

THE CAYLEY-HAMILTON THEOREM FOR SUPERMATRICES

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ABSTRACT

Starting from the expression for the superdeterminant of $(xI - M)$, where M is an arbitrary supermatrix, we propose a definition for the corresponding characteristic polynomial and we prove that each supermatrix satisfies its characteristic equation. Depending upon the factorization properties of the basic polynomials whose ratio defines the above mentioned superdeterminant we are able to construct polynomials of lower degree which are also shown to be annihilated by the supermatrix. Two examples are presented.

1. Introduction

The Cayley-Hamilton theorem is a powerful theorem in the sense that produces n^2 null identities among the matrix elements. This theorem, in its usual form, has recently found interesting applications in $2 + 1$ dimensional Chern-Simons (CS) theories ¹. Pure CS theories are of topological nature and the fundamental degrees of freedom are the traces of the group elements constructed as the holonomies (or Wilson lines, or integrated connections) of the gauge connection around oriented closed curves on the manifold. The observables are the expectation values of the Wilson lines which turned out to be realized as the various knot polynomials known to mathematicians ². Since CS theories are also exactly soluble and possess a finite number of degrees of freedom ³, another aspect of interest is the reduction of the initially infinite-dimensional phase space to the subspace of the true degrees of freedom. The Cayley-Hamilton theorem has played an important role in the construction of the so called skein relations ⁴, which are relevant to the calculation

of expectation values, and also in the process of reduction of the phase space. To illustrate the basic ideas related to the last point let us consider the simple case of two matrices M_1 and M_2 which belong to $SL(2, R)$. In this case the characteristic polynomial is $P(x) = x^2 - \text{Tr}(M_1)x + 1$ and we have the Cayley-Hamilton matrix identity

$$(M_1)^2 - \text{Tr}(M_1)M_1 + I = 0. \quad (1.1)$$

By multiplying Eq.(1.1) by M_2M^{-1} and tracing we obtain the following relation among the traces

$$\text{Tr}(M_2M_1^{-1}) + \text{Tr}(M_1M_2) = \text{Tr}(M_1)\text{Tr}(M_2). \quad (1.2)$$

We recall also that for any $SL(2, R)$ matrix we have $\text{Tr}(M) = \text{Tr}(M^{-1})$. The expression (1.2) finds a very useful application in the discussion of the reduced phase space of the de Sitter gravity in $2+1$ dimensions, which is equivalent to the CS theory of the group $SO(2, 2)$ ³. This theory can be more easily described in terms of two copies of the group $SL(2, R)$, which is the spinorial group of $SO(2, 2)$. The gauge invariant degrees of freedom associated to one genus of an arbitrary genus g two-dimensional surface turn out to be traces of any product of powers of two $SL(2, R)$ matrices M_1 and M_2 , which correspond to the holonomies (or integrated connections) of the two homotopically distinct trajectories on one genus. Nevertheless, because CS theories have a finite number of degrees of freedom, one should be able to reduce this infinite set of traces to a finite one. This task can in fact be accomplished by virtue of the relation (1.2). In other words, the trace $\text{Tr}(M_1^{p_1}M_2^{q_1}M_1^{q_1}M_2^{q_2}\dots M_1^{p_n}M_2^{q_n}\dots)$, for any p_i, q_i in $\mathbb{Z}$, can be shown to be reducible and can be expressed as a function of three traces only: $\text{Tr}(M_1)$, $\text{Tr}(M_2)$ and $\text{Tr}(M_1M_2)$ ⁵. In general, the non-linear constraints among the traces that need to be solved in order to reduce the phase space in CS theories are usually obtained from the so called Mandelstam identities ⁶. The discussion of the relation among these identities and the Cayley-Hamilton identities, together with the construction of the former identities in the case of supermatrices is reported in Ref.7.

In this work we review the general construction of Cayley-Hamilton type identities for supermatrices. This is an interesting problem in its own, besides the possible application in the study of the reduced phase space in CS theories defined over a supergroup ⁸.

2. The Characteristic and Null Polynomials for Supermatrices

A $(p+q) \times (p+q)$ supermatrix is a block matrix of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (2.1)$$

where A, B, C and D are $p \times p$, $p \times q$, $q \times p$, $q \times q$ matrices respectively. The distinguishing feature with respect to an ordinary matrix is that the matrix elements M_{RS} , $R =$

$(i, \alpha), S = (j, \beta)$ are elements of a Grassmann algebra with the property that A_{ij} ($i, j = 1 \dots p$) and $D_{\alpha\beta}$ ($\alpha, \beta = 1 \dots q$) are even elements, while $B_{i\alpha}$ and $C_{\beta j}$ are odd elements of such algebra.

Let us recall that the ordinary matrix addition and the ordinary matrix product of two supermatrices is again a supermatrix. Nevertheless, such concepts as the trace and the determinant need to be redefined, because of the odd component piece of the supermatrix. The basic invariant under similarity transformations for supermatrices is the supertrace, defined by

$$Str(M) = Tr(A) - Tr(D). \quad (2.3)$$

The generalization of the determinant, called the superdeterminant, is obtained from (2.3) by defining

$$\delta \ln(Sdet M) = Str(M^{-1} \delta M) \quad (2.4)$$

with appropriate boundary conditions, which produces the following equivalent two forms of calculating the superdeterminant ⁹

$$Sdet(M) = \frac{\det(A - BD^{-1}C)}{\det D} = \frac{\det A}{\det(D - CA^{-1}B)}. \quad (2.5)$$

All the matrices involved now are even in the Grassmann algebra and $\det$ has its usual meaning. Let us consider now the characteristic function

$$h(x) = Sdet(xI - M) = \frac{\tilde{F}(x)}{\tilde{G}(x)} = \frac{F(x)}{G(x)}, \quad (2.6)$$

each form arising from the two alternatives (2.5) of calculating the superdeterminant. The explicit expressions for the numerators and denominators are

$$\tilde{F}(x) = \det(d(x)(xI - A) - B \text{adj}(xI - D)C), \quad \tilde{G}(x) = (d(x))^{p+1}, \quad (2.7a)$$

$$F(x) = (a(x))^{q+1}, \quad G(x) = \det(a(x)(xI - D) - C \text{adj}(xI - A)B), \quad (2.7b)$$

where $a(x) = \det(xI - A)$ and $d(x) = \det(xI - D)$.

We are able to prove that a characteristic (i.e. null) polynomial $\mathcal{P}(x)$ for an arbitrary supermatrix is given by ¹⁰

$$\mathcal{P}(x) = \tilde{F}(x)G(x) = F(x)\tilde{G}(x) = a(x)^{q+1}d(x)^{p+1}. \quad (2.8)$$

Nevertheless, and motivated by the work of Ref.11, we have realized that there are some cases in which we can construct null polynomials of lower degree according to the factorization properties of the basic polynomials $\tilde{F}, \tilde{G}, F, G$. At this point it is important to observe that we do not have a unique factorization theorem for polynomials defined over a Grassmann algebra. The construction of such null polynomials of lower degree starts from finding the divisors of maximum degree of the pairs $\tilde{F}, \tilde{G}$,

and F, G which we denote by R and S respectively. This means that we are able to write

$$\tilde{F} = R\tilde{f}, \quad \tilde{G} = Rg, \quad F = Sf, \quad G = Sg, \quad (2.9)$$

where all the polynomials are monic, and $\tilde{f}, \tilde{g}, f, g$ are of minimum degree by construction. They must satisfy

$$\frac{\tilde{f}}{\tilde{g}} = \frac{f}{g}. \quad (2.10)$$

For each family of possible factorizations written in Eq.(2.9) we can prove that ¹⁰

$$P(x) = \tilde{f}(x)g(x) = f(x)\tilde{g}(x) \quad (2.11)$$

is also a null polynomial, which is clearly of lower degree than $\mathcal{P}(x)$. The characteristic polynomial $\mathcal{P}(x)$ is just a particular case of the latter null polynomials, when $R = S = 1$.

3. Characteristic Polynomials of $(1+1) \times (1+1)$ Supermatrices

3.1. The Polynomials $a(x)$ and $d(x)$ are Coprime

In this section we consider a particular example of null polynomials of minimum degree for supermatrices, according to the definitions given in the previous section. Our general procedure for constructing such null polynomials is based on the factorization properties of the polynomials $\tilde{F}, \tilde{G}, F, G$ introduced in Section 2. At this point we emphasize that when dealing with polynomials over a Grassmann algebra, the existence of a maximum common divisor of two polynomials is not in one to one correspondence with fact that these polynomials are not coprime. In fact, we can find polynomials which are not coprime and nevertheless do not have a common factor, as shown in the next case 3.2.

The simplest example of null polynomials of minimum degree corresponds to the supermatrix given by

$$M = \begin{pmatrix} \bar{p} & \alpha \\ \beta & \bar{q} \end{pmatrix}, \quad (3.1)$$

with $\bar{p} \neq \bar{q}$ in such way that $a = x - p$ and $d = x - q$ are coprime polynomials according to the lemma (3.3) in Ref.11. The bar over a number denotes its body. Here we have

$$\begin{aligned} \tilde{F} &= (x - q)(x - p) - \alpha\beta, & \tilde{G} &= (x - q)^2, \\ F &= (x - p)^2, & G &= (x - q)(x - p) + \alpha\beta. \end{aligned} \quad (3.2)$$

The modified Euclidean algorithm of Ref. 11 applied to each pair $\tilde{F}, \tilde{G}$, F, G leads to the following factorizations

$$\begin{aligned}
\tilde{F} &= (x - p + \frac{\alpha\beta}{q-p})(x - q - \frac{\alpha\beta}{q-p}), \\
\tilde{G} &= (x - q + \frac{\alpha\beta}{q-p})(x - q - \frac{\alpha\beta}{q-p}), \\
F &= (x - p + \frac{\alpha\beta}{q-p})(x - p - \frac{\alpha\beta}{q-p}), \\
G &= (x - q + \frac{\alpha\beta}{q-p})(x - p - \frac{\alpha\beta}{q-p}).
\end{aligned} \tag{3.3}$$

Thus we can identify $R = (x - q - \frac{\alpha\beta}{q-p})$, $S = (x - p - \frac{\alpha\beta}{q-p})$ together with $\tilde{f} = f = x - p + \frac{\alpha\beta}{q-p}$ and $g = \tilde{g} = x - q + \frac{\alpha\beta}{q-p}$ in the notation of Section 2. The null polynomial of minimum degree is then ¹⁰

$$P(x) = fg = x^2 - x(p + q - \frac{\alpha\beta}{q-p}) + pq - \frac{q+p}{q-p}\alpha\beta, \tag{3.4}$$

which can be verified by direct substitution.

3.2. The Polynomials $a(x)$ and $d(x)$ are not Coprime

An example of this kind is provided by the $(1+1) \times (1+1)$ supermatrix

$$M = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.5}$$

where σ is an even element of the Grassmann algebra such that $\bar{\sigma} = 0$ and $\sigma^2 = 0$. In this case our procedure will produce a family of null polynomials. Here, $a = x - \sigma$ and $d = x$ which are not coprime polynomials according to the definition of Ref.11. Again, we emphasize the unintuitive fact that even though a and d are not coprime, they do not have a common factor. The basic polynomials are

$$\begin{aligned}
\tilde{F} &= x(x - \sigma), & F &= (x - \sigma)^2 = x^2 - 2x\sigma, \\
\tilde{G} &= x^2, & G &= x(x - \sigma)
\end{aligned} \tag{3.6}$$

and we need to consider the corresponding factorization properties. It is obvious, for example, that $\tilde{F}$ and $\tilde{G}$ have x as a common factor. Surprisingly, this result can not be obtained by applying the Euclidean algorithm to $\tilde{F}$ and $\tilde{G}$. Besides, the non existence of a unique factorization theorem is clearly shown here in the identity $x^2 = (x + z\sigma)(x - z\sigma)$, with z being an arbitrary complex number. Choosing $z = 1$ leads to the conclusion that $\tilde{F}$ and $\tilde{G}$ have two common factors of maximum degree which are x and $(x - \sigma)$. The same happens with F and G . Thus, after each cancellation is made, we are left with four possible combinations of the reduced ratios

$$\frac{\tilde{f}_i}{\tilde{g}_i} = \frac{f_j}{g_j}, \quad i, j = 1, 2, \tag{3.7}$$

where

$$\begin{aligned}\tilde{f}_1 &= x - \sigma, & \tilde{f}_2 &= x, & f_1 &= x - \sigma, & f_2 &= x - 2\sigma, \\ \tilde{g}_1 &= x, & \tilde{g}_2 &= x + \sigma, & g_1 &= x, & g_2 &= x - \sigma.\end{aligned}\tag{3.8}$$

For each possibility we can verify that Eq. (3.7) is indeed correct. According to Eq. (2.11) we obtain four null polynomials given by $P_{ij}(x) = \tilde{f}_i g_j$. They are

$$\begin{aligned}P_{11} &= x^2 - \sigma x, & P_{12} &= x^2 - 2\sigma x \\ P_{21} &= x^2, & P_{22} &= x^2 - \sigma x.\end{aligned}\tag{3.9}$$

Since any linear combination of the above polynomials will be also annihilated by the supermatrix, we finally obtain two basic null polynomials which are

$$P_1 = x^2, \quad P_2 = \sigma x.\tag{3.10}$$

4. References

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