$, set $p_\mu(\mathbf{x})=\prod_{i=1}^{h}p_{\mu_i}(\mathbf{x})$. For $\vec{\mu}\in\mathcal{P}^L$, and $\vec{\mathbf{x}}=(\mathbf{x}^1,\ldots,\mathbf{x}^L)$, let $p_{\vec{\mu}}(\vec{\mathbf{x}})=\prod_{\alpha=1}^{L}p_{\mu^\alpha}(\mathbf{x}^\alpha)$. The total free energy and partition function of open topological string on $(X,\mathcal{D})$ are expressed in the following forms:

$$F_{\rm str}^{(X,\mathcal{D})}(g_s,\omega,\vec{\mathbf{x}})=\sum_{g\geq0}\sum_{\vec{\mu}\in\mathcal{P}^L\backslash\{0\}}\frac{1}{|Aut(\vec{\mu})|}g_s^{2g-2+l(\mu)}\sum_{Q\neq0}K_{\vec{\mu},g,Q}e^{-Q\cdot\omega}p_{\vec{\mu}}(\vec{\mathbf{x}})$$
$$Z_{\rm str}^{(X,\mathcal{D})}(g_s,\omega,\vec{\mathbf{x}})=\exp(F_{\rm str}^{(X,\mathcal{D})}(g_s,\omega,\vec{\mathbf{x}})).$$

The central problem in open topological string theory is how to calculate the partition function $Z_{\rm str}^{(X,\mathcal{D})}(g_s,\omega,\vec{\mathbf{x}})$ or the open Gromov-Witten invariants $K_{\vec{\mu},g,Q}$. For the case of compact Calabi-Yau 3-folds, such as the quintic X_5, there are only a few works devoted to the study of its open Gromov-Witten invariants, for example, a complete calculation of the disk invariants of X_5 with boundary in a real Lagrangian was given in [68].

Suppose X is a toric Calabi-Yau 3-fold, and $\mathcal{D}$ is a special Lagrangian submanifold named as Aganagic-Vafa A-brane in the sense of [1, 3]. The open string partition function $Z_{\rm str}^{(X,\mathcal{D})}(g_s,\omega,\mathbf{x})$ can be computed by the method of topological vertex [4, 41] and the method of topological recursion developed by Eynard and Orantin [15]. The second approach was first proposed by Mariño [58], and studied further by Bouchard, Klemm, Mariño and Pasquetti [9], the equivalence of the two methods was proved in [16, 20]

In the following, we only need to consider the case of L=1. It is also useful to introduce the generating function of $K_{\mu,g,Q}$ in the fixed genus g as follow:

$$F_{(g,l)}^{(X,\mathcal{D})}=\sum_{\mu\in\mathcal{P}_+}\sum_{Q\neq0}K_{\mu,g,Q}e^{-Q\cdot\omega}x_1^{\mu_1}\cdots x_l^{\mu_l}.$$

2.3 Integrality structures and LMOV invariants

We introduce the new variables $q = e^{\sqrt{-1}g_s}$, $a = e^{-\omega}$, and let $f_\lambda(q,a)$ be a function determined by the following formula

$$F_{\rm str}^{(X,\mathcal{D})}(g_s,a,\mathbf{x}) = \sum_{d=1}^{\infty}\frac{1}{d}\sum_{\lambda\in\mathcal{P}^+} f_\lambda(q^d,a^d)s_\lambda(\mathbf{x}^d),$$

where $s_\lambda(\mathbf{x})$ is the Schur symmetric functions [57].

Just as in the closed string case [26], the open topological strings compute the partition function of BPS domain walls in a related superstring theory [66]. It follows that $F^{(X,\mathcal{D})}$ also has an integral expansion. This integrality structure was further refined in [45–47] which showed that $f_\lambda(q,a)$ has the following integral expansion

$$f_\lambda(q,a) = \sum_{g=0}^{\infty}\sum_{Q\neq 0}\sum_{|\mu|=|\lambda|} M_{\lambda\mu}(q)N_{\mu,g,Q}\left(q^{\frac{1}{2}}-q^{-\frac{1}{2}}\right)^{2g-2}a^Q,$$

where $N_{\mu,g,Q}$ are integers which compute the net number of BPS domain walls and $M_{\lambda\mu}(q)$ is defined by

$$M_{\lambda\mu}(q) = \sum_{\mu}\frac{\chi_\lambda(C_\nu)\chi_\mu(C_\nu)}{\mathfrak{z}_\nu}\prod_{j=1}^{l(\nu)}(q^{-\nu_j/2}-q^{\nu_j/2}), \tag{2.1}$$

where $\chi_\nu(C_\mu)$ is the character of an irreducible representation of the symmetric group and $\mathfrak{z}_\mu = |Aut(\mu)|\prod_{i=1}^{l(\mu)}\mu_i$.

For convenience, we usually introduce the invariant

$$n_{\mu,g,Q} = \sum_{\nu}\chi_\nu(C_\mu)N_{\nu,g,Q}. \tag{2.2}$$

Definition 2.1. These predicted integers $N_{\mu,g,Q}$ and $n_{\mu,g,Q}$ are both called LMOV invariants.

Therefore,

$$f_\lambda(q,a) = \sum_{g\geq 0}\sum_{Q\neq 0}\sum_{\mu\in\mathcal{P}}\frac{\chi_\lambda(C_\mu)}{\mathfrak{z}_\mu}n_{\mu,g,Q}\prod_{j=1}^{l(\mu)}\left(q^{-\frac{\mu_j}{2}}-q^{\frac{\mu_j}{2}}\right)\left(q^{-\frac{1}{2}}-q^{\frac{1}{2}}\right)^{2g-2}a^Q$$

By applying the orthogonal relation $\sum_\lambda\frac{\chi_\lambda(C_\mu)\chi_\lambda(C_\nu)}{\mathfrak{z}_\mu} = \delta_{\mu,\nu}$, we obtain the following multiple covering formula for open topological string:

$$\begin{aligned}&\sum_{g\geq 0}\sum_{Q\neq 0} g_s^{2g-2+l(\mu)}K_{\mu,g,Q}a^Q \\ &= \sum_{g\geq 0}\sum_{Q\neq 0}\sum_{d|\mu}\frac{(-1)^{l(\mu)+g}}{\prod_{i=1}^{l(\mu)}\mu_i}d^{l(\mu)-1}n_{\mu/d,g,Q}\prod_{j=1}^{l(\mu)}\left(2\sin\frac{\mu_j g_s}{2}\right)\left(2\sin\frac{dg_s}{2}\right)^{2g-2}a^{dQ}.\end{aligned} \tag{2.3}$$

Hence we have the following integrality structure conjecture which is referred as the Labastida-Mariño-Ooguri-Vafa (LMOV) conjecture for open topological string.

Conjecture 2.2 (LMOV conjecture for open topological string)**.** *Let $F_\mu^{(X,\mathcal{D})}$ be the generating function defined by*

$$F_{\rm str}^{(X,\mathcal{D})}(g_s,a,\mathbf{x}) = \sum_{\mu\in\mathcal{P}_+} F_\mu^{(X,\mathcal{D})} p_\mu(\mathbf{x}),$$

then $F_\mu^{(X,\mathcal{D})}$ has the integral expansion as in the righthand side of the formula (2.3).

There is no general definition for the open Gromov-Witten invariants $K_{\mu,g,Q}$. However, just as mentioned in the previous subsection, when X is a toric Calabi-Yau 3-fold, and $\mathcal{D}$ is the Aganagic-Vafa A-brane [1], the open string partition function $Z_{\rm str}^{(X,\mathcal{D})}$ can be fully determined by using the method of topological vertex [4, 41], and the open Gromov-Witten invariants $K_{\mu,g,Q}$ can also be computed by the topological recursion formula [9]. It is natural to ask how to prove the Conjecture 2.2 for the case of toric Calabi-Yau 3-fold. Actually, this paper is devoted to this conjecture for the resolved conifold with one AV-brane of framing τ.

2.4 Lower genus cases

We illustrate some lower genus cases for the above multiple covering formula (2.3). By using the expansion $\sin x = \sum_{k\geq 1}\frac{x^{2k-1}}{(2k-1)!}$, and taking the coefficients of $g_s^{2g-2+l(\mu)}a^Q$ in formula (2.3), we obtain

$$K_{\mu,0,Q} = \sum_{d|\mu}(-1)^{l(\mu)} d^{l(\mu)-3} n_{\frac{\mu}{d},0,\frac{Q}{d}}, \tag{2.4}$$

$$K_{\mu,1,Q} = \sum_{d|\mu}(-1)^{l(\mu)+1}\left(d^{l(\mu)-1}n_{\frac{\mu}{d},1,\frac{Q}{d}} + \left(\frac{\sum_{j=1}^{l(\mu)}\mu_j^2}{24}d^{l(\mu)-3} - \frac{1}{12}d^{l(\mu)-1}\right)n_{\frac{\mu}{d},0,\frac{Q}{d}}\right)$$

$$K_{\mu,2,Q} = \sum_{d|\mu}(-1)^{l(\mu)}\left(d^{l(\mu)+1}n_{\frac{\mu}{d},2,\frac{Q}{d}} + \frac{\sum_{j=1}^{l(\mu)}\mu_j^2}{24}d^{l(\mu)-1}n_{\frac{\mu}{d},1,\frac{Q}{d}}\right.$$
$$\left. + \left(\frac{\sum_{j=1}^{l(\mu)}\mu_j^4}{1920}d^{l(\mu)-3} + \frac{\sum_{i<j}\mu_i^2\mu_j^2}{576}d^{l(\mu)-3} - \frac{\sum_{j=1}^{l(\mu)}\mu_j^2}{288}d^{l(\mu)-1} + \frac{1}{240}d^{l(\mu)+1}\right)n_{\frac{\mu}{d},0,\frac{Q}{d}}\right)$$

for $g=0$, $g=1$ and $g=2$ respectively. These formulae were firstly illustrated in [62].

Therefore

$$\begin{aligned} F_{(0,l)} &= \sum_{|\mu|=l}\sum_{Q} K_{\mu,0,Q}a^Q x_1^{\mu_1}\cdots x_l^{\mu_l} \\ &= \sum_{|\mu|=l}\sum_{Q}\sum_{d|\mu}(-1)^{l(\mu)}d^{l(\mu)-3}n_{\frac{\mu}{d},0,\frac{Q}{d}}a^Q x_1^{\mu_1}\cdots x_l^{\mu_l} \\ &= (-1)^l\sum_{|\mu|=l}\sum_{Q}\sum_{d\geq 1}d^{l-3}n_{\mu,0,Q}a^Q x_1^{d\mu_1}\cdots x_l^{d\mu_l}. \end{aligned} \tag{2.5}$$

In particular

$$F_{(0,1)} = -\sum_{m\geq 1}\sum_{d\geq 1}\sum_{Q}\frac{n_{m,0,Q}}{d^2}a^{dQ}x^{dm}, \tag{2.6}$$

and for $g=1, l=1$,

$$
\begin{aligned}
F_{(1,1)} &= \sum_{m\geq 0}\sum_{Q} K_{(m),1,Q} a^Q x^m \\
&= \sum_{m\geq 0}\sum_{Q}\left(\sum_{d|m} n_{m/d,1,Q/d} + \left(\frac{m^2}{24}d^{-2} - \frac{1}{12}\right) n_{m/d,0,Q/d}\right) a^Q x^m \\
&= \sum_{m\geq 0}\sum_{Q}\sum_{d\geq 1}\frac{1}{d}\left(n_{m,1,Q} + \left(\frac{m^2}{24} - \frac{1}{12}\right)\right) a^{dQ} x^{dm}.
\end{aligned}
$$

3 Chern-Simons theory and large N duality

3.1 Quantum invariants

In his seminal paper [76], E. Witten introduced a new topological invariant of a 3-manifold M as a partition function of quantum Chern-Simons theory. Let G be a compact gauge group which is a Lie group, and M be an oriented three-dimensional manifold. Let $\mathcal{A}$ be a $\mathfrak{g}$-valued connection on M where $\mathfrak{g}$ is the Lie algebra of G. The Chern-Simons [11] action is given by

$$
S(\mathcal{A}) = \frac{k}{4\pi}\int_M Tr\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)
$$

where k is an integer called the level.

Chern-Simons partition function is defined as the path integral in quantum field theory

$$
Z^G(M;k) = \int e^{iS(A)} D\mathcal{A}
$$

where the integral is over the space of all $\mathfrak{g}$-valued connections $\mathcal{A}$ on M. Although it is not rigorous, Witten [76] developed some techniques to calculate such invariants.

If the three-manifold M contains a link $\mathcal{L}$, we let $\mathcal{L}$ be an L-component link denoted by $\mathcal{L} = \bigsqcup_{j=1}^{L}\mathcal{K}_j$. Define

$$
W_{R_j}(\mathcal{K}_j) = Tr_{R_j} Hol_{\mathcal{K}_j}(\mathcal{A})
$$

which is the trace of holomony along $\mathcal{K}_j$ taken in representation R_j. Then Witten's invariant of the pair $(M,\mathcal{L})$ is given by

$$
Z^G(M,\mathcal{L};\{R_j\};k) = \int e^{iS(\mathcal{A})}\prod_{j=1}^{L} W_{R_j}(\mathcal{K}_j) D\mathcal{A}.
$$

When $M = S^3$ and the Lie algebra of G is semisimple, Reshetikhin and Turaev [72, 73] developed a systematic way to constructed the above invariants by using the representation theory of quantum groups. Their construction led to the definition of colored HOMFLY-PT invariants [46, 53], which can be viewed as the large N limit of the quantum $U_q(sl_N)$ invariants. Usually, we use the notation $W_{\lambda^1,\ldots,\lambda^L}(\mathcal{L};q,a)$ to denote the (framing-independent) colored HOMFLY-PT invariants for a (oriented) link $\mathcal{L} = \bigsqcup_{j=1}^{L}\mathcal{K}_j$, where each component

$\mathcal{K}_j$ is colored by an irreducible representation V_{λ^j} of $U_q(sl_N)$. Some basic structures for $W_{\lambda^1,\ldots,\lambda^L}(\mathcal{L};q,a)$ were proved in [48, 49, 81]. It is difficult to obtain an explicit formula of a given link for any irreducible representations λ. We refer to [53] for an explicit formula for torus links, and a series of works due to Morozov et al. [59] and Nawata et al. [64] for some conjectural formulae of twist knots. In particular, we have the following explicit formula for a trivial knot (unknot) U:

$$W_\lambda(U;q,a) = \prod_{x\in\lambda} \frac{a^{-1/2}q^{cn(x)/2} - a^{1/2}q^{-cn(x)/2}}{q^{h(x)/2} - q^{-h(x)/2}}. \tag{3.1}$$

For a box $x = (i,j) \in \lambda$, the hook length and content are defined to be $hl(x) = \lambda_i + \lambda_j^t - i - j + 1$ and $cn(x) = j - i$ respectively.

3.2 Large N duality

In another fundamental work of Witten [78], the SU(N) Chern-Simons gauge theory on a three-manifold M was interpreted as an open topological string theory on T^*M with N topological branes wrapping M inside T^*M. Furthermore, Gopakumar and Vafa [27] conjectured that the large N limit of SU(N) Chern-Simons gauge theory on S^3 is equivalent to the closed topological string theory on the resolved conifold. Furthermore, Ooguri and Vafa [66] generalized the above construction to the case of a knot $\mathcal{K}$ in S^3. They introduced the Chern-Simons partition function $Z_{\text{CS}}^{(S^3,\mathcal{K})}(q,a,\mathbf{x})$ for $(S^3,\mathcal{K})$ which is a generating function of the colored HOMFLY-PT invariants in all irreducible representations.

$$Z_{\text{CS}}^{(S^3,\mathcal{K})}(q,a,\mathbf{x}) = \sum_{\lambda\in\mathcal{P}} W_\lambda(\mathcal{L},q,a)s_\lambda(\mathbf{x}). \tag{3.2}$$

Ooguri and Vafa [66] conjectured that for any knot $\mathcal{K}$ in S^3, there exists a corresponding Lagrangian submanifold $\mathcal{D}_\mathcal{K}$, such that the Chern-Simons partition function is equal to the open topological string partition function on $(X,\mathcal{D}_\mathcal{K})$. They have established this duality for the case of a trivial knot U in S^3, and the link case was further discussed in [47].

In general, we first should find a way to construct the Lagrangian submanifold $\mathcal{D}_\mathcal{L}$ corresponding to the link $\mathcal{L}$ in geometry. See [13, 34, 47, 75] for the constructions for some special links. Furthermore, if the Lagrangian submanifold $\mathcal{D}_\mathcal{L}$ is constructed, then we need to compute the open sting partition function under this geometry. For a trivial knot in S^3, the dual open string partition function was computed by J. Li and Y. Song [51] and S. Katz and C.-C.M. Liu [35].

On the other hand side, Aganagic and Vafa [1] introduced the special Lagrangian submanifold in toric Calabi-Yau 3-fold which we call Aganagic-Vafa A-brane (AV-brane) and studied its mirror geometry, then they computed the counting of holomorphic disc end on AV-brane by using the idea of mirror symmetry. Moreover, Aganagic and Vafa surprisingly found the computation by using mirror symmetry and the result from Chern-Simons knot invariants [66] are matched. Furthermore, in [3], Aganagic, Klemm and Vafa investigated the integer ambiguity appearing in the disc counting and discovered that the corresponding ambiguity in Chern-Simons theory was described by the framing of the knot.

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They checked that the two ambiguities match for the case of the unknot, by comparing the disk amplitudes on both sides.

Then, Mariño and Vafa [62] generalized the large N duality to the case of knots with arbitrary framing. They studied carefully and established the large N duality between a framed unknot in S^3 and the open string theory on resolved conifold with AV-brane by using the mathematical approach in [35]. By comparing the coefficient of the highest degree of the Kähler parameter in this duality, they derived a remarkable Hodge integral identity which now is called the Mariño-Vafa formula. Two mathematical proofs for the Mariño-Vafa formula were given in [42] and [65] respectively. We describe this duality in more details. For a framed knot $\mathcal{K}_\tau$ with framing $\tau \in \mathbb{Z}$, we define the framed colored HOMFLYPT invariants $\mathcal{K}_\tau$ as follow,

$$\mathcal{H}_\lambda(\mathcal{K}_\tau, q, a) = (-1)^{|\lambda|\tau} q^{\frac{\kappa_\lambda \tau}{2}} W_\lambda(\mathcal{K}, q, a), \tag{3.3}$$

where $\kappa_\lambda = \sum_{i=1}^{l(\lambda)} \lambda_i(\lambda_i - 2i + 1)$.

The Chern-Simon partition function for $(S^3, \mathcal{K}_\tau)$ is given by

$$Z_{\text{CS}}^{(S^3,\mathcal{K}_\tau)}(q, a; \mathbf{x}) = \sum_{\lambda \in \mathcal{P}} \mathcal{H}_\lambda(\mathcal{K}_\tau, q, a) s_\lambda(\mathbf{x}). \tag{3.4}$$

We let $\hat{X} := \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ be the resolved conifold, and D_τ be the corresponding AV-brane. The open string partition function for $(\hat{X}, D_\tau)$ has the structure

$$Z_{\text{str}}^{(\hat{X},D_\tau)}(g_s, a; \mathbf{x}) = \exp\left(-\sum_{g\geq 0,\mu} \frac{\sqrt{-1}^{l(\mu)}}{|Aut(\mu)|} g_s^{2g-2+l(\mu)} F^\tau_{\mu,g}(a) p_\mu(\mathbf{x})\right) \tag{3.5}$$

where $F^\tau_{\mu,g}(a) = \sum_{Q\in\mathbb{Z}/2} K^\tau_{\mu,g,Q} a^Q$ and $K^\tau_{\mu,g,Q}$ is the open Gromov-Witten invariants

$$K^\tau_{\mu,g,Q} = \int_{[\mathcal{M}_{g,l(\mu)}(D^2,S^1|2Q,\mu_1,\ldots,\mu_l)]} e(\mathcal{V}),$$

which is defined by S. Katz and C.-C. Liu [35]. In particular, when $Q = \frac{|\mu|}{2}$, the computations in [35] shows

$$\begin{aligned} K^\tau_{\mu,g,\frac{|\mu|}{2}} &= (-1)^{|\mu|\tau}(\tau(\tau+1))^{l(\mu)-1} \\ &\prod_{i=1}^{l(\mu)} \frac{\prod_{j=1}^{\mu_i-1}(\mu_i\tau + j)}{(\mu_i-1)!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_j\psi_j)} \end{aligned} \tag{3.6}$$

where $\Lambda_g^\vee(\tau) = \tau^g - \lambda_1\tau^{g-1} + \cdots + (-1)^g\lambda_g$. Therefore, the large N duality in this case is given by the following identity:

$$Z_{\text{CS}}^{(S^3,U_\tau)}(q, a; \mathbf{x}) = Z_{\text{str}}^{(\hat{X},D_\tau)}(g_s, a; \mathbf{x}) \tag{3.7}$$

where $q = e^{ig_s}$. By taking the coefficients of $a^{\frac{|\mu|}{2}}$ of the following equality:

$$[p_\mu(\mathbf{x})g_s^{2g-2+l(\mu)}] \log Z_{\text{CS}}^{(S^3,U_\tau)}(q, a; \mathbf{x}) = [p_\mu(\mathbf{x})g_s^{2g-2+l(\mu)}] \log Z_{\text{str}}^{(\hat{X},D_\tau)}(g_s, a; \mathbf{x}),$$

we obtain the Mariño-Vafa formula which is a Hodge integral identity with triple λ-classes. The Mariño-Vafa formula provides a very powerful tool to study the intersection theory of moduli space of curves. From it, we can derive the Witten conjecture [30, 77], ELSV formula [14], and various interesting Hodge integral identities, see [40, 43, 80].

Combining the idea of dualities shown above, and together with several new technical ingredients, Aganagic, Klemm, Mariño and Vafa finally developed a systematic method, gluing the topological vertices, to compute all loop topological string amplitudes on toric Calabi-Yau manifolds [4, 5]. The mathematical theory for topological vertex was finally established in [41]. This method provides an effective way to compute both the closed and open string partition function for a toric Calabi-Yau 3-fold with AV-brane. Therefore, we have an explicit formula for the partition function of resolved conifold $Z_{\text{str}}^{(\hat{X},D_\tau)}(g_s,a,\mathbf{x})$, by comparing to the explicit formula $Z_{\text{CS}}^{(S^3,U_\tau)}(q,a,\mathbf{x})$ of Chern-Simons partition function describe above. Finally, J. Zhou proved the identity (3.7) in [79] based on the results in their previous works [41, 42, 44].

3.3 Integrality of the quantum invariants

Now, let us collect the above discussions together. Let $\mathcal{L}$ be a link in S^3, the large N duality predicts there exists a Lagrangian submanifold $\mathcal{D}_\mathcal{L}$ in the resolved confold $\hat{X}$, and provides us the identity (3.7). Since the topological string partition function $Z_{\text{str}}^{(\hat{X},\mathcal{D}_\mathcal{L})}(g_s,a,\mathbf{x})$ has the integrality structures by the discussions in section 2.3, it implies that the Chern-Simons partition function $Z_{\text{CS}}^{(S^3,\mathcal{L})}(q,a,\mathbf{x})$ also inherits this integrality structure. Usually, this integrality structure is called the LMOV conjecture for link in [48]. Furthermore, as mentioned previously, the large N duality was generalized to the case of framed knot $\mathcal{K}_\tau$ [62], where the Chern-Simons partition $Z_{\text{CS}}^{(S^3,\mathcal{K}_\tau)}$ for framed knot $\mathcal{K}_\tau$ is given by formula (3.4). For convenience, we only formulate the LMOV conjecture for framed knot $\mathcal{K}_\tau$ in the following, although the conjecture should also holds for any framed link, see [50].

Conjecture 3.1 (LMOV conjecture for framed knot or framed LMOV conjecture)**.** *Let*

$$F_{\text{CS}}^{(S^3,\mathcal{K}_\tau)}(q,a,\mathbf{x}) = \log Z_{\text{CS}}^{(S^3,\mathcal{K}_\tau)}(q,a,\mathbf{x})$$

be the Chern-Simons free energy for a framed knot $\mathcal{K}_\tau$ in S^3. Then there exist functions $f_\lambda(\mathcal{K}_\tau;q,a)$ such that

$$F_{\text{CS}}^{(S^3,\mathcal{K}_\tau)}(q,a,\mathbf{x}) = \sum_{d=1}^{\infty}\frac{1}{d}\sum_{\lambda\in\mathcal{P},\lambda\neq 0} f_\lambda(\mathcal{K}_\tau;q^d,a^d)s_\lambda(\mathbf{x}^d).$$

Let $\hat{f}_\mu(\mathcal{K}_\tau;q,a) = \sum_\lambda f_\lambda(\mathcal{K}_\tau;q,a)M_{\lambda\mu}(q)^{-1}$, where $M_{\lambda\mu}(q)$ is defined in the formula (2.1). Denote $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, then for any $\mu \in \mathcal{P}^+$, there are integers $N_{\mu,g,Q}(\tau)$ such that

$$\hat{f}_\mu(\mathcal{K}_\tau;q,a) = \sum_{g\geq 0}\sum_Q N_{\mu,g,Q}(\tau)z^{2g-2}a^Q \in z^{-2}\mathbb{Z}[z^2,a^{\pm\frac{1}{2}}].$$

Therefore,

$$\mathfrak{z}_\mu \hat{g}_\mu(\mathcal{K}_\tau; q, a) = \sum_\nu \chi_\nu(C_\mu) \hat{f}_\nu(\mathcal{K}_\tau; q, a)$$
$$= \sum_{g\geq 0} \sum_Q n_{\mu,g,Q}(\tau) z^{2g-2} a^Q \in z^{-2}\mathbb{Z}[z^2, a^{\pm\frac{1}{2}}],$$

where $n_{\mu,g,Q}(\tau) = \sum_\nu \chi_\nu(C_\mu) N_{\nu,g,Q}(\tau)$.

K. Liu and P. Peng [48] first studied the mathematical structures of LMOV conjecture for general links without framing contribution (i.e. as to the Chern-Simons partition (3.2)), which is equivalent to the framed LMOV conjecture for any links in framing zero. They provided a proof for this case by using cut-and-join analysis and the cabling technique [48]. Motivated by the work [62], K. Liu and P. Peng [50] formulated the framed LMOV conjecture for any links(as to the Chern-Simons partition function (3.4). In [10], the author together with Q. Chen, K. Liu and P. Peng, developed the ideas in [50] to study the mathematical structures of framed LMOV conjecture and discovered the new structures named congruence skein relations for colored HOMFLY-PT invariants.

4 LMOV invariants for framed unknot U_τ

In section 3.2, we have showed that, for a framed unknot U_τ in S^3, the large N duality holds [79]:

$$Z_{\text{CS}}^{(S^3,U_\tau)}(q, a; \mathbf{x}) = Z_{\text{str}}^{(\hat{X},D_\tau)}(g_s, a; \mathbf{x}),\ q = e^{\sqrt{-1}g_s}.$$

So one can compute LMOV invariants completely by using the colored HOMFLY-PT invariants of the framed unknot U_τ. On the other hand side, by using mirror symmetry, one can also compute the partition function $Z_{\text{str}}^{(\hat{X},D_\tau)}(g_s, a; \mathbf{x})$ from B-model. The mirror geometry information of $(\hat{X}, D_\tau)$ is encoded in a mirror curve $\mathcal{C}_{\hat{X}}$. The disc counting information of $(\hat{X}, D_\tau)$ is given by the superpotential related to the mirror curve [1, 3], and this fact was proved in [19].

Furthermore, the open Gromov-Witten invariants of higher genus with more holes can be computed by using Eynard-Orantin topological recursions [15]. This approach named as BKMP conjecture, was proposed by Bouchard, Klemm, Mariño and Pasquetti [9], and was fully proved in [16, 20] for any toric Calabi-Yau 3-fold with AV-brane, so one can also use the BKMP method to compute the LMOV invariants for $(\hat{X}, D_\tau)$. To determine the mirror curve of $(\hat{X}, D_\tau)$, there are standard methods in toric geometry. However, in [2], Aganagic and Vafa proposed another effective way to compute the mirror curve, their method can be applied to more general large N geometry of an arbitrary knot in S^3 [6]. The rest contents of this section will be organized as follow, we first illustrate the computations of the mirror curve of $(\hat{X}, D_\tau)$ by using the method in [2]. Then, we compute the explicit formulae for genus 0 LMOV invariants. Next, we obtain the higher genus LMOV invariants with one hole.

4.1 a-deformed A-polynomial as the mirror curve

The method used in [2] to compute the mirror curve is based on the fact that, colored HOMFLY-PT invariants colored by a partition with a single row is a q-holonomic function, this fact was conjectured and used in many literatures, such as [17, 18], and was finally proved in [23]. In fact, such idea can go back to [22].

Now, we illustrate such computations for framed unknot U_τ. We first compute the noncommutative a-deformed A-polynomial (it is called the Q-deformed A-polynomial in [2], the variable Q in [2] is the variable a here) for U_τ.

By formula (3.1), the colored HOMFLY-PT invariants colored by partition (n) for the unknot U is given by

$$W_n(U;q,a) = \frac{a^{\frac{1}{2}} - a^{-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \cdots \frac{a^{\frac{1}{2}}q^{\frac{n-1}{2}} - a^{-\frac{1}{2}}q^{\frac{-n-1}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}$$

It gives the recursion

$$(q^{n+1}-1)W_{n+1}(U;q,a) - \left(a^{\frac{1}{2}}q^{n+\frac{1}{2}} - a^{-\frac{1}{2}}q^{\frac{1}{2}}\right) W_n(U;q,a) = 0.$$

By formula (3.3), the framed colored HOMFLY-PT invariants for the framed unknot with framing $\tau \in \mathbb{Z}$ is

$$\mathcal{H}_n(U_\tau;q,a) = (-1)^{n\tau} q^{\frac{n(n-1)}{2}\tau} W_n(U;q,a).$$

Then we obtain the recursion for $\mathcal{H}_n(U_\tau;q,a)$ as follow

$$(-1)^\tau (q^{n+1}-1)\mathcal{H}_{n+1}(U_\tau;q,a) - \left(a^{\frac{1}{2}}q^{n+\frac{1}{2}} - a^{-\frac{1}{2}}q^{\frac{1}{2}}\right) q^{n\tau}\mathcal{H}_n(U_\tau;q,a) = 0. \tag{4.1}$$

For a general series $\{\mathcal{H}_n(q,a)\}_{n\geq 0}$, we introduce two operators M and L act on $\{\mathcal{H}_n(q,a)\}_{n\geq 0}$ as follow:

$$M\mathcal{H}_n = q^n\mathcal{H}_n, \; L\mathcal{H}_n = \mathcal{H}_{n+1},$$

then $LM = qML$.

Definition 4.1. The noncommutative a-deformed A-polynomial for series $\{\mathcal{H}_n(q,a)\}_{n\geq 0}$ is a polynomial $\hat{A}(M,L;q,a)$ of operators M, L, such that

$$\hat{A}(M,L;q,a)\mathcal{H}_n(q,a) = 0, \text{for } \; n \geq 0,$$

and $A(M,L;a) = \lim_{q\to 1} \hat{A}(M,L;q,a)$ is called the a-deformed A-polynomial.

Therefore, from the recursion (4.1), we obtain the noncommutative a-deformed A-polynomial for U_τ as follow:

$$\hat{A}_{U_\tau}(M,L,q;a) = (-1)^\tau(qM-1)L - M^\tau \left(a^{\frac{1}{2}}q^{\frac{1}{2}}M - a^{-\frac{1}{2}}q^{\frac{1}{2}}\right).$$

And the a-deformed A-polynomial is given by

$$A_{U_\tau}(M,L;a)=\lim_{q\to 1}\hat{A}(M,L,q;a)=(-1)^\tau(M-1)L-M^\tau\left(a^{\frac{1}{2}}M-a^{-\frac{1}{2}}\right).$$

In order to get the mirror curve of U_τ, we need the following general result which is written in the following lemma. Let $Z(x)=\sum_{k\geq 0}\mathcal{H}_k(q,a)x^k$ be a generating function of the series $\{\mathcal{H}_k(q,a)|k\geq 0\}$. We also introduce two operators $\hat{x},\hat{y}$ act on $Z(x)$ as follow:

$$\hat{x}Z(x)=xZ(x),\ \hat{y}Z(x)=Z(qx),$$

then $\hat{y}\hat{x}=q\hat{x}\hat{y}$. It is easy to obtain the following result (see Lemma 2.1 in [21] for the similar statement).

Proposition 4.2. *Given a noncommutative A-polynomial* $\hat{A}(M,L,q,a)=\sum_{i,j}c_{i,j}M^iL^j$ *for the series* $\{\mathcal{H}_k(q,a)|k\geq 0\}$*, then we have*

$$\hat{A}(\hat{y},\hat{x}^{-1},q,a)Z(x)=\sum_{i,j}\sum_{-j\leq k\leq -1}\mathcal{H}_{k+j}q^{ki}x^k. \tag{4.2}$$

Proof. Since

$$\begin{aligned}\hat{A}(\hat{y},\hat{x}^{-1},q,a)Z(x)&=\sum_{i,j}c_{i,j}\hat{y}^i\hat{x}^{-j}Z(x)\\&=\sum_{i,j}c_{i,j}q^{-ij}x^{-j}Z(q^ix)\\&=\sum_{i,j}c_{i,j}\sum_{n\geq 0}\mathcal{H}_nq^{(n-j)i}x^{n-j}\\&=\sum_{i,j}c_{i,j}\sum_{k\geq 0}\mathcal{H}_{k+j}q^{ki}x^k+\sum_{i,j}\sum_{-j\leq k\leq -1}a_{k+j}q^{ki}x^k,\end{aligned}$$

and by the definitions of the operators M,L, $\hat{A}(M,L,q,a)\mathcal{H}_k=0$ gives

$$\sum_{i,j}c_{i,j}q^{ki}\mathcal{H}_{k+j}=0,\text{for }k\geq 0.$$

We obtain the formula (4.2). □

Finally, the mirror curve is given by

$$A(y,x^{-1};a)=\lim_{q\to 1}\hat{A}(\hat{y},\hat{x}^{-1};q,a)=0.$$

Therefore, in our case, the mirror curve is:

$$A_{U_\tau}(y,x^{-1};a)=y-1-a^{-\frac{1}{2}}(-1)^\tau xy^\tau(ay-1)=0. \tag{4.3}$$

4.2 Disc countings

For convenience, we let $X = a^{-\frac{1}{2}}(-1)^\tau x$, and $Y = 1-y$, then the mirror curve (4.3) is changed to the functional equation

$$Y = X(1-Y)^\tau(1-a(1-Y)). \tag{4.4}$$

In order to solve the above equation, we introduce the following Lagrangian inversion formula [74].

Lemma 4.3. *Let $\phi(\lambda)$ be an invertible formal power series in the indeterminate λ. Then the functional equation $Y = X\phi(Y)$ has a unique formal power series solution $Y = Y(X)$. Moreover, if f is a formal power series, then*

$$f(Y(X)) = f(0) + \sum_{n\geq 1}\frac{X^n}{n}\left[\frac{df(\lambda)}{d\lambda}\phi(\lambda)^n\right]_{\lambda^{n-1}} \tag{4.5}$$

Remark 4.4. In the following, we will frequently use the binomial coefficient $\binom{n}{k}$ for all $n\in\mathbb{Z}$. That means for $n<0$, we define $\binom{n}{k} = (-1)^k\binom{-n+k-1}{k}$.

In our case, we take $\phi(Y) = (1-Y)^\tau(1-a(1-Y))$. Let $f(Y) = 1-Y$, by formula (4.5), we obtain

$$y(X) = 1-Y(X) = 1+\sum_{n\geq 1}\frac{X^n}{n}\sum_{i\geq 0}(-1)^{n+i}\binom{n}{i}\binom{n\tau+i}{n-1}a^i$$

since $\phi(\lambda)^n$ has the expansion

$$\begin{aligned}\phi(\lambda)^n &= (1-\lambda)^{n\tau}(1-a(1-\lambda))^n\\ &= \sum_{i\geq 0}\binom{n}{i}(-a)^i(1-\lambda)^{n\tau+i}\\ &= \sum_{i,j\geq 0}\binom{n}{i}(-1)^{i+j}\binom{n\tau+i}{j}a^i\lambda^j.\end{aligned}$$

Moreover, if we let $f(Y(X)) = \log(1-Y(X))$, then

$$\begin{aligned}\left[\frac{df(\lambda)}{d\lambda}\phi(\lambda)^n\right]_{\lambda^{n-1}} &= \sum_{i\geq 0}(-1)^i\binom{n}{i}\sum_{j=0}^{n-1}(-1)^{j+1}\binom{n\tau+i}{j}a^i\\ &= \sum_{i\geq 0}(-1)^i\binom{n}{i}(-1)^n\binom{n\tau+i-1}{n-1}a^i\end{aligned}$$

where we have used the combinatoric identity:

$$\sum_{j=0}^{n-1}(-1)^{j+1}\binom{m}{j} = (-1)^n\binom{m-1}{n-1}.$$

Formula (4.5) gives

$$\log(y(X)) = \log(1-Y(X)) = \sum_{n\geq 1}\frac{X^n}{n}\sum_{i\geq 0}(-1)^{n+i}\binom{n}{i}\binom{n\tau+i-1}{n-1}a^i,$$

i.e.

$$\log(y(x)) = \sum_{n\geq 1}\frac{x^n}{n}\sum_{i\geq 0}(-1)^{n\tau+n+i}\binom{n}{i}\binom{n\tau+i-1}{n-1}a^{i-\frac{n}{2}}.$$

By BKMP's construction in genus 0 with one hole, one obtains

$$\begin{aligned}F_{(0,1)} &= \int \log(y(x))\frac{dx}{x} \\ &= \sum_{n\geq 1}\frac{x^n}{n^2}\sum_{i\geq 0}(-1)^{n\tau+n+i}\binom{n}{i}\binom{n\tau+i-1}{n-1}a^{i-\frac{n}{2}}.\end{aligned} \tag{4.6}$$

By formula (2.6), and if we let $n_{m,l}(\tau) = n_{m,0,l-\frac{m}{2}}(\tau)$, then

$$F_{(0,1)} = -\sum_{m\geq 1}\sum_{d|m,d|l} d^{-2} n_{\frac{m}{d},\frac{l}{d}}(\tau)x^m a^{l-\frac{m}{2}}. \tag{4.7}$$

Set

$$c_{m,l}(\tau) = -\frac{(-1)^{m\tau+m+l}}{m^2}\binom{m}{l}\binom{m\tau+l-1}{m-1},$$

by comparing the coefficients of $x^m a^{l-\frac{m}{2}}$ in (4.7) and (4.6), we have

$$c_{m,l}(\tau) = \sum_{d|m,d|l}\frac{n_{m/d,l/d}(\tau)}{d^2}.$$

By Möbius inversion formula, we obtain

$$n_{m,l}(\tau) = \sum_{d|m,d|l}\frac{\mu(d)}{d^2}c_{\frac{m}{d},\frac{l}{d}}(\tau). \tag{4.8}$$

In [56], we prove that $n_{m,l}(\tau)\in\mathbb{Z}$ by using the basic method of number theory. Recently, Panfil and Sulkowski [69] generalized the above disc counting formula (4.8) to a class of toric Calabi-Yau manifolds without compact four-cycles which is also referred to as strip geometries. In our notations, their formula (cf. formula (4.19) in [69]) can be formulated as follow.

Given two integers $r,s\geq 0$, set $\mathbf{l}=(l_1,\ldots,l_r)$, $\mathbf{k}=(k_1,\ldots,k_s)$, and $|\mathbf{l}|=\sum_{j=1}^r l_j$, $|\mathbf{k}|=\sum_{j=1}^s k_j$. We define

$$c_{m,\mathbf{l},\mathbf{k}}(\tau) = \frac{(-1)^{m(\tau+1)+|\mathbf{l}|}}{m^2}\binom{m\tau+|\mathbf{l}|+|\mathbf{k}|-1}{m-1}\prod_{j=1}^r\binom{m}{l_j}\prod_{j=1}^s\frac{m}{m+k_j}\binom{m+k_j}{k_j}.$$

Then, we have the disc counting formula

$$n_{m,\mathbf{l},\mathbf{k}}(\tau) = \sum_{d|\gcd(m,\mathbf{l},\mathbf{k})}\frac{\mu(d)}{d^2}c_{m/d,\mathbf{l}/d,\mathbf{k}/d}(\tau) \tag{4.9}$$

It is obvious that formula (4.8) is just the special case of (4.9) by taking $r=1$ and $s=0$.

In [84], we will generalize the method used in [56] to show that $n_{m,\mathbf{l},\mathbf{k}}(\tau)\in\mathbb{Z}$.

4.3 Annulus counting

The Bergmann kernel of the curve (4.4) is

$$B(X_1, X_2) = \frac{dY_1 dY_2}{(Y_1 - Y_2)^2}.$$

By the construction of BKMP [9], the annulus amplitude is calculated by the integral

$$\int \left(B(X_1, X_2) - \frac{dX_1 dX_2}{(X_1 - X_2)^2} \right) = \ln \left(\frac{Y_2(X_2) - Y_1(X_1)}{X_2 - X_1} \right)$$

More precisely, for $m_1, m_2 \geq 1$, the coefficients $\left[\ln \left(\frac{Y_2(X_2) - Y_1(X_1)}{X_2 - X_1} \right) \right]_{x_1^{m_1} x_2^{m_2} a^l}$ gives the annulus Gromov-Witten invariants $K_{(m_1,m_2),0,l}$.

Let $b_{n,i} = \frac{(-1)^{n+i}}{n+1} \binom{n+1}{i} \binom{(n+1)\tau + i}{n}$ and $b_n = \sum_{i \geq 0} b_{n,i} a^i$. In particular $b_0 = 1 - a$. Then

$$Y(X) = \sum_{n \geq 1} b_n X^n,$$

and

$$\frac{Y_2(X_2) - Y_1(X_1)}{X_2 - X_1} = (1 - a) + \sum_{n \geq 1} b_n \left(\sum_{i=0}^{n} X_1^i X_2^{n-i} \right).$$

Let $\tilde{b}_{m,l} = \sum_{i=0}^{l} b_{m,i}$ and $\tilde{b}_m = \sum_{l=0} \tilde{b}_{m,l} a^l$. For $m_1 \geq 1, m_2 \geq 1$, the coefficients $c_{(m_1,m_2)}$ of $[X_1^{m_1} X_2^{m_2}]$ in the expansion

$$\ln \left(1 + \sum_{n \geq 1} \tilde{b}_n \left(\sum_{i=0}^{n} X_1^i X_2^{n-i} \right) \right)$$

is given by

$$c_{(m_1,m_2)} = \sum_{|\mu| = m_1 + m_2} \frac{(-1)^{l(\mu)-1} (l(\mu) - 1)! \tilde{b}_\mu}{|Aut(\mu)|} |S_\mu(m_1)|$$

where $S_\mu(m_1)$ is the set

$$S_\mu(m_1) = \{(i_1, \ldots, i_{l(\mu)}) \in \mathbb{Z}^{l(\mu)} | \sum_{k=1}^{l(\mu)} i_k = m_1, \text{ where } 0 \leq i_k \leq \mu_k, \text{ for } k = 1, \ldots, l(\mu)\},$$

by this definition, $S_\mu(m_1) = S_\mu(m_2)$.

We write $c_{(m_1,m_2)} = \sum_{l \geq 0} c_{(m_1,m_2),l} a^l$, then the annulus amplitude is

$$F_{(0,2)} = \sum_{m_1 \geq 1, m_2 \geq 1} \sum_{l \geq 0} (-1)^{(m_1+m_2)\tau} c_{(m_1,m_2),l} a^{l - \frac{m_1+m_2}{2}} x_1^{m_1} x_2^{m_2}.$$

Set $n_{(m_1,m_2),l} = n_{(m_1,m_2),0,l-\frac{m_1+m_2}{2}}$, the multiple covering formula (2.5) for $l = 2$ gives

$$F_{(0,2)} = \sum_{m_1\geq 1, m_2\geq 1} \sum_{l\geq 0} \sum_{d|m_1,d|m_2,d|l} \frac{1}{d} n_{(\frac{m_1}{d},\frac{m_2}{d}),\frac{l}{d}} a^{l-\frac{m_1+m_2}{2}} x_1^{m_1} x_2^{m_2}$$

we have

$$(-1)^{(m_1+m_2)\tau} c_{(m_1,m_2),l} = \sum_{d|m_1,d|m_2,d|l} \frac{1}{d} n_{(\frac{m_1}{d},\frac{m_2}{d}),\frac{l}{d}},$$

so

$$n_{(m_1,m_2),l} = \sum_{d|m_1,d|m_2,d|l} \frac{\mu(d)}{d} (-1)^{\frac{(m_1+m_2)\tau}{d}} c_{(\frac{m_1}{d},\frac{m_2}{d}),\frac{l}{d}}.$$

In particular, when $l = \frac{m_1+m_2}{2}$, we only need to consider the curve $Y = X(1-Y)^\tau$. With the help of the following formula proved in [80]

Lemma 4.5 (Lemma 2.3 of [80])**.**

$$\ln\left(\frac{Y_1(X_1)-Y_2(X_2)}{X_1-X_2}\right) = \sum_{m_1,m_2\geq 1} \frac{1}{m_1+m_2} \binom{m_1\tau+m_1-1}{m_1} \binom{m_2\tau+m_2}{m_2} X_1^{m_1} X_2^{m_2} \\ -\tau\left(\ln(1-Y_1(X_1))+\ln(1-Y_2(X_2))\right). \tag{4.10}$$

We obtain

$$c_{(m_1,m_2),\frac{m_1+m_2}{2}}(\tau) = \frac{1}{m_1+m_2} \binom{m_1\tau+m_1-1}{m_1} \binom{m_2\tau+m_2}{m_2}. \tag{4.11}$$

For brevity, we let $n_{(m_1,m_2)}(\tau) := n_{(m_1,m_2),\frac{m_1+m_2}{2}}(\tau)$ which is defined through formula (4.11). Then we obtain

$$n_{(m_1,m_2)}(\tau) = \frac{1}{m_1+m_2} \sum_{d|m_1,d|m_2} \mu(d)(-1)^{(m_1+m_2)(\tau+1)/d} \\ \cdot \binom{(m_1\tau+m_1)/d-1}{m_1/d} \binom{(m_2\tau+m_2)/d}{m_2/d}.$$

In [56], we have also proved that $n_{(m_1,m_2)}(\tau) \in \mathbb{Z}$ for any $m_1, m_2 \geq 1$ and $\tau \in \mathbb{Z}$.

4.4 Genus $g = 0$ with more holes

By formula (3.6), we have

$$K^\tau_{\mu,g,\frac{|\mu|}{2}} = (-1)^{|\mu|\tau}[\tau(\tau+1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Gamma_g(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} \\ = (-1)^{|\mu|\tau}[\tau(\tau+1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \binom{\mu_i(\tau+1)-1}{\mu_i-1} \sum_{b_i\geq 0} \prod_{i=1}^{l(\mu)} \mu_i^{b_i} \langle \prod_{i=1}^{l(\mu)} \tau_{b_i}\Gamma_g(\tau)\rangle_{g,l(\mu)}$$

When $g = 0$ and $l \geq 3$, then $\Gamma_0(\tau) = 1$ and we have the Hodge integral identity:

$$\langle \tau_{b_1} \cdots \tau_{b_l} \rangle_{0,l} = \binom{l-3}{b_1, \ldots, b_l}.$$

Hence, we obtain

$$K^{\tau}_{\mu,0,\frac{|\mu|}{2}} = (-1)^{|\mu|\tau} [\tau(\tau+1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \binom{\mu_i(\tau+1)-1}{\mu_i - 1} \left(\sum_{i=1}^{l(\mu)} \mu_i \right)^{l(\mu)-3}. \tag{4.12}$$

By using formula (2.4), we get

$$n_{\mu,0,\frac{|\mu|}{2}}(\tau) = (-1)^{l(\mu)} \sum_{d|\mu} \mu(d) d^{l(\mu)-1} K^{\tau}_{\frac{\mu}{d},0,\frac{|\mu|}{2d}}. \tag{4.13}$$

It is clear that $K^{\tau}_{\mu,0,\frac{|\mu|}{2}} \in \mathbb{Z}$ from formula (4.12), and since $l(\mu) \geq 3$, it is clear that

$$n_{\mu,0,\frac{|\mu|}{2}}(\tau) \in \mathbb{Z}$$

for any partition μ with $l(\mu) \geq 3$.

4.5 Genus $g \geq 1$ with one hole

As discussed in the introduction, we have three approaches to the LMOV invariants $n_{\mu,g,Q}(\tau)$ for the open topological string model $(\hat{X}, \mathcal{D}_\tau)$, as to the higher genus LMOV invariants $n_{\mu,g,Q}(\tau)$, we would like to use the identity (1.2) of large N duality to change all the computations from topological string to Chern-Simons theory for knot invariants. Since the large N duality of topological string and Chern-Simons theory was conjectured for any framed knots (even links), we formulate the LMOV integrality conjecture for any framed knots first, and then we focus on the special case of framed unknot U_τ in S^3, and give an explicit formula for the LMOV invariants $n_{(m),g,Q}(\tau)$ whose integrality properties was proved in [56].

4.5.1 Revist LMOV integrality conjecture for framed knot $\mathcal{K}_\tau$

We introduce the following notations first. Let $n \in \mathbb{Z}$ and λ, μ, ν denote the partitions. Set

$$\{n\}_x = x^{\frac{n}{2}} - x^{-\frac{n}{2}}, \ \{\mu\}_x = \prod_{i=1}^{l(\mu)} \{\mu_i\}_x. \tag{4.14}$$

For brevity, we denote $\{n\} = \{n\}_q$ and $\{\mu\} = \{\mu\}_q$. Let $\mathcal{K}_\tau$ be a knot with framing $\tau \in \mathbb{Z}$. The framed colored HOMFLYPT invariant $\mathcal{H}(\mathcal{K}_\tau; q, a)$ of $\mathcal{K}_\tau$ is defined by formula (3.3). Let

$$\mathcal{Z}_\mu(\mathcal{K}_\tau) = \sum_\lambda \chi_\lambda(C_\mu) \mathcal{H}_\lambda(\mathcal{K}_\tau),$$

then the Chern-Simons partition function is given by

$$Z_{\mathrm{CS}}^{(S^3,\mathcal{K}_\tau)} = \sum_{\lambda\in\mathcal{P}} \mathcal{H}_\lambda(\mathcal{K}_\tau)s_\lambda(x) = \sum_{\mu\in\mathcal{P}} \frac{\mathcal{Z}_\mu(\mathcal{K}_\tau)}{\mathfrak{z}_\mu} p_\mu(x).$$

We define $F_\mu(\mathcal{K}_\tau)$ though the expansion formula

$$F_{\mathrm{CS}}^{(S^3,\mathcal{K}_\tau)} = \log(Z_{\mathrm{CS}}^{(S^3,\mathcal{K}_\tau)}) = \sum_{\mu\in\mathcal{P}^+} F_\mu(\mathcal{K}_\tau)p_\mu(x),$$

then we have

$$F_\mu(\mathcal{K}_\tau) = \sum_{n\geq 1} \sum_{\cup_{i=1}^n \nu^i=\mu} \frac{(-1)^{n-1}}{n} \prod_{i=1}^n \frac{\mathcal{Z}_{\nu^i}(\mathcal{K}_\tau)}{\mathfrak{z}_{\nu^i}}.$$

Remark 4.6. For two partitions ν^1 and ν^2, the notation $\nu^1 \cup \nu^2$ denotes the new partition by combing all the parts in ν^1, ν^2. For example $\mu = (2,2,1)$, then the set of pairs (ν^1, ν^2) such that $\nu^1 \cup \nu^2 = (2,2,1)$ is

$$(\nu^1 = (2), \nu^2 = (2,1)), \quad (\nu^1 = (2,1), \nu^2 = (2)),$$
$$(\nu^1 = (1), \nu^2 = (2,2)), \quad (\nu^1 = (2,2), \nu^2 = (1)),$$

For a rational function $f(q,a) \in \mathbb{Q}(q^\pm, a^\pm)$, we define the adams operator

$$\Psi_d(f(q,a)) = f(q^d, a^d).$$

Then, we set

$$\hat{g}_\mu(\mathcal{K}_\tau) = \sum_{d|\mu} \frac{\mu(d)}{d} \Psi_d(\hat{F}_{\mu/d}(\mathcal{K}_\tau)), \tag{4.15}$$

where

$$\hat{F}_\mu(\mathcal{K}_\tau) = \frac{F_\mu(\mathcal{K}_\tau)}{\{\mu\}}.$$

The LMOV integrality conjecture for framed knot $\mathcal{K}_\tau$ states that

Conjecture 4.7. *$\mathfrak{z}_\mu \hat{g}_\mu(\mathcal{K}_\tau)$ is a polynomial of the LMOV invariants $n_{\mu,g,Q}(\tau)$, more precisely,*

$$\mathfrak{z}_\mu \hat{g}_\mu(\mathcal{K}_\tau) = \sum_{g\geq 0} \sum_Q n_{\mu,g,Q}(\tau) z^{2g-2} a^Q \in z^{-2}\mathbb{Z}[z^2, a^{\pm\frac{1}{2}}],$$

where $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}} = \{1\}$.

4.5.2 LMOV integrality invariants for U_τ

Now we apply the above computations to the case of framed unknot U_τ. By the large N duality formula in this special case (1.2) proved by [79], the LMOV integrality Conjecture 2.2 for the open string model $(\hat{X}, \mathcal{D}_\tau)$ and LMOV integrality conjecture for the Chern-Simon theory (S^3, U_τ) are same. In other words, LMOV invariants for the $(\hat{X}, \mathcal{D}_\tau)$ and (S^3, U_τ) are the same one.

For convenience, we define the function

$$\phi_{\mu,\nu}(x) = \sum_\lambda \chi_\lambda(C_\mu)\chi_\lambda(C_\nu)x^{\kappa_\lambda}.$$

By Lemma 5.1 in [10], for $d \in \mathbb{Z}_+$, we have

$$\phi_{(d),\nu}(x) = \frac{\{d\nu\}_{x^2}}{\{d\}_{x^2}}.$$

By using the formula of colored HOMFLYPT invariant for unknot (3.1), we obtain

$$\begin{aligned}
\mathcal{Z}_\mu(U_\tau) &= \sum_\lambda \chi_\lambda(C_\mu)\mathcal{H}_\lambda(U_\tau) \\
&= (-1)^{|\mu|\tau} \sum_\lambda \chi_\lambda(C_\mu) q^{\frac{\kappa_\lambda \tau}{2}} \sum_\nu \frac{\chi_\lambda(C_\nu)}{z_\nu} \frac{\{\nu\}_a}{\{\nu\}} \\
&= (-1)^{|\mu|\tau} \sum_\nu \frac{1}{\mathfrak{z}_\nu} \phi_{\mu,\nu}(q^{\frac{\tau}{2}}) \frac{\{\nu\}_a}{\{\nu\}}.
\end{aligned}$$

In particular, for $\mu = (m)$, $m \in \mathbb{Z}$, we have

$$\mathcal{Z}_m(U_\tau) = (-1)^{m\tau} \sum_{|\nu|=m} \frac{1}{\mathfrak{z}_\nu} \frac{\{m\nu\tau\}}{\{m\tau\}} \frac{\{\nu\}_a}{\{\nu\}}.$$

For brevity, we set $\mathcal{Z}_m(q,a) = \frac{1}{\{m\}}\mathcal{Z}_m(U_\tau) = (-1)^{m\tau}\sum_{|\nu|=m} \frac{1}{\mathfrak{z}_\nu} \frac{\{m\nu\tau\}}{\{m\}\{m\tau\}} \frac{\{\nu\}_a}{\{\nu\}}$ and $g_m(q,a) = \mathfrak{z}_{(m)}\hat{g}_m(U_\tau)$. Then, by formula (4.15), we obtain

$$g_m(q,a) = \sum_{d|m} \mu(d)\mathcal{Z}_{m/d}(q^d, a^d). \tag{4.16}$$

In [56], we have proved that $g_m(q,a)$ is a polynomial of the higher genus with one hole LMOV invariants $n_{m,g,Q}(\tau)$. More precisely,

$$g_m(q,a) = \sum_{g \geq 0} \sum_Q n_{m,g,Q}(\tau) z^{2g-2} a^Q \in z^{-2}\mathbb{Z}[z^2, a^{\pm\frac{1}{2}}],$$

where $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}} = \{1\}$. In other words, we have

$$n_{m,g,Q}(\tau) = \text{Coefficient of term } z^{2g-2}a^Q \text{ in the polynomial } g_m(q,a). \tag{4.17}$$

5 Conclusions and related works

In this finial section, we mention some related works and problems which are deserved to study further.

- Applications of our explicit formulae. In A. Mironov et al's work [61], they made a lot of numerical computations for a large variety of LMOV invariants by using their recent works on knot invariants. By experimental observation, they proposed a conjecture that the absolute values of the LMOV invariants $N^{\mathcal{K}}_{\mu,g,Q}$ for big enough representations μ approach Gaussian/binomial distribution in g with just three μ, Q-dependent parameters (cf. Conjecture in section 5 of [61] and note that the meanings of the symbols μ, g, Q used here are corresponding to Q, g, n in [61]). The LMOV invariant $n_{\mu,g,Q}$ are related to $N_{\mu,g,Q}$ through a character transformation formula (2.2). So we expect that our explicit formulae could provide a rigid proof of their conjecture at least for the case of framed unknot U_τ.

- Interpretations of the integrality of LMOV invariants. In [55], we found a relation between the open string partition of $\mathbb{C}^3$ with AV brane D_τ and the Hilbert-Poincaré series of the Cohomological Hall algebra of the $|\tau|$-loop quiver in the sense of [32]. This is the first example of toric Calabi-Yau and quiver correspondence. Then in [37, 38, 83], a general knot-quiver correspondence was proposed. Especially, Sulkowski et al. [38, 70] established this correspondence for a large class of knot, and links.

- Compute the explicit formulae for LMOV invariants in more general settings. In the recent work of Panfil and Sulkowski [69], they found a direct relation between quiver representation theory and open topological string theory on a class of toric Calabi-Yau 3-folds referred as strip geometries. With the help of the relation to quivers they also derive explicit expressions for classical open BPS invariants for an arbitrary strip geometry, which lead to a large set of number theoretic integrality statements. In [84], we generalize our current work to the case of torus knots and other settings, more explicit formulae for corresponding LMOV invariants are obtained. We will develop a general number theory method to prove integrality properties of LMOV invariants.

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