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On the nature of correlation between neutrino-SM CP phase and unitarity violating new physics parameters

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ABSTRACT: To perform leptonic unitarity test, understanding the system with the three-flavor active neutrinos with non-unitary mixing is required, in particular, on its evolution in matter and the general features of parameter correlations. In this paper, we discuss the nature of correlation between ν SM CP phase δ and the α parameters, where α 's are to quantify the effect of non-unitarity. A question arose on whether it is real and physical when the same authors uncovered, in a previous paper, the $\delta - \alpha$ parameter correlation of the form $[e^{-i\delta}\bar{\alpha}_{\mu e}, e^{-i\delta}\bar{\alpha}_{\tau e}, \bar{\alpha}_{\tau\mu}]$ using the PDG convention of the flavor mixing matrix U_{MNS} . This analysis utilizes a perturbative framework which is valid at around the atmospheric MSW enhancement. In fact, the phase correlation depends on the convention of U_{MNS} , and the existence of the SOL convention ($e^{\pm i\delta}$ attached to s_{12}) in which the correlation is absent triggered a doubt that it may not be physical. We resolve the controversy of whether the phase correlation is physical by examining the correlation in completely different kinematical phase space, at around the solar-scale enhancement. It reveals a dynamical δ –(blob of α parameter) correlation in the SOL convention which prevails in the other conventions of U_{MNS} . It indicates that the phase correlation seen in this and the previous papers is physical and cannot be an artifact of U_{MNS} convention.

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1 Introduction

The discovery of neutrino oscillation and hence neutrino mass [1, 2] under the framework of three-generation lepton flavor mixing [3] created a new field of research in particle physics. It leads to construction of the next-generation accelerator and underground experiments with the massive detectors, Hyper-Kamiokande [4] and DUNE [5]. They are going to establish CP violation due to the lepton Kobayashi-Maskawa phase [6], and determine the neutrino mass ordering by utilizing the earth matter effect [7, 8]. Of course, the flagship projects will be challenged by the ongoing and the other upcoming experiments, for example [9–17], which compete for the same goals.

Toward establishing the three-flavor mixing scheme, in particular in the absence of confirmed anomaly beyond the neutrino-mass embedded Standard Model (ν SM),¹ one of the most important topics in the future would be the high-precision paradigm test. In this context, leptonic unitarity test, either by closing the unitarity triangle [19], or by an alternative method of constraining the models of unitarity violation (UV)² at high-energy [20, 21], or at low-energy scales [22–24] are discussed. For the subsequent developments, see a partial list of the references [25–33]. A summary of the current constraints on UV is given e.g., in refs. [24, 34].

It was observed that in the 3×3 active neutrino subspace the evolution of the system can be formulated in the same footing in low-scale as well as high-scale UV scenarios [23, 24]. Nonetheless, dynamics of the three neutrino system with non-unitary mixing in matter has not been investigated in a sufficient depth. Apart from numerically implemented calculation done in some of the aforementioned references, only a very limited effort was devoted for analytical understanding of the system so far. It has a sharp contrast to the fact that great amount of efforts were devoted to understand the three-flavor neutrino oscillation.³ A general result known so far is the exact S matrix with non-unitarity in matter with constant density [23] calculated by using the KTY-type construction [35]. It allows us to obtain the exact expression of the oscillation probability with non-unitarity.

In a previous paper [36], we have started a systematic investigation of analytic structure of the three neutrino evolution in matter with non-unitarity. We have used so called the α parametrization [21] to implement non-unitarity in the three neutrino system. Using a perturbative framework dubbed as the “helio-UV perturbation theory” (UV extended version of [37]) with the two kind of expansion parameters, the helio-to-terrestrial Δm^2 ratio $\epsilon \approx \Delta m_{21}^2 / \Delta m_{31}^2$ and the α parameters, we computed the oscillation probability to first-order in the expansion parameters. The perturbative framework is valid at around the atmospheric MSW enhancement [37] and for the α parameters $\ll 1$. The expression of the probability revealed a dynamical symmetry in which the ϕ , the matter-affected angle θ_{13} , is involved, in a parallelism to the similar symmetry with matter-affected angle θ_{12} found

¹ For possible candidates of the anomalies which suggest physics beyond the ν SM see e.g., ref. [18].

² It is appropriate to mention that in the physics literature UV usually means “ultraviolet”. But, in this paper UV is used as an abbreviation for “unitarity violation” or “unitarity violating”.

³ Here, we give a cautious remark that when the term “neutrino oscillation” is used in this paper, or often in many other literatures, it may imply not only the original meaning, but also something beyond such as “neutrino flavor transformation”, or “neutrino flavor conversion”, depending upon the contexts.

in ref. [38]. The formulas not only provide reasonable first order approximation for the oscillation probabilities, but also serve for the first explicit demonstration of perturbative unitarity, which stems from the unitary nature of neutrino evolution even with non-unitary mixing matrix.

An even more interesting outcome of the structure revealing expression of the oscillation probability in ref. [36] is that it displays the intriguing correlation between ν SM CP phase δ and the complex α parameters. It has a form $[e^{-i\delta}\alpha_{\mu e}, \alpha_{\tau e}, e^{i\delta}\alpha_{\tau\mu}]$, always the same combination in all the oscillation channels. However, the definition of the α parameters, and consequently the form of the correlation between the CP phases, depends on the phase convention of the lepton flavor mixing MNS matrix. The above result is obtained under the ATM phase convention in which $e^{\pm i\delta}$ is attached to s_{23} . In the Particle Data Group (PDG) convention of U_{MNS} [39] the universal phase combination takes the form $[e^{-i\delta}\bar{\alpha}_{\mu e}, e^{-i\delta}\bar{\alpha}_{\tau e}, \bar{\alpha}_{\tau\mu}]$. A puzzling feature is that the correlation between δ and α parameter phases looks absent in the SOL convention of U_{MNS} in which $e^{\pm i\delta}$ is attached to s_{12} . It leads to the two alternative interpretations of the phase correlation we uncovered:

1. Existence of the SOL phase convention of U_{MNS} in which δ and α phase correlation is absent implies that the CP phase correlation is not physical, but an artifact of inadequate choice of U_{MNS} phase convention.
2. Physics must be independent of which convention is taken for U_{MNS} . In all the other phase convention of U_{MNS} except for the SOL, there exists $\delta - \alpha$ parameter phase correlation. Therefore, the existence of phase correlation is generic and it must be physical.

The primary purpose of this paper is to settle the issue of whether the phase correlation observed in the previous paper is physical, or an artifact of the convention of U_{MNS} . If the interpretation 1 and the reasoning behind it are correct, δ and α phase correlation must be absent under the SOL convention of U_{MNS} everywhere in the allowed kinematical regions. Fortunately, it is a falsifiable statement. Namely, if we see the phase correlation in the oscillation probability calculated with the SOL convention somewhere, it implies that the interpretation 1 cannot be true. Thus, the investigation of neutrino evolution outside the region of validity of the helio-UV perturbation theory used in ref. [36] would clearly resolve the controversy between the above two interpretations.

Very recently, timely enough for this purpose, we have formulated another perturbative framework whose validity is around the region of solar-scale enhanced oscillation, called the “solar resonance perturbation theory” [38]. In this paper, we use this perturbative framework to investigate dynamics of the three neutrino evolution under the influence of non-unitary mixing matrix and the matter effect under the constant matter density approximation. We first extend the perturbative framework to include the effect of UV, and examine the feature of neutrino oscillation with focusing in on the issue of phase correlation. As we will see in section 4, our results support the interpretation 2 above.

We emphasize that though we pay special attention to the phase correlation, our present study is interesting by itself, extending the frontier of analytic investigation of neutrino

oscillation in matter with non-unitarity. As it is said, it naturally leads to our second purpose of this paper. We will show that the system of three-flavor neutrino oscillation in matter with non-unitarity displays a rich, new phenomenon of clustering of the ν SM and the UV variables. We hope that this study, if followed by further, more elaborate analyses, sheds light on the dynamics of the three-flavor neutrino system with 6+9 degrees of freedom. It will eventually help analyzing data for leptonic unitarity test.

In section 2, we introduce the concept of parameter correlations by describing a pedagogical example of the three-neutrino system with the non-standard interactions. In section 3, we give a step-by-step formulation of the perturbative framework we use to analyze the three-neutrino evolution with non-unitary mixing matrix. The prescription for computing S matrix elements is given with the help of the tilde basis $\tilde{S}$ matrix elements summarized in appendix B, and the general formula for the oscillation probability is presented. In section 4, we discuss the characteristic features of the correlation between the ν SM CP phase and UV α parameters in the region of validity of our perturbative framework. In section 5, we give the concluding remarks. In appendix D, the explicit expression of the oscillation probability in the $\nu_\mu \rightarrow \nu_e$ channel is given to first order in expansion parameters.

2 Parameter correlation in neutrino oscillation with beyond- ν SM extended settings

It may be useful to start the description of this paper by briefly recollecting some known features of parameter correlation in neutrino oscillation, in particular, in an extended setting that includes physics beyond the ν SM. In this context, a general framework that is most frequently discussed is the one that includes the neutrinos' non-standard interactions (NSI)

$$H_{\text{NSI}} = \frac{a}{2E} \begin{bmatrix} \varepsilon_{ee} & \varepsilon_{e\mu} & \varepsilon_{e\tau} \\ \varepsilon_{e\mu}^* & \varepsilon_{\mu\mu} & \varepsilon_{\mu\tau} \\ \varepsilon_{e\tau}^* & \varepsilon_{\mu\tau}^* & \varepsilon_{\tau\tau} \end{bmatrix}, \quad (2.1)$$

to the flavor basis Hamiltonian, where the ε parameters describe flavor dependent strengths of NSI and a denotes the matter potential, see (3.4). We discuss only so called the “propagation NSI”. For a review of physics of NSI in wider contexts, see e.g., refs. [40–42]. We note that inclusion of the NSI Hamiltonian (2.1) brings the extra nine parameters into the ν SM Hamiltonian with six degrees of freedom, the two Δm^2 , the three mixing angles, and the unique CP phase, under the influence of the matter potential background.

2.1 Emergence of collective variables involving ν SM and NSI parameters

With more than doubled, a large number of the parameters, it is conceivable that dynamics of neutrino oscillation naturally involves rich correlations among these variables.⁴ Here, we describe only one particular example discussed in ref. [43] because, we believe, it illuminates the point. In this reference, the authors formulated a perturbative framework of the system

⁴ They include the correlations between the NSI variables themselves. The examples include the $\varepsilon_{ee} - \varepsilon_{e\tau} - \varepsilon_{\tau\tau}$ correlation discussed in refs. [45, 46].

with NSI by using the three (the latter two assumed to be) small expansion parameters, $\epsilon \equiv \Delta m_{21}^2/\Delta m_{31}^2$, $s_{13} \equiv \sin \theta_{13}$, and the ε parameters. They derived the formulas of the oscillation probability to second order (third order in $\nu_\mu \rightarrow \nu_e$ channel) in the expansion parameters, which is nothing but an extension of the Cervera *et. al.* formulas [44]⁵ to include NSI. In this calculation the PDG convention of U_{MNS} [39] is used.

An interesting and unexpected feature of the NSI-extended formulas is the emergence of the two sets of “collective variables”

$$\begin{aligned}\Theta_{13} &\equiv s_{13} \frac{\Delta m_{31}^2}{a} + e^{i\delta} (s_{23}\varepsilon_{e\mu} + c_{23}\varepsilon_{e\tau}), \\ \Theta_{12} &\equiv \left(c_{12}s_{12} \frac{\Delta m_{21}^2}{a} + c_{23}\varepsilon_{e\mu} - s_{23}\varepsilon_{e\tau} \right) e^{i\delta},\end{aligned}\tag{2.2}$$

where an overall $e^{-i\delta}$ is factor out from the matrix element $S_{e\mu}$ to make the s_{13} term δ free through which $e^{i\delta}$ dependences in (2.2) result. That is, if we replace $s_{13} \frac{\Delta m_{31}^2}{a}$ and $c_{12}s_{12} \frac{\Delta m_{21}^2}{a}$ in the original formulas by Θ_{13} and Θ_{12} , respectively, the extended second-order formulas with full inclusion of NSI effects automatically appear [43]. In fact, the procedure works for the third order formula for $P(\nu_\mu \rightarrow \nu_e)$ as well. We note that the second order computation of ref. [43] includes the $\nu_\mu - \nu_\tau$ sector, and the additional corrective variables are identified. But, for simplicity, we do not discuss them here and refer the interested readers ref. [43].

Appearance of the cluster variables composed of the ν SM and NSI parameters in (2.2) implies that there exists strong correlations between the ν SM variables s_{13} - δ and the NSI $\varepsilon_{e\mu}$ - $\varepsilon_{e\tau}$ parameters in such a way that they form the collective variable Θ_{13} to convert the Cervera *et. al.* formula to its NSI extended version. The similar statement can be made for the other cluster variable Θ_{12} as well. The NSI-extended second order formula derived in this way serves for understanding the $s_{13} - \varepsilon_{e\mu}$ confusion uncovered in ref. [50] in a more complete manner in such a way that the effects of $\varepsilon_{e\tau}$ and CP phase δ are also included. It also predicts occurrence of the similar correlation among the variables to produce the collective variable Θ_{12} , whose feature could be confirmed by experiments at low energies, $\frac{\Delta m_{21}^2}{a} \sim \mathcal{O}(1)$, the possibility revisited recently [38].

Therefore, there is nothing strange in the parameter correlations among the ν SM and new physics parameters. It appears that the phenomenon arises generically, at least under the environment that the matter effect is comparable to the vacuum effect.

2.2 Dynamical nature of the parameter correlation

We must point out, however, that the features of the parameter correlation depend on the values of the parameters involved, and also on the kinematical region of neutrino energy and baseline with background matter density. Therefore, depending upon the region of validity of the perturbative framework which is used to derive the correlation, the form of

⁵ The Cervera *et. al.* formula is the most commonly used probability formula in the standard three flavor mixing in matter for many purposes, e.g., in the discussion of parameter degeneracy [47–49].

parameter correlation changes. We call this feature as “dynamical” nature of the parameter correlation.⁶

We want to see explicitly whether a change in features of the correlation occurs when the values of the parameters involved are varied, or its effect is incorporated into the framework of perturbation theory. For this purpose let us go back to the collective variable correlation in (2.2). We know now the value of θ_{13} is larger than what is assumed at the time the Cervera *et. al.* formula was derived [39]. The latest value from Daya Bay is $s_{13} = 0.148$ [51], which is of the order of $\sqrt{\epsilon} = 0.176$. Then, we need higher order corrections of s_{13} up to the fourth order terms to match to the second order accuracy in ϵ [52, 53]. When it is carried out with inclusion of NSI [53], it is seen that part of the additional terms generated do not fit to the form of collective variables given in (2.2). Therefore, when we make θ_{13} larger, the parameter correlation which produced the collective variables (2.2) is started to dissolve.

Thus, the analysis of this particular example revealed the dynamical nature of the parameter correlation in the neutrino propagation with NSI. We expect that overseeing the results of computations of the oscillation probabilities in this and the previous paper [36] will reveal the similar dynamical behavior of the parameter correlation in the three-flavor neutrino evolution in matter with non-unitary mixing.

2.3 Phase correlation through NSI-UV parameter correspondence?

Can we extract information of the $\delta - \alpha$ parameter phase correlation from the collective variables (2.2)? The answer is *Yes* if we assume a “uniform chemical composition model” of the matter. As far as the propagation NSI is concerned there is a one to one mapping between NSI ε parameters and the UV α parameters, as noticed by Blennow *et. al.* [24] under the assumption $N_n = N_e$, an equal neutron and proton number densities in charge-neutral medium. For the purpose of the present discussion, one also has to “approve” the procedure by which the $e^{i\delta}$ dependence of the collective variables (2.2) is fixed. That is, removing an overall phase from the matrix element $S_{e\mu}$ to make the s_{13} term δ free, as done in ref. [43].

Assuming the two conditions above, it leads to the collective variables in (2.2) written by the UV α parameters,

$$\begin{aligned}\Theta_{13} &= s_{13} \frac{\Delta m_{31}^2}{a} + \frac{1}{2} \left\{ s_{23} \left(\bar{\alpha}_{\mu e} e^{-i\delta} \right)^* + c_{23} \left(\bar{\alpha}_{\tau e} e^{-i\delta} \right)^* \right\}, \\ \Theta_{12} &= c_{12} s_{12} e^{i\delta} \frac{\Delta m_{21}^2}{a} + \frac{1}{2} \left\{ c_{23} \left(\bar{\alpha}_{\mu e} e^{-i\delta} \right)^* - s_{23} \left(\bar{\alpha}_{\tau e} e^{-i\delta} \right)^* \right\},\end{aligned}\quad (2.3)$$

where we have to use the α parameters defined in the PDG convention of U_{MNS} . The emerged correlation between δ and the α parameters is consistent with the canonical phase combination obtained in ref. [36] in the PDG convention. For the relationships between the α parameters with the various U_{MNS} conventions, see section 3.1. It is not unreasonable because the regions of validity of the perturbative frameworks in refs. [43] and [36] overlaps.

⁶ One must be aware that our terminology of “dynamical” correlation may be different from those used in condensed matter physics or many body theory. In our case the correlated parameters are not the dynamical variables in quantum theory and there is no direct interactions between them.

3 Formulating perturbation theory around the solar-scale enhancement with non-unitarity

3.1 Neutrino evolution in the vacuum mass eigenstate basis

As is customary in our formulation of the three active neutrino evolution in matter with unitarity violation (UV) [36], we start from the evolution equation in the vacuum mass eigenstate basis, whose justification is given in refs. [23, 24].⁷ With use of the “check basis” for the vacuum mass eigenstate basis, it takes the form of Schrödinger equation

$$i\frac{d}{dx}\tilde{\nu} = \tilde{H}\tilde{\nu} \quad (3.1)$$

with Hamiltonian

$$\tilde{H} \equiv \frac{1}{2E} \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{bmatrix} + N^\dagger \begin{bmatrix} a-b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -b \end{bmatrix} N \right\} \quad (3.2)$$

where E is neutrino energy and $\Delta m_{ji}^2 \equiv m_j^2 - m_i^2$. A usual phase redefinition of neutrino wave function is done to leave only the mass squared differences. N denotes the non-unitary flavor mixing matrix which relates the flavor neutrino states to the vacuum mass eigenstates as

$$\nu_\beta = N_{\beta i} \tilde{\nu}_i. \quad (3.3)$$

where β runs over e, μ, τ , and the mass eigenstate index i runs over 1, 2, and 3. It must be noticed that the neutrino evolution described by eq. (3.1) is unitary, as emphasized in ref. [36]. Notice that due to limited number of appropriate symbols the notations for the various basis may not be always the same in our series of papers.

The functions $a(x)$ and $b(x)$ in (3.13) denote the Wolfenstein matter potential [7] due to CC and NC reactions, respectively.

$$\begin{aligned} a &= 2\sqrt{2}G_F N_e E \approx 1.52 \times 10^{-4} \left(\frac{Y_e \rho}{\text{g cm}^{-3}} \right) \left(\frac{E}{\text{GeV}} \right) \text{eV}^2, \\ b &= \sqrt{2}G_F N_n E = \frac{1}{2} \left(\frac{N_n}{N_e} \right) a. \end{aligned} \quad (3.4)$$

Here, G_F is the Fermi constant, N_e and N_n are the electron and neutron number densities in matter. ρ and Y_e denote, respectively, the matter density and number of electron per nucleon in matter. We define the following notations for simplicity to be used in the discussions hereafter in this paper:

$$\Delta_{ji} \equiv \frac{\Delta m_{ji}^2}{2E}, \quad \Delta_a \equiv \frac{a}{2E}, \quad \Delta_b \equiv \frac{b}{2E}. \quad (3.5)$$

⁷ In a nutshell, the equation (3.1) with (3.2) describes evolution of the active three neutrinos in the 3×3 sub-space in the $(3 + N_s)$ model (as a model for low-scale UV) [22, 23], or just the three ν system in high-scale UV [24].

For simplicity and clarity we will work with the uniform matter density approximation in this paper. But, it is not difficult to extend our treatment to varying matter density case if adiabaticity holds.

Throughout this paper, due to the reasoning mentioned in section 1, we use the SOL convention of the U_{MNS} matrix, the standard 3×3 unitary flavor mixing matrix

$$U_{\text{SOL}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12}e^{i\delta} & 0 \\ -s_{12}e^{-i\delta} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv U_{23}U_{13}U_{12}, \quad (3.6)$$

where we have used the obvious notations $s_{ij} \equiv \sin \theta_{ij}$ etc. and δ denotes the lepton KM phase [6], or the νSM CP violating phase. Our terminology ‘‘SOL’’ is because the phase factor $e^{\pm i\delta}$ is attached to the ‘‘solar angle’’ s_{12} . It is physically equivalent to the commonly used PDG convention [39] in which the phase factor is attached to s_{13} .

We use the α parametrization of non-unitary mixing matrix [21] defined in the U_{SOL} convention

$$N = (\mathbf{1} - \tilde{\alpha}) U_{\text{SOL}} = \left\{ \mathbf{1} - \begin{bmatrix} \tilde{\alpha}_{ee} & 0 & 0 \\ \tilde{\alpha}_{\mu e} & \tilde{\alpha}_{\mu\mu} & 0 \\ \tilde{\alpha}_{\tau e} & \tilde{\alpha}_{\tau\mu} & \tilde{\alpha}_{\tau\tau} \end{bmatrix} \right\} U_{\text{SOL}} \quad (3.7)$$

As seen in (3.7), and discussed in detail in ref. [36], the definition of the α matrix depends on the phase convention of the flavor mixing matrix U_{MNS} . In consistent with the notation used in ref. [36], we denote the α matrix elements in the SOL convention as $\tilde{\alpha}_{\beta\gamma}$.

The other convention of the MNS matrix which is heavily used in ref. [36] is the ‘‘ATM’’ convention in which $e^{\pm i\delta}$ is attached to the ‘‘atmospheric angle’’ s_{23} :

$$U_{\text{ATM}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{i\delta} \\ 0 & -s_{23}e^{-i\delta} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (3.8)$$

The α parameters defined in the ATM and PDG conventions of U_{MNS} are denoted as $\alpha_{\beta\gamma}$ and $\bar{\alpha}_{\beta\gamma}$, respectively, in ref. [36], the notation we follow in this paper. Then, we recapitulate here the relationships between the α parameters defined with the PDG ($\bar{\alpha}$), ATM (α) and the SOL ($\tilde{\alpha}$) conventions of U_{MNS} :

$$\begin{aligned} \tilde{\alpha}_{\mu e} &= \bar{\alpha}_{\mu e} e^{-i\delta} = \alpha_{\mu e} e^{-i\delta}, \\ \tilde{\alpha}_{\tau e} &= \bar{\alpha}_{\tau e} e^{-i\delta} = \alpha_{\tau e}, \\ \tilde{\alpha}_{\tau\mu} &= \bar{\alpha}_{\tau\mu} = \alpha_{\tau\mu} e^{i\delta}. \end{aligned} \quad (3.9)$$

3.2 The region of validity and the expansion parameters

We aim at constructing the perturbative framework which is valid at around the solar oscillation maximum, $\Delta m_{21}^2 L/4E \sim \mathcal{O}(1)$. Given the formula

$$\frac{\Delta m_{21}^2 L}{4E} = 0.953 \left(\frac{\Delta m_{21}^2}{7.5 \times 10^{-5} \text{eV}^2} \right) \left(\frac{L}{1000 \text{km}} \right) \left(\frac{E}{100 \text{MeV}} \right)^{-1}, \quad (3.10)$$

it implies neutrino energy $E = (1 - 5) \times 100$ MeV and baseline $L = (1 - 10) \times 1000$ km. In this region, the matter potential is comparable in size to the vacuum effect represented by Δm_{21}^2 ,

$$\frac{a}{\Delta m_{21}^2} = 0.609 \left(\frac{\Delta m_{21}^2}{7.5 \times 10^{-5} \text{ eV}^2} \right)^{-1} \left(\frac{\rho}{3.0 \text{ g/cm}^3} \right) \left(\frac{E}{200 \text{ MeV}} \right) \sim \mathcal{O}(1). \quad (3.11)$$

Hence, our perturbative framework must fully take into account the MSW effect caused by the earth matter effect. A more detailed discussion of the region of validity without UV effect is given in ref. [38].

To formulate our perturbative framework in the following sections, though we follow basically the same procedure as in ref. [38], we give a step-by-step presentation of the formulation because of the additional complexities associated with inclusion of UV, and to make this paper self-contained.

We start by assuming that our solar resonance perturbation theory with UV has the unique expansion parameter $\tilde{\alpha}_{\beta\gamma}$ defined in eq. (3.7). We assume that deviation from unitarity is small. Therefore, $\tilde{\alpha}_{\beta\gamma} \ll 1$ holds for all β and γ , and we set the target accuracy of unitarity test of less than 1% level, $\tilde{\alpha}_{\beta\gamma} \lesssim 10^{-2}$. In fact, we will have one more “effective” expansion parameter in the ν SM sector, $A_{\text{exp}} = c_{13}s_{13} (a/\Delta m_{31}^2) \sim 10^{-3}$, as will be discussed in section 3.8. The reason for having such a small another expansion parameter is due to the special structure of our perturbative Hamiltonian.

3.3 Transformation to the tilde basis

We transform to a different basis to formulate our perturbation theory for solar-scale enhancement. It is the tilde basis

$$\tilde{\nu}_i = (U_{12})_{ij} \check{\nu}_j \quad (3.12)$$

with Hamiltonian

$$\tilde{H} = U_{12} \check{H} U_{12}^\dagger, \quad \text{or} \quad \check{H} = U_{12}^\dagger \tilde{H} U_{12}. \quad (3.13)$$

Notice that the term “tilde basis” has no connection to our notation of $\tilde{\alpha}$ parameters in the SOL convention. The Hamiltonian in the tilde basis is given by

$$\tilde{H} = \tilde{H}_{\nu\text{SM}} + \tilde{H}_{\text{UV}}^{(1)} + \tilde{H}_{\text{UV}}^{(2)} \quad (3.14)$$

where each term of the right-hand side of (3.14) is given by

$$\tilde{H}_{\nu\text{SM}} = \begin{bmatrix} s_{12}^2 \Delta_{21} & c_{12}s_{12}e^{i\delta} \Delta_{21} & 0 \\ c_{12}s_{12}e^{-i\delta} \Delta_{21} & c_{12}^2 \Delta_{21} & 0 \\ 0 & 0 & \Delta_{31} \end{bmatrix} + \begin{bmatrix} c_{13}^2 \Delta_a & 0 & c_{13}s_{13} \Delta_a \\ 0 & 0 & 0 \\ c_{13}s_{13} \Delta_a & 0 & s_{13}^2 \Delta_a \end{bmatrix}, \quad (3.15)$$

$$\tilde{H}_{\text{UV}}^{(1)} = \Delta_b U_{13}^\dagger U_{23}^\dagger \begin{bmatrix} 2\tilde{\alpha}_{ee} \left(1 - \frac{\Delta_a}{\Delta_b}\right) & \tilde{\alpha}_{\mu e}^* & \tilde{\alpha}_{\tau e}^* \\ \tilde{\alpha}_{\mu e} & 2\tilde{\alpha}_{\mu\mu} & \tilde{\alpha}_{\tau\mu}^* \\ \tilde{\alpha}_{\tau e} & \tilde{\alpha}_{\tau\mu} & 2\tilde{\alpha}_{\tau\tau} \end{bmatrix} U_{23} U_{13}, \quad (3.16)$$

$$\tilde{H}_{\text{UV}}^{(2)} = -\Delta_b U_{13}^\dagger U_{23}^\dagger \begin{bmatrix} \tilde{\alpha}_{ee}^2 \left(1 - \frac{\Delta_a}{\Delta_b}\right) + |\tilde{\alpha}_{\mu e}|^2 + |\tilde{\alpha}_{\tau e}|^2 & \tilde{\alpha}_{\mu e}^* \tilde{\alpha}_{\mu\mu} + \tilde{\alpha}_{\tau e}^* \tilde{\alpha}_{\tau\mu} & \tilde{\alpha}_{\tau e}^* \tilde{\alpha}_{\tau\tau} \\ \tilde{\alpha}_{\mu e} \tilde{\alpha}_{\mu\mu} + \tilde{\alpha}_{\tau e} \tilde{\alpha}_{\tau\mu}^* & \tilde{\alpha}_{\mu\mu}^2 + |\tilde{\alpha}_{\tau\mu}|^2 & \tilde{\alpha}_{\tau\mu}^* \tilde{\alpha}_{\tau\tau} \\ \tilde{\alpha}_{\tau e} \tilde{\alpha}_{\tau\tau} & \tilde{\alpha}_{\tau\mu} \tilde{\alpha}_{\tau\tau} & \tilde{\alpha}_{\tau\tau}^2 \end{bmatrix} U_{23} U_{13}. \quad (3.17)$$

In this paper, we restrict ourselves to the perturbative correction to first order in the expansion parameters. There is a number of reasons for this limitation. It certainly simplifies our discussion of the δ - α parameter phase correlation, though we will make a brief comment on effect of $\tilde{H}_{\text{UV}}^{(2)}$ to the correlation in section 4.1. Unfortunately, the expression of the first order UV correction to the oscillation probability is sufficiently complex at this order, as we will see in section D. We do not consider our restriction to first order in $\tilde{\alpha}_{\beta\gamma}$ a serious limitation because the framework anticipates a precision era of neutrino experiment for unitarity test in which the condition $\tilde{\alpha}_{\beta\gamma} \ll 1$ should be justified.

3.4 Definitions of F and K matrices

To make expressions of the S matrix and the oscillation probability as compact as possible, it is important to introduce the new matrix notations F and K :

$$F \equiv \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} = U_{23}^\dagger \begin{bmatrix} 2\tilde{\alpha}_{ee} \left(1 - \frac{\Delta_a}{\Delta_b}\right) & \tilde{\alpha}_{\mu e}^* & \tilde{\alpha}_{\tau e}^* \\ \tilde{\alpha}_{\mu e} & 2\tilde{\alpha}_{\mu\mu} & \tilde{\alpha}_{\tau\mu}^* \\ \tilde{\alpha}_{\tau e} & \tilde{\alpha}_{\tau\mu} & 2\tilde{\alpha}_{\tau\tau} \end{bmatrix} U_{23}, \quad (3.18)$$

$$K = U_{13}^\dagger F U_{13} \equiv \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} = \begin{bmatrix} c_{13}^2 F_{11} + s_{13}^2 F_{33} - c_{13} s_{13} (F_{13} + F_{31}) & c_{13} F_{12} - s_{13} F_{32} & c_{13}^2 F_{13} - s_{13}^2 F_{31} + c_{13} s_{13} (F_{11} - F_{33}) \\ c_{13} F_{21} - s_{13} F_{23} & F_{22} & s_{13} F_{21} + c_{13} F_{23} \\ c_{13}^2 F_{31} - s_{13}^2 F_{13} + c_{13} s_{13} (F_{11} - F_{33}) & s_{13} F_{12} + c_{13} F_{32} & s_{13}^2 F_{11} + c_{13}^2 F_{33} + c_{13} s_{13} (F_{13} + F_{31}) \end{bmatrix}. \quad (3.19)$$

The explicit expressions of the elements F_{ij} and K_{ij} defined in eqs. (3.18) and (3.19), respectively, are given in appendix A. By using these notations the first order Hamiltonian in the tilde basis (3.16) can be written as

$$\tilde{H}_{\text{UV}}^{(1)} = \Delta_b K. \quad (3.20)$$

3.5 Formulating perturbation theory with the hat basis

We use the “renormalized basis” such that the zeroth-order and the perturbed Hamiltonian takes the form $\tilde{H} = \tilde{H}_0 + \tilde{H}_1$. $\tilde{H}_0$ is given by (we discuss $\tilde{H}_1$ later)

$$\tilde{H}_0 = \begin{bmatrix} s_{12}^2 \Delta_{21} + c_{13}^2 \Delta_a & c_{12} s_{12} e^{i\delta} \Delta_{21} & 0 \\ c_{12} s_{12} e^{-i\delta} \Delta_{21} & c_{12}^2 \Delta_{21} & 0 \\ 0 & 0 & \Delta_{31} + s_{13}^2 \Delta_a \end{bmatrix}. \quad (3.21)$$

To formulate the solar-resonance perturbation theory with UV, we transform to the “hat basis”, which diagonalizes $\tilde{H}_0$:

$$\hat{\nu}_i = (U_\varphi^\dagger)_{ij} \tilde{\nu}_j \quad (3.22)$$

with Hamiltonian

$$\hat{H} = U_\varphi^\dagger \tilde{H} U_\varphi \quad (3.23)$$

where U_φ is parametrized as

$$U_\varphi = \begin{bmatrix} \cos \varphi & \sin \varphi e^{i\delta} & 0 \\ -\sin \varphi e^{-i\delta} & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.24)$$

U_φ is determined such that $\hat{H}_{(0)}$ is diagonal, which leads to

$$\begin{aligned} \cos 2\varphi &= \frac{\cos 2\theta_{12} - c_{13}^2 r_a}{\sqrt{(\cos 2\theta_{12} - c_{13}^2 r_a)^2 + \sin^2 2\theta_{12}}}, \\ \sin 2\varphi &= \frac{\sin 2\theta_{12}}{\sqrt{(\cos 2\theta_{12} - c_{13}^2 r_a)^2 + \sin^2 2\theta_{12}}}, \end{aligned} \quad (3.25)$$

where

$$r_a \equiv \frac{a}{\Delta m_{21}^2} = \frac{\Delta_a}{\Delta_{21}}. \quad (3.26)$$

The three eigenvalues of the zeroth order Hamiltonian $\tilde{H}_0$ in (3.21) is given by⁸

$$\begin{aligned} h_1 &= \sin^2(\varphi - \theta_{12}) \Delta_{21} + \cos^2 \varphi c_{13}^2 \Delta_a, \\ h_2 &= \cos^2(\varphi - \theta_{12}) \Delta_{21} + \sin^2 \varphi c_{13}^2 \Delta_a, \\ h_3 &= \Delta_{31} + s_{13}^2 \Delta_a. \end{aligned} \quad (3.28)$$

Then, the Hamiltonian in the hat basis is given by $\hat{H} = \hat{H}_0 + \hat{H}_{\nu\text{SM}1} + \hat{H}_{\text{UV}1}$ where

$$\begin{aligned} \hat{H}_0 &= \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix}, \quad \hat{H}_1^{\nu\text{SM}} = \begin{bmatrix} 0 & 0 & c_\varphi c_{13} s_{13} \Delta_a \\ 0 & 0 & s_\varphi c_{13} s_{13} e^{-i\delta} \Delta_a \\ c_\varphi c_{13} s_{13} \Delta_a & s_\varphi c_{13} s_{13} e^{i\delta} \Delta_a & 0 \end{bmatrix}, \\ \hat{H}_1^{\text{UV}} &= \Delta_b U_\varphi^\dagger K U_\varphi, \end{aligned} \quad (3.29)$$

where the K matrix is defined in (3.19), and the simplified notations are hereafter used: $c_\varphi = \cos \varphi$ and $s_\varphi = \sin \varphi$. Notice that we have omitted the second order $\hat{H}_{\text{UV}2}$ as one can compute from (3.17).

⁸ Notice that one can show that

$$\begin{aligned} h_1 &= \frac{\Delta_{21}}{2} \left[(1 + c_{13}^2 r_a) - \sqrt{(\cos 2\theta_{12} - c_{13}^2 r_a)^2 + \sin^2 2\theta_{12}} \right] \\ h_2 &= \frac{\Delta_{21}}{2} \left[(1 + c_{13}^2 r_a) + \sqrt{(\cos 2\theta_{12} - c_{13}^2 r_a)^2 + \sin^2 2\theta_{12}} \right] \end{aligned} \quad (3.27)$$

3.6 Calculation of $\hat{S}$ and $\tilde{S}$ matrices

To calculate $\hat{S}(x)$ we define $\Omega(x)$ as

$$\Omega(x) = e^{i\hat{H}_0 x} \hat{S}(x). \quad (3.30)$$

Then, $\Omega(x)$ obeys the evolution equation

$$i \frac{d}{dx} \Omega(x) = H_1 \Omega(x) \quad (3.31)$$

where

$$H_1 \equiv e^{i\hat{H}_0 x} \hat{H}_1 e^{-i\hat{H}_0 x}. \quad (3.32)$$

Notice that $\hat{H}_1 = \hat{H}_1^{\nu\text{SM}} + \hat{H}_1^{\text{UV}}$ as in (3.29). Then, $\Omega(x)$ can be computed perturbatively as

$$\Omega(x) = 1 + (-i) \int_0^x dx' H_1(x') + (-i)^2 \int_0^x dx' H_1(x') \int_0^{x'} dx'' H_1(x'') + \dots, \quad (3.33)$$

and the $\hat{S}$ matrix is given by

$$\hat{S}(x) = e^{-i\hat{H}_0 x} \Omega(x). \quad (3.34)$$

Using $\hat{H}_1 = \hat{H}_1^{\nu\text{SM}} + \hat{H}_1^{\text{UV}}$ in (3.29), $\hat{S}$ matrix of the νSM part is given to the zeroth and the first orders in the effective expansion parameter $s_{13} \frac{\Delta_a}{h_3 - h_1}$ by

$$\begin{aligned} & \hat{S}_{\nu\text{SM}}^{(0+1)}(x) = e^{-i\hat{H}_0 x} \Omega_{\nu\text{SM}}(x) \\ = & \begin{bmatrix} e^{-ih_1 x} & 0 & c_\varphi c_{13} s_{13} \frac{\Delta_a}{h_3 - h_1} \{e^{-ih_3 x} - e^{-ih_1 x}\} \\ 0 & e^{-ih_2 x} & s_\varphi c_{13} s_{13} e^{-i\delta} \frac{\Delta_a}{h_3 - h_2} \{e^{-ih_3 x} - e^{-ih_2 x}\} \\ c_\varphi c_{13} s_{13} \frac{\Delta_a}{h_3 - h_1} \{e^{-ih_3 x} - e^{-ih_1 x}\} & s_\varphi c_{13} s_{13} e^{-i\delta} \frac{\Delta_a}{h_3 - h_2} \{e^{-ih_3 x} - e^{-ih_2 x}\} & e^{-ih_3 x} \end{bmatrix}, \end{aligned} \quad (3.35)$$

where we have used the fact that Δ_b is spatially constant as a consequence of the uniform matter density approximation.

Then, νSM part of the tilde basis $\tilde{S}$ matrix is given by

$$\tilde{S}_{\nu\text{SM}}^{(0+1)} = U_\varphi \hat{S}_{\nu\text{SM}}^{(0+1)} U_\varphi^\dagger = \tilde{S}_{\nu\text{SM}}^{(0)} + \tilde{S}_{\nu\text{SM}}^{(1)}, \quad (3.36)$$

where

$$\tilde{S}_{\nu\text{SM}}^{(0)} = \begin{bmatrix} c_\varphi^2 e^{-ih_1 x} + s_\varphi^2 e^{-ih_2 x} & c_\varphi s_\varphi e^{i\delta} (e^{-ih_2 x} - e^{-ih_1 x}) & 0 \\ c_\varphi s_\varphi e^{-i\delta} (e^{-ih_2 x} - e^{-ih_1 x}) & s_\varphi^2 e^{-ih_1 x} + c_\varphi^2 e^{-ih_2 x} & 0 \\ 0 & 0 & e^{-ih_3 x} \end{bmatrix}. \quad (3.37)$$

$\tilde{S}_{\nu\text{SM}}^{(1)}$ can be written in the form

$$\tilde{S}_{\nu\text{SM}}^{(1)} = \begin{bmatrix} 0 & 0 & X \\ 0 & 0 & Y e^{-i\delta} \\ X & Y e^{i\delta} & 0 \end{bmatrix}, \quad (3.38)$$

where

$$\begin{aligned} X &= c_{13}s_{13} \left\{ \frac{\Delta_a}{h_3 - h_1} c_\varphi^2 \left(e^{-ih_3x} - e^{-ih_1x} \right) + \frac{\Delta_a}{h_3 - h_2} s_\varphi^2 \left(e^{-ih_3x} - e^{-ih_2x} \right) \right\}, \\ Y &= c_{13}s_{13}c_\varphi s_\varphi \left\{ -\frac{\Delta_a}{h_3 - h_1} \left(e^{-ih_3x} - e^{-ih_1x} \right) + \frac{\Delta_a}{h_3 - h_2} \left(e^{-ih_3x} - e^{-ih_2x} \right) \right\}. \end{aligned} \quad (3.39)$$

Notice that $\tilde{S}_{\nu\text{SM}}$ respects the generalized T invariance.

Now, we must compute $\hat{S}$ and $\tilde{S}$ matrices of the UV part. By remembering $\hat{H}_1^{\text{UV}} = \Delta_b U_\varphi^\dagger K U_\varphi$, the UV part of H_1 in (3.32), is given by

$$\begin{aligned} H_1^{\text{UV}} &= e^{i\hat{H}_0x} \hat{H}_1^{\text{UV}} e^{-i\hat{H}_0x} = \Delta_b e^{i\hat{H}_0x} U_\varphi^\dagger K U_\varphi e^{-i\hat{H}_0x} \\ &= \Delta_b U_\varphi^\dagger \left(U_\varphi e^{i\hat{H}_0x} U_\varphi^\dagger \right) K \left(U_\varphi e^{-i\hat{H}_0x} U_\varphi^\dagger \right) U_\varphi. \end{aligned} \quad (3.40)$$

Due to frequent usage of the factors in the parenthesis above we give the formula for them

$$\begin{aligned} S_\varphi^{(\pm)} &\equiv \left(U_\varphi e^{\pm i\hat{H}_0x} U_\varphi^\dagger \right) \\ &= \begin{bmatrix} c_\varphi^2 e^{\pm ih_1x} + s_\varphi^2 e^{\pm ih_2x} & c_\varphi s_\varphi e^{i\delta} (e^{\pm ih_2x} - e^{\pm ih_1x}) & 0 \\ c_\varphi s_\varphi e^{-i\delta} (e^{\pm ih_2x} - e^{\pm ih_1x}) & s_\varphi^2 e^{\pm ih_1x} + c_\varphi^2 e^{\pm ih_2x} & 0 \\ 0 & 0 & e^{\pm ih_3x} \end{bmatrix}. \end{aligned} \quad (3.41)$$

Notice that $\tilde{S}_{\nu\text{SM}}^{(0)}$ is nothing but $S_\varphi^{(-)}$. Then, H_1^{UV} takes a simple form

$$H_1^{\text{UV}} = \Delta_b U_\varphi^\dagger S_\varphi^{(+)} K S_\varphi^{(-)} U_\varphi \equiv \Delta_b U_\varphi^\dagger \Phi U_\varphi = \Delta_b U_\varphi^\dagger \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{bmatrix} U_\varphi \quad (3.42)$$

where we have introduced another simplifying matrix notation $\Phi \equiv S_\varphi^{(+)} K S_\varphi^{(-)}$ and its elements Φ_{ij} . The explicit expressions of Φ_{ij} are given in appendix A.

Since U_φ rotation back to the tilde basis removes $U_\varphi^\dagger$ and U_φ in (3.42), it is simpler to go directly to the calculation of the tilde basis $\tilde{S}$ matrix

$$\begin{aligned} \tilde{S}(x)_{\text{UV}}^{(1)} &= U_\varphi \hat{S}(x)_{\text{UV}}^{(1)} U_\varphi^\dagger = U_\varphi e^{-i\hat{H}_0x} \Omega(x)_{\text{UV}}^{(1)} U_\varphi^\dagger = \Delta_b U_\varphi e^{-i\hat{H}_0x} U_\varphi^\dagger \left[(-i) \int_0^x dx' \Phi(x') \right] \\ &= \Delta_b S_\varphi^{(-)} (-i) \int_0^x dx' \begin{bmatrix} \Phi_{11}(x') & \Phi_{12}(x') & \Phi_{13}(x') \\ \Phi_{21}(x') & \Phi_{22}(x') & \Phi_{23}(x') \\ \Phi_{31}(x') & \Phi_{32}(x') & \Phi_{33}(x') \end{bmatrix}. \end{aligned} \quad (3.43)$$

The computed results of the elements of $\tilde{S}(x)_{\text{UV}}^{(1)}$ are given in appendix B. Notice that again $\tilde{S}(x)_{\text{UV}}^{(1)}$ respects the generalized T invariance.

Thus, we have computed all the tilde basis S matrix elements to first order as

$$\tilde{S} = \tilde{S}_{\nu\text{SM}}^{(0)} + \tilde{S}_{\nu\text{SM}}^{(1)} + \tilde{S}_{\text{UV}}^{(1)}. \quad (3.44)$$

The first and the second terms are given, respectively, in (3.37) and (3.38) with (3.39), and the third in appendix B.

3.7 The relations between various bases and computation of the flavor basis S matrix

We first summarize the relationship between the flavor basis, the check (vacuum mass eigenstate) basis, the tilde, and the hat (zeroth order diagonalized hamiltonian) basis. Only the unitary transformations are involved in changing from the hat basis to the tilde basis, and from the tilde basis to the check basis:

$$\begin{aligned}\hat{H} &= U_\varphi^\dagger \tilde{H} U_\varphi, & \text{or} & & \tilde{H} &= U_\varphi \hat{H} U_\varphi^\dagger, \\ \tilde{H} &= U_{12} \check{H} U_{12}^\dagger, & \text{or} & & \check{H} &= U_{12}^\dagger \tilde{H} U_{12} = U_{12}^\dagger U_\varphi \hat{H} U_\varphi^\dagger U_{12}\end{aligned}\quad (3.45)$$

The non-unitary transformation is involved from the check basis to the flavor basis:

$$\nu_\beta = N_{\beta i} \check{\nu}_i = \{(1 - \tilde{\alpha})U\}_{\beta i} \check{\nu}_i. \quad (3.46)$$

The relationship between the flavor basis Hamiltonian H_{flavor} and the hat basis one $\hat{H}$ is

$$\begin{aligned}H_{\text{flavor}} &= \{(1 - \tilde{\alpha})U\} \check{H} \{(1 - \tilde{\alpha})U\}^\dagger = (1 - \tilde{\alpha})U U_{12}^\dagger U_\varphi \hat{H} U_\varphi^\dagger U_{12} U^\dagger (1 - \tilde{\alpha})^\dagger \\ &= (1 - \tilde{\alpha})U_{23}U_{13}U_\varphi \hat{H} U_\varphi^\dagger U_{13}^\dagger U_{23}^\dagger (1 - \tilde{\alpha})^\dagger.\end{aligned}\quad (3.47)$$

Then, the flavor basis S matrix is related to $\hat{S}$ and $\tilde{S}$ matrices as

$$\begin{aligned}S_{\text{flavor}} &= (1 - \tilde{\alpha})U_{23}U_{13}U_\varphi \hat{S} U_\varphi^\dagger U_{13}^\dagger U_{23}^\dagger (1 - \tilde{\alpha})^\dagger \\ &= (1 - \tilde{\alpha})U_{23}U_{13}\tilde{S} U_{13}^\dagger U_{23}^\dagger (1 - \tilde{\alpha})^\dagger.\end{aligned}\quad (3.48)$$

Using the formula eq. (3.48), it is straightforward to compute the flavor basis S matrix elements. Notice, however, that U_{13} is free from CP phase δ due to our choice of the SOL convention of the U_{MNS} matrix in (3.6).

The flavor basis S matrix has a structure $S_{\text{flavor}} = (1 - \tilde{\alpha})S^{\text{prop}}(1 - \tilde{\alpha})^\dagger$ where $S_{\text{prop}} \equiv U_{23}U_{13}\tilde{S}U_{13}^\dagger U_{23}^\dagger$ describes the unitary evolution despite the presence of non-unitary mixing [36]. The factors $(1 - \tilde{\alpha})$ and $(1 - \tilde{\alpha})^\dagger$, parts of the N matrix which project the flavor states to the mass eigenstates and vice versa, may be interpreted as the ones analogous to the “production NSI” and “detection NSI” [54].

3.8 Effective expansion parameter with and without the UV effect

As announced in section 3.2, the expression of $\tilde{S}_{\nu\text{SM}}^{(1)}$ in (3.38) with (3.39) tells us that we have another expansion parameter [38]

$$A_{\text{exp}} \equiv c_{13}s_{13} \left| \frac{a}{\Delta m_{31}^2} \right| = 2.78 \times 10^{-3} \left(\frac{\Delta m_{31}^2}{2.4 \times 10^{-3} \text{ eV}^2} \right)^{-1} \left(\frac{\rho}{3.0 \text{ g/cm}^3} \right) \left(\frac{E}{200 \text{ MeV}} \right), \quad (3.49)$$

which is very small. The reason for such a “generated by the framework” expansion parameter is the special feature of the perturbed Hamiltonian in (3.29).

In fact, our perturbative framework is peculiar from the beginning in the sense that the key non-perturbed part of the Hamiltonian (3.21), its top-left 2×2 sub-matrix, is smaller

in size than the 33 element by a factor of ~ 30 , and is comparable with $\hat{H}_1^{\nu\text{SM}}$ in (3.29). The secret for emergence of the very small effective expansion parameter (3.49) is that the 33 element decouples in the leading order and appear in the perturbative corrections only in the energy denominator, making them *smaller for the larger ratio* of $\Delta m_{31}^2/\Delta m_{21}^2$. The latter property holds because of the special structure of perturbative Hamiltonian $\hat{H}_1^{\nu\text{SM}}$ with non-vanishing elements only in the third row and third column.

With inclusion of the UV Hamiltonian (3.20), however, the size of the first order correction is controlled not only by A_{exp} in (3.49) but also the magnitudes of $\tilde{\alpha}_{\beta,\gamma}$. In computing the higher order corrections the energy denominator suppression does not work for all the terms because the last property, “non-vanishing elements in the third row and third column only”, ceases to hold in the first order Hamiltonian. It can be confirmed in looking into the formulas of the oscillation probabilities in appendix D.

3.9 Neutrino oscillation probability to first order in expansion

The oscillation-probability can be calculated by using the formula

$$P(\nu_\beta \rightarrow \nu_\alpha) = |(S_{\text{flavor}})_{\alpha\beta}|^2. \quad (3.50)$$

We denote the flavor basis S matrices corresponding to $\tilde{S}_{\nu\text{SM}}^{(0)}$, $\tilde{S}_{\nu\text{SM}}^{(1)}$, and $\tilde{S}_{\text{UV}}^{(1)}$ as $S_{\nu\text{SM}}^{(0)}$, $S_{\nu\text{SM}}^{(1)}$, and $S_{\text{UV}}^{(1)}$, respectively, as they are related through (3.48). To first order we have

$$S_{\text{flavor}} = S_{\nu\text{SM}}^{(0)} + S_{\nu\text{SM}}^{(1)} + S_{\text{UV}}^{(1)} - \tilde{\alpha} S_{\nu\text{SM}}^{(0)} - S_{\nu\text{SM}}^{(0)} \tilde{\alpha}^\dagger. \quad (3.51)$$

Then, we are ready to calculate the expressions of the oscillation probabilities using the formula (3.50) to first order in the expansion parameters. Following ref. [36], we categorize $P(\nu_\beta \rightarrow \nu_\alpha)$ into the three types of terms:

$$P(\nu_\beta \rightarrow \nu_\alpha) = P(\nu_\beta \rightarrow \nu_\alpha)_{\nu\text{SM}}^{(0+1)} + P(\nu_\beta \rightarrow \nu_\alpha)_{\text{int-UV}}^{(1)} + P(\nu_\beta \rightarrow \nu_\alpha)_{\text{ext-UV}}^{(1)}, \quad (3.52)$$

where

$$\begin{aligned} P(\nu_\beta \rightarrow \nu_\alpha)_{\nu\text{SM}}^{(0+1)} &= \left| (S_{\nu\text{SM}}^{(0)})_{\alpha\beta} \right|^2 + 2\text{Re} \left[(S_{\nu\text{SM}}^{(0)})_{\alpha\beta}^* (S_{\nu\text{SM}}^{(1)})_{\alpha\beta} \right], \\ P(\nu_\beta \rightarrow \nu_\alpha)_{\text{int-UV}}^{(1)} &= 2\text{Re} \left[(S_{\nu\text{SM}}^{(0)})_{\alpha\beta}^* (S_{\text{UV}}^{(1)})_{\alpha\beta} \right], \\ P(\nu_\beta \rightarrow \nu_\alpha)_{\text{ext-UV}}^{(1)} &= -2\text{Re} \left[(S_{\nu\text{SM}}^{(0)})_{\alpha\beta}^* (\tilde{\alpha} S_{\nu\text{SM}}^{(0)} + S_{\nu\text{SM}}^{(0)} \tilde{\alpha}^\dagger)_{\alpha\beta} \right]. \end{aligned} \quad (3.53)$$

The subscripts “int-UV” and “ext-UV” refer the intrinsic and extrinsic UV contributions, terminology defined in ref. [36].

The first term in eq. (3.53), $P(\nu_\beta \rightarrow \nu_\alpha)_{\nu\text{SM}}^{(0+1)}$, is already computed in ref. [38]. Hence, we do not repeat the calculation, but urge the readers to go to this reference. Notice that use of the SOL convention of U_{MNS} does not alter the expression of the oscillation probabilities. The rest of the terms in eq. (3.53) can be computed straightforwardly by using the expressions of the tilde basis S matrix which are given explicitly in appendix B, and the $\tilde{\alpha}$ matrix defined in (3.7).

To simplify the expressions of the probabilities, we define the reduced Jarlskog factor in matter [38] as

$$J_{mr} \equiv c_{23}s_{23}c_{13}^2s_{13}c_\varphi s_\varphi = J_r \left[(\cos 2\theta_{12} - c_{13}^2 r_a)^2 + \sin^2 2\theta_{12} \right]^{-1/2}, \quad (3.54)$$

which is proportional to the reduced Jarlskog factor in vacuum, $J_r \equiv c_{23}s_{23}c_{13}^2s_{13}c_{12}s_{12}$ [55]. We have used eq. (3.25) in the second equality in (3.54).

We defer our computation of $P(\nu_\beta \rightarrow \nu_\alpha)_{\text{int-UV}}^{(1)}$ and $P(\nu_\beta \rightarrow \nu_\alpha)_{\text{ext-UV}}^{(1)}$ to appendix D. The reason is that the nature of correlation between νSM phase and the α parameters, our primary concern in this paper, can be addressed without computing the oscillation probabilities. If any significant physical property is present in our system, it would be revealed at the amplitude level before going to calculation of the oscillation probability.⁹

For the same reason we do not try to compute the oscillation probabilities in the other channels, in particular the ones in the $\nu_\mu - \nu_\tau$ sector. If one needs these expressions of the probabilities, one can readily calculate them following the above instruction.¹⁰ If any demand exists for an accurate oscillation probability formula in matter with non-unitarity under the uniform matter density approximation, the exact but fairly simple formula is readily available to use [23].

3.10 Symmetry of the oscillation probability

As discussed in ref. [38], the oscillation probability respects “dynamical symmetry”, an invariance under the transformation

$$\varphi \rightarrow \varphi + \frac{\pi}{2}, \quad (3.55)$$

which induces the following transformations simultaneously

$$\begin{aligned} h_1 &\rightarrow h_2, & h_2 &\rightarrow h_1, \\ c_\varphi &\rightarrow -s_\varphi, & s_\varphi &\rightarrow +c_\varphi, & \cos 2\varphi &\rightarrow -\cos 2\varphi, & \sin 2\varphi &\rightarrow -\sin 2\varphi. \end{aligned} \quad (3.56)$$

Hence, $J_{mr} \rightarrow -J_{mr}$ under the transformation. It is interesting to observe explicitly that the symmetry is respected by $P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)}$ and $P(\nu_\mu \rightarrow \nu_e)_{\text{ext-UV}}^{(1)}$ which will be presented in the rest of this section.

We remark that the symmetry under the transformation (3.55) involves φ , the matter-affected mixing angle θ_{12} . It is interesting to notice that the similar symmetry exists in the helio-UV perturbation theory with ϕ , the matter-affected θ_{13} [36].

⁹ Even though the “canonical phase combination” $[e^{-i\delta}\alpha_{\mu e}, \alpha_{\tau e}, e^{i\delta}\alpha_{\tau\mu}]$ (ATM conventions of U_{MNS}) can only be revealed after computing the oscillation probability, it has a footprint at the amplitude level as the “lozenge position $e^{\pm i\delta}$ structure” [36].

¹⁰ For general readers, we recommend to use e.g., mathematica software to perform computation of the oscillation probability using (3.53) due to its complexity even at first order. It may be the necessity for the computation in the $\nu_\mu - \nu_\tau$ sector.

4 Dynamical correlation between ν SM phase δ and the α parameters, and their clustering

In this section, we discuss correlations between ν SM phase δ and the UV α parameters, and the clustering of various α parameters, which are manifested in the oscillation probabilities calculated in the previous section D. For the former, however, the expressions of the flavor basis S matrix, S_{flavor} , are sufficiently informative. Furthermore, one can readily notice that it suffices to discuss $\tilde{S}_{\text{UV}}^{(1)}$ matrix for this purpose. It is because neither the external multiplication of $(1 - \tilde{\alpha})$ and $(1 - \tilde{\alpha})^\dagger$ nor the $U_{13}U_{23}$ rotations in the SOL convention of U_{MNS} introduce $\delta - \alpha$ parameter correlation. The former carries the phases of the α parameters and the latter no phase.¹¹

4.1 Correlations between ν SM phase δ and the α parameters

To extract the $\delta - \alpha$ parameter correlation in a well defined way, we use the same prescription as in ref. [36]. That is, we factorize the $e^{\pm i\delta}$ factor, whenever possible, from each matrix element such that the same $\delta - \alpha$ parameter correlations are obtained in all the $\tilde{S}_{\text{UV}}^{(1)}$ matrix elements universally as seen in appendix B.¹² Then, we identify the following correlated pairs:

$$K_{12}e^{-i\delta}, \quad \text{and} \quad K_{23}e^{i\delta}, \quad (4.1)$$

where the blobs of the α parameters K_{12} and K_{23} are given by (see (3.19) for definition)

$$\begin{aligned} K_{12} &= c_{13} (c_{23}\tilde{\alpha}_{\mu e}^* - s_{23}\tilde{\alpha}_{\tau e}^*) - s_{13} [2c_{23}s_{23}(\tilde{\alpha}_{\mu\mu} - \tilde{\alpha}_{\tau\tau}) + c_{23}^2\tilde{\alpha}_{\tau\mu} - s_{23}^2\tilde{\alpha}_{\tau\mu}^*], \\ K_{23} &= s_{13} (c_{23}\tilde{\alpha}_{\mu e} - s_{23}\tilde{\alpha}_{\tau e}) + c_{13} [2c_{23}s_{23}(\tilde{\alpha}_{\mu\mu} - \tilde{\alpha}_{\tau\tau}) + c_{23}^2\tilde{\alpha}_{\tau\mu}^* - s_{23}^2\tilde{\alpha}_{\tau\mu}]. \end{aligned} \quad (4.2)$$

Notice that $K_{21}e^{i\delta} = (K_{12}e^{-i\delta})^*$ and $K_{32}e^{-i\delta} = (K_{23}e^{i\delta})^*$, and therefore they do not introduce correlations independent of those in (4.2). In fact, the feature of the $e^{\pm i\delta}$ - K blob correlation in (4.1) can be traced back to the form of Φ_{ij} given in appendix A. For readers' convenience we also summarize the explicit expressions of K_{ij} in appendix A.

As expected, the correlations between δ and the α parameter phases (4.2) appear in the first order oscillation probability in the intrinsic part $P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)}$, but not in the extrinsic part $P(\nu_\mu \rightarrow \nu_e)_{\text{ext-UV}}^{(1)}$. See the explicit expressions in sections D.1 and D.2. This feature is not shared in the phase correlation discussed in ref. [36]. Since the correlation between δ and the K_{12} - K_{23} cluster variables lives in Φ matrix elements, which are the building block of the perturbation series, it is obvious that the correlation prevails to higher order in perturbation theory in inside the intrinsic contributions. We do not discuss this point further in this paper.

¹¹ We refer the UV parameters, in generic contexts, as the “ α parameters”, but use the notation “ $\tilde{\alpha}$ ” in making the statements about the formulas and the results obtained by using the SOL convention of U_{MNS} .

¹² In carrying out this procedure, one notices that Φ and $\tilde{S}_{\text{UV}}^{(1)}$ matrix elements possess the “lozenge position $e^{\pm i\delta}$ form” which played the key role for the stability of the form of phase correlation called the canonical phase combination in our treatment with the ATM conventions of U_{MNS} [36]. However, this property does not appear to play any crucial role in the present treatment of the phase correlation in the U_{SOL} convention.

One may ask the question: “What happens in the $\delta - \alpha$ parameter correlation if we include the second order perturbed Hamiltonian $\tilde{H}_{UV}^{(2)}$?”. It is obvious that a qualitatively similar $\delta - \alpha$ parameter correlation exists, but with a tiny magnitude of order $\lesssim 10^{-4}$ under our target accuracy of unitarity test. If necessary, one can even simply read off the form of correlation by extracting $\tilde{H}_{UV}^{(2)}$ versions of the F and the K matrices. However, we do not enter into this discussion because it is unlikely that we can find a qualitatively new feature of the $\delta - \alpha$ parameter correlation which would influence the system with non-unitarity in a visible way.

4.2 Features of the $\delta - \alpha$ parameter correlation: Are they real?

We now understand that the $\delta - \alpha$ parameter correlations seen in this and the previous paper [36] are all physical. There is no U_{MNS} phase convention in which the phase correlation is absent in the two different regions of validity, one of the helio-UV perturbation theory and the other of the solar resonance perturbation theory extended to include UV. The former (the latter) is quite wide regions around the atmospheric-scale (solar-scale) oscillation maximum, which will be referred to as the “atmospheric region” (“solar region”) below.

We also understand that the features of parameter correlation in the both regions are different. In the “atmospheric region”, it is a direct $\delta - \alpha$ parameter correlation [36]. Whereas in the “solar region”, it is a δ –(cluster of the α parameters) correlation, as in (4.1). But, this is entirely normal. As we have learned in section 2, the nature of the parameter correlation in neutrino evolution with inclusion of outside- ν SM ingredients are dynamical. Their features depend on the values of the relevant parameters as well as the kinematical regions where different degrees of freedom play the dominant role.

4.3 Parameter correlation in the other conventions of U_{MNS}

We want to make a brief remark on what would be the feature of the $\delta - \alpha$ parameter correlation in the other conventions of U_{MNS} . A preliminary investigation of this problem with the ATM convention of U_{MNS} reveals that the same δ –(cluster of the α parameters) correlation as in (4.1) survives, but inside K_{ij} $\tilde{\alpha}_{\beta\gamma}$ must be transformed to $\alpha_{\beta\gamma}$ (α matrix elements in the ATM convention) by the transformation rule (3.9).

When we use the translation rule of the α parameters inside K_{ij} to make them the ones in the PDG conventions, the way $\tilde{\alpha}_{\tau\mu}$ transforms is different from the way $\tilde{\alpha}_{\mu e}$ and $\tilde{\alpha}_{\tau e}$ transform. In the transformation to the ATM convention, everybody transform differently. Therefore, inside K_{ij} we have an “enhanced” $\delta - \alpha$ parameter correlation in the ATM or the PDG conventions compared to the SOL case. Though we did not go through the explicit computation in the PDG convention, it is extremely unlikely that the phase correlation can be eliminated by choosing the different U_{MNS} convention. Thus, the $\delta - \alpha$ parameter correlation prevails in all the U_{MNS} phase conventions in the “solar region”.

4.4 Clustering of the α parameters

In addition to the δ –(blob of the α parameters) correlation, we observe a feature which may be called the “clustering of the α parameters” in the first order intrinsic UV corrections

to the oscillation probability $P(\nu_\mu \rightarrow \nu_e)$ calculated in appendix D. We can identify the following “clustering variables” at the level of the $\tilde{S}_{UV}^{(1)}$ matrix elements:

$$K_{12}e^{-i\delta} + K_{21}e^{i\delta}, \quad c_\varphi^2 K_{13} - c_\varphi s_\varphi K_{23}e^{i\delta}, \quad c_\varphi s_\varphi K_{13} + c_\varphi^2 K_{23}e^{i\delta}, \quad (4.3)$$

where we have not listed $(s_\varphi^2 K_{13} + c_\varphi s_\varphi K_{23}e^{i\delta})$ and $(c_\varphi s_\varphi K_{13} - s_\varphi^2 K_{23}e^{i\delta})$. They are not dynamically independent from the ones in (4.3) because they can be generated by the symmetry transformation (3.55) from the second and the third in (4.3). Also there exists the exceptional, isolated one $K_{12}e^{-i\delta}$ in eq. (D.3).

In (4.3), we did not quote the diagonal variables which come as a form of the difference, for example $(K_{22} - K_{11})$, because these combinations are enforced by re-phasing invariance. But, these diagonal α parameter differences often come with the particular combination with the other cluster variables, e.g., as $[\cos 2\varphi (K_{22} - K_{11}) + \sin 2\varphi (K_{12}e^{-i\delta} + K_{21}e^{i\delta})]$, or $[\sin 2\varphi (K_{22} - K_{11}) - \cos 2\varphi (K_{12}e^{-i\delta} + K_{21}e^{i\delta})]$. Moreover, the other blobs of variables $(c_{13}^2 K_{13} - s_{13}^2 K_{31})$, and $(c_{13}^2 K_{23}e^{i\delta} - s_{13}^2 K_{32}e^{-i\delta})$, which are not visible at the level of the $\tilde{S}_{UV}^{(1)}$ matrix, shows up in the oscillation probability. See eqs. (D.2), (D.3) - (D.5) for all the above examples of blobs.

It appears that appearance of such cluster variables as well as the correlation between δ and the α parameter blobs are worth attention though we do not quite understand the cause of this phenomenon.

5 Concluding remarks

In this paper, we have addressed the question of whether the $\delta - \alpha$ parameter correlation is real and physical, or it is merely an artifact of phase convention of the MNS flavor mixing matrix U_{MNS} . The question arose being triggered by the result of our previous paper [36], in which we investigated the three flavor neutrino evolution in matter with non-unitarity in region around the atmospheric-scale enhanced oscillations, for short the “atmospheric region”. We have observed that the form of the $\delta - \alpha$ parameter correlation we have uncovered depends upon the phase convention of U_{MNS} used, and furthermore there is a U_{MNS} convention called SOL ($e^{\pm i\delta}$ attached to s_{12}) in which the phase correlation is absent. On the other hand, one can show that the $\delta - \alpha$ parameter correlation generically exist in all the U_{MNS} convention except for the SOL. It leads to the two different interpretations (see section 1) which are quite contrary with each other on whether the phase correlation is physical.

To answer the question of whether the phase correlation is physical in an unambiguous way, we have formulated in this paper a new perturbative framework, an extension of the solar resonance perturbation theory [38], to include the effect of UV (unitarity violation). Its validity covers the region around the solar-scale enhanced oscillations, the “solar region” for short. We have used the same SOL convention of U_{MNS} , which might be a “phase correlation killer”, to derive the flavor basis S matrix and to calculate the oscillation probability. The results of this computation which is reported in this paper demonstrate that the $\delta - \alpha$ parameter correlation *does exist* in the U_{SOL} convention. In fact, the form of the correlation

we observe is more like a δ –(cluster of the α parameters) correlation, as shown in (4.1). Furthermore, we have argued in section 4.3 that the $\delta - \alpha$ parameter correlation prevails in all the conventions of U_{MNS} in region where the solar oscillation is enhanced.

By summarizing the obtained informations in this and the previous [36] papers, we can make the following statements:

- In the “atmospheric region” there is no conventions of U_{MNS} , except for the SOL convention, in which the phase correlation vanishes.
- In the “solar region” the phase correlation exists in all of the three conventions of U_{MNS} . It is shown to be the case by explicit calculation in the SOL convention and agreed on a reasonably firm ground for the ATM and the PDG conventions.
- The choice of the SOL convention of U_{MNS} does not eliminate the $\delta - \alpha$ parameter correlation throughout entire kinematical region of energy and baseline.

We conclude that there is no U_{MNS} convention in which the $\delta - \alpha$ parameter correlation disappears in the entire kinematical phase space. We believe that it is a fair statement that the $\delta - \alpha$ parameter correlation is physical, not an artifact of choice of convention of U_{MNS} . Then, the absence of correlation in the “atmospheric region” with the SOL convention of U_{MNS} is likely to be a very special accidental phenomena.

Finally, apart from the phase correlation so much discussed in this paper, we should emphasize in more generic contexts the importance of understanding the dynamical nature of parameter correlations in the three neutrino evolution in matter with new ingredients beyond the νSM . As discussed in section 2, occurrence of dynamical correlations between the parameters in system with the many degree of freedom is very common. A rich variety of correlations we encountered in our system with non-unitarity adds another example to this list. If one chooses the way of testing leptonic unitarity in this way, namely, setting up a class of models with UV and confrontation of them with the experiments, understanding the system with UV would be indispensable step to performing the task.

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A Explicit expressions of F_{ij} , K_{ij} and Φ_{ij}

The explicit expressions of the elements F_{ij} , K_{ij} and Φ_{ij} defined, respectively, in eqs. (3.18), (3.19) and (3.42) are given as follows:

$$\begin{aligned}
F_{11} &= 2\tilde{\alpha}_{ee} \left(1 - \frac{\Delta_a}{\Delta_b} \right), \\
F_{12} &= c_{23}\tilde{\alpha}_{\mu e}^* - s_{23}\tilde{\alpha}_{\tau e}^*, \\
F_{13} &= s_{23}\tilde{\alpha}_{\mu e}^* + c_{23}\tilde{\alpha}_{\tau e}^*, \\
F_{21} &= c_{23}\tilde{\alpha}_{\mu e} - s_{23}\tilde{\alpha}_{\tau e} = (F_{12})^*, \\
F_{22} &= 2 \left[c_{23}^2\tilde{\alpha}_{\mu\mu} + s_{23}^2\tilde{\alpha}_{\tau\tau} - c_{23}s_{23}\text{Re}(\tilde{\alpha}_{\tau\mu}) \right], \\
F_{23} &= \left[2c_{23}s_{23}(\tilde{\alpha}_{\mu\mu} - \tilde{\alpha}_{\tau\tau}) + c_{23}^2\tilde{\alpha}_{\tau\mu}^* - s_{23}^2\tilde{\alpha}_{\tau\mu} \right], \\
F_{31} &= s_{23}\tilde{\alpha}_{\mu e} + c_{23}\tilde{\alpha}_{\tau e} = (F_{13})^*, \\
F_{32} &= \left[2c_{23}s_{23}(\tilde{\alpha}_{\mu\mu} - \tilde{\alpha}_{\tau\tau}) + c_{23}^2\tilde{\alpha}_{\tau\mu} - s_{23}^2\tilde{\alpha}_{\tau\mu}^* \right] = (F_{23})^*, \\
F_{33} &= 2 \left[s_{23}^2\tilde{\alpha}_{\mu\mu} + c_{23}^2\tilde{\alpha}_{\tau\tau} + c_{23}s_{23}\text{Re}(\tilde{\alpha}_{\tau\mu}) \right]. \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
K_{11} &= 2c_{13}^2\tilde{\alpha}_{ee} \left(1 - \frac{\Delta_a}{\Delta_b} \right) + 2s_{13}^2 \left[s_{23}^2\tilde{\alpha}_{\mu\mu} + c_{23}^2\tilde{\alpha}_{\tau\tau} + c_{23}s_{23}\text{Re}(\tilde{\alpha}_{\tau\mu}) \right], \\
&\quad - 2c_{13}s_{13}\text{Re}(s_{23}\tilde{\alpha}_{\mu e} + c_{23}\tilde{\alpha}_{\tau e}) \\
K_{12} &= c_{13} \left(c_{23}\tilde{\alpha}_{\mu e}^* - s_{23}\tilde{\alpha}_{\tau e}^* \right) - s_{13} \left[2c_{23}s_{23}(\tilde{\alpha}_{\mu\mu} - \tilde{\alpha}_{\tau\tau}) + c_{23}^2\tilde{\alpha}_{\tau\mu} - s_{23}^2\tilde{\alpha}_{\tau\mu}^* \right] = (K_{21})^*, \\
K_{13} &= 2c_{13}s_{13} \left[\tilde{\alpha}_{ee} \left(1 - \frac{\Delta_a}{\Delta_b} \right) - (s_{23}^2\tilde{\alpha}_{\mu\mu} + c_{23}^2\tilde{\alpha}_{\tau\tau}) \right] \\
&\quad + c_{13}^2 \left(s_{23}\tilde{\alpha}_{\mu e}^* + c_{23}\tilde{\alpha}_{\tau e}^* \right) - s_{13}^2 \left(s_{23}\tilde{\alpha}_{\mu e} + c_{23}\tilde{\alpha}_{\tau e} \right) - 2c_{23}s_{23}c_{13}s_{13}\text{Re}(\tilde{\alpha}_{\tau\mu}) = (K_{31})^*, \\
K_{22} &= 2 \left[c_{23}^2\tilde{\alpha}_{\mu\mu} + s_{23}^2\tilde{\alpha}_{\tau\tau} - c_{23}s_{23}\text{Re}(\tilde{\alpha}_{\tau\mu}) \right], \\
K_{23} &= s_{13} \left(c_{23}\tilde{\alpha}_{\mu e} - s_{23}\tilde{\alpha}_{\tau e} \right) + c_{13} \left[2c_{23}s_{23}(\tilde{\alpha}_{\mu\mu} - \tilde{\alpha}_{\tau\tau}) + c_{23}^2\tilde{\alpha}_{\tau\mu}^* - s_{23}^2\tilde{\alpha}_{\tau\mu} \right] = (K_{32})^*, \\
K_{33} &= 2s_{13}^2\tilde{\alpha}_{ee} \left(1 - \frac{\Delta_a}{\Delta_b} \right) + 2c_{13}^2 \left[s_{23}^2\tilde{\alpha}_{\mu\mu} + c_{23}^2\tilde{\alpha}_{\tau\tau} + c_{23}s_{23}\text{Re}(\tilde{\alpha}_{\tau\mu}) \right] \\
&\quad + 2c_{13}s_{13}\text{Re}(s_{23}\tilde{\alpha}_{\mu e} + c_{23}\tilde{\alpha}_{\tau e}). \tag{A.2}
\end{aligned}$$

$$\begin{aligned}
\Phi_{11} &= K_{11} + 2c_\varphi^2 s_\varphi^2 (K_{22} - K_{11}) - c_\varphi^2 s_\varphi^2 (K_{22} - K_{11}) \left\{ e^{i(h_2-h_1)x} + e^{-i(h_2-h_1)x} \right\} \\
&\quad - c_\varphi s_\varphi \cos 2\varphi \left(K_{12}e^{-i\delta} + K_{21}e^{i\delta} \right) \\
&\quad + c_\varphi s_\varphi \left\{ - \left(s_\varphi^2 K_{12}e^{-i\delta} - c_\varphi^2 K_{21}e^{i\delta} \right) e^{i(h_2-h_1)x} + \left(c_\varphi^2 K_{12}e^{-i\delta} - s_\varphi^2 K_{21}e^{i\delta} \right) e^{-i(h_2-h_1)x} \right\}, \\
\Phi_{12} &= e^{i\delta} \left[c_\varphi s_\varphi \cos 2\varphi (K_{22} - K_{11}) + c_\varphi s_\varphi \left\{ s_\varphi^2 e^{i(h_2-h_1)x} - c_\varphi^2 e^{-i(h_2-h_1)x} \right\} (K_{22} - K_{11}) \right. \\
&\quad + 2c_\varphi^2 s_\varphi^2 \left(K_{12}e^{-i\delta} + K_{21}e^{i\delta} \right) + s_\varphi^2 \left(s_\varphi^2 K_{12}e^{-i\delta} - c_\varphi^2 K_{21}e^{i\delta} \right) e^{i(h_2-h_1)x} \\
&\quad \left. + c_\varphi^2 \left(c_\varphi^2 K_{12}e^{-i\delta} - s_\varphi^2 K_{21}e^{i\delta} \right) e^{-i(h_2-h_1)x} \right], \\
\Phi_{13} &= \left(s_\varphi^2 K_{13} + c_\varphi s_\varphi K_{23}e^{i\delta} \right) e^{-i(h_3-h_2)x} + \left(c_\varphi^2 K_{13} - c_\varphi s_\varphi K_{23}e^{i\delta} \right) e^{-i(h_3-h_1)x},
\end{aligned}$$

$$\begin{aligned}
\Phi_{21} &= e^{-i\delta} \left\{ c_\varphi s_\varphi \cos 2\varphi (K_{22} - K_{11}) - c_\varphi s_\varphi \left\{ c_\varphi^2 e^{i(h_2-h_1)x} - s_\varphi^2 e^{-i(h_2-h_1)x} \right\} (K_{22} - K_{11}) \right. \\
&+ 2c_\varphi^2 s_\varphi^2 (K_{12} e^{-i\delta} + K_{21} e^{i\delta}) + c_\varphi^2 (c_\varphi^2 K_{21} e^{i\delta} - s_\varphi^2 K_{12} e^{-i\delta}) e^{i(h_2-h_1)x} \\
&+ \left. s_\varphi^2 (s_\varphi^2 K_{21} e^{i\delta} - c_\varphi^2 K_{12} e^{-i\delta}) e^{-i(h_2-h_1)x} \right\}, \\
\Phi_{22} &= K_{22} - 2c_\varphi^2 s_\varphi^2 (K_{22} - K_{11}) + c_\varphi^2 s_\varphi^2 (K_{22} - K_{11}) \left\{ e^{i(h_2-h_1)x} + e^{-i(h_2-h_1)x} \right\} \\
&+ c_\varphi s_\varphi \left[\cos 2\varphi (K_{12} e^{-i\delta} + K_{21} e^{i\delta}) + (s_\varphi^2 K_{12} e^{-i\delta} - c_\varphi^2 K_{21} e^{i\delta}) e^{i(h_2-h_1)x} \right. \\
&- \left. (c_\varphi^2 K_{12} e^{-i\delta} - s_\varphi^2 K_{21} e^{i\delta}) e^{-i(h_2-h_1)x} \right], \\
\Phi_{23} &= e^{-i\delta} \left[(c_\varphi s_\varphi K_{13} + c_\varphi^2 K_{23} e^{i\delta}) e^{-i(h_3-h_2)x} - (c_\varphi s_\varphi K_{13} - s_\varphi^2 K_{23} e^{i\delta}) e^{-i(h_3-h_1)x} \right], \\
\Phi_{31} &= (s_\varphi^2 K_{31} + c_\varphi s_\varphi K_{32} e^{-i\delta}) e^{i(h_3-h_2)x} + (c_\varphi^2 K_{31} - c_\varphi s_\varphi K_{32} e^{-i\delta}) e^{i(h_3-h_1)x}, \\
\Phi_{32} &= e^{i\delta} \left[(c_\varphi s_\varphi K_{31} + c_\varphi^2 K_{32} e^{-i\delta}) e^{i(h_3-h_2)x} - (c_\varphi s_\varphi K_{31} - s_\varphi^2 K_{32} e^{-i\delta}) e^{i(h_3-h_1)x} \right], \\
\Phi_{33} &= K_{33}. \tag{A.3}
\end{aligned}$$

B The tilde basis $\tilde{S}$ matrix elements: First order UV part

$$\begin{aligned}
&\tilde{S}(x)_{11}^{\text{UV}} \\
&= \Delta_b \left\{ K_{11} + 2c_\varphi^2 s_\varphi^2 (K_{22} - K_{11}) - c_\varphi s_\varphi \cos 2\varphi (K_{12} e^{-i\delta} + K_{21} e^{i\delta}) \right\} (-ix) (c_\varphi^2 e^{-ih_1x} + s_\varphi^2 e^{-ih_2x}) \\
&+ c_\varphi s_\varphi \left\{ c_\varphi s_\varphi \cos 2\varphi (K_{22} - K_{11}) + 2c_\varphi^2 s_\varphi^2 (K_{12} e^{-i\delta} + K_{21} e^{i\delta}) \right\} (-ix) (e^{-ih_2x} - e^{-ih_1x}) \\
&+ c_\varphi s_\varphi \left[-2c_\varphi s_\varphi (K_{22} - K_{11}) + \cos 2\varphi (K_{12} e^{-i\delta} + K_{21} e^{i\delta}) \right] \frac{1}{h_2 - h_1} (e^{-ih_2x} - e^{-ih_1x}). \tag{B.1}
\end{aligned}$$

$$\begin{aligned}
&\tilde{S}(x)_{12}^{\text{UV}} \\
&= e^{i\delta} \Delta_b \left[\left\{ c_\varphi s_\varphi \cos 2\varphi (K_{22} - K_{11}) + 2c_\varphi^2 s_\varphi^2 (K_{12} e^{-i\delta} + K_{21} e^{i\delta}) \right\} (-ix) (c_\varphi^2 e^{-ih_1x} + s_\varphi^2 e^{-ih_2x}) \right. \\
&+ c_\varphi s_\varphi \left\{ K_{22} - 2c_\varphi^2 s_\varphi^2 (K_{22} - K_{11}) + c_\varphi s_\varphi \cos 2\varphi (K_{12} e^{-i\delta} + K_{21} e^{i\delta}) \right\} (-ix) (e^{-ih_2x} - e^{-ih_1x}) \\
&+ \left. \left\{ -c_\varphi s_\varphi \cos 2\varphi (K_{22} - K_{11}) + \left\{ K_{12} e^{-i\delta} - 2c_\varphi^2 s_\varphi^2 (K_{12} e^{-i\delta} + K_{21} e^{i\delta}) \right\} \right\} \frac{1}{h_2 - h_1} (e^{-ih_2x} - e^{-ih_1x}) \right]. \tag{B.2}
\end{aligned}$$

$$\begin{aligned}
\tilde{S}(x)_{13}^{\text{UV}} &= \Delta_b \left[(s_\varphi^2 K_{13} + c_\varphi s_\varphi K_{23} e^{i\delta}) \frac{1}{h_3 - h_2} (e^{-ih_3x} - e^{-ih_2x}) \right. \\
&+ \left. (c_\varphi^2 K_{13} - c_\varphi s_\varphi K_{23} e^{i\delta}) \frac{1}{h_3 - h_1} (e^{-ih_3x} - e^{-ih_1x}) \right]. \tag{B.3}
\end{aligned}$$

$$\begin{aligned}
& \tilde{S}(x)_{21}^{\text{UV}} \\
&= e^{-i\delta} \Delta_b \left[\left\{ c_\varphi s_\varphi \cos 2\varphi (K_{22} - K_{11}) + 2c_\varphi^2 s_\varphi^2 (K_{12}e^{-i\delta} + K_{21}e^{i\delta}) \right\} (-ix) (s_\varphi^2 e^{-ih_1x} + c_\varphi^2 e^{-ih_2x}) \right. \\
&+ c_\varphi s_\varphi \left\{ K_{11} + 2c_\varphi^2 s_\varphi^2 (K_{22} - K_{11}) - c_\varphi s_\varphi \cos 2\varphi (K_{12}e^{-i\delta} + K_{21}e^{i\delta}) \right\} (-ix) (e^{-ih_2x} - e^{-ih_1x}) \\
&+ \left. \left\{ -c_\varphi s_\varphi \cos 2\varphi (K_{22} - K_{11}) + \left\{ K_{21}e^{i\delta} - 2c_\varphi^2 s_\varphi^2 (K_{12}e^{-i\delta} + K_{21}e^{i\delta}) \right\} \right\} \frac{1}{h_2 - h_1} (e^{-ih_2x} - e^{-ih_1x}) \right]. \quad (\text{B.4})
\end{aligned}$$

$$\begin{aligned}
& \tilde{S}(x)_{22}^{\text{UV}} \\
&= \Delta_b \left[c_\varphi s_\varphi \left\{ c_\varphi s_\varphi \cos 2\varphi (K_{22} - K_{11}) + 2c_\varphi^2 s_\varphi^2 (K_{12}e^{-i\delta} + K_{21}e^{i\delta}) \right\} (-ix) (e^{-ih_2x} - e^{-ih_1x}) \right. \\
&+ \left\{ K_{22} - 2c_\varphi^2 s_\varphi^2 (K_{22} - K_{11}) + c_\varphi s_\varphi \cos 2\varphi (K_{12}e^{-i\delta} + K_{21}e^{i\delta}) \right\} (-ix) (s_\varphi^2 e^{-ih_1x} + c_\varphi^2 e^{-ih_2x}) \\
&+ \left. \left\{ 2c_\varphi^2 s_\varphi^2 (K_{22} - K_{11}) - c_\varphi s_\varphi \cos 2\varphi (K_{12}e^{-i\delta} + K_{21}e^{i\delta}) \right\} \frac{1}{h_2 - h_1} (e^{-ih_2x} - e^{-ih_1x}) \right]. \quad (\text{B.5})
\end{aligned}$$

$$\begin{aligned}
& \tilde{S}(x)_{23}^{\text{UV}} = e^{-i\delta} \Delta_b \left\{ \left[c_\varphi s_\varphi K_{13} + c_\varphi^2 K_{23}e^{i\delta} \right] \frac{1}{h_3 - h_2} (e^{-ih_3x} - e^{-ih_2x}) \right. \\
&- \left. \left[c_\varphi s_\varphi K_{13} - s_\varphi^2 K_{23}e^{i\delta} \right] \frac{1}{h_3 - h_1} (e^{-ih_3x} - e^{-ih_1x}) \right\}. \quad (\text{B.6})
\end{aligned}$$

$$\begin{aligned}
& \tilde{S}(x)_{31}^{\text{UV}} = \Delta_b \left[\left(s_\varphi^2 K_{31} + c_\varphi s_\varphi K_{32}e^{-i\delta} \right) \frac{1}{h_3 - h_2} (e^{-ih_3x} - e^{-ih_2x}) \right. \\
&+ \left. \left(c_\varphi^2 K_{31} - c_\varphi s_\varphi K_{32}e^{-i\delta} \right) \frac{1}{h_3 - h_1} (e^{-ih_3x} - e^{-ih_1x}) \right]. \quad (\text{B.7})
\end{aligned}$$

$$\begin{aligned}
& \tilde{S}(x)_{32}^{\text{UV}} = e^{i\delta} \Delta_b \left[\left(c_\varphi s_\varphi K_{31} + c_\varphi^2 K_{32}e^{-i\delta} \right) \frac{1}{h_3 - h_2} \left\{ e^{-ih_3x} - e^{-ih_2x} \right\} \right. \\
&- \left. \left(c_\varphi s_\varphi K_{31} - s_\varphi^2 K_{32}e^{-i\delta} \right) \frac{1}{h_3 - h_1} \left\{ e^{-ih_3x} - e^{-ih_1x} \right\} \right]. \quad (\text{B.8})
\end{aligned}$$

$$\tilde{S}(x)_{33}^{\text{UV}} = (-ix \Delta_b) e^{-ih_3x} K_{33}. \quad (\text{B.9})$$

C The zeroth-order ν SM S matrix elements

Here, we give the expressions of the flavor basis S matrix elements of ν SM part at zeroth order. The superscript “ ν SM” is abbreviated.

$$\begin{aligned}
S_{ee}^{(0)} &= c_{13}^2 \left(c_\varphi^2 e^{-ih_1x} + s_\varphi^2 e^{-ih_2x} \right) + s_{13}^2 e^{-ih_3x}, \\
S_{e\mu}^{(0)} &= c_{23}c_{13}c_\varphi s_\varphi e^{i\delta} \left(e^{-ih_2x} - e^{-ih_1x} \right) - s_{23}c_{13}s_{13} \left(c_\varphi^2 e^{-ih_1x} + s_\varphi^2 e^{-ih_2x} - e^{-ih_3x} \right), \\
S_{e\tau}^{(0)} &= -c_{23}c_{13}s_{13} \left(c_\varphi^2 e^{-ih_1x} + s_\varphi^2 e^{-ih_2x} - e^{-ih_3x} \right) - s_{23}c_{13}c_\varphi s_\varphi e^{i\delta} \left(e^{-ih_2x} - e^{-ih_1x} \right), \\
S_{\mu e}^{(0)} &= c_{23}c_{13}c_\varphi s_\varphi e^{-i\delta} \left(e^{-ih_2x} - e^{-ih_1x} \right) - s_{23}c_{13}s_{13} \left(c_\varphi^2 e^{-ih_1x} + s_\varphi^2 e^{-ih_2x} - e^{-ih_3x} \right) = S_{e\mu}(-\delta), \\
S_{\mu\mu}^{(0)} &= c_{23}^2 \left(s_\varphi^2 e^{-ih_1x} + c_\varphi^2 e^{-ih_2x} \right) + s_{23}^2 \left\{ s_{13}^2 \left(c_\varphi^2 e^{-ih_1x} + s_\varphi^2 e^{-ih_2x} \right) + c_{13}^2 e^{-ih_3x} \right\} \\
&\quad - 2c_{23}s_{23}s_{13}c_\varphi s_\varphi \cos \delta \left(e^{-ih_2x} - e^{-ih_1x} \right), \\
S_{\mu\tau}^{(0)} &= s_{13}c_\varphi s_\varphi \left(s_{23}^2 e^{i\delta} - c_{23}^2 e^{-i\delta} \right) \left(e^{-ih_2x} - e^{-ih_1x} \right) \\
&\quad + c_{23}s_{23} \left[s_{13}^2 \left(c_\varphi^2 e^{-ih_1x} + s_\varphi^2 e^{-ih_2x} \right) + c_{13}^2 e^{-ih_3x} - \left(s_\varphi^2 e^{-ih_1x} + c_\varphi^2 e^{-ih_2x} \right) \right], \\
S_{\tau e}^{(0)} &= -c_{23}c_{13}s_{13} \left(c_\varphi^2 e^{-ih_1x} + s_\varphi^2 e^{-ih_2x} - e^{-ih_3x} \right) - s_{23}c_{13}c_\varphi s_\varphi e^{-i\delta} \left(e^{-ih_2x} - e^{-ih_1x} \right) = S_{e\tau}(-\delta), \\
S_{\tau\mu}^{(0)} &= s_{13}c_\varphi s_\varphi \left(s_{23}^2 e^{-i\delta} - c_{23}^2 e^{i\delta} \right) \left(e^{-ih_2x} - e^{-ih_1x} \right) \\
&\quad + c_{23}s_{23} \left[s_{13}^2 \left(c_\varphi^2 e^{-ih_1x} + s_\varphi^2 e^{-ih_2x} \right) + c_{13}^2 e^{-ih_3x} - \left(s_\varphi^2 e^{-ih_1x} + c_\varphi^2 e^{-ih_2x} \right) \right] = S_{\mu\tau}(-\delta), \\
S_{\tau\tau}^{(0)} &= s_{23}^2 \left(s_\varphi^2 e^{-ih_1x} + c_\varphi^2 e^{-ih_2x} \right) + c_{23}^2 \left\{ s_{13}^2 \left(c_\varphi^2 e^{-ih_1x} + s_\varphi^2 e^{-ih_2x} \right) + c_{13}^2 e^{-ih_3x} \right\} \\
&\quad + 2c_{23}s_{23}s_{13}c_\varphi s_\varphi \cos \delta \left(e^{-ih_2x} - e^{-ih_1x} \right). \tag{C.1}
\end{aligned}$$

D Neutrino oscillation probability in the $\nu_\mu \rightarrow \nu_e$ channel

In this appendix, we calculate the oscillation probability of the appearance channel $\nu_\mu \rightarrow \nu_e$ to first order in the expansion parameters, A_{exp} and $\tilde{\alpha}_{\beta\gamma}$. Since the ν SM part is already computed in ref. [38], we concentrate on the first order UV correction terms, the second and the third terms in (3.52). We present the results of computations in the following two subsections.

In this paper, we do not attempt to examine numerical accuracy of the first order oscillation probability formulas for the two reasons: (1) we focus on revealing dynamical properties of correlation between ν SM and the UV variables in an analytic way, and (2) the accuracy of the first order formula with UV is controlled by the magnitude of the α parameters alone because the ν SM part is very accurate to first order, as shown in ref. [38]. The property (2) allows us to estimate the accuracy of the first order formula.¹³

¹³ Notice that the statement (2) applies for $\alpha_{\beta\gamma} \gtrsim A_{\text{exp}} \sim 10^{-3}$, because if $\alpha_{\beta\gamma} \lesssim A_{\text{exp}}$ the accuracy of the ν SM controls the accuracy of the whole formula. We set the target accuracy of unitarity test at a % level, $\alpha_{\beta\gamma} \lesssim 10^{-2}$ in section 3.2. Then, in the relevant region $10^{-3} \lesssim \alpha_{\beta\gamma} \lesssim 10^{-2}$, the second order UV corrections are of order of $\sim \alpha_{\beta\gamma}^2 \lesssim 10^{-4}$, and therefore, the first order formula must be accurate to this order.

D.1 $P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)}$

We start from the defining formula for $P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)}$. After computation of all the terms, we assemble them according to the types of the K_{ij} variables involved. See eq. (3.19) for definition of the K_{ij} . For bookkeeping purpose we decompose $P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)}$ into the following four terms:

$$\begin{aligned}
P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)} &= P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)}|_{\text{D-OD}} \\
&\quad + P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)}|_{\text{OD1}} P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)}|_{\text{OD2}} + P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)}|_{\text{OD3}},
\end{aligned}
\tag{D.1}$$

where the subscripts “D” and “OD” refer to the diagonal and the off-diagonal K_{ij} variables. Each term in (D.1) is presented in order below with some appropriate comments. The organization inside each term is largely determined by making $\varphi \rightarrow \varphi + \frac{\pi}{2}$ symmetry [38]

manifest. See section 3.10. The first term in (D.1) reads

$$\begin{aligned}
& P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)}|_{\text{D-OD}} \\
&= 4J_{mr} \sin \delta \cos 2\varphi (K_{22} - K_{11}) (\Delta_b x) \sin^2 \frac{(h_2 - h_1)x}{2} \\
&+ 2(K_{33} - K_{11}) (\Delta_b x) \left[J_{mr} \cos \delta \sin(h_2 - h_1)x - 2s_{23}^2 c_{13} s_{13} \{c_\varphi^2 \sin(h_3 - h_1)x + s_\varphi^2 \sin(h_3 - h_2)x\} \right] \\
&+ 4J_{mr} (K_{33} - K_{22}) (\Delta_b x) \\
&\times \left[2 \cos \delta \sin \frac{(h_3 - h_2)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_1 - h_3)x}{2} + \sin \delta \left\{ \sin^2 \frac{(h_3 - h_2)x}{2} - \sin^2 \frac{(h_3 - h_1)x}{2} \right\} \right] \\
&+ \left[\cos 2\varphi (K_{22} - K_{11}) + \sin 2\varphi (K_{12} e^{-i\delta} + K_{21} e^{i\delta}) \right] (\Delta_b x) \\
&\times \left[2c_{13} c_\varphi^2 s_\varphi^2 \left\{ c_{23}^2 c_{13} \sin(h_2 - h_1)x - 4s_{23}^2 s_{13} \sin \frac{(h_3 - h_2)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_1 - h_3)x}{2} \right\} \right. \\
&+ 2J_{mr} \cos \delta [c_\varphi^2 \sin(h_3 - h_1)x + s_\varphi^2 \sin(h_3 - h_2)x] \\
&- 4J_{mr} \sin \delta \left\{ c_\varphi^2 \sin^2 \frac{(h_3 - h_1)x}{2} + s_\varphi^2 \sin^2 \frac{(h_3 - h_2)x}{2} - 4c_\varphi^2 s_\varphi^2 \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \Big] \\
&+ 2 \left[\sin 2\varphi (K_{22} - K_{11}) - \cos 2\varphi (K_{12} e^{-i\delta} + K_{21} e^{i\delta}) \right] \\
&\times \left[s_{23}^2 c_{13} s_{13} c_\varphi s_\varphi \left\{ (\Delta_b x) [c_\varphi^2 \sin(h_3 - h_1)x + s_\varphi^2 \sin(h_3 - h_2)x] \right. \right. \\
&+ 2 \frac{\Delta_b}{h_2 - h_1} \left[\sin^2 \frac{(h_3 - h_2)x}{2} - \sin^2 \frac{(h_3 - h_1)x}{2} - \cos 2\varphi \sin^2 \frac{(h_2 - h_1)x}{2} \right] \Big\} \\
&+ 4J_{mr} \cos \delta c_\varphi s_\varphi \frac{\Delta_b}{h_2 - h_1} \sin^2 \frac{(h_2 - h_1)x}{2} \Big] \\
&- 4c_{23} c_{13}^2 c_\varphi s_\varphi \left[\cos 2\varphi (K_{22} - K_{11}) + \sin 2\varphi (K_{12} e^{-i\delta} + K_{21} e^{i\delta}) \right] \\
&\times \frac{\Delta_b}{h_2 - h_1} \left[-s_{23} \cos \delta \left\{ \sin^2 \frac{(h_3 - h_2)x}{2} - \sin^2 \frac{(h_3 - h_1)x}{2} - \cos 2\varphi \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \right. \\
&+ c_{23} \sin 2\varphi \sin^2 \frac{(h_2 - h_1)x}{2} + 2s_{23} \sin \delta \sin \frac{(h_3 - h_2)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_1 - h_3)x}{2} \Big] \\
&+ 2J_{mr} c_\varphi s_\varphi \left[\sin 2\varphi (K_{22} - K_{11}) - \cos 2\varphi (K_{12} e^{-i\delta} + K_{21} e^{i\delta}) \right] (\Delta_b x) \\
&\times \left[- \left\{ \cos \delta \sin(h_2 - h_1)x + 2 \sin \delta \cos 2\varphi \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \right. \\
&+ 2 \sin \delta \left\{ \sin^2 \frac{(h_3 - h_2)x}{2} - \sin^2 \frac{(h_3 - h_1)x}{2} - \cos 2\varphi \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \\
&+ 4 \cos \delta \sin \frac{(h_3 - h_2)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_1 - h_3)x}{2} \Big]. \tag{D.2}
\end{aligned}$$

We note that the diagonal K_{jj} elements organize themselves into the form of difference, $(K_{22} - K_{11})$ type combinations, as is seen in $P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)}|_{\text{D-OD}}$ (D.2), as it should be. The only difference between the diagonal elements in the Hamiltonian affects physics due to the re-phasing freedom of neutrino wave functions, the well known fact.¹⁴

¹⁴ For an example of explicit demonstration of this feature in the NSI case, see e.g., ref. [43].

Now, we present the remaining three “OD” terms:

$$\begin{aligned}
& P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)}|_{\text{OD1}} \\
&= 4c_{23}c_{13}^2 \text{Re} \left(K_{12}e^{-i\delta} \right) \\
&\times \frac{\Delta_b}{h_2 - h_1} \left[-s_{23} \cos \delta \left\{ \sin^2 \frac{(h_3 - h_2)x}{2} - \sin^2 \frac{(h_3 - h_1)x}{2} - \cos 2\varphi \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \right. \\
&+ c_{23} \sin 2\varphi \sin^2 \frac{(h_2 - h_1)x}{2} + 2s_{23} \sin \delta \sin \frac{(h_3 - h_2)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_1 - h_3)x}{2} \left. \right] \\
&+ 4c_{23}s_{23}c_{13}^2 \text{Im} \left(K_{12}e^{-i\delta} \right) \\
&\times \frac{\Delta_b}{h_2 - h_1} \left[\sin \delta \left\{ \sin^2 \frac{(h_3 - h_2)x}{2} - \sin^2 \frac{(h_3 - h_1)x}{2} - \cos 2\varphi \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \right. \\
&+ 2 \cos \delta \sin \frac{(h_3 - h_2)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_1 - h_3)x}{2} \left. \right]. \tag{D.3}
\end{aligned}$$

$$\begin{aligned}
& P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)}|_{\text{OD2}} \\
&= -4c_{23}^2c_{13}s_{13}c_\varphi s_\varphi \text{Re} \left(c_\varphi s_\varphi K_{31} + c_\varphi^2 K_{32}e^{-i\delta} \right) \\
&\times \frac{\Delta_b}{h_3 - h_2} \left\{ \sin^2 \frac{(h_3 - h_2)x}{2} - \sin^2 \frac{(h_3 - h_1)x}{2} + \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \\
&+ 4c_{23}^2c_{13}s_{13}c_\varphi s_\varphi \text{Re} \left(c_\varphi s_\varphi K_{31} - s_\varphi^2 K_{32}e^{-i\delta} \right) \\
&\times \frac{\Delta_b}{h_3 - h_1} \left\{ \sin^2 \frac{(h_3 - h_2)x}{2} - \sin^2 \frac{(h_3 - h_1)x}{2} - \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \\
&+ 4c_{23}s_{23}c_{13}s_{13}^2 \left\{ \cos \delta \text{Re} \left(c_\varphi s_\varphi K_{31} + c_\varphi^2 K_{32}e^{-i\delta} \right) - \sin \delta \text{Im} \left(c_\varphi s_\varphi K_{31} + c_\varphi^2 K_{32}e^{-i\delta} \right) \right\} \\
&\times \frac{\Delta_b}{h_3 - h_2} \left[c_\varphi^2 \left\{ \sin^2 \frac{(h_3 - h_1)x}{2} - \sin^2 \frac{(h_2 - h_1)x}{2} \right\} + (1 + s_\varphi^2) \sin^2 \frac{(h_3 - h_2)x}{2} \right] \\
&- 4c_{23}s_{23}c_{13}s_{13}^2 \left\{ \cos \delta \text{Re} \left(c_\varphi s_\varphi K_{31} - s_\varphi^2 K_{32}e^{-i\delta} \right) - \sin \delta \text{Im} \left(c_\varphi s_\varphi K_{31} - s_\varphi^2 K_{32}e^{-i\delta} \right) \right\} \\
&\times \frac{\Delta_b}{h_3 - h_1} \left[s_\varphi^2 \left\{ \sin^2 \frac{(h_3 - h_2)x}{2} - \sin^2 \frac{(h_2 - h_1)x}{2} \right\} + (1 + c_\varphi^2) \sin^2 \frac{(h_3 - h_1)x}{2} \right] \\
&- 8c_{23}c_{13}s_{13} \left[(s_{23}s_{13}c_\varphi^2 \cos \delta + c_{23}c_\varphi s_\varphi) \text{Im} \left(c_\varphi s_\varphi K_{31} + c_\varphi^2 K_{32}e^{-i\delta} \right) \right. \\
&+ s_{23}s_{13}c_\varphi^2 \sin \delta \text{Re} \left(c_\varphi s_\varphi K_{31} + c_\varphi^2 K_{32}e^{-i\delta} \right) \left. \right] \frac{\Delta_b}{h_3 - h_2} \sin \frac{(h_3 - h_2)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_1 - h_3)x}{2} \\
&- 8c_{23}c_{13}s_{13} \left[(s_{23}s_{13}s_\varphi^2 \cos \delta - c_{23}c_\varphi s_\varphi) \text{Im} \left(c_\varphi s_\varphi K_{31} - s_\varphi^2 K_{32}e^{-i\delta} \right) \right. \\
&+ s_{23}s_{13}s_\varphi^2 \sin \delta \text{Re} \left(c_\varphi s_\varphi K_{31} - s_\varphi^2 K_{32}e^{-i\delta} \right) \left. \right] \frac{\Delta_b}{h_3 - h_1} \sin \frac{(h_3 - h_2)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_1 - h_3)x}{2}. \tag{D.4}
\end{aligned}$$

$$\begin{aligned}
& P(\nu_\mu \rightarrow \nu_e)_{\text{int-UV}}^{(1)}|_{\text{OD3}} \\
&= -4c_{23}s_{23}c_{13}c_\varphi s_\varphi \left\{ \cos \delta \text{Re} \left[s_\varphi^2 (c_{13}^2 K_{13} - s_{13}^2 K_{31}) + c_\varphi s_\varphi (c_{13}^2 K_{23} e^{i\delta} - s_{13}^2 K_{32} e^{-i\delta}) \right] \right. \\
&+ \sin \delta \text{Im} \left[s_\varphi^2 (c_{13}^2 K_{13} - s_{13}^2 K_{31}) + c_\varphi s_\varphi (c_{13}^2 K_{23} e^{i\delta} - s_{13}^2 K_{32} e^{-i\delta}) \right] \Big\} \\
&\times \frac{\Delta_b}{h_3 - h_2} \left\{ \sin^2 \frac{(h_3 - h_2)x}{2} - \sin^2 \frac{(h_3 - h_1)x}{2} + \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \\
&- 4c_{23}s_{23}c_{13}c_\varphi s_\varphi \left\{ \cos \delta \text{Re} \left[c_\varphi^2 (c_{13}^2 K_{13} - s_{13}^2 K_{31}) - c_\varphi s_\varphi (c_{13}^2 K_{23} e^{i\delta} - s_{13}^2 K_{32} e^{-i\delta}) \right] \right. \\
&+ \sin \delta \text{Im} \left[c_\varphi^2 (c_{13}^2 K_{13} - s_{13}^2 K_{31}) - c_\varphi s_\varphi (c_{13}^2 K_{23} e^{i\delta} - s_{13}^2 K_{32} e^{-i\delta}) \right] \Big\} \\
&\times \frac{\Delta_b}{h_3 - h_1} \left\{ \sin^2 \frac{(h_3 - h_2)x}{2} - \sin^2 \frac{(h_3 - h_1)x}{2} - \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \\
&- 4s_{23}^2 c_{13} s_{13} \text{Re} \left[s_\varphi^2 (c_{13}^2 K_{13} - s_{13}^2 K_{31}) + c_\varphi s_\varphi (c_{13}^2 K_{23} e^{i\delta} - s_{13}^2 K_{32} e^{-i\delta}) \right] \\
&\times \frac{\Delta_b}{h_3 - h_2} \left[c_\varphi^2 \left\{ -\sin^2 \frac{(h_3 - h_1)x}{2} + \sin^2 \frac{(h_2 - h_1)x}{2} \right\} - (1 + s_\varphi^2) \sin^2 \frac{(h_3 - h_2)x}{2} \right] \\
&- 4s_{23}^2 c_{13} s_{13} \text{Re} \left[c_\varphi^2 (c_{13}^2 K_{13} - s_{13}^2 K_{31}) - c_\varphi s_\varphi (c_{13}^2 K_{23} e^{i\delta} - s_{13}^2 K_{32} e^{-i\delta}) \right] \\
&\times \frac{\Delta_b}{h_3 - h_1} \left[s_\varphi^2 \left\{ -\sin^2 \frac{(h_3 - h_2)x}{2} + \sin^2 \frac{(h_2 - h_1)x}{2} \right\} - (1 + c_\varphi^2) \sin^2 \frac{(h_3 - h_1)x}{2} \right] \\
&+ 8s_{23}c_{13} \left\{ (c_{23}c_\varphi s_\varphi \cos \delta - s_{23}s_{13}c_\varphi^2) \text{Im} \left[s_\varphi^2 (c_{13}^2 K_{13} - s_{13}^2 K_{31}) + c_\varphi s_\varphi (c_{13}^2 K_{23} e^{i\delta} - s_{13}^2 K_{32} e^{-i\delta}) \right] \right. \\
&+ c_{23}c_\varphi s_\varphi \sin \delta \text{Re} \left[s_\varphi^2 (c_{13}^2 K_{13} - s_{13}^2 K_{31}) + c_\varphi s_\varphi (c_{13}^2 K_{23} e^{i\delta} - s_{13}^2 K_{32} e^{-i\delta}) \right] \Big\} \\
&\times \frac{\Delta_b}{h_3 - h_2} \sin \frac{(h_3 - h_2)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_1 - h_3)x}{2} \\
&+ 8s_{23}c_{13} \left\{ (c_{23}c_\varphi s_\varphi \cos \delta + s_{23}s_{13}s_\varphi^2) \text{Im} \left[c_\varphi^2 (c_{13}^2 K_{13} - s_{13}^2 K_{31}) - c_\varphi s_\varphi (c_{13}^2 K_{23} e^{i\delta} - s_{13}^2 K_{32} e^{-i\delta}) \right] \right. \\
&+ c_{23}c_\varphi s_\varphi \sin \delta \text{Re} \left[c_\varphi^2 (c_{13}^2 K_{13} - s_{13}^2 K_{31}) - c_\varphi s_\varphi (c_{13}^2 K_{23} e^{i\delta} - s_{13}^2 K_{32} e^{-i\delta}) \right] \Big\} \\
&\times \frac{\Delta_b}{h_3 - h_1} \sin \frac{(h_3 - h_2)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_1 - h_3)x}{2}. \tag{D.5}
\end{aligned}$$

D.2 $P(\nu_\mu \rightarrow \nu_e)_{\text{ext-UV}}^{(1)}$

To calculate $P(\nu_\mu \rightarrow \nu_e)_{\text{ext-UV}}^{(1)}$ defined in the last line in eq. (3.53), we need the expressions of zeroth-order elements of ν SM matrix $S_{\nu\text{SM}}^{(0)}$, which are given in appendix C. They can be easily obtained from the tilde basis S matrix in (3.37). Using the $S^{(0)}$ matrix elements

$P(\nu_\mu \rightarrow \nu_e)_{\text{ext-UV}}^{(1)}$ can be readily calculated as

$$\begin{aligned}
P(\nu_\mu \rightarrow \nu_e)_{\text{ext-UV}}^{(1)} &= -2(\tilde{\alpha}_{ee} + \tilde{\alpha}_{\mu\mu})|S_{e\mu}^{(0)}|^2 - 2\text{Re} \left[\tilde{\alpha}_{\mu e}(S_{ee}^{(0)})^* S_{e\mu}^{(0)} \right] \\
&= -2(\tilde{\alpha}_{ee} + \tilde{\alpha}_{\mu\mu}) \left[c_{23}^2 c_{13}^2 \sin^2 2\varphi \sin^2 \frac{(h_2 - h_1)x}{2} \right. \\
&\quad + s_{23}^2 \sin^2 2\theta_{13} \left\{ c_\varphi^2 \sin^2 \frac{(h_3 - h_1)x}{2} + s_\varphi^2 \sin^2 \frac{(h_3 - h_2)x}{2} - c_\varphi^2 s_\varphi^2 \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \\
&\quad + 4J_{mr} \cos \delta \left\{ \cos 2\varphi \sin^2 \frac{(h_2 - h_1)x}{2} - \sin^2 \frac{(h_3 - h_2)x}{2} + \sin^2 \frac{(h_3 - h_1)x}{2} \right\} \\
&\quad + 8J_{mr} \sin \delta \sin \frac{(h_3 - h_2)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_1 - h_3)x}{2} \Big] \\
&\quad + \text{Re}(\tilde{\alpha}_{\mu e}) \left[2c_{23}c_{13} \sin 2\varphi \cos \delta \left(c_{13}^2 \cos 2\varphi \sin^2 \frac{(h_2 - h_1)x}{2} + s_{13}^2 \left\{ \sin^2 \frac{(h_3 - h_2)x}{2} - \sin^2 \frac{(h_3 - h_1)x}{2} \right\} \right) \right. \\
&\quad - c_{23}c_{13} \sin 2\varphi \sin \delta \left(c_{13}^2 \sin(h_2 - h_1)x - s_{13}^2 \{ \sin(h_3 - h_2)x - \sin(h_3 - h_1)x \} \right) \\
&\quad - s_{23} \sin 2\theta_{13} \left(c_{13}^2 \sin^2 2\varphi \sin^2 \frac{(h_2 - h_1)x}{2} - 2 \cos 2\theta_{13} \left\{ c_\varphi^2 \sin^2 \frac{(h_3 - h_1)x}{2} + s_\varphi^2 \sin^2 \frac{(h_3 - h_2)x}{2} \right\} \right) \Big] \\
&\quad - \text{Im}(\tilde{\alpha}_{\mu e}) \left[c_{23}c_{13} \sin 2\varphi \cos \delta \left(c_{13}^2 \sin(h_2 - h_1)x - s_{13}^2 \{ \sin(h_3 - h_2)x - \sin(h_3 - h_1)x \} \right) \right. \\
&\quad + 2c_{23}c_{13} \sin 2\varphi \sin \delta \left(c_{13}^2 \cos 2\varphi \sin^2 \frac{(h_2 - h_1)x}{2} + s_{13}^2 \left\{ \sin^2 \frac{(h_3 - h_2)x}{2} - \sin^2 \frac{(h_3 - h_1)x}{2} \right\} \right) \\
&\quad + s_{23} \sin 2\theta_{13} \{ c_\varphi^2 \sin(h_3 - h_1)x + s_\varphi^2 \sin(h_3 - h_2)x \} \Big]. \tag{D.6}
\end{aligned}$$

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