Gravitating Q-balls

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Abstract

We study properties of Q-balls in flat spacetime and in curved spacetime. (1) By energy analysis with catastrophe theory we obtain a clear picture of stability change of equilibrium solutions. (2) Numerical analysis of dynamical equations as a whole confirms the stability obtained by the energy analysis. However, even if we give perturbed initial conditions with the same charge, a part of charge is radiated away and approaches a different equilibrium solution with lower charge. (3) We study gravitating Q-balls as well. If the mass of the scalar filed is close to Planck mass, equilibrium solutions are nonexistent; a Q-ball either approaches a stable configuration or collapses to a black hole. We also argue that Q-ball inflation does not occur.

1 Introduction

Q-balls [1] are natural consequences of many models of a scalar field and could be dark matter [2]. To understand basic properties of Q-balls, we address the following issues.

(1) In flat spacetime stability against infinitesimal perturbations is well understood both in the thin-wall limit and in the thick-wall limit by energy analysis [1]-[6]. Here we investigate how stability changes in between the two limits by numerical analysis and catastrophe theory.

(2) To explore the fate of Q-balls with finite perturbations, we numerically solve dynamical field equations. We argue limitations of energy analysis in discussing finite perturbations.

(3) If the mass of the scalar field is so large, gravitational effects are not negligible. Therefore, we extend our investigations to gravitating Q-balls in curved spacetime.

2 Q-balls in flat spacetime

Consider an SO(2)-symmetric scalar field, whose action is given by

$$\mathcal{S} = \int d^4x \left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu \boldsymbol{\phi}\cdot\partial_\nu\boldsymbol{\phi} - V(\phi)\right], \tag{1}$$

with $\boldsymbol{\phi} = (\phi_1,\ \phi_2)$ and $\phi \equiv \sqrt{\boldsymbol{\phi}\cdot\boldsymbol{\phi}} = \sqrt{\phi_1^2+\phi_2^2}$. Due to the symmetry there is a conserved charge:

$$Q \equiv \int_{\Sigma_t} d^3x(\phi_1\partial_t\phi_2 - \phi_2\partial_t\phi_1), \tag{2}$$

where Σ_t is the 3-hyperspace at $t =$const. Supposing homogeneous phase rotation,

$$\boldsymbol{\phi} = \phi(r)(\cos\omega t,\ \sin\omega t), \tag{3}$$

we obtain the field equation,

$$\frac{d^2\phi}{dr^2} = -\frac{2}{r}\frac{d\phi}{dr} - \omega^2\phi + \frac{dV}{d\phi}, \tag{4}$$

which is equivalent to the static field equation of a single scalar field with the potential $V - (\omega^2/2)\phi^2$. Monotonically decreasing solutions $\phi(r)$ with the boundary condition $d\phi/dr(0) = 0,\ \phi(\infty) = 0$ exist if

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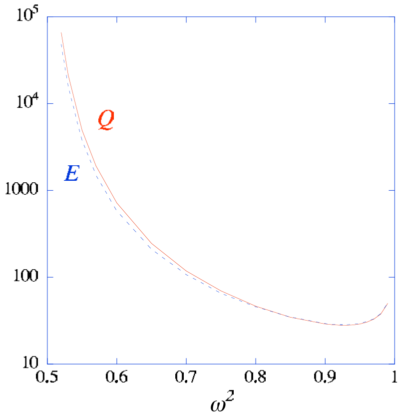


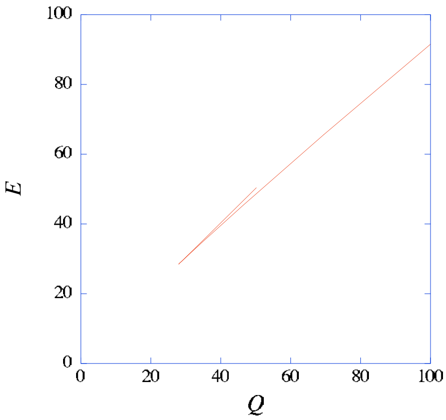


Figure 1: Relation among ω^2, Q and E for equilibrium solutions in flat spacetime.

$$\omega_{\min}^2 < \omega^2 < m^2, \quad \text{with} \quad \omega_{\min}^2 \equiv \min\left(\frac{2V}{\phi^2}\right), \quad m^2 \equiv \frac{d^2V}{d\phi^2}(0). \tag{5}$$

The condition $\omega_{\min}^2 < m^2$ is not so severe because it is satisfied if the potential has the form,

$$V = \frac{m^2}{2}\phi^2 - \lambda\phi^n + O(\phi^{n+1}), \quad m^2 > 0, \quad \lambda > 0, \quad n \geq 3 \tag{6}$$

In the literature the stability of equilibrium solutions has been studied by energetics argument as follows. The total energy of the system for equilibrium solutions is given by

$$E = \frac{Q^2}{2I} + \int_{\Sigma_t} d^3x \left\{\frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 + V\right\}, \quad Q = \omega I, \quad I \equiv \int_{\Sigma_t} d^3x\, \phi^2 \tag{7}$$

Its first variation by fixing the integral boundary and charge yields the field equation (4). Analysis of the second variation gives the stability of the equilibrium solutions; the main results which have already been obtained are as follows.

- In the thin wall limit ($\omega^2 \to \omega_{\min}^2$) they are stable [1, 5, 6].
- In the thick wall limit ($\omega^2 \to m^2$) they are stable if $n = 3$ [3, 5, 6] and unstable if $n \geq 4$ [5, 6].
- For any ω, if $\dfrac{\omega}{Q}\dfrac{dQ}{d\omega}$ is negative (positive), equilibrium solutions are stable (unstable) [6].

Here we analyze equilibrium solutions for the whole range $\omega_{\min}^2 < \omega^2 < m^2$ numerically and then discuss their stability. For definiteness we assume a sextic function,

$$V(\phi) = \frac{\phi^6}{M^2} - \lambda\phi^4 + \frac{m^2}{2}\phi^2 \quad \text{with} \quad 0 < \lambda,\ M < \infty. \tag{8}$$

By rescaling the field variables, we can set $M = \lambda = 1$ without loss of generality. Then the existing condition (5) reduces to $m^2 - 1/2 < \omega^2 < m^2$. Fixing $m^2 = 1$, we numerically solve (4) for $0.5 < \omega^2 < 1$, and obtain a series of equilibrium solutions $\phi(r)$. For each equilibrium solution we calculate charge (2) and energy (7). We depict the relation among ω^2, Q and E in Figure 1. There is a minimum charge, which we denote by $Q_{\min}$, and near the minimum there are two equilibrium solutions for each Q.

To discuss the stability near $Q = Q_{\min}$ by analogy with a mechanical system, let us consider the one-parameter family of field configuration $\phi_{\xi;Q}(r)$ for each Q in such a way that $\phi_{\xi;Q}(r)$ contains all equilibrium solutions. Note that we do not impose any restriction on perturbation type. Then the

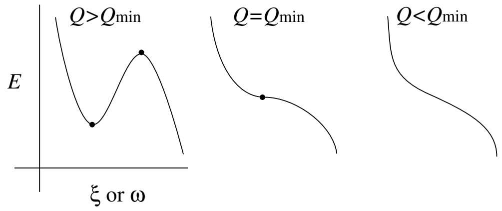


Figure 2: Behavior of the "potential" E for $Q \approx Q_{\min}$.

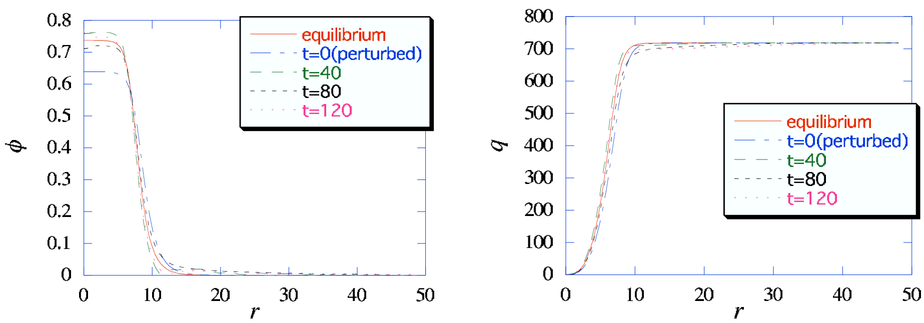


Figure 3: Dynamics of a Q-ball with a perturbed initial condition. $\omega^2 = 0.6$ and $Q = 718$. The right panel shows the evolution of the local charge, which is defined by $q \equiv 4\pi \int_0^r dr r^2(\phi_1 \partial_t \phi_2 - \phi_2 \partial_t \phi_1)$.

energy is written as $E_Q(\xi) \equiv E[\phi_{\xi;Q}]$. Equilibrium solutions are realized when $dE_Q(\xi)/d\xi = 0$, and their stability is determined by the sign of $d^2E_Q(\xi)/d\xi^2$. Therefore, the system completely corresponds to a mechanical system with the "potential" $E_Q(\xi)$, where ξ is a dynamical variable (or a "behavior variable" in catastrophe theory) and Q is a "control parameter" which is given by hand. Because ω is a function of ξ through $Q = \omega I[\phi_{\xi;Q}]$ for fixed Q, unless $d\xi/d\omega = 0$, ω also can be regarded as a behavior variable. Figure 2 shows how equilibrium points of $E_Q(\xi)$ change as Q varies near $Q = Q_{\min}$. This behavior is just a "fold catastrophe" in catastrophe theory.

Next, to explore the dynamics of Q-balls with finite deformation, we analyze numerically the dynamical field equations with perturbed initial conditions. Figure 3 shows an example of dynamical solutions. Although we give the initial configuration with the same Q and $\partial_t \phi = 0$, a part of charge is radiated away together with energy dispersion, and the Q-ball approaches to a different equilibrium solution with smaller Q. This shows a limitation of energy analysis with fixing Q when we discuss the dynamics of Q-balls with finite perturbations.

Coleman claimed that Q-balls with large Q are absolutely stable, not just stable under small deformation. Mathematically his statement is correct within energy-variation analysis for fixed Q. In a physical situation, however, charge is conserved but not necessarily confined in a local system. Therefore, we should not discuss finite perturbations by energy analysis with fixing Q.

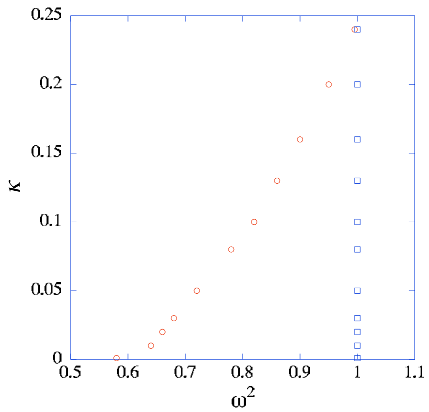
3 Q-balls in curved spacetime

In this section we consider gravitating Q-balls. The action is given by

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{{m_{\rm Pl}}^2}{16\pi} \mathcal{R} - \frac{1}{2} g^{\mu\nu} \partial_\mu \boldsymbol{\phi} \cdot \partial_\nu \boldsymbol{\phi} - V(\phi) \right]. \tag{9}$$

To obtain equilibrium solutions, we assume a spherically symmetric and static spacetime,

$$ds^2 = -\alpha^2(r)dt^2 + A^2(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{10}$$



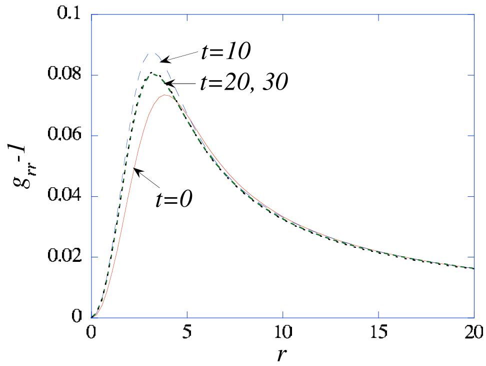


Figure 4: (left) Parameters which allow equilibrium solutions. A square denotes the maximum of ω^2 for a fixed κ, and a circle denotes the minimum.

Figure 5: Evolution of a Q-ball for $\kappa = 0.3$. In this case the metric approaches a stable configuration.

Supposing homogeneous phase rotation (3) again, we numerically solve the field equations, which follows from (9) and (10), with regularity conditions at the origin and at infinity. The model contains three independent parameters: m^2, ω^2 and $\kappa \equiv 8\pi\lambda M^2/{m_{\rm Pl}}^2$. Fixing $m^2 = 1$, we survey equilibrium solutions. The parameter range $(\omega^2,\ \kappa)$ which allow equilibrium solutions are summarized in Fig. 3. As κ becomes larger, the range of ω^2 which allows equilibrium solutions becomes smaller. If $\kappa \gtrsim 0.24$, equilibrium solutions are nonexistent regardless of ω^2.

Next, we investigate the stability of the equilibrium solutions. Becuase the energy defined by (7) does not have a definite meaning in curved spacetime, we adopt the gravitational (Misner-Sharp) mass, which is defined by $E_g \equiv {m_{\rm Pl}}^2 r(1-a^2)/2$. We find the relation among ω^2, Q and E_g of equilibrium solutions are similar to that in Fig. 1. Therefore, we can understand their stability in the same way.

Finally, we consider the fate of Q-balls for $\kappa > 0.24$, where equilibrium solutions are nonexistent. For several initial conditions we solve the dynamical field equations numerically. We obtain two types of solutions: a Q-ball either approaches a stable solution (as shown in Fig. 5) or collapses to a black hole.

Contrary to the claim in [7], Q-ball inflation does not occur. In the core of an equilibrium Q-ball, the effective potential $V - \omega^2\phi^2/2$ must be negative, and accordingly $-T_t^t + T_i^i \propto \omega^2\phi^2 - V > 0$, which induces attractive nature of gravity. Although inflation may occur if $\omega^2\phi^2$ is sufficiently small and the slow-roll condition is satisfied, it is perhaps inappropriate to call such a configuration a Q-ball.

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