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ON THE LOW-ENERGY RAMIFICATIONS AND A MATHEMATICAL EXTENSION OF LOOP QUANTUM GRAVITY

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Abstract

In this thesis we address two remaining open questions in loop quantum gravity. The first deals with the low-energy limit of the theory. We illustrate some of the conceptual difficulties and their resolution through the study of a toy model: the quantum mechanics of a point particle. We then find that this model can also be applied to the quantum mechanics of spatially isotropic, homogeneous cosmology within the framework of loop quantum cosmology (LQC). This leads us to extend our results to investigate, for the quantum constraint in LQC, the effective classical dynamics of the quantum theory. We find that we can calculate an effective Hamiltonian constraint, and we employ this to calculate the modifications to Friedmann's equations for a dust filled, spatially flat, isotropic universe.

We then turn to a mathematical question, investigating the extension of integration theory on spaces of connections to connections with non-compact structure group. For groups that are the direct product of a compact group with a non-compact Abelian group, we demonstrate a fully satisfactory theory based on the almost periodic compactification of the group. This approach fails for other non-compact groups, and for the case of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ we present a partial 'no-go' theorem that demonstrates that any successful integration theory for such spaces of connections with these gauge groups will of necessity be different in essential structure from the theory for compact and non-compact, Abelian groups.

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Chapter 1

Introduction

Both quantum mechanics and general relativity have radically reshaped our conception of the physical world. Both were born in the first half of the twentieth century, and almost since their birth there have been efforts to unify them into a single theory that could be properly called a quantum theory of gravity.

Such a unification has turned out to be extremely elusive. Naive attempts to unite the theories along the approach normally taken in quantum field theories fail; in particular, general relativity is by now well known to be perturbatively non-renormalizable. On a closer examination of the foundations of each theory, deep reasons for this discord become apparent. Quantum theory in its usual framework expects a background geometry on which the dynamics of the theory unfolds; general relativity, on the other hand, insists that geometry itself should be dynamical. Moreover, as we have already alluded, any quantum theory of gravity should possess the usual features of a quantum field theory, since the classical theory has an infinite number of degrees of freedom. Yet quantum field theories are well known to possess many difficulties of their own even in the presence of a background geometry. Removing this background might make the task seem entirely hopeless.

Remarkably, however, over the past fifteen years or so an approach has emerged that takes seriously the central lesson of general relativity: *gravity is geometry*. This approach (for a recent review, see [1]), pioneered by a number of researchers, starts from a reformulation of general relativity as a theory of connections, rather than a theory of metrics. The resulting theory, called variously non-perturbative quantum general relativity, background-independent quantum gravity, or loop quantum gravity, preserves the central feature of general relativity mentioned above: the absence of any *a priori* background geometry. This theory has had a number of remarkable successes:

- The theory is rigorously well-defined, developing a functional analysis on spaces of connections, a Hilbert space, and well-defined operators on that space, all without relying on any background metric [2, 3].
- A number of interesting geometrical observables have been defined in the resulting quantum theory, among them an area operator [4, 5] and volume operator [4, 6], and these operators have discrete spectrum. Thus, a picture emerges of *quantum geometry*,

in which observables that classically may have continuous spectrum in the quantum theory turn out to have discrete spectrum. Moreover, this discreteness extends to the basic nature of space itself: the fundamental excitations of the theory are not three dimensional, but one-dimensional *spin-networks*.

- The constraints of canonical general relativity can be promoted to well-defined operators in the quantum theory, both the spatial diffeomorphism constraint [7], and the scalar constraint [8–11]. The theory can also be coupled to matter fields [12].

- There are interesting and tantalizing connections to mathematics, in particular to knot theory [13].

- The theory appears to admit not only the Hamiltonian formulation in which it was originally derived, but also a covariant formulation, the *spin foam* approach (for a recent review, see [14]).

- There is an explanation of the microscopic states of a black hole, and a resulting derivation of the entropy [15] that (for appropriate choice of a parameter of the theory) agrees with the Hawking-Bekenstein value.

- There is a notion of 'symmetry reduced sectors' of the theory which in particular have allowed concrete results in the subject of loop quantum cosmology. The most striking of these is the avoidance of the initial singularity in big bang cosmologies [16].

Despite these successes, however, many outstanding issues remain. This thesis will address two of them. The first is the question of the low-energy limit of the theory: does it resemble, in any appropriate limit, *classical* general relativity, and can the corrections induced by the quantum theory on the classical theory be controlled in a systematic fashion? The second question deals with the mathematical structure of the theory: is it possible to formulate it in terms of connections based on non-compact gauge groups, in particular the gauge group $\mathrm{SL}(2,\mathbb{C})$ that naturally emerges from the classical theory?

1.1 Contact with low-energy physics

Among all of the successes listed above, we noted that loop quantum gravity provides a rigorous, background-independent theory in which basic geometric operators are well defined, and in particular there are operators corresponding to the quantized constraints of the theory. Thus, in a mathematical sense it does provide a quantum theory of gravity.

To be physically viable, however, more is needed. In general there is no guarantee that a 'quantization' of a given classical system will reduce to the classical behavior of

that system in the low-energy limit; that depends upon the existence of suitable semi-classical states with appropriate dynamical behavior in the quantum theory. Thus, a central question is whether or not loop quantum gravity admits states that are semi-classical, well approximating classical solutions to Einstein's equation. However, we would like an even stronger result, since we would like to reproduce not only the classical behavior of general relativity, but also the low-energy quantum behavior of perturbative quantum theory. That is, we should like also to find a suitable sector of our theory that contains states that approximate the graviton Fock space.

At first sight, such a hope may seem unrealistic. The basic excitations of quantum geometry are one-dimensional and polymer like, whereas in conventional low-energy quantum physics on a (commonly flat) background geometry the basic excitations are three dimensional. In quantum geometry a convenient basis is the spin network basis, whereas in ordinary quantum field theory one generally works with Fock states. However, because the perturbative approach to quantum gravity (based on the aforementioned Fock states) breaks down quickly because of graviton loops, the low energy results are not likely to emerge as first terms in a systematic expansion of a finite, full theory. Hence the differences between the frameworks of the two theories need not be fatal, but one expects it to be a subtle matter to extract semi-classical or Fock-like states—if they exist—from the full quantum theory. Thus, the central question remains: what is the precise sense in which low-energy states arise from the full theory?

As explained in more detail in the next chapter, this disparity between conventional perturbative quantum gravity and quantum geometry becomes sharper when one examines the theories in mathematical detail: the basic sets of operators of the two theories seem disjoint. Thus, one might worry that this is an indication that the theories describe different phases, and hence we must also ask: can we find the well-tested, macroscopic Coulomb phase of low-energy gravity emerging from the Planck scale discreteness inherent to quantum geometry?

These questions are being systematically tackled in loop quantum gravity, beginning with an approach outlined in [17], which in turn built on the work of [18, 19]. In this thesis we will examine in detail the first step of this program: an application of the basic ideas and constructions of [17] to the quantum mechanics of a point particle. Specifically, we shall exhibit a representation of a quantized point particle (that we call the polymer particle representation) that is unitarily inequivalent to the standard Schrödinger representation, but which in certain key respects mimics the structure of loop quantum gravity. In particular we will see a similar disparity between the basic operators of this model and those of the usual Schrödinger representation as is found between the basic operators of quantum geometry and those of perturbative quantum field theory. Given this disparity, and the inequivalence of the representations, we may justly wonder whether they can describe similar physics at

any energy scale.

Remarkably, we shall find that they do. We examine this from a kinematical standpoint in chapter 2, where we also introduce the polymer particle representation. Key to the proof that the low-energy kinematics of this representation coincides with that of Schrödinger quantum mechanics is the notion of a *shadow state*. This is a basic mathematical tool introduced in [17] and expected to play a central role in the investigation of the low-energy limit of loop quantum gravity itself, and we demonstrate the utility of shadow states in analyzing the low-energy limit of the polymer particle representation.

In chapter 3 we consider the relationship between the dynamics of the polymer particle and Schrödinger representations. In particular, we find that the machinery of shadow states lets us conclude that, in a precise sense, the eigenstates and eigenvalues of the two representations are close to each other at low energies. However, we will find that there are small corrections that grow to become large at sufficiently high energies, where heuristically one is probing the fundamental discreteness of space at finer and finer scales. Thus, while in the regime of validity of non-relativistic quantum mechanics the two representations are experimentally indistinguishable, at sufficiently high energies the fundamental discreteness of space manifests itself and the physical predictions of the two theories diverge. This is analogous to what one expects and indeed desires in full quantum gravity: because the standard perturbative treatment breaks down at high energies, the fundamental theory *should* diverge from it in this regime. Moreover, we shall also see in this chapter that there are some important subtleties arising in the construction of the Hamiltonian for the polymer particle, pointing to the fact that even in this simple model there are *a priori* reasonable choices (from a purely mathematical standpoint) that lead to physically wrong predictions in the low energy limit; it is therefore all the more nontrivial that there is *any* choice of Hamiltonian that coincides with the Schrödinger representation at low energies.

In chapter 4 we shall find that the polymer particle model considered thus far is of more than just illustrative value: it has physically relevant predictions of its own to make. These occur in the context of loop quantum cosmology, a symmetry reduced model of the full theory. In the particular case of a spatially homogeneous and isotropic model, the degrees of freedom of general relativity become finite dimensional—in fact, two dimensional. This means that at some level the quantization of this theory should lead to a simply quantum mechanical system. Remarkably, in the context of loop quantum gravity, one is led to *precisely* the polymer particle model considered in this thesis. The key change over what was done in chapters 2 and 3 is in the Hamiltonian: this is now dictated by considerations from the quantization of the scalar constraint in the full theory. The kinematical considerations of chapter 2 therefore go over unchanged, as was found in [20]. That paper also began the investigation of the low-energy dynamics of loop quantum cosmology, but here we develop that much more fully. Specifically, we look for an effective classical description

of the quantum dynamics that can therefore be thought of as the quantum corrections to Friedmann's equation. Such a description can be found from the expectation value of the scalar constraint. Unlike [20], we do not limit ourselves to just the leading classical term but find the corrections to all orders. We use these corrections to find, for the case of a dust-filled universe, the corrections to the equations of motion themselves. Thus, the results of this section do justify the claim that the polymer particle itself contains information on the low-energy ramifications of loop quantum gravity.

1.2 Integration on spaces of connections with non-compact gauge group

In the final chapter of this thesis, we turn from the more physical questions considered in earlier chapters to a question of a more mathematical nature. As we have already mentioned, general relativity is traditionally seen as a theory of a dynamical metric. In particular, this has often been the viewpoint adopted in attempted quantizations of general relativity. This puts gravity in apparently stark contrast to the other fundamental forces of nature, all of which are described in terms of a quantum theory of connections for gauge fields.

A fundamental tenet of the loop quantum gravity program is that this apparent contrast is in fact illusory. General relativity *can* be formulated as a theory of connections, and indeed the impetus for the entire loop quantum program was the observation of [21, 22] that general relativity may be recast as a theory of self-dual (or anti-self-dual) $\mathrm{SL}(2,\mathbb{C})$ connections. The use of such (anti-) self-dual connections is particularly appealing not only because of the resulting simplification of the mathematical structure of the theory, but also because of the appearance of (only) left-handed spinors in the standard model, and the similarity to the use of self-dual and anti-self dual fields in Penrose's nonlinear graviton construction on twistor space [23], as well as to the $\mathcal{H}$-space construction of [24].

However, none of the achievements in the quantum domain listed above were made for a theory based on such (anti-) self-dual connections. That is because thus far the functional analysis on spaces of connections which lies at the heart of the successes of loop quantum gravity has been rigorously developed only for the case of compact gauge groups. It is possible to make a canonical transformation on the phase space of general relativity and recast the theory in terms of $\mathrm{SU}(2)$ connections [25], and while the resulting theory is indeed still a theory of Lorentzian gravity, the physical interpretation of the connection is less direct. For instance, the connection (which is defined on a three-dimensional hypersurface) is no longer the pull-back to that hypersurface of a four-dimensional connection [26]. Nonetheless, the physical interpretation can be obtained and the above cited results of loop quantum gravity are based upon it; however one would still like to be able to construct the quantum theory using the more natural $\mathrm{SL}(2,\mathbb{C})$ connections.

Another reason for this, aside from the aesthetic considerations above, is that the canonical transformation which enables the use of SU(2) connections also introduces a real parameter, the Barbero-Immirzi parameter, into the theory. This parameter enters into the physical predictions of the theory, such as the spectrum of the area and volume operators and the value of the entropy of a black hole. The latter calculations, when compared with the Hawking-Bekenstein value of the entropy, indeed suggest a particular value of this parameter—$\ln 2/\sqrt{3}\pi$—but this particular value is certainly rather mysterious. In the case of $\mathrm{SL}(2,\mathbb{C})$ connections, there is a natural choice of the corresponding ambiguity: the one leading to either self-dual or anti-self-dual connections. If one had a theory in terms of such connections, then, one would certainly like to investigate the corresponding ambiguity: are physically viable results obtained for the mathematically preferred connections? Finally, there has also been much work lately in the development of spin foam models with $\mathrm{SL}(2,\mathbb{C})$ gauge groups, and one would like to understand how these relate to the canonical picture in terms of spin-networks. This seems impossible without an understanding of the integration theory on spaces of connections with non-compact gauge group, in particular with gauge group $\mathrm{SL}(2,\mathbb{C})$.

Thus, in the final chapter of this thesis we take up the extension of this integration theory to connections with non-compact gauge groups. Such an extension has already been attempted [27], but the resulting construction has major physical drawbacks [28]. We outline a construction that is based on a certain algebraic compactification of the gauge group, and which therefore allows one to handle non-compact gauge groups on the same footing as compact groups. The particular construction we examine first works only for groups that are the direct product of a compact group with a non-compact Abelian group, and so do not include $\mathrm{SL}(2,\mathbb{C})$. We examine several alternate compactifications and find them also to be unsuitable, and indeed we are able to produce a partial 'no-go' theorem. Thus, while the complete picture for such groups is still not available, it seems increasingly likely that the canonical transformation to compact groups is in fact *necessary* and not just mathematically convenient.

Chapter 2

Kinematics of the polymer particle and shadow states

In this chapter and the next we take up the general discussion of the polymer particle representation as a toy model of the low-energy limit of loop quantum gravity. The work reported in these chapters is joint with Abhay Ashtekar and Stephen Fairhurst, and has appeared in [29].

2.1 Overview of the basic issues

As we have already explained in the introduction, a central feature of loop quantum gravity is that it preserves, in the quantum theory, the distinguishing feature of classical general relativity: the absence of any background geometry. This means that in its fundamental structure the full quantum theory will use physical concepts and mathematical tools that are quite different from those normally used in low energy quantum physics. A major challenge, then, is to show that this low energy description does arise from the pristine, Planckian world in an appropriate sense.

Given the importance and difficulty of this question, it is being approached in a series of stages. In the work described in this chapter and the next, we illustrate, through a simple example, both the tension between the two frameworks and the new physical notions and mathematical techniques that are being used to resolve it. This work follows the program outlined in [17], which itself was motivated by [18, 19]. For related ideas see also [30–33]. The program described in these two chapters will be significantly extended in a series of papers [34–36]; the present work is just the first step on that road.

There are two reasons for beginning with this kind of 'toy model.' First, in embarking on a complex program such as the present program to examine the low-energy limit of loop quantum gravity, one expects that there will be subtleties arising that are not easily anticipated at the outset. The more complex the theory, the more of these subtleties that are expected to arise, and the more technically demanding will be their resolution. It thus makes sense to begin with a simple system to try and gain some insight into the nature of such issues.

Second, we are proposing new techniques and physical concepts in order to address the central question—the low-energy limit of loop quantum gravity. Any time one introduces such new techniques, it is wise to explore their application and success in simpler systems where, in some sense, one already knows the answer. This is to develop confidence, when the same methods are applied to more complex systems, that results obtained from these methods are believable. We stress this point because at various places in the next two chapters the reader may have the feeling that the results are all 'exactly what one would expect.' But that is precisely the point: if the methods used for the quantum mechanics of a point particle lead to semi-classical states or operators with no obvious connection to the corresponding states and operators in standard quantum mechanics, then we would hardly have confidence that the approach could yield meaningful insights into as complex a theory as non-perturbative, quantum general relativity.

Of course, any time one studies a toy model, it must not be *too* simple: it must still preserve essential aspects of the difficulties of the complex theory one is ultimately interested in, and must allow application of the techniques to be applied in that theory. Thus, to understand in what sense the present toy model meets these criteria, we must first review the outstanding conceptual and technical issues present in analyzing the low-energy limit of loop quantum gravity itself.

Before discussing briefly some of the central issues surrounding the low energy limit of loop quantum gravity, let us say a little bit more about the general problem of the low energy or semi-classical limit of a quantum theory. We pointed out in the introduction that loop quantum gravity already provides a rigorous quantum theory in which the dynamical equations of classical general relativity are promoted to operators of the quantum theory, so that, at 'zeroth order' it does provide a quantization of gravity. How then might it be that the classical limit of the theory is incorrect?

We give two examples where just such a phenomenon can occur. The first is elementary. Consider ordinary non-relativistic quantum mechanics of a point particle in three dimensions. From the three coordinate functions and their conjugate momenta we may construct in the usual fashion the angular momentum observables, which as is well known form a closed Lie algebra under the Poisson brackets. Suppose now that we take *this* algebra as our starting point, rather than the usual canonical algebra among the coordinate observables and their conjugate momenta, and suppose that we look for representations of this algebra. As is well known, we obtain both integral and half integral spin representations. In the integral spin representations there is also a representation of the basic canonical variables and their conjugate momenta as operators, but in the half integral representations there are no operators corresponding to, for instance, $\hat{x}$ and $\hat{p}_x$. Thus in the half integral representations there is no limit of the theory which would behave like the classical theory with its usual canonical variables.

Of course, one can argue that this is an artifact of choosing the wrong variables as the basic variables for quantization. But the point is that we may be doing an analogous thing in loop quantum gravity by choosing fluxes and holonomies are our basic observables; we shall comment more on the disparity between these observables and the observables one needs for a semi-classical limit below.

The second example comes from constructive quantum field theory. In four dimensions, classically the $\lambda\phi^4$ theory is readily seen to be non-trivial (that is, interacting), and yet it appears that in constructive quantum field theory quantizations the theory is free (see, for instance, [37]). Again, most would argue that this shows simply that there is something 'wrong' about the quantization, and again, that is precisely the point. Without examination of the low energy limit of loop quantum gravity, we cannot be sure that we have not made a fundamental mistake in quantizing general relativity.

Thus, do we have reason to believe that similar phenomena to those we have just illustrated may be happening in the loop quantization of general relativity? At first sight, it might appear that we do. That is because, as already mentioned in the introduction, there is manifest disparity just between the conceptual frameworks of loop quantum gravity and conventional perturbative quantum field theory. Loop quantum gravity is based on *quantum geometry*, the essential discreteness of which permeates all constructions and results. The fundamental excitations are 1-dimensional and polymer-like. A convenient basis of states is provided by spin networks. Low energy physics, on the other hand, is based on quantum field theories which are rooted in a flat space continuum. The fundamental excitations of these fields are 3-dimensional, typically representing wavy undulations on the background Minkowskian geometry. The convenient Fock-basis is given by specifying the occupation number in one particle states labeled by momenta and helicities. As we have already explained, this difference in frameworks indicates that it will be a nontrivial matter to extract semi-classical states from loop quantum gravity. Nonetheless, one would hope that the polymer description admits semi-classical states which approximate classical space-times as well as fluctuations on them represented by gravitons and other fields, and so one long term goal of this program is to understand how such semi-classical states arise from the full theory, if indeed they do.

There are further discrepancies between conventional low energy quantum gravity and quantum geometry than just the difference in the interpretation of the fundamental states highlighted above. From a mathematical physics perspective, the basic variables of quantum geometry are holonomies (or Wilson loops) of the gravitational connection A along 1-dimensional curves and fluxes of the conjugate momenta (the triads) E across 2-surfaces. In the final quantum theory, the connection A fails to be a well-defined operator(-valued distribution); only the holonomies are well-defined. By contrast, in Fock space the vector potential operators are distributions, whence, a priori, their holonomies fail to be well-defined

operators. Similarly, fluxes of electric field operators across 2-surfaces also fail to be well-defined on the Fock space of photons. Heuristically, then, it would appear that, even at a kinematic level, loop quantum gravity describes a 'phase' of gauge theories which is distinct from the one used in electrodynamics. Since it is generally believed that distinct phases carry distinct physics, it is natural to ask whether or not the full quantum theory contains a macroscopic 'Coulomb phase.' If so, in what sense; how does it emerge from loop quantum gravity? Given the apparent deep differences, the procedure of extracting the 'Coulomb phase' from the fundamental Planckian description should be rather subtle.

Finally, a further technical but important complication arises from the detailed treatment of dynamics. Solutions to the quantum Einstein equations (i.e. quantum constraints) do not belong to the so-called kinematical Hilbert space $\mathcal{H}_{\rm Poly}$. This is not surprising: a similar situation occurs already for simple, quantum mechanical constrained systems. The kinematical Hilbert space provides the mathematical framework to construct well-defined operators which can be regarded as the quantum analogs of the classical constraint functions. If zero lies in the continuous part of the spectrum of these operators, none of the solutions to the quantum constraints are normalizable with respect to the kinematic inner product. (This is the case even for the simple constraint $p_x = 0$ in $\mathbb{R}^3$, and for the constraint $g^{ab}p_a p_b - \mu^2 = 0$ satisfied by a free particle in Minkowski space-time.) The solutions are distributional; they belong to the *dual* of a sub-space of 'nice' quantum states (e.g. the Schwartz space). The situation is completely analogous in quantum gravity. The 'nice' quantum states are typically taken to be finite linear combinations of spin network states and their space is denoted by Cyl (the space of '*cylindrical*' functions of connections). Solutions to the quantum Einstein equations belong to its dual, Cyl*. There is an inclusion relation (providing a 'Gel'fand-type' triplet) Cyl $\subset \mathcal{H}_{\rm Poly} \subset$ Cyl*. While the kinematical spin network states belong to Cyl, the physical states belong to Cyl*. Therefore, semi-classical states, capturing the low energy physics, should also be in Cyl*. The problem is that, as of now, Cyl* does not have a physically justified inner product; a definite Hilbert space structure is not yet available. Can one nonetheless hope to extract low energy physics already at this stage? In particular, can one test a candidate state in Cyl* for semi-classicality without access to expectation values?

Thus, our goal in the next two chapters is to provide a simple model system that reflects in essential fashion the issues, both conceptual and technical, outlined above. We shall find such a model in the simple example of a non-relativistic particle, and we shall find that the issues raised above can be resolved satisfactorily. (For an analysis with similar motivation, but which emphasizes the role of constraints and discrete time evolutions, see [38].)

For readers who are not familiar with quantum geometry, this example can also serve as an introduction to the mathematical techniques used in that framework. However, as is typically the case with toy models, one has to exercise some caution. First, motivations

behind various construction often become obscure from the restrictive perspective of the toy model, whence the framework can seem cumbersome if one's only goal is to describe a non-relativistic particle. Secondly, even within mathematical constructions, occasionally external elements have to be brought in to mimic the situation in quantum geometry. Finally, because the toy model fails to capture several essential features of general relativity, there are some key differences between the treatment of the Hamiltonian and other constraints in the full theory and that of the Hamiltonian operator in the toy model. With these caveats in mind, the toy model can be useful in understanding the essential differences between our background independent approach and the Fock-space approach used in Minkowskian, perturbative quantum field theory.

We will begin with the usual Weyl algebra generated by the exponentiated position and momentum operators. The standard Schrödinger representation of this algebra will play the role of the Fock representation of low energy quantum field theories and we will construct a new, unitarily inequivalent representation—called the *polymer particle representation*—in which states are mathematically analogous to the polymer-like excitations of quantum geometry. The mathematical structure of this representation mimics various features of quantum geometry quite well; in particular there are clear analogs of holonomies of connections and fluxes of electric fields, non-existence of connection operators, fundamental discreteness, spin networks, and the spaces Cyl and $\mathrm{Cyl}^\star$.[1] At the basic mathematical level, the two descriptions are quite distinct and, indeed, appear to be disparate. Yet, we will show that states in the standard Schrödinger Hilbert space define elements of the analog of $\mathrm{Cyl}^\star$. As in quantum geometry, the polymer particle $\mathrm{Cyl}^\star$ does *not* admit a natural inner product. Nonetheless, as indicated in [17], we *can* extract the relevant physics from elements of $\mathrm{Cyl}^\star$ by examining their *shadows*, which belong to the polymer particle Hilbert space $\mathcal{H}_{\mathrm{Poly}}$. This physics is indistinguishable from that contained in Schrödinger quantum mechanics in its domain of applicability.

These results will show that, in principle, one could adopt the viewpoint that the polymer particle representation is the 'fundamental one'—it incorporates the underlying discreteness of spatial geometry—and the standard Schrödinger representation corresponds only to the 'coarse-grained' sector of the fundamental theory in the continuum approximation. Indeed, this viewpoint is viable from a purely mathematical physics perspective, i.e., if the *only* limitation of Schrödinger quantum mechanics were its failure to take into account the discrete nature of the Riemannian geometry. In the real world, however, the corrections to non-relativistic quantum mechanics due to special relativity and quantum field theoretic effects largely overwhelm the quantum geometry effects, whence the above viewpoint is not

[1] Of course, since this is only a simple, 'toy example', it does not capture all the subtleties. In particular, we will see that a number of distinct notions in quantum geometry often coalesce to a single notion in the example. We shall remark on a few examples of this phenomenon throughout the next two chapters

physically tenable. Nonetheless, the results for this toy model illustrate why an analogous viewpoint can be viable in the full theory: Although the standard, low energy quantum field theory seems disparate from quantum geometry, it can arise, in a systematic way, as a suitable semi-classical sector of loop quantum gravity.

This chapter is organized as follows. Section 2.2 recalls a few essential notions from quantum geometry which motivate our construction of the polymer particle representation. This representation is constructed in detail in section 2.3. In section 2.4 we show that the standard coherent states of the Schrödinger theory can be regarded as elements of $\mathrm{Cyl}^\star$, introduce the notion of 'shadow states' and use them to show that the elements of $\mathrm{Cyl}^\star$ defined by the coherent states are, in a precise sense, semi-classical from the perspective of the 'fundamental' polymer particle representation. Thus, at a kinematical level, the relationship between the low-energy physics of the two representations is entirely satisfactory. We take up the important question of the relationship between the dynamics of the two representations in chapter 3.

Since the present work is just the first in a series that move progressively closer to analyzing the low-energy limit of loop quantum gravity, we also introduce some technical material that, though it can be defined in the present example, is of interest chiefly because of its anticipated utility when the shadow state framework is extended to quantum field theories. Because it therefore lies outside the main scope of the next two chapters, this material is presented in appendix A, along with an important proof that it would be too disruptive to include in the body of this chapter.

2.2 Quantum geometry

This summary of quantum geometry will enable the reader to see the parallels between quantum geometry and the polymer particle representation constructed in section 2.3. It will be used primarily to motivate our constructions in subsequent sections. Our discussion will be rather brief and, in particular, we will omit all proofs. (These can be found, e.g., in [2–7, 39–44].)

In diffeomorphism invariant theories of connections, the phase space consists of pairs of fields (A, E) on a 3-manifold Σ, where A_a^i are connection 1-forms which take values in the Lie-algebra of the structure group G, and E_i^a are 'electric fields' which are vector densities with values in the dual of the Lie algebra. For the purpose of this chapter and the next, it suffices to restrict ourselves to two special cases: i) $G = \mathrm{SU}(2)$, used in quantum geometry, and, ii) $G = \mathrm{U}(1)$ used in quantum Maxwell theory. In either case, the 'elementary' classical observables are taken to be holonomies A_e along paths e defined by A and fluxes E_S of electric fields across 2-surfaces S. From the perspective of the standard Hamiltonian formulation of field theories, these functions are 'singular': Since they are supported on 1-dimensional

curves and 2-dimensional surfaces, respectively, we are in effect using distributional smearing functions. Nonetheless, the symplectic structure on the classical phase space endows them with a natural Lie bracket and the resulting Lie-algebra is taken as the point of departure in quantum theory.

The Hilbert space of states can be constructed in two ways. In the first, one uses the fact that, as usual, the configuration variables A_e give rise to an Abelian $C^\star$ algebra $\mathcal{HA}$, called the holonomy algebra. One then introduces a natural (diffeomorphism invariant) positive linear functional on it and uses the Gel'fand-Naimark-Segal (GNS) construction to obtain a Hilbert space $\mathcal{H}_{\rm Poly}$ of states and a representation of $\mathcal{HA}$ on it. Finally, self-adjoint electric flux operators are introduced on $\mathcal{H}_{\rm Poly}$ using the heuristic idea that E should be represented by $-i\hbar\,\delta/\delta A$.[2] The second approach is more explicit. One begins by specifying the space Cyl of 'nice' functions of connections. Fix a graph γ on the 3-manifold Σ with N edges. A connection A associates to each edge e a holonomy $A_e \in G$. The space of N-tuples $(A_1, \ldots, A_N)$ defines a configuration of the gauge theory restricted to the graph γ and will be denoted by $\mathcal{A}_\gamma$. Clearly, $\mathcal{A}_\gamma$ is isomorphic with G^N. Now, given a smooth, complex-valued function ψ on G^N, we can define a function Ψ of connections in an obvious fashion:

$$\Psi(A) = \psi(A_1, \ldots, A_N).$$

The space of these functions is denoted Cyl_γ. Elements of Cyl_γ have knowledge only of the connection *restricted to* γ. The space Cyl of *all* cylindrical functions is obtained by simply considering all possible graphs γ:

$$\mathrm{Cyl} = \bigcup_\gamma \mathrm{Cyl}_\gamma.$$

Thus, each element of Cyl depends only on holonomies of the connection along edges of *some* finite graph γ but the graph can vary from one function to another. Had we restricted ourselves to a fixed graph γ, the theory would have been equivalent to a lattice gauge theory on a (generically irregular) 'lattice' γ. However, since we allow *all* possible graphs, we are dealing with a *field* theory, with an infinite number of degrees of freedom, of all connections on Σ.

The next step is to introduce an inner product on Cyl. For this, we simply use the induced Haar measure $\mu_H^{(N)}$ on $\mathcal{A}_\gamma \approx G^N$: Given any two functions Ψ_1 and Ψ_2 on Cyl_γ, we set

$$(\Psi_1, \Psi_2) = \int_{\mathcal{A}_\gamma} \bar{\psi_1}\, \psi_2 \; d\mu_H^{(N)}. \qquad (2.2.1)$$

[2] From the viewpoint of the algebraic approach, which has been so successful in quantum field theory in curved space-times, working with a specific Hilbert space representation may seem restrictive. However, the algebraic approach is not so well-suited for systems, like general relativity, with non-trivial constraints. More importantly, there is no loss of generality in working with the above representation because it is singled out essentially by the requirement of diffeomorphism covariance [45].

Using properties of the Haar measure one can verify that this definition is unambiguous, i.e., if Ψ_1 and Ψ_2 are cylindrical with respect to another graph γ', the right side of (2.2.1) is unchanged if we replace γ with γ'. This prescription provides us with an Hermitian inner product on all of Cyl because, given any $\Psi_1, \Psi_2 \in \mathrm{Cyl}$, there exists a (sufficiently large) graph γ such that $\Psi_1, \Psi_2 \in \mathrm{Cyl}_\gamma$. The Cauchy completion of Cyl with respect to this inner product provides the required Hilbert space $\mathcal{H}_{\mathrm{Poly}}$ of all quantum states, obtained in the first method via the GNS construction.

Because we consider *all* possible graphs on Σ in its construction, $\mathcal{H}_{\mathrm{Poly}}$ is very large. However, it can be decomposed into convenient *finite* dimensional sub-spaces. Each of these subspaces is associated with a labeling of edges of a graph γ by non-trivial irreducible representations of G. Thus, in the case when $G = \mathrm{SU}(2)$, let us label each edge e of γ with a non-zero half-integer (i.e., spin) j_e. Then, there is a finite dimensional sub-space $\mathcal{H}_{\gamma,\vec{j}}$ such that

$$\mathcal{H}_{\mathrm{Poly}} = \bigoplus_{\gamma,\vec{j}} \mathcal{H}_{\gamma,\vec{j}}. \tag{2.2.2}$$

This is called the *spin network decomposition* of $\mathcal{H}_{\mathrm{Poly}}$. Although $\mathcal{H}_{\mathrm{Poly}}$ is very large, practical calculations are feasible because each of the sub-spaces $\mathcal{H}_{\gamma,\vec{j}}$ can be identified with the Hilbert space of a spin-system which is extremely well understood. In the case when $G = \mathrm{U}(1)$, we label each edge e with a non-zero integer n_e. The Hilbert space $\mathcal{H}_{\gamma,\vec{n}}$ is now *1-dimensional*, spanned by the function

$$\Psi(A) = e^{in_1\theta_1} \cdots e^{in_N\theta_N}$$

where $e^{i\theta_m}$ is the holonomy of the connection A along the edge e_m. These functions are called *flux network states* and by replacing $\vec{j}$ by $\vec{n}$ in (2.2.2) one now obtains a decomposition of $\mathcal{H}_{\mathrm{Poly}}$ in terms of 1-dimensional orthonormal subspaces.

As in any Schrödinger description, quantum states in $\mathcal{H}_{\mathrm{Poly}}$ can be regarded as square integrable function on the quantum configuration space. In systems with finite number of degrees of freedom, the quantum configuration space is normally the same as the classical one. However, for systems with an infinite number of degrees of freedom, there is typically a significant enlargement: while classical configurations are represented by smooth fields, quantum configurations are *distributional*. This occurs also in our case: $\mathcal{H}_{\mathrm{Poly}} = L^2(\overline{\mathcal{A}}, d\mu_o)$, where $\overline{\mathcal{A}}$ is a suitable completion of the space $\mathcal{A}$ of smooth connections and μ_o, a regular measure on it. An element $\bar{A}$ of $\overline{\mathcal{A}}$ is called a *generalized connection*. It associates with every oriented path e in Σ an element $\bar{A}(e)$ of G, the holonomy along e subject only to two conditions: i) $\bar{A}(e_1 \circ e_2) = \bar{A}(e_1)\,\bar{A}(e_2)$; and, ii) $\bar{A}(e^{-1}) = [\bar{A}(e)]^{-1}$. Note that the assignment $e \longrightarrow \bar{A}(e)$ can be *arbitrarily discontinuous*, whence the quantum configuration space $\overline{\mathcal{A}}$ is a genuine extension of the classical configuration space $\mathcal{A}$. Nonetheless, in a natural topology,

$\mathcal{A}$ is dense in $\overline{\mathcal{A}}$, whence $\overline{\mathcal{A}}$ can be regarded as a suitable completion of $\mathcal{A}$. However, as is typically the case in field theories, the measure μ_o is concentrated on genuinely generalized connections; all the smooth configurations in $\mathcal{A}$ are contained in a set of zero measure.

The measure μ_o is completely defined by the family of measures $\mu_H^{(N)}$ on $\mathcal{A}_\gamma \approx G^N$: because $\mu_H^{(N)}$ are mutually consistent in a precise sense, they can be 'glued together' to obtain μ_o. Indeed, *every* measure on $\overline{\mathcal{A}}$ arises as a *consistent* family of measures on $\mathcal{A}_\gamma$. More generally, structures in the full quantum theory are constructed as *consistent* families of structures on $\mathcal{A}_\gamma$ or Cyl_γ. In particular, many of the physically interesting operators on $\mathcal{H}_{\mathrm{Poly}}$—such as the holonomies $\hat{A}_e$, the fluxes $\hat{E}_S$ of $\hat{E}$ across S, area operators $\hat{A}_S$ associated with 2-surfaces S, and volume operators $\hat{V}_R$ associated with spatial regions R—arise as consistent families of operators on Cyl_γ. Therefore, their properties can be explored in terms of their actions on finite dimensional spaces $\mathcal{H}_{\gamma,\vec{j}}$ (or $\mathcal{H}_{\gamma,\vec{n}}$).

While the above structures suffice to discuss quantum kinematics, as pointed out in the Introduction, an additional notion is needed in the discussion of quantum dynamics: solutions to the quantum Einstein's equations do not belong $\mathcal{H}$ because they fail to be normalizable. Their natural home is $\mathrm{Cyl}^\star$, the algebraic dual of Cyl. We have a natural inclusion:

$$\mathrm{Cyl} \subset \mathcal{H}_{\mathrm{Poly}} \subset \mathrm{Cyl}^\star.$$

To discuss physical states and explore the physically relevant semi-classical sector, then, we are led to focus on $\mathrm{Cyl}^\star$.

We will see in section 2.3 that the essential features of these constructions and results are mirrored in a transparent way in the 'polymer particle representation' of a non-relativistic point particle.

2.3 Schrödinger and polymer particle frameworks

The physical system we wish to consider is a particle moving on the real line $\mathbb{R}$. (It is straightforward to extend our discussion to $\mathbb{R}^n$.) The kinematics of this system are ordinarily described in terms of a position operator[3] x and a momentum operator p satisfying the canonical commutation relation

$$[x, p] = i\hbar. \tag{2.3.1}$$

From a technical standpoint, however, the commutation relation (2.3.1) is inconvenient as a starting point for two main reasons. First, in a concrete representation the x and p operators will be self-adjoint operators that are unbounded, and hence only densely defined.

[3]Our conventions are that only operators with a concrete representation on some Hilbert space will be denoted with a hat. Thus, at this point the x and p operators are unhatted since they are considered as abstract operators not tied to any particular representation.

Thus, it is not obvious that the composition of these operators needed to define the commutator in (2.3.1) is well defined. Second, and for our purposes ultimately more important, we shall soon be looking at a representation for which the p operator does not exist, and the canonical commutation relation (2.3.1) is then certainly nonexistent.

Therefore, we will take instead as out starting point an algebra that is derived from exponentiating the x and p operators. Accordingly, define:

$$U(\lambda) := e^{i\lambda x} \quad \text{and} \quad V(\mu) := e^{i\frac{\mu}{\hbar}p}. \tag{2.3.2}$$

Then the canonical commutation relations (2.3.1) imply the following relations among the U and V operators:

$$\begin{aligned} U(\lambda_1)U(\lambda_2) = U(\lambda_1+\lambda_2), \quad V(\mu_1)V(\mu_2) = V(\mu_1+\mu_2), \\ U(\lambda)V(\mu) = e^{-i\lambda\mu}\, V(\mu)\, U(\lambda)\,. \end{aligned} \tag{2.3.3}$$

Since we expect the x and p operators to be self-adjoint, the $U(\lambda)$ and $V(\mu)$ operators should be unitary. But as we are not yet considering any particular representation of the algebra of (2.3.3), such a requirement has no meaning as yet. What we should instead say is that we wish to make the algebra into a star algebra by introducing an involution $\star$ on it. We then require that $[U(\lambda)]^\star = U(-\lambda),\ [V(\mu)]^\star = V(-\mu)$ in this star algebra. This in turn will mean that in any concrete representation of the U and V operators on a Hilbert space, they are unitary.

We can abstract this still further, to arrive at the algebra normally used in the mathematics literature. Observe that the parameter λ must have the physical dimensions of inverse length, and the parameter μ the dimensions of length. If we introduce a length scale d, then we may combine these two parameters into a single, complex dimensionless number ζ, defined to be $\lambda d + i(\mu/d)$. We then make the following definitions. To each complex number ζ associate an abstract operator $W(\zeta)$ and consider the free vector space $\mathbf{W}$ generated by them. Introduce a product on $\mathbf{W}$ via:

$$W(\zeta_1)W(\zeta_2) = e^{\frac{i}{2}\,\mathrm{Im}\zeta_1\bar{\zeta_2}}\ W(\zeta_1+\zeta_2), \tag{2.3.4}$$

and an involution $\star$ via

$$[W(\zeta)]^\star = W(-\zeta)\,. \tag{2.3.5}$$

These two equations define the Weyl-Heisenberg $\star$-algebra of non-relativistic quantum mechanics.

This algebra is completely equivalent to the algebra of the U and V operators, since we can undo the above construction by introducing a length scale d and 'splitting' the operators

$W(\zeta)$ by setting

$$W(\zeta) = e^{\frac{i}{2}\lambda\mu}\, U(\lambda)\, V(\mu)$$

with (as before) $\zeta = \lambda d + i(\mu/d)$. Thus, $U(\lambda) = W(\lambda d)$ and $V(\mu) = W(i\mu/d)$. The involution and product rule (2.3.4) then imply that the $U(\lambda)$ and $V(\mu)$ operators satisfy (2.3.3) and the involution given above for those operators.

Thus, from a mathematical perspective we shall be looking at different representations of the Weyl-Heisenberg algebra as defined above, and examining the similarities and differences in the physical systems described by those different representations. We begin by considering the Schrödinger representation.

2.3.1 The Schrödinger representation

Why is it that most introductory textbooks on quantum mechanics make no mention of the construction of the Weyl-Heisenberg algebra as we have sketched it above? Aside perhaps from a desire not to over burden students with functional analytic details, a more significant reason is the Stone-von Neumann theorem. This celebrated result ensures us that every irreducible representation of $\mathbf{W}$ *which is weakly continuous in the parameter* ζ is unitarily equivalent to the standard Schrödinger representation, where the Hilbert space is the space $L^2(\mathbb{R}, d\underline{x})$ of square integrable functions on $\mathbb{R}$ (where $\underline{x}$ is dimensionless). $W(\zeta)$ are represented via:

$$\hat{W}(\zeta)\psi(\underline{x}) = e^{\frac{i}{2}\alpha\beta}\, e^{i\alpha\underline{x}}\, \psi(\underline{x} + \beta), \tag{2.3.6}$$

where $\zeta = \alpha + i\beta$. This is an irreducible representation of $\mathbf{W}$. Furthermore, the $\hat{W}(\zeta)$ are all unitary (i.e., satisfy $[\hat{W}(\zeta)]^\dagger = [\hat{W}(\zeta)]^{-1}$) and weakly continuous in ζ (i.e., all matrix elements of $\hat{W}(\zeta)$ are continuous in ζ).

In physics terms, the Hilbert space $\mathcal{H}_{\text{Sch}}$ is the space of square integrable functions of $x = \underline{x}d$ and the action of these operators is given by

$$\hat{U}(\lambda)\,\psi(x) = e^{i\lambda x}\,\psi(x) \quad \text{and} \quad \hat{V}(\mu)\,\psi(x) = \psi(x+\mu) \tag{2.3.7}$$

for all $\psi \in \mathcal{H}_{\text{Sch}}$. Now, the 1-parameter unitary groups $\hat{U}(\lambda)$ and $\hat{V}(\mu)$ are weakly continuous in the parameters λ, μ. This ensures that there exist self-adjoint operators $\hat{x}$ and $\hat{p}$ on $\mathcal{H}_{\text{Sch}}$ such that

$$\hat{U}(\lambda) := e^{i\lambda\hat{x}} \qquad \text{and} \qquad \hat{V}(\mu) = e^{i\frac{\mu}{\hbar}\hat{p}}\,. \tag{2.3.8}$$

The action of those operators (which follows from (2.3.8)) is the familiar one:

$$\hat{x}\,\psi(x) = x\,\psi(x), \qquad \hat{p}\,\psi(x) = -i\hbar\frac{d}{dx}\psi(x). \tag{2.3.9}$$

We conclude with two remarks:

1. The Schrödinger representation can be obtained using the Gel'fand-Naimark-Segal (GNS) construction with the positive linear (or, 'expectation-value') functional $F_{\rm Sch}$ on $\mathbf{W}$:

$$F_{\rm Sch}(W(\zeta)) = e^{-\frac{1}{2}|\zeta|^2}. \tag{2.3.10}$$

The expectation values of $\hat{U}$ and $\hat{V}$ are given by:

$$F_{\rm Sch}(U(\lambda)) = e^{-\frac{1}{2}\lambda^2 d^2} \qquad \text{and} \qquad F_{\rm Sch}(V(\mu)) = e^{-\frac{1}{2}\frac{\mu^2}{d^2}}. \tag{2.3.11}$$

The corresponding GNS 'vacuum' (i.e., cyclic) state $\psi_{\rm Sch}$ is simply

$$\psi_{\rm Sch}(x) = (\pi d^2)^{-\frac{1}{4}}\ e^{-\frac{x^2}{2d^2}}\ ,$$

i.e., the ground state of the simple harmonic oscillator with fundamental length scale d.

2. For definiteness, we have presented the Schrödinger representation using position wave functions $\psi(x)$. In terms of momentum wave functions $\psi(k)$, which will be more useful in the next subsection, we have:

$$\hat{U}(\lambda)\psi(k) = \psi(k-\lambda), \quad \text{and} \quad \hat{V}(\mu)\psi(k) = e^{i\mu k}\,\psi(k) \tag{2.3.12}$$

and the GNS cyclic state is given by:

$$\psi_{\rm Sch}(k) = \left(\frac{\pi}{d^2}\right)^{-\frac{1}{4}}\ e^{-\frac{k^2 d^2}{2}}.$$

2.3.2 The polymer particle representation

We are now ready to introduce the polymer particle representation of the Weyl-Heisenberg algebra which is unitarily *inequivalent* to the Schrödinger. This construction must, of course, violate one or more assumptions of the Stone-von Neumann uniqueness theorem. It turns out that only one assumption is violated: in the new representation, the operator $V(\mu)$ will not be weakly continuous in μ, whence there will be no self-adjoint operator $\hat{p}$ such that $V(\mu) = \exp(i\mu\hat{p})$. While the unavailability of the standard momentum operator seems alarming at first, this is just what one would expect physically in the absence of a spatial continuum. More precisely, if the spatial Riemannian geometry is to be discrete (as, for example, in loop quantum gravity), one would *not* expect the operator $p = -i\hbar\, d/dx$ to exist at a fundamental level. The key question is whether one can nonetheless do quantum mechanics and reproduce the well-tested results. This is a difficult question with many technical subtleties. But, as we will see in section 2.4 and chapter 3, the answer is affirmative:

by adopting the viewpoint that the natural arena for quantum theory is the analog of $\mathrm{Cyl}^\star$, one can recover results of Schrödinger quantum mechanics in the domain of its validity.

To bring out the similarity and differences with quantum geometry, we will construct the Hilbert space of states, $\mathcal{H}_{\mathrm{Poly}}$, in steps, using the same terminology. A *graph* γ will consist of a countable set $\{x_i\}$ of points on the real line $\mathbb{R}$ with the following two properties:

1. The x_i do not contain sequences with accumulation points in $\mathbb{R}$.
2. There exist constants ℓ_γ and ρ_γ such that the number $n(I)$ of points in any interval I of length $\ell(I) \geq \ell_\gamma$ is bounded by $n(I) \leq \rho_\gamma \ell(I)$.

The two technical conditions will ensure convergence of certain series; see section 2.4 for the application and appendix A.1 for the needed proofs.[4]

Denote by Cyl_γ the space of complex valued functions $f(k)$ of the type:

$$f(k) = \sum_j f_j \, e^{-ix_j k} \tag{2.3.13}$$

on $\mathbb{R}$, where x_j are real and f_j are complex numbers with a suitable fall-off. To simplify the later specification of domains of operators, we will choose the fall-off to be such that $\sum_j |x_j|^{2n}|f_j|^2 < \infty$ for all n. Cyl_γ is a vector space (which is infinite dimensional if the number of points in γ is infinite). We will say that functions $f(k)$ in Cyl_γ are *cylindrical* with respect to γ. Thus, each cylindrical state is a *discrete* sum of plane waves; it fails to belong to the Schrödinger Hilbert space. The real number k is the analog of connections in quantum geometry and the plane wave $\exp(-ikx_j)$ can be thought of as the 'holonomy of the connection k along the edge x_j'.

Next, let us consider all possible graphs, where the points (and even their number) can vary from one graph to another, and denote by Cyl the *infinite* dimensional vector space of functions on $\mathbb{R}$ which are cylindrical with respect to *some* graph. Thus, any element $f(k)$ of Cyl can be expanded as in (2.3.13), where the uncountable basis $\exp(-ix_j k)$ is now labeled by *arbitrary* real numbers x_j. Let us introduce a natural, Hermitian inner product on Cyl by demanding that $\exp(-ix_j k)$ are orthonormal:

$$< e^{-ix_i k} | e^{-ix_j k} > = \delta_{x_i, x_j}. \tag{2.3.14}$$

(Note that the right side is the Kronecker δ and not the Dirac distribution.) Denote by $\mathcal{H}_{\mathrm{Poly}}$ the Cauchy completion of Cyl. This is the Hilbert space underlying our representation.

To summarize, $\mathcal{H}_{\mathrm{Poly}}$ is the Hilbert space spanned by countable linear combinations

[4] In an earlier version of this work, I only had condition 1. I thank Jacob Yngvasson for pointing out that it does not suffice and Chris Fewster and Jerzy Lewandowski for the precise formulation of 2.

$\sum_1^\infty f_j \exp(-ix_jk)$ of plane waves in the momentum space, subject to the condition

$$\sum_1^\infty |f_j|^2 < \infty,$$

where $\{x_j\}$ is an arbitrary countable set of real numbers, which can vary from one state to another. Even more succinctly, $\mathcal{H}_{\rm Poly} = L^2(\mathbb{R}_{\rm d}, d\mu_{\rm d})$, where $\mathbb{R}_{\rm d}$ is the real line equipped with discrete topology and $\mu_{\rm d}$ is the natural discrete measure on it.

The Weyl-Heisenberg algebra **W** is represented on $\mathcal{H}_{\rm Poly}$ in the same manner as in the Schrödinger representation:

$$\hat{W}(\zeta)f(k) = [e^{\frac{i}{2}\lambda\mu}\ U(\lambda)\,V(\mu)]\,f(k) \tag{2.3.15}$$

where, as before, $\zeta = \lambda d + i(\mu/d)$ and the action of $\hat{U}$ and $\hat{V}$ is given by (see (2.3.12))

$$\hat{U}(\lambda)f(k) = f(k-\lambda) \quad \text{and} \quad \hat{V}(\mu)f(k) = e^{i\mu k}\ f(k). \tag{2.3.16}$$

It is straightforward to check that these operators provide a faithful, irreducible representation of **W** on $\mathcal{H}_{\rm Poly}$. Each $\hat{U}(\lambda)$ and $\hat{V}(\mu)$ is unitary.

The structure of this representation becomes more transparent in terms of eigenkets of $\hat{U}(\lambda)$. Let us associate with the basis elements $\exp(-ix_jk)$ a ket $|x_j\rangle$ and, using the textbook heuristic notation, express $\exp(-ix_jk)$ as a generalized scalar product:

$$(k, x_j) = e^{-ix_jk}$$

Then, $\{\,|x_j\rangle\,\}$ is an orthonormal basis and the action of the basic operators $\hat{U}$ and $\hat{V}$ is given by:

$$\hat{U}(\lambda)|x_j\rangle\ =\ e^{i\lambda x_j}|x_j\rangle \quad \text{and} \quad \hat{V}(\mu)|x_j\rangle\ =\ |x_j - \mu\rangle. \tag{2.3.17}$$

One may easily verify that $\hat{U}(\lambda)$ is weakly continuous in λ whence there exists a self-adjoint operator $\hat{x}$ on $\mathcal{H}_{\rm Poly}$ with $\hat{U}(\lambda) = \exp(i\lambda\hat{x})$. Its action can now be expressed as:

$$\hat{x}|x_j\rangle\ =\ x_j|x_j\rangle \tag{2.3.18}$$

just as one would expect. However, there is an important difference from the Schrödinger representation: The eigenkets of $\hat{x}$ are now *normalizable*, and hence elements of the Hilbert space itself. In this sense the eigenvalues are 'discrete'.

By contrast, although the family $\hat{V}(\mu)$ provides a 1-parameter unitary group on $\mathcal{H}_{\rm Poly}$, it *fails to be weakly continuous* in the parameter μ. This follows from the fact that, no matter

how small μ is, $|x_j\rangle$ and $\hat{V}(\mu)|x_j\rangle$ are orthogonal to one another, whence

$$\lim_{\mu\mapsto 0}\langle x_j|\hat{V}(\mu)|x_j\rangle \,=0\,,$$

while $\hat{V}(\mu=0)=1$ and $\langle x_j\,|\,x_j\rangle \,=\,1$. Thus, there is no self-adjoint operator $\hat{p}$ on $\mathcal{H}_{\rm Poly}$ satisfying the second of eqs. (2.3.8).

Finally, this representation can be obtained via Gel'fand-Naimark-Segal construction, using the following positive linear (or expectation value) functional on the Weyl-Heisenberg algebra **W**:

$$F_{\rm Poly}(W(\zeta)) = \begin{cases} 1 & \text{if } {\rm Im}\,\zeta = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.19}$$

In terms of $U(\lambda)$ and $V(\mu)$, we have:

$$\begin{aligned} F_{\rm Poly}(U(\lambda)) &= 1 \qquad \forall\,\lambda, \\ F_{\rm Poly}(V(\mu)) &= \begin{cases} 1 & \text{if } \mu = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.3.20}$$

The corresponding cyclic state is simply $|\psi_o\rangle \,=\, |x_o=0\rangle$. Note that, in contrast to the Schrödinger positive linear functional $F_{\rm Sch}$, no scale had to be introduced in the definition of $F_{\rm Poly}$. This is the analog of the fact that the corresponding positive linear functional in quantum geometry is diffeomorphism invariant.

We conclude this section with a few remarks.

1. The step by step procedure used above brings out the fact that the polymer particle description captures many of the mathematical features of quantum geometry, but now for a very simple physical system. Our notation is geared to reflect the analogies. Thus, sets $\gamma = \{x_k\}$ are the analogs of graphs of quantum geometry; individual points x_j, the analogs of edges; the continuous momentum variable k, the analog of connections; $\exp(-ix_jk)$ the analog of the holonomy along an edge; ${\rm Cyl}_\gamma$ the analog of the space of cylindrical functions associated with a graph and Cyl the space of all cylindrical functions of quantum geometry; and the $|x_j\rangle$ the analogs of spin network states. Indeed, we again have the Hilbert space decomposition analogous to (2.2.2):

$$\mathcal{H}_{\rm Poly} = \bigoplus_x \mathcal{H}_x$$

where $\mathcal{H}_x$ are the 1-dimensional subspaces spanned by our basis vectors $|x\rangle$. (The decomposition is thus analogous to that in the U(1) case).

2. What is the situation with operators? The basic operators of quantum geometry—

holonomies and fluxes of the electric field—are respectively analogous to the operators $\hat{V}(\mu)$ and $\hat{x}$ on $\mathcal{H}_{\rm Poly}$. The commutator between $\hat{x}$ and $\hat{V}(\mu)$,

$$[\hat{x}, \hat{V}(\mu)] = -\mu \hat{V}(\mu), \tag{2.3.21}$$

is completely parallel to the commutator between electric fields and holonomies in quantum geometry. Just as $\hat{V}(\mu)$ are unitary but $\hat{p}$ does not exist, holonomies are unitarily represented but the connection operator does not exist. Like $\hat{x}$, the electric flux operators are unbounded self-adjoint operators with discrete eigenvalues. (However, in the case of electric fluxes, the set of eigenvalues is a discrete subset of the real line, equipped with its standard topology.) It is this discreteness that leads to the loss of continuum in the quantum Riemannian geometry which in turn 'justifies' the absence of the standard momentum operator $-i\hbar\, d/dx$ in the polymer particle example.

Note also that the analogy between the $\hat{x}$ operators in the polymer particle and Schrödinger frameworks and the flux operators in quantum geometry provides us with an example of how concepts that are distinct in the full theory coalesce in the toy model. In quantum geometry, the flux operator corresponds to an integration of the electric field on a two-dimensional surface; this operator is not well defined in Fock space, as we have mentioned. Conversely, the three-dimensionally smeared electric field operators of Fock space are not well defined as operators in $\mathcal{H}_{\rm Poly}$. There are therefore two distinct notions of electric field operators between the two frameworks. In the polymer particle and Schrödinger representations, however, the $\hat{x}$ operator has the same action on each space.

3. Recall that in quantum geometry, elements of $\mathcal{H}_{\rm Poly}$ can be represented as *functions* on a compact space $\overline{\mathcal{A}}$, the quantum configuration space obtained by a suitable completion of the classical configuration space $\mathcal{A}$. What is the situation with respect to $\mathcal{H}_{\rm Poly}$? Now, the classical configuration space is just the real line $\mathbb{R}$ (of momenta k). The quantum configuration space turns out to be the Bohr compactification $b\mathbb{R}$ of $\mathbb{R}$ (discovered and analyzed by the mathematician Harald Bohr, Niels' brother). All quantum states are represented by square integrable functions on $b\mathbb{R}$ with respect to a natural measure μ_o; $\mathcal{H}_{\rm Poly} = L^2(b\mathbb{R}, d\mu_o)$. Finally, as in quantum geometry, $b\mathbb{R}$ is also the Gel'fand spectrum of the *Abelian* $C^\star$-algebra of 'holonomy' operators $V(\mu)$. (For details on the Bohr compactification, see [46, 47].)

2.4 Relationship between the Schrödinger and polymer particle descriptions: kinematics

Elements of the polymer Hilbert space $\mathcal{H}_{\rm Poly}$ consist of *discrete* sums

$$f(k) = \sum_j f_j \, \exp\left(-ix_j k\right)$$

of plane waves. Therefore, it follows that the intersection of $\mathcal{H}_{\rm Poly}$ with $\mathcal{H}_{\rm Sch}$ consists just of the zero element. While each provides an irreducible, unitary representation of the Weyl-Heisenberg algebra, the two Hilbert spaces are 'orthogonal'. Therefore, one might first think that the standard physics contained in the Schrödinger representation cannot be recovered from the polymer framework. We will now show that this is *not* the case.

As explained in the introduction, the key idea is to focus on Cyl*, the algebraic dual[5] of Cyl. Since Cyl $\subset \mathcal{H}_{\rm Poly}$, it follows that we have:

$$\mathrm{Cyl} \subset \mathcal{H}_{\rm Poly} \subset \mathrm{Cyl}^\star \,.$$

We will denote the elements of Cyl* by upper case letters, e.g., $(\Psi|$, and their action on elements $|f\rangle$ of Cyl simply with a juxtaposition, e.g. $(\Psi|$ maps $|f\rangle$ to the complex number $(\Psi|f\rangle$.

The Weyl-Heisenberg algebra has a well-defined action on Cyl, and hence by duality, on Cyl*:

$$\left[(\Psi|\hat{W}(\zeta)\right] \, |f\rangle = (\Psi| \, \left[(\hat{W}(\zeta))^\dagger |f\rangle\right] \tag{2.4.1}$$

However, this representation is far from being irreducible. In particular, $\mathcal{H}_{\rm Poly}$ is contained in Cyl* and provides us with an irreducible representation. More importantly for what follows, the Schwartz space $\mathcal{S}$, a dense subspace of $\mathcal{H}_{\rm Sch}$ consisting of smooth functions on $\mathbb{R}$ which, together with all their derivatives fall off faster than any inverse polynomial in x, is also embedded in Cyl*. (This follows from the two technical conditions in the definition of a graph and, of course, the definition of Cyl.) Since all coherent states belong to $\mathcal{S}$ and they form an over-complete basis in $\mathcal{H}_{\rm Sch}$, Schrödinger quantum mechanics is somehow encoded in Cyl*. Our task is to analyze this encoding.

We will often use the fact that $\mathcal{S}$ is stable under Fourier transform; i.e., $\psi(x) \in \mathcal{S}$ if and only if its Fourier transform $\tilde{\psi}(k) \in \mathcal{S}$. The embedding of $\mathcal{S}$ in Cyl* is given just by the

[5] As in quantum geometry, we are taking the algebraic dual just for simplicity. When the framework is further developed, one would introduce an appropriate topology on Cyl (which is finer than that of $\mathcal{H}_{\rm Poly}$) and define Cyl* as the space of linear functions on Cyl which are continuous in this topology. The algebraic dual is 'too large' but this fact is not relevant here: since our main goal is to represent all semi-classical Schrödinger states by elements of Cyl* we can just ignore the fact that the algebraic dual also contains other 'unwanted' states.

Schrödinger scalar product: each element $\psi \in \mathcal{S}$ defines an element $(\Psi|$ of Cyl* via

$$(\Psi| \left[\sum_j f_j \, |e^{-ix_j k}\rangle\right] = \frac{1}{\sqrt{2\pi}} \sum_j f_j \int dk \, \tilde{\bar{\psi}}(k) e^{-ix_j k} = \sum_j f_j \, \bar{\psi}(x_j) \tag{2.4.2}$$

where $\tilde{\psi}(k)$ is the Fourier transform of $\psi(x)$. Thus, although elements of Cyl fail to be normalizable in the Schrödinger Hilbert space, their Schrödinger inner product with elements of $\mathcal{S}$ is well-defined and naturally leads to a linear map from Cyl to $\mathbb{C}$.

Can we exploit the fact that $\mathcal{S}$ is embedded in Cyl* to extract the physics of Schrödinger quantum mechanics from Cyl*? At first sight, there appears to be a key problem: Cyl* is not equipped with a scalar product. We could restrict ourselves just to $\mathcal{S} \subset$ Cyl* and introduce on it the Schrödinger scalar product by hand. But this would just be an unnecessarily complicated way of arriving at the Schrödinger representation. More importantly, in non-perturbative quantum gravity, we do not have the analog of the Schrödinger Hilbert space and, furthermore, indications are that its perturbative substitute, the graviton Fock space, is 'too small'. Therefore, for our polymer particle toy model to be an effective tool, we should not restrict ourselves to a 'small' subspace of it such as $\mathcal{S}$. Rather, we should work with the full Cyl* and use only that structure which is naturally available on it. Thus, our challenge is to show that standard quantum physics can be extracted from Cyl* directly, without making an appeal to the Schrödinger Hilbert space. Known facts about the Schrödinger representation can be used only to motivate various constructions, but not in the constructions themselves.

In quantum gravity, a principal open problem is that of the existence of semi-classical states. Therefore, in the rest of this chapter we will focus on the problem of isolating elements of Cyl* which correspond to the standard coherent states of Schrödinger quantum mechanics and extracting their physics using only those structures which are naturally available in the polymer framework. Hamiltonians and their various properties will be discussed in the next chapter.

2.4.1 Isolating semi-classical states

Fix a classical state, i.e., a point (x_o, p_o) in the classical phase space. In Schrödinger quantum mechanics, the corresponding semi-classical states are generally represented by coherent states peaked at this point. In these states, the product of uncertainties in the basic observables $\hat{x}$ and $\hat{p}$ is minimized, $(\Delta\,\hat{x})\,(\Delta\,\hat{p}) = \hbar/2$, and furthermore, in suitable units, these uncertainties are distributed 'equally' among the two observables. To obtain a specific coherent state, one has to specify these units, or, in physical terms, 'tolerance'—the uncertainties in x and p we can tolerate. Let us therefore introduce a length scale d and ask that the uncertainty $\Delta\, x$ in $\hat{x}$ be $d/\sqrt{2}$ and that in $\hat{p}$ be $\hbar/(\sqrt{2}\, d)$. (In the case of an harmonic

oscillator, d is generally taken to be $\sqrt{\hbar/m\omega}$. However, in this section on kinematics, it is not necessary to restrict ourselves to a specific system.) Set

$$\zeta_o = \frac{1}{\sqrt{2}d}\left(x_o + i\frac{d^2}{\hbar}p_o\right) = \frac{1}{\sqrt{2}d}\left(x_o + ik_o d^2\right)$$

where, from now on, we will use $k_o := p_o/\hbar$. Then, the standard coherent state ψ_{ζ_o} is generally obtained by solving the eigenvalue equation

$$\hat{a}\,\psi_{\zeta_o}(x) \equiv \frac{1}{\sqrt{2}\,d}\left(\hat{x} + i\frac{d^2}{\hbar}\hat{p}\right)\,\psi_{\zeta_o}(x) = \zeta_o\,\psi_o(x), \tag{2.4.3}$$

whose solution is

$$\psi_{\zeta_o}(x) = c\,e^{-\frac{(x-x_o)^2}{2d^2}}\;e^{ik_o(x-x_o)}, \tag{2.4.4}$$

where $\hat{a}$ is the annihilation operator and c is a normalization constant. Since $\psi_{\zeta_o} \in \mathcal{S}$, it canonically defines an element Ψ_{ζ_o} of Cyl*. Our first task is to isolate this Ψ_{ζ_o} using just the polymer framework. The second task, that of analyzing its properties and specifying the sense in which it is a semi-classical state also from the polymer perspective, will be taken up in the next subsection.

Now, in the polymer framework, the operator $\hat{p}$ fails to be well-defined. Therefore, we cannot introduce the creation and annihilation operators used in the above construction. However, recall that the operators $\hat{U}(\lambda)$, $\hat{V}(\mu)$ and $\hat{x}$ *are* well-defined on Cyl and hence also on Cyl*. We can therefore reformulate (2.4.3) by an equivalent eigenvalue equation in terms of these operators. Since the equation is now to be imposed on Cyl*, we have to replace the annihilation operator $\hat{a}$ by its adjoint, $\hat{a}^\dagger$, the creation operator. Now, using the Baker-Hausdorff-Campbell identity in $\mathcal{H}_{\rm Sch}$, we have:

$$e^{\sqrt{2}\alpha\hat{a}^\dagger} = e^{\frac{\alpha}{d}\hat{x}}\,V(-\alpha d)\,e^{-\frac{\alpha^2}{2}}.$$

where the factor of $\sqrt{2}$ is introduced just for technical simplification and α is an arbitrary real number. Note that the operators on the right side are all well-defined on Cyl*.

Collecting these ideas motivated by results in the Schrödinger representation, we are now led to seek the analog of coherent states in Cyl* by solving the eigenvalue equation:

$$(\Psi_{\zeta_o}|\,\left[e^{\frac{\alpha}{d}\hat{x}}\,V(-\alpha d)\,e^{-\frac{\alpha^2}{2}}\right] \;=\; e^{\sqrt{2}\alpha\,\bar{\zeta_o}}\,(\Psi_{\zeta_o}|. \tag{2.4.5}$$

for all real numbers α. Note that, to capture the full content of the original eigenvalue equation (2.4.3), it is essential to allow arbitrary α in the exponentiated version (2.4.5).

To obtain the solution, it is convenient to use a basis in Cyl*. Recall first that any element f of Cyl can be expanded out as a discrete sum, $f = \sum_j f_j\,|x_j\rangle$, where the f_j are

complex coefficients and the x_j real numbers. Therefore, the action of any element $(\Psi|$ of $\mathrm{Cyl}^\star$ is completely determined by the action $(\Psi|x\rangle = \overline{\Psi}(x)$ of $(\Psi|$ on all basis vectors $|x\rangle$. That is, $(\Psi|$ can be expanded as a continuous sum

$$(\Psi| = \sum_x \overline{\Psi}(x)(x| \tag{2.4.6}$$

where the dual basis $(x|$ in $\mathrm{Cyl}^\star$, labeled by real numbers x, is defined in an obvious fashion:

$$(x|x_j\rangle \,=\, \delta_{x,x_j}.$$

Note that, although there is a continuous sum in (2.4.6), when operating on any element of Cyl only a countable number of terms are non-zero.

Using (2.4.6) in (2.4.5), it is straightforward to show that the coefficients $\Psi_{\zeta_o}(x)$ must satisfy:

$$\overline{\Psi}_{\zeta_o}(x+\alpha d) \,=\, \exp\left[\sqrt{2}\alpha\bar{\zeta}_o - \frac{\alpha x}{d} + \frac{\alpha^2}{2}\right]\overline{\Psi}(x) \tag{2.4.7}$$

for all real numbers α. It is easy to verify that this equation admits a solution which is unique up to a normalization factor. The general solution is given by:

$$(\Psi_{\zeta_o}| = \bar{c}\,\sum_x \left[e^{-\frac{(x-x_o)^2}{2d^2}}\; e^{-ik_o(x-x_o)}\right](x|\,. \tag{2.4.8}$$

As we would desire, the coefficients in this expansion are the same as the expression (2.4.4) of the coherent state wave function in the Schrödinger representation. Note that, to obtain a unique solution (up to a multiplicative constant), it is essential to impose (2.4.7) for *all* real numbers α.

To summarize, by using the standard procedure in the Schrödinger representation as motivation, we wrote down an eigenvalue equation directly in $\mathrm{Cyl}^\star$ to single out a candidate semi-classical state $(\Psi_{\zeta_o}|$ peaked at a generic point (x_o, p_o) of the classical phase space. Since this is a linear equation, one cannot hope to restrict the overall normalization of the solution. Up to this trivial ambiguity, however, the solution is unique. We will refer to it as a *polymer coherent state*. As one might have hoped, this polymer coherent state is just the element $(\Psi_{\zeta_o}|$ of $\mathrm{Cyl}^\star$ defined by the standard coherent state $\psi_{\zeta_o} \in \mathcal{S}$ in $\mathcal{H}_{\mathrm{Sch}}$. Note, though, that this is not an assumption, but the result of a self-contained calculation that was carried out entirely in $\mathrm{Cyl}^\star$. However, at this stage, it is not a priori obvious that $(\Psi_{\zeta_o}|$ is a semi-classical state *from the polymer perspective*, especially because we no longer have access to the Schrödinger scalar product. This issue will be discussed in the next subsection.

2.4.2 Shadow states

For simplicity, in this subsection we will restrict ourselves to the candidate semi-classical state $(\Psi_o|$ corresponding to $\zeta = 0$. (The general case is completely analogous and discussed in subsection 2.4.3.) Our task is to show that this state is sharply peaked at x=0 and p=0 using only the polymer framework. However, right at the outset we encounter two difficulties. Firstly, the operator $\hat{p}$ is not defined in the polymer framework. We will therefore have to define a 'fundamental operator' on $\mathcal{H}_{\rm Poly}$ which is approximated by $\hat{p}$ of the Schrödinger representation. The second difficulty is that, since there is no inner product on Cyl*, the required expectation values cannot be defined on it. To overcome this obstacle, we will use graphs as 'probes' to extract physical information from elements $(\Psi|$ of Cyl*. More precisely, we will 'project' each $(\Psi|$ to an element $|\Psi^{\rm shad}_\gamma\rangle$ in Cyl$_\gamma$ and analyze properties of $(\Psi|$ in terms of its shadows $|\Psi^{\rm shad}_\gamma\rangle$. Each shadow captures only a part of the information contained in our state, but the collection of shadows can be used to determine the properties of the full state in Cyl*.

Let us begin by defining the required projection $\hat{P}_\gamma$ from Cyl* to Cyl$_\gamma$:

$$(\Psi|\,\hat{P}_\gamma := \sum_{x_j\in\gamma} \Psi(x_j)\,|x_j\rangle \;\equiv |\Psi^{\rm shad}_\gamma\rangle\,. \tag{2.4.9}$$

The ket $|\Psi^{\rm shad}_\gamma\rangle$ defines the shadow cast by the element $(\Psi|$ of Cyl* on the graph γ in the sense that

$$(\Psi|f_\gamma\rangle = \langle\Psi^{\rm shad}_\gamma\,|\,f_\gamma\rangle$$

where the left side is the result of the action of an element of Cyl* on an arbitrary element f_γ of Cyl$_\gamma$ and the right side is the scalar product on Cyl$_\gamma$. Our task is to analyze properties of the shadows

$$|\Psi^{\rm shad}_{o,\gamma}\rangle := (\Psi_o|\,\hat{P}_\gamma\,.$$

of our candidate semi-classical state. The essential idea is to say that $(\Psi_o|$ is semi-classical if physical observables of interest have expected mean values with small uncertainties in its shadows $|\Psi^{\rm shad}_{o,\gamma}\rangle$ on sufficiently refined graphs γ.

To make this notion precise, we need to select: i) A suitable family of graphs; ii) a class of observables of interest; and, iii) acceptable 'tolerances' for mean-values and uncertainties of these observables. We will restrict ourselves to shadows on *regular* lattices [6] with sufficiently small lattice spacing (as discussed below). For definiteness, as in Schrödinger quantum mechanics, the class $\mathcal{C}$ of observables of interest will consist just of position and momentum

[6] Quantum geometry considerations imply that, to probe semi-classicality, we should only use those graphs in which the number of points in *any* macroscopic interval is proportional to the length of the interval. Regular lattices offer the simplest way to achieve this. A priori one may be concerned that this is 'too small a class'. But the results of this section show that it suffices.

operators. Tolerances τ will be determined by the physical parameters of the system under consideration (i.e., the length scale d of subsection 2.4.1).

We will say that a state $(\Psi| \in \mathrm{Cyl}^\star$ is semi-classical with respect to $\mathcal{C}$ and peaked at a point (x,p) of the classical phase space, if within specified tolerances τ, the 'expectation values' of any operator $\hat{A} \in \mathcal{C}$ equals the classical value $A(x,p)$ and the fluctuations are small; i.e., if

$$\frac{(\Psi|\hat{A}|\Psi^{\mathrm{shad}}_\gamma\rangle}{\|\Psi^{\mathrm{shad}}_\gamma\|^2} = A(x,p)(1+\tau^{(1)}_A) \;\text{ and }\; \frac{(\Psi|\hat{A}^2|\Psi^{\mathrm{shad}}_\gamma\rangle}{\|\Psi^{\mathrm{shad}}_\gamma\|^2} - \left(\frac{(\Psi|\hat{A}|\Psi^{\mathrm{shad}}_\gamma\rangle}{\|\Psi^{\mathrm{shad}}_\gamma\|^2}\right)^2 \leq \tau^{(2)}_A \quad (2.4.10)$$

for all sufficiently refined graphs γ. Here $\|f\|$ is the norm of the state $|f\rangle$ in $\mathcal{H}_{\mathrm{Poly}}$, and $\tau^{(i)}_A$ are the tolerances assigned to the observable A. The meaning of the equation is clearer if the operators are thought as acting on the candidate semi-classical state $(\Psi|$ in $\mathrm{Cyl}^\star$ by duality. Thus, in the first equality, the 'expectation value' of $\hat{A}$ in the candidate semi-classical state $(\psi|$ is evaluated by the action of $(\Psi|\hat{A}$ $(\in \mathrm{Cyl}^\star)$ on the shadow $|\Psi^{\mathrm{shad}}_\gamma\rangle$ of $(\Psi|$ on the graph γ. If the action of the operator $\hat{A}$ leaves Cyl_γ invariant, as one might hope, this 'expectation value' reduces to the more familiar expression $\langle\Psi^{\mathrm{shad}}_\gamma|\hat{A}|\Psi^{\mathrm{shad}}_\gamma\rangle$. However, for more general operators, the two expressions do not agree and $(\Psi|\hat{A}|\Psi^{\mathrm{shad}}_\gamma\rangle$ turns out to be the better measure of the expectation value.

Let us then work with infinite regular lattices with spacing ℓ, where ℓ is chosen to be sufficiently small (see below). The shadow of our candidate semi-classical state $(\Psi_o|$ on the regular graph is given by:

$$|\Psi^{\mathrm{shad}}_{o,\ell}\rangle = c \sum_{n\in\mathbb{Z}} e^{-\frac{n^2\ell^2}{2d^2}} |n\ell\rangle\,, \quad (2.4.11)$$

where c is an arbitrary constant. We can now compute the expectation values and fluctuations of various operators in detail and examine if the state can be regarded as semi-classical. On general grounds, one would hope to obtain good agreement with the standard coherent state of Schrödinger quantum mechanics provided the lattice spacing ℓ is much smaller than the length scale d that defines our tolerance. We will show that, although there are subtleties, this expectation is borne out. However, let us first pause to examine whether this requirement is physically reasonable. As an example, consider the vibrational oscillations of a carbon monoxide molecule. These are well described by a harmonic oscillator with parameters

$$m \approx 10^{-26}\,\mathrm{kg} \quad \text{and} \quad \omega \approx 10^{15}\,\mathrm{Hz}$$

The textbook treatment of the harmonic oscillator implies that we cannot require the tolerance d for $\hat{x}$ to be smaller than

$$d_{\mathrm{min}} = \sqrt{\frac{\hbar}{m\omega}} \approx 10^{-12}\,\mathrm{m}\,;$$

if we did, the resulting state would spread out quickly under quantum evolution. On the other hand, since no evidence of spatial discreteness has been observed at particle accelerators, the quantum geometry viewpoint requires us to choose $\ell < 10^{-19}$m, and we may even wish to move ℓ all the way down to the Planck scale ($\ell_p = 1.6 \times 10^{-35}$ m). Thus, our assumption that $\ell \ll d$ is well justified. Working in this regime, we will now show that the quantities computed using (2.4.11) agree to leading order with the standard Schrödinger coherent state. The corrections are of order $\ell^2/d^2 < 10^{-14}$ and, furthermore, appear in the regime in which Schrödinger quantum mechanics is inapplicable due to, e.g., relativistic effects.

Let us begin with the norm of the shadow of the polymer coherent state:

$$\langle \Psi^{\rm shad}_{o,\ell} \,|\, \Psi^{\rm shad}_{o,\ell} \rangle \;=\; |c|^2 \;\sum_{n=-\infty}^{\infty} e^{-\frac{n^2\ell^2}{d^2}} \,. \tag{2.4.12}$$

Here, we have used the fact that $\langle x_i \,|\, x_j \rangle = \delta_{x_i,x_j}$ to simplify the double sum to a single one. Now, since $\ell \ll d$, the exponential on the right hand side of (2.4.12) decays very slowly, whence we can not estimate the norm by keeping just a few terms in the sum. Fortunately, however, we can use the Poisson re-summation formula:

$$\sum_n g(x+n) = \sum_{n=-\infty}^{\infty} e^{2\pi i\, x\, n} \int_{-\infty}^{\infty} g(y) e^{-2\pi i\, y\, n} dy \,, \tag{2.4.13}$$

for all functions $g(y)$ which are suitably well behaved for the sums to converge. This re-summation will allow us to replace the slowly convergent sum in (2.4.12) by a rapidly convergent one; it is an important technique that we shall use frequently. We will take

$$g(y) = e^{-\frac{y^2\ell^2}{d^2}} \quad \text{and} \quad x = 0 \,.$$

Then it is straightforward to calculate

$$\langle \Psi^{\rm shad}_{o,\ell} \,|\, \Psi^{\rm shad}_{o,\ell} \rangle \;=\; |c|^2 \,\frac{\sqrt{\pi} d}{\ell} \;\sum_{n=-\infty}^{\infty} e^{-\frac{\pi^2 n^2 d^2}{\ell^2}} \;\approx\; |c|^2 \,\frac{\sqrt{\pi} d}{\ell} \left(1 + 2e^{-\frac{\pi^2 d^2}{\ell^2}}\right) , \tag{2.4.14}$$

where we have used $(d/\ell) \gg 1$ to truncate the series after the second term.

Next we turn to the expectation value of and fluctuations in $\hat{x}$. For semi-classicality, the expectation value should be close to zero and the fluctuations of the order $d/\sqrt{2}$. For expectation values, we obtain

$$(\Psi_o|\hat{x}|\Psi^{\rm shad}_{o,\ell}\rangle \;=\; |c|^2 \,\sum_n (n\ell)\, e^{-\frac{n^2\ell^2}{d^2}} = 0 \,, \tag{2.4.15}$$

due to antisymmetry in n. This result agrees exactly with that obtained from the Schrödinger coherent state. Let us turn to the calculation of fluctuations. We have

$$\begin{aligned}(\Psi_o|\hat{x}^2|\Psi^{\rm shad}_{o,\ell}\rangle &= |c|^2\sum_n (n\ell)^2\, e^{-\frac{n^2\ell^2}{d^2}} \\ &= |c|^2\,\frac{\sqrt{\pi}d^3}{2\ell}\sum_n e^{-\frac{\pi^2n^2d^2}{\ell^2}}\left(1-\frac{2\pi^2n^2d^2}{\ell^2}\right), \end{aligned} \tag{2.4.16}$$

where we have once again made use of the Poisson re-summation formula. By combining the results of (2.4.16) and (2.4.14), we can obtain the fluctuations in $\hat{x}$,

$$(\Delta x)^2 := \frac{(\Psi_o|\hat{x}^2|\Psi^{\rm shad}_{o,\ell}\rangle}{\|\Psi^{\rm shad}_{o,\ell}\|^2} - \left(\frac{(\Psi_o|\hat{x}|\Psi^{\rm shad}_{o,\ell}\rangle}{\|\Psi^{\rm shad}_{o,\ell}\|^2}\right)^2 \approx \frac{d^2}{2}\left(1-\frac{4\pi^2d^2}{\ell^2}\,e^{-\frac{\pi^2d^2}{\ell^2}}\right), \tag{2.4.17}$$

where we have made use of the fact that the expectation value of $\hat{x}$ is zero. Hence, we see that the fluctuations in $\hat{x}$ satisfy our 'tolerance' requirement. Indeed, to leading order, they agree with the those in the standard coherent states of the Schrödinger framework and the sub-leading terms are extremely small, going to zero as ℓ/d tends to zero. Interestingly, these corrections actually *decrease* the uncertainty in x for the discrete case.

Thus, we see that our candidate semi-classical state $(\Psi_o|$ is indeed sharply peaked at $x = 0$. What about the momentum? As mentioned above, there is no natural analog of the Schrödinger momentum operator $\hat{p}$ on $\mathcal{H}_{\rm Poly}$. Thus, the viewpoint is that the standard $\hat{p}$ operator is a 'low energy' construct. There are several operators in the 'fundamental description' whose action on 'low lying states' is approximated by $\hat{p}$. Here, we will choose one and test for semi-classicality of $(\Psi_o|$. As one might hope, the difference between candidate choices is manifest only at such high energies that the Schrödinger quantum mechanics is inapplicable there.

To define an analog of the Schrödinger momentum operator, we will use a standard strategy from lattice gauge theories. We first note that, classically, if $k\mu$ is small then we can expand $\exp(-ik\mu)$ as

$$\exp(-ik\mu) = 1 - ik\mu - \frac{k^2\mu^2}{2} + \cdots \tag{2.4.18}$$

whence

$$\frac{\exp(-ik\mu) - \exp(ik\mu)}{-2i\mu} = k + O(k^2\mu)\,. \tag{2.4.19}$$

In quantum theory, then, it seems natural to define the analog of the momentum operator in a similar way. Choose a sufficiently small value μ_o of μ (with $\ell \leq \mu_o \ll d$) and define the

momentum operator on $\mathcal{H}_{\rm Poly}$ as $\hat{p} = \hbar \hat{K}_{\mu_o}$, with

$$\hat{K}_{\mu_o} := \frac{i}{2\mu_o} \left(\hat{V}(\mu_o) - \hat{V}(-\mu_o) \right) . \tag{2.4.20}$$

(The simpler definition $\hat{K}_{\mu_o} = (i/2\mu_o)(\hat{V}(\mu_o) - 1)$ is not viable because this operator fails to be self-adjoint.) With this definition in hand, let us examine the expectation value and fluctuations in $\hat{K}_{\mu_o}$. $(\Psi|$ will be semi-classical also for momentum if the expectation value of $\hat{K}_{\mu_o}$ is close to zero and the fluctuation is of the order $1/\sqrt{2}d$.

Now, a direct calculation in the polymer Hilbert space yields

$$\langle \hat{V}(\mu) \rangle := \frac{(\Psi_o | \hat{V}(\mu) | \Psi^{\rm shad}_{o,\ell} \rangle}{\| \Psi^{\rm shad}_{o,\ell} \|^2} \approx e^{-\frac{\mu^2}{4d^2}} \left(1 + 2\, e^{-\frac{\pi^2 d^2}{\ell^2}} \left[\cos\left(\frac{\pi \mu}{\ell} \right) - 1 \right] \right) , \tag{2.4.21}$$

for any value of μ. Using this result, it is straightforward to show that

$$\langle \hat{K}_{\mu_o} \rangle = 0 \tag{2.4.22}$$

because of the antisymmetry between $\hat{V}(\mu_o)$ and $\hat{V}(-\mu_o)$ in our definition (2.4.20). Next, let us analyze the fluctuations

$$\langle \hat{K}^2_{\mu_o} \rangle = \frac{1}{4{\mu_o}^2} \left(2 - \langle \hat{V}(2\mu_o) \rangle - \langle \hat{V}(-2\mu_o) \rangle \right) . \tag{2.4.23}$$

Substituting $\mu = \pm 2\mu_o$ in the above expression (2.4.21), we obtain

$$\langle \hat{K}^2_{\mu_o} \rangle \;\approx\; \frac{1}{2{\mu_o}^2} \left[1 - e^{-\frac{{\mu_o}^2}{d^2}} \right] \;\approx\; \frac{1}{2d^2} \left[1 - \left(\frac{{\mu_o}^2}{2d^2} \right) \right] , \tag{2.4.24}$$

where we have used the fact $\mu_o \ll d$ to expand in powers of (μ_o/d) in the last step. Recalling that the expectation value of $\hat{K}$ in the state $|\Psi^\ell_o\rangle$ is zero, we obtain the fluctuations in $\hat{K}$ as

$$(\Delta K_{\mu_o})^2 \approx \frac{1}{2d^2} \left[1 - \left(\frac{{\mu_o}^2}{2d^2} \right) \right] . \tag{2.4.25}$$

Since the approximate momentum operator is given by $\hbar \hat{K}_{\mu_o}$, we conclude that the state is sharply peaked at $p = 0$ and the fluctuations are within the specified tolerance.

Finally, collecting the results for $\Delta\, x$ and $\Delta\, k$, we obtain the uncertainty relations for the shadow of the polymer semi-classical state:

$$(\Delta x)^2 (\Delta K_{\mu_o})^2 = \frac{1}{4} \left[1 - \left(\frac{{\mu_o}^2}{2d^2} \right) + \mathcal{O}\left(\frac{{\mu_o}^4}{d^4} \right) \right] . \tag{2.4.26}$$

Note that the corrections to the standard uncertainty relations at order $(\mu_o/d)^2$ *decrease*

the uncertainty. This can occur because the commutator between the position and the approximate momentum operator is not simply a multiple of identity. Such modifications of the uncertainty relations have also been obtained in string theory. Our discussion shows that the effect is rather generic.

To summarize, in subsection 2.4.1, we found candidate semi-classical states $(\Psi_{\zeta_o}|$ in Cyl* working entirely in the polymer particle framework. In this sub-section, we showed that the polymer coherent state $(\Psi_o|$ is semi-classical in the polymer sense: its shadows on sufficiently refined lattices are sharply peaked at the point (x=0, p=0) of the classical phase space. Furthermore, the fluctuations in x and p are essentially the same as those in the Schrödinger coherent state ψ_o of (2.4.4). There *are* deviations, but in the regime of applicability of Schrödinger quantum mechanics, they are too small to violate experimental bounds.

2.4.3 General coherent states

Let us now analyze the properties of general polymer coherent states $(\Psi_\zeta|$ with

$$\zeta = \frac{1}{\sqrt{2}d}\left(x + id^2k\right).$$

Calculations of expectation values and fluctuations proceed in a very similar manner to those described above for $(\Psi_o|$. (The only difference arises from the fact that we may not have a point in our graph at the position x. However, this affects only the sub-leading terms.) Therefore, we will simply state the final results:

1. The norm of the state is given by

$$\langle \Psi^{\rm shad}_{\zeta,\ell} \,|\, \Psi^{\rm shad}_{\zeta,\ell}\rangle = |c|^2\, \frac{\sqrt{\pi}d}{\ell}\, \left(1 + \mathcal{O}\left(e^{-\frac{\pi^2 d^2}{\ell^2}}\right)\right). \tag{2.4.27}$$

2. The expectation value of the $\hat{x}$ operator is

$$\langle \hat{x}\rangle \,:=\, \frac{(\Psi_\zeta \,|\, \hat{x} \,|\, \Psi^{\rm shad}_{\zeta,\ell}\rangle}{\|\Psi^{\rm shad}_{\zeta,\ell}\|^2} = x\, \left[1 + \mathcal{O}\left(e^{-\frac{\pi^2 d^2}{\ell^2}}\right)\right]. \tag{2.4.28}$$

   Thus, the expectation value of position is x within the tolerance $\tau^{(1)}_x = e^{-\frac{\pi^2 d^2}{\ell^2}}$.

3. The fluctuation in x is

$$(\Delta x)^2 = \frac{d^2}{2}\, \left[1 + \mathcal{O}\left(e^{-\frac{\pi^2 d^2}{\ell^2}}\right)\right]. \tag{2.4.29}$$

   So, the leading term, $d/\sqrt{2}$, in the fluctuation in x is the same as in the Schrödinger

coherent states. Also, the sub-leading terms are independent of ζ, i.e., are the same for all polymer coherent states.

4. One can evaluate the $\hat{K}_{\mu_o}$ operator on an arbitrary coherent state. The result is,

$$\langle \hat{K}_{\mu_o} \rangle = k \left(1 + \mathcal{O}(k^2\mu_o^2) + \mathcal{O}\Big(\frac{\ell^2}{d^2}\Big)\right). \tag{2.4.30}$$

Thus, we now encounter a new situation. The tolerance $\tau^{(1)}_{K_{\mu_o}}$ is acceptably small only if $k\mu_o \ll 1$. In this case, we obtain an uncertainty relation similar to the one in (2.4.26). However, for $k\mu_o \sim 1$ our states do not satisfy the semi-classicality requirement. But note that the non-relativistic approximation —and hence the motivation for including $\hat{K}_{\mu_o}$ in the list $\mathcal{C}$ of observables— breaks down long before one reaches such high momenta. (In the case of the CO molecule, for example, this would correspond to the energy level $n \geq 10^{14}$.)

To summarize, we have introduced polymer coherent states $(\Psi_\zeta|$ and investigated their properties using their shadows $|\Psi^\ell_\zeta\rangle$. Given a tolerance d for $\hat{x}$, an uniform graph can serve as a suitable 'probe' provided the lattice spacing ℓ is chosen so that $\ell/d \ll 1$. As far as semi-classical states are concerned, systems which can be treated adequately within non-relativistic quantum mechanics can also be well-described by the polymer particle framework, without any reference to the Schrödinger Hilbert space.

Remark: Recall that the normalized Schrödinger coherent states $|\psi_\zeta\rangle$ form an overcomplete basis in $\mathcal{H}_{\rm Sch}$ providing a convenient resolution of the identity:

$$\int dk \int dx\, |\psi_\zeta\rangle\langle\psi_\zeta| \;=\; I\,. \tag{2.4.31}$$

Does a similar result hold for the shadow coherent states $|\Psi^{\rm shad}_{\zeta,\ell}\rangle$ in the Hilbert space $\mathcal{H}^\ell_{\rm Poly}$ restricted to the graph? A priori, it would appear that there is a potential problem. Since

$$|\Psi^{\rm shad}_{\zeta,\ell}\rangle = c \sum_n \Big(e^{(n\ell - x)^2}\, e^{ik(n\ell - x)} \Big)\;\; |n\ell\rangle$$

where $\zeta = \frac{1}{\sqrt{2}d}(x + id^2k)$, it follows that the projection operators

$$P_\zeta := \frac{|\Psi^{\rm shad}_{\zeta,\ell}\rangle\langle\Psi^{\rm shad}_{\zeta,\ell}|}{\|\Psi^{\rm shad}_{\zeta,\ell}\|^2}$$

defined by the shadow coherent states are periodic: $P_\zeta = P_{\zeta'}$ where $k' = k + (2\pi N)/\ell$. Thus, while the label k took values on the entire real line in (2.4.31), with shadow coherent states in $\mathcal{H}^\ell_{\rm Poly}$, it can only take values in $[-\pi/\ell, \pi/\ell]$. Therefore, one might be concerned

that, because of the 'effective momentum cut-off' we may not have 'sufficient' coherent states for the standard over-completeness to hold. However, it turns out that this concern is misplaced. $\mathcal{H}^{\ell}_{\rm Poly}$ is sufficiently small because of the restriction to a fixed lattice for an *exact* over-completeness of the desired type to hold [48, 49] :

$$\int_{-\pi/\ell}^{\pi/\ell} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dx\, P_\zeta = \sum_n |n\ell\rangle\, \langle n\ell| \;=\; I_\ell\,, \tag{2.4.32}$$

where I_ℓ is the identity operator on $\mathcal{H}^{\ell}_{\rm Poly}$.

2.5 Discussion

We began this chapter by raising three conceptual issues of a rather general nature that arise in relating background independent approaches to quantum gravity with low energy physics: i) What is the precise sense in which semi-classical states arise in the full theory? ii) Is the fundamental Planck scale theory, with an in-built fundamental discreteness, capable of describing also the low energy physics rooted in the continuum, or, does it only describe an entirely distinct phase? iii) Can one hope to probe semi-classical issues without a canonical inner product on the space of physical states Cyl*? To probe these issues in a technically simpler context, we introduced the 'polymer framework' in a toy model—a non-relativistic particle—where the same questions arise naturally. In the context of the model, we found encouraging answers to all three questions: although at first the polymer description seems far removed from the standard Schrödinger one, the second can be recovered from the former in detail.

Specifically, we have, so far in this thesis, given a criterion to select the coherent states entirely within the polymer framework and, using their shadows, demonstrate in detail that they are sharply peaked about the corresponding classical states. Thus, at the level of kinematics, we have a concrete mathematical model, inspired by loop quantum gravity, which realizes the idea that a fundamental theory can be radically different from the continuum theory *both conceptually and technically* and yet reproduce the familiar low energy results.

While it may seem 'obvious' that calculations on sufficiently refined graphs should reproduce the continuum answers, even at the kinematical level that we have addressed in this chapter we have seen that this is a subtle point. A priori, it is not at all obvious that any calculation to select semi-classical states ($\Psi|$ in Cyl*, carried out entirely within the polymer framework, will reproduce the *standard* coherent states. One could indeed be working in an inequivalent 'phase' of the theory and thus find that there are no semi-classical states at all or discover states which are semi-classical in a certain well-defined sense but distinct from the standard coherent states (as in, e.g., [30]). The fact that the Schrödinger semi-classical states can be recovered in the polymer framework is thus non-trivial and suggests

how *standard* low energy physics could emerge from the polymer framework.

To be confident of this, however, we must also investigate dynamical questions. If it should happen that our semi-classical states, though kinematically well suited for low-energy physics, quickly evolve away from semi-classicality, or evolve in a manner that is radically different from the standard low-energy behavior, then we should be forced to conclude that the low-energy behavior of this model is in fact not compatible with observed physics.

To actually perform such evolutions is a complicated matter. We shall make steps toward this in chapter 4, in the somewhat more limited context of an effective classical description of the quantum dynamics. However, even before we tackle this there is a more tractable approach to answering dynamical questions. That is to examine the Hamiltonians for both the polymer particle and Schrödinger representations, and compare their eigenstates and eigenvalues. We turn therefore to that question in the next chapter.

Chapter 3

Dynamics of the polymer particle: Hamiltonians and eigenstates

We saw in the last chapter that the coherent states in Cyl* are indeed kinematically semi-classical: they have expectation values peaked around the corresponding classical values with small fluctuations. This allowed us to conclude that, kinematically, the low-energy behavior of the polymer particle representation is experimentally indistinguishable from that of the standard Schrödinger representation, within the regime of validity of non-relativistic quantum mechanics.

We now turn in this chapter to a comparison of the dynamics of the two representations. Since we again wish our simple model to parallel the more complicated situation in quantum geometry, to define the kinetic energy term in the Hamiltonian, we mimic the procedure used to define the Hamiltonian constraint operator in quantum general relativity. However, in the toy model, this requires the introduction of a new structure by hand, namely a fundamental length scale, which can be regarded as descending from an underlying quantum geometry. The resulting dynamics is indistinguishable from the standard Schrödinger mechanics in the domain of applicability of the non-relativistic approximation. Deviations arise only at energies which are sufficiently high to probe the quantum geometry scale. In particular, shadows of the Schrödinger energy eigenstates are excellent approximations to the 'more fundamental' polymer eigenstates.

3.1 Motivation and framework

Since a secondary goal of this chapter and the previous is to illustrate strategies used in loop quantum gravity, let us begin by recalling the situation with the Hamiltonian constraint of quantum general relativity [8–11]. The main term in the classical constraint is of the form $\mathrm{Tr}\, E^a E^b F_{ab}$, where, as explained in section 2.2, the triad fields E are the analogs of x in the polymer particle example and F_{ab} is the curvature of the gravitational connection A, the analog of k. While E's and holonomies of A are well-defined operators on the quantum geometry Hilbert space, connections are not. Therefore, F_{ab} has to be expressed in terms of holonomies. Given a spin network state, at each vertex, one introduces new

edges, creating 'small' loops and expresses F_{ab} in terms of holonomies along these small loops (taking care of appropriate 'area factors'). The resulting operator initially depends on the choice these new edges. However, while acting on *diffeomorphism invariant states (in* $\mathrm{Cyl}^\star$*), the dependence on the details of these edges drops out.* Thus, on states of physical interest, the final Hamiltonian constraint does not make explicit reference to details such as the lengths and 'positions' of the new edges.

Let us now turn to the polymer particle. Now, the classical Hamiltonian is of the form

$$H = \frac{p^2}{2m} + \mathcal{V}(x)\,, \tag{3.1.1}$$

where $\mathcal{V}(x)$ is a potential. Since the operator $\hat{x}$ is well-defined in the polymer framework, the main technical problem is that of defining the operator analog of p^2. Thus the situation is analogous to that with the Hamiltonian constraint, described above. Again, we will need to introduce some extra structure (which is invisible classically), this time to define the analog of p^2 in terms of 'holonomies' $V(\mu)$ of the 'connection' k on the full Hilbert space $\mathcal{H}_{\mathrm{Poly}}$. From a mathematical viewpoint, the obvious choice is an 'elementary length' μ_o. Physically, this is motivated by the expectation that such a scale will be provided by a deeper theory (such as quantum geometry) through an underlying discreteness. *From now on, we will adopt the viewpoint that this discreteness is fundamental, whence observationally only those* $V(\mu)$ *are relevant for which* $\mu = N\mu_o$*, for an integer* N*.*

Given μ_o, we will set

$$\hat{H} = \frac{\hbar^2}{2m}\widehat{K^2_{\mu_o}} + \mathcal{V}(\hat{x}) \quad \text{where} \quad \widehat{K^2_{\mu_o}} = \frac{1}{\mu_o^2}\left(2 - V(\mu_o) - V(-\mu_o)\right) \tag{3.1.2}$$

Unfortunately, in this example, we do not have a useful analog of the diffeomorphism invariance of loop quantum gravity which can help remove the dependence on this extra structure. Therefore, the final Hamiltonian operator on $\mathcal{H}_{\mathrm{Poly}}$ will continue to depend on μ_o; the reference to the additional structure does not go away. This is simply a consequence of the fact that a toy model can not mimic all aspects of the richer, more complicated theory, whence, to carry out constructions which are analogous to those in the full theory, certain structures have to be introduced 'externally'. However, we will see that, if one chooses the discreteness scale $\mu_o < 10^{-19}m$ as in Section 2.4.2, in the domain of validity of non-relativistic quantum mechanics, predictions derived from (3.1.2) are indistinguishable from those of Schrödinger quantum mechanics and therefore in agreement with experiments. *In contrast to results of section* 2.4*, this holds for fully quantum mechanical results, not just the semi-classical ones.* At first sight, this may seem obvious. However, the detailed analysis will reveal that certain subtleties arise and have to be handled carefully. These issues provide concrete hints for the precise procedure required to compare the polymer and continuum theories in the much

more complicated context of quantum geometry. Thus, while the toy model is constructed to mimic the situation in the full theory, its concrete results, in turn, provide guidance for the full theory.

Since the key difficulties in the polymer description involve the kinetic term, to illustrate the similarity and differences between the polymer and Schrödinger dynamics it suffices to work with a fixed potential. To facilitate the detailed comparison, in this chapter we will focus on the harmonic oscillator potential. We shall study another Hamiltonian—not of the form $p^2/2m + \mathcal{V}(x)$—in the next chapter when we consider the dynamics of loop quantum cosmology.

We close this section with two remarks:

1. In the semi-classical considerations of the last section, we had to find a 'fundamental' operator on $\mathcal{H}_{\rm Poly}$ which is the analog of the Schrödinger momentum operator. Technically, the situation with the kinetic term in the Hamiltonian, discussed above, is completely analogous. However, there is a conceptual difference: whereas the operator $\hat{K}_{\mu_o}$ was used only for semi-classical purposes, $\hat{H}$ is to govern 'fundamental dynamics' on $\mathcal{H}_{\rm Poly}$. Therefore, it has to be constructed and analyzed more carefully. In particular, $\widehat{K^2_{\mu_o}} \neq \hat{K}^2_{\mu_o}$; we will see that the latter choice gives an unwanted degeneracy in the eigenvalues of $\hat{H}$.

2. Since the *final* Hamiltonian now depends on μ_o, in the polymer description, μ_o now has a fundamental significance. This strengthens the viewpoint that the algebra of physical observables is generated only by $\hat{V}(N\mu_o)$ and $\hat{x}$.

3.2 Eigenvalues and eigenstates of $\hat{H}$ in $\mathcal{H}_{\rm Poly}$

Recall that a general element $|\Psi\rangle$ of $\mathcal{H}_{\rm Poly}$ can be expanded out as $|\Psi\rangle = \sum_x \Psi(x)|x\rangle$ (where $\Psi(x)$ is non-zero only at a countable set of points). Therefore, the eigenvalue equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$ becomes a difference equation on the coefficients $\Psi(x)$:

$$\Psi(x+\mu_o) + \Psi(x-\mu_o) = \left[2 - \frac{2E}{\hbar\omega}\frac{\mu_o^2}{d^2} + \frac{x^2}{d^2}\frac{\mu_o^2}{d^2}\right]\, \Psi(x)\,. \tag{3.2.1}$$

The form of this equation implies that a basis of solutions is given by states of the type

$$|\Psi_{x_o}\rangle = \sum_{m=-\infty}^{\infty} \Psi^{(m)}_{x_o} |x_o + m\mu_o\rangle \,\in {\rm Cyl}_{\alpha^{x_o}}\,,$$

where α^{x_o} is the regular lattice consisting of points $x_o + m\mu_o$ with $x_o \in [0, \mu_o)$. For these states, the difference equation reduces to a recursion relation

$$\Psi^{(m+1)}_{x_o} + \Psi^{(m-1)}_{x_o} = \left[2 - \frac{2E}{\hbar\omega}\frac{\mu_o^2}{d^2} + \frac{(x_o + m\mu_o)^2}{d^2}\frac{\mu_o^2}{d^2}\right] \Psi^{(m)}_{x_o}\,. \tag{3.2.2}$$

The full polymer Hilbert space $\mathcal{H}_{\rm Poly}$ can be decomposed as a direct sum of separable Hilbert spaces $\mathcal{H}^{x_o}_{\rm Poly}$,

$$\mathcal{H}_{\rm Poly} = \bigoplus_{x_o\in[0,\mu_o)} \mathcal{H}^{x_o}_{\rm Poly}\,,$$

and the above energy eigenstates belong to the sub-space $\mathcal{H}^{x_o}_{\rm Poly}$ of $\mathcal{H}_{\rm Poly}$. Note that since the observable algebra is now generated by $\hat{x}$ and $\hat{V}(N\mu_o)$, observables can not mix states belonging to distinct $\mathcal{H}^{x_o}_{\rm Poly}$; each of these Hilbert spaces is superselected. Hence, we can focus on one at a time and find all eigenvalues and eigenstates of the Hamiltonian in it.

3.2.1 The case $x_o = 0$

Let us consider the $x_o = 0$ case first. If E is to be an eigenvalue of $\hat{H}$, the coefficients $\Psi^{(m)}_0$ must fall off sufficiently fast for $|\Psi_0\rangle$ to be normalizable. It turns out that the simplest way to get a control on this fall-off is to make a 'Fourier transform' and go to the momentum representation. Recall from section 2.3.2 that for each real number k, there is an element $(k|$ of Cyl* defined by: $(k|x\rangle = e^{-ikx}$. Given any energy eigenstate $|\Psi_0\rangle \in \mathcal{H}^0_{\rm Poly}$, we can evaluate its 'Fourier transform'

$$\psi(k) := (k|\Psi_0\rangle = \sum_{m=-\infty}^{\infty} \Psi^{(m)}_0\, e^{-ikm\mu_o} \tag{3.2.3}$$

where $k \in (-\frac{\pi}{\mu_o}, \frac{\pi}{\mu_o})$; by construction $\psi(k)$ is periodic in k with period $2\pi/\mu_o$. The eigenvalue equation (3.2.2) now becomes

$$\frac{d^2\psi_0(k)}{dk^2} + 2d^2\left(\frac{E}{\hbar\omega} + \frac{d^2}{\mu_o^2}\Big[\cos(k\mu_o) - 1\Big]\right)\psi_0(k) = 0\,. \tag{3.2.4}$$

Thus, the difference equation (3.2.1) in the position space becomes a *differential equation* in the momentum space. Setting

$$\phi := \frac{k\mu_o + \pi}{2}, \quad h := \frac{4d^2}{\mu_o^2}, \quad \text{and} \quad b := h\cdot\frac{2E}{\hbar\omega}, \tag{3.2.5}$$

the equation simplifies to:

$$\frac{d^2\psi_0(\phi)}{d\phi^2} + \big(b - h^2\cos^2(\phi)\big)\,\psi_0(\phi) = 0\,. \tag{3.2.6}$$

This is precisely the well-studied Mathieu's equation. From basic theory of differential equations we conclude that (3.2.6) does admit solutions. However, since the Fourier transforms (3.2.3) of states in the position representation are necessarily periodic, the question is whether the solutions $\psi_0(\phi)$ are *periodic* (with period π, in terms of the variable ϕ, as is easily checked). If they are, we may take the inverse Fourier transform and recover a state $|\Psi_0\rangle \in \mathcal{H}^0_{\rm Poly}$; by Parseval's theorem this state must be normalizable. Thus, the question of whether $\psi_0^{(m)}$ have appropriate fall-off reduces to whether solutions $\psi_0(\phi)$ to Mathieu's equation are periodic.

We can now appeal to the general theory of ordinary differential equations with periodic coefficients—specifically, Floquet's theorem—to conclude that: i) there is a discrete infinity of periodic solutions with the required period π; and ii) each of the corresponding energy eigenvalues is non-degenerate. (See [50] for the general theory; [51] for application to Mathieu's equation. We also discuss this issue further in the next subsection, and [52] is a comprehensive reference on Mathieu's equation.) Let us denote the allowed eigenvalues by $E_{0,n}$ and the corresponding eigenstates in $\mathcal{H}^0_{\rm Poly}$ by $|\Psi_{0,n}\rangle = \sum \Psi^{(m)}_{0,n}|m\mu_o\rangle$. The question now is how these eigenvalues and eigenstates are related to those of the Schrödinger theory.

By examining our definition of parameters in (3.2.5), we see that the ratio of μ_o/d in which we are interested corresponds to very large values of h. We can then employ an asymptotic formula [51] for the b coefficients:

$$b \sim (2n+1)h - \frac{2n^2+2n+1}{4} + \mathcal{O}(h^{-1}). \tag{3.2.7}$$

By substituting this expansion back into our definition (3.2.5) of the b coefficients we obtain the following expansion for the energy eigenvalues $E_{0,n}$:

$$E_{0,n} \sim (2n+1)\frac{\hbar\omega}{2} - \frac{2n^2+2n+1}{16}\left(\frac{\mu_o}{d}\right)^2\frac{\hbar\omega}{2} + \mathcal{O}\left(\frac{\mu_o^4}{d^4}\right). \tag{3.2.8}$$

Thus, in the limit $\mu_o/d \to 0$, the $E_{0,n}$ reduce to the Schrödinger eigenvalues, but for the 'physical' nonzero value of μ_o/d, there is a correction introduced by the 'fundamental' discreteness. We see from this equation that the first correction to the eigenvalue is negative and of order μ_o^2/d^2. Using the very conservative value 10^{-19}m of μ_o, for a carbon monoxide molecule we conclude that these corrections are significant only when $n \approx 10^7$, i.e., when the vibrational energy of the oscillator is $\approx 10\,\text{MeV}$, or, in classical terms, the average vibrational velocity is $10^{14}\,\text{ms}^{-1}$. Thus, while the corrections are conceptually important, in the domain of validity of non-relativistic quantum mechanics they are too small to have been observed.

Next, let us compare the eigenstate $|\Psi_{0,n}\rangle$ with the shadow $|\Psi^{\rm shad}_{0,n}\rangle$ of the nth Schrödinger eigenstate on the graph α^0. Unfortunately, we cannot carry out this task analytically be-

cause closed form expressions for the Mathieu functions are not available. Therefore, let us calculate the norm of

$$|\Delta\Psi_{0,n}\rangle := |\Psi_{0,n}^{\text{shad}}\rangle - |\Psi_{0,n}\rangle \tag{3.2.9}$$

numerically and study its behavior as a function of n and μ_o/d.

This turns out to be computationally feasible for small values of n and relatively large μ_o/d. Let us begin with the ground state. Figure 3.1 shows a plot of this norm as a function of ℓ/d for the ground state, on a log-log scale. From the highly linear behavior we can infer that the norm follows a power law, at least over the range of μ_o/d studied, and from the least squares fit to the data in the figure we can estimate:

$$\langle \Delta\Psi_{0,0} \,|\, \Delta\Psi_{0,0} \rangle^{\frac{1}{2}} \sim 0.069 \left(\frac{\mu_o}{d}\right)^{1.10}. \tag{3.2.10}$$

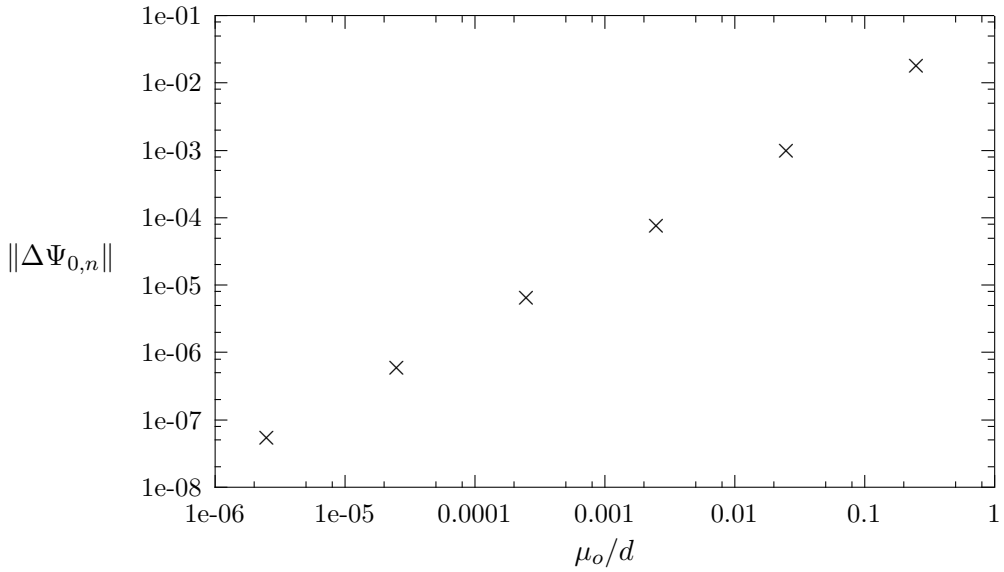

Figure 3.1: Plot of the norm of $|\Delta\Psi_{0,0}\rangle$ *vs.* the dimensionless ratio μ_o/d.

If we perform a similar analysis for the lowest few excited states, we will again see very linear behavior on a log-log plot and it is therefore natural to study the coefficients m_n and t_n of the least squares fits to these graphs. Thus, when m_n and t_n are defined as the best fit values such that

$$\frac{1}{2} \log \langle \Delta\Psi_{0,n} \,|\, \Delta\Psi_{0,n} \rangle \sim m_n \log (\mu_o/d) + t_n \tag{3.2.11}$$

then we observe the behavior that is shown in Figures 3.2 and 3.3.

We can see from these figures that m_n scales linearly with n, and t_n linearly with $\ln(n+1)$. When we calculate least squares fits to m_n as a function of n and t_n as a function

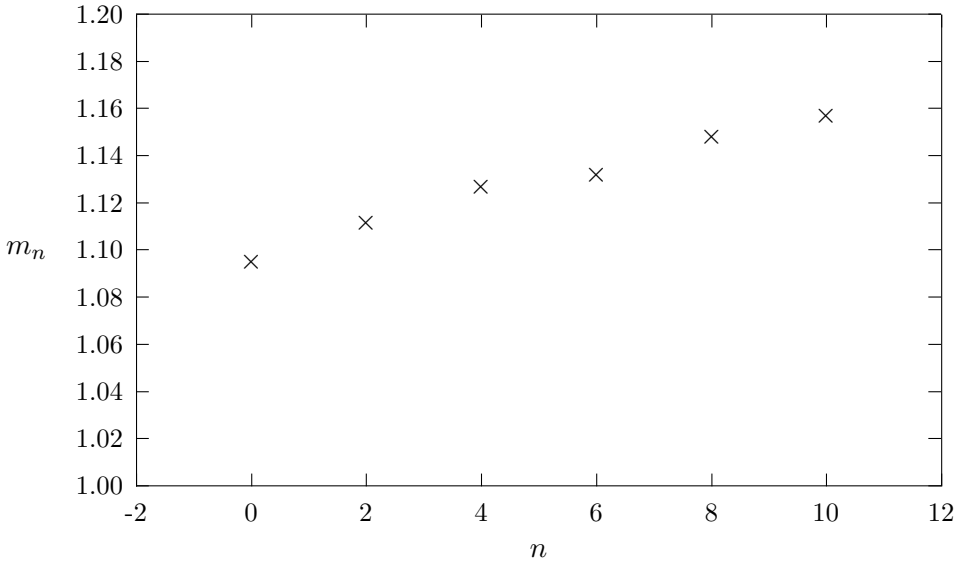

Figure 3.2: Plot of the coefficients m_n *vs.* energy level number n.

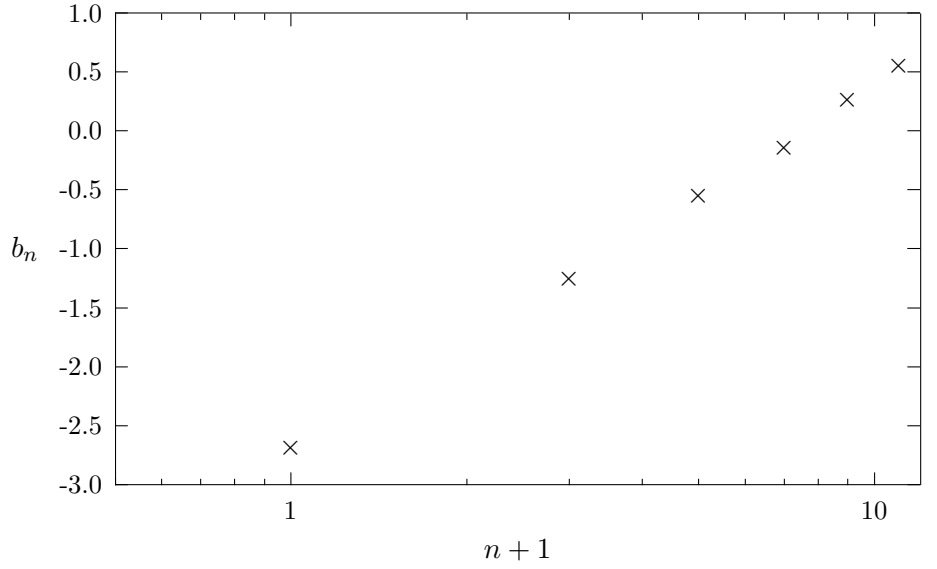

Figure 3.3: Plot of the coefficients t_n *vs.* energy level number $n+1$

of $\ln(n+1)$ we obtain

$$m_n \sim 0.0060\, n + 1.099 \tag{3.2.12}$$

$$t_n \sim 1.345\, \ln(n+1) - 2.692 \tag{3.2.13}$$

Since the coefficient of n in the fit for m_n is so small, we in fact assume that m_n is roughly constant with n. We can then infer that at least for $10^{-6} < (\mu_o/d) < 1$ and $n \leq 10$,

$$\langle \Delta\Psi_{0,n} \,|\, \Delta\Psi_{0,n} \rangle^{\frac{1}{2}} \sim (n+1)^{1.35} \left(\frac{\mu_o}{d}\right)^{1.10}. \tag{3.2.14}$$

These numerical results together with the analytic knowledge that the difference equation (3.2.1) with which we began is itself a standard discretization of Hermite's equation strongly suggest that for $n \ll 10^7$, the exact eigenstates are experimentally indistinguishable from the shadows of the Schrödinger eigenstates on the graph α^0. However, since this evidence is not as mathematically clear-cut as other results of this chapter, we will examine this issue from a different angle in sub-section 3.3.

3.2.2 General x_o

Let us now consider the energy eigenstates in the Hilbert space $\mathcal{H}^{x_o}$ for a general value of $x_o \in (0, \mu_0)$. Now, the momentum wave function is given by

$$\psi_{x_o}(k) := (k|\Psi_{x_o}\rangle = \sum_{m=-\infty}^{\infty} \Psi_{x_o}^{(m)}\, e^{-ikx_o}\, e^{-ikm\mu_o} \tag{3.2.15}$$

Thus the momentum space wave function $\psi_{x_o}(k)$ is no longer periodic in k but instead satisfies:

$$\psi_{x_o}\left(k + \frac{2\pi}{\mu_o}\right) = e^{-2\pi i x_o/\mu_o}\, \psi_{x_o}(k)\ . \tag{3.2.16}$$

The differential equation that an energy eigenstate must satisfy continues to be (3.2.4) and using the same definitions (3.2.5) of our parameters we are again led to Mathieu's equation (3.2.6). Thus, the only difference between the $x_o = 0$ and $x_o \neq 0$ cases lies in the boundary conditions that the solutions are to satisfy. Again, thanks to the very exhaustive literature available on Mathieu's equation [50–52], we conclude that: i) there is a discrete infinity of solutions satisfying (3.2.16); ii) each of the corresponding energy eigenvalues is non-degenerate; and iii) the eigenvalues are very close to those in the Schrödinger theory with corrections which become $O(1)$ only for energy levels corresponding to $n \approx 10^7$.

Let us elaborate a little on the reasoning behind these statements. In so doing we closely follow [52]. As we shall need a figure from that reference, we must introduce the notation

of that reference as well. It considers Mathieu's equation in the form

$$\psi'' + (a - 2q\cos(2\phi))\psi = 0 \tag{3.2.17}$$

so that the parameters a and q are related to our parameters b and h through:

$$a = b - \frac{1}{2}h^2, \qquad q = \frac{1}{4}h^2. \tag{3.2.18}$$

In order to fully explain the case of general x_o, we must back up somewhat and say a little more about the case $x_o = 0$. That case corresponds, in the terminology of [52], to solutions of Mathieu's equation of integral order. These are by definition solutions to Mathieu's equation with period either π or 2π. For a given value of q, there is, as we have mentioned, a discrete infinity of values of a for which (3.2.17) has a (normalizable) solution that is periodic with period π. In the limit $q \to 0$, the eigenvalues a tend toward m^2 for integral m and the corresponding solutions tend toward either $\sin(m\phi)$ or $\cos(m\phi)$. Reference [52] somewhat confusingly decides to call the corresponding eigenvalues of a either a_m if the eigenfunction is even, reducing to $\cos(m\phi)$ when $q = 0$, or b_m if the eigenfunction is odd and becomes $\sin(m\phi)$ whenever $q = 0$. These b_m should not be confused with b as we have defined it in equations (3.2.5) and (3.2.6).

While it is true that $a_m = b_m = m^2$ when $q = 0$, for $q > 0$ this no longer holds, and the a_m and b_m are distinct. Asymptotically, however, $a_m \approx b_{m+1}$ in the limit where $q \to \infty$. At first sight this might seem to imply a double degeneracy of eigenfunctions; however, it can be shown that only one of a_m, b_{m+1} corresponds to an eigenfunction periodic with period π in ϕ; the other has period 2π. So as we stated in the previous subsection, there is a one-to-one correspondence between the eigenvalues of the oscillator in the Schrödinger representation and the eigenvalues in the polymer particle representation, with the corresponding eigenvalues approaching one another as $\mu_o \to 0$. This is demonstrated graphically in figure 3.4

Thus, we have so far fleshed out somewhat the behavior of solutions to Mathieu's equation that are relevant when $x_o = 0$, but we have still not addressed the behavior of solutions for non-zero x_o. We have, however, laid the necessary background. As noted above, we require solutions that have the boundary condition (3.2.16). In terms of the form of Mathieu's equation as given in equation (3.2.17) and in terms of the variable ϕ, this means that we require

$$\psi(\phi + \pi) = e^{-2\pi i x_o/\mu_o}\psi(\phi) \tag{3.2.19}$$

where we remind the reader that x_o is constrained to lie in an interval of length μ_o.

Now, chapter 4 of [52] is concerned exclusively with the solution of Mathieu's equation

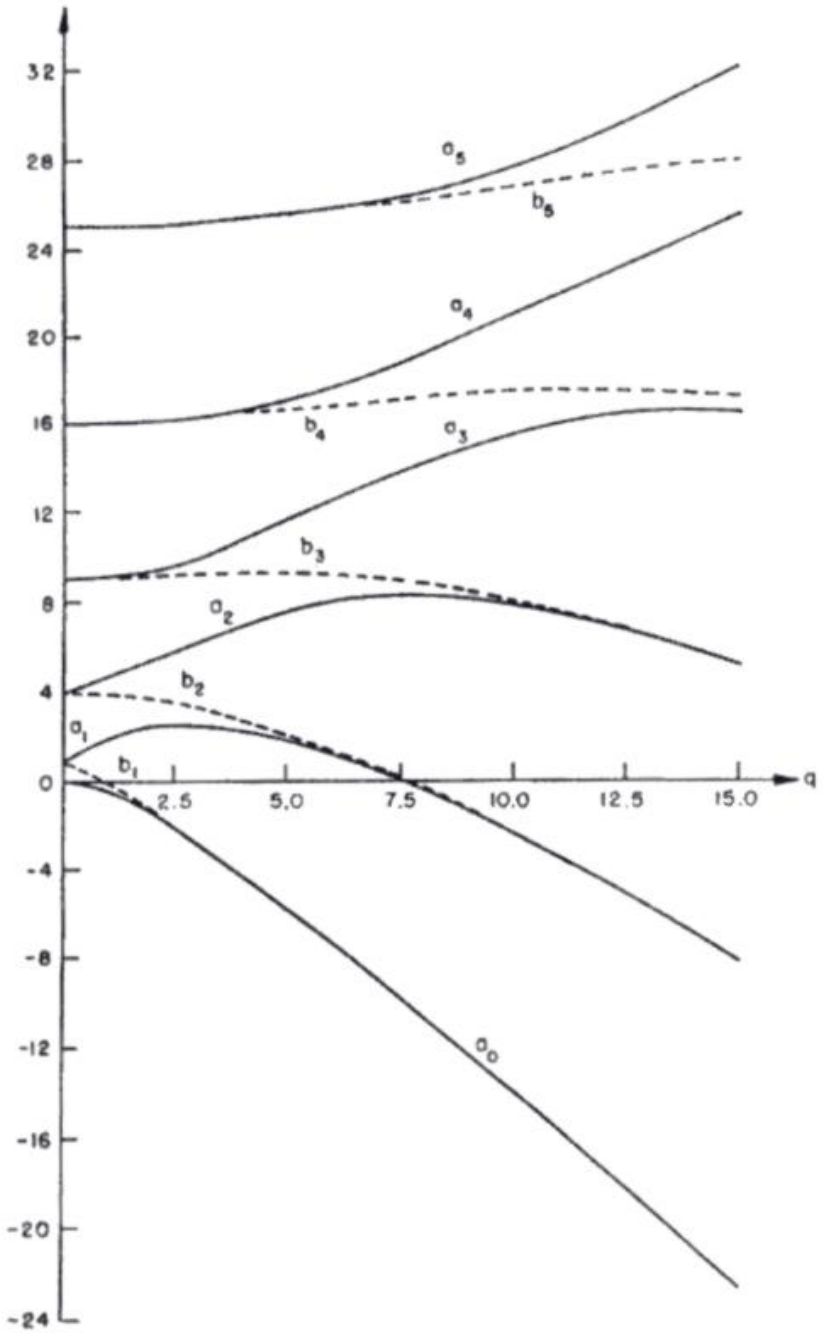

Figure 3.4: Characteristic values of a as a function of q for integral order solutions to Mathieu's equation (taken from [51], figure 20.1)

subject to the condition

$$\psi(\phi + \pi) = e^{\pi\lambda}\,\psi(\phi) \tag{3.2.20}$$

for some complex number λ. When λ has a nonzero real part, the resulting solution diverges at infinity, and hence cannot be normalizable. These are referred to as unstable solutions of the equation, and for certain points in the (a, q) plane they are the only solutions to the equation. However, it may be shown that there are always solutions (termed stable solutions) satisfying (3.2.20) with λ purely imaginary, in which case following the notation of [52] we define $\lambda = i\beta$. There are such solutions for each value of β between zero and one. Comparing to (3.2.19) we see that we must have $\beta = -2x_o/\mu_o$. We observe right away that as x_o ranges over an interval of length μ_o, β ranges over an interval of length two rather than one. We return to this subtlety in a moment, but for now focus on what happens as we allow β to range from zero to one, corresponding to x_o ranging from (say) zero to $-\mu_o/2$.

Then it is possible to show that there is a stable solution of Mathieu's equation subject to the condition (3.2.20) for each value of β between zero and one between the two curves in the a–q plane that correspond to a_m and b_{m+1}. This is shown graphically in figure 3.5. These curves of constant β never cross each other and it therefore follows that at any fixed q there is an eigenvalue corresponding to each value of x_o between zero and $-\mu_o/2$ that lies between $a_m(q)$ and $b_{m+1}(q)$. But we also know that as $q \to \infty$, $a_m(q)$ and $b_{m+1}(q)$ approach each other and are given by the formula (3.2.7) (when the latter is adjusted to the different form (3.2.17) of Mathieu's equation used in reference [52]). Hence it follows that for sufficiently large q—that is, sufficiently small μ_o as compared to d—that the eigenvalues for the non-zero x_o, though distinct, nonetheless approach each other and the corresponding Schrödinger eigenvalues.

What of values of x_o between $-\mu_o$ and $-\mu_0/2$? First observe what happens for $x_o = \mu_0/2$, corresponding to $\beta = 1$. Inspection of equation (3.2.20) shows that then we are simply looking for a solution that is periodic, but with period 2π rather than π; in other words, one of the integral solutions we have discussed already. The other values of x_o occur in the following manner. As a second order differential equation, Mathieu's equation for any given values of its parameters must always have exactly two independent solutions, and it can be shown that the other independent solution that occurs in conjunction with the solution satisfying (3.2.20) (for $\lambda = i\beta$) is a solution that satisfies

$$\psi(\phi + \pi) = e^{-i\pi\beta}\,\psi(\phi) \tag{3.2.21}$$

and therefore corresponds to a value of x_o in the interval $(-\mu_o, -\mu_0/2)$. For integral order solutions the second solution is never periodic, and so there also we have no degeneracy of the eigenvalues.

Thus, we have found that for each allowed value of x_o there is a non-degenerate eigenvalue

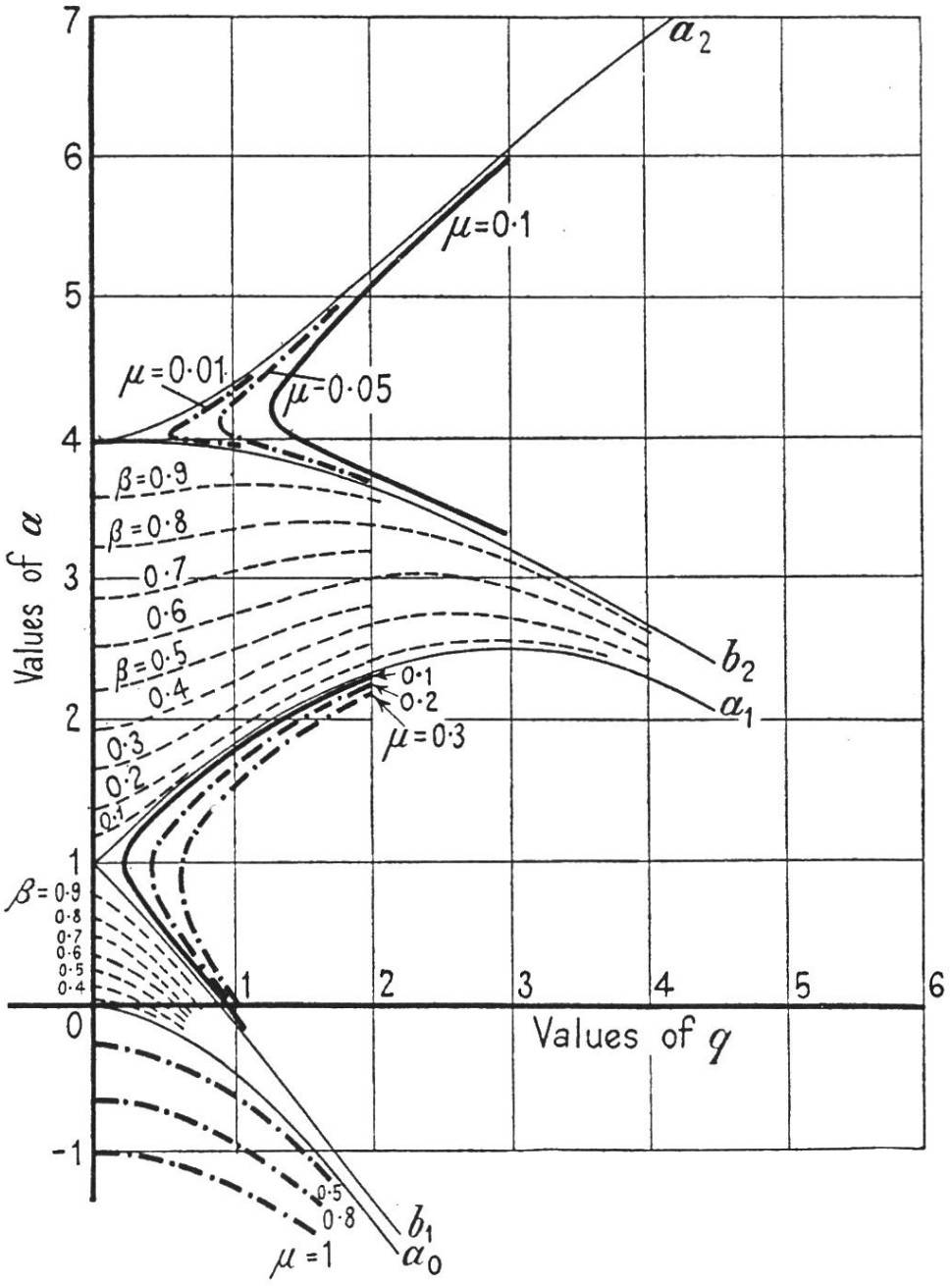

Figure 3.5: Characteristic values of a as a function of q for non-integral order solutions of Mathieu's equation (taken from [52], figure 11)

and corresponding normalizable eigenstate such that in the limit of large d/μ_o, the eigenvalue is indistinguishable from one of the eigenvalues of the simple harmonica oscillator in the Schrödinger representation. This is true for each eigenvalue of the Schrödinger oscillator.

To summarize, the full polymer Hilbert space $\mathcal{H}_{\rm Poly}$ can be decomposed in to orthogonal, separable subspaces $\mathcal{H}^{x_o}_{\rm Poly}$, each left invariant by the algebra of observables. The energy eigenvalue equation can therefore be solved on these subspaces independently. In all cases, there is a discrete infinity of eigenvalues; they are very close to the eigenvalues of the Schrödinger theory in its domain of validity; and each eigenvalue is non-degenerate. There is numerical evidence that the eigenstates in $\mathcal{H}^{x_o}_{\rm Poly}$ are very close to the shadows of the Schrödinger eigenstates (which naturally belong to Cyl*) on graphs α^{x_o}.

Let us close with several important observations:

1. Recall that for the kinetic energy term $\hat{H}_{\rm kin}$ in the Hamiltonian, we used the operator $\widehat{K^2_\ell}$ of (3.1.2) rather than the square $\hat{K}^2_{\mu_o}$ of the operator $\hat{K}_{\mu_o}$ of (2.4.20). Both alternatives appear to be viable from the classical standpoint. However, had we chosen $\hat{K}^2_{\mu_o}$ in place of $\widehat{K^2_\ell}$, we would have found a 2-fold degeneracy in the eigenvectors irrespective of how small n is because, in effect, the coefficients $\psi^{(m)}_{x_o}$ for even and odd m would have decoupled in the eigenvalue equation. Hence, from a quantum mechanical perspective, only the choice $\widehat{K^2_\ell}$ is experimentally viable. This situation is familiar from lattice gauge theories but brings out the fact that the polymer framework has to be set up rather delicately; small μ_o/d does *not* automatically ensure that the polymer results would be close to the continuum ones.

2. While all 'low lying' eigenvalues are very close to $\hbar\omega(n+\frac{1}{2})$, eigenvalues in different sectors $\mathcal{H}^{x_o}_{\rm Poly}$ differ from one another slightly. Suppose for a moment that the only limitation of Schrödinger quantum mechanics comes from the fact that it ignores the inherent discreteness implied by quantum geometry. Since the polymer particle model accounts for this discreteness, it would then be the 'fundamental' theory underlying Schrödinger quantum mechanics. Then, we would be led to conclude that the detailed energy levels of physical harmonic oscillators would be sensitive to the physical, underlying quantum geometry; i.e. depend on the graph which best captures the fundamental discreteness along the line of motion of the oscillator.

3. Our construction was motivated by the way the Hamiltonian constraint is treated in full general relativity. Note however, a qualitative difference. Solutions to the Hamiltonian constraints fail to belong to the polymer Hilbert space of quantum geometry because zero fails to be a discrete eigenvalue of those operators. In the case of a harmonic oscillator, by contrast, the full spectrum is discrete. Therefore, now the eigenvectors belong to the polymer Hilbert space and Cyl* is relevant only in making contact with the Schrödinger quantum mechanics. Had we considered a free particle instead, as in

the Schrödinger theory, the spectrum of the polymer Hamiltonian operator

$$\hat{H}_{\rm Free} = (\hbar^2/2m)\,\widehat{K^2_{\mu_o}}$$

would have been continuous. The eigenvectors would no longer be normalizable in $\mathcal{H}_{\rm Poly}$ but belong to Cyl*. For energies $E \ll \hbar^2/(2m\mu_o^2)$, they are practically indistinguishable from plane waves in the sense that their shadows on sufficiently refined regular graphs (with separation $\sim \mu_o$) are very close to those of plane waves. However, as one would expect, for higher energies, the 'fundamental description' introduces major corrections.

3.3 Shadows of Schrödinger eigenstates

We saw in section 3.2.1 that there is strong numerical evidence that for sufficiently small μ_o/d, the eigenstates of the fundamental polymer particle Hamiltonian differ only slightly from the shadows of the eigenstates of the continuum Hamiltonian. In this sub-section we will further explore that relationship further, by asking to what extent the shadows of the continuum eigenstates are eigenstates of the polymer Hamiltonian. We are thus in a sense asking the 'converse' question to that asked in section 3.2.1. For definiteness we restrict ourselves to the $x_o = 0$ case, i.e., to the Hilbert space $\mathcal{H}^0_{\rm Poly}$ associated with the graph α^0.

The shadows on α^0 of the Schrödinger eigenstates $(\Psi_n|$ are given by

$$|\Psi^{\rm shad}_{0,n}\rangle = c_n \sum_m H_n\Big(\frac{m\mu_o}{d}\Big)\, e^{-\frac{m^2{\mu_0}^2}{2d^2}}\, |m\mu_o\rangle. \tag{3.3.1}$$

where c_n is the normalization constant. The main result of this sub-section is that these shadows satisfy the eigenvalue equation of the polymer Hamiltonian (3.1.2) to an excellent degree of approximation if $n \ll 10^7$.

The action of the Hamiltonian (3.1.2) on an arbitrary state $|\Psi\rangle = \sum_m \psi(m)|m\mu_o\rangle$ can be easily calculated. The result is:

$$\hat{H}\,|\Psi\rangle = \frac{\hbar\omega d^2}{2{\mu_o}^2} \sum_m \left[\left(2 + \frac{m^2{\mu_o}^4}{d^4}\right)\psi(m) - \Big(\psi(m+1) + \psi(m-1)\Big)\right]\,|m\mu_o\rangle\,. \tag{3.3.2}$$

Let us begin with the shadow ground state $|\Psi^{\rm shad}_{0,0}\rangle$. We have:

$$\hat{H}\,|\Psi^{\rm shad}_{0,0}\rangle = \frac{\hbar\omega d^2}{2{\mu_o}^2}\, c_0 \sum_m e^{-\frac{m^2{\mu_o}^2}{2d^2}} \left[\left(2 + \frac{m^2{\mu_o}^4}{d^4}\right) - e^{-\frac{{\mu_o}^2}{2d^2}}\left(e^{-\frac{m{\mu_o}^2}{d^2}} + e^{\frac{m{\mu_o}^2}{d^2}}\right)\right]\,|m\mu_o\rangle\,. \tag{3.3.3}$$

To make the structure of the right side more transparent, let us expand the last three

exponentials and keep the lowest few terms:

$$\begin{aligned}\hat{H}\,|\Psi^{\rm shad}_{0,0}\rangle &= \frac{\hbar\omega d^2}{2{\mu_o}^2}\,c_0\sum_m e^{-\frac{m^2{\mu_o}^2}{2d^2}}\times \\ &\qquad\left[\left(2+\frac{m^2{\mu_o}^4}{d^4}\right)-\left(2-\frac{{\mu_o}^2}{d^2}+\frac{{\mu_o}^4}{4d^4}+\frac{m^2{\mu_o}^4}{d^4}-\frac{{\mu_o}^6}{24d^6}-\frac{m^2{\mu_o}^6}{2d^6}+\cdots\right)\right]|m\mu_o\rangle \\ &= \frac{\hbar\omega}{2}\,|\Psi^{\rm shad}_{0,\mu_o}\rangle+\frac{\hbar\omega}{2}\,\frac{c_0}{4}\sum_m e^{-\frac{m^2{\mu_o}^2}{2d^2}}\left[-\frac{{\mu_o}^2}{d^2}+\frac{{\mu_o}^4}{6d^4}+\frac{2m^2{\mu_o}^4}{d^4}-\cdots\right]|m\mu_o\rangle\,.\end{aligned} \tag{3.3.4}$$

Thus, we have

$$\hat{H}\,|\Psi^{\rm shad}_{0,0}\rangle = \frac{\hbar\omega}{2}\,\left[|\Psi^{\rm shad}_{0,0}\rangle+|\delta\Psi_{0,0}\rangle\right] \tag{3.3.5}$$

where $|\delta\Psi_0\rangle$ is $c_0/4$ times the last sum in (3.3.4). Since this 'remainder' proportional to ${\mu_o}^2/d^2$, in the series in square brackets only terms with large m make significant contributions and these terms are severely damped by the exponential multiplicative factor. Hence it is plausible that $\langle\delta\Psi_0\,|\,\delta\Psi_0\rangle \ll 1$, i.e., that the shadow state is very nearly an eigenstate of $\hat{H}$. We will first establish that the situation is similar for all excited states and then show that the expectation on smallness of the remainder term is correct *for all* eigenstates.

Let us then act on the shadow (3.3.1) of the n-th excited state with the Hamiltonian. We obtain:

$$\begin{aligned}\hat{H}\,|\Psi^{\rm shad}_{0,n}\rangle = \frac{\hbar\omega}{2}\left(\frac{\mu_o}{d}\right)^{-2}c_n\sum_m e^{-\frac{m^2\mu_o^2}{2d^2}}\,\left\{\left[2+m^2\left(\frac{\mu_o}{d}\right)^4\right]H_n\Big(\frac{m\mu_o}{d}\Big)\right. \\ \left.-\,e^{-\frac{\mu_o^2}{2d^2}}\left[e^{-\frac{m\mu_o^2}{d^2}}H_n\Big(\frac{m\mu_o}{d}+\frac{\mu_o}{d}\Big)+e^{\frac{m\mu_o^2}{d^2}}H_n\Big(\frac{m\mu_o}{d}-\frac{\mu_o}{d}\Big)\right]\right\}|m\mu_o\rangle\end{aligned} \tag{3.3.6}$$

This expression can be simplified using the basic recurrences satisfied by the Hermite polynomials and by expanding the exponentials using Taylor's theorem. As with the ground state, we can then conclude

$$\hat{H}\,|\Psi^{\rm shad}_{0,n}\rangle = \frac{2n+1}{2}\,\hbar\omega\,|\Psi^{\rm shad}_{0,n}\rangle+\frac{\hbar\omega}{2}\,|\delta\Psi_n\rangle. \tag{3.3.7}$$

where the state $|\delta\Psi_n\rangle$ is given by the following (rather unpleasant) expression:

$$\begin{aligned}|\delta\Psi_n\rangle = \left(\frac{\mu_o}{d}\right)^{-2}c_n\sum_m e^{-\frac{m^2\mu_o^2}{2d^2}}\,\left\{\left[2+m^2\left(\frac{\mu_o}{d}\right)^4-(2n+1)\left(\frac{\mu_o}{d}\right)^2\right]H_n\left(\frac{m\mu_o}{d}\right)\right. \\ \left.-e^{-\frac{\mu_o^2}{2d^2}}\left[e^{-\frac{m\mu_o^2}{d}}H_n\left(\frac{m\mu_o}{d}+\frac{\mu_0}{d}\right)+e^{\frac{m\mu_o^2}{d}}H_n\left(\frac{m\mu_o}{d}-\frac{\mu_o}{d}\right)\right]\right\}|m\mu_o\rangle\end{aligned} \tag{3.3.8}$$

We may fix the normalization constant c_n by requiring that the norm of $|\Psi^{\rm shad}_{0,n}\rangle$ be one,

at least in the limit where $\mu_o/d \to 0$. This then implies that

$$c_n = \left(\frac{\mu_o}{d} \frac{1}{\sqrt{\pi}\, 2^n n!} \right)^{\frac{1}{2}} \tag{3.3.9}$$

With this value of c_n, we then find that we may write:

$$\langle \delta\Psi_n \,|\, \delta\Psi_n \rangle = \left(\frac{\mu_o}{d} \right)^{-4} \sum_m f\left(\frac{m\mu_o}{d} \right) \frac{\mu_o}{d}. \tag{3.3.10}$$

We have not written out the exact expression for the function f, though it may readily be calculated from (3.3.8). Our main point is that it depends on m only in the combination $m\mu_o/d$. It is for this reason that we have pulled an additional factor μ_o/d inside the sum in (3.3.10), for then we see that the sum (ignoring the prefactor of $(\mu_o/d)^{-4}$) is a Riemann sum for the integral $\int f(x)\, dx$. Since we have an explicit expression for for f, we can perform this integral and find the limiting value (as $\mu_o/d \to 0$) of the sum.

Of course, the expression for the norm of $|\delta\Psi_n\rangle$ *does* contain the prefactor of $(\mu_o/d)^{-4}$, and that obviously diverges as $\mu_o/d \to 0$. Thus, to conclude that the norm of $|\delta\Psi_n\rangle$ is small in this limit it must happen that $\int f(x)\, dx$ has a multiplicative factor of $(\mu_o/d)^n$ with $n > 4$. This is a delicate condition, and it is here that we are really putting to the test the hope that the shadows of Schrödinger eigenstates are themselves very closely eigenstates of the exact polymer Hamiltonian.

Remarkably, this condition is satisfied and so our hope is met. Specifically, we find that:

$$\langle \delta\Psi_{n,\mu_o} \,|\, \delta\Psi_{n,\mu_o} \rangle^{\frac{1}{2}} \;=\; \frac{\sqrt{35}}{48} \left(2n^4 + 4n^3 + 10n^2 + 8n + 3 \right)^{\frac{1}{2}} \left(\frac{\mu_o}{d} \right)^2 + \mathcal{O}\left(n^3 \left(\frac{\mu_o}{d} \right)^4 \right). \tag{3.3.11}$$

We see immediately that $\langle \delta\Psi_n \,|\, \delta\Psi_n \rangle^{1/2}$ approaches zero if we let μ_o/d approach zero. For finite μ_o/d, its value depends on n and, as one would expect on physical grounds, is of order unity when $n \sim d/\mu_o$. In the case of the molecular vibrations of carbon monoxide considered above, this corresponds to $n \sim 10^7$. It is obvious that (among other things) the approximation that $V(x)$ can be described by the simple harmonic oscillator potential will break down long before this energy level n is reached.

For the bound (3.3.11) to be useful, we must know when the $\mathcal{O}(n^3\,(\mu_o/d)^4)$ term is negligible. This can easily be investigated numerically, again for moderate values of n and μ_o/d.

We see in figure 3.6 a representative plot in which n is held fixed (here at $n = 0$) and μ_o/d is allowed to vary. Conversely, in figure 3.7 we hold μ_o/d fixed at 0.001 and allow n to vary. In each case we see that the asymptotic behavior (3.3.11) is attained almost as soon as $\mu_o/d < 1$. To give some examples, one finds that for the ground state, even when μ_o/d

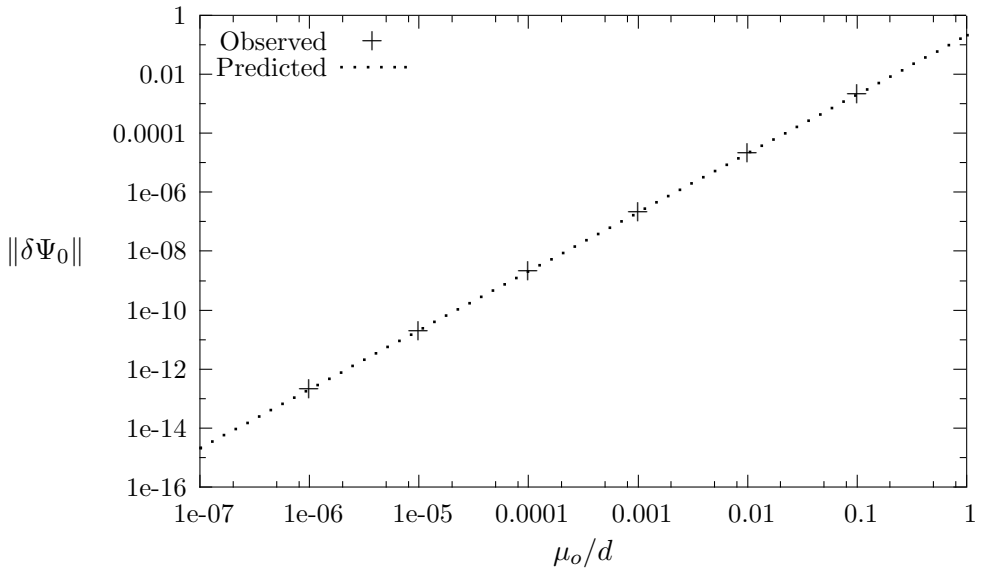

Figure 3.6: Norm of $|\delta\Psi_0\rangle$ *vs.* μ_o/d

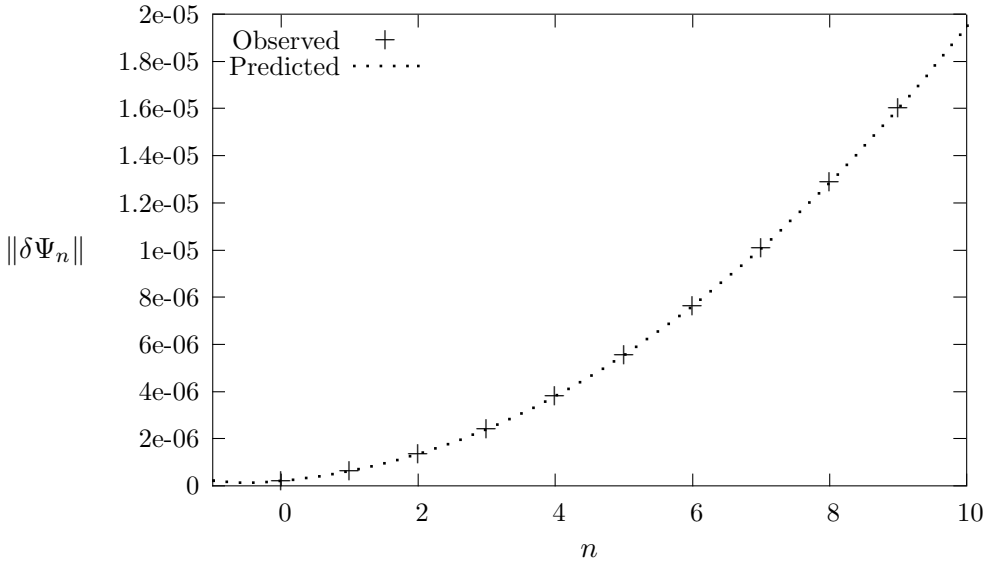

Figure 3.7: Norm of $|\delta\Psi_n\rangle$ *vs.* n for $\mu_o/d = 0.001$

is as large as 0.1, equation (3.3.11) is accurate to less than a percent and the magnitude of the norm of $|\delta\Psi_n\rangle$ is about 2.2×10^{-3}. For $n = 9$ and $\mu_o/d = 10^{-3}$, equation (3.3.11) is accurate to one part in 3×10^{-5} when $\mu_o/d = 10^{-3}$, and the magnitude of the norm of $|\delta\Psi_n\rangle$ is 2.13×10^{-13}. Thus, not only does the norm of $|\delta\Psi_n\rangle$ approach zero as μ_o/d approaches zero, but it also quickly approaches the asymptotic behavior of equation (3.3.11).

To summarize, we have shown that the shadows of the Schrödinger energy eigenstates on the graph α^0 are eigenstates of the polymer Hamiltonian $\hat{H}$ to a high degree of approximation at 'low' energies. Quantum geometry effects manifest themselves only at energy levels as high as $n \sim 10^7$, i.e., long beyond the validity of non-relativistic approximation. This result complements our findings in Section 3.2 where we compared the exact eigenstates of the polymer Hamiltonian with the shadows of the Schrödinger Hamiltonian.

3.4 Time evolution of states

We have titled this chapter 'Dynamics of the Polymer Particle,' and yet thus far we have not actually discussed the time evolution of any states. As we mentioned at the close of the last chapter, this is in general a quite difficult question, as it does not seem easy to evolve the polymer particle states in time exactly in closed form.

However, having found that the shadows of Schrödinger eigenstates are close to the exact polymer eigenstates, and having found that the corresponding eigenvalues of the two representations are close for low enough quantum number n, we might expect that we can draw some conclusions about time evolution. After all, the eigenstates are stationary states, and any state in the Hilbert space may be expanded in terms of them. Thus, if both states and eigenvalues are close to each other, we might hope that the time evolutions would also agree, at least for some length of time.

Let us make this more precise. What we would like to see is whether it is true that if we take a certain state $|\Psi\rangle \in L^2(\mathbb{R})$ and evolve it under the Schrödinger time evolution, and then take its shadow state $|\Psi^{\rm shad}(t)\rangle$, the result is close to what we obtain if we first take the shadow state $|\Psi^{\rm shad}\rangle$ and evolve that under the polymer particle evolution. For definiteness, choose $x_o = 0$, and consider the shadow $|\Psi^{\rm shad}_{0,0}\rangle$ of the ground state of the Schrödinger Hamiltonian. Under the unitary time evolution operator $\hat{U}_{\rm Schr}(t) = \exp(-it\hat{H}_{\rm Schr}/\hbar)$ this state is stationary, so we obtain:

$$\hat{U}_{\rm Schr}(t)|\Psi^{\rm shad}_{0,0}(t)\rangle = e^{-iE_0 t/\hbar}|\Psi^{\rm shad}_{0,0}\rangle \tag{3.4.1}$$

Here we use $E_n = (2n+1)\hbar\omega/$ to denote the eigenvalues of the Schrödinger Hamiltonian. We denote the eigenvalues of the polymer Hamiltonian by E'_n, to distinguish them from these; the E'_n are given by equation (3.2.8), where we referred to them as $E_{0,n}$.

How does this compare to the evolution of the shadow state in the Schrödinger representation? We found that we could relate the polymer eigenstates to the shadows of Schrödinger eigenstates through

$$|\Psi_{0,n}\rangle = |\Psi^{\rm shad}_{0,n}\rangle - |\Delta\Psi_{0,n}\rangle \tag{3.4.2}$$

where the norm of $|\Delta\Psi_{0,n}\rangle$ is small compared to the norm of $|\Psi_{0,n}\rangle$. For convenience we fix the norm of each $|\Psi_{0,n}\rangle$ to be one. Then we have:

$$\begin{aligned}
\hat{U}_{\rm Poly}(t)|\Psi^{\rm shad}_{0,0}\rangle &= \hat{U}_{\rm Poly}(t)\Big(|\Psi_{0,0}\rangle + |\Delta\Psi_{0,0}\rangle\Big) \\
&= e^{-iE'_0t/\hbar}|\Psi_{0,0}\rangle + \hat{U}_{\rm Poly}(t)|\Delta\Psi_{0,0}\rangle \\
&= e^{-iE'_0t/\hbar}\left(|\Psi^{\rm shad}_{0,0}\rangle - |\Delta\Psi_{0,0}\rangle\right) + \hat{U}_{\rm Poly}(t)|\Delta\Psi_{0,0}\rangle \\
&= e^{-iE'_0t/\hbar}|\Psi^{\rm shad}_{0,0}\rangle + \left(\hat{U}_{\rm Poly}(t) - e^{-iE'_0t/\hbar}\right)|\Delta\Psi_{0,0}\rangle.
\end{aligned} \tag{3.4.3}$$

Equations (3.4.1) and (3.4.3) allow us to calculate the difference in the evolution of the states that we are looking for. We have:

$$\begin{aligned}
\left(\hat{U}_{\rm Poly}(t) - \hat{U}_{\rm Schr}(t)\right)|\Psi^{\rm shad}_{0,0}\rangle \quad &= \quad 2\left(e^{-iE'_0t/\hbar} - e^{-iE_0t/\hbar}\right)|\Psi^{\rm shad}_{0,0}\rangle \\
&\quad + \left(\hat{U}_{\rm Poly}(t) - e^{-iE'_0t/\hbar}\right)|\Delta\Psi_{0,0}\rangle.
\end{aligned} \tag{3.4.4}$$

Thus, we see there are two contributions to the difference in evolution. The second term on the right of (3.4.4) may be bounded by a term that is always of small norm. The first term, however, has small norm only so long as $E_0t \sim E'_0t$, which in turn implies (from (3.2.8)) that $t \ll 32(d/\mu_o)^2/\omega$. For our continuing example of the carbon monoxide molecule, and with the conservative value of $\mu_o = 10^{-19}\,\mathrm{m}$, this gives $t \ll 1\,\mathrm{s}$. While this may seem like a very short time, note that it corresponds to 10^{15} oscillations of the system; for a molecule, one second is practically an eternity. Nonetheless, such differences might be observable experimentally and it would perhaps be interesting to look for them. Moreover, if we adopt not the largest allowed value of μ_o but instead the Planck scale, then we find that (3.4.4) requires $t \ll 10^{32}\,\mathrm{s}$, which is roughly 10^{15} times the present age of the universe. Thus, the time evolution of the polymer ground state will be indistinguishable from its evolution under the Schrödinger time evolution for long times scales, provided we choose the fundamental length scale μ_o sufficiently small.

What about states that are truly semi-classical? Consider for instance a macroscopic oscillator, for instance a mass on a spring. Then reasonable 'laboratory' values correspond to taking $m \sim 1\,\mathrm{kg}$ and $\omega \sim 1\,\mathrm{Hz}$. Note that then the 'fundamental' length scale of this oscillator is of order $10^{-17}\,\mathrm{m}$. We should certainly *not* take the state of the system to be the ground state of this system, since that would imply that we have localized our macroscopic

mass to much less than the size of a proton. Instead, we should choose a semi-classical shadow state corresponding to a Gaussian with uncertainty in x that is much larger, say 10^{-7} m, roughly the wavelength of visible light. We must then expand this wavefunction in terms of the eigenstates of our simple harmonic oscillator. Unfortunately, though it is possible to find this expansion analytically, it is not easy to examine the time evolution in a straightforward fashion, since there are many states of high quantum number contributing to the sum.

Instead, we shall perform such an analysis for a different Hamiltonian: the free particle Hamiltonian. It is well known [53] that in the Schrödinger representation coherent state wavefunctions as we are considering spread out over a time scale of order

$$t_c \sim \frac{md^2}{\hbar}. \tag{3.4.5}$$

For instance, if we choose a particle of mass 10^{-3} kg and a value of d corresponding to 10^{-7} m (roughly the wavelength of visible light), then (3.4.5) implies that the state is sharply peaked in both momentum and position for a timescale of order 10^{10} years, roughly the present age of the universe. We would like to verify that a similar result holds in the polymer particle representation.

The Hamiltonian we are considering is simply $(\hbar^2/2m)\,\widehat{K^2_{\mu_o}}$, and we first would like to identify the eigenstates and eigenvalues of this Hamiltonian. As we note in a remark above, the spectrum of this Hamiltonian will be continuous, and so its eigenstates will not be normalizable. The eigenstates and eigenvalues are again most easily found in the momentum representation. Then the eigenvalue equation corresponding to (3.2.4) becomes:

$$\left(E_\lambda + \frac{\hbar^2}{m\mu_o^2}\left[\cos\left(k\mu_o\right) - 1\right]\right)\psi_\lambda(k) = 0 \tag{3.4.6}$$

and indeed we see that it has no normalizable, non-zero solutions. However, we also see that $\delta(k - k_\lambda)$ will be a solution, for each $k_\lambda \in (-\frac{\pi}{\mu_o}, \frac{\pi}{\mu_o})$, provided the coefficient vanishes at $k = k_\lambda$. Thus, the eigenstates are $\delta(k - k_\lambda)$, and the corresponding eigenvalues are

$$E_\lambda = \frac{\hbar^2}{m\mu_0^2}\left[1 - \cos\left(k_\lambda\mu_o\right)\right]. \tag{3.4.7}$$

In the limit where k_λ is small compared to π/μ_o, this result agrees with the usual formula of $\hbar^2 k_\lambda^2/2m$, but as $|k_\lambda| \to \pi/\mu_o$, the discrepancy in eigenvalues approaches 60 %.

Now we must expand our coherent state in the basis of eigenstates of the Hamiltonian; this just amounts to finding the Fourier transform of our general coherent state peaked at

$(x_o = \mu_o N, k_o)$, where N is some integer. Then, again using Poisson re-summation we find:

$$\begin{aligned}\psi_{(N\mu_o,k_o)}(k) &:= \sum_{n=-\infty}^{\infty} \Psi_{(N,k_o)}(n) e^{-ikn\mu_o} \\ &= \frac{d}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} e^{-\frac{(n-N)^2\mu_o^2}{2d^2}} e^{ik_o(n-N)\mu_o} e^{-ikn\mu_o} \\ &= \frac{d}{\sqrt{\pi}} \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{(y-N)^2\mu_o^2}{2d^2}} e^{ik_o(y-N)\mu_o} E^{-ik\mu_o y} e^{-2\pi i l y} dy \\ &= \sqrt{2} e^{-ik\mu_o N} e^{-\frac{(k-k_o)^2 d^2}{2}} + \mathcal{O}\left(e^{-\frac{\pi^2 d^2}{2\mu_o^2}}\right).\end{aligned} \tag{3.4.8}$$

In reaching the last equality, we have had to assume that $k_o \ll \pi/\mu_o$, since only then can we conclude that the dominant term in the Poisson re-summation is the $l = 0$ term. We also note that the formula (3.4.8) is, in its leading term, *exactly* the same as the Fourier transform[1] of the Schrödinger coherent state.

We can now trivially time evolve the states in the momentum representation, and Fourier transform back to the position representation to find that (dropping terms of order $e^{-\pi^2 d^2/2\mu_o^2}$):

$$\Psi_{(N\mu_o,k_o)}(n,t) = \frac{1}{\sqrt{2}\pi} \int_{-\pi/\mu_o}^{\pi/\mu_o} e^{ik(n-N)\mu_o} e^{-\frac{(k-k_o)^2 d^2}{2}} e^{-\frac{i\hbar t}{m\mu_o^2}(1-\cos(k\mu_o))} dk \tag{3.4.9}$$

whereas the evolution of the same state in the Schrödinger representation is:

$$\Psi_{(x_o,k_o)}(x,t) = \frac{1}{\sqrt{2}\pi} \int_{-\infty}^{\infty} e^{ik(x-x_o)} e^{-\frac{(k-k_o)^2 d^2}{2}} e^{-\frac{i\hbar k^2 t}{2m}} dk. \tag{3.4.10}$$

Because $\mu_o \ll d$, the Gaussian term in the integrand of (3.4.9) will be suppressed by an order of $e^{-\pi^2 d^2/2\mu_o^2}$ at the limits, and so we may to that order of accuracy extend the limits of integration to $(-\infty, \infty)$. Thus, we see that the two evolutions will agree for so long as the difference in their phases is negligible for the k dominating the integral. Hence we conclude that the evolutions agree so long as

$$t \ll \frac{md^2}{\hbar} \left(\frac{d}{\mu_o}\right)^2. \tag{3.4.11}$$

Thus, comparing with (3.4.5), we see that the two evolutions agree for much longer than the coherence time scale; in particular, for our earlier example of a 1 kg mass with $d = 10^{-7}$ m, we find that the evolutions agree for roughly 10^{30} years, even for the very conservative value

[1] Provided we define the Fourier transform as $\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$, so that the inverse Fourier transform is given by $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$.

of $\mu_o = 10^{-19}\,\mathrm{m}$.

Hence, we find that not just eigenstates and eigenvalues, but the time evolutions as well of the polymer particle and Schrödinger representations agree, for time scales long compared to the typical time scale of the quantum system under consideration. In particular, for the free particle evolution we have shown that semi-classical states for macroscopic objects remain semi-classical and indistinguishable from the corresponding states in standard quantum mechanics, for at least as long as those states remain semi-classical under the Schrödinger time evolution.

3.5 Discussion

In the previous chapter we demonstrated the utility of shadow states in probing the kinematics of the low-energy limit of the polymer particle, showing how we could extract coherent states entirely from within the polymer particle representation, and then study their expectation values and fluctuations using shadow states find them to be kinematically indistinguishable from the Schrödinger coherent states. In this chapter we have extended our analysis of the relationship between the two theories questions centered on their dynamics. In particular, we found that we could introduce the Hamiltonian operators in the polymer framework and show that their eigenvalues and eigenfunctions are indistinguishable from those of the continuum, Schrödinger theory within its domain of validity. Taken together with the results of the previous chapter, this implies that logically one can forgo the continuum theory, work entirely with the polymer description, and compare the theoretical predictions with experimental results. However, since we already know that Schrödinger quantum mechanics reproduces the experimental results within its domain of validity, it is simpler in practice to verify agreement with the Schrödinger results. As one might physically expect, since the polymer framework 'knows' about the underlying discreteness, it predicts corrections to the Schrödinger framework which become significant when the energies involved are sufficiently high to probe that discreteness.

The broad strategy we followed, including the use of shadow states, was already outlined in the general program [17]. Notions needed in this analysis are all available in field theories as well as full quantum gravity. The details of the polymer particle toy model provide concrete hints for these more complicated theories. The study of dynamics illustrates further that it is nontrivial to reproduce the results of the continuum, even when working with a very refined lattice. For instance, naively, one would have used the operator $(\hbar^2/2m)\hat{K}^2$ as the kinetic part of the quantum Hamiltonian. However, this choice would have given a two-fold degeneracy for all eigenstates of the polymer Hamiltonian including the 'low energy' ones, while in the Schrödinger theory, all eigenstates are non-degenerate. This is a concrete illustration of how the requirement that the theory should reproduce predictions

of well-established theories in the *low energy regime* can be used to discriminate between choices available in the construction of the 'fundamental' framework. A second example arises from a cursory examination of the form of the polymer particle Hamiltonian. While the potential continues to be unbounded as in the Schrödinger theory, the kinetic part of the Hamiltonian is now bounded. Therefore, at first, it seems that the kinetic term will not be able to 'catch up' for large x to produce normalizable solutions to the eigenvalue equation $((\hbar^2/2m)\widehat{K_\ell^2} + x^2)|\Psi\rangle = E|\Psi\rangle$. Furthermore, this expectation can be 'confirmed' by numerical solutions to the difference eigenvalue equation (3.2.2). However the careful examination of section 3.2, involving the Mathieu equation in the momentum representation, showed that the expectation is incorrect and the divergence of $|\Psi\rangle$ one encounters in computer calculations is just a numerical artifact. Thus, our analysis provides useful guidelines for more realistic theories, pointing out potential pitfalls where care is needed and suggesting technical strategies.

Finally, there are also some conceptual lessons. First, we saw concretely that recovery of semi-classical physics entails two things: isolation of suitable states *and* a suitable coarse graining. In the toy model, the coarse graining scale was set by our tolerance d and continuum physics emerges only when we coarse grain on this scale. A second lesson is that the availability of a scalar product on the space of physical states is not essential at least for semi-classical considerations: The framework of shadow states—with its Wilsonian overtones—provides an effective strategy to recover low energy physics. This will be further borne out in the next chapter, where we shall see how these same ideas and constructions can be applied to loop quantum cosmology.

Chapter 4

Quantum corrections to Friedmann's equations from loop quantum cosmology

In the previous two chapters we have found that even as simple a system as a quantized non-relativistic point particle can serve as a useful model of as complex a system as quantum general relativity. For its purposes as a model, essentially all of the likely benefit has been examined already, and the next step in understanding the low energy limit of loop quantum gravity is to move on to more complex systems that mirror more features of general relativity. There is indeed work in progress on such models, presently a quantized version of Maxwell theory that parallels the quantization in loop quantum gravity, as well as an examination of linearized general relativity.

Remarkably, however, the polymer particle model is useful not simply as a toy model of more complicated systems. There is a regime of quantum general relativity whose dynamics seems to reduce to that of the polymer particle, and thus a study of this model can tell us something about aspects of quantum gravity that may have direct physical relevance.

This occurs in the context of symmetry reduced models, specifically in loop quantum cosmology. That is a program pioneered by Bojowald to study loop quantum gravity after imposing symmetries at the quantum level. Classically, such an approach is well-known: most of the solutions to general relativity that are known are solutions with a high degree of symmetry. In particular the usual Friedmann-Robertson-Walker universes are solutions with a high degree of symmetry, so much so that the number of degrees of freedom becomes *finite*. Thus, one would expect that on quantization such a symmetric model would, in some sense, correspond to quantum mechanics. Indeed, this approach has been studied for many years, and leads to the concept of a midi-superspace model, in which one first reduces the degrees of freedom of general relativity classically, and then quantizes the resulting model. What is unique about loop quantum cosmology, however, is that the symmetry reduction is performed at the quantum level, through the concept of symmetric states. Moreover, it is consciously patterned after loop quantum gravity, to mimic the procedures employed there.

What is important for our purposes is that the particular form of quantum mechanics encountered in loop quantum cosmology is not the conventional Schrödinger representation, but rather the polymer particle representation we have studied in earlier chapters. Therefore

in particular one expects that the shadow state framework may be able to tell us something about the low energy limit of loop quantum cosmology.

That approach was already taken as a part of the work in [20]. The work in this present chapter extends it in several directions. We shall comment in some detail on these as we progress, but we briefly summarize the main extensions here:

- In [20], only the lowest order of the expectation value of the Hamiltonian constraint was explicitly considered; here we find that expectation value to all orders.

- It was stated in [20] that one expects the series obtained for the expectation value of the Hamiltonian constraint to be asymptotic, but this was not proven. In this work that is rigorously shown.

- Only the Hamiltonian constraint was considered in [20]; the expectation values of the time derivatives of the canonical variables and the need to go beyond Erhenfest's theorem in order to establish semi-classicality was not addressed. Here we outline in section 4.1.3 a general framework for effective classical dynamics (taken largely from [54]) that does include such considerations, and we apply that framework to loop quantum cosmology.

- Finally, the Hamiltonian constraint operator used in [20] was that generally used in loop quantum cosmology. However, that operator is not symmetric, and its expectation value is not real. Here we work with a symmetrized version of that operator, to obtain a real expectation value. The particular choice of operator used is dictated by the requirement that one still avoid the singularity in loop quantum cosmology, and this in turn has implications for the Hamiltonian constraint in full quantum gravity.

The outline of this chapter is as follows. In the first section we review basic material we shall need about loop quantum cosmology. In particular we review the connections to the polymer particle representation, and also the Hamiltonian formulation of cosmology in general, since the Hamiltonian framework is not typically what one sees in textbook treatments of cosmological models. We then close our overview by defining what we mean by an effective classical dynamics for a quantum system, and when we might expect such a thing to exist. Then in the following section we take up the corrections to Friedmann's equations. First we calculate the expectation values of the quantum constraint and the time derivatives of the canonical variables, giving asymptotic series for all of these. In the following subsection we combine these results with the considerations of section 4.1.3 to derive modifications to Friedmann's equations for a dust-filled FRW universe. Since we shall find there are many interesting questions still to be addressed, we close this chapter with a discussion of the outlook for this approach.

4.1 Overview

In this section we review the background material needed for an understanding of the main results of this chapter. This material is divided into three parts: a review of loop quantum cosmology as a whole, a summary of Hamiltonian cosmology and the Hamiltonian constraint in loop quantum cosmology, and a general overview of effective classical dynamics for a quantum system. In the middle subsection we shall also discuss the need to adopt a different form of the Hamiltonian constraint than that ordinarily employed in loop quantum cosmology.

4.1.1 Review of loop quantum cosmology

Let us begin by reviewing the central ideas of loop quantum cosmology. In so doing we shall closely follow [20], to which we refer the reader for more details. We shall be particularly interested in making clear the connection between loop quantum cosmology and the polymer particle model considered in chapters 2 and 3.

As we have already mentioned in section 2.2, loop quantum gravity is built on a classical phase space in which the basic canonical variables are a connection A_a^i and an electric field E_i^a. When one passes to the quantum theory, certain functions of these basic variables become fundamental operators of the theory, in terms of which all other operators are constructed. These basic operators are the holonomies of the connection along curves, and the flux of the electric fields across two-surfaces. How does one introduce a symmetry reduction into this framework? Clearly we must first examine what it means for a connection and an electric field to be symmetric, and then we must examine the implications of this for our basic observables.

We shall in this chapter, just like reference [20], restrict ourselves to the case of a spatially flat, homogeneous, and isotropic universe, all though more general cases can be considered within loop quantum cosmology. Now, in the metric based approach to general relativity as exemplified, for instance, in [55], one typically defines symmetric solutions as those possessing a certain group as their isometry group. In other words, for some symmetry group it must be the case that every action on the manifold by that symmetry group carries the metric into itself. When we work with connections and electric fields, however, we do not want to require that they be exactly invariant under the symmetry group, only that every element of the symmetry group carries the connection and electric field variables into connection and electric field variables that are related to those with which we started by a gauge transformation. That is, if $\mathcal{S}$ is our symmetry group, if for every $s \in \mathcal{S}$ we have

$$(s^*A, s^*E) = (g^{-1}Ag + g^{-1}dg, g^{-1}Eg) \tag{4.1.1}$$

for some local gauge transformation g on the spatial manifold M, then we will say that the pair (A, E) is symmetric, with respect to $\mathcal{S}$. For our purposes in this chapter (since we focus on spatially flat, homogeneous, and isotropic cosmologies) we are assuming that $\mathcal{S}$ is the group of Euclidean motions of the plane, and that it acts simply and transitively on M.

This defines for us a certain subset of the phase space of full general relativity; the space of all symmetric solutions for a certain symmetry group, in our case the Euclidean group. For further progress we need a more concrete description of this space than that afforded by equation (4.1.1). This is easy to obtain. The Euclidean group of motions selects on M an equivalence class of positive definite, flat metrics, all related to one another by a multiplicative constant. Choose some particular element ${}^{o}q_{ab}$ in this class as a fiducial flat metric; we shall take care that no physical quantities depend upon which metric we choose. We also pick a constant orthonormal triad ${}^{o}e^{a}_{i}$ and its dual co-triad ${}^{o}\omega^{i}_{a}$, both compatible with ${}^{o}q_{ab}$. Finally, for reasons that we shall explain momentarily, we also pick within M a fiducial 'cell' $\mathcal{V}$ which for simplicity we assume to be cubical with respect to ${}^{o}q_{ab}$. Let V_o be its volume with respect to ${}^{o}q_{ab}$. Then it is possible to show that every symmetric pair (A', E') that satisfies the gauge and diffeomorphism constraints is equivalent to a pair (A, E) of the form:

$$A_a = c\, V_o^{-\frac{1}{3}}\, {}^{o}\omega^{i}_{a}\tau_i, \qquad E^a = p\, V_o^{-\frac{2}{3}} \sqrt{{}^{o}q}\, {}^{o}e^{a}_{i}\tau^i. \tag{4.1.2}$$

In these equations the matrices $\{\tau_i\}$ form a representation of SU(2) and are normalized so that $[\tau_i, \tau_j] = \epsilon_{ijk}\tau^k$; that is, $\tau_i = \frac{1}{2i}\sigma_i$ in terms of the Pauli matrices σ_i. We have also absorbed the density weight of E into the determinant ${}^{o}q$ of the fiducial metric.

We have thus solved the Gauss and diffeomorphism constraints of Hamiltonian general relativity, and as a consequence we are left with only two degrees of freedom, parameterized by the variables c and p. That is, the phase space of our theory is two dimensional: all physical predictions of the theory may be cast in terms of the variables c and p. In fact, it was partly to achieve this that we introduced the cell $\mathcal{V}$. The first reason for doing so, however, is that the symplectic structure of the full theory is defined as an integral of various fields constructed from A and E over M; because of the homogeneity this integral will diverge for our case. By introducing the cell $\mathcal{V}$ we may take the needed integral over this cell, and one may then show that the symplectic structure of our reduced phase space is given by

$$\Omega = \frac{3}{\kappa\gamma}\, dc \wedge dp. \tag{4.1.3}$$

In this equation we have defined $\kappa = 8\pi G$, where G is Newton's constant. The symplectic structure (4.1.3) in turn means that the Poisson brackets of any two functions f and g on

our phase space is given by:

$$\{f, g\} = \frac{\kappa\gamma}{3}\left(\frac{\partial f}{\partial c}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial c}\frac{\partial f}{\partial p}\right). \tag{4.1.4}$$

It was because we introduced the powers of V_o into the definitions (4.1.2) that the symplectic structure does not depend on V_o. Another desirable consequence of the precise definition in (4.1.2) is that the variables c and p do not change if we rescale the fiducial flat metric; they therefore have physical meaning. For instance, it is not hard to verify that the physical volume of the cell $\mathcal{V}$ is $|p|^{\frac{3}{2}}$; i.e., it does not depend at all on the volume V_o of that cell relative to the fiducial metric ${}^o q_{ab}$.

Before considering the quantization of this phase space, we pause to make some brief comparisons between the variables we have introduced here and the variables more usual in cosmology. Any spatially flat, isotropic, and homogeneous spacetime metric may be expressed in the form [55]:

$$ds^2 = -d\tau^2 + a^2(\tau)(dx^2 + dy^2 + dz^2). \tag{4.1.5}$$

Here a is the scale factor, and in solving Einstein's equation starting from (4.1.5) the task is to determine a as a function of τ (note that τ is the proper time as measured by observers co-moving with the cosmic flow). How does a relate to the variables c and p as we have defined them? If we choose for our spatial manifold M surfaces of constant τ, then it is possible to show directly from the usual definitions of the Ashtekar connection in terms of the Levi-Civita connection and extrinsic curvature that $c = \gamma\dot{a}/2$, where γ is the Barbero-Immirzi parameter. Likewise, starting from the definition of the electric field E as a partially densitized triad one may show that $|p| = a^2$. Thus, starting from a single variable that is a function of time, we obtain two canonical variables, one determined by the value of the variable at a given instant, the other by its time derivative.

How is the quantization of this symmetric phase space accomplished? As we have already explained, in loop quantum gravity states are complex valued functions of the connection. In a symmetry reduced model, we seek states that have support only on symmetric connections; those satisfying (4.1.1). Because of the decomposition (4.1.2), this means that any such state is in fact a function of c.

However, it cannot be any function of c: since our approach is to mimic loop quantum gravity we should consider functions that arise as holonomies of connections of the form (4.1.2). In the full theory, we must consider holonomies along all possible paths. In our symmetry reduced model, however, it is sufficient to consider only holonomies along straight lines in M (where 'straight' is defined relative to the fiducial metric ${}^o q_{ab}$). One can then

show that the holonomy of a symmetric connection along such an edge is given by:

$$h_e(A) = \cos\frac{\mu c}{2} + 2\sin\frac{\mu c}{2}(\dot{e}^{a\,o}\omega_a^i)\tau^i. \tag{4.1.6}$$

In this equation the oriented length of the edge e is given by $\mu V_o^{\frac{1}{3}}$. Then the algebra generated by sums and products of matrix elements of such holonomies is simply the algebra of functions of c of the form:

$$f(c) = \sum_j f_j e^{\frac{i\mu_j c}{2}}. \tag{4.1.7}$$

This is precisely the algebra of almost periodic functions of momentum k considered in chapters 2 and 3 for the polymer particle: thus we see already a strong similarity to the polymer particle representation. However, in the polymer particle model, we found that the functions of the form (4.1.7) were *analogous* to holonomies; here we find that they actually *are* holonomies.

For the relation to the polymer particle to be complete, we must examine the canonically conjugate variable as well. From the Poisson brackets in (4.1.4) we see that this variable is just p, as we might expect. When we come to the quantum theory, loop quantum gravity dictates that we should choose as our functions of momentum the electric field smeared across two surfaces. As with holonomies above, we find that because of homogeneity and isotropy we need not consider all possible surfaces, but rather may restrict ourselves to squares (relative to ${}^o q_{ab}$) that are tangent to the fiducial triad ${}^o e_i^a$, and may use smearing functions f that are constant over those squares. One then finds that the electric flux operators are of the form

$$E(S,f) = p\, V_o^{-\frac{2}{3}} A_{S,f} \tag{4.1.8}$$

where $A_{S,f}$ is the oriented area of the square S relative to ${}^o q_{ab}$, with the orientation dependent on f. Thus, just as for the polymer particle, the fundamental classical observables depending on the canonically conjugate variable, which will be promoted to operators in the quantum theory, are simply multiples of that observable. Here, it is p; in the polymer particle representation it was x.

We see therefore that the representation of our basic observables will be the same as it was for the polymer particle. States are functions of the form (4.1.7), with the inner product given by

$$\langle f \,|\, g\rangle = \sum_j \overline{f}_j g_j \tag{4.1.9}$$

On the Hilbert space all functions of the form (4.1.7) act unitarily by multiplication, but there is no self-adjoint operator corresponding to the classical phase space variable c. The

$\hat{p}$ operator has the action:

$$\hat{p}|f\rangle = \frac{\gamma\ell_{\mathrm{Pl}}^2}{6}\sum_j f_j\mu_j e^{\frac{i\mu_j c}{2}} \tag{4.1.10}$$

It is evident from (4.1.10) that functions of the form $\exp(i\mu c/2)$ are eigenstates of the $\hat{p}$ operator, and we denote these eigenstates by $|\mu\rangle$. We can also use the relation (mentioned earlier) $V = |p|^{\frac{3}{2}}$ to define an operator corresponding to the volume of our fiducial cell $\mathcal{V}$, and the $|\mu\rangle$ will be eigenstates of this operator as well:

$$\hat{V}|\mu\rangle = \left(\frac{\gamma|\mu|}{6}\right)^{\frac{3}{2}}\ell_{\mathrm{Pl}}^3|\mu\rangle \equiv V_\mu|\mu\rangle. \tag{4.1.11}$$

The volume eigenvalues defined by this equation will be important to us shortly.

What can we say about coherent states for loop quantum cosmology? Because we are in fact working with the polymer particle representation, it follows immediately that all of the conclusions of section 2.4 carry over. In particular, given any point (c', p') of the classical phase space, we can using only operators defined in loop quantum cosmology select a coherent state $(\Psi_{(c',p')}|$ peaked about (c', p'):[1]

$$(\Psi_{(c',p')}| = \sum_{\mu\in\mathbb{R}} e^{-\frac{(\mu-\mu')^2}{2d^2}} e^{\frac{i(\mu-\mu')c'}{2}}(\mu|. \tag{4.1.12}$$

Here μ' is related to p' through $p' = \frac{1}{6}\gamma\ell_{\mathrm{Pl}}^2\mu'$; note that this implies that μ and hence also d is dimensionless. We should like to check that the state $(\Psi_{(c',p')}|$ is indeed sharply peaked around the point (c', p'). As in section 2.4, in order to make statements about the expectation value of c we must define a replacement for the nonexistent c operator in the quantum theory; this replacement is given by:

$$\hat{c}_{\mu_o} = \frac{1}{i\hbar}\left(e^{\frac{i\mu_o c}{2}} - e^{-\frac{i\mu_o c}{2}}\right) \tag{4.1.13}$$

so that the action of $\hat{c}_{\mu_o}$ on our volume eigenstates is given by:

$$\hat{c}_{\mu_o}|\mu\rangle = \frac{1}{i\hbar}\left(|\mu+\mu_o\rangle - |\mu-\mu_o\rangle\right). \tag{4.1.14}$$

At the moment, μ_o is simply a parameter that we are free to choose, but in the next subsection we shall see that once we incorporate dynamics, it acquires a fundamental significance, just as it did for the polymer particle.

We can then verify, through shadow state calculations similar to those of section 2.4,

[1]The relative minus sign between the imaginary exponentials of (4.1.12) and (2.4.8) arises because we define $|\mu\rangle$ to be $e^{i\mu c/2}$ whereas in chapter 2 we defined $|x\rangle$ to be e^{-ikx}.

that the following are true (all expectation values are taken in the state $(\Psi_{(c',p')}|$):

$$\langle p\rangle = p', \tag{4.1.15}$$

$$\langle c\rangle = \frac{2}{\mu_o} e^{-\frac{\epsilon^2}{4}} \sin\left(\frac{\mu_o c'}{2}\right), \tag{4.1.16}$$

$$\Delta p^2 = \frac{\gamma^2 \mu_o^2 \ell_{\rm Pl}^4}{72\epsilon^2}, \tag{4.1.17}$$

$$\Delta c^2 = \frac{2}{\mu_o^2}\left[(1-e^{-\frac{\epsilon^2}{2}}) + (e^{-\frac{\epsilon^2}{2}} - e^{-\epsilon^2})\cos(\mu_o c')\right] \approx \frac{2\epsilon^2}{\mu_o^2}. \tag{4.1.18}$$

Here, as we did in section 2.4, we have defined the quantity ϵ to be the ratio μ_o/d, where d is as before the width of the Gaussian. Then so long as ϵ is much smaller than one, we find as before that our coherent states are indeed sharply peaked about the point (c', p') of the classical phase space, so long as $\mu_o c' \ll 1$. When $\mu_o c'$ approaches (or exceeds) unity, however, the expectation value of our replacement operator no longer agrees with the value of c' at the phase space point. In the context of the polymer particle this reflected a difference between expectation values in the polymer particle and Schrödinger representations for phase space points with high momenta; we shall comment on the physical meaning of the discrepancy in the context of quantum cosmology below.

We have therefore shown that kinematically there is a well-defined notion of the low energy limit of loop quantum cosmology, but we need also to investigate the dynamics. Since we have already solved both the gauge and diffeomorphism constraints of general relativity, we need only to impose the scalar constraint. For our goal of understanding the effective classical dynamics of the quantum theory, it is important to have a good understanding of this constraint and the method by which the equations of motion arise from it, both classically and quantum mechanically. We therefore turn to a careful review of these questions in the next subsection.

4.1.2 Hamiltonian cosmology

In textbook treatments of cosmological models (see for instance [55]), one typically shows that the assumptions of spatial flatness and isotropy lead to the form (4.1.5) for the spacetime metric. It then becomes the task of Einstein's equation (as we have already mentioned) to determine a as a function of τ.

In order to complete this approach, some choice must be made for the form of matter appearing in the stress-energy tensor that stands on the right hand side of Einstein's equation. For cosmological models, one typically focuses on perfect fluids, whose stress tensor is given by:

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b). \tag{4.1.19}$$

In this equation u_a is the four velocity of the matter field, which by homogeneity and isotropy is easily seen to be proportional to ∂_τ. It is then possible to show that Einstein's equation implies the following two equations for a [55] (recall that we have defined $\kappa = 8\pi G$):

$$\frac{3\dot{a}^2}{a^2} = \kappa\rho, \tag{4.1.20}$$

$$\frac{3\ddot{a}}{a} = -\frac{\kappa}{2}(\rho + 3P) \tag{4.1.21}$$

which are Friedmann's equations. It is modifications to these equations induced by quantum gravity effects that we wish ultimately to study.

In fact, one typically specializes even further to two particular forms of perfect fluid: dust (for which $P = 0$) and radiation (for which $P = \rho/3$). It is then easy to integrate the equations (4.1.20) and (4.1.21) to find:

$$a(\tau) = \left(\frac{9C}{4}\right)^{\frac{1}{3}} \tau^{\frac{2}{3}} \qquad \text{for dust,} \tag{4.1.22}$$

$$a(\tau) = \left(4C'\right)^{\frac{1}{4}} \tau^{\frac{1}{2}} \qquad \text{for radiation.} \tag{4.1.23}$$

The constants C and C' are related to the energy density ρ of the matter through $C = \kappa a^3/3$ and $C' = \kappa a^4/3$; we thus see that for dust the density scales inversely to the volume, while for radiation there is an additional factor of the scale factor a because of the redshift.

We will in this thesis also restrict ourselves to these two forms of matter sources. We only expect to have a valid semi-classical description of loop quantum cosmology at times late compared to the Planck time, and so we need only consider matter sources applicable at such times. One might therefore also like to consider a cosmological constant term, in addition to dust and radiation. However, with a positive cosmological constant $\dot{a}$ does not remain bounded with increasing time, and therefore neither would c, in terms of the variables we are using. This makes it a more subtle matter to apply our coherent states since as we have just seen they are only closely peaked around c' when that quantity is small. We therefore defer investigation of this important question to future work.

We would like now to understand how the results above for dust and radiation filled universes arise classically in a Hamiltonian framework. We can derive this by using the Hamiltonian form of general relativity. Loop quantum gravity is derived from a Hamiltonian framework in terms of connection and triad variables, but it will be possible (and easier) for us to use the ADM formulation commonly encountered in numerical relativity, which uses as basic variables the spatial metric and a field constructed from the extrinsic curvature. In the ADM formulation one finds that the gravitational part of the Hamiltonian constraint

may be written as (for the case of vanishing shift vector, as we have here):

$$C_{\text{grav}} = -\int_M d^3x N\sqrt{h}({}^3R + K^2 - K_{ab}K^{ab}). \tag{4.1.24}$$

In this equation, h is the determinant of the spatial metric h_{ab}, and ${}^3R$ is the Ricci curvature scalar of that three dimensional metric. K_{ab} is the extrinsic curvature of the three manifold M, K is the trace of K_{ab}, and N is the lapse function.

For the metric (4.1.5), we choose our time evolution to be that given by the vector field ∂_τ. This leads, as we have already said, to a vanishing shift vector; it also leads to a constant lapse function N. We will then find that the integral over M in (4.1.24) diverges, so we make it finite by restricting it to our fiducial cell $\mathcal{V}$. Since our spatial metric is flat, ${}^3R = 0$. We also clearly have $h_{ab} = a^2\delta_{ab}$ if we choose Cartesian coordinates on M, and from this we find the extrinsic curvature directly from its definition in terms of the Lie derivative of h_{ab} along ∂_τ:

$$K_{ab} := \frac{1}{2}\mathcal{L}_\tau h_{ab} = a\dot{a}\delta_{ab} = \left(\frac{\dot{a}}{a}\right) h_{ab} \tag{4.1.25}$$

Thus, $\sqrt{h} = a^3$ and $K = 3\dot{a}/a$. Using these we find:

$$C_{\text{grav}} = -6a\dot{a}^2. \tag{4.1.26}$$

Now, this expresses only the gravitational part of the Hamiltonian constraint; the full constraint is the sum of the gravitational and matter terms of the constraint. How do we find the matter terms? This is most easily accomplished by noting that

$${}^3R + K^2 - K_{ab}K^{ab} = -2G_{ab}n^a n^b \tag{4.1.27}$$

where G_{ab} is the Einstein tensor and n^a the unit normal to our spatial slice (for us, it is just ∂_τ^2). Thus, since Einstein's equation tells us that $G_a b - \kappa T_{ab} = 0$, we see that the full constraint should be given by:

$$C = -6\kappa^{-1}a\dot{a}^2 + 2\int_{\mathcal{V}} d^3x\sqrt{h}T_{ab}n^a n^b. \tag{4.1.28}$$

We have here divided by κ, since it will turn out that this is necessary to obtain the correct equations of motion from the Hamiltonian. Since $n^a = u^a$—i.e., the unit normal to M is parallel to the four velocity of the fluid—we may use (4.1.19) to obtain:

$$C = -6\kappa^{-1}a\dot{a}^2 + 2a^3\rho \tag{4.1.29}$$

Finally, we may use the relations $|p| = a^2, c = \gamma\dot{a}/2$ to rewrite this in terms of our basic

canonical variables, c and p:

$$C = -6\kappa^{-1}\gamma^{-2}c^2\sqrt{|p|} + \frac{1}{2}|p|^{\frac{3}{2}}\rho. \tag{4.1.30}$$

In arriving at (4.1.30), we have divided the expression of (4.1.29) by four. This of course does not change the constraint surface (i.e., the points (c, p) for which $H(c, p) = 0$), but it *does* affect the dynamics, which depends on the Poisson brackets of the constraint with observables. We have chosen our normalization of the constraint so as to obtain the correct dynamics with the Poisson brackets of equation (4.1.4); we shall show momentarily that our normalization is correct. We could in fact have derived this normalization by checking how the relationship between our canonical variables c and p and the ADM variables relates the Poisson bracket of our phase space to that of the ADM phase space; it is simply easier to adjust the normalization at the end so as to obtain the correct equations of motion.

How do we obtain the equations of motion? In order to do this we must make one more improvement on our expression (4.1.30) for the Hamiltonian constraint. It contains the density ρ, but that is not a constant on phase space. So we must express it in terms of something that is constant, and it proves most convenient to choose the total energy E_0 'initially' in our fiducial cell; i.e., at some fixed time τ_0. Then we obtain:

$$C = -6\kappa^{-1}\gamma^{-2}c^2\sqrt{|p|} + \frac{1}{2}E_0 \qquad \text{for dust,} \tag{4.1.31}$$

$$C = -6\kappa^{-1}\gamma^{-2}c^2\sqrt{|p|} + \frac{1}{2}E_0\left(\frac{p_0}{p}\right)^{\frac{1}{2}} \qquad \text{for radiation,} \tag{4.1.32}$$

where p_0 is the value of p at $\tau = \tau_0$.

With these definitions of the Hamiltonian constraint, we may calculate the equations of motion. Let us illustrate this for the case of a dust filled universe. We know that we must impose the vanishing of the constraint, and we must also express $\dot{c}$ and $\dot{p}$ using the Poisson brackets. Since for any observable O we have $\dot{O} = \{O, C\}$, we have three relations that must hold:

$$C(c, p) = 0, \tag{4.1.33}$$

$$\dot{c} = \{c, C\} = \frac{\kappa\gamma}{3}\frac{\partial C}{\partial p}, \tag{4.1.34}$$

$$\dot{p} = \{p, C\} = -\frac{\kappa\gamma}{3}\frac{\partial C}{\partial c}. \tag{4.1.35}$$

Consider equation (4.1.35) first. In the regime $p > 0$ it gives:

$$\dot{p} = 4\gamma c\sqrt{|p|}. \tag{4.1.36}$$

But using $p = \operatorname{sgn} a\, a^2$ we find that this simplifies to $c = \gamma\dot{a}/2$. This illustrates an important point: in general, c is not *defined* by any particular relation to the time derivative of a; rather, this relationship comes from the equations of motion. That is because the Hamiltonian equations of motion have 'built in' the definition of what time with respect to which they are evolving. When we come to the modifications of Friedmann's equations induced by loop quantum cosmology, this will be important because we shall find that the relation $c = \gamma\dot{a}/2$ no longer holds exactly. What this means, therefore, is that we have quantum corrections not just to the equations of motion, but to the lapse function as well.

Continuing on, we can use this expression for c in equation (4.1.33). That equation then becomes:

$$-\frac{3}{2}\kappa^{-1}\dot{a}^2 a + \frac{1}{2}E_0 = 0. \tag{4.1.37}$$

Upon some rearrangement, this becomes

$$\frac{3\dot{a}^2}{a^2} = \kappa\frac{E_0}{a^3} \tag{4.1.38}$$

which we easily see to be equivalent to the first of Friedmann's equations, (4.1.20). When we use our expression for c in equation (4.1.34), we find that it in turn is equivalent to the second of Friedmann's equations, (4.1.21).

A similar analysis to that just outlined shows that the three equations (4.1.33–4.1.35) also give us the expression for c in terms of $\dot{a}$ (which is the same) and the two Friedmann's equations when we use the constraint (4.1.32) appropriate for a radiation filled universe. Thus, summarizing we have found that classically the dynamics of a spatially flat, isotropic universe can be incorporated into the single Hamiltonian constraint[2]

$$C = -6\kappa^{-1}\gamma^{-2}c^2 \operatorname{sgn} p\sqrt{|p|} + C_{\text{matter}} \tag{4.1.39}$$

where:

$$C_{\text{matter}} = \begin{cases} \frac{1}{2}E_0 & \text{for dust,} \\ \frac{1}{2}E_0\sqrt{\frac{|p_0|}{|p|}} & \text{for radiation.} \end{cases} \tag{4.1.40}$$

We have not derived the factor of $\operatorname{sgn} p$ in (4.1.39) since in our analysis we only considered positive p, but it is not hard to verify. Note that negative values of p in the evolution correspond to electric fields oriented opposite to that of the fiducial triad.

We now turn to the quantization of this dynamics, which amounts to a quantization of

[2]Note that [20] incorrectly has this as

$$C = -6\gamma^{-2}c^2 \operatorname{sgn} p\sqrt{|p|} + \kappa C_{\text{matter}}$$

in equation (11) of that reference. But this normalization is inconsistent with the Poisson brackets used in [20], which are the same as those we use here.

the constraint (4.1.39). We may be tempted to follow the lines used in the polymer particle, defining a substitute operator for c^2 and attempting to define an operator for $\sqrt{p}$ directly in terms of its action on the volume eigenstates $|\mu\rangle$. However, this approach would be rather *ad hoc* from the point of view of quantum geometry: we are no longer trying just to model general features of the full theory (as we were in the previous two chapters) but to follow as closely as possible the strategies used in the full theory.

In fact, it is possible to apply the procedure used in defining the operator corresponding to the gravitational part of the constraint directly to the case at hand; this is carried out in section 4.1 of [20]. Since our concern in this chapter is not so much with the precise origins of the quantum constraint as in the implications of that constraint for low energy physics, we will not repeat those arguments here but will instead just quote the result. One finds for the operator:

$$\hat{C}^{(\mu_o)}_{\rm grav} = 96i(\kappa\gamma^3\mu_o^3\ell_{\rm Pl}^2)^{-1}\sin^2\frac{\mu_o c}{2}\cos^2\frac{\mu_o c}{2} \times\Big(\sin\frac{\mu_o c}{2}\hat{V}\cos\frac{\mu_o c}{2} - \cos\frac{\mu_o c}{2}\hat{V}\sin\frac{\mu_o c}{2}\Big). \quad (4.1.41)$$

In this equation $\hat{V}$ is the volume operator introduced in (4.1.11) above. The action of this constraint operator on eigenstates of $\hat{p}$ is given by:

$$\hat{C}^{(\mu_o)}_{\rm grav}|\mu\rangle = 3(\kappa\gamma^3\mu_o^3\ell_{\rm Pl}^2)^{-1}(V_{\mu+\mu_o} - V_{\mu-\mu_o})\big(|\mu+4\mu_o\rangle - 2|\mu\rangle + |\mu-4\mu_o\rangle\big). \quad (4.1.42)$$

This operator depends upon a parameter μ_o. The origin of this parameter is as a regulator in the derivation of the constraint and has an analogue in the full theory as well, discussed in [20]. For our purposes in this chapter what is important is that once again we will have a restriction of our lattice to integer multiples of μ_o, so that we consider the Hilbert space spanned by the kets $|n\mu_o\rangle$ for all natural numbers n.

Also important to us, however, is that comparing the origin of μ_o to the corresponding ambiguity in the full theory (the 'j' ambiguity as to what spin to assign to the new edges introduced by the constraint) one is led to the conclusion that there is a natural choice for its value, unlike the parameter (denoted by the same symbol) that we considered in chapter 3. That is because the value of μ_o affects the value of areas of squares considered as part of the constraint regularization. While in the reduced model itself these areas may assume arbitrarily small (positive) values, in the full theory the discreteness of the area spectrum in quantum geometry dictates a smallest area. If we use this same value as the smallest area in the regularization of the constraint for the reduced model, then we are led to the conclusion that μ_o can be no smaller than $\sqrt{3}/4$. Henceforth we shall assume this value.

The constraint operator (4.1.40) is the operator normally used in the loop quantum cosmology literature, and in particular in [20], where the semi-classical limit of loop quantum

cosmology was previously considered. However, it suffers from a serious limitation, whose full significance has not been previously appreciated: it is not symmetric. What this means is that in general its expectation value in a given state need not be real. Since we would like to interpret just such an expectation value as an effective classical Hamiltonian, we must ask if we know how to handle a situation where this expectation value is not real. This problem was not noticed in [20] because it turns out that the leading order term of the expectation value—which is all that was calculated explicitly in that paper—*is* real. This in itself is rather remarkable, as there is no *a priori* reason that it need have happened.

However, we are still left with the problem of what to do with a complex effective Hamiltonian. While some attempt was made to understand such a thing (for instance, in terms of particle creation) no successful strategy was found. Therefore we must ask if there is any modification of the constraint so that it is symmetric.

Now, given *any* operator on a Hilbert space, we may form a symmetric operator from it by adding it to its adjoint. Certainly this is the simplest strategy to employ here as well. While that does indeed give us a symmetric operator, the resulting operator has a serious deficiency of it own. One of the reasons that the operator (4.1.41) has been used in the literature is that it ensures that the classical singularity is avoided (see, for instance, [16]). What is meant by this is two things. First, as we shall see in a moment, quantities that diverge classically remain bounded in the quantum theory as the initial singularity is approached (note that the initial singularity corresponds to $p = 0$); in particular this holds for the inverse scale factor to which classically the curvature is proportional. Second, and more important for the considerations at hand, the quantum evolution does not break down at the singularity. That is because when (4.1.42) is applied to a state and then interpreted as a difference equation for that state, the coefficient of $|\mu = 0\rangle$ in the expansion of the state decouples from the rest of the coefficients of the state.

This decoupling is very sensitive to the precise form of the constraint; while it holds for the constraint (4.1.41) it does not hold for the constraint formed by adding this operator to its adjoint[3] . There is however a way to form a constraint that is singularity avoiding and yet also is symmetric. To do so, take the full constraint (i.e., $\hat{C}_{\rm grav} + \hat{C}_{\rm matter}$), multiply it by $\operatorname{sgn} p$, and add *this* operator to its adjoint. Then the resulting operator is symmetric, and the singularity decouples from the quantum evolution.

While the strategy sketched above is successful in the context of loop quantum cosmology, we must also take care that we are not making an adjustment in the symmetry reduced theory that we shall be unable to make in the full theory. After all, we were led to the (non-symmetric) constraint (4.1.41) in the first place by following the construction used for the scalar constraint in the full theory. Fortunately, the remedy we have prescribed may be

[3] I am grateful to Abhay Ashtekar and Martin Bojowald for pointing out this difficulty to me, as well as the resolution discussed below.

carried over in natural fashion to the full theory as well: one simply multiplies the constraint by the sign of the determinant of the electric field operator E.

We saw already in chapter 3 that certain operators that seemed mathematically plausible as part of the Hamiltonian had to be eliminated as they would give the wrong low-energy limit for the theory. Here we find something similar, only stronger: the low energy considerations of loop quantum cosmology predict an adjustment to a central aspect of the *full* theory. We thus have a concrete example of how studying a (very) reduced model can nonetheless tell us something nontrivial about the full theory.

We have therefore adequately covered the gravitational term of the Hamiltonian constraint; what about the matter terms? For a dust filled universe, the matter Hamiltonian is trivial to implement: it is simply a constant multiple of the identity. For the radiation filled universe, we must be more careful, since the classical expression contains $|p|^{-\frac{1}{2}}$. Fortunately, this corresponds precisely to the inverse scale factor whose quantum operator analogue has been well studied in loop quantum cosmology. As we alluded to above, the main reason for studying this operator thus far in loop quantum cosmology is because classically it is related to the curvature scalar and becomes unbounded as the initial singularity is approached. A certain quantum operator has been developed which is bounded as an operator; the construction of this operator also parallels the construction of the Hamiltonian constraint in the full theory. The resulting operator is:

$$\left[\frac{\widehat{\mathrm{sgn}\, p}}{\sqrt{|p|}}\right] = -\frac{12i}{\gamma\ell_{\mathrm{Pl}}^2}\left(\sin\frac{c}{2}\hat{V}^{\frac{1}{3}}\cos\frac{c}{2} - \cos\frac{c}{2}\hat{V}^{\frac{1}{3}}\sin\frac{c}{2}\right). \tag{4.1.43}$$

Even though this operator contains both configuration and momentum operators, it commutes with $\hat{p}$, so its eigenstates are also $|\mu\rangle$ and the action of the operator (4.1.43) on such states is:

$$\left[\frac{\widehat{\mathrm{sgn}\, p}}{\sqrt{|p|}}\right]|\mu\rangle = \frac{6}{\gamma\ell_{\mathrm{Pl}}^2}\left(V_{\mu+1}^{\frac{1}{3}} - V_{\mu-1}^{\frac{1}{3}}\right)|\mu\rangle. \tag{4.1.44}$$

One might be tempted to work instead with an operator that is simply defined to act on eigenstates of $\hat{p}$ and return $|p|^{-1/2}$ as an eigenvalue, on the grounds that in the late universe regime that we are interested in, the difference between such an operator and (4.1.43) would be small. If all we were interested in were finding that the theory has the correct classical limit, such reasoning would indeed be justified. However, we have a more ambitious goal: besides studying the leading order classical terms emerging from the quantum theory, we also wish to find the (hopefully) small corrections arising from quantum effects. Thus, 'small' differences can indeed be important, and we should stick with the operator (4.1.43) that is well defined in all regimes of loop quantum cosmology.

Summarizing, we have arrived at the following prescription for defining the Hamiltonian constraint as a quantum operator in loop quantum cosmology. First, start with the operator

given by (4.1.41) and add to it the corresponding operator for $\hat{C}_{\text{matter}}$. Then multiply that sum by $\operatorname{sgn} p$, add the result to its adjoint as an operator, and take half. This is a rather unwieldy description in the way we have described it. As a practical calculational matter it is generally simpler to calculate quantities for the non-symmetrized constraint and then simply take the real part of the resulting expression at the end of the calculation, and that is what we shall do for the remainder of this chapter. Thus, (4.1.41) really does serve as the fundamental calculational starting point.

So, now that we have kinematically semi-classical states and a natural prescription for the scalar constraint as a quantum operator, we must combine those two elements to derive the effective classical dynamics for our quantum theory. But before we can do that, we must define what we mean by that, and to that we turn in the next and final introductory subsection.

4.1.3 The general framework for the effective Hamiltonian

We saw in the first part of the last subsection that classically we can generate the full dynamics of the FRW cosmological models by prescribing a Hamiltonian constraint function $C(c, p)$ on phase space. The constraint surface consists of those (c, p) such that $C(c, p) = 0$, and the time evolution of c and p is determined in the usual fashion by taking the Poisson brackets of these observables with the constraint.

We should like now to find a similar, though in general slightly modified, constraint that does the same thing for the effective dynamics of the quantum theory. Since we have a quantum operator that represents the constraint, and since we have a candidate semi-classical state, a natural approach is simply to take the expectation value of this constraint in that state, and see if the result is close to the classical value. This is in fact what was done in [20], where it was found that the lowest order expression for the expectation value was indeed the classical constraint function for the gravitational term of the constraint.

On closer reflection, however, it is not so clear what this means. We are finding a constraint *function* on phase space; that means that we are taking the expectation value of the quantum operator not in one semi-classical state, but a family of them: one for each point in at least some region of phase space. How do we know that we should use the same coherent state for each point of phase space (as was effectively done in [20])? More importantly, what quantum process—if any—does the dynamics of this Hamiltonian function correspond to?

In order to better understand this, it is helpful to back up somewhat. What is usually meant by the statement that ordinary quantum mechanics has the correct semi-classical limit? In many textbook treatments, an argument is made based on Ehrenfest's theorem. For a particle with a Hamiltonian of the form $p^2/2m + V(x)$ for some potential $V(x)$, this

is the result [53] that:

$$m\frac{d^2}{dt^2}\langle x\rangle = -\left\langle \frac{\partial V(x)}{\partial x}\right\rangle. \tag{4.1.45}$$

However, despite the apparent similarity to Newton's second law, as pointed out in [56], this is really not sufficient to establish semi-classicality; for one thing, it is true for *all* states, not just semi-classical ones. Instead, what one needs is for the state to be such that instead we have:

$$m\frac{d^2}{dt^2}\langle x\rangle \approx -\frac{\partial V(\langle x\rangle)}{\partial x}. \tag{4.1.46}$$

In other words, we may take the derivative outside of the expectation value.

In fact, what we would like is something even stronger. We would like to know:

$$m\frac{d^2}{dt^2}\langle x\rangle = -\frac{\partial V(\langle x\rangle)}{\partial x} - \frac{\partial(\delta V(\langle x\rangle))}{\partial x} + \text{smaller corrections.} \tag{4.1.47}$$

In other words, we would like to be able to incorporate the corrections from the exact classical result into a small 'corrected potential.'

It is not at all clear whether we can do this. We are asking, in effect, whether the full quantum dynamics can be encapsulated in dynamics on the classical phase space, expressed in terms of expectation values. Even for semi-classical states, there is in general no reason to believe that the full quantum dynamics can be described on the classical phase space.

There is, however, a framework in which the full quantum dynamics *can* be described in terms of symplectic dynamics on an appropriate phase space. This is the geometric formulation of quantum mechanics, and not only does it allow us to describe full quantum mechanics in fashion structurally similar to the classical theory, but it also provides the appropriate framework to answer semi-classical questions. Therefore, we shall spend some time in this subsection reviewing a few key elements of this framework; for more details the reader is referred to [54].

The geometrical formulation of quantum mechanics starts by describing the Hilbert space $\mathcal{H}$ of quantum theory as a symplectic space itself.[4] This is possible because the imaginary part of the inner product is in fact a symplectic form on the phase space. Moreover, the real part forms a Riemannian metric, and the two structures are related so that together they comprise a Kähler structure on the Hilbert space. Explicitly, what we have for any two states $|\Psi\rangle$ and $|\Phi\rangle$ is:

$$\langle\Psi\,|\,\Phi\rangle =: \frac{1}{2\hbar}G(\Psi,\Phi) + \frac{i}{2\hbar}\Omega(\Psi,\Phi) \tag{4.1.48}$$

with G the Riemannian metric and Ω the symplectic form. The Kähler structure is completed by selecting a preferred linear operator J on $\mathcal{H}$ such that when $\mathcal{H}$ is regarded as a *real* vector

[4] In fact, the true quantum phase space is not $\mathcal{H}$ itself, but rather a projective space constructed from it. This difference is not crucial for our considerations, however, so we gloss over; for a complete discussion the reader is again referred to [54]

space, $J^2 = -I$ for the identity operator I. The triple (J, G, Ω) makes $\mathcal{H}$ a Kähler space because

$$G(\Psi, \Phi) = \Omega(\Psi, J\Phi). \tag{4.1.49}$$

While the Riemannian metric is important for measurement theory, as indeed is the Kähler space nature of $\mathcal{H}$, for our limited considerations in this thesis all we shall need is the fact that $\mathcal{H}$ is a symplectic space, and henceforth that is all we shall refer to.

Now, the nature of $\mathcal{H}$ as a symplectic space is important because we may naturally incorporate the evolution under Schrödinger's equation as Hamiltonian evolution. First, we observe that any self-adjoint observable $\hat{F}$ defines a vector field on $\mathcal{H}$, since it assigns to each point in $\mathcal{H}$ a vector (where we use the canonical identification of the tangent space of $\mathcal{H}$ with $\mathcal{H}$ itself). Schrödinger's equation then, in terms of the Kähler space structure on $\mathcal{H}$, becomes:

$$\dot{\Psi} = -\frac{1}{\hbar} J\hat{F}\Psi. \tag{4.1.50}$$

What is important for us is that the vector field defined by the right hand side is globally Hamiltonian, and moreover the Hamiltonian function that generates it is simply *the expectation value of* $\hat{F}$ *on* $\mathcal{H}$ [54]. Moreover, it is also possible to show that the Poisson brackets (with respect to the symplectic structure Ω) between two observables are given by the expectation value of $1/i\hbar$ times the commutator of the corresponding observables. Thus, these two facts together trivially imply Ehrenfest's theorem.

We emphasize that despite the strong similarities what is going on here is much more than the usual classical dynamics: the phase space manifold and symplectic structure are quite different, in general, from the phase space of the analogous classical system. Already, however, we see that we are justified in calculating the expectation value of the Hamiltonian and treating it as the Hamiltonian function on a phase space. However, in general it is not a Hamiltonian on the classical phase space.

We are making progress though. To complete our understanding of the role of semi-classical states, we rely on another set of related observations from [54]. Consider the expectation values of the basic canonical observables that coordinatize the classical phase space. These provide a natural projection from the quantum phase space to the classical phase space, and it is possible then to show that the quantum phase space has the structure of a bundle whose base space is the classical phase space. Vertical vectors are those that are tangent to the fibers; that is, which have only components in directions along which the expectation values of the canonical observables does not change. The symplectic form allows us to construct a notion of horizontal vectors (those orthogonal, with respect to Ω, to the vertical vectors), and hence horizontal sections. The key property of these horizontal sections is that if we pull back the symplectic form to any such section, it gives precisely the classical symplectic form. Since each horizontal section is coordinatized entirely by the canonical

observables, this means that each horizontal section may be viewed as an embedding of the classical phase space into the quantum phase space. There is additionally a close connection between such horizontal sections and generalized coherent states; but as we shall not need this connection we refer the interested reader instead to [54]. The one fact we do take away from that analysis is that every horizontal section has the property that the uncertainties in each of the canonical variables is constant on the section.

So, we see then that since horizontal sections can provide us a natural embedding of the classical phase space in the quantum phase space, from which we can recover the classical symplectic structure as simply the horizontal part of the quantum symplectic structure, that we are in an excellent position to ask about effective semi-classical dynamics. Since (Gaussian) coherent states of any (given) fixed width have the property that the quantum uncertainties of each of the basic canonical observables are constant, it follows that each such collection of coherent states (one coherent state for each point in the classical phase space) comprises a horizontal section that is naturally viewed as an embedding of the classical phase space. At the moment, of course, there are many such embeddings; one of course for each possible width of a Gaussian, but also for all other generalized coherent states. This ambiguity comes about because we have so far considered only the kinematics of the problem. Once we consider the dynamics, however, there is a natural criterion to apply: we look for horizontal sections that are preserved by the Hamiltonian flow. Such sections will provide not only an embedding of the classical phase space into the quantum phase space, but the quantum dynamics on such a section can be described through an effective classical Hamiltonian on the classical phase space; the value of that effective classical Hamiltonian will be simply the expectation value of the quantum Hamiltonian operator.

Of course, we have *a priori* no guarantee that there will be any horizontal sections that are preserved under the quantum evolution. In [54] the existence of such sections is considered for the simplest possible Hamiltonian: the simple harmonic oscillator. There, it is found that there is precisely one such horizontal section: the section consisting of Gaussian coherent states whose width is that given by the length scale d of the Hamiltonian. All other sets of Gaussian coherent states of fixed width, while necessarily forming horizontal sections, are not preserved by the dynamical flow since the Hamiltonian flow on such sections has a vertical as well as horizontal component. On the unique preferred section, where the dynamical flow is purely horizontal, the expectation value of the simple harmonic oscillator Hamiltonian turns out to be exactly equal to the classical harmonic oscillator Hamiltonian, yielding the expected result that the effective classical dynamics for the quantum system in fact corresponds precisely with the classical dynamics.

For so simple a system as the simple harmonic oscillator, this is not a surprising result (indeed, the only problem would have been to not obtain this result, as it is well known). Our real question is how far we can carry this program forward for more complicated Hamil-

tonians, particularly those relevant to loop quantum cosmology. That is, is it the case that we can find a horizontal section of the quantum phase space (viewed as a bundle over the classical phase space) that is preserved by the quantum evolution of the system? There either may be no such section, or it may be difficult to determine whether there is. In either case, as a first attempt at addressing the problem we may seek some form of approximation. Heuristically, we seek either a section that is almost horizontal and preserved by the quantum evolution, or a horizontal section that is almost preserved by the evolution, or both. In each case, we have not quantified what we mean by 'almost,' and doing so remains an important open question. For loop quantum cosmology we shall, however, present some evidence to support the claim that there is an almost horizontal section that is almost preserved by the Hamiltonian evolution; specifically, we shall find a consistent approximation scheme (at least for dust filled universes) where this is true.

Also, we wish to emphasize that the failure of some semi-classical state to lie on a horizontal section preserved under evolution does *not* mean that the quantum evolution of such a state is far from semi-classical. It only means that the full quantum evolution cannot be incorporated entirely in terms of an effective Hamiltonian on just the classical phase space. In terms of equation (4.1.47), a horizontal section for that is preserved by the quantum evolution is one for which the 'smaller corrections' on the right hand side vanish entirely: the complete dynamics can be encoded into a Hamiltonian (in this case, more specifically a potential) defined on the classical phase space. But even if these smaller corrections are not exactly zero, it can still be the case that they are very much smaller than the leading classical term (at least) and so we still have approximately classical behavior. It may even be the case that the corrections are smaller than the 'corrected potential,' in which case we can even meaningfully find corrections to the classical equations of motion that are induced by quantum effects. In effect, this is what we are claiming occurs when we make the approximations sketched above.

Indeed, exactly such behavior was found in at least one other investigation of the semi-classical dynamics of quantum cosmology. In [56] it was found, within the framework of the consistent histories approach to quantum mechanics, that the quantum dynamics in suitable regimes of quantum mechanics did follow the classical dynamics, plus the addition of small effective forces. However, these forces were generally non-local in time, depending upon the entire history of the quantum state. Such non-local forces are simply an indication that the true phase space of the system is not the classical one, but a larger space. As long as such terms are small, however, we may still be near a domain where we can use and effective Hamiltonian. Part of our goal, therefore, is to 'split' these terms into corrections that come from an effective Hamiltonian on the classical phase space, and still smaller corrections that cannot be so incorporated.

4.2 Calculating the corrections

Having outlined in the last section the general strategy that we wish to follow, we turn now to implementing it in the specific case of loop quantum cosmology.

While the general strategy requires us to find a horizontal section of the quantum phase space that is preserved by the quantum evolution, we shall, as we indicated at the end of the last section, not satisfy this requirement exactly. Instead, following [20], we shall first restrict attention to sections comprised of Gaussian coherent states of the form (4.1.12). We see immediately that unless these coherent states remain coherent under the quantum evolution, there is no hope that the section is exactly preserved by the quantum evolution. Clearly this is at best an approximation; the question is just how good the approximation is.

It will not be perfect, but for the case of the Hamiltonian constraint in loop quantum cosmology we have particular reason to think it may be better than average. That comes from the following. If we multiply the expectation values given in (4.1.17) and (4.1.18) together, we find that to lowest order our coherent states satisfy:

$$\Delta c \Delta p = \frac{\gamma\kappa}{3}\frac{\hbar}{2}, \tag{4.2.1}$$

that is, they saturate the Heisenberg uncertainty bound. This is not surprising, as it is true in the Schrödinger representation as well and indeed is a well known property of coherent states. What *is* surprising is that if we time evolve this product of uncertainties, to lowest order the time derivative of the left hand side of (4.2.1) vanishes, for our particular Hamiltonian (for the proof of this result, see appendix B.3). This strongly suggests that coherent states evolve to other coherent states, at least to a high level of approximation.

Thus, our task becomes deciding how to choose the width of those coherent states as a function of phase space. To be an exactly horizontal section, it must be the case that this width is in fact constant. So here again we shall make another approximation, whose validity we examine later, and we will allow the width to vary as a function on the (classical) phase space, in order to have evolutions valid for all time (we will see why this is necessary shortly). Such sections therefore cannot be exactly horizontal, but we shall argue in section 4.2.2 that they are 'approximately' horizontal. Quantifying and proving this assertion is again an important question deserving of more study than we shall be able to give it here.

So once we have decided to allow the width of the Gaussian to vary over phase space, what restrictions can we place on it? Because we are now considering the dynamics of the Hamiltonian constraint, we no longer work in Cyl* but, as in chapter 3, restrict ourselves to the Hilbert space associated to a regular lattice. This time we focus on the regular lattice $|n\mu_o\rangle$ determined by the fundamental length scale μ_o. Suppose we pick some point $(c, N\mu_o)$

of the classical phase space; then the appropriate coherent state in the Hilbert space is:

$$|\psi\rangle = \sum_n e^{-\frac{1}{2}\epsilon^2(n-N)^2} e^{-\frac{i\mu_o c}{2}(n-N)} |n\mu_0\rangle \tag{4.2.2}$$

Here we have defined as before the parameter $\epsilon := \mu_o/d$, where d is the characteristic 'width' of the coherent state.

What restrictions do we want on N, c, and ϵ? We do not expect even approximately semi-classical behavior for all regions of phase space, so we must have some restrictions on c and N. We first list all of these restrictions (they are the same as those used in [20]) and then comment individually on their physical motivation:

1. $N \gg 1$
2. $c \ll 1$
3. $N\epsilon \gg 1$
4. $\epsilon \ll c$.

Because of the definitions of our variables, the first requirement physically corresponds to $p \gg \ell_{\mathrm{Pl}}^2$. That is, we want a large scale factor. This may seem intuitively obvious given that we are concerned with times late compared to the Planck scale, but recall that because we are in an open universe, the total volume is infinite and it is not as clear what this requirement means. However, returning to (4.1.11) we see that this is equivalent to requiring that the volume of our fiducial cell be much larger than the Planck volume.

The second requirement is that the extrinsic curvature integrated over the fiducial cell should be small; this again corresponds to late times. We already know that if we do *not* have $c \ll 1$, then in general the expectation value of $\hat{c}$ will not be close to its classical value. We can gain a little more insight into this requirement by noting that for a dust or radiation filled universe, this requirement reduces to requiring $\dot{a} \ll c_{\mathrm{light}}$. In terms of the scale factor at some initial time τ_0, this in turn means that $a_0 \ll \tau_0$, since $\dot{a}_0 = a_0/\tau_0$ for both dust and radiation. Thus, we see that for dust and radiation filled universes requirement two is equivalent to the requirement that the total volume of the fiducial cell be much less than the volume of the observable universe. This, together with requirement one, tells us that the time τ_0 must be much greater than the Planck time. Moreover, for the evolutions of dust or radiation filled universes, if requirements one and two hold at some initial time, then they continue to hold at all later times.

In requirements three and four we first meet restrictions on ϵ. These two requirements can be motivated by noting that they are equivalent to requiring

$$\frac{\Delta N}{N} \ll 1, \qquad \frac{\Delta c}{c} \ll 1. \tag{4.2.3}$$

That is, we want the uncertainties in our canonical variables to be much less than the expectation values of those variables. We can meaningfully apply this criteria because unlike the case, for instance, of a simple harmonic oscillator, at no time in our evolution (in a semi-classical domain) do either of N or c become zero. The variable c asymptotically approaches zero for late times in dust and radiation filled universes, but still never reaches it.

We can combine these restrictions into the single series of inequalities:

$$\frac{1}{N} \ll \epsilon \ll c \ll 1 \tag{4.2.4}$$

In this form we see that in particular we must have $1/N \ll c \ll 1$.

We also note that it is this equation which tells us that we cannot have an exactly horizontal section if we want to consider evolution for all times, because c approaches zero as the time approaches infinity, and so there is no constant value we can assign for ϵ that satisfies (4.2.4) for all sufficiently late times.

With these restrictions in mind, we may now make the necessary calculations of our needed expectation values, and use these restrictions as we make our approximations.

4.2.1 Asymptotic expansions for $\langle \hat{C} \rangle$, $\langle \dot{c} \rangle$, and $\langle \dot{p} \rangle$

We know that we need the expectation value of the Hamiltonian constraint, since that is what gives us our effective classical Hamiltonian. However, in order to apply our criteria that the vector field on phase space generated by our assignment of semi-classical states be Hamiltonian, we shall also need the expectation values of $\dot{c}$ and $\dot{p}$. We calculate all three in this subsection, but we only outline the calculation of $\langle \hat{C}_{\text{grav}} \rangle$ in any detail, since the other calculations are similar.

A word on notation: for simplicity, from here onward we drop the notation to indicate that $\hat{C}_{\text{grav}}$ depends on μ_o, but of course the dependence is still there.

We remind the reader that the way we calculate the necessary expectation values is to use the non-symmetrized constraint, and then take the real part of the resulting expression at the end of the calculation. Starting from the state (4.2.2) and the result (2.4.14) that $\langle \Psi \,|\, \Psi \rangle = \sqrt{\pi}/\epsilon$, we find from our expression for the action of the Hamiltonian constraint

(4.1.42) (with the appropriate factor of $\operatorname{sgn} p = \operatorname{sgn} n$ included) that:

$$\begin{aligned}
\langle \hat{C}_{\mathrm{grav}} \rangle &= \frac{3\epsilon}{\sqrt{\pi}} (\kappa\gamma^3 \mu_o^3 \ell_{\mathrm{Pl}}^2)^{-1} \sum_{n,n'} e^{-\frac{1}{2}\epsilon^2\left[(n'-N)^2+(n-N)^2\right]} e^{\frac{i\mu_o c}{2}(n'-n)} \operatorname{sgn} n \\
&\quad \times \left(V_{(n+1)\mu_o} - V_{(n-1)\mu_o}\right) \left[\langle n'\mu_o \,|\, (n+4)\mu_o\rangle - 2\langle n'\mu_o \,|\, n\mu_o\rangle + \langle n'\mu_o \,|\, (n-4)\mu_o\rangle\right] \\
&= \frac{3\ell_{\mathrm{Pl}}\epsilon}{\kappa\sqrt{\pi}} (6\gamma\mu_o^3)^{-\frac{3}{2}} \left[e^{2i\mu_o c} \sum_n e^{-\frac{1}{2}\epsilon^2\left[(n+4-N)^2+(n-N)^2\right]} \left(|n+1|^{\frac{3}{2}} - |n-1|^{\frac{3}{2}}\right) \operatorname{sgn} n \right. \\
&\quad + \sum_n e^{-\epsilon^2(n-N)^2} \left(|n+1|^{\frac{3}{2}} - |n-1|^{\frac{3}{2}}\right) \operatorname{sgn} n \\
&\quad \left. + e^{-2i\mu_o c} \sum_n e^{-\frac{1}{2}\epsilon^2\left[(n-4-N)^2+(n-N)^2\right]} \left(|n+1|^{\frac{3}{2}} - |n-1|^{\frac{3}{2}}\right) \operatorname{sgn} n \right] \\
&= \frac{3\ell_{\mathrm{Pl}}\epsilon}{\kappa\sqrt{\pi}} (6\gamma\mu_o^3)^{-\frac{3}{2}} \left[e^{2i\mu_o c} e^{-4\epsilon^2} \sum_n e^{-\epsilon^2(n-N)^2} \left(|n-1|^{\frac{3}{2}} - |n-3|^{\frac{3}{2}}\right) \operatorname{sgn}(n+2) \right. \\
&\quad + \sum_n e^{-\epsilon^2(n-N)^2} \left(|n+1|^{\frac{3}{2}} - |n-1|^{\frac{3}{2}}\right) \operatorname{sgn} n \\
&\quad \left. + e^{-2i\mu_o c} e^{-4\epsilon^2} \sum_n e^{-\epsilon^2(n-N)^2} \left(|n+3|^{\frac{3}{2}} - |n+1|^{\frac{3}{2}}\right) \operatorname{sgn}(n-2) \right] \\
&= \frac{3\ell_{\mathrm{Pl}}\epsilon}{\kappa\sqrt{\pi}} (6\gamma\mu_o^3)^{-\frac{3}{2}} \left[e^{-4\epsilon^2 + 2i\mu_0 c} \big(S_{-1,2}(N,\epsilon) - S_{-3,2}(N,\epsilon)\big) \right. \\
&\quad \left. - 2\big(S_{1,0}(N,\epsilon) - S_{-1,0}(N,\epsilon)\big) + e^{-4\epsilon^2 - 2i\mu_0 c} \big(S_{3,-2}(N,\epsilon) - S_{1,-2}(N,\epsilon)\big) \right]
\end{aligned} \tag{4.2.5}$$

In going from the first equality to the second we have used the value (4.1.11):

$$V_\mu = \ell_{\mathrm{Pl}}^3 \left(\frac{\gamma}{6}\right)^{\frac{3}{2}} |\mu|^{\frac{3}{2}}. \tag{4.2.6}$$

of the volume eigenvalues V_μ; in going from the second equality to the third we have completed the squares in the real exponentials and shifted the indices of summation; and in going from the third equality to the last we have defined:

$$S_{m,k}(N,\epsilon) := \sum_n e^{-\epsilon^2(n-N)^2} |n+m|^{\frac{3}{2}} \operatorname{sgn}(n+k). \tag{4.2.7}$$

We emphasize that the expression (4.2.5) is *exact*; no approximations have been made, nor have we had to use any of our assumptions (4.2.4) about what states and phase space points $(c, N\mu_o)$ should correspond to semi-classical behavior. If we do use the assumption that ϵ is small (as we must, for a sharply peaked state) then we see from (4.2.5) and (4.2.7) that it is not at all clear what the 'leading order' behavior of this expression is. That is because for small ϵ, the sums of (4.2.7) converge very slowly, and so once again we cannot

approximate them well by keeping only the first few terms.

We have encountered this problem several times before in this thesis, and we shall apply the solution that has been so successful before: Poisson re-summation to convert the slowly convergent sums of (4.2.7) into rapidly convergent sums that we may easily approximate by using the first few terms (actually, simply the first).

Now, however, we will encounter several new technical complications when we attempt to apply the Poisson re-summation formula. We recall that this formula is premised on the identity

$$\sum_n f(n) = \sum_n \int_{-\infty}^{\infty} e^{2\pi i n y} f(y)\, dy, \tag{4.2.8}$$

valid whenever the sums and integrals all converge. An inspection of (4.2.7) shows that now we will obtain integrals that we will not be able to evaluate in any (useful) closed form; we shall have to approximate these integrals. We will do that by using the method of steepest descents, and that in turn introduces an additional complication, because to apply that technique (which requires us to move the contour of integration) we must have an analytic integrand and $f(n)$ for the sums (4.2.7) is not analytic *anywhere*, because of the absolute values in its definition.

Thus, we shall proceed in two steps. We break up each sum $S_{m,k}$ into a sum of a function that *is* analytic, and another sum whose magnitude will turn out to be exponentially small in comparison. We then apply Poisson re-summation to the sum with analytic summand, and evaluate the integrals appearing in the re-summation using the method of steepest descents. This in turn will lead us to an asymptotic expansion for the analytic sum, and hence ultimately for $\langle \hat{C}_{\text{grav}} \rangle$ as well.

Thus, we begin by defining:

$$S_{m,k}(N,\epsilon) = S_m(N,\epsilon) + \delta S_{m,k}(N,\epsilon) \tag{4.2.9}$$

where:

$$S_m(N,\epsilon) = \sum_{n=-\infty}^{\infty} e^{-\epsilon^2 (n-N)^2} (n+m)^{\frac{3}{2}} \tag{4.2.10}$$

and:

$$\begin{aligned}\delta S_{m,k}(N,\epsilon) &= \sum_{n=-\infty}^{\infty} e^{-\epsilon^2(n-N)^2}|n+m)|^{\frac{3}{2}}\,\mathrm{sgn}\,(n+k) \\ &\quad - \sum_{n=-\infty}^{\infty} e^{-\epsilon^2(n-N)^2}(n+m)^{\frac{3}{2}} \\ &= \sum_{n=-\infty}^{\max(-m,-k)} \left(e^{-\epsilon^2(n-N)^2}|n+m)|^{\frac{3}{2}}\,\mathrm{sgn}\,(n+k) - e^{-\epsilon^2(n-N)^2}(n+m)^{\frac{3}{2}}\right)\end{aligned} \tag{4.2.11}$$

We show in appendix B.2 that we may bound $|\delta S_{m,k}|$ by an exponentially suppressed term, so we set aside that contribution to $S_{m,k}$ for the moment and focus on calculating $S_m(N,\epsilon)$.

Applying Poisson re-summation to $S_m(N,\epsilon)$ we get:

$$\begin{aligned}S_m(N,\epsilon) &= \sum_n \int_{-\infty}^{\infty} e^{-\epsilon^2(y-N)^2} e^{2\pi i n x}(y+m)^{\frac{3}{2}}\,dy \\ &= N^{\frac{5}{2}} \sum_n \int_{-\infty}^{\infty} e^{-N^2\epsilon^2(x-1)^2} e^{2\pi i n N x}\left(x+\frac{m}{N}\right)^{\frac{3}{2}}\,dx\end{aligned} \tag{4.2.12}$$

In order to evaluate this integral we apply the method of steepest descents, which in our case is the same as completing the square in the (complex) exponential:

$$-N^2\epsilon^2(x-1)^2 + 2\pi i n N x = -N^2\epsilon^2\left(x-1-\frac{i\pi n}{N\epsilon^2}\right)^2 + 2\pi i n N - \frac{\pi^2 n^2}{\epsilon^2}, \tag{4.2.13}$$

so that:

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{-N^2\epsilon^2(x-1)^2} e^{2\pi i n N x}\left(x+\frac{m}{N}\right)^{\frac{3}{2}}\,dx \\ &\qquad = e^{-\frac{\pi^2 n^2}{\epsilon^2}} \int_{-\infty}^{\infty} e^{-N^2\epsilon^2\left(x-1-\frac{m}{N}-\frac{i\pi n}{N\epsilon^2}\right)^2} x^{\frac{3}{2}}\,dx\end{aligned} \tag{4.2.14}$$

We are therefore seeking to move the contour from the real axis to the line:

$$z = 1 + x + \frac{i\pi n}{N\epsilon^2} \tag{4.2.15}$$

for real x. Now, equation (4.2.10)—and hence equation (4.2.11)—requires us to make a choice of branch cut in order to complete its definition. We now see that this choice of branch cut is dictated by the requirement that we be able to move the contour to the steepest path without crossing the branch cut. Thus, inspection of (4.2.15) shows us that

we must choose the branch cut below the real axis if $n > 0$, and above the real axis if $n < 0$. Accordingly, we choose the branch cut to be the negative imaginary axis for positive n, and the positive imaginary axis for negative n. When $n = 0$ there is no need to move the contour at all, and we choose the branch cut to lie along the negative real axis.

With this convention, we see that

$$S_m(N, \epsilon) = N^{\frac{5}{2}} \sum_n e^{-\frac{\pi^2 n^2}{\epsilon^2}} I_n(N, m, \epsilon) \tag{4.2.16}$$

where:

$$I_n(N, m, \epsilon) := \int_{-\infty}^{\infty} e^{-N^2\epsilon^2\left(x-1-\frac{m}{N}-\frac{i\pi n}{N\epsilon^2}\right)^2} x^{\frac{3}{2}}\, dx \tag{4.2.17}$$

and our task is now to evaluate, or at least estimate, the integrals $I_n(N, m, \epsilon)$.

After we move the contour of integration to the steepest path,[5] these integrals are:

$$I_n(N, m, \epsilon) = \int_{-\infty}^{\infty} e^{-N^2\epsilon^2\left(x-\frac{m}{N}\right)^2} \left(x + 1 + \frac{i\pi n}{N\epsilon^2}\right)^{\frac{3}{2}} dx. \tag{4.2.18}$$

These integrals do not have any (useful) closed form expression. What we seek instead is an asymptotic series expansion of this integral, in terms of the parameter $N\epsilon$, which we know to be large. We can obtain such an asymptotic series by Taylor series expanding the function

$$g_n(x) := \left(x + 1 + \frac{i\pi n}{N\epsilon^2}\right)^{\frac{3}{2}} \tag{4.2.19}$$

in x and integrating the resulting series term by term.

As we prove in appendix B.1, this approach does indeed lead to an asymptotic series for $I_n(N, m, \epsilon)$ for each n. However, inspection of equation (4.2.16) shows us that all but the $n = 0$ terms will be strongly suppressed, provided that the integral $I_n(N, m, \epsilon)$ does not grow too rapidly with n. We therefore also prove in appendix B.1 that we may bound $I_n(N, m, \epsilon)$ by:

$$|I_n(N, m, \epsilon)| \leq \frac{\sqrt{2\pi}}{N\epsilon}\left(2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right) e^{27\epsilon^2}. \tag{4.2.20}$$

Thus, since we will obtain a power series in ϵ for $I_0(N, m, \epsilon)$, we see that we may safely neglect the contributions from all non-zero values of n, since they are suppressed exponentially compared to any finite power of ϵ. Hence we do not use the asymptotic series expansion of $I_n(N, m, \epsilon)$ for non-zero n, even though it is available.

[5] It proves more convenient to keep the term $\frac{m}{N}$ in the exponential rather than the radical, as this eliminates the need to further expand the Taylor series coefficients in order to combine the sums appearing in the expectation value of the constraint.

We do however need the asymptotic series for $I_0(N, m\epsilon)$. We find:

$$I_0(N, m, \epsilon) \sim \frac{\sqrt{\pi}}{N\epsilon} \sum_{l=0}^{\infty} \frac{g_l}{(2iN\epsilon)^l} H_l(im\epsilon). \tag{4.2.21}$$

Here the $H_l(x)$ are the Hermite polynomials, and the g_l are the Taylor series coefficients of the expansion of $(1+x)^{\frac{3}{2}}$. Explicitly, the latter are:

$$g_l = \begin{cases} 1 & \text{for } l = 0, \\ \frac{\frac{3}{2} \cdot \frac{1}{2} \cdots (\frac{3}{2} - l + 1)}{l!} & \text{for } l \geq 1. \end{cases} \tag{4.2.22}$$

We now return to the question of the correction term $\delta S_{m,k}$. As we have mentioned, we show it in appendix B.2 to be exponentially suppressed, compared to any finite power of $N\epsilon$. This allows us to make use of a property of asymptotic series: they do not uniquely identify a function, since two functions may have the same asymptotic series if their difference goes to zero faster than any polynomial in the asymptotic parameter. This is clearly the case for us, and thus we can simply *drop* the term $\delta S_{m,k}$; it is subsumed by the '$\sim$' relationship.

Hence, when we plug back in all of our expansions, we obtain:

$$\begin{aligned} \langle \hat{C}_{\text{grav}} \rangle \sim \frac{3\ell_{\text{Pl}}}{\kappa} (6\gamma\mu_o^3)^{-\frac{3}{2}} N^{\frac{3}{2}} \sum_{l=0}^{\infty} \frac{g_l}{(2iN\epsilon)^l} \Bigg\{ & e^{-4\epsilon^2 + 2i\mu_0 c} \big[H_l(-i\epsilon) - H_l(-3i\epsilon) \big] \\ & - 2\big[H_l(i\epsilon) - H_l(-i\epsilon) \big] + e^{-4\epsilon^2 - 2i\mu_0 c} \big[H_l(3i\epsilon) - H_l(i\epsilon) \big] \Bigg\} \end{aligned} \tag{4.2.23}$$

Then making use of the parity of the Hermite polynomials:

$$H_l(-x) = (-1)^l H_l(x) \tag{4.2.24}$$

we can re-express this as:

$$\begin{aligned} \langle \hat{C}_{\text{grav}} \rangle \sim \frac{3\ell_{\text{Pl}}}{\kappa} (6\gamma\mu_o^3)^{-\frac{3}{2}} N^{\frac{3}{2}} \Bigg\{ & \sum_{l=0}^{\infty} \frac{i g_{2l}}{(2iN\epsilon)^{2l}} \Big[e^{-4\epsilon^2} \sin(2\mu_0 c) \Big(H_{2l}(i\epsilon) - H_{2l}(3i\epsilon) \Big) \Big] \\ & + \sum_{l=0}^{\infty} \frac{g_{2l+1}}{(2iN\epsilon)^{2l+1}} \Big[e^{-4\epsilon^2} \cos(2\mu_0 c) \Big(H_{2l+1}(3i\epsilon) - H_{2l+1}(i\epsilon) \Big) - 2H_{2l+1}(i\epsilon) \Big] \Bigg\} \end{aligned} \tag{4.2.25}$$

We see explicitly from (4.2.25) that because we have not used the symmetrized constraint, we have obtained a complex expectation value. However, as we have mentioned the solution to this is simply to drop the imaginary terms; this corresponds to replacing the operator $\hat{C}_{\text{grav}}$ with which we calculated (4.2.25) with the operator $\frac{1}{2}(\hat{C}_{\text{grav}} + \hat{C}^{\dagger}_{\text{grav}})$, which is manifestly

symmetric. When this is done we obtain for our final expression:

$$\langle \hat{C}_{\rm grav} \rangle \sim \frac{3\ell_{\rm Pl}}{\kappa}(6\gamma\mu_o^3)^{-\frac{3}{2}}N^{\frac{3}{2}}\left\{\sum_{l=0}^{\infty}\frac{g_{2l+1}}{(2iN\epsilon)^{2l+1}} \times\left[e^{-4\epsilon^2}\cos(2\mu_0 c)\Big(H_{2l+1}(3i\epsilon)-H_{2l+1}(i\epsilon)\Big)-2H_{2l+1}(i\epsilon)\right]\right\} \quad (4.2.26)$$

Equation (4.2.26) gives us a systematic asymptotic expansion of $\langle \hat{C}_{\rm grav}\rangle$ to all orders. As we shall soon see, its lowest order term is in fact the classical term, as we should desire. Before examining this leading order behavior, though, we summarize the results one obtains not just for $\langle \hat{C}_{\rm grav}\rangle$, but for $\langle \hat{C}\rangle$ itself, as well as $\langle \dot{c}\rangle$ and $\langle \dot{p}\rangle$, for the cases of both dust and radiation. To find the time derivative expectation values, we calculate the commutator of the operator in question with the full quantum constraint, find the expectation value of this quantity, and then multiply that by $-i/\hbar$. The sums appearing in the expectation values can be handled in the same manner as those for $\langle \hat{C}_{\rm grav}\rangle$ above, by splitting off an exponentially small, non-analytic sum, applying Poisson re-summation to the remaining sum, and asymptotically expanding the resulting integral. For completeness the needed asymptotic expansions are all listed at the end of appendix B.1.

When this is done, we obtain for dust:

$$\langle C\rangle \sim \frac{3\ell_{\rm Pl}}{\kappa}(6\gamma\mu_o^3)^{-\frac{3}{2}}N^{\frac{3}{2}}\left\{\sum_{l=0}^{\infty}\frac{g_{2l+1}}{(2iN\epsilon)^{2l+1}} \times\left[e^{-4\epsilon^2}\cos(2\mu_0 c)\Big(H_{2l+1}(3i\epsilon)-H_{2l+1}(i\epsilon)\Big)-2H_{2l+1}(i\epsilon)\right]\right\}+\frac{1}{2}E_0, \quad (4.2.27)$$

$$\langle \dot{c}\rangle \sim \frac{N^{\frac{3}{2}}}{\mu_0\ell_{\rm Pl}\sqrt{6}(\gamma\mu_o)^{\frac{3}{2}}}\left\{\left[e^{-\frac{25\epsilon^2}{4}}\cos\left(\frac{5\mu_o c}{2}\right)+e^{-\frac{9\epsilon^2}{4}}\cos\left(\frac{3\mu_o}{2}\right)\right] \times\left(\sum_{l=0}^{\infty}\frac{g_{2l}}{(2iN\epsilon)^{2l}}\left[H_{2l}\left(\frac{7i\epsilon}{2}\right)-H_{2l}\left(\frac{5i\epsilon}{2}\right)-H_{2l}\left(\frac{3i\epsilon}{2}\right)+H_{2l}\left(\frac{i\epsilon}{2}\right)\right]\right) -4e^{-\frac{\epsilon^2}{4}}\cos\left(\frac{\mu_o c}{2}\right)\sum_{l=0}^{\infty}\frac{g_{2l}}{(2iN\epsilon)^{2l}}\left[H_{2l}\left(\frac{3i\epsilon}{2}\right)-H_{2l}\left(\frac{i\epsilon}{2}\right)\right]\right\}, \quad (4.2.28)$$

$$\langle \dot{p}\rangle \sim \frac{2\ell_{\rm Pl}N^{\frac{3}{2}}e^{-4\epsilon^2}}{3(6\gamma\mu_o)^{\frac{1}{2}}}\sin(2\mu_o c)\sum_{l=0}^{\infty}\frac{g_{2l+1}}{(2iN\epsilon)^{2l+1}}\left[H_{2l+1}(3i\epsilon)-H_{2l+1}(i\epsilon)\right]. \quad (4.2.29)$$

For notational simplicity, we have omitted the hats over operators, and generally shall for the rest of this section.

To obtain the results for radiation we must use for the matter constraint $\frac{1}{2}E_0\sqrt{p_0}$ times

the inverse scale factor operator, as discussed in section 4.1.2. When we do this, we obtain:

$$\begin{aligned}
\langle C\rangle \;\sim\; & \frac{3\ell_{\rm Pl}}{\kappa}(6\gamma\mu_o^3)^{-\frac{3}{2}}N^{\frac{3}{2}}\left\{\sum_{l=0}^{\infty}\frac{g_{2l+1}}{(2iN\epsilon)^{2l+1}}\right. \\
& \left.\times\left[e^{-4\epsilon^2}\cos\left(2\mu_0 c\right)\Big(H_{2l+1}(3i\epsilon)-H_{2l+1}(i\epsilon)\Big)-2H_{2l+1}(i\epsilon)\right]\right\} \\
& + E_0\sqrt{p_0}\left(\frac{6\mu_o}{\gamma\ell_{\rm Pl}^2}\right)^{\frac{1}{2}}N^{\frac{1}{2}}\sum_{l=0}^{\infty}\frac{f_{2l+1}}{(2iN\epsilon)^{2l+1}}H_{2l+1}\left(\frac{i\epsilon}{\mu_o}\right), \qquad (4.2.30)\\
\langle\dot{c}\rangle \;\sim\; & \frac{N^{\frac{3}{2}}}{\mu_0\ell_{\rm Pl}\sqrt{6}(\gamma\mu_o)^{\frac{3}{2}}}\left\{\left[e^{-\frac{25\epsilon^2}{4}}\cos\left(\frac{5\mu_o c}{2}\right)+e^{-\frac{9\epsilon^2}{4}}\cos\left(\frac{3\mu_o}{2}\right)\right]\right. \\
& \times\left(\sum_{l=0}^{\infty}\frac{g_{2l}}{(2iN\epsilon)^{2l}}\left[H_{2l}\left(\frac{7i\epsilon}{2}\right)-H_{2l}\left(\frac{5i\epsilon}{2}\right)-H_{2l}\left(\frac{3i\epsilon}{2}\right)+H_{2l}\left(\frac{i\epsilon}{2}\right)\right]\right) \\
& \left.-4e^{-\frac{\epsilon^2}{4}}\cos\left(\frac{\mu_o c}{2}\right)\sum_{l=0}^{\infty}\frac{g_{2l}}{(2iN\epsilon)^{2l}}\left[H_{2l}\left(\frac{3i\epsilon}{2}\right)-H_{2l}\left(\frac{i\epsilon}{2}\right)\right]\right\} \\
& + \frac{12E_0\sqrt{p_0}}{\gamma\mu_o\ell_{\rm Pl}\hbar}\left(\frac{\gamma\mu_o}{6}\right)^{\frac{1}{2}}N^{\frac{1}{2}}e^{-\frac{\epsilon^2}{2}}\cos\frac{\mu_o c}{2} \\
& \times\sum_{l=0}^{\infty}\frac{f_{2l}}{(2iN\epsilon)^{2l}}\left[H_{2l}\left(\frac{i\epsilon}{2}+\frac{i\epsilon}{\mu_o}\right)-H_{2l}\left(\frac{i\epsilon}{2}-\frac{i\epsilon}{\mu_o}\right)\right] \qquad (4.2.31)\\
\langle\dot{p}\rangle \;\sim\; & \frac{2\ell_{\rm Pl}N^{\frac{3}{2}}e^{-4\epsilon^2}}{3(6\gamma\mu_o)^{\frac{1}{2}}}\sin\left(2\mu_o c\right)\sum_{l=0}^{\infty}\frac{g_{2l+1}}{(2iN\epsilon)^{2l+1}}\left[H_{2l+1}(3i\epsilon)-H_{2l+1}(i\epsilon)\right]. \qquad (4.2.32)
\end{aligned}$$

In these equations, f_l are the coefficients of the Taylor expansion of $f(x)=(1+x)^{\frac{1}{2}}$ about $x=0$. Observe from (4.2.32) that the expression for $\langle\dot{p}\rangle$ is the same for both dust and radiation filled universes; this is because the matter term of the Hamiltonian constraint commutes with $\hat{p}$.

Equations (4.2.27) to (4.2.29) and (4.2.30) to (4.2.32) represent a complete solution to the problem of systematically expanding the leading classical terms and corrections to all orders of the expectation values of the constraint and time derivatives of the canonical variables. Of course, to actually verify that the leading order term is the classical one, and to gain some insight into the corrections, we need to look not at these complete expansions, but just the lowest order terms.

Moreover, as they stand these expansions are not in terms of very useful variables. It is easy enough to convert the variable N to the canonical variable p using $p:=\frac{1}{6}\gamma\mu_0\ell_{\rm Pl}^2N$. However, this still leaves us with expressions in terms of p and c, and neither of these is directly observable. That is because according to our strategy, what we should be treating as basic observables for the corrected equations of motion are not c and p themselves, but rather their expectation values. So we define $\overline{p}=\langle p\rangle$ and $\overline{c}=\langle c\rangle$; then from (4.1.15) and

(4.1.16) we see that $\overline{p}$ and $\overline{c}$ are related to the (unobservable) p and c through:

$$\overline{p} = p, \qquad \overline{c} = \frac{2}{\mu_o} e^{-\frac{\epsilon^2}{4}} \sin\Big(\frac{\mu_o c}{2}\Big). \tag{4.2.33}$$

We can then solve the second of these for c in terms of $\overline{c}$, and use various trigonometric identities to replace the trigonometric functions of c appearing in the asymptotic expansions above with algebraic functions of $\overline{c}$.

When we do this, and keep only the leading order and next to leading order terms in our expansions, we obtain for dust filled universes:

$$\begin{aligned}
\overline{C} &= -6\kappa^{-1}\gamma^{-2} e^{-\frac{7\epsilon^2}{2}} \overline{c}^2 \sqrt{\overline{p}} + \frac{3}{2}\kappa^{-1}\gamma^{-2}\mu_o^2 e^{-3\epsilon^2}\overline{c}^4\sqrt{\overline{p}} \\
&\qquad + 3\kappa^{-1}\gamma^{-2}\mu_o^{-2}(e^{-4\epsilon^2}-1)\sqrt{\overline{p}} + \frac{1}{2}E_0, \qquad (4.2.34)\\
\dot{\overline{c}} &= \frac{e^{-\frac{\epsilon^2}{4}}}{4\gamma\mu_o^2\sqrt{\overline{p}}}\sqrt{1-\frac{\mu_o^2}{4}e^{\frac{\epsilon^2}{2}}\overline{c}^2}\Bigg[-\left(3e^{-\frac{11\epsilon^2}{2}}+e^{-\frac{3\epsilon^2}{2}}\right)\mu_o^2\overline{c}^2 \\
&\qquad + \mu_0^4 e^{-5\epsilon^2}\overline{c}^4 + \left(e^{-6\epsilon^2}+e^{-2\epsilon^2}-2\right)\Bigg], \qquad (4.2.35)\\
\dot{\overline{p}} &= 4\gamma^{-1}e^{-\frac{15\epsilon^2}{4}}\sqrt{\overline{p}}\sqrt{1-\frac{\mu_o^2}{4}e^{\frac{\epsilon^2}{2}}\overline{c}^2}\;\overline{c}\left(1-\frac{\mu_o^2}{2}e^{\frac{\epsilon^2}{2}}\overline{c}^2\right); \qquad (4.2.36)
\end{aligned}$$

whereas for radiation filled we obtain:

$$\begin{aligned}
\overline{C} &= -6\kappa^{-1}\gamma^{-2} e^{-\frac{7\epsilon^2}{2}} \overline{c}^2 \sqrt{\overline{p}} + \frac{3}{2}\kappa^{-1}\gamma^{-2}\mu_o^2 e^{-3\epsilon^2}\overline{c}^4\sqrt{\overline{p}} \\
&\qquad + 3\kappa^{-1}\gamma^{-2}\mu_o^{-2}(e^{-4\epsilon^2}-1)\sqrt{\overline{p}} + \frac{1}{2}E_0\sqrt{p_0}\;\overline{p}^{-\frac{1}{2}} \\
&\qquad + \frac{E_0\sqrt{p_0}\gamma^2\mu_o^2\ell_{\mathrm{Pl}}^4}{384}\frac{\overline{p}^{-\frac{5}{2}}}{\epsilon^2}, \qquad (4.2.37)\\
\dot{\overline{c}} &= \frac{e^{-\frac{\epsilon^2}{4}}}{4\gamma\mu_o^2\sqrt{\overline{p}}}\sqrt{1-\frac{\mu_o^2}{4}e^{\frac{\epsilon^2}{2}}\overline{c}^2}\Bigg[-\left(3e^{-\frac{11\epsilon^2}{2}}+e^{-\frac{3\epsilon^2}{2}}\right)\mu_o^2\overline{c}^2 \\
&\qquad + \mu_0^4 e^{-5\epsilon^2}\overline{c}^4 + \left(e^{-6\epsilon^2}+e^{-2\epsilon^2}-2\right)\Bigg] \\
&\qquad - \frac{\kappa\gamma E_o\sqrt{p_0}}{6}e^{-\frac{\epsilon^2}{4}}\sqrt{1-\frac{\mu_o^2}{4}e^{\frac{\epsilon^2}{2}}\overline{c}^2}\left[\frac{1}{2}\overline{p}^{-\frac{3}{2}} + \frac{5}{384}\gamma^2\mu_0^2\ell_{\mathrm{Pl}}^4\frac{\overline{p}^{-\frac{7}{2}}}{\epsilon^2}\right], \qquad (4.2.38)\\
\dot{\overline{p}} &= 4\gamma^{-1}e^{-\frac{15\epsilon^2}{4}}\sqrt{\overline{p}}\sqrt{1-\frac{\mu_o^2}{4}e^{\frac{\epsilon^2}{2}}\overline{c}^2}\;\overline{c}\left(1-\frac{\mu_o^2}{2}e^{\frac{\epsilon^2}{2}}\overline{c}^2\right). \qquad (4.2.39)
\end{aligned}$$

Observe that there are no terms coming from higher orders in the asymptotic expansion for dust; it it possible to show that all terms from the next order in the asymptotic expansion

are much smaller than the corrections already included. The same is not true for radiation, however; there are terms proportional to ϵ^{-2} that come from the next order in the asymptotic expansion and are not obviously smaller than the terms of order ϵ^2 or $\overline{c}^4$ that also appear as corrections.

It is also possible using these expansions to see that the leading order behavior is indeed the classical behavior. So long as $\epsilon \ll c$, the leading term in $\langle \hat{C}_{\rm grav} \rangle$ is the term of order $\overline{c}^2\sqrt{\overline{p}}$. When we keep only such terms, we find exactly the classical expressions of (4.1.39) and (4.1.40), only written in terms of $\overline{c}$ and $\overline{p}$ instead of c and p. Moreover, the time derivatives are given exactly by (4.1.34) and (4.1.35), again provided we make the substitutions of $\overline{c}$ for c and $\overline{p}$ for p. Thus, to leading order the Poisson brackets are not modified either.

It was also shown in [20] that the correct classical equations of motion arise as the leading order, but that analysis was less thorough than what we have just shown in several respects. First, that work did not consider the expressions in terms of $\overline{c}$ and $\overline{p}$, but rather c and p, which as we have noted is not a consistent approach, if one is taking expectation values to be what a 'classical' observer sees. Second, that reference considered only the expectation value of the Hamiltonian constraint, and did not consider at all the time derivatives or verify that the Poisson brackets were unmodified. Finally, there was no explicit calculation of the next to leading order terms, nor verification that the series was in fact asymptotic.

Our ultimate goal, however, is not simply a technically better proof that the correct classical behavior is obtained from the leading order equations of motion, but a calculation of the corrections to that behavior as well. That is a more subtle calculation, and we turn to it next.

4.2.2 Effective classical dynamics: results for a dust filled universe

In this subsection we take up the question of whether the next to leading order behavior of the dynamics of loop quantum cosmology affords an effective classical description, according to the program sketched in section 4.1.3. For simplicity, we focus here only on the case of a dust filled universe, leaving radiation filled universes to future work (we shall comment below on the difficulty that prevents the radiation filled universe from being straightforward).

As we have already explained, we shall be choosing a section that corresponds to choosing a Gaussian coherent state for each point in the classical phase space, with ϵ allowed to vary from point to point in our phase space, so long as it satisfies all of the restrictions of (4.2.4). We wish to find a consistent order of approximation such that to that order of approximation both of the following are true:

1. The section is horizontal,

2. The section is preserved by the quantum evolution.

For our purposes, we shall say that the section is approximately horizontal if we can (to our order of approximation) consistently use the classical symplectic structure on the phase space; as noted earlier it is always the case that the pull back of the quantum symplectic structure to a *horizontal* section agrees exactly with the classical symplectic structure.

We shall say that the second criterion holds to our order of approximation if it is true that

$$\dot{\overline{c}} = \{\overline{c}, \overline{C}\}, \qquad \dot{\overline{p}} = \{\overline{p}, \overline{C}\}. \tag{4.2.40}$$

In these equations, $\overline{C}$ is the expectation value of the constraint operator, and the Poisson brackets are calculated using the classical brackets. On the left hand side we use the expressions found from the exact expectation values of the time derivatives of our basic variables, as calculated in the preceding subsection.

Now, the order of approximation we wish to make is the following: we will simply neglect all terms of order ϵ^2 or higher. We must verify that we can do this consistently and still satisfy the requirements of (4.2.4). For a dust filled universe we shall see that we can; for radiation we cannot and that is why we defer examination of the radiation filled universe to future work.

To see that this is consistent for a dust filled universe, we observe that inspection of equations (4.2.34) through (4.2.36) tells us that we may consistently neglect ϵ^2 terms compared to all others if, in addition to the restrictions contained in (4.2.4), we have $\epsilon \ll \overline{c}^2$. But comparing this with (4.2.4), we see that we would then need $1/N \ll \overline{c}^2$, and so our consistency boils down to verifying that if this inequality holds at an initial time, it is preserved under time evolution.

But this is easy to check using the classical equations of motion for a dust filled universe. These imply (setting the constraint for a dust filled universe to zero) that $\overline{c}^2 \propto 1/\sqrt{N}$, and so if we have $1/N \ll 1/\sqrt{N}$ initially, then since N grows without bound as τ increases, this inequality will continue to hold.

For a radiation filled universe we would again need to have $\epsilon \ll \overline{c}^2$, and we would again find that if this criterion holds initially it continues to hold for all time. The problem, however, is that from equations (4.2.37) to (4.2.39) we see that we have another term appearing in the constraint, proportional to $\overline{p}^{-5/2}/\epsilon^2$. If we require this term to be smaller than the correction term proportional to $\overline{c}^4\sqrt{\overline{p}}$, we arrive at an inequality that even if satisfied initially, will not continue to hold for all time, under the classical evolution.

So, we obtain results in this thesis only for the case of dust filled universes. We are almost ready to verify that equation (4.2.40) holds, and thus that we have a consistent approximation, but we have one further subtlety to deal with. As we have mentioned, we are assuming that we may use the classical symplectic structure to determine the Poisson brackets. However, this symplectic structure is naturally expressed in terms of the variables

c and p, rather than the variables $\overline{c}$ and $\overline{p}$ that we have available to us, so we should like to re-express it in the latter variables. Since $\overline{p} = p$ to within our order of approximation, that is no problem; we can then calculate:

$$\begin{aligned}\Omega &= \frac{3}{\kappa\gamma} dc \wedge dp \\ &= \frac{3}{\kappa\gamma}\left(\frac{\partial c}{\partial \overline{c}}\right) d\overline{c} \wedge d\overline{p} \\ &= \frac{3}{\kappa\gamma}\frac{1}{\sqrt{1-\frac{\mu_o^2}{4}\overline{c}^2}} d\overline{c} \wedge d\overline{p}.\end{aligned} \tag{4.2.41}$$

where we have used equation (4.2.33) to calculate the necessary partial derivative. We then have for the Poisson brackets between any two functions:

$$\{f, g\} = \frac{\kappa\gamma}{3}\sqrt{1-\frac{\mu_o^2}{4}\overline{c}^2}\left(\frac{\partial f}{\partial \overline{c}}\frac{\partial g}{\partial \overline{p}} - \frac{\partial g}{\partial \overline{c}}\frac{\partial f}{\partial \overline{p}}\right). \tag{4.2.42}$$

Thus, we now have all of the ingredients in place to try to verify that equation (4.2.40) holds, if in fact it does. If we return to equations (4.2.34) to (4.2.36) and insert our approximation, we find:

$$\overline{C} = \kappa^{-1}\gamma^{-2}\sqrt{\overline{p}}\left(-6\overline{c}^2 + \frac{3}{2}\mu_o^2\overline{c}^4 - 12\mu_o^{-2}\epsilon^2\right) + \frac{1}{2}E_0. \tag{4.2.43}$$

$$\dot{\overline{c}} = \frac{1}{4\gamma\sqrt{\overline{p}}}\left(-4\overline{c}^2 + \mu_o^2\overline{c}^4\right)\sqrt{1-\frac{\mu_0^2}{4}\overline{c}^2} \tag{4.2.44}$$

$$\dot{\overline{p}} = 4\gamma^{-1}\sqrt{\overline{p}}\,\overline{c}\left(1-\frac{\mu_o^2}{2}\overline{c}^2\right)\sqrt{1-\frac{\mu_0^2}{4}\overline{c}^2} \tag{4.2.45}$$

Then, using these equations and the definition (4.2.42), we see that (4.2.40) does indeed hold. Thus, we can make a systematic approximation, and for the dust filled universe, equation (4.2.43) gives us our effective Hamiltonian constraint function. Note that it is the classical constraint function plus a small correction term, proportional to $\overline{c}^4$. Also observe that this correction comes with a factor of μ_o^2, so that it vanishes in the limit where $\mu_o \to 0$. This corresponds precisely to the limit in which quantum geometry effects are ignored, so we see that our corrections do indeed represent an effect of the underlying quantum geometry.

Finally, we should say something about the solution of the system given by equations (4.2.43) to (4.2.45). If we set equation (4.2.43) equal to zero we can solve for $\overline{c}^2$; when we do this and keep only the leading and next to leading terms, we can then substitute the result into (4.2.45) and integrate that to find $\overline{p}$ as a function of time. Again keeping

only the lowest and next to lowest order terms, we find when we do so that:

$$\sqrt{\overline{p}} \approx \left(\frac{3}{4}\beta\right)^{\frac{2}{3}} t^{\frac{2}{3}} - \frac{3\gamma\mu_o}{8}\beta \tag{4.2.46}$$

where in this equation:

$$\beta = 4\sqrt{\frac{\kappa E_0}{6}} \tag{4.2.47}$$

Comparing to equation (4.1.22), we see that we have the same leading order time dependence, plus a small correction that is constant in time and negative. Again we note that as $\mu_o \to 0$, this correction vanishes, implying its origin in quantum geometry.

Thus, we see that for dust filled universes we are able to carry through our program and obtain small corrections to the classical equations of motion that are the corrections induced by quantum gravity effects, within the framework of loop quantum cosmology. However, there is much that still needs to be understood about this result. Can we quantify—and verify—the assertions that the section we have studied is 'approximately' horizontal and 'approximately' preserved by the quantum dynamics? If so, can we improve on the approximations, and perhaps find higher order corrections? Finally, can similar techniques be applied to other forms of matter, particularly a radiation dominated universe? These questions form the basis of ongoing research.

4.3 Outlook

We have, in this chapter, seen that the polymer particle representation considered in the preceding two chapters is in fact the representation at the basis of loop quantum cosmology. Thus, we can apply the techniques and results of those chapters to study the low energy limit of loop quantum cosmology. We find, for instance, that there is a large supply of kinematically semi-classical states, as indeed was already noted in [20].

Our main focus, however, has been to study the dynamics of such semi-classical states. In particular, we found that we could:

- Calculate the expectation value of the quantum Hamiltonian constraint, expanding it as an asymptotic series whose terms were found to all orders. This was also done for the time derivatives of the expectation values of the two canonical variables coordinatizing the phase space.

- Introduce a general framework of when the quantum dynamics of a system admits an effective semi-classical Hamiltonian description.

- Apply, within certain approximations, this framework to the case of a dust filled universe to verify both that the leading order effective dynamics coincides with the clas-

sical dynamics, and also to find the next leading order correction.

It seems widely believed that because loop quantum gravity proceeds by quantizing Einstein's equations exactly, there are no corrections to the classical equations of motion induced by the quantum theory. We have shown concretely that this is *not* the case. Moreover, we also learned an important lesson for the full theory: in order to have a real expectation value for the constraint, it was necessary to use a different quantum operator than that ordinarily used in loop quantum cosmology, and this in turn suggests a modification of the scalar constraint in the full theory. Thus, we see even more directly than in the previous chapters that even an apparently simple finite dimensional system can have concrete implications for the full theory.

There are also important questions still unanswered, which will be the subject of ongoing research. The derivation of the quantum correction to the classical equation of motion relied on certain approximations; how good are these approximations, and how far can the correction be trusted? It would also be useful to know how precisely the general definition of effective Hamiltonian dynamics given relates, for instance, to more familiar notions such as an effective action: does the definition given allow the construction of an effective action, and if so does it agree with what is normally obtained? Additionally, we have considered thus far the corrected equations of motion only for a dust filled universe; can this be extended to a radiation filled, or to a universe with cosmological constant?

Finally, and most importantly, if the corrections can be robustly established as theoretical predictions of loop quantum cosmology, then the key question is what observational consequences these might have.

Chapter 5

Integration theory on $\overline{\mathcal{A}}$ for non-compact groups

In this chapter we turn from the questions of the low-energy limit of LQG which have comprised most of this thesis to questions of a more mathematical nature. In particular, we look at ways of extending the definition of integration theory on $\overline{\mathcal{A}}$ to the situation where the gauge group G is non-compact. In doing this we are concerned not only to develop an integration theory, but more specifically an integration theory that supports a diffeomorphism invariant measure on $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ such that the flux operators are well defined, essentially self-adjoint operators.

Although the motivation of the material in this chapter is substantially different from that in the rest of this thesis, one key construction will carry over: the use of the Bohr compactification. In fact, for general topological groups a slightly different compactification known as the *almost periodic* compactification turns out to be much more relevant. We will briefly discuss the difference between the two compactifications in subsection 5.2.1; for more details the reader should consult, for instance, chapter 4 of [57]. We shall not be greatly concerned with these differences and in particular it is important to observe that for the only group considered thus far in this thesis—that of the additive real numbers—the two compactifications coincide.

We begin with a brief review of the construction for the case of compact G, and then move into a terse review of previous work on this subject, with the central aim of motivating the use of an algebraic compactification $\overline{G}$ when G itself is non-compact. Then we illustrate a viable construction (making use of the almost periodic compactification of the group) in some detail for two particular cases: the real line $(\mathbb{R}, +)$ and the complex unitary group $\mathrm{U}_{\mathbb{C}}(1)$.

Next, we turn to more general questions. We show that the construction carried out for $\mathbb{R}$ and $\mathrm{U}_{\mathbb{C}}(1)$ is possible for any group of the form $G \times \mathbb{R}^n$ (for compact G), again using the almost periodic compactification of the group, and that this is the most general case for which the almost periodic compactification is useful. We then consider the use of other compactifications for more general non-compact groups, with particular emphasis on the groups $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SL}(2, \mathbb{C})$ of relevance to 2+1 and self-dual 3+1 gravity respectively. We find that of several compactifications commonly considered by algebraists, none is effective in this case, and we outline a set of criteria that any potential compactification must have

in order to be both physically and mathematically plausible. Finally, we conclude with a partial 'no-go' theorem for the case of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$, which shows that any definition of measure theory on $\overline{\mathcal{A}}$ with these gauge groups (by way of a compactification) that also supports flux operators must at the least be quite different from the construction for both compact gauge groups (using Haar measure on the group) and the construction for $G \times \mathbb{R}^n$ (using Haar measure on the almost periodic compactification).

Certain of these ideas have been obtained independently by Lewandowski and Okołów; also the original motivation for this idea comes from the letter by Ashtekar, Lewandowski, and Sahlmann [34]. We also include a comparison of our construction with that of Freidel and Livine [27] for non-compact groups and that of Baez [40] for amenable groups.

5.1 Background material

In this section we first review the construction of $\overline{\mathcal{A}}$ and measure theory on it for the case of compact gauge groups G, and then next consider previous work on extending the theory to non-compact gauge groups.

5.1.1 Measure theory on $\overline{\mathcal{A}}$ for compact gauge groups

In section 2.2 we have already briefly reviewed the construction of the space $\overline{\mathcal{A}}$ for quantum geometry. Here we recall some aspects of that overview, as well as some other considerations in the construction of measure theory on a compact gauge group that will be important as we examine ways to extend this construction to non-compact gauge groups. Our purpose is *not* to give a thorough review of all aspects of the construction, nor even all major aspects of the construction. For a thorough but pedagogical review we instead refer the reader to section IV of [1]; for representative earlier work see [2, 3, 40].

So, let G be a compact, connected, semi-simple Lie group, and consider a smooth manifold M with a G-bundle over it. For simplicity (and because it is the case relevant to quantum gravity) we assume that the bundle is globally trivial and choose some global trivialization (the case of non-trivial bundles may also be handled [40, 58]). In this case, we may equate smooth connections to smooth $\mathfrak{g}$-valued one forms A_a^i globally defined on M. Here i is an internal index labeling a basis of $\mathfrak{g}$ and a is a spacetime index. The canonically conjugate variables will then be smooth $\mathfrak{g}$-valued[1] vector densities E_i^a on M (often called *electric fields* as they are the analogue of the Yang-Mills electric field), and the Poisson

[1]Technically, the electric field takes values in the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. However, because the Killing form on $\mathfrak{g}$ is non-degenerate (since we have assumed G semisimple) we can move freely between $\mathfrak{g}$ and $\mathfrak{g}^*$ and shall frequently do so.

bracket between any two classical observables f_1 and f_2 is given by:

$$\{f_1(A,E), f_2(A,E)\} = \int_M d^3x \left(\frac{\delta f_1}{\delta A_a^i} \frac{\delta f_2}{\delta E_i^a} - \frac{\delta f_2}{\delta A_a^i} \frac{\delta f_1}{\delta E_i^a} \right). \tag{5.1.1}$$

The gauge transformations can be identified with the group of G-valued functions g on M, and they act in the usual ways on our canonical variables:

$$(A \cdot g, E \cdot g) = (g^{-1}Ag + g^{-1}dg, g^{-1}Eg). \tag{5.1.2}$$

Our goal is to construct a Hamiltonian quantum theory preserving (for at least a distinguished set of basic observables) the commutation relations (5.1.1). Moreover, we are looking for a 'connection representation,' where the Hilbert space may be thought of as a space of functions on the space $\mathcal{A}$ of connections, with observables that classically are functions of connections acting by multiplication in this representation. We would then expect that observables that classically are functions of the canonically conjugate field will act by differentiation.

As in section 2.2, we proceed by choosing as our basic 'connection' observables functions of the holonomies of a connection along the edges of a graph embedded in M. Let γ be such a graph, with N analytic edges that meet only in zero-dimensional vertices. Orient these edges arbitrarily. Every connection $A \in \mathcal{A}$ determines for each edge $e \in \gamma$ a holonomy $A_e \in G$ (recall that we have fixed a global trivialization). A function $\psi(A)$ is said to be cylindrical with respect to the graph γ if it depends on A only through the values of the N holonomies $\{A_{e_n}\}$; i.e., it is of the form

$$\Psi(A) = \psi(A_1, \ldots, A_n). \tag{5.1.3}$$

As before, the space of all such functions for a particular γ is denoted by Cyl_γ: those functions cylindrical with respect to the graph γ. We can form the space of *all* cylindrical functions simply as the union of the Cyl_γ over all closed, piecewise analytic graphs with a finite number of edges:

$$\mathrm{Cyl} = \bigcup_\gamma \mathrm{Cyl}_\gamma. \tag{5.1.4}$$

Again, as we found in section 2.2, we may use the Haar measure on the group G to introduce a natural inner product on Cyl; for any two $\Psi_1, \Psi_2 \in \mathrm{Cyl}$ we put:

$$(\Psi_1, \Psi_2) = \int_\gamma \overline{\psi}_1 \psi_2 \, d\mu_H^{(N)}. \tag{5.1.5}$$

On the right hand side, we are integrating over some graph γ that contains both the of the graphs γ_1 and γ_2 on which Ψ_1 and Ψ_2, respectively, are based. To extend the definition of

Ψ_1 and Ψ_2 to such a graph, we simply define them to be constant on the edges which must be added to γ_1 or γ_2 to obtain γ. The right hand side of (5.1.5) is independent of *which* such graph γ precisely because we are using the normalized Haar measure; it is possible to normalize the Haar measure precisely because the constant functions are integrable. This crucial difference between the case of compact groups (that we consider in this subsection) and non-compact groups (to which we wish to extend the construction) underlies much of what we do in the remainder of this chapter.

When we complete Cyl with respect to the inner product (5.1.5) we obtain the polymer Hilbert space $\mathcal{H}_{\rm Poly}$ already considered in this thesis. It is then easy to see that every cylindrical function $\Psi(A)$ gives rise to a bounded operator $\hat{\Psi}(A)$ that acts on $\mathcal{H}_{\rm Poly}$ by multiplication; the set of all such operators forms the Abelian subalgebra of holonomy observables. Moreover, it is possible to regard this Hilbert space as $L^2(\overline{\mathcal{A}}, d\mu_{\rm AL})$ for a compact, Hausdorff space $\overline{\mathcal{A}}$ and a regular Borel measure $\mu_{\rm AL}$ on that space. Since the space of connections on a *fixed* graph is isomorphic to G^N, it is possible to show that the space $\overline{\mathcal{A}}$ is in fact an uncountable direct product of copies of the gauge group G, one for each edge in each possible graph. Despite the fact that the product is uncountable, since each space is compact and Hausdorff the resulting product space is, by Tychonoff's theorem, itself compact and Hausdorff.

Also important for our considerations are the operators canonically conjugate to the connection operators. These, as we indicated in section 2.2, are the flux operators. They depend upon a smooth two dimensional surface S and associated smooth, $\mathfrak{g}^*$-valued function f_i on S. The action of any such flux operator on a cylindrical function Ψ based on a graph γ depends upon the nature of the intersection of γ and S and the value of f_i at isolated points of intersection of γ and S. We will not discuss the construction of these operators in any further detail (see [1] for such details), except to note one crucial fact: they are constructed out of left and right invariant vector fields on the group G. One such operator $\hat{J}_e$ acts on each edge e of a given graph, with the choice of left or right invariant operator determined by the orientation of the edge.

These left and right invariant vector fields may be defined by their action on an arbitrary smooth function f on G. That action is:

$$L_i(f(g)) = \frac{d}{dt} f(g e^{t\tau_i}), \qquad R_i(f(g)) = \frac{d}{dt} f(e^{-t\tau_i} g) \tag{5.1.6}$$

and one readily checks that either of $\{L_i\}$ or $\{R_i\}$ is isomorphic to $\mathfrak{g}$. For the Hilbert space $L^2(G, d\mu_H)$ on the group (note that the Hilbert space $\mathcal{H}_\gamma$ for a given graph is simply the direct sum of one copy of $L^2(G, d\mu_H)$ for each edge in the graph), the corresponding quantized operators are easily constructed; one simply includes an i on the right hand side

of (5.1.6):

$$\hat{L}_i(f(g)) = i\frac{d}{dt}f(ge^{t\tau_i}), \qquad \hat{R}_i(f(g)) = i\frac{d}{dt}f(e^{-t\tau_i}g). \tag{5.1.7}$$

The important fact for us is that the operators of (5.1.7) are self-adjoint when acting on the Hilbert space $L^2(G, d\mu_H)$. This in turn is what allows the flux operators (acting on $\mathcal{H}_{\rm Poly}$) to be self-adjoint. It is easy to check this directly, but for our purposes a more useful verification is to note that these operators exponentiate to continuous, unitary operators implementing left and right translations on the group. Unitarity follows because we are using Haar measure to define our Hilbert space. Continuity follows from the following. Let f be any smooth function on G. Then the Haar measure defines a continuous linear functional on the algebra $C(G)$ of all continuous functions on G via:

$$\mu_H(f) := \int_G f\, d\mu_H. \tag{5.1.8}$$

Now, for an arbitrary element $g \in G$ let ${}_gf$ denote the left translate of f, and f_g the right translate. Weak continuity of the unitary action of left and right translations is equivalent to the assertion that for any one-parameter subgroup g_t of G, the following are continuous functions of t:

$$\mu_H(\overline{h}\,{}_{g_t}f) \qquad \text{and} \qquad \mu_H(\overline{h}\,f_{g_t}) \tag{5.1.9}$$

for any continuous functions h and f on G. This continuity holds because for a compact group, any continuous function is in fact uniformly continuous under both left and right translations [59], and the mapping $\mu : C(G) \to \mathbb{C}$ is continuous by the Riesz representation theorem.

Finally, we note one more important fact about the construction of integration theory on $\overline{\mathcal{A}}$: the spin-network decomposition. This is a decomposition of the polymer Hilbert space $\mathcal{H}_{\rm Poly}$ into orthogonal subspaces. We have already noted that this can be done in section 2.2 for the specific case of $G = \mathrm{SU}(2)$; there we found the decomposition

$$\mathcal{H}_{\rm Poly} = \bigoplus_{\gamma,\vec{j}} \mathcal{H}_{\gamma,\vec{j}} \tag{5.1.10}$$

where the symbol $\vec{j}$ denotes a labeling of the edges of γ by irreducible unitary representations of SU(2), i.e., by spins. For a general compact group G, a similar decomposition applies in terms of the irreducible unitary representations of G, or what will be more useful for us, the matrix elements of such unitary representations. This decomposition for $L^2(\overline{\mathcal{A}}, d\mu_{\rm AL})$ follows from the decomposition of $L^2(G, d\mu_H)$ that is given by the Peter-Weyl theorem [1].

We have thus briefly reviewed the salient features of the construction of $L^2(\overline{\mathcal{A}}, d\mu_{\rm AL})$ and important operators on it. For convenience, we now list the crucial properties of integration on the gauge group G itself that were used in this construction. We formulate these in an

'algebraic' form that will be better suited to our use in the rest of this chapter. This requires the concept of a mean, which may be defined for any regular, positive Borel measure μ on G in the same manner as μ_H was defined in (5.1.8):

$$\mu(f) = \int_G f \, d\mu \tag{5.1.11}$$

This mapping is usually termed a positive linear functional in the physics literature, but we shall use the terminology mean both because it is common in the mathematical literature on which we shall rely, but more importantly because it is more specifically suited to our case, being a special case of a positive linear functional. That is because a positive linear functional is defined for any C^*-algebra, whereas the term mean is generally reserved for Abelian C^* sets of functions. Moreover, a mean is required to be bounded; i.e., to satisfy $\inf f \leq \mu(f) \leq \sup f$.[2] This in turn means it is automatically normalized if the C^*-algebra in question is unital.

The important properties of the mean μ_H and the algebra $C(G)$ of all continuous functions on the group G (on which μ_H is defined) are therefore the following:

1. The constant function 1_G is an element of $C(G)$, and the mean is normalized, $\mu_H(1) = 1$ (as we have noted the latter is a consequence of the definition of a mean, but we repeat it for emphasis). This is needed to allow the definition in (5.1.5) of the inner product between functions defined on different graphs, and to ensure that that inner product does not depend on the choice of the larger graph γ containing both γ_1 and γ_2.

2. The mean is invariant under both left and right translations; $\mu_H({}_g f) = \mu_H(f) = \mu_H(f_g)$ for all $f \in C(G)$ and all $g \in G$. This is needed both to ensure unitarity of left and right translations in the Hilbert space (and therefore ultimately the self-adjointness of flux operators on $\mathcal{H}_{\rm Poly}$), and also to ensure that the construction does not depend on the choice of trivialization made at the outset.

3. For any $f, h \in C(G)$, and any one-parameter subgroup g_t in G, both $\mu_H(\overline{h}\, {}_{g_t} f)$ and $\mu_H(\overline{h}\, f_{g_t})$ are continuous functions of t. The importance of this property is that it guarantees that the unitary representations of left and right translations provided in item 2 are in fact continuous, and therefore by Stone's theorem have self-adjoint generators. Therefore, this property is also necessary for the existence of the flux operators on $\mathcal{H}_{\rm Poly}$.

[2] In fact, the complete definition of a mean on a linear subspace $\mathcal{F}$ of the algebra $C_b(X)$ of all bounded, continuous functions on some topological space X is that it is a linear functional from $\mathcal{F}$ to $\mathbb{C}$ with the further property that $\inf f \leq \mu(f) \leq \sup f$ whenever $f \in \mathcal{F}$ is real-valued.

We shall soon see that analogues of all three of these properties seem necessary for any successful construction of an integration theory on $\overline{\mathcal{A}}$ when G is non-compact.

5.1.2 Previous work

Now we consider what goes wrong when we try to extend the construction outlined in the previous subsection to spaces of connections with non-compact gauge group.

The construction of $\overline{\mathcal{A}}$ itself goes through without difficulty: it is still the case that for each graph we can identify the space of connections restricted to that graph with G^N. We can then consider the projective limit of all graphs [2, 3] which still exists as a topological space. It is no longer compact, but that is *a priori* not a problem; after all, why should $\overline{\mathcal{A}}$ be compact when G itself is not?

The difficulties occur, however, when we attempt to construct a measure on $\overline{\mathcal{A}}$. For a given, fixed graph γ there is still no difficulty: we can construct the Hilbert space $\mathcal{H}_\gamma$ of all square integrable functions with respect to $(d\mu_H)^N$ on G^N. Note, however, that the constant functions are no longer integrable, since G is not compact. And therein lies the problem: when we attempt to define an inner product between functions based on distinct graphs γ_1 and γ_2, as we did in the compact case in equation (5.1.5), we can no longer ensure that the inner product is independent of the larger graph γ (such that $\gamma_1, \gamma_2 \subset \gamma$); indeed, we can no longer even *define* the product for arbitrary γ containing γ_1 and γ_2.

A major attempt around this difficulty was made in [27] by Freidel and Livine. We will not review this work in any real depth, but will just highlight the central ideas and then comment on the difficulties associated with their construction.

First, they abandoned the idea of constructing the total Hilbert space as a measure space—that is, something of the form $L^2(\overline{\mathcal{A}}, d\mu)$ for some suitable $\overline{\mathcal{A}}$ and μ—and instead aim to directly generate a Hilbert space by generalizing the spin network construction. Second, they work only and directly with the space of gauge invariant functions on the space of connections, so that at the level of a single graph they consider not $\mathcal{A}_\gamma$ but $\mathcal{A}_\gamma/\mathcal{G}_\gamma$.

On $\mathcal{A}_\gamma/\mathcal{G}_\gamma$ they can indeed construct a Hilbert space and that Hilbert space is an L^2 space, at least for $G = \mathrm{SL}(2,\mathbb{C})$ or $G = \mathrm{SL}(2,\mathbb{R})$. This space is defined as the completion in an L^2 norm of the space of continuous functions on $\mathcal{A}_\gamma/\mathcal{G}_\gamma$ of compact support. As we have just seen (and passing to the quotient space $\mathcal{A}_\gamma/\mathcal{G}_\gamma$ does not affect this, as that space is still non-compact), the difficulty comes in patching together functions based on different graphs. If all we are aiming for is a Hilbert space, then all we need find is the inner product between states based on *different* graphs; it is only by considering different graphs that we probe the full infinite dimensional configuration space of general relativity. What Freidel and Livine do is to set all such inner products between distinct graphs to *zero*; one then obtains a well defined Hilbert space that is proposed as the analogue of $\mathcal{H}_{\mathrm{Poly}}$ for non-compact groups.

However, there is a serious defect remaining. We must now consider the action of our

basic observables in this Hilbert space. When one considers the action of the Abelian part of the algebra—corresponding to multiplication by cylindrical functions—one finds [28, 60] that the action is *trivial* unless the function f is based on the same graph as the state Ψ on which it acts. Thus, in this representation the holonomy operators leave the states based on a particular graph invariant; as the flux operators also have this property the resulting representation of the basic algebra of observables is very reducible: any single graph suffices.

This problem can be directly traced to the assumption that distinct graphs give rise to orthogonal Hilbert spaces, and one might think that by relaxing this one could eliminate the reducibility of the representation. However, other problems remain. In particular, it is possible to show the following [28, 60]:

1. If one assumes that the total Hilbert space may be decomposed as an orthogonal direct sum of the form $\mathcal{H} = \oplus_\gamma \mathcal{H}_\gamma$, and if further there is a mild assumption on the representation π of the $\star$-algebra of configuration observables, then $\pi(f_{\gamma_1})f_{\gamma_2} = 0$ if γ_1 and γ_2 are distinct graphs. Note that no assumption is made that the inner product comes from a measure on $\mathcal{A}_\gamma/\mathcal{G}_\gamma$, nor that π acts by multiplication.

2. If instead one *does* assume that the inner product on $\mathcal{H}_\gamma$ comes from a measure (satisfying a mild condition), and further that π acts by multiplication, then without assuming orthogonality of the Hilbert spaces corresponding to different graphs, one may show that the resulting inner product is not positive definite when G is non-compact.

3. Furthermore, if one drops the assumption that π acts by multiplication, but does assume that the measure is diffeomorphism covariant, then it is still possible to show that the inner product is not positive definite.

The arguments establishing both of the last two points rely crucially on the assumption that the measure on $\mathcal{A}_\gamma/\mathcal{G}_\gamma$ is not normalized; i.e., that there are sets with measure greater than one. As long as the constant functions are not integrable, there seems no natural way to normalize the measure, and simply choosing some probability measure on G will in general also fail, as such a measure will not have the right invariance properties.

Thus, at each turn a critical difficulty is the inability to integrate the constant functions with respect to Haar measure when the group is non-compact, and so it seems natural to seek for some modification that would allow us to do so.

5.2 Constructing $\overline{\mathcal{A}}$ for non-compact, Abelian groups

In this section we present a construction that can overcome the difficulties mentioned at the close of the last section, for a class of non-compact Abelian groups. The construction is based on the use of the almost-periodic functions on the group G, and a resulting compactification

of the group. It may indeed seem odd to compactify the gauge group if it is initially non-compact. However, as we saw in the last section, most of the difficulties in constructing a successful integration theory seem to stem from the fact that the constant functions are not integrable with respect to the Haar measure. If we seek some algebra of functions that does include the constant functions, then we are lead, by the Gel'fand-Naimark theorem, to a compact space, whether we sought such a space or not.

In this section we therefore outline the construction based on the almost periodic compactification of the gauge group. For clarity, we begin slowly, with two simple groups: the real line, and the complexified circle group, $\mathrm{U}_{\mathbb{C}}(1)$. We next discuss the approach in its greatest generality, and then in the final subsection make a comparison with earlier approaches in the literature.

5.2.1 The real line

Consider first the situation where the gauge group is simply the real line. Before moving directly to the space of generalized connections, we first review the Bohr compactification of $\mathbb{R}$ and the structures on it that we will need.

We have used the Bohr compactification of $\mathbb{R}$ repeatedly in this thesis. As pointed out earlier, it can be viewed as the spectrum of the C^*-algebra of almost-periodic ($\mathcal{AP}$) functions on the real line, $\mathcal{AP}(\mathbb{R})$. We have not, however, given a general definition of almost periodic functions, and for completeness we do so now.

In fact, the Bohr compactification is technically the compactification resulting from the algebra of *Bohr almost periodic functions* $\mathcal{BAP}$; it just so happens that for Abelian topological groups this algebra coincides with the $\mathcal{AP}$ functions [57]. The compactification resulting from the almost periodic functions $\mathcal{AP}$ in the general case (where $\mathcal{AP}$ and $\mathcal{BAP}$ do not coincide) is then normally called the $\mathcal{AP}$-compactification; we shall often use this terminology for the Bohr compactification as well in cases where the two compactifications are the same.

So, let us give definitions of both classes of functions. A function f on a topological group G (in complete generality, we could even consider a semitopological semigroup [57]) is said to be *left almost periodic* if the set of its left translates is relatively compact in the norm topology on the algebra $C_b(G)$ of all continuous bounded functions on the group. Since $C_b(G)$ with the uniform norm is a complete metric space, this definition of left almost periodicity is equivalent to the assertion that the set of left translates of f is totally bounded. This in turn means that for any $\epsilon > 0$ there exists a *finite* set of points $\{g_i\}$ such that

$$\min_{g_i} |{}_g f - {}_{g_i} f| < \epsilon \tag{5.2.1}$$

for every point $g \in G$; that is, that each left translate of f is within ϵ of at least one of the

left translates of f by an element of the finite set $\{g_i\}$. Right almost periodicity is defined analogously, and a function is almost periodic if it is both left and right almost periodic.

The definition of Bohr almost periodic functions is quite similar, except that now instead of requiring the set $\{g_i\}$ to be finite, we merely require that it be compact. The set of almost periodic functions is always a C^*-algebra (satisfying moreover some stronger conditions such as admissibility, which we shall define later in this chapter); in general the Bohr almost periodic functions do not even form a linear space [57].

However, as we have indicated, for Abelian topological groups the two algebras (and hence the corresponding compactifications) can be shown to coincide; historically, therefore, that is why Bohr almost periodic functions were studied first even though in general they are not well behaved: they were first studied in the context of the additive group of the reals where they do form a C^*-algebra.

Moreover, for Abelian groups it can be shown [61] that the $\mathcal{AP}$ functions are the same as the algebra generated by the *characters* of the group; that is, the set of continuous homomorphisms from the group G to the group U(1). Since the 'algebra generated' by a subset of $C_b(G)$ is simply the set of all functions that can be uniformly approximated by finite linear combinations of sums and products from that subset, and since for the reals the characters are precisely the functions of the form $\exp i\lambda x$, we are led back to the definition of the almost periodic functions that we have used before: they are those bounded continuous functions whose Fourier series converge *uniformly*. Thus, they are functions that are the limit, in the sup norm, of finite linear combinations of functions of the form:

$$e_\lambda(x) = e^{i\lambda x} \tag{5.2.2}$$

where λ is any *real* number. These exponential basis functions clearly separate the points of $\mathbb{R}$, so the canonical continuous map $\phi : \mathbb{R} \to b\,\mathbb{R}$ is in fact one-to-one (here $b\,\mathbb{R}$ is the Bohr compactification of $\mathbb{R}$).

The general theory of almost periodic functions assures us that the almost periodic compactification of any topological group is itself a compact, Hausdorff topological group. As we are attempting to define a generalization of integration theory on $\overline{\mathcal{A}}$ from the compact case based on the almost periodic compactification, it is reassuring to know that for a compact group the almost periodic functions consist precisely of *all* continuous functions on the group, and the resulting 'compactification' simply gives us back the original group itself. Hence the approach based on almost periodic functions that we shall develop in this chapter will automatically give us the familiar construction of measure theory on $\overline{\mathcal{A}}$ if it is applied when the gauge group is compact.

Since for $\mathbb{R}$ the map ϕ is one-to-one, we know that $b\,\mathbb{R}$ is not trivial. By virtue of being a compact topological group, $b\,\mathbb{R}$ has a unique, normalized Haar measure μ. We can use this

measure to construct the Hilbert space $L^2(b\,\mathbb{R}, d\mu_H)$ of square integrable functions on $b\,\mathbb{R}$. For practical purposes, however, the following two ways of characterizing the measure tend to be more convenient.

First, let $f \in \mathcal{AP}(\mathbb{R})$ be any almost-periodic function on the real line, and let $\mu(f)$ denote the mean on $\mathcal{AP}(\mathbb{R})$ that corresponds to integration against $d\mu$. Then one may show that [57]:

$$\mu(f) = \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} f(x)\, dx. \tag{5.2.3}$$

From this definition it is obvious that μ is indeed positive; that is, that $\mu(f) \geq 0$ whenever $f \geq 0$. Moreover, note that the constant functions are integrable and that the measure is normalized: $\mu(1) = 1$. This is in sharp contrast to the situation where one uses $L^2(\mathbb{R}, dx)$; there none of the constant functions are integrable. This in turn will have important implications when we consider the space of connections in the next subsection.

Alternatively, one may specify μ by giving its action on the basis functions $e_\lambda(x)$:

$$\mu(e_\lambda) = \delta_{0,\lambda}. \tag{5.2.4}$$

It is not difficult to see that this definition agrees with that in (5.2.3). This definition is often more convenient for calculations though it is not as straightforward to see that μ is indeed a mean on the algebra (though of course it is still true).

This mean then enables us to define an inner product between any two $\mathcal{AP}$ functions f and g:

$$\langle f \,|\, g\rangle = \mu(\overline{f} g). \tag{5.2.5}$$

For the exponential basis functions e_λ this yields:

$$\langle e_{\lambda_1} \,|\, e_{\lambda_2}\rangle = \delta_{\lambda_1,\lambda_2}. \tag{5.2.6}$$

When we complete $\mathcal{AP}(\mathbb{R})$ with this inner product we obtain the Hilbert space $L^2(b\,\mathbb{R}, d\mu_H)$.

Having now a Hilbert space, we naturally look for operators on that Hilbert space. It is not hard to see that any $f \in \mathcal{AP}(\mathbb{R})$ gives rise to an operator $\hat{f}$ that acts by multiplication. Moreover, the action of these operators respects the $\star$-structure of the algebra $\mathcal{AP}(\mathbb{R})$: for any $f \in \mathcal{AP}(\mathbb{R})$ we have $\hat{f}^\dagger = \hat{\overline{f}}$. In particular, for any real λ one can define the 'holonomy' operator $\hat{h}_\lambda$ to be multiplication by the function e_λ. These $\{\hat{h}_\lambda\}$ operators are then readily verified to be unitary.

We can also define a derivation operator through its action on the basis functions e_λ:

$$\hat{k}\, e_\lambda = \lambda\, e_\lambda. \tag{5.2.7}$$

This definition is what one would obtain from the definition on the space $L^2(\mathbb{R}, dx)$, where

$\hat{k} = (1/i)\,(d/dx)$. We can now extend the definition (5.2.7) to all of $\mathcal{AP}(\mathbb{R})$ by linearity, and so obtain an operator that is densely defined on $L^2(b\,\mathbb{R}, d\mu)$. This operator is easily seen to be symmetric, and a simple calculation shows that it is essentially self adjoint and therefore has a unique self adjoint extension to $L^2(b\,\mathbb{R}, d\mu)$. Finally, it is easy to check that the commutation relations between $\hat{h}_\lambda$ and $\hat{k}$ are:

$$[\hat{k}, \hat{h}_\lambda] = \lambda\,\hat{h}_\lambda \tag{5.2.8}$$

just as they are when the Hilbert space is $L^2(\mathbb{R}, dx)$.

Thus, we can construct a suitable harmonic analysis on $b\,\mathbb{R}$ to support a non-trivial Hilbert space with operators corresponding to our basic canonical variables. We must now 'piece these together' in a consistent fashion in order to obtain a Hilbert space space that we may think of as $L^2(\overline{\mathcal{A}}, d\mu)$.

Accordingly, let M be a manifold with a real line bundle sitting over it. Let γ be an analytic graph with N edges immersed in M. Given any connection $A \in \mathcal{A}$ we can associate a corresponding holonomy, which is an element of $\mathbb{R}$, to each edge of γ. We now want to define a certain class of functions on $\mathcal{A}$ determined by the graph γ. We call a function f *cylindrical with respect to the graph* γ if it is of the form

$$f(A) = f(h_{e_1}(A), \ldots, h_{e_N}(A)). \tag{5.2.9}$$

Here $h_{e_i}(A)$ is the holonomy determined by the connection A along the edge e_i of γ. However, it is not sufficient simply for f to be of the form (5.2.9); it must further have the property that it is smooth[3] and moreover that, if all of the holonomies except h_{e_i} are fixed at any real values and f is regarded as a function only of h_{e_i}, then f is an AP function. This must hold for any way of fixing the other $N-1$ holonomies and for each edge e_i in γ.

This definition is precisely the same as used in the compact case, except that in the compact case there is no need to insist that the restrictions of cylindrical functions to each edge of a graph be almost periodic. However, as we have already noted, in the compact case that is because *all* continuous functions on the group are almost periodic; the usual construction of $\overline{\mathcal{A}}$ in the compact case coincides with the construction outlined here. Note that because the constant functions are in $\mathcal{AP}(\mathbb{R})$, our cylindrical functions include functions which are constant on any subset of the edges of γ; this will be important momentarily. We can now use the inner product on $L^2(b\,\mathbb{R}, d\mu)$ to define an inner product on our cylindrical functions. For any two cylindrical functions f and g put:

$$\langle f \,|\, g \rangle = \int_{(b\,\mathbb{R})^N} \overline{f}\, g\, (d\mu)^N \tag{5.2.10}$$

[3]Since we have not defined a smooth structure on $b\,\mathbb{R}$, we shall call a function $f \in \mathcal{AP}(\mathbb{R})$ smooth if $(\hat{k})^n\, f \in \mathcal{AP}(\mathbb{R})$ for each nonnegative integer n.

We can complete the space of cylindrical functions with respect to this inner product to obtain a space that we shall refer to as $L^2(\mathcal{A}_\gamma, d\mu)$. It is then easy to check that any smooth cylindrical function f defines an operator that acts by multiplication and preserves the $\star$-algebra structure of the space of cylindrical functions. Moreover, in complete analogy with the compact case we may define momentum operators. The operator $\hat{J}_e$ is defined to act as the identity on the copy of the gauge group corresponding to every edge except the edge e; on that edge it has the action of the $\hat{k}$ operator defined in the previous subsection. It is therefore easy to see that this operator is self adjoint acting on $L^2(\mathcal{A}_\gamma, d\mu)$.

We note that there is an alternative way of viewing this construction. We have in the previous paragraphs emphasized an 'algebraic' point of view in which we focus on certain algebras and means defined on them, and then the operators on the corresponding Hilbert space. However we can instead view things in terms of the Bohr compactification. Then we would say the following. To each edge we are associating not a copy of the gauge group $\mathbb{R}$, but rather a copy of its Bohr compactification $b\,\mathbb{R}$. Since $\mathbb{R}$ is canonically embedded in $b\,\mathbb{R}$, for any connection on our graph the holonomy will in fact naturally lie also in $b\,\mathbb{R}$. But we see now that, unlike the compact case, we must consider 'generalized connections' for a single graph, since we can have holonomies along some edges which lie in $b\,\mathbb{R} \setminus \mathbb{R}$. We are, from this point of view, restricting ourselves to functions in $\mathcal{AP}(\mathbb{R})$ because it is only these functions that have continuous extensions to $b\,\mathbb{R}$. We are then using the Haar measure on $b\,\mathbb{R}$ an proceeding exactly as one does for the case of a compact gauge group.

We therefore finally come to the case of the full Hilbert space of functions on $\overline{\mathcal{A}}$. To define this we first define a function f to be cylindrical if it is cylindrical with respect to *some* graph γ. We then define the inner product between two cylindrical functions f and g by finding a graph γ with respect to which both are cylindrical and setting:

$$\langle f \,|\, g \rangle = \int_{\mathcal{A}_\gamma} \overline{f}\, g \, (d\mu)^N \tag{5.2.11}$$

Because of the invariance and normalization of the Haar measure, just as for the compact case this definition is independent of the graph γ (so long as both f and g are cylindrical with respect to it). It is then immediate to see that we can define multiplication operators corresponding to any cylindrical function and likewise electric flux operators constructed from the momentum operators $\hat{J}_e$ defined on each graph. The electric flux operators are essentially self adjoint because the momentum operators $\hat{J}_e$ are; and the multiplication operators corresponding to holonomies are unitary operators.

We emphasize that this construction is possible only because the constant functions are in $\mathcal{AP}(\mathbb{R})$. This means that a cylindrical function with respect to one graph is also cylindrical with respect to any larger graph, and that we can integrate such functions when defining our inner product.

Thus, it appears that the use of the almost periodic compactification does allow one to overcome significant obstacles encountered in other ways of defining integration on $\overline{\mathcal{A}}$ for non-compact groups, at least for the simplest case of $\mathbb{R}$ that we have considered here. We turn next therefore to another non-compact Abelian group, $\mathrm{U}_{\mathbb{C}}(1)$, to see how general this success might be.

5.2.2 The case of $\mathrm{U}_{\mathbb{C}}(1)$

The complexified U(1) group $\mathrm{U}_{\mathbb{C}}(1)$ is the group of the complex numbers minus the origin under multiplication; i.e., it is $\mathbb{C}^*$. It is isomorphic to the product of the group $\mathbb{R}^*_+$ of positive reals under multiplication with the circle group U(1). Paralleling our treatment of $\mathbb{R}$, we first begin with an examination of the almost periodic compactification of $\mathbb{C}^*$.

In order to find this compactification, we need to know the almost periodic functions on this group. As we have already observed, this is most easily found by noting that for an Abelian group, the almost periodic functions are always the closure in the sup norm of the algebra generated by the continuous characters of the group. Thus, we need to find the characters of $\mathbb{C}^*$; that is, continuous homomorphisms from $\mathbb{C}^*$ to U(1) [62]. We do not wish to go into the general theory here but we note that these can be found by using the notion of the 'dual' of the group, and using the fact that the group $\mathbb{C}^*$ factors as $\mathbb{R}^*_+ \times \mathrm{U}(1)$. If we express an arbitrary nonzero complex number z as $\rho e^{i\theta}$ for $\rho \in (0, \infty)$ and $\theta \in [0, 2\pi)$ then one finds that any character of the group $\mathbb{C}^*$ is labeled by a real number α and an integer n and is of the form

$$e_{\alpha,n}(z) = e^{i\alpha \ln \rho} e^{in\theta} = e^{i\alpha \ln |z|} e^{in \arg z} \tag{5.2.12}$$

Note in particular that $e_{0,0} = 1$, so the constant functions will again be among the almost periodic functions on the group, as indeed is always the case for any topological group.

We can then define the inner product between two such functions as

$$\langle e_{\alpha_1,n_1} \,|\, e_{\alpha_2,n_2} \rangle = \delta_{\alpha_1,\alpha_2}\, \delta_{n_1,n_2}. \tag{5.2.13}$$

It is not hard to verify that this is indeed a mean on the algebra $\mathcal{AP}(\mathbb{C}^*)$, and hence we may complete the algebra in this norm to obtain a Hilbert space. On this Hilbert space we will again find that to any $\mathcal{AP}$ function there corresponds an operator acting by multiplication and preserving the $\star$-structure of the algebra; the operators corresponding to the basis functions $e_{\alpha,n}$ are all unitary.

We lack therefore only a momentum operator. In order to see what momentum operator we should use, we must look for invariant vector fields on the group. In our case it is simple to apply the definition $X_\xi f(g) = (d/dt) f(g e^{t\xi})$ to the basis functions to find, in the (ρ, θ) coordinate system, the two operators $\hat{k}_\rho$ and $\hat{k}_\theta$ by defining their action on the basis

functions:

$$\hat{k}_\rho \, e_{\alpha,n} = \alpha \, e_{\alpha,n} \tag{5.2.14}$$

$$\hat{k}_\theta \, e_{\alpha,n} = n \, e_{\alpha,n} \tag{5.2.15}$$

We thus see that the basis functions $e_{\alpha,n}$ are eigenstates of the $\hat{k}_\rho$ and $\hat{k}_\theta$ operators. It is simple to check that these operators are essentially self adjoint and therefore have a unique self adjoint extension to the entire Hilbert space. This Hilbert space is again of the form $L^2(b\mathbb{C}^*, d\mu)$ for the Haar measure μ on the compact topological group $b\mathbb{C}^*$.

The construction of a Hilbert space with relevant operators for the space of $\mathbb{C}^*$ connections exactly parallels the construction for the real line considered earlier and we will not dwell on it in detail. Once again we are considering connections whose holonomies take values in the $\mathcal{AP}$-compactification of $\mathbb{C}^*$ rather than just in $\mathbb{C}^*$ itself. This has the consequence that since we deal with a compact space and have a unique invariant measure, we can construct the projective limit just as is done for compact gauge groups. We again get a representation of the $\star$-algebra of the cylindrical functions which is non-trivial and irreducible. Our electric flux operators are again self adjoint; they now have an internal index since the group is two dimensional.

5.2.3 General applicability of the $\mathcal{AP}$-compactification

With two successful applications of the almost periodic compactification to the construction of $L^2(\overline{\mathcal{A}}, d\mu_H)$ for connections with non-compact structure group, we are naturally led to ask how far this success can be extended.

Note first that since $\mathbb{R}^*_+$ is isomorphic as a topological group to $\mathbb{R}$ (under the homomorphism ln: $\mathbb{R}^*_+ \to \mathbb{R}$), in our construction for $\mathrm{U}_\mathbb{C}(1)$ we were actually looking at a group of the form $G \times \mathbb{R}$ for a compact group G. In fact, the most general group to which the almost periodic construction can be applied is any group of the form $G \times \mathbb{R}^n$, for some integer n and any compact (not necessarily Abelian) group G. This is because of the following result[4] [63]: every connected locally compact group for which the $\mathcal{AP}$ functions separate the points of the group is of the form $G \times \mathbb{R}^n$, for compact G. In particular, it is known that this class includes all connected, non-compact, locally compact Abelian groups [62].

Thus, such groups of the form $G \times \mathbb{R}^n$ are clearly the largest for which we can apply the $\mathcal{AP}$-compactification to construct $L^2(\overline{\mathcal{A}}, d\mu_H)$, and it is not difficult to see that we may apply this construction for every such group. The construction of $L^2(\overline{\mathcal{A}}, d\mu_H)$ itself is a trivial extension of what we have already covered; likewise the $\star$-algebra of holonomy operators acts by multiplication. The only real question is to verify that we can construct flux operators on this space.

[4] I am grateful to Andrzej Okołów for pointing out this reference to me.

That in turn depends upon the existence of momentum operators on the $\mathcal{AP}$ compactification. Given the form of the gauge group we are considering these are straightforward to exhibit. Since any $\mathcal{AP}$ function on $G \times \mathbb{R}^n$ is the uniform limit of a sum of functions of the form $f(g)e^{i\alpha_1 x_1} \cdots e^{i\alpha_n x_n}$, we may define our momentum operators acting on these. We then get left and right invariant vector fields on G acting as usual on the first factor; for each $\mathbb{R}$ factor we have a momentum operator $\hat{k_l}$ such that

$$\hat{k_l} f(g)e^{i\alpha_1 x_1} \cdots e^{i\alpha_l x_l} \cdots e^{i\alpha_n x_n} = \alpha_l f(g)e^{i\alpha_1 x_1} \cdots e^{i\alpha_l x_l} \cdots e^{i\alpha_n x_n}. \tag{5.2.16}$$

Thus, we have succeeded in giving a construction for all locally compact gauge groups that are of the form $G \times \mathbb{R}^n$ for compact (and possibly non-Abelian) groups G. However, the class of gauge groups for which we have successfully applied the $\mathcal{AP}$-compactification does not include either of the two groups we are most interested in from the point of view of quantum gravity: $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$. We turn to such groups in the next section, but first we pause to make a comparison of our work to that of earlier authors.

5.2.4 Comparison with earlier work

We begin by comparing the approach just developed to the approach of Freidel and Livine [27] that we discussed briefly in section 5.1.2. First, recall that Freidel and Livine's construction is based around the use of continuous functions of compact support, $C_0(G)$, whereas ours is based on the algebra of almost periodic functions on a group G, $\mathcal{AP}(G)$. What is the relation between these two algebras? One can show that for any non-compact, locally compact topological group $C_0(G) \cap \mathcal{AP}(G) = \emptyset$ [57]. Thus, the approaches are based on 'orthogonal' sets of functions.

More significantly, in the Freidel and Livine approach the constant functions are not among the integrable functions. This means that a function which is cylindrical with respect to one graph will not be cylindrical with respect to a larger graph. Likewise, when defining the inner product, functions can only be integrated on the graph to which they 'belong.' These features are absent in our approach based on the almost periodic compactification; as we pointed out, the definition (5.2.11) is independent of the graph used in its definition, so long as it is sufficiently large that both f and g are cylindrical with respect to it. This means that we are able to form the projective limit in exactly the same fashion as is done for the case of compact gauge groups. Similarly, the representation of the $\star$-algebra is non-trivial for our construction even when we consider two functions that are cylindrical with respect to different graphs; this is not true for the Freidel and Livine approach and moreover appears to be a general feature of any approach similar to theirs.

In summary, the success of the approach described here seems based on the use of an algebra of functions (the almost periodic functions) that separates the points of the

group and possesses a mean satisfying all three of the properties 1– 3 listed in section 5.1.1 above. One naturally wonders if there are other algebras and means on them satisfying these properties. This in turn brings us to a comparison to another previous work that dealt with non-compact gauge groups, that of Baez [40]. He considered the definition of $\overline{\mathcal{A}}$ for *amenable groups*. These are groups for which the C^*-algebra of *all* continuous, bounded functions possesses an invariant mean. This class is known to include all compact groups, all Abelian groups, and all direct products of amenable groups. In particular, therefore, it contains all of the groups just considered in section 5.2.3. One may therefore justly ask whether anything new has been accomplished in the present work.

In fact, the treatment of amenable groups in [40] is quite brief. All that is directly shown is that for amenable groups the limits of $\mathcal{G}$-invariant cylinder functions on $\mathcal{A}$ are precisely the same as the $\mathcal{G}$-invariant functions on $\mathcal{A}$ that are limits of cylinder functions. A measure itself on $\mathcal{A}$ for non-compact groups is not actually constructed.

Nonetheless, such a construction is in some sense implicit. It is in fact easy to use any invariant mean on $C(G)$ to construct a measure on $\overline{\mathcal{A}}$ along the lines we have developed in this section; one is simply replacing the $\mathcal{AP}$-compactification with the compactification generated by the C^*-algebra of all bounded continuous functions (the Stone-Čech compactification, denoted by βG). For any Lie group, at least, the algebra $C(G)$ certainly suffices to separate the points of G and therefore we have a one-to-one continuous mapping from G onto βG. Moreover, this algebra will always contain the constant functions and since it is amenable, the action of left and right translations will be unitary; the first two conditions of section 5.1.1 are met.

However, two difficulties remain, and the first of these is with the last condition of section 5.1.1. We have no guarantee that an invariant mean on $C(G)$ (whose existence is guaranteed by the hypothesis of amenability) will be such that the mappings analogous to those in Eq. (5.1.9) will be continuous. If they are not, then the action of left and right translations on G, though unitary, will have no self-adjoint generators. Hence we have no obvious way to construct flux operators. We emphasize that this is not an idle point: in particular, if the algebra in question is not a subalgebra of the uniformly continuous functions, then there is no guarantee that the unitary semigroups of left and right translations have self-adjoint generators.

The second key difference is that in the general case of amenable groups, there is in general not just *one* invariant mean, but many [57]. This contrasts sharply with the case of the $\mathcal{AP}$-compactification, whose invariant mean is guaranteed to be unique. While non-uniqueness is clearly a much less serious problem in this context than non-existence, when uniqueness is available, it seems desirable to take advantage of it. Moreover, as we have already alluded, the $\mathcal{AP}$-compactification itself is a compact topological group, and it is in fact the largest compactification of a given topological group with this property (that is,

any other group compactification is a quotient of the $\mathcal{AP}$-periodic compactification). In particular, the Stone-Čech compactification will not even be a semigroup, for the case of the real line. Again, while the physical necessity of having a group compactification is not clear, it does at least suggest that the $\mathcal{AP}$-compactification is the more natural of the two.

Thus, it appears that the use of the almost periodic compactification allows significant advances over previous attempts to construct $\mathcal{H}_{\text{Poly}}$ for spaces of connections with certain non-compact gauge groups. Unfortunately, it has not allowed us to construct the space for the gauge groups we are most interested in: $\text{SL}(2,\mathbb{R})$ and $\text{SL}(2,\mathbb{C})$. So in the next section we consider some generalizations of the ideas considered in this section, to see what they can tell us about such gauge groups.

5.3 Extension to more general non-compact groups

We know from section 5.2.3 that for groups such as $\text{SL}(2,\mathbb{R})$ and $\text{SL}(2,\mathbb{C})$ that are not direct products of a compact group with $\mathbb{R}^n$, the almost periodic functions fail to separate the points of the group, and so the $\mathcal{AP}$-compactification itself is not useful in constructing $\overline{\mathcal{A}}$.

In fact, for the groups $\text{SL}(n,\mathbb{R})$ and $\text{SL}(n,\mathbb{C})$, with $n \geq 2$, this failure is spectacular: the almost periodic functions consist *entirely* of the constant functions [57], and so the $\mathcal{AP}$-compactification of these groups is trivial, consisting of just a single point.

The success we have achieved for locally compact Abelian groups and the nature of the difficulties with previous work that we outlined in section 5.1.2 strongly suggest that we should continue to work with a compactification of the gauge group; in other words, we should insist that the constant functions be among the integrable functions so that we can 'patch together' different graphs appropriately. Thus, our task becomes one of identifying a suitable unital C^*-algebra of functions on G.

That is the task we take up in this section. In the first subsection, we consider several commonly studied C^*-algebras of functions on topological groups, and find that for various reasons they are not suitable for the groups $\text{SL}(2,\mathbb{R})$ and $\text{SL}(2,\mathbb{C})$. We then identify a list of requirements that any workable choice of algebra would appear to need. In the second subsection we prove a partial 'no-go' theorem that shows that for these groups, at least, something quite different must happen, if the compactification approach can be made to work at all.

5.3.1 Difficulties with standard compactifications for $\text{SL}(2,\mathbb{R})$ and $\text{SL}(2,\mathbb{C})$

Since the algebra of almost periodic functions is too small for the groups $\text{SL}(2,\mathbb{R})$ and $\text{SL}(2,\mathbb{C})$, we need to try a bigger algebra and therefore one possible choice is the biggest of them all: the algebra $C_b(G)$ of all bounded, continuous functions on the group. Though we

noted difficulties in using this algebra at the end of section 5.2.4, we may prefer to try to live with those difficulties rather than have no measure theory on $\overline{\mathcal{A}}$ at all.

Unfortunately, for the groups $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{C})$, with $n \geq 2$, this approach is not viable for another reason: these groups are not amenable [57]. They have *no* invariant means at all on the algebra $C_b(G)$. Thus, this algebra is too big.

Is there another algebra large enough to separate the points of the group, and yet small enough to possess an invariant mean? There is at least one: the algebra of *weakly almost periodic functions*. Since it would take us too far afield, we do not present the definition of this algebra (which may be found among other places in [57]), but rather note important properties that it has for our purposes.

We need two useful facts (both of which may be found in [57]) about the algebra of weakly almost periodic functions ($\mathcal{WAP}$) on a locally compact, non-compact group. The first is that for any such group, the algebra $\mathcal{WAP}$ contains all of the continuous functions vanishing at infinity, $C_0(G)$. The second is that $\mathcal{WAP}$ possesses a unique invariant mean.

This first property ensures that $\mathcal{WAP}(G)$ separates the points of the group, and since it is easy to show that $\mathcal{WAP}(G)$ always contains the constant functions, by virtue of the second fact the first two at least of the properties of section 5.1.1 are satisfied.

However, for the groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$, at least, there is a catch. Because the center of these groups is finite (consisting just of ± 1) and the groups themselves are semisimple, the main result of [64] applies: the $\mathcal{WAP}$ functions on these groups consist entirely of the algebra $C_0(G)+\mathbb{C}$; i.e., the functions approaching a constant value at infinity. The problem with this is that the unique invariant mean on this algebra consists simply of evaluation at the point at infinity [64]. Hence, even though the algebra $\mathcal{WAP}(G)$ separates the points of G, if $\overline{G}$ is the resulting compactification, we have found that the Hilbert space $L^2(\overline{G}, d\mu_H)$ is *one-dimensional*. Physically, this is simply unworkable.

Notice that the $\mathcal{AP}$-compactification of these groups consisted of a single point, and that the invariant mean on the $\mathcal{WAP}$-compactification consisted of evaluation at a single point. This is not a coincidence. In general, for any topological group G, the invariant mean on $\mathcal{WAP}(G)$ consists of integration over a group isomorphic to the $\mathcal{AP}$-compactification of G [65]. Thus, the approach based on $\mathcal{WAP}$ functions was in fact doomed as soon as the approach based on the $\mathcal{AP}$-compactification failed.

We may therefore consider other standard algebras. We shall here be even more terse than we have been for the $\mathcal{WAP}$ algebra: we shall simply state the problem with each choice, providing a reference to the results needed to establish the difficulty. Definitions of all of these algebras and a much fuller discussion may be found in chapter 4 of [57]; see particularly section 4.11 for a useful summary of results and inclusion relations between the various algebras.

We have seen that for $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ none of $\mathcal{AP}(G)$, $\mathcal{WAP}(G)$, or $C_b(G)$ are

usable algebras for our purpose. For other algebras described in chapter 4 of [57] we have the following results (throughout, G denotes either of $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$):

1. **The strongly almost periodic functions $\mathcal{SAP}$.** For a topological group, these coincide with the $\mathcal{AP}$-functions, and so fail to separate the points of G

2. **The uniformly continuous functions $\mathcal{U}$.** These are functions uniformly continuous with respect to *both* the left and right uniform structures of G. By Theorem 2.4.7 of [57] and the non-amenability of G, this algebra has no invariant mean.

3. **The algebras $\mathcal{LC} \cap \mathcal{RC}$, $\mathcal{LMC} \cap \mathcal{RMC}$, and $\mathcal{WLC} \cap \mathcal{WRC}$.** By Remark 4.4.2 and Theorem 4.5.7 of [57], for a locally compact topological group (as G is) each of these algebras coincides with $\mathcal{U}$, and hence possesses no invariant mean.

4. **The set $\mathcal{MIN}$ of minimal functions**. By Theorem 4.8.10 of [57], this set separates the points of G. However, it is in general not even a vector space, much less an algebra, and hence provides us with no useful compactification.

5. **The almost automorphic functions $\mathcal{AA}$.** By Lemma 4.7.10 of [57] and the triviality of the $\mathcal{AP}$-compactification of G, the $\mathcal{AA}$-compactification of G is also trivial.

6. **The Bohr almost periodic functions $\mathcal{BAP}$.** We need functions that are both left and right Bohr almost periodic, and by Problem 4.10.12 of [57] and the triviality of the $\mathcal{AP}$-compactification of G, the only such functions are the constant functions.

Several times above we have dealt with the intersections of algebras defined with respect to some property for both left and right translations. That is because for our constructions, we need an algebra that is closed under both left and right translations and possesses a mean invariant under both types of translation.

In the above list, we have dealt with almost all of the algebras of functions considered in chapter 4 of [57], and found them to be unsatisfactory for one reason or another. This certainly does not give one much hope for the prospect of creating a useful integration theory on $\overline{\mathcal{A}}$ when $G = \mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$ by means of an algebraic compactification, but it is still far from a complete impossibility proof. However, we can first attempt to summarize what we have learned by listing the properties any algebra A on G must have in order to provide us with a physically plausible construction. It is assumed throughout that A contains the constant functions and is closed under complex conjugation and both left and right translation.

1. A must separate the points of G.

2. A must possess an invariant mean μ.

3. The Hilbert space $L^2(\overline{G}, d\mu)$ must be infinite dimensional. (Here $\overline{G}$ is the compactification of G resulting from the C^*-algebra A.)

We emphasize that all of these conditions are necessary; there are examples of algebras satisfying any two of them which are each unusable for constructing a suitable integration theory. In addition, we should also require that the unitary action of left and right translations (guaranteed by 2 and 3 above) be weakly continuous. This will always be the case if A is a subalgebra of the uniformly continuous functions; it is not clear if it need hold in more general settings. However, without it we have no momentum operators and hence no flux operators on $\mathcal{H}_{\rm Poly}$. It is perhaps conceivable that a theory could be constructed without this (after all, we do not directly have a connection operator), but such a theory would certainly be very different from the theory so successfully developed for compact gauge groups.

Thus, we find discouraging results for the gauge groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$, but still no definitive proof that a construction along these lines is impossible. Such a proof—or alternatively, a successful construction—is still lacking, but there is a partial 'impossibility' proof available, and we turn to that next.

5.3.2 A partial 'no-go' theorem

As the results of the previous subsection indicate that none of the commonly studied algebras of functions on G give a good construction when $G = \mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$, we next consider what functions we should *like* to see in out algebra, if we had the choice.

That question is fairly easy to answer. Recall from section 5.1.1 that when dealing with measure theory on $\overline{\mathcal{A}}$ for a compact gauge group G we could, by virtue of the Peter-Weyl theorem, decompose the total Hilbert space into a direct, orthogonal sum over the Hilbert spaces of each graph and each labeling of a given graph by irreducible unitary representations of G. This in turn reflected the fact that an arbitrary element of $L^2(G, d\mu_H)$ can be decomposed into a sum of matrix elements of unitary irreducible representations of G. These matrix elements have the further property that they are eigenfunctions of the Casimir operator on the group (for SU(2) there is only one Casimir) and this has important implications for quantum geometry: it is these eigenfunctions which determine the eigenstates of the area operator, and the eigenvalues of the area operator are determined from the eigenvalues of the SU(2) Casimir.

Thus, it would certainly be desirable—and would mimic closely the structure of $\overline{\mathcal{A}}$ for compact gauge groups—if in the non-compact case as well we found that these matrix elements were among our integrable functions; that is, elements of the C^*-star algebra with which we compactify the group. What we shall show in this subsection is that for $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ they cannot be.

This in and of itself is a cause for concern, because a part of the motivation for con-

sidering these groups comes from work on spin foams, where there are proposals to define Lorentzian spin foams [66–68], and those are all premised on a labeling of edges with unitary representations of $\mathrm{SL}(2,\mathbb{C})$.

Let us comment a little more on this. In both [66] and [68] it is argued that in Lorentzian spin foam models the labels of faces by representations of $\mathrm{SL}(2,\mathbb{C})$ dictate the area eigenvalues of the corresponding faces. So long as one stays within the confines of spin foam models, one may take this essentially as a definition, strengthened by the derivation of spin foam model vertex amplitudes through geometric quantization of the space of assignments of areas to the faces of a tetrahedron (see [69] for this derivation in the case of Euclidean models).

As soon as one seeks to make contact with a Hamiltonian theory, however, it is no longer sufficient to simply postulate such an interpretation, but instead one must construct an appropriate Hilbert space of states on which there is a densely defined area operator; this is just another way of looking at our goal of an extension of spin networks to non-compact gauge groups. Starting from the proposal of Freidel and Livine, in [70] just such a definition was made for the length and area operators in 2+1-dimensional Lorentzian gravity. However, as we have already noted there are major physical difficulties with the construction of [27], and consequently with the conclusions of [70] regarding area and length spectra.

The result of this subsection can be interpreted as showing that such difficulties are rather generic, for even if one has a definition of a spin foam model for a Lorentzian signature in which the faces are labeled by irreducible representations of $\mathrm{SL}(2,\mathbb{C})$, one cannot find a corresponding Hamiltonian theory in which there are normalizable spin network states given by matrix elements of the irreducible representations of $\mathrm{SL}(2,\mathbb{C})$.

Before discussing the main result of this subsection and its proof, we must mention several subtleties that arise in harmonic analysis on a non-compact Lie group. For simplicity, we shall restrict ourselves to semisimple Lie groups, which in particular are always unimodular; that is, the left and right Haar measures coincide. This is a fairly broad class that in particular includes $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$, our groups of interest.

First, consider the right regular representation of the group. This is the representation of the group acting on the Hilbert space $L^2(G, d\mu_H)$ of square integrable functions on the group by $g \mapsto U(g)$ where $U(g_0)f(g) = f(gg_0)$ for all $f \in L^2(G, d\mu_H)$. As in the case of a compact group, this representation is unitary on account of the invariance of the Haar measure. It can also be decomposed into a direct sum or direct integral over irreducible unitary representations. Unlike the compact case, however, in general this decomposition will not use *all* irreducible unitary representations, but only a certain subset, those of the so-called *principal series*. Henceforth, we shall be exclusively concerned with the principal series representations; these also are the only ones used in the existing Lorentzian spin foam models.

Another subtlety is that the Casimir operators on the group (the bi-invariant elements of the enveloping algebra) will in general have a continuous spectrum; we already alluded to this in the previous paragraph when we mentioned that the decomposition of the right regular representation can involve a direct integral and not just a direct sum. A more serious consequence of the continuous spectrum than just the appearance of a direct integral is that in general the eigenfunctions of the bi-invariant operators may only be distributions, and not functions on the group. Again, however, we are fortunate in the case of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$: for these groups, and more generally for any semisimple Lie group with a maximal compact subgroup, the eigenvectors *are* actual functions on the group [59].

Let us introduce some notation for these eigenfunctions; in so doing we follow [59] and [71]. We denote an arbitrary eigenfunction by $D^{\lambda}_{pq}(g)$; here λ denotes the eigenvalues of the commuting, invariant elements of the enveloping algebra, and p and q label non-invariant eigenvalues corresponding to the left and right representations. For semisimple Lie groups with maximal compact subgroup, we then have the following [59]: any function $f \in L^2(G, d\mu_H)$ can be decomposed as:

$$f(g) = \int_{\Lambda} d\rho(\lambda) \sum_{p,q=1}^{\dim\mathcal{H}(\lambda)} \hat{f}_{pq}(\lambda) D^{\lambda}_{pq} \tag{5.3.1}$$

where the Fourier transform $\hat{f}$ is given by

$$\hat{f}_{pq}(\lambda) = \int_G f(g) \overline{D}^{\lambda}_{pq}(g)\, d\mu_H. \tag{5.3.2}$$

Here $\rho(\lambda)$ is the Plancherel measure on the spectrum of the commuting invariant operators; it vanishes for λ outside of the principal series, by definition.

These formulas generalize the usual harmonic analysis on compact groups. Note that while our notation would tend to indicate that the set of eigenvalues $\{\lambda\}$ is continuous and the sets $\{p\}$ and $\{q\}$ are discrete, in general any of these sets may be continuous, discrete, or mixed. For the particular case of $\mathrm{SL}(2,\mathbb{C})$, for example, λ corresponds to a pair (ν,ρ), where ν is a half integer and ρ a nonnegative real number. In this case we then also have $p = (J, M)$ and $q = (J', M')$ with $J, J' = \nu, \nu+1, \ldots$ and $-J \le M \le J$, $-J' \le M' \le J'$. The D^{λ}_{pq} then correspond to matrix elements of the unitary representations of the principal series in the so-called canonical basis [66, 67, 72].

Two facts about the $D^{\lambda}_{pq}(g)$ will be very important for us. First, they satisfy the relations:

$$D^{\lambda}_{pq}(gg') = \sum_r D^{\lambda}_{pr}(g) D^{\lambda}_{rq}(g') \tag{5.3.3}$$

$$D^{\lambda}_{pq}(g^{-1}) = \overline{D}^{\lambda}_{pq}(g) \tag{5.3.4}$$

Second, at least for $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$, the $D^{\lambda}_{pq}(g)$ go to zero as g goes to infinity. This fact can be checked directly from an asymptotic expression of the D^{λ}_{pq} for the principal series [73].

That concludes the preliminary information we need about the $D^{\lambda}_{pq}(g)$ themselves; now we need a few more definitions in order to state and prove our theorem.

We have already defined a mean on a C^*-algebra A of bounded continuous functions on a group as a positive linear functional on that algebra satisfying $\inf f \leq \mu(f) \leq \sup f$ when f is real. Among the means are also a particular class that we shall need: the *multiplicative means*, those for which $\mu(fg) = \mu(f)\mu(g)$ for all $f, g \in A$. Such means, for example, may be found by evaluating functions at particular points of G. In fact, more generally the multiplicative means correspond precisely to the spectrum of A [57]; they thus form a compact Hausdorff space that is the canonical compactification of G by the algebra A.

Next, note that for any mean on A and any function $f \in A$ we can define a new function on G by way of the *left introversion operator* L_μ:

$$L_\mu f(g) := \mu({}_g f) \tag{5.3.5}$$

The right introversion operator R_μ is defined similarly, in terms of the right translates of f. An algebra A is then said to be *left introverted* if $L_\mu(f) \in A$ for every mean μ on A and every $f \in A$. Right introversion is defined analogously. An algebra A is then said to be simply *introverted* if it is both left and right introverted. It is said to be *left m-introverted* if it is closed under the left introversion operators for multiplicative means only; right m-introversion and m-introversion are defined in the obvious fashion.

The concept of introversion is a central part of the definition of admissibility that we shall need, and therefore now state:

5.3.1 Definition (Admissibility, m-admissibility) *A subspace of the C^*-algebra $C_b(G)$ of all bounded, continuous functions on a topological group G is said to be an* admissible subspace *if it satisfies each of the following properties:*

1. *It contains the constant functions,*
2. *It is closed under complex conjugation,*
3. *It is closed under the norm of $C_b(G)$,*
4. *It is invariant under left and right translations, and*
5. *It is introverted.*

If a subalgebra satisfies all of the above except that in the last requirement m-introversion is substituted for introversion, then it is said to be an m-admissible subalgebra.

Our need for this definition is the following theorem, which is 4.2.14 in [57]. We have simplified its statement considerably as our needs are quite limited.

5.3.2 Theorem *Let A be an m-admissible subalgebra of the $\mathcal{WAP}$ functions on a topological group. Then A has a unique invariant mean.*

From this theorem, we can deduce the following, the main result of this subsection:

5.3.3 Theorem *Let G be either $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$. Let A be the C^*-algebra generated by the constant functions and the matrix elements D^λ_{pq} of the principal series. If $\mathcal{F}$ is any C^*-algebra containing A and possessing an invariant mean μ, then $\mu(\overline{D}^\lambda_{pq} D^\lambda_{pq}) = 0$ for all matrix elements in the principal series. In other words, they have norm zero in the Hilbert space $L^2(\overline{G}, d\mu)$, with $\overline{G}$ the $\mathcal{F}$-compactification of G.*

Put another way, if the matrix elements are to have non-zero norm, then they must in fact have *infinite* norm, and so cannot be elements of the Hilbert space $L^2(\bar{G}, d\mu)$.

Proof: We prove this theorem by showing that the algebra A defined in the hypothesis is an m-admissible subalgebra of $\mathcal{WAP}(G)$. This suffices to establish the theorem because we already know that for $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ the unique invariant mean on $\mathcal{WAP}$ consists of evaluation at the point at infinity. Theorem 5.3.2 assures us that the mean on A must then be the same, and we already know that under that mean $\mu(\overline{D}^\lambda_{pq} D^\lambda_{pq}) = 0$, since we have noted above that the matrix elements D^λ_{pq} go to zero at infinity.

Because of this asymptotic behavior, and because the $\mathcal{WAP}$ functions on a locally compact group always contain the functions that are constant at infinity, we know that $A \subset \mathcal{WAP}(G)$; we need only verify that A as defined is m-admissible. That it contains the constant functions, is closed under conjugation, and is norm closed is immediate from its definition. By virtue of equation (5.3.3), it is also invariant under left and right translations.

So we need only establish m-introversion. But this too follows from (5.3.3), since for any multiplicative mean μ we have:

$$\mu\left(D^{\lambda_1}_{p_1q_1}(g_0g)\cdots D^{\lambda_l}_{p_lq_l}(g_0g)\right) = \left(\sum_{r_1} D^{\lambda_1}_{p_1r_1}(g_0)\,\mu\Big(D^{\lambda_1}_{r_1q_1}\Big)\right)\cdots\left(\sum_{r_l} D^{\lambda_l}_{p_lr_l}(g_0)\,\mu\Big(D^{\lambda_l}_{r_lq_l}\Big)\right). \quad (5.3.6)$$

Viewed as a function of g_0, the right hand side is again in A. Since every element of A is the uniform limit of expressions of the form on the left hand side of (5.3.6), this is sufficient to establish left-introversion. Right introversion is handled similarly. ∎

As a consequence of this theorem, any C^*-algebra of continuous functions that we might try to use in constructing a compactification of G should in fact *not* contain the matrix

elements D^{λ}_{pq}, since if it does they would have norm zero in the Hilbert space. This in turn would imply that all of the area eigenstates of $\mathcal{H}_{\mathrm{Poly}}$ had zero norm, a physically unacceptable conclusion. But if the matrix elements are not integrable functions, then the eigenstates of the area operator will not be normalizable either. Since we are considering a non-compact group, this is perhaps not surprising; after all, typically when the spectrum of an operator is a continuous segment of the real line, that segment inherits the usual topology of $\mathbb{R}$ and so the eigenstates of that operator will include generalized eigenstates. However, that is not what happens with the non-compact Abelian groups considered in section 5.2; even though the momentum operator had all real numbers in its spectrum, its eigenstates were still normalizable because the spectrum was in fact $\mathbb{R}$ with the discrete topology, not its usual topology.

5.4 Discussion

Motivated by the desire to formulate loop quantum gravity with directly with $\mathrm{SL}(2,\mathbb{C})$ as its gauge group, in this chapter we have attempted to extend the construction of measure theory to spaces of generalized connections with non-compact gauge group. Because of the difficulties encountered in existing approaches with the failure of the constant functions to be integrable, we were led to look instead at compactifications of the gauge group that arise as the spectrum of unital C^*-algebras of continuous functions on the gauge group.

For locally compact, connected Abelian groups this strategy is completely successful if we adopt the almost periodic compactification. More generally, we may use this compactification whenever the gauge group is of the form $G \times \mathbb{R}^n$ for some compact G. However, this strategy fails for the groups we are most interested in, since the almost periodic functions fail to separate the points of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$. We likewise found that various other standard algebraic compactifications failed, and formulated criteria that any successful compactification must meet. Moreover, we were able to prove a partial 'no-go' theorem that establishes that the most obvious generalizations of spin-network states cannot be normalizable.

Thus, a key open question is to see whether the no-go result can be extended to completely exclude any integration theory on $\overline{\mathcal{A}}$ based on compactifying $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$, or whether in fact there is a workable approach along these lines. If the former should turn out to be true, then in conjunction with the existing constraints on other approaches to constructing measure theory on $\overline{\mathcal{A}}$ (that do not rely on compactifying G) it would begin to seem plausible that the passage to the real $\mathrm{SU}(2)$ variables is in fact unavoidable.

Appendix A

Additional proofs and constructions for the polymer particle

In this appendix we collect together various proofs and concepts related to the polymer particle model studied in chapters 2 and 3. While useful, these concepts largely fall outside the main development of the text. They are likely, however, to be of more importance as the shadow state framework is extended to more complex systems, such as Maxwell fields and linearized gravity.

A.1 Restrictions on graphs for the polymer particle

In this section we demonstrate that the two restrictions on the allowed graphs in the polymer particle Cyl ensure that the functions in Schwartz space $\mathcal{S}$ are all in Cyl*. We therefore recall the definition:

A.1.1 Definition *A graph γ is a countable collection of points $\{x_i\}$ in the real line $\mathbb{R}$ having no accumulation point. Furthermore, there exists a length ℓ_γ such that if I is any interval whose length $\ell(I)$ is greater than or equal to ℓ_γ, then the number of points $n(I)$ of γ lying in I satisfies*

$$n(I) \leq \rho_\gamma \, \ell(I) \tag{A.1.1}$$

for some constant ρ_γ.

We would now like to show that this definition implies that all functions $\phi \in \mathcal{S}$ are in fact in Cyl*. Therefore, we assume that $\phi \in \mathcal{S}$; we now need to show that for any cylindrical function f the inner product $\langle \phi \,|\, f \rangle$ is a finite complex number.

To do this, let $\gamma = \{x_j\}$ be the graph with which f is associated. We use our restriction (A.1.1) to group the points $\{x_j\}$ of our graph. We partition the real line into intervals of the form $[n\,\ell_\gamma, (n+1)\ell_\gamma)$. Then each point x_j lies in precisely one such interval, and there are at most $\rho_\gamma \, \ell_\gamma$ such points in any given interval. We may therefore alternately label each point x_j of γ as $x_{n,k}$, where the index n identifies in which of the intervals of length ℓ_γ the point lies, and the index k specifies which of the points of the graph in this interval we have.

Thus, in particular we always have:

$$k \leq \rho_\gamma \, \ell_\gamma. \tag{A.1.2}$$

By the assumption that f is a cylindrical function over our graph we must also have:

$$\sum_{j \in \mathbb{N}} |f(x_j)|^2 < \infty. \tag{A.1.3}$$

This in turn implies that there exists some natural number N such that

$$|f(x_{n,k})| < 1 \tag{A.1.4}$$

for all $|n| \geq N$.

Finally, since $\phi \in \mathcal{S}$, it must be the case [74] that there exists some positive real number M_ϕ such that

$$|\phi(x)| < \frac{M_\phi}{1 + x^2} \tag{A.1.5}$$

for every real number x.

With this observation we now have enough to prove our claim that $\langle \phi \,|\, f \rangle$ is a finite complex number. Indeed, we have:

$$\begin{aligned} |\langle \phi \,|\, f \rangle| &:= \left| \sum_{j \in \mathbb{N}} \overline{\phi(x_j)} \, f(x_j) \right|, \\ &\leq \sum_{j \in \mathbb{N}} |f(x_j)| \, |\phi(x_j)|, \qquad \text{by the Schwarz inequality} \\ &= \sum_n \sum_k |f(x_{n,k})| \, |\phi(x_{n,k})|, \\ &= \sum_{|n| < N} \sum_k |f(x_{n,k})| \, |\phi(x_{n,k})| + \sum_{|n| \geq N} \sum_k |f(x_{n,k})| \, |\phi(x_{n,k})|. \end{aligned} \tag{A.1.6}$$

Since the sum over $|n| < N$ is over a finite number of points and both f and ϕ are finite everywhere, there is no question of convergence of the first sum on the last line above; it is

manifestly of the form $|C_{N,f,\phi}|$ for some complex number $C_{N,f,\phi}$ and therefore we have:

$$\begin{aligned}
|\langle\phi\,|\,f\rangle| &\leq |C_{N,f,\phi}| + \sum_{|n|\geq N}\sum_{k} |f(x_{n,k})|\,|\phi(x_{n,k})|, && \text{from (A.1.6)}\\
&< |C_{N,f,\phi}| + \sum_{|n|\geq N}\sum_{k} |\phi(x_{n,k})|, && \text{by (A.1.4)}\\
&< |C_{N,f,\phi}| + \sum_{|n|\geq N}\sum_{k} \frac{M_\phi}{1+x_{n,k}^2}, && \text{by (A.1.5)}\\
&\leq |C_{N,f,\phi}| + \rho_\gamma\,\ell_\gamma \sum_{|n|\geq N} \frac{M_\phi}{1+x_{n,0}^2}, && \text{by (A.1.2)}\\
&\leq |C_{N,f,\phi}| + \rho_\gamma\,\ell_\gamma\,M_\phi\left(1+2\sum_{n>0}\frac{1}{n^2\ell_\gamma^2}\right),\\
&= |C_{N,f,\phi}| + \rho_\gamma\,\ell_\gamma\,M_\phi\left(1+\frac{\pi^2}{3\,\ell_\gamma^2}\right).
\end{aligned} \tag{A.1.7}$$

In arriving at the last equality we have used $\sum_{n>0} n^{-2} = \pi^2/6$.

Thus, $\langle\phi\,|\,f\rangle$ is indeed finite, as we needed to show.

By entirely similar reasoning we may prove that for any $\phi \in \mathcal{S}$ and any graph γ that meets all of the conditions of Definition (A.1.1), the shadow state $|\phi_\gamma^{\text{shad}}\rangle$ has finite norm with respect to the inner product on $\mathcal{H}_\gamma$. Specifically, we have:

$$\begin{aligned}
|\langle\phi_\gamma^{\text{shad}}\,|\,\phi_\gamma^{\text{shad}}\rangle| &:= \sum_n\sum_k |\phi(x_{n,k})|^2,\\
&< \sum_n\sum_k \left|\frac{M_\phi}{1+x_{n,k}^2}\right|^2,\\
&\leq \rho_\gamma\,\ell_\gamma \sum_n \frac{M_\phi^2}{(1+x_{n,0}^2)^2},\\
&\leq \rho_\gamma\,\ell_\gamma\,M_\phi^2\left(1+2\sum_{n>0}\frac{1}{n^4\ell_\gamma^4}\right),\\
&\leq \rho_\gamma\,\ell_\gamma\,M_\phi^2\left(1+\frac{\pi^4}{45\,\ell_\gamma^4}\right),
\end{aligned} \tag{A.1.8}$$

and this time we have used the sum $\sum_{n>0} n^{-4} = \pi^4/90$.

Finally, we would like to show that the definition of Cyl using graphs that satisfy Definition A.1.1 is independent of the length parameter ℓ_γ. In order to do this, we show that if a graph γ meets the criteria of Definition A.1.1 for one value of ℓ_γ, then it does so for any other (positive) value.

So, let γ be a graph meeting all the conditions of Definition A.1.1 for some particular ℓ_γ. This means that of necessity there exists the positive number ρ_γ of Equation (A.1.1). Consider now some other length scale $\ell'_\gamma > \ell_\gamma$; we want to show that there exists some number ρ'_γ such that

$$n(I) \leq \rho'_\gamma \, \ell(I) \tag{A.1.9}$$

whenever $\ell(I) \geq \ell'_\gamma$. But if $\ell(I) \geq \ell'_\gamma$ and $\ell'_\gamma > \ell_\gamma$, then certainly $\ell(I) \geq \ell_\gamma$ and thus (A.1.9) holds with $\rho'_\gamma = \rho_\gamma$.

So next we turn to the case where $\ell'_\gamma < \ell_\gamma$. Obviously (A.1.9) will continue to hold with $\rho'_\gamma = \rho_\gamma$ for any interval I such that $\ell(I) \geq \ell_\gamma$; the only case we need have any worry about is $\ell'_\gamma \leq \ell(I) < \ell_\gamma$. But for this case we can always find another interval I' containing I that is of length ℓ_γ. Then we know that $\ell(I') \leq \rho_\gamma \, \ell_\gamma$. But, since we must also have $n(I) \leq n(I')$ as I is contained in I', we know that:

$$\begin{aligned} n(I) &\leq \rho_\gamma \, \ell_\gamma \\ &= \frac{\rho_\gamma \, \ell_\gamma}{\ell(I)} \, \ell(I) \\ &\leq \frac{\rho_\gamma \, \ell_\gamma}{\ell'_\gamma} \, \ell(I). \end{aligned} \tag{A.1.10}$$

Thus, (A.1.9) indeed holds, with

$$\rho'_\gamma = \frac{\ell_\gamma}{\ell'_\gamma} \, \rho_\gamma. \tag{A.1.11}$$

Hence, the space Cyl is independent of the value of ℓ_γ used in Definition A.1.1.

A.2 The displacement operator $\hat{V}(\mu)$ and holonomies

Recall that the displacement operators $\hat{V}(\mu)$ are the analogs of holonomy operators in Maxwell theory and quantum geometry. In this section we describe some properties of displacement operators which will be useful in the discussion of holonomies in subsequent work extending the shadow state framework to theories with holonomies.

We begin by recalling the commutator between $\hat{x}$ and $\hat{V}(\mu)$:

$$[\hat{x}, \hat{V}(\mu)] = -\mu \hat{V}(\mu) \, . \tag{A.2.1}$$

This equation gives rise to interesting uncertainty relations, even though $\hat{V}(\mu)$ are unitary rather than self-adjoint [75]. To obtain these, let us decompose $\hat{V}$ into the sum of two Hermitian operators,

$$\hat{V}(\mu) = \hat{C}(\mu) + i\hat{S}(\mu) \, . \tag{A.2.2}$$

It is straightforward to obtain the commutation relations between $\hat{C}$, $\hat{S}$ and $\hat{x}$ from (A.2.1) as

$$[\hat{x}, \hat{C}(\mu)] = -i\mu\hat{S}(\mu) \qquad \text{and} \qquad [\hat{x}, \hat{S}(\mu)] = i\mu\hat{C}(\mu)\,.$$

Therefore, we can obtain uncertainty relations between $\hat{C}$, $\hat{S}$ and $\hat{x}$:

$$(\Delta x)^2(\Delta C(\mu))^2 \geq \frac{\mu^2}{4}\langle\hat{S}(\mu)\rangle^2 \qquad \text{and} \qquad (\Delta x)^2(\Delta S(\mu))^2 \geq \frac{\mu^2}{4}\langle\hat{C}(\mu)\rangle^2 \tag{A.2.3}$$

Now, it is natural to define the uncertainty in $\hat{V}$ as

$$(\Delta V)^2 := \langle V^\dagger V\rangle - |\langle V\rangle|^2 = 1 - (\langle C\rangle^2 + \langle S\rangle^2)\,, \tag{A.2.4}$$

where the second expression follows from the unitarity of $\hat{V}$ and the definitions of $\hat{C}$ and $\hat{S}$ (A.2.2). Finally, combining (A.2.4) and (A.2.3) we obtain the desired uncertainty relation

$$(\Delta x)^2\frac{(\Delta V(\mu))^2}{1-(\Delta V(\mu))^2} \geq \frac{\mu^2}{4}\,. \tag{A.2.5}$$

It is natural to ask how close the semi-classical states of section 2.4.1 come to saturating this bound. Let us begin by considering the state $(\Psi_o|$ peaked at (x=0, k=0). The 'expectation value' of $\hat{V}(\mu)$ in $(\Psi_o|$ and its shadow $|\Psi^{\rm shad}_{o,\ell}\rangle$ on a regular lattice with spacing ℓ is given in (2.4.21) as

$$\langle\hat{V}(\mu)\rangle \approx e^{-\frac{\mu^2}{4d^2}}\left(1 + e^{-\frac{\pi^2 d^2}{\ell^2}}\left[2\cos\left(\frac{\pi\mu}{\ell}\right) - 2\right]\right)\,.$$

Then, it is straightforward to evaluate the fluctuations of $\hat{V}(\mu)$ as

$$(\Delta\hat{V}(\mu))^2 := 1 - |\langle\hat{V}(\mu)\rangle|^2 \approx 1 - e^{-\frac{\mu^2}{2d^2}}\,, \tag{A.2.6}$$

where we have neglected corrections of order $\exp(-\frac{\pi^2 d^2}{l^2})$. Combining (A.2.6) with the fluctuations in x (2.4.17), we obtain:

$$(\Delta x)^2\cdot\frac{(\Delta V(\mu))^2}{1-(\Delta V(\mu))^2} \approx \left(\frac{d^2}{2}\right)\cdot\left(\frac{1-e^{-\frac{\mu^2}{2d^2}}}{e^{-\frac{\mu^2}{2d^2}}}\right)\,. \tag{A.2.7}$$

Thus, for a general μ, we are not close to saturation. However, if $\mu \ll d$, we can expand in powers of μ/d to obtain:

$$(\Delta x)^2\frac{(\Delta V(\mu))^2}{1-(\Delta V(\mu))^2} = \left(\frac{\mu^2}{4}\right)\left(1 + \mathcal{O}\left(\frac{\mu^2}{d^2}\right)\right)\,. \tag{A.2.8}$$

Thus, in this case, the uncertainty relation (A.2.5) is indeed saturated, modulo terms of the order $(\mu/d)^2$. If $\mu \sim \ell$, a similar result can be obtained for general coherent states peaked at any value of momentum k, *even when* k *approaches* π/ℓ. Note that this in marked contrast to the uncertainty relation between $\hat{x}$ and $\hat{K}_{\mu_o}$ which is similarly saturated only if $k\mu_o \ll 1$.

Finally, a natural question is whether the 'expectation value' of $\hat{V}(\mu)$ can be used to determine the momentum of the system when it is in a semi-classical state. In a semi-classical state labeled by $\zeta = \frac{1}{\sqrt{2}d}(x + id^2k)$, the 'expectation value' of $\hat{V}(\mu)$ is given by

$$\langle \hat{V}(\mu) \rangle = e^{-\frac{\mu^2}{4d^2}}\, e^{-ik\mu}\, \left[1 + \mathcal{O}\left(e^{-\frac{\pi^2 d^2}{\ell^2}}\right)\right] . \tag{A.2.9}$$

An obvious strategy is to just define the 'expected momentum' $\tilde{k}$ in the quantum state $(\Psi_\zeta|$ to be:

$$\langle \hat{V}(\mu) \rangle = |\langle \hat{V}(\mu) \rangle|\, e^{-i\mu\,\tilde{k}}\,, \tag{A.2.10}$$

i.e., to associate the momentum $\tilde{k}$ with the phase of the $\hat{V}$ operator. Clearly, modulo corrections $\mathcal{O}(\exp -\pi^2 d^2/\ell^2)$, $\tilde{k}$ equals k. Moreover, this result holds even if $k \sim \pi/\ell$. The $|\langle \hat{V}(\mu) \rangle|$ factor in our expression (A.2.10) may seem surprising. However, it does *not* arise because of the polymer nature of the Hilbert space we are considering; it is necessary also in the Schrödinger representation. Note also that our expression (A.2.6) for the variation of V, implies that $|\langle \hat{V}(\mu) \rangle|$ must be less than one if $\mu \neq 0$. Otherwise the fluctuation in V will vanish, i.e., we will have a state of definite momentum and the uncertainty relation (A.2.5) would imply that the state must have infinite spread in x.

Techniques introduced in this appendix will be useful when it comes to examining expectation values and fluctuations of holonomies in Maxwell and gravitational semi-classical states.

A.3 Approximate consistency

In the main body of the thesis we introduced 'fundamental operators' such as $\hat{K}_{\mu_o}$ and $\hat{H}$ on the entire polymer Hilbert space $\mathcal{H}_{\rm Poly}$ and analyzed their properties. In field theories, by contrast, one often ties operators to the energy scale under consideration and constructs from them 'an effective field theory' a la Wilson. Such constructions are likely to play an important role in relating quantum field theories on quantum geometries with low energy physics. Therefore, in this section, will extend some of the considerations of chapters 2 and 3 by allowing operators which are tied to the lattice spacing under consideration. For example, by setting $\hat{K}_\ell = (i/2\ell)(\hat{V}(\ell) - \hat{V}(-\ell))$, we obtain a family of 'momentum' operators $\hat{K}_\ell$, one for each regular lattice. The dependence of such operators on ℓ in the limit $\ell \to 0$ will enable us to relate our constructions to the Wilsonian renormalization group flow. We will now

examine properties of such families of operators and introduce the notion of 'approximate consistency in the low energy regime', which will be useful in the analysis of field theories in later papers.

As mentioned in section 2.2, operators on the full Hilbert space $\mathcal{H}_{\rm Poly}$ in quantum geometry often arise from *consistent families* of operators on the Hilbert spaces $\{\mathcal{H}_\gamma\}$ associated to graphs γ [3]. However, since we will be interested in 'low energy' states that lie in Cyl* but not in $\mathcal{H}_{\rm Poly}$, we will use as our starting point the consistency of families of operators on Cyl*.[1] This concept is defined naturally using the duality between Cyl and Cyl*. Specifically, if we are given a family of operators $\{\hat{O}_\gamma\}$ defined on each Hilbert space $\mathcal{H}_\gamma$, then this family is said to be *consistent on* Cyl* if, given any state $(\Psi| \in {\rm Cyl}^\star$, any two graphs γ and γ' such that $\gamma \subseteq \gamma'$, and any state $|\phi_\gamma\rangle \in {\rm Cyl}_\gamma$, the following holds:

$$(\Psi \,|\, \hat{O}_\gamma \,|\, \phi_\gamma\rangle = (\Psi \,|\, \hat{O}_{\gamma'} \Pi^*_{\gamma\gamma'} \,|\, \phi_\gamma\rangle \,. \tag{A.3.1}$$

Here $\Pi^*_{\gamma\gamma'}$ denotes the pull-back from ${\rm Cyl}_\gamma$ to the larger Hilbert space ${\rm Cyl}_{\gamma'}$. This condition serves to ensure that the matrix elements of the operator are independent of the graph γ used to calculate them, i.e., that there is a single operator $\hat{O}$ on Cyl* such that $(\Psi \,|\, \hat{O} \,|\, \phi_\gamma\rangle = (\Psi \,|\, \hat{O}_\gamma \,|\, \phi_\gamma\rangle$ for all graphs γ. In the polymer particle example, several important operators are consistent on Cyl*, including the position operator $\hat{x}$ and the displacement operator $\hat{V}(\mu)$.

However, the new families of operators such as $\hat{K}_\ell$, defined above, do *not* form a consistent family. Neither do the family of Hamiltonian operators $\hat{H}_\ell$ on $\mathcal{H}_\ell$, if their definitions are similarly tied to the lattice spacing. (The Hamiltonians defined in lattice gauge theory are typically of this type.) To examine such families of operators, we must weaken our definition of consistency on Cyl*.

We do so in two directions. First, since the momentum operators are intimately connected to differentiation, we cannot expect a weakened form of (A.3.1) to hold for arbitrary states in Cyl*, but only for 'low energy ones', i.e., states that are elements of $\mathcal{S}$. Second, we do not require expectation values in (A.3.1) to be exactly equal, but instead only that the norm of their difference should be small. Finally, as in the main text, we will only consider regular lattices. We then say that a family of operators $\{\hat{O}_\gamma\}$ defined on *regular* lattices γ is *approximately consistent on low energy states* if, given a constant o_0 (with same dimensions as $\hat{O}$), an $\epsilon > 0$, and any two states $\psi(x)$, $\phi(x) \in \mathcal{S}$, there exists a regular lattice γ such that for any regular lattice γ' that is a refinement of γ, the following holds:

$$\frac{|(\Psi \,|\, \hat{O}_\gamma \,|\, \Phi^{\rm shad}_\gamma\rangle - (\Psi \,|\, \hat{O}_{\gamma'} \Pi^*_{\gamma\gamma'} \,|\, \Phi^{\rm shad}_\gamma\rangle|}{\|\Psi^{\rm shad}_\gamma\| \, \|\Phi^{\rm shad}_\gamma\|} \;<\; o_0\,\epsilon. \tag{A.3.2}$$

[1] I am grateful to Jerzy Lewandowski for pointing out the utility of this definition.

Note that, in this definition, it is essential that we divide by the appropriate norms since states in $\mathrm{Cyl}^\star$ are not normalized. The constant o_0 is needed purely for dimensional reasons.

It is obvious from this definition that any consistent family of operators is automatically approximately consistent on low energy states. However, important families of operators that are *not* exactly consistent nevertheless *are* approximately consistent on low-energy states. In particular, this holds for the family of momentum operators $\hat{K}_\ell$ and the Hamiltonians $\hat{H}_\ell$. We present the proof here for the case of the momentum operators $\hat{K}_\ell$; the proof for the Hamiltonian operators is similar. The proof, though somewhat tedious, involves nothing more than elementary real analysis.

On a regular lattice γ, with spacing ℓ, the momentum operator $\hat{K}_\gamma$ is defined as

$$\hat{K}_\gamma = \frac{i}{2\ell}\left[\hat{V}(l) - \hat{V}(-l)\right]. \tag{A.3.3}$$

Using this definition, it is straightforward to calculate:

$$\begin{aligned}(\Psi|\hat{K}_\gamma|\Phi^{\mathrm{shad}}_\gamma\rangle &= \sum_{x_i\in\gamma}\bar{\psi}(x_i)\left(-\frac{i}{2\ell}\right)[\phi(x_{i+1}) - \phi(x_{i-1})] \\ &= \sum_{x_i\in\gamma}\left[\bar{\psi}(x_{i+1}) - \bar{\psi}(x_{i-1})\right]\left(\frac{i}{2\ell}\right)\phi(x_i)\,. \end{aligned}\tag{A.3.4}$$

Next, we can make use of the mean value theorem to write

$$\frac{\bar{\psi}(x_{i+1}) - \bar{\psi}(x_{i-1})}{2\ell} = \bar{\psi}'(\xi_i)$$

for some $\xi_i \in (x_{i-1}, x_{i+1})$. Thus,

$$(\Psi|\hat{K}_\gamma|\Phi^{\mathrm{shad}}_\gamma\rangle = i\sum_{x_i\in\gamma}\bar{\psi}'(\xi_i)\phi(x_i)\,. \tag{A.3.5}$$

We now make the following simple observation:

$$\frac{(\Psi\,|\,\hat{K}_\gamma\,|\,\Phi^{\mathrm{shad}}_\gamma\rangle}{\|\Psi^{\mathrm{shad}}_\gamma\|\,\|\Phi^{\mathrm{shad}}_\gamma\|} = \frac{\ell\,(\Psi\,|\,\hat{K}_\gamma\,|\,\Phi^{\mathrm{shad}}_\gamma\rangle}{\|\sqrt{\ell}\,\Psi^{\mathrm{shad}}_\gamma\|\,\|\sqrt{\ell}\,\Phi^{\mathrm{shad}}_\gamma\|}. \tag{A.3.6}$$

This is important because the various factors in the numerator and denominator on the right hand side are (as we shall see) Riemann sums for suitable integrals.

Indeed, it is obvious that the two sums in the denominator form Riemann sums for

suitable integrals, namely we have (as $\ell \to 0$):

$$\|\sqrt{\ell}\,\Psi_\gamma^{\text{shad}}\| \;=\; \left(\sum_{x_i\,\in\,\gamma} |\psi(x_i)|^2\;\ell\right)^{\frac{1}{2}} \;\to\; \left(\int_{\mathbb{R}} |\psi(x)|^2\,dx\right)^{\frac{1}{2}}, \tag{A.3.7}$$

$$\|\sqrt{\ell}\,\Phi_\gamma^{\text{shad}}\| \;=\; \left(\sum_{x_i\,\in\,\gamma} |\phi(x_i)|^2\;\ell\right)^{\frac{1}{2}} \;\to\; \left(\int_{\mathbb{R}} |\phi(x)|^2\,dx\right)^{\frac{1}{2}}. \tag{A.3.8}$$

Thus, by elementary results of analysis for each $\epsilon > 0$ there exists a length $\tilde{\ell}_{\epsilon,\Psi,\Phi}$ such that for every lattice γ with lattice spacing less than $\tilde{\ell}_{\epsilon,\Psi,\Phi}$ both of the following are true:

$$\left|\left(\sum_{x_i\,\in\,\gamma} |\psi(x_i)|^2\;\ell\right)^{\frac{1}{2}} - \left(\int_{\mathbb{R}} |\psi(x)|^2\,dx\right)^{\frac{1}{2}}\right| \;<\; \epsilon, \tag{A.3.9}$$

$$\left|\left(\sum_{x_i\,\in\,\gamma} |\phi(x_i)|^2\;\ell\right)^{\frac{1}{2}} - \left(\int_{\mathbb{R}} |\phi(x)|^2\,dx\right)^{\frac{1}{2}}\right| \;<\; \epsilon. \tag{A.3.10}$$

The sum in the numerator of (A.3.6) is trickier: it is equal to

$$i \sum_{i\in\mathcal{N}} \bar{\psi}'(\xi_i)\phi(x_i)\,\ell \tag{A.3.11}$$

and it is not *a priori* clear that this is a Riemann sum for anything. There are two difficulties. First, because we do not have $\xi_i = x_i$ in general, the summands in equation (A.3.11) are not the function we seek to integrate—in this case $\overline{\psi}'(x)\,\phi(x)$—evaluated at some point in our intervals, but rather the product of two functions evaluated at two different points in our intervals. Second, since we know only that $\xi_i \in (x_{i-1}, x_{i+1})$, we must have 2ℓ and not ℓ as the length of the intervals in our Riemann sum, however we have ℓ and not 2ℓ appearing in (A.3.2). But we are also summing over 'twice as many' points, and this will cure that difficulty. Specifically, we have:

$$\begin{aligned} i \sum_{i\,\in\,\mathbb{N}} \overline{\psi}'(x_i)\,\phi(\xi_i)\,\ell \;&=\; \frac{i}{2} \sum_{i\,\in\,\mathbb{N}} \overline{\psi}'(x_i)\,\phi(\xi_i)\,2\ell \\ &=\; \frac{i}{2}\left(\sum_{i\,\in\,\mathbb{N}} \overline{\psi}'(x_{2i})\,\phi(\xi_{2i})\,2\ell \;+\; \sum_{i\,\in\,\mathbb{N}} \overline{\psi}'(x_{2i+1})\,\phi(\xi_{2i+1})\,2\ell\right). \end{aligned} \tag{A.3.12}$$

Each of the two sums on the right hand side of (A.3.12) is separately a Riemann sum for $i\int \overline{\psi}'(x)\phi(x)\,dx$, as we shall soon show, and therefore the left hand side is as well.

To show that the sums appearing on the right hand side of (A.3.12) are indeed Riemann sums, we must somehow deal with the fact that the we have a product of two distinct

functions, each evaluated at different points in our interval. Indeed this is tractable, but we must make a brief foray into some basic facts from real analysis. All of these may be found in [76], or indeed just about any other introductory undergraduate analysis text.

What we wish to show is that a sum of the form $\sum_i f(x_i)\, g(\xi_i)\, \Delta x_i$ is in fact a Riemann sum for the integral $\int f(x)\, g(x)\, dx$.

In order to do this, we will quickly review some of the techniques and terminology from the Riemann definition of the integral, just so that we can more easily pass back and forth between quoting theorems from texts and proving other statements.

First some terminology. Given a fixed closed interval $I = [a, b]$, a *partition* of I is collection of points $x_0, \ldots, x_n$ with

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

In our usage, we would just call such a partition a graph. If we are given one partition P of $[a, b]$ and we then consider another partition P' such that $P \subseteq P'$ as sets, then we say that P' is a refinement of P; this is exactly the same terminology we use for the polymer particle. In obvious fashion we define $\Delta x_i = x_{i+1} - x_i$, and we define the *mesh* of the partition P to be the largest of all of the Δx_i, for the given P. Given a partition P of some interval I, we *mark* the partition if we select precisely one point t_i inside each interval of the form $[x_i, x_{i+1}]$. Given a function f and a marked partition P, then we define the Riemann sum $S(P, f)$ to be:

$$S(P, f) = \sum_i f(t_i)\, \Delta x_i. \tag{A.3.13}$$

Closely related to this Riemann sum $S(P, f)$ are the *upper sum* $U(P, f)$ and *lower sum* $L(P, f)$. These depend on the partition P and the function f, but not on how P is marked (note that our notation does not distinguish between a marked and unmarked partition, but the definition of a Riemann sum $S(P, f)$ requires a marking of P). These sums are defined as:

$$U(P, f) = \sum_i \sup_{t \in [x_i, x_{i+1}]} f(t_i)\, \Delta x_i, \tag{A.3.14}$$

$$L(P, f) = \sum_i \inf_{t \in [x_i, x_{i+1}]} f(t_i)\, \Delta x_i. \tag{A.3.15}$$

Thus, the upper and lower sums are just special cases of Riemann sums where the points in the marking of the partition are determined by the requirement that they be at maxima and minima, respectively, of f over each interval.

The actual Riemann integral is defined in terms of convergence of these upper and lower sums towards a common value in the limit as we refine the partition. It is then not hard (and quite standard) to show that this definition is equivalent to the convergence of Riemann

sums, regardless of how we mark the partition, and again as we take the limit of successively refined partitions. In this form the standard theorems are almost identical to what we do most of the time with the polymer particle, since usually we are taking successively refined graphs. However, for the particular statements we prove in connection with approximate consistency, this approach is not actually the most convenient. That is because we we do not want to prove Cauchy convergence of Riemann sums only when one graph is a refinement of another, but any time the lattice spacing of both is less than a certain level. For this it is necessary to have some sort of convergence that depends not on refinement of lattices, but rather on partitions whose mesh is less than a certain size. Fortunately, just such a theorem is available. We quote a simplified version of Theorem 5.6 of [76]:

A.3.1 Theorem *Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and integrable, with integral $\int_a^b f$. Then for each $\epsilon > 0$ there exists a $\delta > 0$ such that for any partition P of $[a, b]$ which has mesh less than δ,*

$$\left| S(P, f) - \int_a^b f \right| \; < \; \epsilon, \tag{A.3.16}$$

regardless of how P is marked.

We can use Theorem A.3.1 to prove that $\sum_i f(s_i)\, g(t_i)\, \Delta x_i$ converges to $\int_a^b f(x)\, g(x)\, dx$, so long as both f and g are bounded and integrable, and each of the two points s_i and t_i lies in the interval $[x_i, x_{i+1}]$, for each i. Since f is bounded by hypothesis, there exists some constant $M \geq 1$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$.

Now let some $\epsilon > 0$ be given. One consequence of Theorem A.3.1 is the following: for any $\epsilon > 0$ and any bounded, integrable function $g : [a, b] \to \mathbb{R}$, there is a $\delta > 0$ such that whenever the mesh of a partition P is less than δ,

$$\sum_i \, |g(s_i) - g(t_i)| \, \Delta x_i < \epsilon, \tag{A.3.17}$$

for any points $s_i, t_i \in [x_i, x_{i+1}]$. To see this, we note that by Theorem A.3.1 there exists a δ such that whenever the mesh of P is less than this δ,

$$\left| \sum_i \, g(s_i) \, \Delta x_i \; - \; \int_a^b g \, dx \right| \; < \; \frac{\epsilon}{2} \tag{A.3.18}$$

$$\left| \sum_i \, g(t_i) \, \Delta x_i \; - \; \int_a^b g \, dx \right| \; < \; \frac{\epsilon}{2}. \tag{A.3.19}$$

It is then a straightforward application of the Schwarz inequality to conclude (A.3.17).

Because of this, we know that, for our given ϵ, there exists some $\delta > 0$ such that whenever

the mesh of a partition is less than δ, both of the following are true:

$$\left| \sum_i f(s_i)\, g(s_i)\, \Delta x_i \, - \, \int_a^b f(x)\, g(x)\, dx \right| \; < \; \frac{\epsilon}{2}, \tag{A.3.20}$$

$$\sum_i |g(s_i) - g(t_i)|\, \Delta x_i \; < \; \frac{\epsilon}{2M}. \tag{A.3.21}$$

Using this, we can now prove that $\sum_i f(s_i)\, g(t_i)\, \Delta x_i$ converges to $\int_a^b f(x)\, g(x)\, dx$. Indeed, so long as the mesh of P is less than δ:

$$\begin{aligned}
\left| \sum_i f(s_i)\, g(t_i)\, \Delta x_i - \int_a^b f(x)\, g(x)\, dx \right| \; &= \; \left| \sum_i f(s_i)\, g(s_i)\, \Delta x_i - \int_a^b f(x)\, g(x)\, dx \right. \\
&\qquad \left. + \sum_i f(s_i)\, (g(t_i) - g(s_i))\, \Delta x_i \right| \\
&\leq \; \left| \sum_i f(s_i)\, g(s_i)\, \Delta x_i - \int_a^b f(x)\, g(x)\, dx \right| \\
&\qquad + \sum_i |f(s_i)|\, |g(s_i - g(t_i)|\, \Delta x_i \\
&< \; \frac{\epsilon}{2} + M \sum_i |g(s_i) - g(t_i)|\, \Delta x_i \\
&= \; \epsilon.
\end{aligned} \tag{A.3.22}$$

Armed with this, we now know that in fact $i \sum_{x_i \in \gamma} \overline{\psi}'(\xi_i) \phi(x_i) \ell$ is a Riemann sum for $\int \overline{\psi}'(x) \phi(x)\, dx$, and thus that for any given length scale L (needed simply for dimensional reasons) and any $\epsilon > 0$ there exists some $\tilde{\ell}_{\epsilon,\Psi,\Phi}$ such that for every lattice γ with lattice spacing less than $\tilde{\ell}_{\epsilon,\Psi,\Phi}$ both of the following are true:

$$\left| i \sum_{x_i \in \gamma} \overline{\psi}'(\xi_i) \phi(x_i) \ell - \int \overline{\psi}'(x) \phi(x)\, dx \right| < \frac{\epsilon}{L} \tag{A.3.23}$$

We then need only some basic facts from the algebra of limits to conclude that for each $\epsilon > 0$ and any given length scale L there exists a lattice spacing $\ell_{\epsilon,L,\Psi,\Phi}$ such that whenever the spacing of γ is less than $\ell_{\epsilon,L,\Psi,\Phi}$, the following holds:

$$\left| \frac{(\Psi \,|\, \hat{K}_\gamma \,|\, \Phi_\gamma^{\text{shad}})}{\|\Psi_{\tilde{\gamma}}^{\text{shad}}\|\, \|\Phi_\gamma^{\text{shad}}\|} \; - \; \frac{(1/i) \int_{\mathbb{R}} \overline{\psi}'(x)\, \phi(x)\, dx}{\left(\int_{\mathbb{R}} |\psi(x)|^2\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\phi(x)|^2\, dx\right)^{\frac{1}{2}}} \right| \; < \; \frac{\epsilon}{2L}. \tag{A.3.24}$$

Next, let us consider the refined lattice γ' (whose points are labeled by $\tilde{x}_i$) with spacing

ℓ'. It is straightforward to show that:

$$(\Psi|\hat{K}_{\gamma'}\Pi^*_{\gamma\gamma'}|\Phi^{shad}_{\gamma'}\rangle = \sum_{\tilde{x}_i\in\gamma}\left[\bar{\psi}(\tilde{x}_{i+1}) - \bar{\psi}(\tilde{x}_{i-1})\right]\left(\frac{i}{2\ell'}\right)\phi(\tilde{x}_i)\,. \tag{A.3.25}$$

It is important to notice that the sum in (A.3.25) is only over those $\tilde{x}_i$ which are in the original graph γ. However, it is not necessary that $\tilde{x}_{i\pm1}$ is a point in γ (although it will of course be in γ'). We can once again apply the mean value theorem to conclude that

$$\frac{\bar{\psi}(\tilde{x}_{i+1}) - \bar{\psi}(\tilde{x}_{i-1})}{2\ell'} = \bar{\psi}'(\tilde{\xi}_i)\,. \tag{A.3.26}$$

for some $\tilde{\xi}_i \in (\tilde{x}_{i-1}, \tilde{x}_{i+1})$. Thus, (A.3.25) becomes:

$$(\Psi|\hat{K}_{\gamma}|\Phi^{shad}_{\gamma}\rangle = i\sum_{\tilde{x}_i\in\gamma}\bar{\psi}'(\tilde{\xi}_i)\phi(\tilde{x}_i)\,. \tag{A.3.27}$$

Because $[\tilde{x}_{i-1}, \tilde{x}_{i+1}] \subseteq [x_{i-1}, x_{i+1}]$, (A.3.27) is, like (A.3.5), lacking only a factor of ℓ to make it into a Riemann sum for $i\int \overline{\psi}'(x)\,\phi(x)\,dx$. Hence we may multiply and divide by ℓ to conclude that so long as $\ell < \ell_{\epsilon,L,\Psi,\Phi}$, we have:

$$\left|\frac{(\Psi\,|\,\hat{K}_{\gamma'}\,\Pi^*_{\gamma\gamma'}\,|\,\Phi^{\rm shad}_{\gamma}\rangle}{\|\Psi^{\rm shad}_{\tilde{\gamma}}\|\,\|\Phi^{\rm shad}_{\gamma}\|} - \frac{(1/i)\,\int_{\mathbb{R}}\,\overline{\psi}'(x)\,\phi(x)\,dx}{\left(\int_{\mathbb{R}}\,|\psi(x)|^2\,dx\right)^{\frac{1}{2}}\,\left(\int_{\mathbb{R}}\,|\phi(x)|^2\,dx\right)^{\frac{1}{2}}}\right| \;<\; \frac{\epsilon}{2L}. \tag{A.3.28}$$

Combining this equation with (A.3.24) we then finally conclude that

$$\left|\frac{(\Psi\,|\,\hat{K}_{\gamma}\,|\,\Phi^{\rm shad}_{\gamma}\rangle}{\|\Psi^{\rm shad}_{\tilde{\gamma}}\|\,\|\Phi^{\rm shad}_{\gamma}\|} - \frac{(\Psi\,|\,\hat{K}_{\gamma'}\,\Pi^*_{\gamma\gamma'}\,|\,\Phi^{\rm shad}_{\gamma}\rangle}{\|\Psi^{\rm shad}_{\tilde{\gamma}}\|\,\|\Phi^{\rm shad}_{\gamma}\|}\right| \;<\; \frac{\epsilon}{L} \tag{A.3.29}$$

as desired.

Thus, the key idea of the proof is that while we are dividing both expectation values by the norms with respect to only the smaller graph, because we pull back the shadow on the larger graph we 'kill' the points in the larger graph that are not in the smaller, and both ratios (expectation values divided by product of norms) are close to being equal to each other and to the Schrödinger value, for sufficiently fine regular lattices.

We see that we have generalized the usual notion of consistent families of operators to important families of operators that do *not* form a consistent family, thus allowing us to use techniques in analyzing such operators that are similar to those that have played such an important role in quantum geometry. This generalization will be useful in subsequent work on the relation between 'polymer field theories' on quantum geometry and the familiar low energy field theories in the continuum.

Appendix B

Further proofs for chapter 4

B.1 Asymptotic series expansion of the integral

In this section we shall prove that we can obtain an asymptotic series expansion of the integrals we encounter in the Poisson re-summations of chapter 4 by Taylor expanding one factor of the integrand and integrating term by term. Such techniques commonly lead to asymptotic series, but there do not seem to be any standard theorems applicable to the case that we need, so we shall prove this directly. We first begin with a review of some definitions and a standard result in asymptotic series expansions of integrals, Watson's lemma. We shall base our proof on the proof of this lemma.

First, we recall Poincaré's definition of an asymptotic series[1] [80]. Since it is sufficient for our purposes, we give the definition only for functions of a large, real variable z. If $f(z)$ is the function in question, then the series

$$a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \tag{B.1.1}$$

is said to be asymptotic to $f(z)$ provided that

$$\lim_{z\to\infty} |z^L\big(f(z) - S_L(z)\big)| = 0 \tag{B.1.2}$$

for all integers L. Here $S_L(z)$ is the partial sum of the first $L+1$ terms,

$$S_L(z) := \sum_{l=0}^{L} \frac{a_l}{z^l}. \tag{B.1.3}$$

[1] We distinguish between three related but not identical concepts. *Asymptotic series* are as we define above; related is the more general notion of an *asymptotic expansion* in which a set of functions other than powers of z are allowed as the expansion basis functions. Finally, there is the notion of an *asymptotic approximation*, in which $f(z)$ is said to be asymptotic to $\phi(z)$ as $z \to \infty$ if $\lim_{z\to\infty} f(z)/\phi(z) = 1$. The latter is perhaps more familiar to the reader, but note that it is in general weaker than the other two, and moreover does not allow a concept of a function asymptotic to zero. See [77–79] for more details.

When (B.1.2) holds, we write

$$f(z) \sim \sum_{l=0}^{\infty} \frac{a_l}{z^l}. \tag{B.1.4}$$

Thus, we see that for an asymptotic series, the error in approximating the function by the sum goes to zero as the variable z is allowed to approach infinity. Contrast this with a convergent series, where we hold the variable fixed and instead demand the error go to zero as the number of terms summed goes to infinity. Note, however, that the factor of z^L in equation (B.1.2) means that as we take more and more terms in the sum, the error goes to zero faster as $z \to \infty$, so that for the series to be asymptotic a stronger statement is true than simply that the error goes to zero as $z \to \infty$.

We are interested in an asymptotic series expansion of the integral

$$I_n(N, m, \epsilon) = \int_{-\infty}^{\infty} e^{-N^2\epsilon^2(x-\frac{m}{N})^2} \left(1 + x + \frac{i\pi n}{N\epsilon^2}\right)^{\frac{3}{2}} dx \tag{B.1.5}$$

in terms of the parameter $N\epsilon$, which we know to be large.

In order to achieve this, we first review a theorem which, though not strong enough to prove what we need, will point us in the right direction. This is the result known as Watson's Lemma [80, 81], which states that if an integral function of a is defined by

$$I(a) = \int_0^{\infty} e^{-az} f(z)\, dz \tag{B.1.6}$$

then an asymptotic series for $I(a)$ is obtained by power series expanding $f(z)$ and integrating the resulting series term by term.

Our integral (B.1.5) differs from the form of (B.1.6) in several respects; the limits of integration are different, and also we have a damping factor of the form $e^{-a(z-b)^2}$, rather than e^{-az}. More importantly, however, the proof of Watson's lemma requires that the function $f(z)$ not depend on a, our large parameter. This is so that when $f(z)$ is Taylor expanded, we can be sure that there is no hidden dependence on a lurking in the coefficients of the Taylor expansion. In our case, however, the function we wish to Taylor expand is $(1 + x + i\pi n/N\epsilon^2)^{\frac{3}{2}}$, and this *does* depend on our large parameter, $N\epsilon$.

Thus, while we will follow the same basic strategy as in proving Watson's lemma (following the proof given in [81]), we shall have to be more careful, explicitly calculating how the Taylor expansion of our $f(x)$ depends on $N\epsilon$ to ensure that the needed inequality (B.1.2) still holds.

We begin with two lemmas.

B.1.1 Lemma *For all real x, if*

$$g_n(x) := \left(1 + x + \frac{i\pi n}{N\epsilon^2}\right)^{\frac{3}{2}} \tag{B.1.7}$$

and if $g_n(x)$ is Taylor expanded as $\sum_l g_{n,l}\, x^l$, then the partial sum $S_{n,L}(x)$ of the first $L+1$ terms satisfies the inequality:

$$|g_n(x) - S_{n,L}(x)| \leq c_L \left(2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right) e^{x^2}. \tag{B.1.8}$$

where the constant c_L may vary with L, but does not depend on N, n, or ϵ.

Proof: We begin by establishing that

$$|g_n(x)| \leq \left(2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right) e^{x^2}. \tag{B.1.9}$$

To do this, we note that

$$|g_n(x)| = \left((1+x)^2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right)^{\frac{3}{4}} \tag{B.1.10}$$

and we seek to bound this by a function of the form $K\, e^{x^2}$. This implies that

$$\left((1+x)^2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right) \leq K^{\frac{4}{3}}\, e^{\frac{4}{3}x^2}. \tag{B.1.11}$$

To prove *this*, we note that from the Taylor expansion of e^{x^2} it is sufficient to prove

$$\left((1+x)^2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right) \leq K^{\frac{4}{3}} \left(1 + \frac{4}{3}x^2\right) \tag{B.1.12}$$

and (B.1.11) then follows.

In order to prove (B.1.12), we rearrange it as

$$\left(1 - \frac{4}{3}K^{\frac{4}{3}}\right)x^2 + 2x + \left(1 + \frac{\pi^2 n^2}{N^2\epsilon^4} - K^{\frac{4}{3}}\right) \leq 0. \tag{B.1.13}$$

The left hand side is a quadratic function of x, and it will be negative for all real x if and only if the coefficient of x^2 is negative, and the quadratic has no real roots. The first condition is clearly satisfied whenever $K \geq 1$; the second will be satisfied if the discriminant D of the quadratic is negative.

This discriminant is:

$$\begin{aligned} D &= 4 - 4\left(1 - \frac{4}{3}K^{\frac{4}{3}}\right)\left(1 + \frac{\pi^2 n^2}{N^2\epsilon^4} - K^{\frac{4}{3}}\right) \\ &= \frac{4}{3}\left[-4K^{\frac{8}{3}} + \left(7 + \frac{4\pi^2 n^2}{N^2\epsilon^4}\right)K^{\frac{4}{3}} - \frac{3\pi^2 n^2}{N^2\epsilon^4}\right], \end{aligned} \tag{B.1.14}$$

which is a quadratic function of the variable $K^{\frac{4}{3}}$ with negative leading coefficient. Hence its graph opens downward and it will be negative whenever $K^{\frac{4}{3}}$ is greater than the largest of the roots of this quadratic.

Those roots are:

$$K^{\frac{4}{3}} = \left(\frac{7}{8} + \frac{\pi^2 n^2}{2N^2\epsilon^4}\right) \pm \frac{1}{8}\sqrt{\left(7 + \frac{4\pi^2 n^2}{N^2\epsilon^4}\right)^2 - \frac{48\pi^2 n^2}{N^2\epsilon^4}} \tag{B.1.15}$$

and the largest of these is clearly the one with the plus sign.[2] But this root satisfies the inequality:

$$\begin{aligned} \left(\frac{7}{8} + \frac{\pi^2 n^2}{2N^2\epsilon^4}\right) + \frac{1}{8}\sqrt{\left(7 + \frac{4\pi^2 n^2}{N^2\epsilon^4}\right)^2 - \frac{48\pi^2 n^2}{N^2\epsilon^4}} &< 2\left(\frac{7}{8} + \frac{\pi^2 n^2}{2N^2\epsilon^4}\right) \\ &< \left(2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right). \end{aligned} \tag{B.1.16}$$

Thus, it is sufficient to choose

$$K = \left(2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right) \tag{B.1.17}$$

and then (B.1.12), and hence (B.1.9), will hold.

Next, we show that the terms $g_{n,l}\, x^l$ in the Taylor expansion of $g_n(x)$ satisfy

$$|g_{n,l}\, x^l| \le c_l \left(2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right) e^{x^2}. \tag{B.1.18}$$

for some c_l independent of N, n, and ϵ. It is trivial that for each l there exists a constant c_l' such that $|x^l| \le c_l'\, e^{x^2}$. To bound $g_{n,l}$, we use (for $l \ge 1$):

$$g_{n,l} = \frac{\frac{3}{2}\cdot\frac{1}{2}\cdots\left(\frac{3}{2} - l + 1\right)}{l!}\left(1 + \frac{i\pi n}{N\epsilon^2}\right)^{\frac{3}{2}-l}. \tag{B.1.19}$$

Since

$$\left|1 + \frac{i\pi n}{N\epsilon^2}\right| \ge 1, \tag{B.1.20}$$

[2] Observe that the discriminant may be rewritten as $49 + \frac{8\pi^2 n^2}{N^2\epsilon^4} + \frac{16\pi^4 n^4}{N^4\epsilon^8}$, which is manifestly positive, so that the roots of *this* quadratic are always real.

we have:

$$\left|1+\frac{i\pi n}{N\epsilon^2}\right|^{\frac{3}{2}-l} \leq \left|1+\frac{i\pi n}{N\epsilon^2}\right|^{\frac{3}{2}} < \left(2+\frac{\pi^2 n^2}{N^2\epsilon^4}\right) \tag{B.1.21}$$

and combining this with the bound on $|x^l|$ readily yields (B.1.18).

Finally, we combine (B.1.18) and (B.1.9) through a repeated application of the Schwarz inequality to arrive at (B.1.8), as we needed to show. ∎

For the second of our two lemmas, we prove a stronger bound on the remainder term inside the interval of convergence of the Taylor series.

B.1.2 Lemma *Let $g_n(x)$ and $S_{n,L}(x)$ be as in lemma B.1.1, and let $0 < r < 1$. Then for $x \in [-r, r]$ the following bound holds:*

$$|g_n(x) - S_{n,L}(x)| \leq c_L \left(2+\frac{\pi^2 n^2}{N^2\epsilon^4}\right) |x|^{L+1} \tag{B.1.22}$$

for some c_L that is independent of N, n, and ϵ.

Proof: We use the form of the remainder for Taylor's theorem from Cauchy's proof of the same (see, for instance, [82]), which requires us to consider our function as a function of a complex variable. Note that we are expanding about $x = 0$, since we have defined

$$g_n(x) = \left(1+x+\frac{i\pi n}{N\epsilon^2}\right)^{\frac{3}{2}}. \tag{B.1.23}$$

We will therefore consider a circular contour C about the origin of radius R, where $r < R < 1$, and we have [82]:

$$g_n(x) - S_{n,L}(x) = \frac{1}{2\pi i}\oint_C \frac{x^{L+1} f(z)\, dz}{z^{L+1}(z-x)} \tag{B.1.24}$$

We now wish to bound the magnitude of this remainder. Observe that by virtue of the Schwarz inequality,

$$\begin{aligned} |z-x| &\geq |z| - |x| \\ &\geq R - r \end{aligned} \tag{B.1.25}$$

whence:

$$\begin{aligned} |g_n(x) - S_{n,L}(x)| &\leq \frac{|x|^{L+1}}{2\pi} \cdot \frac{1}{R^{L+1}(R-r)} \cdot 2\pi(\sup_C |g_n(z)|) \\ &= \left(\frac{|x|}{R}\right)^{L+1} \frac{\sup_C |g_n(z)|}{R-r}. \end{aligned} \tag{B.1.26}$$

Thus, we need only bound $\sup_C |g_n(z)|$. Since z is of the form $Re^{i\theta}$ for some θ, we have:

$$\begin{aligned}\sup_C |g_n(z)| &= \sup_\theta \left[(1+R\cos\theta)^2 + \left(R\sin\theta + \frac{\pi n}{N\epsilon^2}\right)^2\right]^{\frac{3}{4}} \\ &= \sup_\theta \left[R^2 + 2R\cos\theta + 1 + 2R\sin\theta\frac{\pi n}{N\epsilon^2} + \frac{\pi^2 n^2}{N^2\epsilon^4}\right]^{\frac{3}{4}} \\ &\leq \left[4\left(2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right)\right]^{\frac{3}{4}} \\ &\leq 4\left(2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right).\end{aligned} \tag{B.1.27}$$

In going from the second to the third line, we have used a quadratic inequality similar to that used in the proof of lemma B.1.1.

Thus, we have:

$$|g_n(x) - S_{n,L}(x)| \leq \frac{4}{(R-r)R^{L+1}}\left(2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right)|x|^{L+1}, \tag{B.1.28}$$

and since R and r are independent of N, n, and ϵ, we have established (B.1.22) as we needed. ∎

With these two lemmas we are now able to establish the main result of this section: the asymptotic series expansion of the integral (B.1.5).

B.1.3 Theorem *Let $I_n(N,m,\epsilon)$ be as above, and likewise $g_n(x)$ and $g_{n,l}$. Then:*

$$I_n(N,m,\epsilon) \sim \sqrt{\pi}\sum_{l=0}^{\infty} g_l \frac{(2i)^{-l} H_l(im\epsilon)}{(N\epsilon)^{l+1}}. \tag{B.1.29}$$

Proof: The key to the proof is that

$$|g_n(x) - S_{n,L}(x)| \leq c_L\left(2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right)|x|^{L+1}e^{x^2} \tag{B.1.30}$$

for all real x, for a constant c_L that (as always) is independent of N, n, and ϵ. Equation (B.1.30) is a direct consequence of lemmas B.1.1 and B.1.2. To see this, let r be as in the proof of B.1.2, and suppose that c'_L and c''_L are such that:

$$|g_n(x) - S_{n,L}(x)| \leq c'_L\left(2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right)|x|^{L+1} \qquad \text{for } x \in [-r,r] \tag{B.1.31}$$

$$|g_n(x) - S_{n,L}(x)| \leq c''_L\left(2 + \frac{\pi^2 n^2}{N^2\epsilon^4}\right)e^{x^2} \qquad \text{for } x \in (-\infty,\infty) \tag{B.1.32}$$

Then since $r < 1$, (B.1.31) trivially implies the bound (B.1.30) for $x \in [-r, r]$ with $c_L = e\, c'_L$. But (B.1.32) implies the same bound for $x \in (-\infty, -r) \cup (r, \infty)$, with $c_L = c''_L / r^{L+1}$. Hence (B.1.30) holds everywhere with $c_L = \max\{e\, c'_L, r^{-(L+1)} c''_L\}$. Note that here it is crucial that $r > 0$, since if r were allowed to become zero then c_L would not be finite. It is therefore essential to have *both* of lemmas B.1.1 and B.1.2 available: lemma B.1.1 by itself is not sufficient for precisely this reason. This point is important because, as the reader may have noticed, the proof of lemma B.1.1 would go through for many series besides just the Taylor series for $g_n(x)$. However, the same is *not* true of lemma B.1.2. Thus, we are able to get the needed inequality (B.1.30) for all real x—and hence a series for $I_n(N, m, \epsilon)$ that is asymptotic—only by using the Taylor expansion of $g(x)$. The intuitive idea of the proof is thus that we are able to approximate the function well *somewhere* with its Taylor series; our inability to do so elsewhere will be compensated for by the Gaussian damping factor in the integrand.

Now we use the integral (see [83], 3.462.4):

$$\int_{-\infty}^{\infty} x^l\, e^{-(x-\beta)^2}\, dx = (2i)^{-l} \sqrt{\pi}\, H_l\,(i\beta), \tag{B.1.33}$$

where $H_l(x)$ is the l-th Hermite polynomial,[3] to write:

$$\begin{aligned} \sum_{l=0}^{L} \int_{-\infty}^{\infty} e^{-N^2\epsilon^2(x-\frac{m}{N})^2}\, g_{n,\,l}\, x^l\, dx &= \sum_{l=0}^{L} g_{n,\,l} (N\epsilon)^{-(l+1)} \int_{-\infty}^{\infty} e^{-(x-m\epsilon)^2}\, x^l\, dx \\ &= \sqrt{\pi} \sum_{l=0}^{L} g_{n,\,l}\, \frac{(2i)^{-l}\, H_l(im\epsilon)}{(N\epsilon)^{(l+1)}}. \end{aligned} \tag{B.1.34}$$

Hence to prove our theorem we need only bound the remainder term appropriately.

We have:

$$\begin{aligned} &\left| I_n(N, m, \epsilon) - \sqrt{\pi} \sum_{l=0}^{L} g_{n,\,l}\, \frac{(2i)^{-l}\, H_l(im\epsilon)}{(N\epsilon)^{(l+1)}} \right| \\ &\qquad\qquad \leq \int_{-\infty}^{\infty} e^{-N^2\epsilon^2(x-\frac{m}{N})^2}\, |g(x) - S_{n,\,L}(x)|\, dx \end{aligned} \tag{B.1.35}$$

[3] Note that the l-th Hermite polynomial has the same parity as l, hence the right hand side of (B.1.33) is always real, as it must be.

and we have the following inequalities for the right hand side:

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-N^2\epsilon^2(x-\frac{m}{N})^2}\,|g(x)-S_{n,\,L}(x)|\,dx \\
&\qquad\qquad \le c_L\left(2+\frac{\pi^2n^2}{N^2\epsilon^4}\right)\int_{-\infty}^{\infty} e^{-N^2\epsilon^2(x-\frac{m}{N})^2}\,|x|^{L+1}\,e^{x^2}\,dx \\
&\qquad\qquad \le c_L\left(2+\frac{\pi^2n^2}{N^2\epsilon^4}\right)\int_{-\infty}^{\infty} e^{-\frac{1}{2}N^2\epsilon^2x^2+27\epsilon^2}\,|x|^{L+1}\,dx \\
&\qquad\qquad = 2\,c_L\,e^{27\epsilon^2}\left(2+\frac{\pi^2n^2}{N^2\epsilon^4}\right)\int_{0}^{\infty} e^{-\frac{1}{2}N^2\epsilon^2x^2}\,x^{L+1}\,dx \\
&\qquad\qquad = \frac{c_L\,e^{27\epsilon^2}\,2^{\frac{L}{2}+1}\,\Gamma\left(\frac{L}{2}+1\right)}{(N\epsilon)^{L+2}}\left(2+\frac{\pi^2n^2}{N^2\epsilon^4}\right).
\end{aligned} \tag{B.1.36}$$

In going from the first to the second inequality we have again used a quadratic inequality, this time[4] that $-N^2\epsilon^2(x-\frac{m}{N})^2+x^2 \le -\frac{1}{2}N^2\epsilon^2x^2+27\epsilon^2$.

With this inequality, then, we have finally that

$$\begin{aligned}
&\left|(N\epsilon)^L\left(I_n(N,m,\epsilon)-\sqrt{\pi}\sum_{l=0}^{L} g_{n,\,l}\,\frac{(2i)^{-l}\,H_l(im\epsilon)}{(N\epsilon)^{(l+1)}}\right)\right| \\
&\qquad\qquad \le \frac{c_L\,e^{27\epsilon^2}\,2^{\frac{L}{2}+1}\,\Gamma\left(\frac{L}{2}+1\right)}{(N\epsilon)^2}\left(2+\frac{\pi^2n^2}{N^2\epsilon^4}\right),
\end{aligned} \tag{B.1.37}$$

and the limit of the right hand side as $N\epsilon \to \infty$ (for fixed ϵ) is indeed zero, so comparing to (B.1.2) we see that we are done. ∎

In passing, we note that in our original re-summation, the integral $I_n(N,m,\epsilon)$ always occurs with a factor of $e^{-\pi^2n^2/\epsilon^2}$. Thus, we will in practice be neglecting all but the $n=0$ terms in the re-summation. Since it is not as straightforward to "neglect" an asymptotic series that is divergent as a series, we note that we can use the results of this section to prove a somewhat more useful bound. Specifically, we may use equation (B.1.9) to get:

$$\begin{aligned}
|I_n(N,m,\epsilon)| &\le \left(2+\frac{\pi^2n^2}{N^2\epsilon^4}\right)\int_{-\infty}^{\infty} e^{-N^2\epsilon^2(x-\frac{m}{N})^2+x^2}\,dx \\
&\le \left(2+\frac{\pi^2n^2}{N^2\epsilon^4}\right)e^{27\epsilon^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}N^2\epsilon^2x^2}\,dx \\
&= \frac{\sqrt{2\pi}}{N\epsilon}\left(2+\frac{\pi^2n^2}{N^2\epsilon^4}\right)e^{27\epsilon^2}
\end{aligned} \tag{B.1.38}$$

Thus, if we neglect the terms in the re-summation for which $n \neq 0$, we are neglecting

[4] In order for this inequality to hold, we must assume $N\epsilon > \sqrt{2}$. Since we are in this proof only concerned with the limit as $N\epsilon \to \infty$, this is valid; moreover, it will in general hold for the values of $N\epsilon$ that we are interested in since we know that $N\epsilon \gg 1$.

terms that are $O((N\epsilon)^{-1}e^{-\pi^2/\epsilon^2}) + O((N\epsilon)^{-3}\epsilon^{-2}e^{-\pi^2/\epsilon^2})$. For small ϵ, these are suppressed more than any finite power of ϵ, and we are thus justified in neglecting these in comparison to all terms of the asymptotic series expansion of the integral for $n = 0$.

Finally, we have in this section dealt in some detail with the asymptotic expansion of the sums appearing in the expectation value of the gravitational part of the Hamiltonian constraint. However, in considering both the matter part of the constraint for radiation, and the time derivatives of our basic canonical variables p and c and their squares, we encounter sums similar to that considered here, but not identical. By reasoning entirely similar to that presented here, one may show through Poisson re-summation and a bound on the resulting integral for $n \neq 0$ that only the $n = 0$ term of the re-summed series need be considered, and that integral may be expanded in an asymptotic series for large $N\epsilon$. For completeness, we list the results here.

First we define the various functions and their Taylor series that we shall need:

$$f(x) := (1+x)^{\frac{1}{2}} = \sum_{l=0}^{\infty} f_l x^l \quad \text{for } |x| < 1, \tag{B.1.39}$$

$$g(x) := (1+x)^{\frac{3}{2}} = \sum_{l=0}^{\infty} g_l x^l \quad \text{for } |x| < 1, \tag{B.1.40}$$

$$h(x) := (1+x)^{\frac{5}{2}} = \sum_{l=0}^{\infty} h_l x^l \quad \text{for } |x| < 1. \tag{B.1.41}$$

Then we have the following sums that appear, and the corresponding expansions:

$$\sum_n e^{-\epsilon^2(n-N)^2} |n\mu_o + m|^{\frac{1}{2}} \operatorname{sgn}(n+k) \sim \frac{\sqrt{\pi\mu_o} N^{\frac{1}{2}}}{\epsilon} \sum_{l=0}^{\infty} \frac{f_l}{(2iN\epsilon)^l} H_l\left(\frac{im\epsilon}{\mu_o}\right), \tag{B.1.42}$$

$$\sum_n e^{-\epsilon^2(n-N)^2} |n+m|^{\frac{3}{2}} \operatorname{sgn}(n+k) \sim \frac{\sqrt{\pi} N^{\frac{3}{2}}}{\epsilon} \sum_{l=0}^{\infty} \frac{g_l}{(2iN\epsilon)^l} H_l(im\epsilon), \tag{B.1.43}$$

$$\sum_n e^{-\epsilon^2(n-N\pm\frac{1}{2})^2} |n+m|^{\frac{3}{2}} \operatorname{sgn}(n+k) \sim \frac{\sqrt{\pi} N^{\frac{3}{2}}}{\epsilon} \sum_{l=0}^{\infty} \frac{g_l}{(2iN\epsilon)^l} H_l\left(im\epsilon \mp \frac{i\epsilon}{2}\right), \tag{B.1.44}$$

$$\sum_n e^{-\epsilon^2(n-N)^2} |n+m|^{\frac{5}{2}} \operatorname{sgn}(n+k) \sim \frac{\sqrt{\pi} N^{\frac{5}{2}}}{\epsilon} \sum_{l=0}^{\infty} \frac{h_l}{(2iN\epsilon)^l} H_l(im\epsilon). \tag{B.1.45}$$

B.2 Bounding the correction to non-analyticity

In this section we consider the correction to analyticity noted in the main text. We recall that this correction is:

$$\delta S_{m,k} := \sum_{n \leq -m} e^{-\epsilon^2 (n-N)^2} \left(|n+m|^{\frac{3}{2}} \operatorname{sgn}(n+k) - (n+m)^{\frac{3}{2}} \right). \tag{B.2.1}$$

In order to complete this definition, we must choose how we make the branch cut for $(n+m)^{\frac{3}{2}}$. We must, for consistency, use the same branch cut as in the sum to which we apply Poisson re-summation. This cut, in turn, is dictated by two requirements: we must be able to move the contour of integration as required for the steepest descents approximation, and the cut must not interfere with the proof of the asymptotic series expansion in appendix B.1. The latter requirement means that the branch cut must be to the left of a vertical line in the complex plane starting at the branch point, since otherwise the integrand will not be analytic on and inside the circles needed in the proof of lemma B.1.2. The first requirement means that we must be able to move the contour upward for positive n, and downward for negative n. This means in turn that for negative n, we are choosing $(n+m)^{\frac{3}{2}} = (-i)^3 |n+m|^{\frac{3}{2}}$.

Here we run into a slight difficulty, because when m is negative (either -1 or -3), this means that the first few terms in of the sum defining $\delta S_{m,k}$ will need to have the contour moved upwards, while all the rest will need to have the contour moved downwards. There will therefore be one root of -1 used for the first one or three terms in the sum defining $\delta\Sigma_m$, and a different root used for all the other terms. This difficulty can be remedied by breaking off these first few terms, whose summed magnitude will be less than some fixed constant times $e^{-N^2\epsilon^2}$, and then adding and subtracting the terms with the same root of -1 as the rest of the series. This same technique allows us to break off any terms for which $-m < n \leq -k$, should there be any. In this way we may use the branch cut appropriate for negative n for any value of m, and we obtain:

$$\delta S_{m,k} = \delta S''_{m,k} + (-1-i) \sum_{n \leq -m} e^{-\epsilon^2 (n-N)^2} |n+m|^{\frac{3}{2}} \tag{B.2.2}$$

where $|\delta S''_m| \leq c\, e^{-N^2\epsilon^2}$ for a constant c that is independent of m, k, N, and ϵ.

Our focus is now on bounding the second sum in (B.2.2), which we call $\delta S'_m$. We have:

$$\begin{aligned} \delta S'_m &= \sum_{n \leq 0} e^{-\epsilon^2 (n-m-N)^2} |n|^{\frac{3}{2}} \\ &= \sum_{n \geq 0} e^{-\epsilon^2 (n+M)^2} n^{\frac{3}{2}} \\ &= \sum_{n \geq 1} e^{-\epsilon^2 (n+M)^2} n^{\frac{3}{2}} \end{aligned} \tag{B.2.3}$$

where for convenience we have defined $M := m + N$.

To bound this sum, we first express it using Euler-Maclaurin summation. This is the result [82] that:

$$\sum_{a \le n < b} f(n) = \int_a^b f(x)\,dx + \sum_{k=1}^{p} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_{x=a}^{b} + (-1)^{p+1} \int_a^b \frac{B_p(\{x\})}{p!} f^{(p)}(x)\,dx. \quad \text{(B.2.4)}$$

Here the B_k are the Bernoulli numbers, and the $B_k(x)$ the Bernoulli polynomials. The notation $\{x\}$ indicates the fractional part of x. This result holds as long as $f(x)$ is analytic everywhere along the path of integration, and it was for this reason that we dropped the $n = 0$ term (which we can do for free as it is zero anyway) in equation (B.2.3).

We now apply this result to $\delta S'_m$, using $f(x) = e^{-\epsilon^2(x-M)^2} x^{\frac{3}{2}}$. We may choose any value of p that we like—equation (B.2.4) is an identity for any $p > 0$—and for us it will prove most convenient to choose $p = 1$. Using the fact that $B_1 = -\frac{1}{2}$ and $f(\infty) = 0$ we obtain:

$$\delta S'_m = \int_1^\infty e^{-\epsilon^2(x-M)^2} x^{\frac{3}{2}}\,dx + \frac{1}{2} f(1) + \int_1^\infty B_1(\{x\})\, f'(x)\,dx. \quad \text{(B.2.5)}$$

Since $f(x)$ is positive over the range we consider, and since $|B_1(\{x\})| \le \frac{1}{2}$ (see [83]) we get:

$$\left|\delta S'_m\right| \le \int_1^\infty e^{-\epsilon^2(x-M)^2} x^{\frac{3}{2}}\,dx + \frac{1}{2} f(1) + \frac{1}{2} \int_1^\infty |f'(x)|\,dx \quad \text{(B.2.6)}$$

We can bound the second integral and $f(1)$ by terms of the form $e^{-N^2\epsilon^2}$. That we can do so for $f(1)$ is obvious. To see that we can do so for the integral of $|f'(x)|$, we note that this integral is equal to the sum, at the zeros $\{x_i\}$ of $f'(x)$, of $\pm f(x_i)$ where the plus or minus sign is used according to whether x_i is the right endpoint of an interval on which $f'(x)$ is positive or negative. Since

$$f'(x) = \frac{1}{2} x^{\frac{1}{2}} \left(-4\epsilon^2 x^2 + 4\epsilon^2 M x + 3\right) e^{-\epsilon^2(x-M)^2}, \quad \text{(B.2.7)}$$

the only zeros of $f'(x)$ on the interval $(1, \infty)$ are at

$$x_\pm := M \pm \frac{1}{\epsilon^2} \sqrt{M^2\epsilon^4 + 3\epsilon^2}. \quad \text{(B.2.8)}$$

Since we readily verify that

$$f(x_\pm) = x_\pm^{\frac{3}{2}} e^{-M^2\epsilon^2 - 3} \quad \text{(B.2.9)}$$

we see that the needed bound on $\int_1^\infty |f'(x)|\,dx$ holds.

Thus, for some positive constant c' independent of N, m, or ϵ, we have:

$$\begin{aligned}\left|\delta S'_m\right| &\leq \int_1^\infty e^{-\epsilon^2(x-M)^2} x^{\frac{3}{2}}\, dx + c' e^{-N^2\epsilon^2} \\ &< \int_0^\infty e^{-\epsilon^2(x-M)^2} x^{\frac{3}{2}}\, dx + c' e^{-N^2\epsilon^2} \\ &= \epsilon^{-\frac{5}{2}} \int_0^\infty e^{-(x-M\epsilon)^2} x^{\frac{3}{2}}\, dx + c' e^{-N^2\epsilon^2}\end{aligned} \tag{B.2.10}$$

where we have made the change of variables $\epsilon x \to x$ in reaching the last line.

We can therefore bound the magnitude of $\delta S'_m$ by fairly simple integral, that, as we shall soon see, is itself very easy to bound. To do so we need only observe that $x^{\frac{3}{2}} < e^{x^2}$ for $x \geq 0$; thus we have:

$$\begin{aligned}\left|\delta S'_m\right| &< \epsilon^{-\frac{5}{2}} \int_0^\infty e^{-(x-M\epsilon)^2 + x^2}\, dx + c' e^{-N^2\epsilon^2} \\ &= \epsilon^{-\frac{5}{2}} e^{-M^2\epsilon^2} \int_0^\infty e^{-2M\epsilon x}\, dx + c' e^{-N^2\epsilon^2} \\ &= \frac{1}{2M\epsilon} \epsilon^{-\frac{5}{2}} e^{-M^2\epsilon^2} + c' e^{-N^2\epsilon^2}.\end{aligned} \tag{B.2.11}$$

Now, these corrections to analyticity will be combined with the integrals that we have shown may be asymptotically expanded in appendix B.1. Those integrals in turn occur in the original problem multiplied by $N^{\frac{5}{2}}$, so we should pull out such a factor from $\delta S''_{m,k}$ and $\delta S'_m$. When we do this, combine our bounds on these sums, and insert the result back into (B.2.2) we obtain:

$$\left|\delta S_{m,k}\right| \leq N^{\frac{5}{2}} \left(c'' N^{-\frac{5}{2}} e^{-N^2\epsilon^2} + \frac{1}{(N\epsilon)^{\frac{5}{2}}(N+m)\epsilon} e^{-(N+m)^2\epsilon^2} \right). \tag{B.2.12}$$

We now come to the crucial point: if we multiply the right hand side of (B.2.12) by *any* positive power of $N\epsilon$ and take the limit as $N\epsilon \to \infty$, that limit will be zero. This means, from the definition of an asymptotic series given in appendix B.1, that $\delta S_{m,k}$ is asymptotic to *zero*, and hence may be omitted entirely when we are combining it with a series that is already only asymptotic, as we will be. Thus, in the main text we can simply drop the corrections to analyticity altogether as soon as we reach the stage that we approximate the integrals in our Poisson re-summation with an asymptotic series.

B.3 Time derivative of $(\Delta c)^2(\Delta p)^2$

In this section we prove the result, remarked on in the main text, that the time derivative of $(\Delta c)^2(\Delta p)^2$ is, to lowest order, identically zero. This holds irrespective of how ϵ may be

chosen as a function of c and p. Since to lowest order all of our coherent states saturate the uncertainty bound on $(\Delta p)^2(\Delta c)^2$, this lends strong credence to our assumption that we may well approximate the time evolution of coherent states by other coherent states, at least for a fairly long portion of the integration.

Let us begin by noting what Δp and Δc are. By definition, we have[5] :

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2, \tag{B.3.1}$$

$$(\Delta c)^2 = \langle c^2\rangle - \langle c\rangle^2. \tag{B.3.2}$$

Using the appropriate Poisson re-summation (there is no need for an asymptotic expansion for these quantities) we then find:

$$(\Delta p)^2 = \frac{\gamma^2\mu_o^2\ell_{\rm Pl}^4}{72\epsilon^2}, \tag{B.3.3}$$

$$(\Delta c)^2 = \frac{2}{\mu_o^2}\left[(1-e^{-\frac{\epsilon^2}{2}}) + (e^{-\frac{\epsilon^2}{2}} - e^{-\epsilon^2})\cos(\mu_o c)\right] \approx \frac{2\epsilon^2}{\mu_o^2}. \tag{B.3.4}$$

In equation (B.3.4) we have, in the last step, written out the lowest order approximation to $(\Delta c)^2$. Note that there are no terms that are powers just of c; all terms with c in them come also with a factor of at least ϵ^2, so that (B.3.4) is indeed the lowest order contribution, even though $c^2 \gg \epsilon^2$, according to our criteria.

If we now multiply together (B.3.3) and (B.3.4) we obtain:

$$(\Delta c)(\Delta p) = \frac{\gamma\kappa}{3}\frac{\hbar}{2} + \mathcal{O}(c^2). \tag{B.3.5}$$

To leading order, this is the Heisenberg bound: the $\hbar/2$ factor is familiar; the factor of $\gamma\kappa/3$ is needed as well because of the commutation relations between c and p. We therefore see that as expected our coherent states saturate the bound, for any value of ϵ, just as they do in the familiar Schrödinger relationship. Note, however, that now this saturation only holds to lowest order (as indeed we also observed in section 2.4.2).

What about the time derivative of this product? We have:

$$\frac{d}{dt}\left[(\Delta c)^2(\Delta p)^2\right] = \left(\langle \dot{c}^2\rangle - 2\langle c\rangle\langle\dot{c}\rangle\right)(\Delta p)^2 + (\Delta c)^2\left(\langle\dot{p}^2\rangle - 2\langle p\rangle\langle\dot{p}\rangle\right). \tag{B.3.6}$$

To evaluate the time derivatives we simply use the fact, also used throughout chapter 4, that for any classical observable O with corresponding quantum operator $\hat{O}$, we have $\frac{d}{dt}\langle O\rangle = -\frac{i}{\hbar}\langle[\hat{O},\hat{C}]\rangle$ where $\hat{C}$ is the constraint operator. Now, this operator is the sum of

[5] A word on notation. Throughout this section, for clarity we shall omit any hats on operators; thus $\langle p\rangle \equiv \langle\hat{p}\rangle$. Moreover, we shall use a dot over the argument of an expectation value to denote the time derivative of that expectation value. Thus, $\langle\dot{p}\rangle = \frac{d}{dt}\langle\hat{p}\rangle$ and $\langle\dot{p}^2\rangle = \frac{d}{dt}\langle\hat{p}^2\rangle$.

the gravitational part of the constraint and the matter constraint, and so we see that the time derivatives will depend on what type of matter we are considering. Let us focus first on dust, since it is simpler: we have just $\hat{C} = \hat{C}_{\rm grav} + \frac{1}{2}E_0$ and since the matter constraint is a constant, it does not contribute to any commutators.

We may then use the following asymptotic expansions, which may be derived in the same manner as that for the expectation value of the constraint in chapter 4 (throughout, we have dropped any exponentially suppressed terms):

$$\langle p\rangle \;=\; p, \tag{B.3.7}$$

$$\langle \dot{p}\rangle \;\sim\; \frac{2\ell_{\rm Pl}N^{\frac{3}{2}}e^{-4\epsilon^2}}{3(6\gamma\mu_o)^{\frac{1}{2}}}\sin\left(2\mu_o c\right)\sum_{l=0}^{\infty}\frac{g_{2l+1}}{(2iN\epsilon)^{2l+1}}\left[H_{2l+1}(3i\epsilon)-H_{2l+1}(i\epsilon)\right], \tag{B.3.8}$$

$$\begin{aligned}\langle \dot{p}^2\rangle \;\sim\; & \frac{2}{9}\left(\frac{\gamma\mu_o}{6}\right)^{\frac{1}{2}}\ell_{\rm Pl}^3N^{\frac{5}{2}}e^{-4\epsilon^2}\sin\left(2\mu_o c\right)\sum_{l=0}^{\infty}\frac{g_{2l+1}}{(2iN\epsilon)^{2l+1}}\Bigg[H_{2l+1}(3i\epsilon)-H_{2l+1}(i\epsilon)\\ & -\frac{8l(2l+1)}{3-4l}H_{2l-1}(3i\epsilon)+\frac{8l(2l+1)}{3-4l}H_{2l-1}(i\epsilon)\Bigg],\end{aligned} \tag{B.3.9}$$

$$\langle c\rangle \;=\; \frac{2}{\mu_o}e^{-\frac{\epsilon^2}{4}}\sin\left(\frac{\mu_o c}{2}\right), \tag{B.3.10}$$

$$\begin{aligned}\langle \dot{c}\rangle \;\sim\; & \frac{N^{\frac{3}{2}}}{\mu_0\ell_{\rm Pl}\sqrt{6}(\gamma\mu_o)^{\frac{3}{2}}}\Bigg\{\left[e^{-\frac{25\epsilon^2}{4}}\cos\left(\frac{5\mu_o c}{2}\right)+e^{-\frac{9\epsilon^2}{4}}\cos\left(\frac{3\mu_o}{2}\right)\right]\\ & \times\left(\sum_{l=0}^{\infty}\frac{g_{2l}}{(2iN\epsilon)^{2l}}\left[H_{2l}\left(\frac{7i\epsilon}{2}\right)-H_{2l}\left(\frac{5i\epsilon}{2}\right)-H_{2l}\left(\frac{3i\epsilon}{2}\right)+H_{2l}\left(\frac{i\epsilon}{2}\right)\right]\right)\\ & -4e^{-\frac{\epsilon^2}{4}}\cos\left(\frac{\mu_o c}{2}\right)\sum_{l=0}^{\infty}\frac{g_{2l}}{(2iN\epsilon)^{2l}}\left[H_{2l}\left(\frac{3i\epsilon}{2}\right)-H_{2l}\left(\frac{i\epsilon}{2}\right)\right]\Bigg\},\end{aligned} \tag{B.3.11}$$

$$\begin{aligned}\langle \dot{c}^2\rangle \;\sim\; & \frac{N^{\frac{3}{2}}}{\mu_o^2\ell_{\rm Pl}\sqrt{6}(\gamma\mu_o)^{\frac{3}{2}}}\Bigg\{e^{-9\epsilon^2}\sin\left(3\mu_o c\right)\sum_{l=0}^{\infty}\frac{g_{2l}}{(2iN\epsilon)^{2l}}\left[H_{2l}(4i\epsilon)-2H_{2l}(2i\epsilon)+H_{2l}(0)\right]\\ & -e^{-\epsilon^2}\sin\left(\mu_o c\right)\sum_{l=0}^{\infty}\frac{g_{2l}}{(2iN\epsilon)^{2l}}\left[H_{2l}(4i\epsilon)+2H_{2l}(2i\epsilon)-3H_{2l}(0)\right]\Bigg\}.\end{aligned} \tag{B.3.12}$$

From these rather horrific expressions we may calculate the necessary time derivatives of $(\Delta c)^2(\Delta p)^2$. Let us begin with the time derivative of $(\Delta p)^2$, which we denote as $\Delta\dot{p}^2$. We keep only the lowest order of the asymptotic expansions above, as well as only the lowest order terms in the Taylor expansions of the exponentials and trigonometric functions. Some algebra then leads us to:

$$\Delta\dot{p}^2 \approx \frac{\gamma\mu_o^2\ell_{\rm Pl}^4}{18}\frac{c}{\epsilon^2\sqrt{p}} \tag{B.3.13}$$

This turned out to be a rather simple expression. For calculating $\Delta\dot{c}^2$, we must proceed a little more cautiously. First, we keep again only the lowest terms in the asymptotic

expansion, but do not as yet approximate the exponentials or trigonometric functions. We then obtain:

$$\Delta\dot{c}^2 \approx \frac{3(\gamma\mu_o)^{-\frac{3}{2}}}{\mu_o^2\ell_{\rm Pl}\sqrt{6N}}\Big(e^{-9\epsilon^2}\sin(3\mu_o c) - 3e^{-\epsilon^2}\sin(\mu_o c) - e^{-\frac{13\epsilon^2}{2}}\sin(3\mu_o c) \\ +(e^{-\frac{13\epsilon^2}{2}} - e^{-\frac{5\epsilon^2}{2}})\sin(2\mu_o c) + (e^{-\frac{5\epsilon^2}{2}} + 2e^{-\frac{\epsilon^2}{2}})\sin(\mu_o c)\Big) \quad \text{(B.3.14)}$$

Now, the key point about this expression is that if we Taylor expand the exponentials in ϵ^2, the leading order constant terms all cancel exactly, without any Taylor expansion of the sine terms. Thus, as with Δc itself, the lowest order terms will be proportional to ϵ^2; there are no c^2 terms. When we then do perform the Taylor expansions we get:

$$\Delta\dot{c}^2 \approx -\frac{8\epsilon^2 c}{\gamma\mu_o^2\sqrt{p}}. \quad \text{(B.3.15)}$$

If we now put equations (B.3.3), (B.3.4), (B.3.13), and (B.3.15) into equation (B.3.6), then we find that to lowest order (as we have been calculating) *the time derivative of* $(\Delta c)^2(\Delta p)^2$ *vanishes identically, independent of* ϵ. This is a highly nontrivial result, that reflects that at least for infinitesimal time evolutions, the saturation property (B.3.5) is preserved. Therefore we expect that even though a state that is initially coherent will not evolve exactly into another coherent state, it will evolve into another state that is well approximated by a coherent state. What chiefly then remains to be found is how the parameter of the coherent states—that is, ϵ—evolves under time. This we restrict first by the requirement that we be able to obtain an effective Hamiltonian formulation of the quantum dynamics, as explained in more detail in the body of the thesis.

The result above was shown for the case where the matter constraint is that corresponding to a dust filled universe. The other type of matter with which we have been concerned in this thesis is radiation. We comment briefly on the changes needed in the above analysis for that case. We know that the matter constraint for radiation is equal to a constant multiple of the inverse scale factor operator. Now, this operator commutes with $\hat{p}$ and so the result (B.3.13) is unchanged for a radiation filled universe. The matter constraint however does *not* commute with $\hat{c}$, and there are corrections to (B.3.15). However, it is possible to show that they are of higher order than the result in (B.3.15), and so working to lowest order it is again true that the time derivative of $(\Delta c)^2(\Delta p)^2$ vanishes identically, irrespective of the value of ϵ.

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Vita

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