Group representations in entanglement theory

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Contents

Chapter 1

Introduction

Entanglement is one of the most striking consequences of quantum mechanics which follows directly from the axioms. In the early days of the development of the theory, manifestations of entanglement caused many physicists to dislike the probabilistic interpretation of quantum theory. The EPR paradox, formulated by Einstein, Podolsky and Rosen intended to show that either quantum theory is not complete, or it contains nonlocal interaction between spacelike separated particles. This conclusion made Einstein to refer to entanglement derisively as "*spukhafte Fernwirkung*"[11] (spooky action at a distance).

During the 20th century it turned out however, that even when some strange result of quantum theory seems to violate fundamental laws of Nature – at least for our minds condemned to think classically –, a closer examination shows that these violations are avoided in some subtle way, like for example in the teleportation of quantum states or in the EPR paradox itself.

In chapter 1 it is intended to give the reader a brief overview of the concepts related to entanglement, highlighting its significance, and discussing the main questions, answers to which would pave the way to a complete understanding of entanglement. The first section is concerned with the differences between the classical and the quantum world related to entanglement. The next two sections deal with the problem of classification of entangled states and quantification of entanglement in general. In the last section, the well understood case of bipartite entanglement in pure states is outlined as an example.

In chapter 2 we study entanglement in multipartite quantum systems with distinguishable constituents. The fundamental notions mentioned in the first chapter as well as more specific notions are made precise in the first few sections. Then we characterize local unitary invariants using the representation theory of the unitary groups, and also suggest a concrete method for constructing them. Using this method we construct an infinite family of local unitary invariants in every even degree. We also give a formula for the stabilized dimensions of subspaces of invariants. After some remarks on invariants under the group of invertible stochastic local operations aided by classical communication, we conclude this chapter with a relationship between the local unitary-equivalence problem of pure and mixed states.

Chapter 3 is devoted to the study of entanglement in quantum systems containing indistinguishable constituents. Using an algebraic construction proposed by Freudenthal in order to obtain representations of some exceptional Lie groups, we are able to describe completely the entanglement classes in some special tripartite quantum systems. These special quantum systems have similar entanglement properties, which enables us to distill a general correspondance between quantum systems with distunguishable and indistinguishable parts, relating local invariants of the two types of systems. This motivates our study of fermionic entanglement measures with the tools of the second chapter.

Some mathematical concepts, definitions and theorems encountered in this thesis are collected in the appendix. These include some facts from linear algebra and representation theory in general, and the representation theory of symmetric, general linear and unitary groups.

New scientific results are presented in sections 2.6.1-2.6.3, 2.7, 2.8, 2.9, 2.10, 3.2.1-3.2.5, 3.3, 3.5, 3.6.1-3.6.3, 3.7 and 3.8.

1.1 Quantum correlations

The world of classical physics can be described within the frames of probability theory. This one deals with Kolmogorov probability spaces, that is, triples $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a set, $\mathcal{F}$ is a sub-σ-algebra of 2^Ω and $\mathbb{P} : \mathcal{F} \to [0,1]$ is a σ-additive function such that $\mathbb{P}(\Omega) = 1$, or in other words, a probability measure. In addition, one has certain measurable functions $\Omega \to T$ where T is a measurable space, most often a topological space with its Borel σ-algebra, and even more often simply $T = \mathbb{R}$ with the σ-algebra of Borel sets. These functions are called random variables and intend to model results of measurements which are probabilistic in nature, because of the lack of our complete knowledge of the state of the physical system in question.

One then tries to draw consequences from a physical theory in the form of, say, expected values, moments and covariances of random variables, in order to be tested experimentally. The testing of such statements means performing an experiment sufficiently many times, and comparing the theoretical predictions with the gathered statistical data.

Suppose now that the following experiment is to be carried out[39]: a pair of particles is prepared in a certain randomly chosen state, then they are sent in separate laboratories. In each laboratory, the scientists are free to choose between two properties which they measure, and which can take any of the two values $\{1, -1\}$ for simplicity. Let the values of these properties (which are random variables) be denoted by A_1 and A_2 for the first, and B_1 and B_2 for the second laboratory. Notice first that the value of

$$A_1 B_1 + A_2 B_1 + A_2 B_2 - A_1 B_2 = (A_1 + A_2) B_1 + (A_2 - A_1) B_2 \tag{1.1}$$

can only be 2 or -2 because on the right side precisely one of the terms vanish. It follows that the upper bound [16]

$$\begin{aligned}
\mathbb{E}(A_1B_1) + \mathbb{E}(A_2B_1) &+ \mathbb{E}(A_2B_2) - \mathbb{E}(A_1B_2) \\
&= \mathbb{E}(A_1B_1 + A_2B_1 + A_2B_2 - A_1B_2) \\
&= \sum_{\substack{a_1,a_2,b_1,\\ b_2\in\{-1,1\}}} \mathbb{P}(A_1 = a_1, A_2 = a_2, B_1 = b_1, B_2 = b_2)\cdot \\
&\qquad \cdot (a_1b_1 + a_2b_1 + a_2b_2 - a_1b_2) \\
&\leq \sum_{\substack{a_1,a_2,b_1,\\ b_2\in\{-1,1\}}} \mathbb{P}(A_1 = a_1, A_2 = a_2, B_1 = b_1, B_2 = b_2)\cdot 2 \\
&= 2
\end{aligned} \tag{1.2}$$

holds for the covariances. This bound is a special case of the *Bell inequalities*[4].

Let us be more specific, and suppose that the two particles mentioned above are in fact spins of spin-$\frac{1}{2}$ particles, described by quantum mechanics. The state before the measurement is then roughly a unit vector in a four dimensional complex Hilbert space, $\mathbb{C}^2 \otimes \mathbb{C}^2$ as a composite system. On each factor we have the usual observables X_i, Y_i, Z_i ($i \in \{1, 2\}$) which are the spin components in the direction of the axes of a given Cartesian coordinate system. Their normalization may be chosen so that their possible values are $\{1, -1\}$. Let the state vector of the two spins be

$$\psi = \frac{e_0 \otimes e_1 - e_1 \otimes e_0}{\sqrt{2}} \tag{1.3}$$

where e_0 and e_1 are eigenvectors of the observable Z_i in the respective factors of the tensor product. Let us choose the observables which are to be measured as follows:

$$
\begin{aligned}
A_1 &= Z_1 \\
A_2 &= X_1 \\
B_1 &= -\frac{X_2 + Z_2}{\sqrt{2}}. \\
B_2 &= \frac{Z_2 - X_2}{\sqrt{2}}
\end{aligned}
\tag{1.4}
$$

It is easy to see that the eigenvalues of these operators (i.e. the possible outcomes of each measurement) are $\{1, -1\}$. The covariances are

$$
\langle A_1 B_1 \rangle = \langle A_2 B_1 \rangle = \langle A_2 B_2 \rangle = \frac{1}{\sqrt{2}} \qquad \langle A_1 B_2 \rangle = -\frac{1}{\sqrt{2}} \tag{1.5}
$$

where we used the notation $\langle \cdot \rangle$ for expected values common in quantum mechanics. But then

$$
\langle A_1 B_1 \rangle + \langle A_2 B_1 \rangle + \langle A_2 B_2 \rangle - \langle A_1 B_2 \rangle = 2\sqrt{2} > 2, \tag{1.6}
$$

contradicting eq. (1.2).

This is a variation of a thought experiment given by Bohm[10] who distilled it from the famous EPR paradox after Albert Einstein, Boris Podolsky and Nathan Rosen, who proposed a setting in their 1935 paper[20]. Their intention was to show that quantum mechanics is unable to give a complete description of reality. The violation of Bell's inequality means that quantum mechanics contradicts "common sense" and their conclusion was that it must also contradict Nature. However, it turned out that it is common sense which contradicts Nature, and the predictions of quantum mechanics agree with experimental results[2].

This finding led physicists to reconsider the derivation of Bell's inequalities and its main assumptions which are called *locality* (the assumption that the measurements in the two laboratories do not influence each other) and *realism* (the assumption that the physical properties to be measured exist independently of any observation). Either one or both of these assumptions must therefore be dropped from the picture we have of the world. On the other hand, the Copenhagen interpretation together with the EPR paradox shows that there exists a more general theory of probability. Indeed, probability theory can be rephrased using orthomodular σ-lattices instead of σ-algebras of subsets. Quantum logic deals with this more general setting[9], in which the distributivity of meets and joins (as in Boole lattices) is replaced by the weaker requirement of orthomodularity (as in lattices of projectors in a Hilbert space). Upon this new logic, a new theory of probability can be built, far richer than the traditional one, leading in turn to a new theory of information, known as quantum information theory.

A large part of quantum information theory deals with states like in eq. (1.3). These states are special in that they cannot be written as tensor products of vectors from the Hilbert spaces of the individual subsystems. Such states are called *entangled*, and these are precisely the states which bear in themselves the possibility to violate Bell's inequalities [55], confirming Schrödinger's opinion on entanglement: "I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought."[43].

1.2 Entanglement classification problem

The set of separable (not entangled) states has the interesting property that it is fixed under the action of the group of invertible *local* transformations, that is, linear transformations of the composite Hilbert space which are tensor products of invertible linear transformations of the individual subsystems' Hilbert spaces. Moreover, thinking projectively (in effect disregarding the norm and the overall phase), this set is also stable under the action of the semigroup of local (not necessarily invertible) tranformations as long as the result is not the 0 vector. This observation can be interpreted as follows: no entanglement can be created via only *local* manipulations of the subsystems, without interaction between the parts.

A separable state can not be converted into an entangled one. But if we start with an entangled state, is it possible to reach any other entangled state with local transformations? If not, then what are the possible obstructions which prevent us from transforming one state into another? Questions like these belong to the subject of entanglement theory, and the answers have only been found for a limited number of quantum systems so far.

Also, the answer depends on our choice of admissible transformations. One possible requirement is that the transformation succeeds with probability 1. Alternatively, we may be satisfied with the weaker requirement that we succeed in transforming one state to another with a positive probability. In either case, we usually enable that manipulations performed on a particle depend on results of measurements on another one. The former approach leads to the LOCC (local operations and classical communication) classification problem, while the latter one to SLOCC (stochastic LOCC).

In both cases, the admissible transformations induce a preorder $\precsim$ on the set of states of the composite system. Preorders[13] are reflexive and transitive relations, in our case reflexivity follows from the fact that "doing nothing can be done locally", and transitivity means that performing one local transformation after another yields again a local transformation. A standard construction[13] in order theory allows us to pass to equivalence classes with respect to $\precsim \cap \precsim^{-1}$, and on the quotient set this induces a partial order.

Thus, the classification problem can be split into two parts: first we would like to identify equivalently entangled states, then we would like to find out if we are able to convert representatives of any two given equivalence classes into each other via local transformations. The first problem may be solved by finding a complete set of *invariant*s with respect to the action of each group of invertible local tranformations, while the solution to the second problem means finding sufficiently many *entanglement monotone*s, that is, functions from the preordered set of states to, say, $([0, 1], \leq)$ which are monotone with respect to the preorder relations.

It turns out that equivalence classes of states are orbits under the action of certain groups: for SLOCC classification of states in multipartite quantum systems with distinguishable constituents, this group is the product of the local general linear groups, called sometimes (by slight abuse of language) the SLOCC group, while for LOCC classification of these states, the group is the product of unitary groups of the subsystems (LU group). General (not necessarily invertible) SLOCC operations correspond to tensor products of endomorphisms of the respective Hilbert spaces [7].

1.3 Quantification of entanglement

It turned out that entangled states have many interesting applications like quantum state teleportation[6] or superdense coding[8], and they also appear during quantum computations[29]. As entangled states are precisely the ones that enable us to outperform classical information-processing protocols[41], we can regard entanglement as a resource like energy or entropy. As such, it is desireable to find means to quantify it. It is not clear, however, what properties must a function of the quantum state have in order to be "useful". For a review of various aspects of entanglement as a resource, we refer the reader to [27].

From a mathematical point of view, the problem of understanding entanglement is completely stated in the previous section. From the physicists' point of view, however, not every invariant or monotone is equally useful. It is preferable for example, if a quantity can be given a clear *operational* meaning, or any other direct phyisical interpretation.

A commonly used approach is that we single out a special state, usually the Bell state

$$\Phi = \frac{e_0 \otimes e_0 + e_1 \otimes e_1}{\sqrt{2}} \tag{1.7}$$

in $\mathbb{C}^2 \otimes \mathbb{C}^2$, and use it as a standard unit of entanglement. There are essentially two ways to fulfil this program: for a given state ψ, either we try to create as many copies of ψ as possible (this number will be denoted by m) with high fidelity from n Bell states using only local transformations, and calculate the limiting value of m/n as n approaches infinity, or one asks how many Bell states can be prepared from m copies of ψ, again with high fidelity, and looking at $\lim m/n$ where n is the maximal achievable number of Bell states. It is a nontrivial fact that the two notions – called *entanglement of formation* and *distillable entanglement* respectively – actually coincide for pure states[5].

Unfortunately, explicit formulae for these and other operationally defined quantities are very hard to obtain. Although entanglement in bipartite pure states is more or less well understood, our knowledge on multipartite quantum systems or mixed state entanglement is very limited, the basic result being the understanding of the mixed states of a two-qubit system [53]. For example, it is not known how the distillable entanglement of a mixed bipartite state can be calculated.

Because of these enormous difficulties, one cannot do better than temporarily abandon the requirement that the quantities have a clear physical meaning, and seeks simply for entanglement monotones or polynomial invariants, which, at least, enable us to distinguish between different types of entanglement and say something about the ability to convert states into each other via local operations. Especially as the problem of finding polynomial invariants (and identifying fundamental ones which generate the algebra of polynomial invariants) is also very difficult in itself for general quantum systems.

1.4 Bipartite quantum systems

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite dimensional Hilbert spaces, corresponding to two subsystems. The state space of the composite system is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. If $\dim \mathcal{H}_i = n_i$ then $\dim \mathcal{H} = n_1 n_2$. A pure state of the composite system is a one dimensional subspace in $\mathcal{H}$, the orthogonal projection on which is a density operator $\varrho \in \operatorname{End} \mathcal{H}$.

The reduced density matrices of the subsystems are then $\mathrm{Tr}_2\, \varrho = \varrho_1 \in \mathrm{End}\, \mathcal{H}_1$ and $\mathrm{Tr}_1\, \varrho = \varrho_2 \in \mathrm{End}\, \mathcal{H}_2$. A characteristic property of entanglement is that a density operator describing an entangled pure state has mixed reduced density operators. One can then hope that the eigenvalues of the reduced operators – which are local unitary invariants – encode entanglement properties of the pure state. Indeed, it turns out that the classification problem for both the LOCC and the SLOCC transformations can be solved using only the multiset of eigenvalues of the reduced density operators.

Let us begin with a definition:

Definition. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ be two vectors, and let $x^\downarrow = (x_1^\downarrow, \ldots, x_n^\downarrow)$ and $y = (y_1^\downarrow, \ldots, y_n^\downarrow)$ be their permutations ordered nonincreasingly. We say that y majorizes x ($x \preccurlyeq y$) if

$$\forall k \in \{1, \ldots, n\} : \sum_{i=1}^{k} x_i^\downarrow \leq \sum_{i=1}^{k} y_i^\downarrow \tag{1.8}$$

holds with equality for $k = n$.

For a self-adjoint operator A, let $\lambda(A)$ denote the vector of eigenvalues of A in nonincreasing order disregarding any 0 eigenvalues. Then, in our setting, Schmidt decomposition[42] implies that $\lambda(\varrho_1) = \lambda(\varrho_2)$ for any pure state ϱ of $\mathcal{H}$. Now we are ready to state a result about the bipartite entanglement of pure states[38]:

Theorem 1.4.1. *Let ψ and φ be pure states of the bipartite quantum system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Then ψ can be converted with probability one to φ by LOCC if and only if* $\lambda(\mathrm{Tr}_2(\psi\psi^*)) \preccurlyeq \lambda(\mathrm{Tr}_2(\varphi\varphi^*))$

In particular, two states are LOCC-equivalent iff the multisets of eigenvalues of their respective reduced density matrices are equal.

Similarly, ability to convert one pure state into another by SLOCC can be decided using the eigenvalues, but this time not the values themselves, but the number of nonzero eigenvalues – called the *Schmidt rank* – is relevant[18]:

Theorem 1.4.2. *Let ψ and φ be pure states of the bipartite quantum system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Then ψ can be converted with nonzero probability to φ by SLOCC if and only if* $\mathrm{rk}\, \mathrm{Tr}_2(\psi\psi^*) \geq \mathrm{rk}\, \mathrm{Tr}_2(\varphi\varphi^*)$

In particular, two states are SLOCC-equivalent iff the ranks of the respective reduced density matrices are equal.

What is left is to enumerate the possible equivalence classes for each classification scheme. For SLOCC, these classes are clearly labelled by the

Schmidt rank, hence there are $\min\{n_1, n_2\}$ equivalence classes in $\mathcal{H}$. In the case of LOCC, however, there are infinitely many equivalence classes labelled by $\lambda(\mathrm{Tr}_2(\psi\psi^*))$. The same information is contained in the characteristic polynomial of $\mathrm{Tr}_2(\psi\psi^*)$, whose coefficients are polynomial functions of ψ, and symmetric polynomials of the eigenvalues.

In this case we can describe the algebra of polynomial LU-invariants explicitely: it is generated by elementary symmetric polynomials of degree at most $\min\{n_1, n_2\}$ of the eigenvalues. Alternatively, we may use power sum symmetric polynomials of them of degree at most $\min\{n_1, n_2\}$, that is, the quantities $\mathrm{Tr}((\varrho_1)^d)$ where $d = \{1, \ldots, \min\{n_1, n_2\}\}$. Of course, $d = 1$ tells us nothing when we restrict ourselves to normalized states, therefore we can conclude that the algebra of polynomial LU-invariants is generated by $\min\{n_1, n_2\} - 1$ algebraically independent invariants, one in each graded part of bidegree (d, d) (degree d both in the coefficients of ψ and in their conjugates) where $d = \{2, \ldots, \min\{n_1, n_2\}\}$.

Chapter 2

Entanglement of distinguishable subsystems

In this chapter we study in detail the theory of entanglement in the case of quantum systems consisting of distinguishable particles. In many applications we encounter localized particles far enough from each other so that the interaction between them is negligable. In this situation, the amount of quantum correlations between the particles cannot increase. Inequivalent types of quantum correlations are encoded in the values of functions on the state space which are invariant under invertible local transformations. The main task is therefore to find such invariants, called entanglement measures in this context.

Section 2.1 summarizes the basic concepts of quantum mechanics, with an emphasis on local observables of composite quantum systems.

In section 2.2 we discuss the possible time evolutions of a quantum system, not neccessarily isolated from its environment, in which case the evolution is not described by unitary transformations any more. We specialize to time evolutions representing local manipulations of a multipartite quantum system. Three semigroups are introduced, the SLOCC semigroup and the LU and SLOCC groups which contain the possible local transformations in various types of protocols.

The purpose of section 2.3 is to mathematically formulate the problem of entanglement classification. It is also shown that this problem leads to the study of invariant fuctions under certain group actions.

In section 2.4 we make the notion of a (real) polynomial on a vector space precise, and we also compare some aspects of the LU and SLOCC classification problems.

Section 2.5 prepares a more detailed study of LU-invariant polynomials by analyzing the extra structure on the symmetric algebra of a Hilbert space induced by the inner product.

In section 2.6 we first collect some general facts about LU-invariants, then calculate all fourth order LU-invariants for a composite quantum system with arbitrary dimensional single particle state spaces. Then analogous invariants are introduced with every even degree. In the special case when all the single particle state spaces are isomorphic, the set of fourth order LU-invariants with permutation symmetry is also described.

In section 2.7 we find the generating function of the sequence of dimensions of the space of degree $2m$ LU-invariants of quantum systems with varying particle number and at least m dimensional single particle state spaces.

In section 2.8 a formula is derived for the dimension of degree $2m$ LU-invariants for a quantum system with k subsystems and sufficiently large single particle state spaces. The result is interpreted as the Hilbert series of the inverse limit of the algebras of LU-invariant polynomials of k-partite quantum systems with state spaces of varying dimension. A conjecture is formulated regarding the structure of this algebra.

In section 2.9 some relationships with SLOCC-invariants are highlighted.

In section 2.10 it is shown that finding LU-invariant polynomials of mixed states is equivalent to finding LU-invariant polynomials of pure states with one extra subsystem added.

2.1 Introduction

In quantum mechanics a complex separable Hilbert space is associated to every physical system, its state space. A Hilbert space is a vector space equipped with an inner product such that the space is complete with respect to the metric induced by the inner product. In quantum information theory one usually deals with finite dimensional state spaces (although the infinite dimensional case is also considered by some authors, for a review on this topic see ref.[1]), removing the need to bother with the last requirement and separability.

At this point, two kinds of states can be distinguished:

Definition. Let $\mathcal{H}$ be the state space of a quantum system. A state is a positive element in $\mathrm{End}(\mathcal{H})$ with trace 1. A state ϱ is pure, if $\mathrm{rk}\,\varrho = 1$, otherwise it is called mixed.

Clearly, a pure state can be identified with its image, a one dimensional subspace of $\mathcal{H}$. If $\psi \in \mathcal{H}\setminus\{0\}$, then ψ determines a unique pure state, namely $\mathbb{C}\psi$, therefore nonzero vectors in $\mathcal{H}$ are also often called pure states.

The state space of a composite quantum system of distinguishable subsystems is the tensor product of the Hilbert spaces of the individual subsystems (see sections A.3 and A.4 in the appendix for the definition and properties of the tensor product). From the vectors of the individual Hilbert spaces one can build elementary tensors which have a distinguished role:

Definition. Let $k \in \mathbb{N}$ and $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_k$ be the state space of a composite quantum system, where $\mathcal{H}_i$ $(1 \leq i \leq k)$ are Hilbert spaces. A pure state $\mathbb{C}\psi$ is separable if it can be written in the form $\psi = \psi_1 \otimes \cdots \otimes \psi_k$ where $\psi_i \in \mathcal{H}_i$ $(1 \leq i \leq k)$. A pure state which is not separable is called entangled.

A mixed state is called separable if it is a convex combination of separable pure states, otherwise it is entangled.

In quantum mechanics, the result of a measurement is described by a random variable which is in this case an element of $\mathrm{End}(\mathcal{H})$:

Definition. Let $\mathcal{H}$ be the state space of a quantum system. A self-adjoint operator $A \in \mathrm{End}(\mathcal{H})$ is called an observable.

Suppose that $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_k$ (where $k \in \mathbb{N}$) is the state space of a composite quantum system. A local observable (or single-particle observable) is a self-adjoint operator of the form

$$A = A_1 \otimes A_2 \otimes \cdots \otimes A_k \in \mathrm{End}(\mathcal{H}_1) \otimes \cdots \otimes \mathrm{End}(\mathcal{H}_k) \simeq \mathrm{End}(\mathcal{H}) \quad (2.1)$$

where at most one of $A_1, A_2, \ldots, A_k$ differs from the identity operator of the respective Hilbert space.

In a composite quantum system to each observable $A_i \in \mathrm{End}(\mathcal{H}_i)$ of a subsystem one can associate a local observable $\mathrm{id}_{\mathcal{H}_1} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}_{i-1}} \otimes A_i \otimes \mathrm{id}_{\mathcal{H}_{i+1}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}_k}$, and this mapping is an algebra homomorphism (see section A.3 in the appendix) $\mathrm{End}(\mathcal{H}_i) \to \mathrm{End}(\mathcal{H})$. This corresponds to the fact that a property of a subsystem which can be measured is also a measurable property of the composite system.

2.2 Time evolution

Quantum mechanics is not only about describing a state and the probability distribution of observables in a particular moment, but also about how the state evolves over time. One usually wants to find out how the state of an isolated quantum system changes over a given time interval. In other cases, one would like to consider a quantum mechanical system which interacts with its environment, and determine its state after some time has elapsed. In the former case, time evolution is described by a one-parameter unitary group generated by the Hamiltonian, an observable with a special role, while in the latter case, one regards the quantum system in question together with its environment as an isolated composite system, and derives how the unitary evolution of the whole is reflected in the evolution of the parts.

In quantum information theory, time evolution is studied in order to be able to determine how the control parameters of a given quantum system need to be tuned to perform the desired manipulations on it. In addition, understanding the general properties of special types of time evolution enables us to tackle the problem of undesirable interactions with the environment which would otherwise ruin the process of computation or information processing.

Suppose that we have an isolated quantum system with state space $\mathcal{H}$, and the Hamiltonian (the observable corresponding to energy) is $H = H^* \in \operatorname{End}(\mathcal{H})$. Let the initial state be $\varrho \in \operatorname{End}(\mathcal{H})$. Then after a time interval t, the state of the system will be

$$\varrho(t) = e^{-\frac{i}{\hbar}tH}\varrho e^{\frac{i}{\hbar}tH} \tag{2.2}$$

From eq. (2.2) we already see that $\varrho(t)$ is unitary equivalent to ϱ. In particular, a pure state remains pure under such an evolution. More generally, if ϱ is a convex combination of pure states, then $\varrho(t)$ will be the convex combination of evolved versions of the same pure states with the same coefficients.

The case of an open quantum system is more subtle. Let $\mathcal{H}$ be the state space of a quantum mechanical system which interacts with its environment, whose state space is $\mathcal{H}_E$. Let the initial state of the joint system be $\varrho\otimes\varrho_E$, and the Hamiltonian governing the evolution of the joint system be H. Without loss of generality we can assume that ϱ_E is a pure state, the projection to the subspace $\mathbb{C}\psi \leq \mathcal{H}_E$ where $\|\psi\| = 1$. Let $\{e_j\}_{j\in J}$ be an orthonormal basis in $\mathcal{H}_E$. Then

$$\begin{aligned}\varrho(t) &= \operatorname{Tr}_E(e^{-\frac{i}{\hbar}tH}\varrho\otimes\varrho_E e^{\frac{i}{\hbar}tH}) \\ &= \sum_{j\in J}\langle e_j, e^{-\frac{i}{\hbar}tH}\varrho\otimes(\psi\psi^*)e^{\frac{i}{\hbar}tH}e_j\rangle_{\mathcal{H}_E} \\ &= \sum_{j\in J} E_j\varrho E_j^*\end{aligned}, \tag{2.3}$$

where $\operatorname{Tr}_E = \operatorname{id}_{\mathcal{H}}\otimes\operatorname{Tr}_{\mathcal{H}_E}$ is the partial trace and $E_j = \langle e_j, e^{-\frac{i}{\hbar}tH}\psi\rangle_{\mathcal{H}_E} \in \operatorname{End}(\mathcal{H})$. (the definition of the partial trace and the "partial matrix element" can be found in the appendix in section A.4)

The operators $\{E_j\}_{j\in J}$ satisfy a certain completeness relation:

$$1 = \mathrm{Tr}\left(\sum_{j\in J} E_j \varrho E_j^*\right) = \mathrm{Tr}\left(\sum_{j\in J} E_j^* E_j \varrho\right) \tag{2.4}$$

for all states ϱ implies that

$$\sum_{j\in J} E_j^* E_j = \mathrm{id}_{\mathcal{H}} \tag{2.5}$$

holds. In order to be able to include more genaral operations in the description – such as discarding the quantum system in the case of a certain measurement outcome – the equality in eq. (2.5) must be weakened to

$$\sum_{j\in J} E_j^* E_j \leq \mathrm{id}_{\mathcal{H}}\,. \tag{2.6}$$

Indeed, suppose that $P \in \mathrm{End}(\mathcal{H})$ is a projection, and we wish to measure if the state is in $\mathrm{ran}\, P$, and continue the process only in this case, but the initial state is only known to be ϱ. Then the state will be

$$\frac{P\varrho P}{\mathrm{Tr}\, P\varrho P} \tag{2.7}$$

after the measurement with probability $\mathrm{Tr}\, P\varrho P$. In general, the probability of success is given by

$$\mathrm{Tr}\left(\sum_{j\in J} E_j \varrho E_j^*\right) \tag{2.8}$$

This process with measurements and other admissible interactions with the environment enables us to perform more general transformations than in eq. (2.2), provided we are satisfied with a smaller but still nonzero probability of success. In particular, if $A \in \mathrm{End}(\mathcal{H})$ is an *arbitrary* operator such that $\mathrm{ran}\, \varrho \not\leq \ker A$ then we are able to implement the transformation

$$\varrho \mapsto \frac{A\varrho A^*}{\mathrm{Tr}(A\varrho A^*)} \tag{2.9}$$

with nonzero probability since $\exists \alpha \in \mathbb{R}^\times$ such that $0 < (\alpha A)^*(\alpha A) \leq \mathrm{id}_{\mathcal{H}}$ and

$$\frac{\alpha^2 A\varrho A^*}{\mathrm{Tr}(\alpha^2 A\varrho A^*)} = \frac{A\varrho A^*}{\mathrm{Tr}(A\varrho A^*)} \tag{2.10}$$

holds.

2.2.1 Semigroups of local transformations

In entanglement theory, our aim is to study *nonlocal* properties of quantum states, that is properties which cannot be created by *local* operations. In the previous section we have seen that any element of $\mathrm{End}(\mathcal{H})$ can be applied to a quantum system with state space $\mathcal{H}$ with a nonzero probability of success, and any unitary operator in $\mathrm{End}(\mathcal{H})$ can be applied with probability one.

Imagine now that $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ ($k \in \mathbb{N}$) is the state space of a composite quantum system, and there can be no (direct or indirect) interaction between the subsystems, for example because they are in spacelike separated laboratories. A manipulation performed on the ith subsystem is still of the form of eq. (2.9), but instead of A being an arbitrary element of $\mathrm{End}(\mathcal{H})$, it can only be a member of the algebra of local observables of the ith subsystem, that is, $A = \mathrm{id}_{\mathcal{H}_1} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}_{i-1}} \otimes A_i \otimes \mathrm{id}_{\mathcal{H}_{i+1}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}_k}$ (see section C.2 for a proof).

It can be shown[7] that if a *reversible* operation of this type can be performed with probability one, then $A_i \in \mathrm{End}(\mathcal{H}_i)$ can also be chosen to be unitary. These considerations motivate the following definitions:

Definition. Let $k \in \mathbb{N}$ and $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ be the state space of a composite quantum system. The SLOCC semigroup (stochastic local operations and classical communication) of $\mathcal{H}$ is the subsemigroup

$$\{A_1 \otimes \cdots \otimes A_k | \forall i \in [k] : A_i \in \mathrm{End}(\mathcal{H}_i)\} \tag{2.11}$$

of $\mathrm{End}(\mathcal{H})$.

The SLOCC group of $\mathcal{H}$ is the group of invertible elements in the SLOCC semigroup.

The LU group (local unitary) is the subgroup of unitary elements in the SLOCC group.

$\mathrm{id}_{\mathcal{H}} = \mathrm{id}_{\mathcal{H}_1} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}_k}$ implies that the SLOCC semigroup is actually a monoid (semigroup with identity).

The kernel of the map $(A_1, \ldots, A_k) \mapsto A_1 \otimes \cdots \otimes A_k$ is the subsemigroup $\{(a_1 \mathrm{id}_{\mathcal{H}_1}, \ldots, a_k \mathrm{id}_{\mathcal{H}_k}) | a_1 a_2 \cdots a_k = 1\}$, therefore the SLOCC semigroup is isomorphic to

$$\{A_1 \otimes \cdots \otimes A_k | \forall i \in [k] : A_i \in \mathrm{End}(\mathcal{H}_i)\} \simeq \mathrm{End}(\mathcal{H}_1) * \mathrm{End}(\mathcal{H}_2) * \cdots * \mathrm{End}(\mathcal{H}_k) \tag{2.12}$$

where $*$ denotes the central product with identified centers.

Similarly, if $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ where $\dim \mathcal{H}_i = n_i$ and $\dim \mathcal{H} = n = n_1 \cdot \cdots \cdot n_k$, then

$$\det(A_1 \otimes \cdots \otimes A_k) = \prod_{i=1}^{k} \det(A_i)^{\frac{\Pi_{j=1}^{k} n_k}{n_i}} \tag{2.13}$$

which implies that the SLOCC group of $\mathcal{H}$ is

$$\{A_1 \otimes \cdots \otimes A_k | \forall i \in [k] : A_i \in GL(\mathcal{H}_i)\} \simeq GL(\mathcal{H}_1) * GL(\mathcal{H}_2) * \cdots * GL(\mathcal{H}_k) \tag{2.14}$$

$*$ denoting again the central product with identified centers (the center of each $GL(\mathcal{H}_i)$ being $\mathbb{C}^\times \operatorname{id}_{\mathcal{H}_i}$, identified with $\mathbb{C}^\times$). In particular, the SLOCC group of $\mathcal{H}$ is a quotient of $GL(\mathcal{H}_1) \times GL(\mathcal{H}_2) \times \cdots \times GL(\mathcal{H}_k)$.

And finally, as $(A_1 \otimes \cdots \otimes A_k)^* = A_1^* \otimes \cdots \otimes A_k^*$ and $A_1 \otimes \cdots \otimes A_k \in \mathbb{C}^\times \operatorname{id}_{\mathcal{H}}$ iff $\forall i \in [k] : A_i \in \mathbb{C}^\times \operatorname{id}_{\mathcal{H}_i}$, it follows that the LU group is

$$\{A_1 \otimes \cdots \otimes A_k | \forall i \in [k] : A_i A_i^* = \operatorname{id}_{\mathcal{H}_i}\} \simeq U(\mathcal{H}_1) * U(\mathcal{H}_2) * \cdots * U(\mathcal{H}_k) \tag{2.15}$$

where $*$ is again the central product with identified centers (those being all isomorphic to $U(1)$). In particular, the LU group is a quotient of $U(\mathcal{H}_1) \times U(\mathcal{H}_2) \times \cdots \times U(\mathcal{H}_k)$.

2.3 Monoid actions

In this section we would like to formalize the notion of "performing a local manipulation on a pure state". This will enable us to express mathematically what it means for two states to be entangled in the same way. Let us start with a definition[14]:

Definition. Let M be a monoid with 1 denoting the identity element, and S a set. We say that a map $M \times S \to S$ sending $(m, s) \mapsto m \cdot s$ is an action of M on S if the following conditions are met:

1. $\forall s \in S : 1 \cdot s = s$
2. $\forall s \in S : \forall m, n \in M : m \cdot (n \cdot s) = (mn) \cdot s$

One can also think of a monoid as a category with one object. Then a monoid action is simply a functor from this category to **Set** (see section A.1 in the appendix for the definition).

Given a set S on which a monoid M acts, we can define a binary relation on S as follows. For $s_1, s_2 \in S$, let $s_2 \precsim s_1$ iff there exists an element $m \in M$

such that $s_2 = m \cdot s_1$. Clearly, $s = 1 \cdot s$ implies that $\precsim$ is reflexive, and $(s_2 = m \cdot s_1, s_3 = n \cdot s_2) \Rightarrow s_3 = (nm) \cdot s_1$ implies that $\precsim$ is transitive, thus $\precsim$ is a preorder on S.

The next step is to define an equivalence relation based on $\precsim$. Let us write $s_1 \sim s_2$ iff $s_1 \precsim s_2$ and $s_2 \precsim s_1$. Indeed, this relation is symmetric by definition, reflexive and transitive by the same properties of $\precsim$. Let $[s] = \{s' \in S | s \sim s'\}$ denote the equivalence class of s. On the set

$$S/\sim = \{[s] | s \in S\} \tag{2.16}$$

of equivalence classes $\precsim$ induces a relation $\leq$ by letting $[s_1] \leq [s_2]$ iff $s_1 \precsim s_2$. By construction this relation is well-defined, reflexive, antisymmetric and transitive, that is, a partial order. In the special case when M is a group, $\leq$ is the identity relation.

Now we return to our special monoids introduced in the previous section. First note that if $M \times S \to S$, $(m, s) \mapsto m \cdot s$ is an action of the monoid M and M' is another monoid, $f : M' \to M$ is a homomorphism, then $(m', s) \mapsto f(m') \cdot s$ defines an action of M' on S. In particular, instead of actions of the SLOCC semigroup, the SLOCC group and the LU group, we can consider actions of $\mathrm{End}(\mathcal{H}_1) \times \cdots \times \mathrm{End}(\mathcal{H}_k)$, $GL(\mathcal{H}_1) \times \cdots \times GL(\mathcal{H}_k)$ and $U(\mathcal{H}_1) \times \cdots \times U(\mathcal{H}_k)$, respectively. The advantage is that these groups have a simpler representation theory than the central products (see sections B.1 and B.4 in the appendix). As the latter two monoids are subsemigroups of the first one, we need only to define the action of the first one.

We will let these monoids act on $\mathcal{H}$ in the obvious way: for $\psi \in \mathcal{H}$ and $A = (A_1, \cdots, A_k) \in \mathrm{End}(\mathcal{H}_1) \times \cdots \times \mathrm{End}(\mathcal{H}_k)$ we let

$$A \cdot \psi = (A_1 \otimes \cdots \otimes A_k)\psi \tag{2.17}$$

that is, induced from the action of $\mathrm{End}(\mathcal{H})$ on $\mathcal{H}$.

Now we are ready to formulate the classification problem of entanglement of pure states:

Definition. Let $k \in \mathbb{N}$ and $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ be the state space of a composite quantum system. Then two states ψ, φ are

- SLOCC-equivalently entangled iff $\psi \sim \varphi$ where $\sim$ is the equivalence relation induced by the action of the SLOCC group
- LU-equivalently entangled iff $\psi \sim \varphi$ where $\sim$ is the equivalence relation induced by the action of the LU group
- ψ is SLOCC reducible to φ iff $\varphi \precsim \psi$ where $\precsim$ is the preorder relation induced by the action of the SLOCC semigroup

Note that we could also define a preorder using not necessarily invertible LOCC operations, but this can only be done if we consider a monoid action on mixed states first, then restrict the induced preorder relation to pure states. We do not wish to do this here, as we will only discuss the problem of equivalence under local transformations, which leads us back to the LU group.

The main problem of entanglement theory is to find the set of equivalence classes with respect to $\sim$ (induced from the actions of both groups) and provide necessary and sufficient criteria for two states $\psi, \varphi \in \mathcal{H}$ to satisfy $\psi \precsim \varphi$.

In the case of the SLOCC semigroup, therefore, for $\psi, \varphi \in \mathcal{H}$ the relation $\varphi \precsim \psi$ holds precisely when there exist operators $\{A_i\}_{i=1}^k$ ($\forall i \in \{1, \ldots, k\}$: $A_i \in \mathrm{End}(\mathcal{H}_i)$) such that $\varphi = (A_1 \otimes \cdots \otimes A_k)\psi$. We would like to express the equivalence relation $\psi \sim \varphi$ defined by $\psi \precsim \varphi \wedge \varphi \precsim \psi$ more explicitely. We have the following lemma, proven in ref.[18](the proof here is a slightly simplified version):

Lemma 2.3.1. *Suppose that* $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$, $\psi, \varphi \in \mathcal{H}$. *Then the following conditions are equivalent:*

1. $\exists (A_1, \ldots, A_k), (B_1, \ldots, B_k) \in \mathrm{End}(\mathcal{H}_1) \times \cdots \times \mathrm{End}(\mathcal{H}_k)$ *such that* $\varphi = (A_1 \otimes \cdots \otimes A_k)\psi$ *and* $\psi = (B_1 \otimes \cdots \otimes B_k)\varphi$.
2. $\exists (C_1, \ldots, C_k) \in GL(\mathcal{H}_1) \times \cdots \times GL(\mathcal{H}_k)$ *such that* $\varphi = (C_1 \otimes \cdots \otimes C_k)\psi$

Proof. Considering one factor at a time, we can successively replace the operators A_i with inverible ones as follows. For $i = 1$, φ and $\psi' := (\mathrm{id}_{\mathcal{H}_1} \otimes A_2 \otimes \cdots \otimes A_k)\psi$ can be considered as elements of $\mathrm{Hom}_{\mathbf{Vect}_\mathbb{C}}(\mathcal{H}_2^* \otimes \cdots \otimes \mathcal{H}_k^*, \mathcal{H}_1)$. Then $A_1 \circ \psi' = \varphi$ and $B_1 \circ \varphi \circ (B_2^* A_2^* \otimes \cdots \otimes B_k^* A_k^*) = \psi'$, therefore

$$\dim \mathrm{ran}\, \varphi = \dim \mathrm{ran}\, A_1 \circ \psi' \leq \dim \mathrm{ran}\, \psi' \leq \dim \mathrm{ran}\, B_1 \circ \varphi \leq \dim \mathrm{ran}\, \varphi \quad (2.18)$$

which implies that $\dim \mathrm{ran}\, \varphi = \dim \mathrm{ran}\, \psi'$. On the other hand, $\dim \mathrm{ran}\, \varphi = \dim \mathrm{ran}\, \psi' - \dim(\ker A_1 \cap \mathrm{ran}\, \psi')$, therefore $\ker A_1 \cap \mathrm{ran}\, \psi' = \{0\}$. Now let U be a direct complement of $\ker A_1$ containing $\mathrm{ran}\, \psi'$, and let V be a direct complement of $\mathrm{ran}\, A_1$. Let $A_1' := A_1|_U : U \to \mathrm{ran}\, A_1$, and let $A_1'' : \ker A \to V$ be an arbitrary isomorphism (recall that $\mathcal{H}_1 / \mathrm{ran}\, A_1 \simeq \ker A_1$). Then $A_1' \oplus A_2'' : \mathcal{H}_1 \to \mathcal{H}_1$ is an invertible operator such that $(A_1' \oplus A_2'') \circ \psi' = \varphi$. We can proceed similarly with the other subsystems, arriving to an operator of the desired form.

Conversely, if $\exists (C_1, \ldots, C_k) \in GL(\mathcal{H}_1) \times \cdots \times GL(\mathcal{H}_k)$ such that $\varphi = (C_1 \otimes \cdots \otimes C_k)\psi$, then for $(C_1, \ldots, C_k) \in \mathrm{End}(\mathcal{H}_1) \times \cdots \times \mathrm{End}(\mathcal{H}_k)$ and $(C_1^{-1}, \ldots, C_k^{-1}) \in \mathrm{End}(\mathcal{H}_1) \times \cdots \times \mathrm{End}(\mathcal{H}_k)$, $\varphi = (C_1 \otimes \cdots \otimes C_k)\psi$ and $\psi = (C_1^{-1} \otimes \cdots \otimes C_k^{-1})\varphi$. $\square$

Thus, equivalence classes of the $\sim$ relation obtained from the action of the SLOCC semigroup are exactly the orbits under the action of the SLOCC group.

Similarly, if we are given two pure states ψ and φ such that $\psi \precsim \varphi$ and $\varphi \precsim \psi$ under LOCC, then there exist unitary operators $(A_1, \ldots, A_k) \in U(\mathcal{H}_1) \times \cdots \times U(\mathcal{H}_k)$ such that $\varphi = (A_1 \otimes \cdots \otimes A_k)\psi$ [7].

We are interested in finding real valued functions on $\mathcal{H}$ in order to be able to characterize $\precsim$ with them. Clearly, if $\varphi \precsim \psi$ and $\psi \not\precsim \varphi$ then the function

$$f_\psi(\psi') := \begin{cases} 1 & \text{if } \psi \precsim \psi' \\ 0 & \text{otherwise} \end{cases} \tag{2.19}$$

is a monotone function from $(\mathcal{H}, \precsim)$ to $(\mathbb{R}, \leq)$ which separates φ and ψ. For if we take two elements $\varphi \precsim \varphi'$ in $\mathcal{H}$ such that $f_\psi(\varphi) = 1$ then $\psi \precsim \varphi \precsim \varphi'$ implies that $f_\psi(\varphi') = 1$. Thus, such functions separate inequivalently entangled states.

In other words, there exists an index set I, and a family of functions $(f_i)_{i\in I}$ such that $f_i : \mathcal{H} \to \mathbb{R}$, and that $\prod_{i\in I} f_i : \mathcal{H} \to \mathbb{R}^I$ is an order-embedding into $\mathbb{R}^I$ with the componentwise partial order. It is an interesting question whether I can be taken to be finite, and whether the functions f_i can be chosen to be "nicer", for example continuous or differentiable.

In this thesis only the problem of finding the equivalence classes of the two $\sim$ relations – that is, orbits under the action of the SLOCC and LU groups – is considered. Therefore, we shall drop the requirement of being monotone, and look instead for functions which satisfy the weaker condition of being constant along the orbits, that is, which are invariant functions.

Various results exist on the invariants under the action of both groups for quantum systems with small dimensional state spaces and for a few constituents, see eg. [52, 47, 48, 33, 34]

2.4 Invariant polynomials

As the set of invariant functions from $\mathcal{H}$ to $\mathbb{R}$ is too large to be handled, in the following we would like to restrict the class of invariant functions considered, while still retaining the property that the subset of functions separate the orbits.

In which direction to proceed, depends on our choice of the group of local transformations to be considered. Both have advantages and disadvantages. As the dimension of the SLOCC group is larger, there are fewer orbits, their number can even be finite in low dimensional cases with a small number of subsystems. Also, the SLOCC group is algebraic over the complexes, an algebraically closed field, while the LU group is only algebraic over the reals. On the other hand, the LU group is a compact Lie group, while the SLOCC group is only locally compact.

Having a regular action of an algebraic group in both cases makes it tempting to concentrate on invariant polynomial functions. However, invariant polynomials do *not* separate every orbit when the SLOCC group is considered, therefore we opt to look for LU-invariants, in which case the situation is easier, but we still have to consider functions which are polynomial in the coefficients of the state and their conjugates. Occasionally, however, we will be able to derive some facts on the SLOCC-invariant polynomials as well, due to the close connection between the representation theories of the two groups (see section B.4 in the appendix for details). The first step will be to make the notion of such polynomials precise, and define it in a coordinate-free manner at the same time.

Let V be a finite dimensional complex vector space. A vector $v \in V$ determines an element

$$\tilde{v} := (1, v, v \otimes v, v \otimes v \otimes v, \ldots) \in \prod_{m=0}^{\infty} \bigotimes_{i=1}^{m} V \tag{2.20}$$

This vector space is the dual of the tensor algebra (see sections A.2 and A.5 in the appendix)

$$T(V^*) = \coprod_{m=0}^{\infty} \bigotimes_{i=1}^{m} V^* = \bigoplus_{m=0}^{\infty} \bigotimes_{i=1}^{m} V^* \tag{2.21}$$

From the pairing

$$\left(\prod_{m=0}^{\infty} \bigotimes_{i=1}^{m} V \right) \times T(V^*) \to \mathbb{C} \tag{2.22}$$

we have that there exists a map $V \times T(V^*) \to \mathbb{C}$ defined by

$$(v, \varphi) \mapsto \varphi((1, v, v \otimes v, v \otimes v \otimes v, \ldots)) = \varphi(\tilde{v}) \tag{2.23}$$

For any fixed $v \in V$, this map is an algebra morphism from $T(V^*)$ to $\mathbb{C}$:

$$\begin{aligned} ((\varphi_1 \otimes \cdots \otimes \varphi_m) \otimes (\omega_1 \otimes \cdots \otimes \omega_{m'}))(\tilde{v}) = \\ = \varphi_1(v) \cdots \varphi_m(v) \omega_1(v) \cdots \omega_{m'}(v) \\ = (\varphi_1 \otimes \cdots \otimes \varphi_m)(\tilde{v})(\omega_1 \otimes \cdots \otimes \omega_{m'})(\tilde{v}) \end{aligned} \tag{2.24}$$

holds for elementary tensors in addition to linearity.

On the other hand, for any fixed vector $\varphi \in T(V^*)$, there exists a map $\hat{p}_\varphi : V \to \mathbb{C}$ defined by $v \mapsto \varphi(\tilde{v})$. Clearly, the map $\hat{p} : T(V^*) \to C(V, \mathbb{C})$ defined by $\varphi \mapsto \hat{p}_\varphi$ is an algebra morphism. As $C(V, \mathbb{C})$ is commutative, the kernel of this latter map must contain the two sided ideal I generated by commutators, and in fact the kernel equals to this ideal. Therefore, there is a unique morphism p making the following diagram commute:

$$\begin{array}{ccc} T(V^*) & \xrightarrow{\hat{p}} & C(V,\mathbb{C}) \\ {\scriptstyle\pi}\big\downarrow & \nearrow_{\exists! p} & \\ T(V^*)/I & & \end{array} \tag{2.25}$$

The algebra $T(V^*)/I$ will be denoted by $S(V^*)$.

Definition. Let V be a finite dimensional vector space. The algebra $S(V^*)$ is called the algebra of holomorphic polynomials over V. For $\varphi = \hat{\varphi} + I \in S(V^*)$, the function $p_\varphi : V \to \mathbb{C}$ defined by $p_\varphi(v) = \hat{\varphi}(v)$ is a holomorphic polynomial function on V.

Let $\{e_1, \dots, e_n\}$ be an arbitrary basis in V, and $\{e_1^*, \dots, e_n^*\}$ be its dual basis in V^* defined by $e_i^*(e_j) = \delta_{ij}$. An element $\hat{\varphi} \in T(V^*)$ can then be written in the form

$$\hat{\varphi} = \sum_{m=0}^{M} \sum_{i_1,\dots,i_m=1}^{n} \varphi_{i_1,\dots,i_m} e_{i_1}^* \otimes \cdots \otimes e_{i_m}^* \tag{2.26}$$

with $M \in \mathbb{N}$. Let $\varphi = \hat{\varphi} + I$. On the vector $v = \sum_{i=1}^n v_i e_i$, the value of p_φ is then

$$\begin{aligned} p_\varphi(v) &= \sum_{m=0}^{M} \sum_{i_1,\dots,i_m=1}^{n} \varphi_{i_1,\dots,i_m} (e_{i_1}^* \otimes \cdots \otimes e_{i_m}^*)(v \otimes \cdots \otimes v) \\ &= \sum_{m=0}^{M} \sum_{i_1,\dots,i_m=1}^{n} \sum_{j_1,\dots,j_m=1}^{n} \varphi_{i_1,\dots,i_m} v_{j_1} v_{j_2} \cdots v_{j_m} e_{i_1}^*(e_{j_1}) \cdots e_{i_m}^*(e_{j_m}) \\ &= \sum_{m=0}^{M} \sum_{i_1,\dots,i_m=1}^{n} \varphi_{i_1,\dots,i_m} v_{i_1} v_{i_2} \cdots v_{i_m} \end{aligned} \tag{2.27}$$

which is polynomial in the coefficients of v, justifying the definition above. Conversely, a function $f : V \to \mathbb{C}$ which is polynomial in the coefficients can be represented uniquely by a vector in $S(V^*)$.

Suppose now that G is a group acting on V linearly (see section B.1 in the appendix for the definition of a group representation), that is, the function $v \mapsto g \cdot v$ is linear for all $g \in G$. This action induces a unique linear action on $\prod_{m=0}^{\infty} \bigotimes_{i=1}^{m} V$ such that the map $v \mapsto \tilde{v}$ is equivariant, explicitely, $g \cdot (1, v, v \otimes v, v \otimes v \otimes v, \ldots) = (1, g \cdot v, g \cdot v \otimes g \cdot v, g \cdot v \otimes g \cdot v \otimes g \cdot v, \ldots)$. This in turn induces an action on $T(V^*)$ such that the pairing $(\prod_{m=0}^{\infty} \bigotimes_{i=1}^{m} V) \times T(V^*) \to \mathbb{C}$ is also equivariant (when $\mathbb{C}$ is regarded as the trivial representation of G), that is, for $g \in G, \varphi \in T(V^*)$ and $v \in V$, the following equation holds:

$$(g \cdot \varphi)(g \cdot \tilde{v}) = \varphi(\tilde{v}) \tag{2.28}$$

In addition, G acts on $T(V^*)$ by graded algebra morphisms (see section A.5 in the appendix). Finally, as the ideal I is homogenous and G-invariant (this follows from the fact that $g \cdot (\varphi_1 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1) = (g \cdot \varphi_1) \otimes (g \cdot \varphi_2) - (g \cdot \varphi_2) \otimes (g \cdot \varphi_1))$, $S(V^*)$ is also graded and can be given the structure of a G-space such that π in the diagram of eq. (2.25) is G-equivariant.

Now that we have a representation of G on $S(V^*)$, it makes sense to look for fixed elements in $S(V^*)$, which form a subalgebra, because G acts also on $S(V^*)$ by algebra morphisms. The set

$$S(V^*)^G = \{\varphi \in S(V^*) | \forall g \in G : g \cdot \varphi = \varphi\} \tag{2.29}$$

of fixed elements is called the algebra of invariant holomorphic polynomials. As G acts on $S(V^*)$ with graded algebra morphisms, homogenous parts of an invariant holomorphic polynomial is also invariant.

However, these tools cannot be directly utilized in order to be able to say anything about the entanglement of any system. The problem is that both the SLOCC group and the LU group contains elements of the form $e^{i\varphi} \operatorname{id}_{\mathcal{H}}$ – and the algebra of invariant polynomials under this subgroup is already only the algebra of constant polynomials. There are two possible solutions to this problem: either one considers only local transformations with determinant 1, or one forgets about the SLOCC group and looks for invariant polynomials over $\mathbb{R}$ or in other words, polynomials in the coefficients and their conjugates.

Let us see how these polynomials can be defined. Let us start with a finite dimensional Hilbert space $\mathcal{H}$ on which G acts and which is equipped with a G-invariant inner product, that is, one satisfying $\langle g \cdot \psi, g \cdot \psi' \rangle = \langle \psi, \psi' \rangle$ for all $\psi, \psi' \in \mathcal{H}, g \in G$. We also have the induced action on $\mathcal{H}^*$ defined by $(g \cdot \varphi)(g \cdot \psi) = \varphi(\psi)$. To a vector $\psi \in \mathcal{H}$ we can associate a linear functional $\psi^* \in \mathcal{H}^*$ with the inner product: $\psi^*(\psi') := \langle \psi, \psi' \rangle$. The semilinear map $\psi \mapsto \psi^*$ is G-equivariant by construction:

$$(g \cdot \psi^*)(g \cdot \psi') = \langle g \cdot \psi, g \cdot \psi' \rangle = \langle \psi, \psi' \rangle = \psi^*(\psi') \tag{2.30}$$

Let $V = \mathcal{H} \oplus \mathcal{H}^*$. Then V carries a representation of G and we have the G-equivariant map $\mathcal{H} \to V$ $v = \psi \mapsto \psi \oplus \psi^*$. The fact that the map

$\psi \mapsto \psi^*$ is only semilinear makes it plausible that holomorphic polynomials on V correspond to polynomials in the coefficients of ψ and their conjugates. Indeed, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathcal{H}$, and $\{e_1^*, \ldots, e_n^*\} \subseteq \mathcal{H}^*$ its dual basis. Then $\{e_1, \ldots, e_n, e_1^*, \ldots, e_n^*\}$ is a basis of V, and the image of $\psi = \sum_{i=1}^n \psi_i e_i$ under the above map is

$$\psi \oplus \psi' = \sum_{i=1}^{n} \psi_i e_i \oplus \sum_{i=1}^{n} \overline{\psi_i} e_i^* \tag{2.31}$$

and thus polynomials in its coefficients are in fact polynomials in the coefficients of ψ and their conjugates. Note that as $2\operatorname{Re} z = z + \overline{z}$ and $2i \operatorname{Im} z = z - \overline{z}$, these are exactly the polynomials in the coefficients of ψ regarding $\mathcal{H}$ as a real vector space.

Definition. Let $\mathcal{H}$ be a finite dimensional Hilbert space with an action of the group G and a G-equivariant inner product, and let $V = \mathcal{H} \oplus \mathcal{H}^*$. The algebra $S(V^*)$ is called the algebra of polynomials over $\mathcal{H}$. For $\varphi = \hat{\varphi} + I \in S(V^*)$, the function $p_\varphi : \mathcal{H} \to \mathbb{C}$ defined by $p_\varphi(\psi) = \hat{\varphi}(\psi \oplus \psi^*)$ is a polynomial function on $\mathcal{H}$. The set

$$S(V^*)^G = \{\varphi \in S(V^*) | \forall g \in G : g \cdot \varphi = \varphi\} \tag{2.32}$$

of fixed elements is called the algebra of invariant polynomials.

This algebra turns out to contain enough elements to separate the orbits under the LU group. This is implied by the following theorem[37]:

Theorem 2.4.1. *Let $\mathcal{H}$ be a finite dimensional Hilbert space with an action of the compact topological group G and a G-equivariant inner product. For two vectors $\psi, \psi' \in \mathcal{H}$, there exists $g \in G$ such that $g \cdot \psi = \psi'$ iff $f(\psi) = f(\psi')$ for all invariant polynomial functions f.*

Proof. Suppose that ψ and ψ' are on different orbits. As $n : \psi \mapsto \|\psi\|^2$ is an invariant polynomial function, we can assume that $\|\psi\| = \|\psi'\|$. The level set $n^{-1}(\|\psi\|^2)$ is a sphere, in particular, it is compact. The orbits of ψ and ψ' are disjoint compact sets, therefore, by Urysohn's theorem, there exists a continuous function $f : \mathcal{H} \to \mathbb{R}$ which is 1 on the orbit of ψ and 0 on the orbit of ψ'.

By the Stone-Weierstrass theorem, for every $\varepsilon > 0$ there exists a polynomial function p such that $|p(v) - f(v)| < \varepsilon$ for all $v \in n^{-1}(\|\psi\|^2)$. Choose an

$\varepsilon < \frac{1}{2}$ and let p be such a polynomial function. Then

$$\begin{aligned}\left|1 - \int_G p(g \cdot \psi) dg\right| &= \left|\int_G (f(\psi) - p(g \cdot \psi)) dg\right| \\ &= \left|\int_G (f(g \cdot \psi) - p(g \cdot \psi)) dg\right| \\ &\leq \int_G |f(g \cdot \psi) - p(g \cdot \psi)| dg \leq \varepsilon\end{aligned} \tag{2.33}$$

together with

$$\begin{aligned}\left|0 - \int_G p(g \cdot \psi') dg\right| &= \left|\int_G (f(\psi') - p(g \cdot \psi')) dg\right| \\ &= \left|\int_G (f(g \cdot \psi') - p(g \cdot \psi')) dg\right| \\ &\leq \int_G |f(g \cdot \psi') - p(g \cdot \psi')| dg \leq \varepsilon\end{aligned} \tag{2.34}$$

implies that the invariant polynomial function

$$\tilde{p} : v \mapsto \int_G p(g \cdot v) dg \tag{2.35}$$

takes different values on ψ and ψ'. □

Notice that the same program would fail in the case of the SLOCC group at several points: when G is the SLOCC group with the usual action, there are no G-invariant inner products on $\mathcal{H}$; we cannot restrict ourselves to a compact subset, and therefore polynomials are not dense in the space of continuous functions; there is no averaging operator, because the SLOCC group is not compact, and hence there is no normalized Haar measure; and even worse, there may exist a dense orbit, in which case even continuous invariant functions are unable to separate the orbits.

2.5 The symmetric algebra of a Hilbert space

Throughout this section $\mathcal{H}$ denotes a finite dimensional complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We regard $\mathcal{H}$ as a representation of $U(\mathcal{H}) = \{\varphi : \mathcal{H} \to \mathcal{H} | \forall v \in \mathcal{H} : \|\varphi v\| = \|v\|\}$.

Let $S(\mathcal{H})$ denote the symmetric algebra (see section A.5) of $\mathcal{H}$ that is, the algebra of polynomials in vectors of $\mathcal{H}$. $S(\mathcal{H})$ has the structure of a graded algebra, its degree m homogenous subspace will be denoted by $S^m(\mathcal{H})$. As $S^1(\mathcal{H}) = \mathcal{H}$, and this subspace generates $S(\mathcal{H})$ as a unital commutative algebra, we have that $U(\mathcal{H})$ acts on $S(\mathcal{H})$ with algebra automorphisms.

The inner product on $\mathcal{H}$ induces one on $S^m(\mathcal{H})$ by the following requirement: for $u, v \in \mathcal{H}$ let $\langle u^m, v^m \rangle = \langle u, v \rangle^m$ (see also section A.4 in the appendix). This turns out to be equivalent to saying that for a unit vector $u \in \mathcal{H}$, $\|u^m\| = 1$. Clearly, this inner product will be preserved by the action of $U(\mathcal{H})$ on $S(\mathcal{H})$, restricted to each homogenous subspace. It is known from the representation theory of the unitary groups (see section B.4 in the appendix) that in this way each $S^m(\mathcal{H})$ becomes an irreducible unitary representation of $U(\mathcal{H})$, and hence the induced inner product is essentially the only one invariant under this group action.

To be more explicit, if we fix an orthonormal basis $\{e_1, \ldots, e_d\}$ in $\mathcal{H}$, then $S^m(\mathcal{H})$ is the space of degree m homogenous polynomials in the basis elements, and the degree m monomials with coefficient 1 form a basis. These monomials are mutually orthogonal, but they are not unit vectors. If $v = \sum_{i=1}^d \alpha_i e_i$ then

$$\begin{aligned} v^m &= \sum_{i_1=1}^{d} \cdots \sum_{i_m=1}^{d} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_m} e_{i_1}\cdots e_{i_m} \\ &= \sum_{\substack{k_1,\ldots,k_d \geq 0 \\ k_1+\ldots+k_d=m}} \binom{m}{k_1, k_2, \ldots, k_d} \alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_d^{k_d} e_1^{k_1} e_2^{k_2}\cdots e_d^{k_d} \end{aligned} \tag{2.36}$$

(where $\binom{m}{k_1,k_2,\ldots,k_d}$ is the multinomial coefficient) hence

$$\|v^m\|^2 = \sum_{\substack{k_1,\ldots,k_d \geq 0 \\ k_1+\ldots+k_d=m}} \binom{m}{k_1, k_2, \ldots, k_d}^2 |\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_d^{k_d}|^2 \|e_1^{k_1} e_2^{k_2}\cdots e_d^{k_d}\|^2 \tag{2.37}$$

Comparing this with

$$(\|v\|^2)^m = \left(\sum_{i=1}^{d} |\alpha_i|^2\right)^m = \sum_{\substack{k_1,\ldots,k_d \geq 0 \\ k_1+\ldots+k_d=m}} \binom{m}{k_1, k_2, \ldots, k_d} |\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_d^{k_d}|^2 \tag{2.38}$$

we conclude that

$$\|e_1^{k_1} e_2^{k_2}\cdots e_d^{k_d}\| = \binom{m}{k_1, k_2, \ldots, k_d}^{-1/2} \tag{2.39}$$

2.6 Local unitary invariants

We return to the more concrete setting of the entanglement classification problem. In the following, $k \in \mathbb{N}$, and $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ is the state space of a composite quantum system, $\dim \mathcal{H}_i = n_i < \infty$. The group we consider is the direct product of the single-particle unitary groups:

$$G = U(\mathcal{H}_1) \times \cdots \times U(\mathcal{H}_k) \tag{2.40}$$

As each homogenous part of an invariant polynomial is also invariant, we need only to consider homogenous polynomials. These are elements of $S^p(\mathcal{H}^* \oplus \mathcal{H})$. We have the isomorphism

$$S^p(\mathcal{H}^* \oplus \mathcal{H}) \simeq \bigoplus_{m=0}^{p} S^i(\mathcal{H}^*) \otimes S^{p-m}(\mathcal{H}) \tag{2.41}$$

of G-spaces. An element λ of $\{e^{i\varphi} | \varphi \in \mathbb{R}\} \simeq U(1) \times \{1\} \times \cdots \times \{1\} \leq Z(G)$ acts on $S^m(\mathcal{H}^*) \otimes S^{p-m}(\mathcal{H})$ as multiplication by $\lambda^{(p-m)-m} = \lambda^{p-2m}$, therefore, invariant polynomials can reside in $S^m(\mathcal{H}^*) \otimes S^{p-m}(\mathcal{H})$ only if $p = 2m$.

From now on, m will denote $\frac{p}{2}$. Observe that $S^m(\mathcal{H}^*) = S^m(\mathcal{H})^*$. Let

$$S^m(\mathcal{H}) \simeq \bigoplus_{\alpha \in \mathrm{Irr}(G)} c_\alpha V_\alpha \tag{2.42}$$

be the decomposition to the orthogonal sum of isotypic subspaces, where c_α are nonnegative integers and $\mathrm{Irr}(G)$ is a set labelling the isomorphism-classes of irreducible representations of G. Then

$$S^m(\mathcal{H})^* \otimes S^m(\mathcal{H}) \simeq \bigoplus_{\alpha, \alpha' \in \mathrm{Irr}(G)} c_\alpha c_{\alpha'} V_\alpha \otimes V_{\alpha'}^* \tag{2.43}$$

The multiplicity of the trivial representation in $V_\alpha \otimes V_{\alpha'}^*$ is 1 if $\alpha = \alpha'$ and 0 if $\alpha \neq \alpha'$. We can conclude that for $\alpha \in \mathrm{Irr}(G)$ there are three possibilities:

1. $c_\alpha = 0$, in this case we do not get any invariants

2. $c_\alpha = 1$, in this case there is a one dimensional subspace of invariant polynomials in $V_\alpha \otimes V_\alpha^* \leq S^m(\mathcal{H}^*) \otimes S^m(\mathcal{H})$, and we can choose a "canonical" element spanning it

3. $c_\alpha > 1$, in this case we get a c_α^2 dimensional space of invariants in which we can not find a "distinguished" basis in any obvious way

In the following, we concentrate only on the $c_\alpha = 1$ case. Although we throw away this way the majority of invariants, we can hope that the remaining ones have a clearer structure and still we can find previously unknown invariants.

For an $\alpha \in \operatorname{Irr}(G)$ such that $c_\alpha = 1$, the distinguished element can be obtained as follows. As we have seen in section 2.5, up to normalization, there exists a unique inner product on $S^m(\mathcal{H})$ which is invariant under the induced action of the full unitary group acting on $\mathcal{H}$. This follows from the fact that $S^m(\mathcal{H})$ carries an irreducible representation of this group. We choose the normalization so that for any $\psi \in \mathcal{H}$, the equation $\|\psi^m\| = \|\psi\|^m$ holds, where $\psi^m = \psi \otimes \cdots \otimes \psi \in S^m(\mathcal{H})$. Now let P_α denote the orthogonal projection onto the irreducible subrepresentation indexed by α. The value of the distinguished invariant polynomial function on $\psi \in \mathcal{H}$ is then $\langle \psi^m, P_\alpha \psi^m \rangle$.

We would like to express this value as a polynomial in the coefficients with respect to a given basis. To this end we first need to find a generating set of the vector space $\operatorname{ran} P_\alpha$, each generating vector expressed as a linear combination of degree m monomials in elements of an orthonormal basis in $\mathcal{H}$.

Once we have a generating set, we orthogonalize it, and for each vector w in the orthogonal set we calculate the value of $|\langle w, \psi^m \rangle|^2 \|w\|^{-2}$. Finally, the sum of these numbers is the value of our invariant evaluated on the state ψ. Explicitely, let $\psi = \sum_I \psi_I e_I$, and $w = \sum_{k_1,k_2,\ldots,k_d} \beta_{k_1,k_2,\ldots,k_d} e_{I_1}^{k_1} \ldots e_{I_d}^{k_d}$, where $d = n_1 \cdot \cdots \cdot n_k$ and $I_1, \ldots, I_d$ are the possible k-element multi-indices, and $k_1, \ldots, k_d$ run over nonnegative integers such that their sum equals m. Then by equation (2.39) we have

$$\begin{aligned} \langle w, \psi^m \rangle &= \Big\langle \sum_{k_1,\ldots,k_d} \beta_{k_1,\ldots,k_d} e_{I_1}^{k_1} \ldots e_{I_d}^{k_d}, \sum_{k'_1,\ldots,k'_d} \binom{m}{k'_1,\ldots,k'_d} \psi_{I_1}^{k'_1} \ldots \psi_{I_d}^{k'_d} e_{I_1}^{k'_1} \ldots e_{I_d}^{k'_d} \Big\rangle \\ &= \sum_{k_1,\ldots,k_d} \overline{\beta_{k_1,\ldots,k_d}} \psi_{I_1}^{k'_1} \ldots \psi_{I_d}^{k'_d} \underbrace{\binom{m}{k_1,\ldots,k_d} \|e_{I_1}^{k_1} \ldots e_{I_d}^{k_d}\|^2}_{1} \end{aligned} \tag{2.44}$$

A set labelling the isomorphism-classes of irreducible representations of the group $U(\mathcal{H}_i) \simeq U(\mathbb{C}, n_i)$ is the set P_i of generalized partitions with at most n_i parts (see section B.4 in the appendix). Then $\operatorname{Irr}(G)$ can be chosen to be $P_1 \times \cdots \times P_k$. A representation $U(\mathbb{C}, n_i) \to GL(V)$ indexed by a partition of m is an mth degree polynomial function (that is, can be identified with an element of $S^m(\operatorname{End}(\mathbb{C}^{n_i})^*) \otimes \operatorname{End}(V)$). Consequently, only those irreducible representations are present in which all the P_is are partitions of the same number m, and for a fixed m, they span $S^m(\mathcal{H})$.

Note that for a fixed m, for sufficiently large single particle state spaces, there is no restriction to the shape of the partitions, as a partition of m cannot have more then m parts. Moreover, the decomposition of $S^m(\mathcal{H})$ into irreducible components does not depend on the dimensions of the single particle state spaces apart from the vanishing of those indexed by a partition with more parts then the dimension of the respective Hilbert space. It would then be desirable (and in principle possible) to express the invariants with a systematic dependence on the dimensions of the state spaces. This way we can eliminate the need to bother about the dimensions, the two relevant parameters are the degree m and the number of subsystems k. The set $\mathrm{Irr}(G)$ may then be taken to be the set of k-tuples of generalized partitions.

2.6.1 Fourth order invariants

The $m = 2$ case is special in that for all $k \in \mathbb{N}$, the multiplicities are all at most 1. The decomposition of $S^2(\mathcal{H})$ can be obtained inductively, using the following isomorphisms of $GL(V) \times GL(W)$-spaces[22]:

$$\begin{aligned} S^2(V \otimes W) &\simeq (S^2V \otimes S^2W) \oplus (\Lambda^2V \otimes \Lambda^2W) \\ \Lambda^2(V \otimes W) &\simeq (S^2V \otimes \Lambda^2W) \oplus (S^2V \otimes \Lambda^2W) \end{aligned} \tag{2.45}$$

In order to be able to handle various representations of $GL(V)$ – including symmetric and alternating powers of V – in a uniform manner, we will express these using Schur functors (see section B.3 in the appendix for the definition). In particular, $S^mV = \mathbb{S}_{(m)}V$ and $\Lambda^mV = \mathbb{S}_{(1^m)}V$. We will need two kinds of subsets of $\mathrm{Irr}(G)$:

$$\begin{aligned} \mathrm{Irr}_{k,m}^{even} &= \{(\lambda_1, \ldots, \lambda_k) \in \{(m), (1^m)\}^k \| \{i | \lambda_i = (1^m)\}| \equiv 0 \pmod 2\} \\ \mathrm{Irr}_{k,m}^{odd} &= \{(\lambda_1, \ldots, \lambda_k) \in \{(m), (1^m)\}^k \| \{i | \lambda_i = (1^m)\}| \equiv 1 \pmod 2\} \end{aligned} \tag{2.46}$$

Proposition 2.6.1. *Let $k \in \mathbb{N}$ and $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ be the state space of a composite quantum system with $\dim \mathcal{H} < \infty$. Then*

$$\begin{aligned} \mathbb{S}_{(2)}\mathcal{H} &\simeq \bigoplus_{(\lambda_i)_{i=1}^k \in \mathrm{Irr}_{k,2}^{even}} \mathbb{S}_{\lambda_1}\mathcal{H}_1 \otimes \cdots \otimes \mathbb{S}_{\lambda_k}\mathcal{H}_k \\ \mathbb{S}_{(1^2)}\mathcal{H} &\simeq \bigoplus_{(\lambda_i)_{i=1}^k \in \mathrm{Irr}_{k,2}^{odd}} \mathbb{S}_{\lambda_1}\mathcal{H}_1 \otimes \cdots \otimes \mathbb{S}_{\lambda_k}\mathcal{H}_k \end{aligned} \tag{2.47}$$

Proof. We prove by induction in k, the $k = 1$ case is clear. Suppose that for $k - 1$ instead of k, the statement is true. Then, using eq. (2.45) and the induction hypothesis, we have that

$$
\begin{aligned}
\mathbb{S}_{(2)}\mathcal{H} &\simeq \mathbb{S}_{(2)}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}) \otimes \mathbb{S}_{(2)}\mathcal{H}_k \oplus \\
&\quad \oplus \mathbb{S}_{(1^2)}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}) \otimes \mathbb{S}_{(1^2)}\mathcal{H}_k \\
&\simeq \bigoplus_{(\lambda_i)_{i=1}^{k-1} \in \mathrm{Irr}_{k,2}^{even}} \mathbb{S}_{\lambda_1}\mathcal{H}_1 \otimes \cdots \otimes \mathbb{S}_{\lambda_{k-1}}\mathcal{H}_{k-1} \otimes \mathbb{S}_{(2)}\mathcal{H}_k \oplus \\
&\quad \oplus \bigoplus_{(\lambda_i)_{i=1}^{k-1} \in \mathrm{Irr}_{k,2}^{odd}} \mathbb{S}_{\lambda_1}\mathcal{H}_1 \otimes \cdots \otimes \mathbb{S}_{\lambda_{k-1}}\mathcal{H}_{k-1} \otimes \mathbb{S}_{(1^2)}\mathcal{H}_k \\
&\simeq \bigoplus_{(\lambda_i)_{i=1}^{k} \in \mathrm{Irr}_{k,2}^{even}} \mathbb{S}_{\lambda_1}\mathcal{H}_1 \otimes \cdots \otimes \mathbb{S}_{\lambda_k}\mathcal{H}_k
\end{aligned} \tag{2.48}
$$

and

$$
\begin{aligned}
\mathbb{S}_{(1^2)}\mathcal{H} &\simeq \mathbb{S}_{(2)}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}) \otimes \mathbb{S}_{(1^2)}\mathcal{H}_k \oplus \\
&\quad \oplus \mathbb{S}_{(1^2)}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}) \otimes \mathbb{S}_{(2)}\mathcal{H}_k \\
&\simeq \bigoplus_{(\lambda_i)_{i=1}^{k-1} \in \mathrm{Irr}_{k,2}^{even}} \mathbb{S}_{\lambda_1}\mathcal{H}_1 \otimes \cdots \otimes \mathbb{S}_{\lambda_{k-1}}\mathcal{H}_{k-1} \otimes \mathbb{S}_{(1^2)}\mathcal{H}_k \oplus \\
&\quad \oplus \bigoplus_{(\lambda_i)_{i=1}^{k-1} \in \mathrm{Irr}_{k,2}^{odd}} \mathbb{S}_{\lambda_1}\mathcal{H}_1 \otimes \cdots \otimes \mathbb{S}_{\lambda_{k-1}}\mathcal{H}_{k-1} \otimes \mathbb{S}_{(2)}\mathcal{H}_k \\
&\simeq \bigoplus_{(\lambda_i)_{i=1}^{k} \in \mathrm{Irr}_{k,2}^{odd}} \mathbb{S}_{\lambda_1}\mathcal{H}_1 \otimes \cdots \otimes \mathbb{S}_{\lambda_k}\mathcal{H}_k
\end{aligned} \tag{2.49}
$$

which finishes the proof. □

Consequently, the irreducible components of $S^2(\mathcal{H})$ can be indexed by even-element subsets of $[k]$, with a bijection sending the set A to the component in which the alternating power of $\mathcal{H}_j$ appears iff $j \in A$. The subspace corresponding to A will be denoted by V_A. It follows that the dimension of the space of fourth order G-invariant polynomials is 2^{k-1}.

Our next aim is to construct an orthonormal basis in V_A for each possible subset A. We would like to express elements of $S^2(\mathcal{H})$ in terms of a computational basis in $\mathcal{H}$. Let $\{e_{j,i} | 1 \leq j \leq k, 1 \leq i \leq n_j\}$ be a set of vectors such that $\{e_{j,i}\}_{1 \leq i \leq n_j}$ is an orthonormal basis in $\mathcal{H}_j$. Let us now introduce the following short notation: $e_{i_1 i_2 \ldots i_k} := e_{1,i_1} \otimes e_{2,i_2} \otimes \ldots \otimes e_{k,i_k} \in \mathcal{H}$, where $1 \leq i_j \leq n_j$ (for all $j \in [k]$). The set of vectors of this form is an orthonormal basis in $\mathcal{H}$. Elements of the symmetric algebra $S(\mathcal{H})$ are polynomials in these vectors, in particular, a vector of $S^m(\mathcal{H})$ is a degree m homogenous polynomial.

Let $i_{0,1}, i_{0,2}, \ldots, i_{0,k}, i_{1,1}, i_{1,2}, \ldots, i_{1,k}$ be fixed integers such that $1 \le i_{0,j} \le i_{1,j} \le n_j$ for all $1 \le j \le k$, and $i_{0,j} \ne i_{1,j}$ whenever $j \in A$. Let us now consider the vector

$$v = \sum_{b_1,\ldots,b_k \in \{0,1\}} (-1)^{|A\cap B|} e_{i_{b_1,1} i_{b_2,2} \ldots i_{b_k,k}} e_{i_{1-b_1,1} i_{1-b_2,2} \ldots i_{1-b_k,k}} \tag{2.50}$$

where $B = \{j \in [k] | b_j = 1\}$. We claim that this is an element of V_A, moreover, vectors of this type form a basis of V_A and are pairwise orthogonal.

Clearly, when we construct two vectors v and v' this way starting from different sets of indices, then not only v and v' are orthogonal, but any term appearing in the above expression of v is orthogonal to any term in v'. It is also easy to see that the span of these vectors is G-invariant. The highest weights can be read off from the vector with smallest possible indices, namely, for $j \notin A$, the highest weight for the jth factor in G is (2), while for $j \in A$ it is $(1,1)$. Therefore, vectors of this type span V_A. Note, that this is consistent with the fact that the number of admissible sets of indices is

$$\prod_{j\in\{1,\ldots,k\}\setminus A} \binom{n_j+1}{2} \prod_{j\in A} \binom{n_j}{2} = \dim V_A \tag{2.51}$$

for a fixed subset A.

We calculate next the norm squared of the elements of this basis. The sum has 2^k terms, but they are not necessarily distinct. More precisely, each term appears with the same multiplicity, which is easily seen to be 2^{c+1} if $c := \{j \in [k] | i_{0,j} = i_{1,j}\} < k$ and 2^k if $c = k$. The latter case can only be realized if $A = \emptyset$. The norm of a single term is 1 if $c = k$, and $\frac{1}{\sqrt{2}}$ otherwise. To sum up, the norm squared of v is

$$\|v\| = \begin{cases} \frac{2^k}{2^{c+1}} (2^{c+1})^2 \frac{1}{2} = 2^{k+c} & \text{if } c < k \\ (2^k)^2 & \text{if } c = k \end{cases} \tag{2.52}$$

A formula which gives back both cases is $\|v\|^2 = 2^{k+c}$.

The invariant associated to the subrepresentation V_A is therefore given by

$$I_A(\psi) = 2^{-k} \sum_{\substack{1 \le i_1^0 \le i_1^1 \le n_i \\ \vdots \\ 1 \le i_k^0 \le i_k^1 \le n_k}} 2^{-c} \left| \sum_{b_1,\ldots,b_k=0}^{1} (-1)^{|A\cap B|} \psi_{i_{b_1,1} i_{b_2,2} \ldots i_{b_k,k}} \psi_{i_{1-b_1,1} i_{1-b_2,2} \ldots i_{1-b_k,k}} \right|^2 \tag{2.53}$$

with B and c as above, and $\psi = \sum \psi_{i_1 i_2 \ldots i_k} e_{i_1 i_2 \ldots i_k}$.

2.6.2 Invariants of higher order

In the general $m \geq 2$ case, a similar treatment would be impractical, as instead of eq. (2.45), we would need one similar isomorphism for each partition of m, and the full decomposition of $S^m(\mathcal{H})$ would be rather complicated. We can derive however invariants in this case which are analogous to the ones found in the $m = 2$ case. We assume that the single-particle state spaces are "large enough", meaning that $\forall i \in [k] : \dim \mathcal{H}_i \geq m$, ensuring that none of the appearing representations are $\{0\}$. For smaller m, we may define the corresponding invariants to be identically zero.

The above mentioned isomorphisms are special cases of the following one[22]:

$$\mathbb{S}_\nu(V \otimes W) \simeq \bigoplus_{\lambda,\mu} C_{\lambda\mu\nu} \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu W \tag{2.54}$$

where ν is a partition of m, λ, μ run over partitions of m and $C_{\lambda\mu\nu}$ are certain coefficients, appearing also in the decomposition of tensor products of irreducible representations of S_m:

$$V_\lambda \otimes V_\mu \simeq \bigoplus_\nu C_{\lambda\mu\nu} V_\nu \tag{2.55}$$

where V_ν is the irreducible representation of S_m labelled by the partition ν. Analogous formulae are valid for more than two factors as well. (see sections B.3 and B.4 in the appendix)

In what follows we only need to consider the two simplest irreducible representations, namely the trivial representation $V_{(m)}$ and the alternating one $V_{(1^m)}$. It is easy to see that

$$V_{(m)} \otimes \cdots \otimes V_{(m)} \otimes \underbrace{V_{(1^m)} \otimes \cdots \otimes V_{(1^m)}}_{b \text{ factors}} \simeq \begin{cases} V_{(m)} & \text{if } b \text{ is even} \\ V_{(1^m)} & \text{if } b \text{ is odd} \end{cases} \tag{2.56}$$

Consequently, in the decomposition of $S^m(\mathcal{H})$ into irreducibles we can find $\mathbb{S}_{\lambda_1}\mathcal{H}_1 \otimes \ldots \otimes \mathbb{S}_{\lambda_k}\mathcal{H}_k$ with multiplicity one (zero) when among the λs only (m) and (1^m) are present and the latter appears an even (odd) number of times (note that the tensor product is commutative up to isomorphism).

This means that again we have a family of local unitary invariants, labelled by even-element subsets of $[k]$, mapped bijectively to the set of irreducible subrepresentations of $S^m(\mathcal{H})$ built up from symmetric and alternating powers of the representations $\mathcal{H}_j$ with an even number of alternating powers. Again, the subspace corresponding to the subset $A \subseteq [k]$ will be denoted by V_A.

For a fixed subset A, let $(i_{j,l})_{1\leq j\leq k,1\leq l\leq m}$ be integers such that if $j\notin A$, then $1\leq i_{j,1}\leq i_{j,2}\leq\ldots\leq i_{j,m}\leq n_j$ and if $j\in A$, then $1\leq i_{j,1}<i_{j,2}<\ldots<i_{j,m}\leq n_j$. From these indices we can form the vector

$$\sum_{\pi_1,\ldots,\pi_k\in S_m}\prod_{j=1}^{m}\chi_{\lambda_j}(\pi_j)e_{i_{1,\pi_1(1)}\cdots i_{k,\pi_k(1)}}e_{i_{1,\pi_1(2)}\cdots i_{k,\pi_k(2)}}\ldots e_{i_{1,\pi_1(m)}\cdots i_{k,\pi_k(m)}} \tag{2.57}$$

where $\lambda_j=(m)$ if $j\notin A$ and $\lambda_j=(1^m)$ if $j\in A$, χ_λ is the character of the corresponding irreducible representation, i.e. constant 1 if $\lambda=(m)$ and the sign of the permutation if $\lambda=(1^m)$. v is then an element of V_A, vectors of this form span V_A and they are pairwise orthogonal for different (multi)sets of indices. As there does not seem to be a simple formula for the norm squared of these vectors, we cannot give the general form of the corresponding invariant, but it can be calculated without any difficulty for every concrete value of m and k and every subset A.

2.6.3 Fourth order invariants with permutation symmetry

We return to the better-understood $m=2$ case. In the special case when all single particle Hilbert spaces have the same dimension, one might want to look for polynomials which are invariant not only under local unitary transformations but also under permutations of the particles. Among the invariants presented above, the ones corresponding to $A=\emptyset$ and $A=[k]$ clearly have this property. From the remaining ones we can easily form permutation invariant polynomials:

$$I_d(\psi):=\frac{1}{\binom{k}{d}}\sum_{A\in\binom{[k]}{d}}I_A(\psi) \tag{2.58}$$

where $0\leq d\leq k$ and $2|d$, and $\binom{[k]}{d}$ denotes the set of d-element subsets of $[k]$.

On the other hand, if I is an arbitrary fourth order invariant, then it can be written uniquely as a linear combination of I_A-s where A runs through even-element subsets of $[k]$:

$$I=\sum_{\substack{A\subseteq[k]\\2||A|}}c_AI_A \tag{2.59}$$

If I is additionally invariant under particle permutations, then

$$\begin{aligned} I = \sum_{\pi \in S_k} I \circ \pi &= \sum_{\pi \in S_k} \sum_{\substack{A \subseteq [k] \\ 2 | |A|}} c_A I_A \circ \pi \\ &= \sum_{\pi \in S_k} \sum_{\substack{A \subseteq [k] \\ 2 | |A|}} c_A I_{\pi^{-1}(A)} = \sum_{\substack{A \subseteq [k] \\ 2 | |A|}} c_A \underbrace{\sum_{\pi \in S_k} I_{\pi^{-1}(A)}}_{k! I_{|A|}} \end{aligned} \tag{2.60}$$

Therefore, the dimension of the vector space of permutation-invariant degree four unitary invariants is $1 + \lfloor \frac{k}{2} \rfloor$.

2.7 Dimensions of subspaces of LU-invariants

Based on the method followed in the previous section, we can find LU-invariants for a given particle number and with a fixed degree in a relatively straightforward way. Although this may be useful when our aim is to find invariants which can be computed directly – low-order invariants are particularly well-suited to this, as the computation time scales with $(\dim \mathcal{H})^d$ for a degree d polynomial –, only little information is gained this way about the structure of the algebra of invariants.

One particularly useful tool to study graded structures (such as the graded algebra of polynomial invariants) is the Hilbert series. This can be thought of as a generalization of dimension to graded vector spaces which are often of infinite dimension, but locally finite dimensional (see also section A.5 in the appendix).

Definition. Let $V = \bigoplus_{i \in \mathbb{N}} V_i$ be a graded vector space such that $\forall i \in I : \dim V_i < \infty$. Then the Hilbert series of V is the formal power series

$$\sum_{i \in \mathbb{N}} (\dim V_i) t^i \tag{2.61}$$

in the indeterminate t.

For example, consider a bipartite quantum system with n dimensional state spaces for both constituents, and let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ denote the state space of a composite system. Then

$$\mathbb{S}_{(m)} \mathcal{H} = \bigoplus_{\lambda \vdash m} \mathbb{S}_\lambda \mathcal{H}_1 \otimes \mathbb{S}_\lambda \mathcal{H}_2 \tag{2.62}$$

therefore, the dimension of the subspace of degree $2m$ invariants – which is equal to the number of terms on the right side – is the number of partitions of m into at most n parts. The Hilbert series is then

$$\sum_{i\in\mathbb{N}} \dim(S^i(\mathcal{H}\oplus\mathcal{H}^*)^{U(\mathcal{H}_1)\times U(\mathcal{H}_2)})t^i = \frac{1}{(1-t^2)(1-t^4)\cdots(1-t^{2n})} \tag{2.63}$$

It turns out that a lot of information is encoded in this function. Note first that the Hilbert series of the tensor product of graded algebras is the product of their Hilbert series. It is easy to see that the Hilbert series of a unital graded algebra with a single generator of degree d is $(1-t^d)^{-1}$ (see eq. (A.16) and section A.5 for a derivation).

Therefore eq. (2.63) suggests that the algebra of invariant polynomials in this bipartite case is generated by n algebraically independent invariants of degrees $2, 4, \ldots, 2n$. As already mentioned in the introduction, this is indeed the case, and the generators can be chosen to be $Tr(\varrho_1^d)$ with $d = 1, 2, \ldots, n$ where ϱ_1 is the reduced density matrix of the first subsystem.

The LU group has the advantage that $\dim(S^m(\mathcal{H}\oplus\mathcal{H}^*))^G$ stabilizes for any fixed number of subsystems as the dimension of the state spaces of individual subsystems is increased, and G is always taken to be the product of their full unitary groups. We can therefore get rid of the dependence on the dimensions of the state spaces if we write the *stabilized* dimensions of the invariant subspaces in eq. (2.63) instead of those for some fixed n. We have

$$\sum_{i\in\mathbb{N}} \dim(S^i(\mathcal{H}\oplus\mathcal{H}^*)^{U(\mathcal{H}_1)\times U(\mathcal{H}_2)})_{stab}t^i = \prod_{i=1}^{\infty}\frac{1}{(1-t^{2i})} = \frac{1}{\phi(t^2)} \tag{2.64}$$

where ϕ is the Euler function given by

$$\phi(q) = \prod_{i=k}^{\infty}(1-q^k) \tag{2.65}$$

Generating functions formed from the dimensions like in eq. (2.61) are important because they contain information about infinitely many subspaces, complementing the data obtained from studying the individual cases. When working with the stabilized dimensions, only two parameters remain: the number of subsystems k and the degree of the polynomials $2m$. The Hilbert series above are built from the dimensions for fixed k and varying m. Another possibility is to fix m and determine the behaviour of the dimension of the subspace of degree $2m$ polynomial invariants in the function of k.

For the general k case, we would like to start from a decomposition analogous to eq. (2.62). In general, however, the multiplicities on the right side are not only 1s, and therefore the dimension of the subspace of invariants in the corresponding space of homogenous polynomials can be calculated as the sum of the squares of the multiplicities.

Let $k, m \in \mathbb{N}$ and $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ be the state space of a composite system with $\dim \mathcal{H}_i \geq m$ $(1 \leq i \leq k)$. The isotypic decomposition of $S^m(\mathcal{H})$ is given by (see eqs. (B.21-B.22) in the appendix):

$$S^m(\mathcal{H}) \simeq \bigoplus_{\lambda_1,\ldots,\lambda_k} C_{(m)\lambda_1\ldots\lambda_k} \mathbb{S}_{\lambda_1}\mathcal{H}_1 \otimes \cdots \otimes \mathbb{S}_{\lambda_k}\mathcal{H}_k \tag{2.66}$$

where $C_{(m)\lambda_1\ldots\lambda_k}$ is the multiplicity of $V_{(m)}$ in the tensor product $V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_k}$ of irreducible representations of S_m. In terms of the characters of irreducible representations of S_m we have that

$$C_{(m)\lambda_1\ldots\lambda_k} = (\chi_{(m)}, \chi_{\lambda_1}\chi_{\lambda_2}\cdots\chi_{\lambda_k}) \tag{2.67}$$

The dimension of the fixed subspace is

$$d_{k,m} := \dim S^{2m}(\mathcal{H} \oplus \mathcal{H}^*)^G = \sum_{\lambda_1,\ldots,\lambda_k \vdash m} C^2_{(m)\lambda_1\ldots\lambda_k} \tag{2.68}$$

Let us introduce the following notation:

$$D_{k,\nu,\nu'} := \sum_{\lambda_1,\ldots,\lambda_k \vdash m} C_{\nu\lambda_1\ldots\lambda_k} C_{\nu'\lambda_1\ldots\lambda_k} \tag{2.69}$$

where $\nu, \nu' \vdash m$. Then clearly $d_{k,m} = D_{k,(m),(m)}$ and we have the recursion

$$\begin{aligned} D_{k,\nu,\nu'} &= \sum_{\lambda_1,\ldots,\lambda_k \vdash m} C_{\nu\lambda_1\ldots\lambda_k} C_{\nu'\lambda_1\ldots\lambda_k} \\ &= \sum_{\substack{\lambda_1,\ldots,\lambda_k \vdash m \\ \mu,\mu' \vdash m}} C_{\nu\mu\lambda_k} C_{\mu\lambda_1\ldots\lambda_{k-1}} C_{\nu'\mu'\lambda_k} C_{\mu'\lambda_1\ldots\lambda_{k-1}} \\ &= \sum_{\substack{\lambda_1,\ldots,\lambda_{k-1}, \\ \mu,\mu' \vdash m}} \underbrace{\left(\sum_{\lambda_k \vdash m} C_{\nu\mu\lambda_k} C_{\nu'\mu'\lambda_k} \right)}_{C_{\nu\nu'\mu\mu'}} C_{\mu\lambda_1\ldots\lambda_{k-1}} C_{\mu'\lambda_1\ldots\lambda_{k-1}} \\ &= \sum_{\mu\mu'} C_{\nu\nu'\mu\mu'} D_{k-1,\mu,\mu'} \end{aligned} \tag{2.70}$$

If we imagine the collection of numbers $C_{\nu\nu'\mu\mu'}$ as a square matrix C whose width and height is the square of the number of partitions of m, the rows being indexed by pairs (ν, ν') and the columns by (μ, μ'), and $D_{k,\nu,\nu'}$

as a column vector D_k, then eq. (2.70) is clearly a matrix multiplication. Therefore, $D_k = C^{k-1}D_1$ where the (ν, ν') entry of D_1 is 1 if $\nu = \nu'$ and 0 otherwise. In principle we can obtain $d_{k,m}$ for any values of k and m, and the generating function

$$\sum_{k=1}^{\infty} d_{k,m} t^k \tag{2.71}$$

is simply the $((m),(m))$ entry of the vector

$$\sum_{k=1}^{\infty} D_k t^k = \sum_{k=1}^{\infty} t(tC)^{k-1} D_1 = t(I - tC)^{-1} D_1 \tag{2.72}$$

where I denotes the identity matrix. In order to evaluate this expression, one needs to know how triple tensor products of irreducible S_m-spaces decompose to irreducibles, or calculate C using the alternative formula

$$C_{\nu\nu'\mu\mu'} = (\chi_\nu \chi_{\nu'}, \chi_{\mu'} \chi_{\mu'}) \tag{2.73}$$

For example the $m = 2$ case yields

$$\sum_{k=1}^{\infty} d_{k,2} t^k = \frac{t}{1 - 2t} \tag{2.74}$$

while for $m = 3$ we have

$$\sum_{k=1}^{\infty} d_{k,3} t^k = \frac{t(1 - 8t + 14t^2)}{(1 - 2t)(1 - 3t)(1 - 6t)} \tag{2.75}$$

2.8 Stable dimensions of the spaces of LU-invariant polynomials

We describe another way to obtain the dimensions of subspaces of LU-invariant polynomials. This method does not give us a generating function directly, but it has the advantage that one does not need to know the decomposition of tensor products of irreducible representations of S_m for which there does not exist a simple formula. The following derivation is a slightly modified version of the one found in ref.[26]

Using the fact that every representation of S_m has real character[22], we can write the dimension as

$$\begin{aligned} d_{k,m} &= \sum_{\lambda_1,\ldots,\lambda_k \vdash m} C^2_{(m)\lambda_1,\ldots,\lambda_k} \\ &= \sum_{\lambda_1,\ldots,\lambda_k \vdash m} (\chi_{(m)}, \chi_{\lambda_1}\cdots\chi_{\lambda_k})^2 \\ &= \sum_{\lambda_1,\ldots,\lambda_k \vdash m} (\chi_{\lambda_1}\cdots\chi_{\lambda_{k-1}}, \chi_{\lambda_k})(\chi_{\lambda_k}, \chi_{\lambda_1}\cdots\chi_{\lambda_{k-1}}) \\ &= \sum_{\lambda_1,\ldots,\lambda_{k-1} \vdash m} (\chi_{\lambda_1}\cdots\chi_{\lambda_{k-1}}, \chi_{\lambda_1}\cdots\chi_{\lambda_{k-1}}) \\ &= \sum_{\lambda_1,\ldots,\lambda_{k-1} \vdash m} (\chi_{(m)}, \chi^2_{\lambda_1}\cdots\chi^2_{\lambda_{k-1}}) \\ &= (\chi_{(m)}, \sum_{\lambda_1,\ldots,\lambda_{k-1} \vdash m} \chi^2_{\lambda_1}\cdots\chi^2_{\lambda_{k-1}}) \\ &= (\chi_{(m)}, \left(\sum_{\lambda \vdash m} \chi^2_\lambda\right)^{k-1}) \end{aligned} \tag{2.76}$$

where we have made use of the fact that irreducible characters form an orthonormal basis in the space of class functions. All we have left is to determine the character of the S_m-representation

$$\bigoplus_{\lambda \vdash m} V_\lambda \otimes V_\lambda \tag{2.77}$$

A general theorem states[21] (see theorem B.2.2 in section B.2) that the representation of a finite group G acting on the group algebra $\mathbb{C}G$ via conjugation decomposes as

$$\bigoplus_{i \in \mathrm{Irr}(G)} V_i \otimes V_i^* \tag{2.78}$$

Again, using that irreducible representations of S_m are isomorphic to their duals, we have that $\sum_\lambda \chi^2_\lambda$ is the character of the conjugation representation on $\mathbb{C}S_m$.

The character of a permutation representation such as the one above is easy to describe: the value on a group element is the number of points fixed by that element. In the case of the conjugation action, this is exactly the number of group elements which the given group element commutes with, that is, the order of its centralizer. By the orbit-stabilizer formula we have that the order of the centralizer of an element multiplied by the size of its conjugacy class yields the order of the group.

According to eq. (2.76), we need to calculate the multiplicity of the trivial character in the conjugation representation. Let R be a set of representatives of the conjugacy classes of G. Then

$$\begin{aligned} \frac{1}{|G|}\sum_{g\in G}\chi_{conj.}(g)^{k-1} &= \frac{1}{|G|}\sum_{g\in R}C(g)\chi_{conj.}(g)^{k-1} \\ &= \frac{1}{|G|}\sum_{g\in R}\frac{|G|}{|Z_G(g)|}|Z_G(g)|^{k-1} \\ &= \sum_{g\in R}|Z_G(g)|^{k-2} \end{aligned} \tag{2.79}$$

where $Z_G(\cdot)$ denotes the centralizer.

Returning to our case, we have to know the order of centralizers in S_m. Conjugacy classes in S_m can be labelled by cycle types (see section B.3 in the appendix), which are m-tuples $a = (a_1, a_2, \ldots, a_m)$ of nonnegative integers having the property

$$\sum_{i=1}^{m} i a_i = m \tag{2.80}$$

We denote this fact by $a \Vdash m$ and say that a is a type of m. Note that a type $a \Vdash m$ is just another way to describe the partition $\lambda = (m^{a_m}, (m-1)^{a_{m-1}}, \ldots, 1^{a_1}) \vdash m$.

The order of the centralizer of an element in the conjugacy class labelled by a is given by[22]

$$|Z_{S_m}(g)| = \prod_{i=1}^{m} i^{a_i} a_i! \tag{2.81}$$

where g is an element with cycle type a, that is, it is the product of disjoint cycles, such that the number of length i factors is a_i. We conclude that

$$d_{k,m} = \sum_{a \Vdash m}\left(\prod_{i=1}^{m} i^{a_i} a_i!\right)^{k-2} \tag{2.82}$$

The dimensions for some small values of k and m are given in table 2.1.

Note that $d_{k,m}$ grows very rapidly in both k and m. Namely, for a fixed m, $d_{k,m}$ is asymptotically equal to $(m!)^{k-2}$.

$d_{k,m}$	$k=1$	2	3	4	5
$m=1$	1	1	1	1	1
2	1	2	4	8	16
3	1	3	11	49	251
4	1	5	43	681	14491

Table 2.1: Stabilized dimensions of spaces of degree $2m$ LU-invariants for a composite quantum system with k subsystems.

So far we have regarded the $d_{k,m}$ as dimensions of subspaces of different but related algebras, but it is possible to introduce a single graded algebra for a fixed k whose graded parts are of dimension $d_{k,m}$, and which is closely related to the algebras of LU-invariant polynomials [50]. In the construction every possible set of single particle state space dimensions appears, therefore we introduce first some notations which show explicitely these dimensions.

Let $k \in \mathbb{N}$ and $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$, and consider the complex Hilbert space $\mathcal{H}_n = \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$ describing the pure states of a composite system with k distinguishable subsystems. The group of local unitary transformations, $LU_n = U(n_1, \mathbb{C}) \times \cdots \times U(n_k, \mathbb{C})$, acts on $\mathcal{H}$ in the obvious way, i.e. regarding $\mathbb{C}^{n_i}$ as the standard representation of $U(n_i, \mathbb{C})$.

Let $I_{k,n}$ denote the graded algebra of polynomials over $\mathcal{H}_n$ which are invariant under the action of LU_n. Suppose that $n, n' \in \mathbb{N}^k$ such that $n \leq n'$ with respect to the componentwise (product) partial order. Then we have the inclusion $\iota_{n,n'} : \mathcal{H}_n \hookrightarrow \mathcal{H}_{n'}$ which is the tensor product of the usual inclusions $\mathbb{C}^{n_i} \hookrightarrow \mathbb{C}^{n'_i}$ sending an n_i-tuple to the first n_i components. We can similarly regard LU_n as a subgroup of $LU_{n'}$ which stabilizes the image of $\iota_{n,n'}$, and thus $\iota_{n,n'}$ is an LU_n-equivariant linear map. Therefore it induces a morphism of graded algebras $\varrho_{n,n'} : I_{k,n'} \to I_{k,n}$ (note that the algebra of polynomials *on* a vector space is the symmetric algebra of its *dual* space, which is a contravariant construction).

Clearly, $\iota_{n,n}$ is the identity and if $n \leq n' \leq n''$ then $\iota_{n',n''} \circ \iota_{n,n'} = \iota_{n,n''}$, which implies that $\varrho_{n,n} = \mathrm{id}_{I_{k,n}}$ and $\varrho_{n,n'} \circ \varrho_{n',n''} = \varrho_{n,n''}$. The central object which we study is the inverse limit of this system of graded algebras and their morphisms (see sections A.1 and A.5 for the definition):

$$I_k := \varprojlim_{n \in \mathbb{N}^k} I_{k,n} = \left\{ (f_n)_{n \in \mathbb{N}^k} \in \prod_{n \in \mathbb{N}^k} I_{k,n} \middle| \forall n \leq n' : f_n = \varrho_{n,n'} f_{n'} \right\} \quad (2.83)$$

Note that as the product is taken in $\mathbb{N} - \mathbf{CAlg}_{\mathbb{C}}$, it consists of sequences with bounded degree. We will call I_k the algebra of LU-invariants.

Lemma 2.8.1. *Suppose that $n, n' \in \mathbb{N}^k$ and $n \leq n'$. Let $m \in \mathbb{N}$ such that* *for all i we have $m \leq n_i$. Then the restriction of $\varrho_{n,n'}$ is an isomorphism (of vector spaces) between the spaces of homogenous degree m elements of $I_{k,n'}$ and $I_{k,n}$.*

Proof. As it was already noted, the dimensions of the two homogenous parts are equal. We will show that $\varrho_{n,n'}$ is injective on elements of degree at most m.

Suppose first that for some $1 \leq i \leq k$ we have $n_i = n'_i - 1$ and for $j \neq i$ we have $n_j = n'_j$. Let $\{e_{1,1,\ldots,1}, \ldots, e_{n'_1,\ldots,n'_k}\}$ be the basis of $\mathcal{H}_{n'}$ formed by tensor products of standard basis elements of the $\mathbb{C}^{n_i}$. Then the algebra of real polynomials is generated by the coordinate functions $\{e^*_{1,1,\ldots,1}, \ldots, e^*_{n'_1,\ldots,n'_k}\}$ and their conjugates $\{\overline{e^*_{1,1,\ldots,1}}, \ldots, \overline{e^*_{n'_1,\ldots,n'_k}}\}$.

Let $f \in I_{k,n'}$ be a degree m homogenous polynomial such that $\varrho_{n,n'} f = 0$. This means that f vanishes on the image of $\iota_{n,n'}$. Denoting by J_a the ideal generated by elements of the form $\{e_{j_1,\ldots,j_k}\}$ such that $j_i = a$, we can reformulate this fact as $f \in J_{n_i}$. Note that f is invariant under the action of LU_n, and we have the subgroup $S_{n_i} \leq LU_n$ which permutes the basis elements in the ith factor $\mathbb{C}^{n_i}$, therefore f is also contained in the ideals $J_1, \ldots, J_{n_i}$.

But the intersection of the ideals $J_1, \ldots, J_{n_1}$ is their product, therefore f is in the ideal generated by n_1-fold products of the coordinate functions. As f is LU_n-invariant, its terms must contain the same number of conjugate coordinate functions, and therefore its homogenous parts of degree less then n_1 vanish. $m \leq n_1$ implies that $f = 0$.

For the general case, observe that if $n \leq n'$ then $\varrho_{n,n'}$ can be written as a composition of the maps considered above (or is the identity in the case of $n = n'$), and hence also injective. □

This lemma means that every element of I_k is represented in some $I_{k,n}$ (it suffices to take n to be $(m, m, \ldots, m)$ with m the degree of the element), and that if $n_{min} = \min\{n_i\}$ then the factors of I_k and $I_{k,n}$ by the ideals generated by homogenous elements of degree at least $n_{min}+1$ are isomorphic. Therefore, the algebras $I_{k,n}$ and I_k are closely related, while the latter seems considerably simpler to study.

The Hilbert series of the algebra I_k is the formal power series

$$\sum_{m \geq 0} d_{k,m} t^m \tag{2.84}$$

Using eq. (2.82) this can be rewritten as

$$\begin{aligned}\sum_{m\geq 0} d_{k,m}t^m &= \sum_{m\geq 0}\sum_{a\Vdash m}\left(\prod_{i=1}^{m} i^{a_i}a_i!\right)^{k-2} t^m \\ &= \sum_{a_1,a_2,\ldots\geq 0}\prod_{i=1}^{m}\left(i^{a_i}a_i!\right)^{k-2}t^{ia_i} \\ &= \prod_{i\geq 1}\left(\sum_{a\geq 0}(i^a a!)^{k-2}t^{ia}\right) \\ &= \prod_{d\geq 1}(1-t^d)^{-u_d(F_{k-1})}\end{aligned} \tag{2.85}$$

where in the last row $u_d(G)$ denotes the number of conjugacy classes of index d subgroups of a group G and F_{k-1} is the free group on $k-1$ generators. This last equality can be found in [45].

The formula obtained suggests the following conjecture[50] (here we return to the usual grading, which differs from the previously used one by a factor of two):

Conjecture 2.8.2. *The algebra of LU-invariants I_k of k-partite quantum systems is free, and the number of degree $2d$ invariants in an algebraically independent generating set equals the number of conjugacy classes of index d subgroups in the free group on $k-1$ generators.*

Note that this conjecture turns out to be true, a proof along with an algebraically independent generating set can be found in ref. [50].

2.9 SLOCC-invariants

At this point we would like to say something about SLOCC invariants as well. As the inner product on $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ is not invariant under the action of the SLOCC group $G = GL(\mathcal{H}_1) \times \cdots \times GL(\mathcal{H}_k)$, there is no element in $\mathcal{H}^*$ corresponding to a vector $\psi \in \mathcal{H}$. As a consequence, we cannot consider polynomials in the complex conjugates of the coefficients of ψ, as this notion would not be G-invariant. But this means also that no nonconstant polynomial can be invariant under G, as $(c\,\mathrm{id}_{\mathcal{H}_1}, \mathrm{id}_{\mathcal{H}_2}, \ldots, \mathrm{id}_{\mathcal{H}_k})$ acts on a homogenous polynomial of degree m with multiplication by c^m.

One possible – and frequently used – solution to this problem is to consider the subgroup of transformations with determinant one. In this case we are essentially working with the group $G = SL(\mathcal{H}_1) \times \cdots \times SL(\mathcal{H}_k)$, and look for invariants under the action of this group. Homogenous invariants span one dimensional representations of the SLOCC group (see section B.4 in the appendix), and hence transform with a power of the determinant. These polynomials are often called SLOCC-invariant, by abuse of language. Although the action of an element of $GL(\mathcal{H}_1)\times\cdots\times GL(\mathcal{H}_k)$ can change the value of such a polynomial, its vanishing or non-vanishing is still an invariant notion.

Generating sets of the algebras of determinant 1 SLOCC-invariants have been obtained for up to four qubits[33], and partial results exist for five qubits[34].

As we have already noted, the determinant 1 SLOCC-invariant polynomials do *not* separate the orbits under this group. It is true however, that these polynomials separate *closed* orbits, and that the closure of any orbit contains exactly one closed orbit. The set of closed orbits has the structure of an affine algebraic variety over $\mathbb{C}$ and its coordinate ring is the algebra of determinant 1 SLOCC-invariant polynomials.

Our task is now to study the algebra of polynomials $S(\mathcal{H})$. For each m, the homogenous part $S^m(\mathcal{H})$ is the direct sum of irreducible G-spaces $\mathbb{S}_{\lambda_1}\mathcal{H}_1\otimes\cdots\otimes\mathbb{S}_{\lambda_1}\mathcal{H}_1$ characterized by k-tuples of partitions of m. Such an irreducible representation is one dimensional iff the factors are all one dimensional, therefore we need that $\forall i\in[k]:\lambda_i=(\frac{m}{n_i},\ldots,\frac{m}{n_i})$ where $n_i=\dim\mathcal{H}_i$, that is the partitions appearing are rectangle-shaped. In particular, m must be a multiple of the least common multiple of the n_i-s.

One can see that the structure of SLOCC-invariants is much more “rigid” than that of LU-invariants: we can imagine how changing the dimensions of the individual state spaces makes invariants vanish and reappear with completely different degrees, according to the varying of the least common multiple of the dimensions. In contrast, an LU-invariant (more precisely its generalizations) remains there when any of the dimensions of the Hilbert spaces is increased.

On the other hand, for a fixed $\mathcal{H}$ and m we have only one k-tuple of partitions such that the corresponding isotypic component contains SLOCC-invariants, and this fact makes it easier to compute the dimensions of the fixed subspaces (that is, subspaces of invariants). Indeed

$$\dim\left(S^m(\mathcal{H})\right)^G=(\chi_{(m)},\prod_{i=1}^{k}\chi_{(\frac{m}{n_i},\ldots,\frac{m}{n_i})})_{S_m}\tag{2.86}$$

with n_i as above.

Note that the special case when the dimensions of the individual Hilbert spaces are equal was studied in ref.[15]. For example, suppose that $\forall i\in[k]:$

$n_i = n$ and $m = n$. In this simple case only (1^n) appears, corresponding to the alternating representation. Thus

$$\dim\left(S^n(\mathcal{H})\right)^G = (\chi_{(n)}, {\chi_{(1^n)}}^k)_{S_n} = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases} \tag{2.87}$$

Let us now look at the $k = 2$ case. The value of $(\chi_{(m)}, \chi_{\lambda_1}\chi_{\lambda_2})$ is 1 if $\lambda_1 = \lambda_2 \vdash m$ and 0 otherwise. Therefore, there are *no* nontrivial SLOCC invariants if $n_1 \neq n_2$ and if $n_1 = n_2 = n$ then the subspace of invariants in each degree which is a multiple of n is 1. But then the algebra of invariants must be generated by a single invariant of degree n, and of course this is the determinant of the matrix built from the coefficients of the state in an arbitrary computational basis.

Finally, let us consider an example which does not fall into any of the previously discussed cases. Again suppose that $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ with dimensions $n_i = \dim \mathcal{H}_i$. Let $d \in \mathbb{N}$ be a fixed number, and suppose that $\{n_1, n_2, \ldots, n_k\} = \{d, d^2\}$, that is, all the single particle state spaces are either of dimension d or d^2, but not all are equal. Then the least common multiple of the dimensions is d^2, so we would like to look at $S^m(\mathcal{H})$ for $m = d^2$.

Two partitions of m are relevant, namely (1^m) and (d^d). The first one corresponds to the alternating representation, while the second to a representation of S_m which is self-conjugate, as the conjugate partition of (d^d) is itself. Consequently, $\chi_{(d^d)}\chi_{(1^m)} = \chi_{(d^d)}$. Therefore in this case

$$\dim\left(S^m(\mathcal{H})\right)^G = (\chi_{(m)}, {\chi_{(d^d)}}^l)_{S_m} \tag{2.88}$$

where $l := |\{i | n_i = d\}|$ is the number of d dimensional state spaces in the tensor product.

The simplest nontrivial concrete example is $d = 2$. According to the character table of S_4 (see table B.4 in section B.3), $\chi_{(2,2)}$ is 2 on the identity element and the conjugacy class of (12)(34) (3 elements), -1 on the conjugacy class of (123) (8 elements) and 0 otherwise. Thus

$$\begin{aligned} \dim\left(S^4(\mathcal{H})\right)^G &= (\chi_{(4)}, {\chi_{(2,2)}}^l)_{S_4} \\ &= \frac{1}{24}(2^l + 3 \cdot 2^l + 8(-1)^l) \\ &= \frac{2^l}{6} + \frac{(-1)^l}{3} \end{aligned} \tag{2.89}$$

which yields $0, 1, 1, 3, 5, \ldots$ for $l = 1, 2, 3, 4, 5, \ldots$ in agreement with (and generalizing the results of) refs. [15, 35].

2.10 Local unitary invariants for mixed states

Observe that the derivation in eq. (2.76) remains valid when we only assume that $\dim \mathcal{H}_k \geq m$ with some modification. If $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ where $\dim \mathcal{H}_i = n_i$ and $n_k \geq m$ then

$$\begin{aligned} d_{(n_1,\ldots,n_{k-1}),m} &:= \dim(S^{2m}(\mathcal{H} \oplus \mathcal{H}^*))^{LU} \\ &= (\chi_{(m)}, \prod_{i=1}^{k-1} \left(\sum_{\substack{\lambda \vdash m \\ |\lambda| \leq n_i}} \chi_\lambda^2 \right)) \end{aligned} \tag{2.90}$$

Moreover, in this case we do not need $n_k \to \infty$ to have stabilization for every m, the condition $n_k \geq n_1 \cdot n_2 \cdot \cdots \cdot n_{k-1}$ is sufficient.

An important special case is when $n_i = 2$ for $i < k$. In this case the above formula reduces to

$$\begin{aligned} d_{(2,\ldots,2),m} &= \dim(S^{2m}(\mathcal{H} \oplus \mathcal{H}^*))^{LU} \\ &= (\chi_{(m)}, \left(\sum_{\substack{\lambda \vdash m \\ |\lambda| \leq 2}} \chi_\lambda^2 \right)^{k-1}) \end{aligned} \tag{2.91}$$

One can also find a physical interpretation of this condition. Regarding $\mathcal{H}_S := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}$ as the Hilbert space of an *open* quantum system interacting with its environment with state space $\mathcal{H}_{ENV} := \mathcal{H}_k$, the condition above ensures that every mixed state over $\mathcal{H}_S$ arises as the reduced state of a pure state of $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_{ENV}$.

Recall that in this case given a mixed state $\varrho \in \mathrm{End}(\mathcal{H}_S)$ we can find a vector $\psi \in \mathcal{H}$ (called a purification of ϱ) such that $\varrho = \mathrm{Tr}_{ENV} \psi\psi^*$ and that ψ is unique up to transformations of the form $id_{\mathcal{H}_S} \otimes U$ where $U \in U(\mathcal{H}_{ENV})$ [39]. As $\mathrm{Tr}_{ENV} : \mathrm{End}(\mathcal{H}_S \otimes \mathcal{H}_{ENV}) \to \mathrm{End}(\mathcal{H}_S)$ is $U(\mathcal{H}_S) \times U(\mathcal{H}_{ENV})$-equivariant, we have that purification gives a bijection between LU-equivalence classes of mixed states over $\mathcal{H}_S$ and pure states in $\mathcal{H}$ with the partial trace as inverse.

Denoting the set of mixed states over a Hilbert space $\mathcal{H}$ by

$$D(\mathcal{H}) := \{A \in \mathrm{End}(\mathcal{H}) | A \geq 0, \mathrm{Tr}\, A = 1\} \tag{2.92}$$

and the set of unit vectors by

$$P(\mathcal{H}) := \{\psi \in \mathcal{H} | \|\psi\|^2 = 1\} \tag{2.93}$$

we can write the commutative diagram

$$
\begin{array}{ccc}
P(\mathcal{H}) & \xrightarrow{\operatorname{Tr}_{ENV} \circ P} & D(\mathcal{H}_S) \\
\downarrow & & \downarrow \\
P(\mathcal{H})/LU & \overset{\sim}{\dashrightarrow} & D(\mathcal{H}_S)/LU
\end{array}
\tag{2.94}
$$

where $P : \mathcal{H} \to \operatorname{End}(\mathcal{H})$ is defined by $\psi \mapsto \psi\psi^*$, $LU = U(\mathcal{H}_1) \times \cdots \times U(\mathcal{H}_{k-1}) \times U(\mathcal{H}_k)$ is the local unitary group acting in the obvious way and the vertical arrows are the quotient maps. Note that when $n_k < n_1 \cdot n_2 \cdot \cdots \cdot n_{k-1}$, the lower horizontal map making this diagram commutative (as well as $\operatorname{Tr}_{ENV} \circ P$) fails to be surjective.

Using the fact that $\operatorname{Tr}_{ENV} \circ P : P(\mathcal{H}) \to D(\mathcal{H}_S)$ is an equivariant polynomial function (of degree 2), from a polynomial invariant $f : D(\mathcal{H}_S) \to \mathbb{C}$ on pure states we can always construct one on mixed states, namely $f \circ \operatorname{Tr}_{ENV} \circ P$. Similarly, if we are given an invariant $g : P(\mathcal{H}) \to \mathbb{C}$, we can pull it back via the isomorphism $D(\mathcal{H}_S)/LU \to P(\mathcal{H})/LU$ to obtain an invariant on mixed states:

$$
\begin{array}{ccc}
\mathbb{C} & \dashleftarrow & D(\mathcal{H}_S) \\
\uparrow g & & \\
P(\mathcal{H}) & \xrightarrow{\operatorname{Tr}_{ENV} \circ P} & D(\mathcal{H}_S) \\
\downarrow & & \downarrow \\
P(\mathcal{H})/LU & \xrightarrow{\sim} & D(\mathcal{H}_S)/LU
\end{array}
\tag{2.95}
$$

The two constructions are clearly inverses of each other, but it is not clear that f is polynomial whenever $f \circ \operatorname{Tr}_{ENV} \circ P$ is polynomial.

To prove this, observe that the map $S^m(\operatorname{End}(\mathcal{H}_S)) \to S^{2m}(\mathcal{H} \oplus \mathcal{H}^*)$ defined by $f \mapsto f \circ \operatorname{Tr}_{ENV} \circ P_2$ is an injective linear map where $P = P_2 \circ P_1$ and $P_1 : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}^*$ is defined by $\psi \mapsto \psi \oplus \psi^*$ while $P_2 : \mathcal{H} \oplus \mathcal{H}^* \to \mathcal{H} \otimes \mathcal{H}^*$ is defined by $\psi \oplus \varphi^* \mapsto \psi\varphi^*$. As the appearing vector spaces are by assumption finite dimensional, we need to show that these dimensions are equal.

Let $\mathcal{H}_S = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}$ be the state space of a composite quantum system, and $\mathcal{H}_{ENV}$ the state space of its environment as before, and let $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_{ENV}$ denote the Hilbert space of the joint system composed from the two. Then

$$\begin{aligned}
S^{2m}(\mathcal{H}\oplus\mathcal{H}^*)^{U(\mathcal{H}_{ENV})} &\simeq \\
&\simeq S^m(\mathcal{H})\otimes S^m(\mathcal{H}^*)^{U(\mathcal{H}_{ENV})} \\
&\simeq \left(\bigoplus_{\lambda,\lambda'\vdash m}\mathbb{S}_\lambda\mathcal{H}_S\otimes\mathbb{S}_\lambda\mathcal{H}_{ENV}\otimes\mathbb{S}_{\lambda'}\mathcal{H}_S^*\otimes\mathbb{S}_{\lambda'}\mathcal{H}_{ENV}^*\right)^{U(\mathcal{H}_{ENV})} \\
&\simeq \bigoplus_{\lambda\vdash m}(\mathbb{S}_\lambda\mathcal{H}_S\otimes\mathbb{S}_\lambda\mathcal{H}_{ENV}\otimes\mathbb{S}_\lambda\mathcal{H}_S^*\otimes\mathbb{S}_\lambda\mathcal{H}_{ENV}^*)^{U(\mathcal{H}_{ENV})} \\
&\simeq \bigoplus_{\lambda\vdash m}\mathbb{S}_\lambda\mathcal{H}_S\otimes\mathbb{S}_\lambda\mathcal{H}_S^* \\
&\simeq S^m(\mathcal{H}_S\otimes\mathcal{H}_S^*) = S^m(\mathrm{End}(\mathcal{H}_S))
\end{aligned} \tag{2.96}$$

as $U(\mathcal{H}_S)\times U(\mathcal{H}_{ENV})$-spaces where we have used that

$$(\mathbb{S}_\lambda\mathcal{H}_{ENV}\otimes\mathbb{S}_\lambda\mathcal{H}_{ENV}^*)^{U(\mathcal{H}_{ENV})}\simeq\begin{cases}\mathbb{C} & \text{if } \dim\mathcal{H}_{ENV}\geq|\lambda| \\ 0 & \text{if } \dim\mathcal{H}_{ENV}<|\lambda|\end{cases} \tag{2.97}$$

with the trivial representation on $\mathbb{C}$ and that $\mathbb{S}_\lambda\mathcal{H}_S\simeq 0$ iff $|\lambda|>\dim\mathcal{H}_S\leq\dim\mathcal{H}_{ENV}$

Similarly to the case of pure state invariants, we may construct the inverse limit of all the algebras of invariant polynomials with a fixed number of subsystems.

For $k\in\mathbb{N}$ and $n\in\mathbb{N}^k$, let $I_{k,n}^{mixed}$ denote the algebra of LU_n-invariant (real) polynomials on $\mathrm{End}(\mathcal{H}_n)$. The inclusions $\iota_{n,n'}:\mathcal{H}_n\to\mathcal{H}_{n'}$ induce also in this case the maps $\varrho_{n,n'}:I_{k,n'}^{mixed}\to I_{k,n}^{mixed}$ with similar composition properties as in the case of pure states. Let us consider the inverse limit of this system:

$$I_k^{mixed}:=\varprojlim_{n\in\mathbb{N}^k}I_{k,n}^{mixed}=\left\{(f_n)_{n\in\mathbb{N}^k}\in\prod_{n\in\mathbb{N}^k}I_{k,n}^{mixed}\middle|\forall n\leq n':f_n=\varrho_{n,n'}f_{n'}\right\} \tag{2.98}$$

As in the case of pure state invariants, this inverse limit also has the property that every element is represented in some $I_{k,n}^{mixed}$, and that there is no difference between $I_{k,n}^{mixed}$ and I_k^{mixed} when we consider only elements with degree at most the minimum of the dimensions $\{n_i\}_{1\leq i\leq k}$.

The above-shown correspondence between mixed and pure state invariants is clearly reflected in the isomorphism $I_k^{mixed}\simeq I_{k+1}$ induced by the isomorphisms

$$I_{k,(n_1,\ldots,n_k)}^{mixed}\simeq I_{k+1,(n_1,\ldots,n_k,n_1\cdot\ldots\cdot n_k)} \tag{2.99}$$

described above. Note that the grading of I_k^{mixed} is the usual one which is to be contrasted with the extra factor of 2 in the grading of I_k. With this convention, the map $f \mapsto f \circ \mathrm{Tr}_{ENV} \circ P$ respects the degree.

We can also formulate our conjecture in terms of mixed state invariants:

Conjecture 2.10.1. *The algebra of mixed state LU-invariants I_k^{mixed} of k-partite quantum systems is free, and the number of degree d invariants in an algebraically independent generating set equals the number of conjugacy classes of index d subgroups in the free group on k generators.*

This conjecture is shown to be true in ref. [50], and an algebraically independent generating set of mixed state invariants is also presented there.

Chapter 3

Entanglement of identical particles

Although quantum information theory traditionally dealt with quantum systems containing distinguishable parts, in the recent years an increasing interest towards entanglement properties of fermionic and bosonic particles appeared [54, 19, 3, 24, 23]. In this chapter we discuss some aspects of entanglement of identical particles as well as connections to the case of distinguishable subsystems.

In section 3.1 we formulate the entanglement classification problem for quantum systems containing identical particles, emphasizing only the differences between these systems and those consisting of distinguishable particles.

Section 3.2 deals with some special tripartite quantum systems containing both identical and distinguishable subsystems. The quantum systems considered are very special, and their SLOCC classification is related to an algebraic construction which was originally introduced in order to obtain representations of exceptional Lie groups. The uniform treatment of these systems results also in a close similarity of their entanglement properties.

In section 3.3 we argue that these similarities are in fact manifestations of a more general phenomenon. We point out a relationship between entanglement measures of different quantum systems, and present an explanation using representation theory.

In sections 3.4-3.5 we outline a method which can be utilized in order to find local unitary invariants for fermionic quantum systems, and show how the simplest invariant of every even degree for any number of fermions can be obtained.

Section 3.6 contains three worked out examples which illustrate some features of this method.

In section 3.7 it is shown how LU-invariants and SLOCC-invariants are related for multifermion systems. The key idea in this section is that a SLOCC invariant can always be generalized into arbitrary dimensional single particle state space, but this generalization only has LU-invariance. The simplest case of this generalization is discussed in more detail.

In section 3.8 we consider fourth order LU-invariant polynomials of quantum systems of femions or bosons. This paralells with the case of fourth order LU-invariant polynomials with an additional permutation symmetry for quantum systems with distinguishable constituents.

3.1 Introduction

So far we have been concerned with composite quantum systems with distinguishable constituents. The situation is somewhat different if there are identical particles in the system, and there are no observables which can distinguish between them (see section C.1 in the appendix). For example, consider a quantum system with k identical particles with single particle state spaces $\mathcal{H}_1$ of dimension n. Then the state space of the composite system can be either

$$S^k(\mathcal{H}_1) \equiv \mathbb{S}_{(k)}\mathcal{H}_1 \tag{3.1}$$

in the case of bosons or

$$\Lambda^k(\mathcal{H}_1) \equiv \mathbb{S}_{(1^k)}\mathcal{H}_1 \tag{3.2}$$

in the case of fermions. (see section B.3 in the appendix for the definition of Schur functors)

These spaces are usually thought of as subspaces of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1$ spanned by symmetrized and antisymmetrized elementary tensors, defined by

$$e_{i_1} \vee e_{i_2} \vee \cdots \vee e_{i_k} = \sum_{\pi \in S_k} e_{i_{\pi(1)}} \otimes \cdots \otimes e_{i_{\pi(k)}} \tag{3.3}$$

and

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} = \sum_{\pi \in S_k} \chi_{(1^k)}(\pi) e_{i_{\pi(1)}} \otimes \cdots \otimes e_{i_{\pi(k)}} \tag{3.4}$$

respectively, where $\chi_{(1^k)}$ is the alternating character of S_k, and $\{e_i\}_{i=1}^n$ is an orthonormal basis of $\mathcal{H}_1$.

The algebra of physically meaningful observables can accordingly be realized as a subspace of $\mathrm{End}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1)$ as follows: the representation of S_k

on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1$ induces an S_k-space structure on $\mathrm{End}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1)$. The set of observables is then the subspace fixed by this action. Indeed, these correspond exactly to those operators that are "insensitive to permutations of identical particles", which can be expressed mathematically as commuting with all particle-exchange operators.

Note that according to our former notions introduced in the case of distinguishable subsystems, states like in eqs. (3.3-3.4) may seem entangled, however, the fact that the set of observables is constrained implies that the correlations resulting from these forms – or the statistics these particles must obey – are uninteresting from the quantum information theoretical point of view. Of course, states of these types already show behaviour which differs from that of distinguishable particles – think of ideal gases of fermions and bosons at low temperatures – but physical processes involving only such states are not capable of outperforming classical information processing protocols. We will call states described by (anti)symmetrized elementary tensors separable [23].

We would like to follow a similar program as in the case of distinguishable subsystems. Many results apply without any change, the first interesting question is that of identifying the analogues of "local transformations". Clearly, we cannot single out particles and manipulate only one of them – that would already imply that the particles are distinguishable. But there is another characterization of local transformations in the distinguishable case: these are precisely the possible time evolutions when no (direct or indirect) interaction between the subsystems is allowed.

This latter approach turns out to be fruitful. Recall that the evolution of noninteracting identical particles is governed by Hamiltonians which are built solely from one-particle operators, that is, they are of the form

$$\begin{aligned} H = A \otimes \mathrm{id}_{\mathcal{H}_1} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}_1} + \mathrm{id}_{\mathcal{H}_1} \otimes A \otimes \mathrm{id}_{\mathcal{H}_1} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}_1} + \cdots \\ \cdots + \mathrm{id}_{\mathcal{H}_1} \otimes \cdots \otimes \mathrm{id}_{\mathcal{H}_1} \otimes A \end{aligned} \tag{3.5}$$

where $A \in \mathrm{End}(\mathcal{H}_1)$.

The terms on the right side commute with each other, therefore we can exponentiate termwise, and then multiply the results, leading to a unitary evolution of the form

$$U(t) = U_1(t) \otimes U_1(t) \otimes \cdots \otimes U_1(t) \tag{3.6}$$

where

$$U_1(t) = e^{\frac{i}{\hbar} tA} \tag{3.7}$$

The case of nonunitary evolutions can be treated similarly. This is the motivation behind the following definitions:

Definition. Let $\mathcal{H}_1$ be a finite dimensional Hilbert space $k \in \mathbb{N}$. Then

1. the fermionic SLOCC semigroup is $\{\mathbb{S}_{(1^k)}g | g \in \mathrm{End}(\mathcal{H}_1)\}$.
2. the fermionic SLOCC group is $\{\mathbb{S}_{(1^k)}g | g \in GL(\mathcal{H}_1)\}$.
3. the fermionic LU group is $\{\mathbb{S}_{(1^k)}g | g \in U(\mathcal{H}_1)\}$.
4. the bosonic SLOCC semigroup is $\{\mathbb{S}_{(k)}g | g \in \mathrm{End}(\mathcal{H}_1)\}$.
5. the bosonic SLOCC group is $\{\mathbb{S}_{(k)}g | g \in GL(\mathcal{H}_1)\}$.
6. the bosonic LU group is $\{\mathbb{S}_{(k)}g | g \in U(\mathcal{H}_1)\}$.

Clearly, these (semi)groups are quotients of $\mathrm{End}(\mathcal{H}_1)$, $GL(\mathcal{H}_1)$ and $U(\mathcal{H}_1)$ respectively, and come with a canonical action on the k-fermion (k-boson) state space.

More generally, a composite quantum system may consist of several different types of indistinguishable particles, and the corresponding semigroup of local transformations is a quotient of the direct product of the End, GL and U (semi)groups of the single particle state spaces of the different types, with its obvious action.

As an example suppose that we have N types of fermionic particles, k_i of the ith type having n_i single particle states. To this composite system we associate the Hilbert space

$$\mathcal{H} = \Lambda^{k_1}\mathcal{H}_1^{(0)} \otimes \cdots \otimes \Lambda^{k_N}\mathcal{H}_N^{(0)} \tag{3.8}$$

where $\dim \mathcal{H}_i^{(0)} = n_i$ and the SLOCC group is a quotient of $G = GL(\mathcal{H}_1^{(0)}) \times \cdots \times GL(\mathcal{H}_N^{(0)})$.

3.2 Special three-particle quantum systems

In this section we analyze entanglement properties of various quantum systems containing three subsystems, among which both distinguishable and identical particles can be found. Our method works only in some very special cases, which are closely related to some algebraic structures called Freudenthal triple systems and Jordan algebras. But at least in these special systems the full SLOCC classificiation of entanglement can be achieved.

First we would like to collect the relevant notions and results from the area of cubic Jordan algebras and Freudenthal triples.

Definition. An algebra (not necessarily associative) $(J, +, \bullet)$ is called a Jordan algebra[28] if for any two elements $A, B \in J$ the equations

$$A \bullet B = B \bullet A \tag{3.9}$$

and

$$(A \bullet A) \bullet (A \bullet B) = A \bullet ((A \bullet A) \bullet B) \tag{3.10}$$

hold.

A Jordan algebra is said to be cubic if every element satisfies a cubic polynomial equation.

One possible way to obtain a cubic Jordan algebra is called the Springer construction [44, 36]. This starts with a vector space V equipped with a suitable cubic form $N : V \to \mathbb{C}$ and a basepoint $c \in V$ such that $N(c) = 1$. One can then define various maps using the linearization

$$\begin{aligned} N(x, y, z) =& \frac{1}{6}\big(N(x + y + z) - N(x + y) - N(x + z) - N(y + z) + \\ &+ N(x) + N(y) + N(z)\big) \end{aligned} \tag{3.11}$$

of N, including the Jordan product, but for our purposes only the following two are needed:

$$\begin{aligned} &(\cdot, \cdot) : V \times V \to \mathbb{C} \\ &\qquad (x, y) \mapsto 9N(c, c, x)N(c, c, y) - 6N(x, y, c) \\ &\quad \cdot^\sharp : V \to V \text{ defined by} \\ &\qquad \forall y \in J : (x^\sharp, y) = 3N(x, x, y) \end{aligned} \tag{3.12}$$

The former is called the trace bilinear form, while the latter is the adjoint or sharp map.

From a Jordan algebra J over $\mathbb{C}$ one can construct the Freudenthal triple system $\mathfrak{M}(J) = \mathbb{C} \oplus \mathbb{C} \oplus J \oplus J$ which is equipped with a skew-symmetric bilinear form and a quartic form defined by:

$$\begin{aligned} \{x, y\} &= \alpha\delta - \beta\gamma + (A, D) - (B, C) \\ q(x) &= 2\big((A, B) - \alpha\beta\big)^2 - 8(A^\sharp, B^\sharp) + 8\alpha N(A) + 8\beta N(B) \end{aligned} \tag{3.13}$$

where $x = (\alpha, \beta, A, B)$ and $y = (\gamma, \delta, C, D)$ are two elements of $\mathfrak{M}(J)$. One also introduces the unique trilinear map $T : \mathfrak{M}(J) \times \mathfrak{M}(J) \times \mathfrak{M}(J) \to \mathfrak{M}(J)$ with the property $\{T(x, y, z), w\} = q(x, y, z, w)$ where $q(\cdot, \cdot, \cdot, \cdot)$ is the linearization of the quartic form $q(\cdot)$.

We are interested in maps preserving these forms:

Definition. We will denote by $\mathrm{Inv}(\mathfrak{M}(J))$ the group of linear transformations which preserve these forms, that is, for all $\sigma \in \mathrm{Inv}(\mathfrak{M}(J))$

$$\{\sigma(\cdot), \sigma(\cdot)\} = \{\cdot, \cdot\} \quad \text{and} \quad q \circ \sigma = q \tag{3.14}$$

holds.

Clearly, the construction yields a $2 + 2\dim J$ dimensional representation of $\mathrm{Inv}(\mathfrak{M}(J))$ and q is a quartic polynomial invariant under the action of this group. In order to study the orbits of this action, we need the following function on $\mathfrak{M}(J)$:

Definition. Let J be a cubic Jordan algebra and $\mathfrak{M}(J)$ the corresponding Freudenthal system. The rank of an element $x \in \mathfrak{M}(J)$ is an integer between 0 and 4 defined uniquely as follows:

- $\mathrm{rk}\, x = 4$ iff $q(x) \neq 0$
- $\mathrm{rk}\, x \leq 3$ iff $q(x) = 0$
- $\mathrm{rk}\, x \leq 2$ iff $T(x, x, x) = 0$
- $\mathrm{rk}\, x \leq 1$ iff $\forall y \in \mathfrak{M}(J) : 3T(x, x, y) + \{x, y\}x = 0$
- $\mathrm{rk}\, x = 0$ iff $x = 0$

As the rank is determined by values of invariant and equivariant forms under $\mathrm{Inv}(\mathfrak{M}(J))$, it is also $\mathrm{Inv}(\mathfrak{M}(J))$-invariant.

It turns out that the rank together with q are enough to separate the orbits [30].

Theorem 3.2.1. *Let J be a cubic Jordan algebra over $\mathbb{C}$, and consider the action of $\mathrm{Inv}(\mathfrak{M}(J))$ on the Freudenthal module $\mathfrak{M}(J)$. Then $\mathrm{Inv}(\mathfrak{M}(J))$ acts transitively on the sets of elements of rank $0, 1, 2, 3$. In the case of rank 4, $\mathrm{Inv}(\mathfrak{M}(J))$ acts transitively on the sets of elements with a given nonzero value of q.*

In the following we give explicitely the Jordan algebras needed for the classification of entangled states in the above mentioned quantum systems. It turns out that all of them can be regarded as a subalgebra of $J_3 = Mat(3, \mathbb{C})$ so we start with this one.

For $A \in J_3$ the cubic form N is simply the determinant of the 3×3 matrix, the basepoint is $c = I_3$ (where I_n denotes the $n \times n$ identity matrix), for $A, B \in J_3$ the trace bilinear form is given by $(A, B) = \mathrm{Tr}(AB)$ and the explicit form of the sharp map is

$$A^\sharp = A^2 - \mathrm{Tr}(A)A + \frac{1}{2}\big(\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2)\big)I_3 \tag{3.15}$$

The simplest nontrivial Jordan algebra is $J_1 = \mathbb{C}$, the cubic norm of $a \in J_1$ is $N(a) = a^3$. It follows that the map $J_1 \to J_3; a \mapsto aI_3$ is an injective morphism of cubic Jordan algebras.

The next Jordan algebra we need is $J_{1+1} = \mathbb{C} \oplus \mathbb{C}$, the value of N on the element $x = (a, b)$ is $N(x) = ab^2$. In this case the image of x in J_3 is

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix} \in Mat(3, \mathbb{C}) \tag{3.16}$$

The third Jordan algebra is $J_{1+1+1} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ Here for $x = (a, b, c)$ the value of N is $N(x) = abc$. This is nothing else but the determinant of the matrix

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \in Mat(3, \mathbb{C}) \tag{3.17}$$

which shows us the isomorphism between J_{1+1+1} and the subalgebra of diagonal matrices in J_3. The last Jordan algebra we consider is $J_{1+2} = \mathbb{C} \oplus Q_4$, where Q_4 is an arbitrary 4 dimensional complex vector space with a nondegenerate quadratic form. It is convinient to let Q_4 be the vector space of 2×2 matrices, and the quadratic form be the determinant. A general element in J_{1+2} is therefore $x = (a, A)$, and its cubic norm is $N(x) = a \det A$. For this Jordan algebra the inclusion map is given by

$$(a, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) \mapsto \begin{bmatrix} a & 0 & 0 \\ 0 & A_{11} & A_{12} \\ 0 & A_{21} & A_{22} \end{bmatrix} \in Mat(3, \mathbb{C}) \tag{3.18}$$

the image being a block diagonal matrix built from a 1×1 and a 2×2 block.

These constructions are useful for studying entanglement because the Inv groups of the respective Freudenthal modules are the (determinant 1) SLOCC groups of various quantum systems. Namely [30],

$$\begin{aligned} \mathrm{Inv}(\mathfrak{M}(J_1)) &\simeq SL(2, \mathbb{C}) \\ \mathrm{Inv}(\mathfrak{M}(J_{1+1})) &\simeq SL(2, \mathbb{C})^2 \\ \mathrm{Inv}(\mathfrak{M}(J_{1+1+1})) &\simeq SL(2, \mathbb{C})^3 \rtimes S_3 \\ \mathrm{Inv}(\mathfrak{M}(J_{1+2})) &\simeq SL(2, \mathbb{C}) \times SL(4, \mathbb{C}) \\ \mathrm{Inv}(\mathfrak{M}(J_3)) &\simeq SL(6, \mathbb{C}) \end{aligned} \tag{3.19}$$

We will see that disregarding the 0 vector, there are three SLOCC orbits in the first case with ranks 1, 3 and 4, four orbits in the second case, one with every possible positive rank. There are six SLOCC orbits in the third case, one for each positive rank except for rank 2 for which there are 3 orbits, which are permuted under the action of S_3. In the fourth case there are 5 orbits, again one for each positive rank except for rank 2 for which there are 2 orbits, and four orbits in the last case, with ranks 1, 2, 3 and 4.

3.2.1 Three fermions with six single particle states

A genuine tripartite entanglement measure for three fermions with six single particle states can be constructed using Freudenthal's construction applied to the cubic Jordan algebra of $J_3 = Mat(3, \mathbb{C})$ as follows [32].

It can be shown[30] that the group of transformations of the Freudenthal triple system $\mathfrak{M} = \mathbb{C} \oplus \mathbb{C} \oplus J_3 \oplus J_3$ preserving its quartic form is precisely $SL(6, \mathbb{C})$ and the representation of this group on $\mathfrak{M}$ is isomorphic to $\Lambda^3 V_6$ where V_n denotes the standard representation of $SL(n, \mathbb{C})$ on the vector space of n-tuples of complex numbers.

An isomorphism can be explicitely given as follows. Let $\{e_1, e_2, \dots, e_6\}$ be an orthonormal basis of $\mathbb{C}^6$, and let e_{ijk} denote the normalized wedge product of the vectors e_i, e_j, e_k:

$$\begin{aligned} e_{ijk} = \frac{1}{\sqrt{6}}(&e_i \otimes e_j \otimes e_k + e_j \otimes e_k \otimes e_i + e_k \otimes e_i \otimes e_j \\ &- e_i \otimes e_k \otimes e_j - e_k \otimes e_j \otimes e_i - e_j \otimes e_i \otimes e_k) \end{aligned} \tag{3.20}$$

Using these notations a three-fermion state may be written as

$$a = \sum_{1 \le a < b < c \le 6} a_{abc} e_{abc} \tag{3.21}$$

with the 20 coefficients satisfying the condition

$$\sum_{1 \le a < b < c \le 6} |a_{abc}|^2 = 1 \tag{3.22}$$

meaning that the norm of the state is 1. The corresponding element of $\mathfrak{M}$ is $x = (\alpha, \beta, A, B)$ where

$$\begin{aligned} &\alpha = a_{123} \quad \beta = a_{456} \\ &A = \begin{bmatrix} a_{156} & a_{164} & a_{145} \\ a_{256} & a_{264} & a_{245} \\ a_{356} & a_{364} & a_{345} \end{bmatrix} \quad B = \begin{bmatrix} a_{423} & a_{431} & a_{412} \\ a_{523} & a_{531} & a_{512} \\ a_{623} & a_{631} & a_{612} \end{bmatrix} \end{aligned} \tag{3.23}$$

Then the quartic polynomial preserved by the action of $SL(6,\mathbb{C})$ is

$$T = 4([\mathrm{Tr}(AB) - \alpha\beta]^2 - 4\,\mathrm{Tr}(A^\sharp B^\sharp) + 4\alpha \det A + 4\beta \det B) \tag{3.24}$$

and the tripartite entanglement measure is $\tau = |T|$. Since under the action of $GL(6,\mathbb{C})$ this quantity takes up a nonzero factor, one immediately concludes that there must be at least two SLOCC equivalence classes of three-fermion states. In fact, using the rank one can complete the classification and it turns out that we have four SLOCC orbits: the separable one, the biseparable one, and two different types of true tripartite entanglement.

3.2.2 One qubit and two fermions with four single particle states

In this case the Jordan algebra we use is $J_{1+2} = \mathbb{C} \oplus M_2(\mathbb{C})$ [51]. Let $x = (\alpha, x_0)$ and $y = (\beta, y_0)$ be two elements of J_{1+2}. The cubic norm form is given by $N(x) = \alpha \det x_0$, the sharp map is $x^\sharp = (\det x_0, \alpha(\mathrm{Tr}\, x_0)I_2 - \alpha x_0)$, and finally the trace bilinear map in this case is $(x, y) \mapsto \alpha\beta + \mathrm{Tr}(x_0 y_0)$. We have seen that this Jordan algebra can be viewed as a subalgebra of J_3 namely it is isomorphic to the subalgebra of block-diagonal matrices with a 1×1 and a 2×2 block in the diagonal.

It is known[30] that the Freudenthal construction applied to J_{1+2} yields a representation of $SL(2,\mathbb{C}) \times SL(4,\mathbb{C})$ isomorphic to $V_2 \otimes \Lambda^2 V_4$. Also we have the inclusion

$$V_2 \otimes \Lambda^2 V_4 \hookrightarrow \Lambda^3(V_2 \oplus V_4) \simeq \Lambda^3 V_4 \oplus V_2 \otimes \Lambda^2 V_4 \oplus \Lambda^2 V_2 \otimes V_4 \tag{3.25}$$

as $SL(2,\mathbb{C}) \times SL(4,\mathbb{C})$-spaces and

$$V_2 \otimes V_2 \otimes V_2 \hookrightarrow V_2 \otimes \Lambda^2(V_2 \oplus V_2) \simeq V_2 \otimes (\Lambda^2 V_2 \oplus V_2 \otimes V_2 \oplus \Lambda^2 V_2) \tag{3.26}$$

as $SL(2,\mathbb{C})^3$-spaces.

The analogy between the bipartite entanglement of two fermions with four single particle states and the two-qubit entanglement[19, 25, 31] suggests that this quantum system might behave much like the three-qubit system. Viewed together with the inclusion above, this fact makes one expect that in a sense this system fits between the three-fermion and the three-qubit one. This expectation is further supported by the fact that we have the inclusions $J_{1+1+1} \hookrightarrow J_{1+2} \hookrightarrow J_3$.

Let us see how this works explicitely. Let $\{e_0, e_1\}$ and $\{f_0, f_1, f_2, f_3\}$ be the canonical basis of $\mathbb{C}^2$ and $\mathbb{C}^4$ respectively. A state $a \in \mathbb{C}^2 \otimes \Lambda^2\mathbb{C}^4$ may be written as

$$a = \sum_{i=0}^{1} \sum_{0 \leq j < k \leq 3} a_{ijk} e_i \otimes (f_j \wedge f_k) \tag{3.27}$$

the coefficients being antisymmetric in the second and third index. The condition of being normalized means that

$$\sum_{i=0}^{1} \sum_{0 \leq j < k \leq 3} |a_{ijk}|^2 = 1 \tag{3.28}$$

Now we relate the six-state labels to the two-state and four-state ones as $(1, 4) \mapsto (0, 1)$ and $(2, 3, 5, 6) \mapsto (0, 1, 2, 3)$ respectively, and keep only the 12 coefficients whose index contains precisely one of 1 and 4. We associate to a the element $x = (\alpha, \beta, A, B) \in \mathfrak{M}$ where

$$\begin{aligned} &\alpha = a_{001} \quad \beta = a_{123} \\ &A = \begin{bmatrix} a_{023} & 0 & 0 \\ 0 & a_{103} & a_{120} \\ 0 & a_{113} & a_{121} \end{bmatrix} \quad B = \begin{bmatrix} a_{101} & 0 & 0 \\ 0 & a_{021} & a_{002} \\ 0 & a_{031} & a_{003} \end{bmatrix} \end{aligned} \tag{3.29}$$

For this state the value of the quartic tripartite entanglement measure is

$$\begin{aligned} T = {} & 4((a_{023}a_{101})^2 + (a_{021}a_{103})^2 + (a_{002}a_{113})^2 \\ & + (a_{031}a_{120})^2 + (a_{003}a_{121})^2 + (a_{001}a_{123})^2) \\ & + 8(a_{002}a_{021}a_{103}a_{113} + a_{021}a_{031}a_{103}a_{120} \\ & + a_{002}a_{003}a_{113}a_{121} + a_{003}a_{031}a_{120}a_{121}) \\ & + 16(a_{003}a_{021}a_{113}a_{120} + a_{001}a_{023}a_{103}a_{121} \\ & + a_{002}a_{031}a_{103}a_{121} + a_{003}a_{021}a_{101}a_{123}) \\ & - 16(a_{001}a_{023}a_{113}a_{120} + a_{002}a_{031}a_{101}a_{123}) \\ & - 8(a_{021}a_{023}a_{101}a_{103} + a_{002}a_{023}a_{101}a_{113} + a_{023}a_{031}a_{101}a_{120} \\ & + a_{002}a_{031}a_{113}a_{120} + a_{003}a_{023}a_{101}a_{121} + a_{003}a_{021}a_{103}a_{121} \\ & + a_{001}a_{023}a_{101}a_{123} + a_{001}a_{021}a_{103}a_{123} + a_{001}a_{002}a_{113}a_{123} \\ & + a_{001}a_{031}a_{120}a_{123} + a_{001}a_{003}a_{121}a_{123}) \end{aligned} \tag{3.30}$$

3.2.3 Three qubits

The SLOCC classification of three-qubit entanglement has already been completed [18, 12], but we present a rederivation of the result using our present methods for completeness, as well as to highlight the connections between the quantum systems. By looking at special three-fermion states one may observe that the space of three-qubit states $\bigotimes_{i=1}^{3} \mathbb{C}^2$ can be injected into our three-fermion one in such a way that the three-tangle[17] defined by Cayley's hyperdeterminant can be viewed as a special case of the quartic above.

To this end we keep only the amplitudes with three different numbers modulo 3 in the subscript. We have 8 such coefficients which is the number of coefficients needed to describe a three-qubit state. Let us choose an orthonormal basis $\{f_0, f_1\} \in \mathbb{C}^2$ and take three-fold tensor products of its elements, forming a computational basis. Now let us map an element $f_i \otimes f_j \otimes f_k$ of this basis to $e_{1+3i} \wedge e_{2+3j} \wedge e_{3+3k} \in \bigwedge^3 \mathbb{C}^6$. To a three-qubit state

$$a = \sum_{i,j,k \in \{0,1\}} a_{ijk} f_i \otimes f_j \otimes f_k \tag{3.31}$$

we associate this way an element $x = (\alpha, \beta, A, B)$ of $\mathfrak{M}$ where

$$\begin{aligned} &\alpha = a_{000} \quad \beta = a_{111} \\ &A = \begin{bmatrix} a_{011} & 0 & 0 \\ 0 & a_{101} & 0 \\ 0 & 0 & a_{110} \end{bmatrix} \quad B = \begin{bmatrix} a_{100} & 0 & 0 \\ 0 & a_{010} & 0 \\ 0 & 0 & a_{001} \end{bmatrix} \end{aligned} \tag{3.32}$$

On this element τ is equal to the three-tangle of a (using decimal notation):

$$\begin{aligned} T = {} & 4((a_0a_7)^2 + (a_1a_6)^2 + (a_2a_5)^2 + (a_3a_4)^2) \\ & - 8(a_0a_7a_1a_6 + a_0a_7a_2a_5 + a_0a_7a_3a_4 \\ & + a_1a_6a_2a_5 + a_1a_6a_3a_4 + a_2a_5a_3a_4) \\ & + 16(a_0a_3a_5a_6 + a_7a_4a_2a_1) \end{aligned} \tag{3.33}$$

Another way to look at the similarity between the two systems [32] is obtained by observing that starting with the cubic Jordan algebra $J_{1+1+1} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ the Freudenthal construction leads to the $V_2^{(1)} \otimes V_2^{(2)} \otimes V_2^{(3)}$ representation of the group $SL(2, \mathbb{C})^3$, and the quartic polynomial preserved by the action of the group is Cayley's hyperdeterminant. In section 3.2 we have seen that J_{1+1+1} is isomorphic to the subalgebra of J_3 of diagonal matrices.

For an element $x = (x_1, x_2, x_3) \in J_{1+1+1}$ we have a cubic norm $N(x) = x_1x_2x_3$, the sharp map assigning $x^\sharp = (x_2x_3, x_1x_3, x_1x_2)$ to x and on J we have a bilinear form whose value is $(x, y) = x_1y_1 + x_2y_2 + x_3y_3$ for $y = (y_1, y_2, y_3)$.

We see that the injection of the space of three-qubit states into the space of three-fermion states can be done at the Jordan algebra level. Moreover the quartic invariant is based entirely on the cubic Jordan algebra structure in both cases. It is not surprising therefore that the three-tangle of a three-qubit state can be obtained also by first taking the associated special three-fermion state and then calculating the value of the quartic invariant on it.

3.2.4 One distinguished qubit and two bosonic qubits

According to the literature on Freudenthal triple systems[30] there are two more Jordan algebras for which the Freudenthal construction yields a representation that has a natural interpretation in quantum information theory. These are $J_1 = \mathbb{C}$ and $J_{1+1} = \mathbb{C} \oplus \mathbb{C}$, and both are isomorphic to subalgebras of J_3.

These correspond to three bosonic qubits and a composite system consisting of one qubit and two other indistinguishable bosonic qubits, respectively [51]. The state space of these systems can be naturally viewed as subspaces of the one describing three qubits so one might expect that these can be injected into the latter much like the three-qubit system is injected in the three-fermion one.

First take a look at J_{1+1}. With the Freudenthal construction we obtain a representation of $SL(2,\mathbb{C})^2$ on $\mathbb{C} \oplus \mathbb{C} \oplus J_{1+1} \oplus J_{1+1}$ that is isomorphic to $V_2^{(1)} \otimes S^2 V_2^{(2)}$. This enables us to classify entangled states in the space of a distinguishable and two bosonic qubits.

Let $\{e_0, e_1\}$ be the computational basis of $\mathbb{C}^2$, and let $f_0 = e_0 \otimes e_0$, $f_1 = e_0 \otimes e_1 + e_1 \otimes e_0$ and $f_2 = e_1 \otimes e_1$. Now a normalized vector in $\mathbb{C}^2 \otimes S^2\mathbb{C}^2$ may be written as

$$a = \sum_{i=0}^{1} \sum_{j=0}^{2} a_{ij} e_i \otimes f_j \tag{3.34}$$

where

$$\sum_{i=0}^{1} (|a_{i0}|^2 + 2|a_{i1}|^2 + |a_{i2}|^2) = 1 \tag{3.35}$$

The corresponding three-fermion state is given by $x = (\alpha, \beta, A, B) \in \mathfrak{M}$ where

$$\begin{aligned} &\alpha = a_{00} \quad \beta = a_{12} \\ &A = \begin{bmatrix} a_{02} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{11} \end{bmatrix} \quad B = \begin{bmatrix} a_{10} & 0 & 0 \\ 0 & a_{01} & 0 \\ 0 & 0 & a_{01} \end{bmatrix} \end{aligned} \tag{3.36}$$

For the state a we have the quartic invariant:

$$
\begin{aligned}
T = 4(a_{00}^2 a_{12}^2 + a_{02}^2 a_{10}^2) + 16(a_{11}^2 a_{00} a_{02} + a_{01}^2 a_{10} a_{12}) \\
- 8 a_{00} a_{02} a_{10} a_{12} - 16(a_{01} a_{02} a_{10} a_{11} + a_{00} a_{01} a_{11} a_{12})
\end{aligned} \tag{3.37}
$$

3.2.5 Three bosonic qubits

Now let us turn to the Jordan algebra $J_1 = \mathbb{C}$ in which the norm of an element is simply its cube, the sharp means taking the square, and the trace bilinear form of two elements $x, y \in J_1$ is $3xy$. Again after some calculation one can show that J_1 is a subalgebra of J_{1+1} the inclusion map being $x \mapsto (x, x)$. The Freudenthal construction in this case leads to a four dimensional representation of $SL(2, \mathbb{C})$ isomorphic to $S^3 V_2$ which is related to the system of three indistinguishable bosonic qubits [51].

A general normalized state in $S^3\mathbb{C}^2$ may be written as

$$
\begin{aligned}
a = a_0 e_0 \otimes e_0 \otimes e_0 + a_3 e_1 \otimes e_1 \otimes e_1 \\
+ a_1(e_1 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_0 + e_0 \otimes e_0 \otimes e_1) \\
+ a_2(e_0 \otimes e_1 \otimes e_1 + e_1 \otimes e_0 \otimes e_1 + e_1 \otimes e_1 \otimes e_0)
\end{aligned} \tag{3.38}
$$

where $|a_0|^2 + |a_3|^2 + 3(|a_1|^2 + |a_2|^2) = 1$. To this we associate the element $x = (a_0, a_3, a_2 I_3, a_1 I_3)$ in $\mathfrak{M}$. For this state the quartic invariant is:

$$
T = 4a_0^2 a_3^2 - 12 a_1^2 a_2^2 - 24 a_0 a_1 a_2 a_3 + 16(a_0 a_2^3 + a_3 a_1^3) \tag{3.39}
$$

To sum up, we have the chain of inclusions of Jordan algebras $J_1 \hookrightarrow J_{1+1} \hookrightarrow J_{1+1+1} \hookrightarrow J_{1+2} \hookrightarrow J_3$ that gives rise via Freudenthal's construction to the chain of inclusions of Hilbert spaces $S^3\mathbb{C}^2 \hookrightarrow \mathbb{C}^2 \otimes S^2\mathbb{C}^2 \hookrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \hookrightarrow \mathbb{C}^2 \otimes \Lambda^2\mathbb{C}^4 \hookrightarrow \Lambda^3\mathbb{C}^6$. These inclusions are compatible with the SLOCC classification of entanglement in the sense that SLOCC orbits of any of these systems are subsets of the intersections of SLOCC orbits of the three-fermion Hilbert space with the appropriate subspace. In order to find representatives of various entanglement classes it is therefore enough to look for them in the smallest possible subspace then interpret them as elements of the larger Hilbert spaces. These representatives can be chosen to be the following ones:

$$
\begin{aligned}
GHZ &= \frac{1}{\sqrt{2}}(1,1,0,0) \\
W &= \frac{1}{\sqrt{3}}(0,0,0,I_3) \\
B_1 &= \frac{1}{\sqrt{2}}(1,0,\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},0) \\
B_2 &= \frac{1}{\sqrt{2}}(1,0,\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},0) \\
B_3 &= \frac{1}{\sqrt{2}}(1,0,\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},0) \\
S &= (1,0,0,0)
\end{aligned}
\tag{3.40}
$$

The GHZ and W states show tripartite entanglement, the B_i are biseparable and S is a separable state. Apart from B_i these can be found in the system of three bosonic qubits, but the relations characterizing states of rank at most two imply separability in this case. Therefore the representative of the biseparable class is chosen from the larger Hilbert space $\mathbb{C}^2 \otimes S^2\mathbb{C}^2$. Of the biseparable subclasses only B_1 is present in the latter, all can be found in the three-qubit case, B_2 and B_3 are equivalent in $\mathbb{C}^2 \otimes \Lambda^2\mathbb{C}^4$, and all three are equivalent in the largest Hilbert space $\Lambda^3\mathbb{C}^6$. Table 3.1 shows these states for each system.

3.3 Entanglement measures for systems with distinguishable particles from fermionic ones

The quantum systems considered in the previous section are the only ones related to cubic Jordan algebras and the Freudenthal construction. Therefore, this method is not suitable for being generalized to a classification of other entangled states. However, we can still learn something from the phenomenon found, namely, that different quantum systems may share some aspects of their entanglement properties, for example having a correspondence between orbits of equivalently entangled states or having invariants which can be regarded as special cases of a single one.

space ($\mathcal{H}$)	representatives
$\bigwedge^3 \mathbb{C}^6$	$GHZ = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6)$ $W = \frac{1}{\sqrt{3}}(e_4 \wedge e_2 \wedge e_3 + e_1 \wedge e_5 \wedge e_3 + e_1 \wedge e_2 \wedge e_6)$ $B_1 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_5 \wedge e_6)$ $S = e_1 \wedge e_2 \wedge e_3$
$\mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^4$	$GHZ = \frac{1}{\sqrt{2}}(e_0 \otimes (f_0 \wedge f_1) + e_1 \otimes (f_2 \wedge f_3))$ $W = \frac{1}{\sqrt{3}}(e_0 \otimes (f_2 \wedge f_3) + e_1 \otimes (f_0 \wedge f_3) + e_1 \otimes (f_2 \wedge f_1))$ $B_1 = \frac{1}{\sqrt{2}} e_0 \otimes (f_0 \wedge f_1 + f_2 \wedge f_3)$ $B_2 = \frac{1}{\sqrt{2}}(e_0 \otimes (f_0 \wedge f_1) + e_1 \otimes (f_0 \wedge f_3))$ $S = e_0 \otimes (f_0 \wedge f_1)$
$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$	$GHZ = \frac{1}{\sqrt{2}}(e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1)$ $W = \frac{1}{\sqrt{3}}(e_1 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_0 + e_0 \otimes e_0 \otimes e_1)$ $B_1 = \frac{1}{\sqrt{2}}(e_0 \otimes (e_0 \otimes e_0 + e_1 \otimes e_1))$ $B_2 = \frac{1}{\sqrt{2}}(e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_0 \otimes e_1)$ $B_3 = \frac{1}{\sqrt{2}}((e_0 \otimes e_0 + e_1 \otimes e_1) \otimes e_0)$ $S = e_0 \otimes e_0 \otimes e_0$
$\mathbb{C}^2 \otimes Sym^2\mathbb{C}^2$	$GHZ = \frac{1}{\sqrt{2}}(e_0 \otimes (e_0 \otimes e_0) + e_1 \otimes (e_1 \otimes e_1))$ $W = \frac{1}{\sqrt{3}}(e_1 \otimes (e_0 \otimes e_0) + e_0 \otimes (e_1 \otimes e_0 + e_0 \otimes e_1))$ $B_1 = \frac{1}{\sqrt{2}}(e_0 \otimes (e_0 \otimes e_0 + e_1 \otimes e_1))$ $S = e_0 \otimes (e_0 \otimes e_0)$
$Sym^3\mathbb{C}^2$	$GHZ = \frac{1}{\sqrt{2}}(e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1)$ $W = \frac{1}{\sqrt{3}}(e_1 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_0 + e_0 \otimes e_0 \otimes e_1)$ $S = e_0 \otimes e_0 \otimes e_0$

Table 3.1: Representatives of SLOCC orbits of quantum mechanical systems classified via Freudenthal's construction.

We can formulate this correspondence more precisely as follows. First note that according to eq. (3.3), the state space of k identical bosons with $\mathcal{H}_1$ as single particle state space can be viewed as a subspace of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1$. From the inclusion

$$S^k(\mathcal{H}_1) \hookrightarrow \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1 \simeq S^k(\mathcal{H}_1) \oplus \bigoplus_{\substack{\lambda \vdash k \\ \lambda \neq (1^k)}} \mathbb{S}_\lambda \mathcal{H}_1 \otimes V_\lambda \tag{3.41}$$

of $GL(\mathcal{H}_1) \times S_k$-spaces we see that this mapping is G-equivariant when $G = GL(\mathcal{H}_1)$ or $G = U(\mathcal{H}_1)$. (see section B.4 in the appendix for details)

Secondly, suppose that we have N types of fermionic particles, k_i of the ith type having $\mathcal{H}_i^{(0)}$ as single particle state space ($\dim \mathcal{H}_i^{(0)} = n_i < \infty$) with the composite state space as in eq. (3.8). Recall that we are interested in finding the orbits under the action of the group $G = GL(\mathcal{H}_1^{(0)}) \times \cdots \times GL(\mathcal{H}_N^{(0)})$ or $G = U(\mathcal{H}_1^{(0)}) \times \cdots \times U(\mathcal{H}_N^{(0)})$.

Let $\mathcal{H}^{(0)} = \mathcal{H}_1^{(0)} \oplus \cdots \oplus \mathcal{H}_N^{(0)}$ and $k = k_1 + \cdots + k_N$. In both cases

$$\mathcal{K} := \Lambda^k(\mathcal{H}_1^{(0)} \oplus \cdots \oplus \mathcal{H}_N^{(0)}) \simeq \bigoplus_{\substack{0 \leq i_1, \ldots, i_N \\ i_1 + \cdots + i_N = k}} \Lambda^{i_1}\mathcal{H}_1^{(0)} \otimes \cdots \otimes \Lambda^{i_N}\mathcal{H}_N^{(0)} \tag{3.42}$$

as G-spaces. One of the terms in the direct sum is

$$\mathcal{H} = \Lambda^{k_1}\mathcal{H}_1^{(0)} \otimes \cdots \otimes \Lambda^{k_N}\mathcal{H}_N^{(0)} \tag{3.43}$$

therefore we have a G-equivariant inclusion map $\mathcal{H} \hookrightarrow \mathcal{K}$ [51].

These homomorphisms are useful because they enable us to relate some of the invariants of the domain with those of the codomain. To see how this works, let us first consider a more abstract setting. Let H be a group and $G \leq H$, and take two spaces $V \in \mathrm{Ob}({}_G\mathbf{Mod})$ and $W \in \mathrm{Ob}({}_H\mathbf{Mod})$. By restriction, one can also think of W as a G-space (Actually, in this way we get a functor ${}_H\mathbf{Mod} \to {}_G\mathbf{Mod}$). Suppose that we have a G-equivariant map $\varphi : V \to W$. Then $\operatorname{ran}\varphi$ is clearly a G-subspace of W.

As taking the dual is a contravariant while constructing the symmetric algebra is a covariant functor, φ induces a map $S(W^*) \to S(V^*)$, that is, we can map a polynomial on W to a polynomial on V G-equivariantly. But then G-invariant vectors in $S(W^*)$ are mapped to G-invariant vectors, so by restriction we have a map

$$\varphi^* : S(W^*)^G \to S(V^*)^G \tag{3.44}$$

It is easy to see that this map is surjective if φ is injective, but in this case we cannot expect $S(W^*)^G$ to be easier to describe than $S(V^*)^G$. On the other hand, the subalgebra $S(W^*)^H \leq S(W^*)^G$ might have a simpler structure than $S(W^*)^G$ for a suitably chosen larger group H, and $\varphi^*(S(W^*)^H)$ can contain many "physically important" invariants.

Returning to our more concrete situation, it is convenient to choose H to be the SLOCC (LU) group $GL(\mathcal{H}_1^{(0)} \oplus \cdots \oplus \mathcal{H}_N^{(0)})$ and $U(\mathcal{H}_1^{(0)} \oplus \cdots \oplus \mathcal{H}_N^{(0)})$ of $\mathcal{K}$ with G as above respectively, and the roles are $V = \mathcal{H}$, $W = \mathcal{K}$ for the SLOCC group and $V = \mathcal{H} \oplus \mathcal{H}^*$, $W = \mathcal{K} \oplus \mathcal{K}^*$ for the LU group.

To sum up, a SLOCC (LU) invariant of the fermionic quantum system $\mathcal{K}$ gives rise to a SLOCC (LU) invariant of the smaller system $\mathcal{H}$ consisting of the same number particles, but which are of different types, can be either bosonic, fermionic or distinguishable, but with fewer single particle states. Notice that this is exactly what we have seen in the previous section.

Similarly, using the isomorphism

$$S^k(\mathcal{H}_1^{(0)} \oplus \cdots \oplus \mathcal{H}_N^{(0)}) \simeq \bigoplus_{\substack{0 \leq i_1,\ldots,i_N \\ i_1+\cdots+i_N=k}} S^{i_1}\mathcal{H}_1^{(0)} \otimes \cdots \otimes S^{i_N}\mathcal{H}_N^{(0)} \tag{3.45}$$

we can regard the state space of a quantum system with distinguishable constituents as a subspace of a bosonic quantum system with the same number of particles and such that the bosonic single particle state space is the direct sum of the original single particle state spaces. In what follows, we will only consider the fermionic case because an orthonormal basis of the fermionic state space is easier to describe than that of a bosonic state space, but it would be possible to translate the results to the bosonic case.

One can view this phenomenon from the opposite direction and ask whether an invariant for a quantum system with distinguishable constituents can be extended to an invariant of a fermionic or bosonic system. For some well-known invariants this can indeed be done. We have already seen one example, the generalization of the usual three-tangle for three qubits has been found in [32]. Another example is a fermionic extension[51] of the invariants given by Wong and Christensen[52].

3.4 Local unitary invariants of fermionic pure states

In this section $\mathcal{H}$ will be a finite dimensional complex Hilbert space playing the role of the single-particle state space of a fermionic quantum system of k particles. If $n = \dim \mathcal{H}$, then the k-particle Hilbert space is isomorphic to

$$\Lambda^k\mathcal{H} \simeq \Lambda^k\mathbb{C}^n \tag{3.46}$$

and hence its dimension is $\binom{n}{k}$. This space also comes equipped with an inner product induced from that of $\mathcal{H}$, and an irreducible unitary representation of $U(\mathcal{H})$ which models local unitary transformations of the k-particle states.

Now let us look at the symmetric algebra of the k-fermion state space. On its homogenous subspaces $S^m(\Lambda^k\mathcal{H})$ we have an action of $U(\mathcal{H})$ which factors through $U(\Lambda^k\mathcal{H})$ and an inner product which is invariant under $U(\Lambda^k\mathcal{H})$ hence also invariant under $U(\mathcal{H})$. This time the representation of $U(\mathcal{H})$ is not irreducible, and $S^m(\Lambda^k\mathcal{H})$ can be split into the orthogonal sum of $U(\mathcal{H})$-invariant subspaces in a non-trivial way:

$$S^m(\Lambda^k\mathcal{H}) = \bigoplus_{\lambda} V_\lambda \tag{3.47}$$

where λ ranges over the partitions of km, and V_λ is the corresponding isotypic component of the representation. Interestingly, this decomposition is independent of n (apart from the vanishing of the subrepresentations associated to partitions involving more than n parts, but for $n \geq km$ this certainly cannot happen). This is essentially due to the fact that a degree j symmetric polynomial in n variables can be reconstructed even if we only now its restriction to a subspace in which only j variables take nonzero values.

This decomposition allows us to introduce unitary invariants, one for each isotypic subspace [49]. Let $\psi \in \Lambda^k\mathcal{H}$ be a k-fermion state vector, and ψ^m its m-th power which is an element of $S^m(\Lambda^k\mathcal{H})$. Let $P_\lambda : S^m(\Lambda^k\mathcal{H}) \to V_\lambda$ denote the orthogonal projection. This commutes with the representation of $U(\mathcal{H})$, therefore the value of $I_\lambda(\psi) := \langle\psi^m, P_\lambda\psi^m\rangle = \|P_\lambda\psi^m\|^2$ is invariant:

$$\begin{aligned} \forall g \in U(\mathcal{H}) : \langle (g\cdot\psi)^m, P_\lambda(g\cdot\psi)^m\rangle &= \langle g\cdot(\psi^m), g\cdot(P_\lambda\psi^m)\rangle \\ &= \langle\psi^m, P_\lambda\psi^m\rangle \end{aligned} \tag{3.48}$$

Note that the number of independent invariants of this type is one less than the number of nonvanishing isotypic components, because

$$\sum_\lambda \langle\psi^m, P_\lambda\psi^m\rangle = \langle\psi^m, \left(\sum_\lambda P_\lambda\right)\psi^m\rangle = \langle\psi^m, \psi^m\rangle = 1 \tag{3.49}$$

Unfortunately, it is in general not an easy task to calculate the projections for all these invariant subspaces for every value of k and m, but some of them are easy enough to be done by hand.

Note also that if the multiplicity of the representation $\mathbb{S}_\lambda\mathcal{H}$ in $S^m(\Lambda^k\mathcal{H})$ is c_λ, then the dimension of the space of degree $2m$ invariants is $\sum_\lambda c_\lambda^2$. However, to obtain $\sum_\lambda c_\lambda^2$ linearly independent invariants we would need to choose a decomposition of the isotypic subspaces into irreducible ones. Our method does not make use of such arbitrary choices.

3.5 Invariant subspaces with maximal highest weight

Let us now fix an ordered orthonormal basis $(e_1, \ldots, e_n)$ in $\mathcal{H}$. This also gives the isomorphisms $\mathcal{H} \simeq \mathbb{C}^n$, and $U(\mathcal{H}) \simeq U(n, \mathbb{C})$. The maximal torus T which acts diagonally in this basis is then identified with the subgroup of diagonal unitary matrices. The set of one dimensional representations $T \to \mathbb{C}^\times$ is a commutative group isomorphic to $\mathbb{Z}^n$. We will use the following identification:

$$\begin{aligned}(r_1, r_2, \ldots, r_n) &: T \to \mathbb{C}^\times \\ (r_1, r_2, \ldots, r_n)(diag(\lambda_1, \ldots, \lambda_n)) &= \prod_{i=1}^{n} \lambda_i^{r_i}\end{aligned} \tag{3.50}$$

On the set of n-tuples of integers we have the usual partial ordering, an n-tuple $(r_1, r_2, \ldots, r_n)$ being positive iff $r_1 + \ldots + r_n = 0$ and r_1, $r_1 + r_2$, $\ldots$, $r_1 + r_2 + \ldots + r_{n-1}$ are nonnegative, and $\lambda \geq \mu$ iff $\lambda - \mu$ is positive. A finite dimensional representation of $U(\mathcal{H})$, when restricted to T, splits into one dimensional subrepresentations. The representations with nonzero multiplicity are called weights, and a vector whose orbit under T spans a one dimensional subspace is called a weight vector. The isomorphism class of an irreducible representation of $U(\mathcal{H})$ is determined by its highest weight.

For $I = \{i_1, i_2, \ldots, i_k\}$ where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ let us introduce the following notation:

$$\begin{aligned}e_I &= e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} \\ &= \frac{1}{\sqrt{k!}} \sum_{\pi \in S_k} \chi_{(1^k)}(\pi) e_{i_{\pi(1)}} \otimes e_{i_{\pi(2)}} \otimes \ldots \otimes e_{i_{\pi(k)}}\end{aligned} \tag{3.51}$$

where S_k is the symmetric group on k elements, and $\chi_{(1^k)} : s_k \to \{1, -1\}$ denotes the alternating representation. The set $\{e_{\{i_1,\ldots,i_k\}} | 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$ forms an orthonormal basis of $\bigwedge^k \mathcal{H}$, and therefore every k-fermion pure state can be expressed uniquely as a linear combination of these vectors:

$$\psi = \sum_{I \in \binom{[n]}{k}} \psi_I e_I \text{ where } \sum_{I \in \binom{[n]}{k}} |\psi_I|^2 = 1 \tag{3.52}$$

(Here we used the short notation $[n] = \{1, 2, \ldots, n\}$ and $\binom{[n]}{k}$ denotes the set of k-element subsets of $[n]$.) For each $m \in \mathbb{N}$, the mth power of ψ is a vector in $S^m(\Lambda^k \mathcal{H})$:

$$\psi^m = \sum_{I_1, \ldots, I_m} \psi_{I_1} \psi_{I_2} \ldots \psi_{I_m} e_{I_1} e_{I_2} \ldots e_{I_m} \tag{3.53}$$

We would like to find a vector in $S^m(\Lambda^k \mathcal{H})$ which generates an irreducible $U(\mathcal{H})$-representation. In general we cannot say much about all the irreducible subrepresentations, but we always have one weight vector, $e^m_{1,2,\ldots,k}$, corresponding to the highest weight, which is easily seen to be (m^k) with the tailing zeros omitted. We now have that $\langle U(\mathcal{H})e^m_{1,2,\ldots,k}\rangle := W$ is irreducible. The next step will be to find an orthonormal basis for W.

Our first goal will be to find a generating set for W as a linear space, then we can orthogonalize it to obtain an orthonormal basis. To this end, we will use the fact that W is also an irreducible representation of $GL(n,\mathbb{C})$ whose action on $S^m(\Lambda^k\mathcal{H})$ is defined in the same way as that of $U(\mathcal{H})$. (see section B.4 in the appendix)

In order to find a generating set which is easy to handle, we will look for one that is the union of orbits under $S_n \leq GL(n,\mathbb{C})$ (possibly up to a nonzero multiple) which permutes the basis elements of $\mathcal{H}$. It turns out that we can require also that the generating set consists of weight vectors. We will call sets with these properties *good*:

Definition. Let $S \subset S^m(\Lambda^k\mathcal{H})$ be a subset, $\{e_1,\ldots,e_n\}$ an orthonormal basis in $\mathcal{H}$ and $S_n \leq U(\mathcal{H})$ the subgroup which permutes these basis elements. The subset S will be called *good* (with respect to this basis) if it has the following two properties:

1. The subset
$$\mathbb{C}S := \bigcup_{w\in S}\mathbb{C}w \subseteq S^m(\Lambda^k\mathcal{H}) \tag{3.54}$$
is fixed under the action of S_n.

2. If v is an element of S then if we write v as a polynomial in the vectors $\{e_I\}_{I\in\binom{[n]}{k}}$ then every index $i\in[n]$ appears the same number of times in every term. Or equivalently: v is a weight vector for the maximal torus fixing the given orthonormal basis.

We can immediately see that $\{e_I^m\}_{I\in\binom{[n]}{k}}$ is the smallest *good* subset containing $e^m_{1,2,\ldots,k}$.

To reach every element in W, we will use the fact that $GL(n,\mathbb{C})$ is generated by matrices of the form $u_{ij}(s) = id + sE_{ij}$ where E_{ij} is a matrix with a 1 at the intersection of the ith row and the jth column, and zeros everywhere else. We need to know how these matrices act on the basis elements of $\Lambda^k(\mathcal{H})$. One can calculate using equation (3.51) that

$$u_{ij}(s)\cdot e_I = \begin{cases} e_I & j\notin I \\ e_I + (-1)^{|I\cap(i,j)|}se_{I\cup\{i\}\setminus\{j\}} & j\in I, i\notin I \\ e_I & i,j\in I \end{cases} \tag{3.55}$$

The first and last cases are not interesting, but the second one allows us to build our generating set step by step starting from the above mentioned elements. Keeping track of the appearing sign could cause some difficulty, but we can overcome this by letting $e_{abc\ldots} = -e_{bac\ldots}$ etc. and simply substituting j with i without reordering the indices.

Observe that when $u_{ij}(s)$ acts on a degree m polynomial in the e_I-s, then we get a polynomial in s with coefficients in $S^m(\Lambda^k\mathcal{H})$. Since W contains this polynomial for any $s \in \mathbb{C}$, and it is a linear subspace, W must also contain the coefficient of s^l for each $0 \le l \le m$ (because of the non-vanishing of a Vandermonde determinant). Using this method, one can calculate in a few steps a generating set for the isotypic (in fact, irreducible) subspace corresponding to the highest weight. The following lemma shows which terms should one concentrate on:

Lemma 3.5.1. *Let $W \le S^m(\Lambda^k\mathcal{H})$ be an invariant subspace and $S \subseteq W$ a* good *subset*

Suppose that $w \in S$ and $i \ne j$ are indices such that i does not appear in w when written in the monomial basis as above. Then

a) The coefficients of every power of s in $u_{ij}(s) \cdot w$ as a polynomial in s are weight vectors.

b) If the degree of this polynomial is d then the one dimensional subspaces spanned by the coefficients of s^r and s^{d-r} are in the same S_n-orbit.

c) The coefficient of the constant and the leading terms is contained in $\mathbb{C}S$.

d) If $\mathbb{C}w = \mathbb{C}\pi \cdot w'$ for some $\pi \in S_n$, then the minimal good *subsets containing S and each coefficient in the polynomial $u_{ij}(s) \cdot w$ or $u_{\pi^{-1}(i)\pi^{-1}(j)}(s) \cdot w'$ generate the same subspace.*

Proof. a) $u_{ij}(s) \cdot e_{I_1} e_{I_2} \ldots e_{I_m} = (e_{I_1} + s e_{I'_1})(e_{I_2} + s e_{I'_2}) \ldots (e_{I_m} + s e_{I'_m})$ where I'_l is obtained from I_l by replacing j with i if I_l contains j and $e_{I'_l} = 0$ else. The coefficient of s^l contains exactly those terms in the expansion in which the number of replaced j-indices is l.

b) d is the (common) number of occurrencies of the index j in each term of w. The coefficient of s^{d-r} term is therefore proportional to the image of the coefficient of s^r under the transposition swapping e_i and e_j.

c) The constant term is w.

d) Let $\pi \in S_n \leq U(\mathcal{H})$ be an element such that $\mathbb{C}w = \mathbb{C}\pi \cdot w'$. Then

$$\begin{aligned} \mathbb{C}u_{ij}(s) \cdot w &= u_{ij}(s)\mathbb{C}w \\ &= u_{ij}(s)\mathbb{C}\pi \cdot w' \\ &= \mathbb{C}u_{ij}(s)\pi \cdot w' \\ &= \mathbb{C}\pi u_{\pi^{-1}(i)\pi^{-1}(j)}(s) \cdot w' \end{aligned} \tag{3.56}$$

□

Corollary. If $S \subseteq S^m(\Lambda^k\mathcal{H})$ is a *good* subset and $w \in S$ such that in each term of w the index j appears exactly once and w does not contain the index i, then $u_{ij}(s) \cdot w \in \langle S \rangle$ for all $s \in \mathbb{C}$.

Proof. In this case, $u_{ij}(s) \cdot w$ is a degree 1 polynomial in s, therefore, by the lemma above, both of its terms are in $\mathbb{C}S$, hence their sum is in $\langle S \rangle$. □

To sum up, we begin with the vector $e^m_{12\ldots k}$, then act on it and the distinct types of obtained coefficients of s successively with the matrices $u_{ij}(s)$, as long as we get new types of vectors. Finally, we take union of the S_n-orbits of the vectors we have met. This will result in a generating set of W.

We would like to remark that if we are to use these invariants as measures of entanglement, then, taking into account the constraint (3.49) and the fact that the mth power of a decomposable state is always in the irreducible subspace generated by $e^m_{12\ldots k}$, we should use $1 - \langle \psi^m, P_W \psi^m \rangle = \langle \psi^m, P_{W^\perp} \psi \rangle$, or the invariants associated to the subspaces other than W.

If we wanted to calculate the projectors of the other isotypic subspaces, then we simply needed to take the orthogonal complement of W, and find the weight vectors corresponding to the highest weight, and proceed with it the same way as we did with $e^m_{12\ldots k}$.

3.6 Examples

3.6.1 $k = m = 2$ case

The first nontrivial case is the space of quadratic polynomials in vectors of the space of two fermions. As we have seen, a weight vector with maximal weight is e^2_{12}, therefore $W := \langle GL(n, \mathbb{C})e^2_{12} \rangle$ contains e^2_{ij} for $1 \leq i < j \leq n$. In the next step we let $u_{kj}(s)$ act on an element:

$$u_{kj}(s)(e^2_{ij}) = (e_{ij} + se_{ik})^2 = e^2_{ij} + 2se_{ij}e_{ik} + s^2 e^2_{ik} \tag{3.57}$$

This shows that we must add $e_{ij}e_{ik}$ for each triple i, j, k, where the appearing indices are distinct. Now as

$$u_{li}(s)(e_{ij}e_{ik}) = (e_{ij}+se_{lj})(e_{ik}+se_{lk}) = e_{ij}e_{ik}+s(e_{lj}e_{ik}+e_{ij}e_{lk})+e_{lj}e_{lk} \quad (3.58)$$

we also have to add $e_{ij}e_{lk} + e_{ik}e_{lj}$ for each combination of indices.

By the corollary after the lemma we are ready, but it is instructive to verify the dimension of the generated subspace. Clearly, the set

$$\{e_{ij}^2 | 1 \le i < j \le n\} \cup \{e_{ij}e_{ik} | 1 \le i \le n, i \ne j < k \ne i\} \quad (3.59)$$

consists of pairwise orthogonal elements. The third type in the generating set is $\{e_{ij}e_{lk} + e_{ik}e_{lj}\}$ which generates a two dimensional space for each set of four indices, and these subspaces are pairwise orthogonal and also orthogonal to the other elements. Therefore,

$$\dim W = \binom{n}{2} + n\binom{n-1}{2} + 2\binom{n}{4} = \frac{n^2(n^2-1)}{12} \quad (3.60)$$

which is exactly the dimension of the irreducible representation of $GL(n, \mathbb{C})$ corresponding to the partition (2^2). (see section B.4 in the appendix)

Orthogonalization needs to be performed only within the two dimensional subspaces, and this leads to the vectors $e_{ij}e_{kl}+e_{ik}e_{jl}$ and $e_{ij}e_{lk}+2e_{il}e_{jk}-e_{ik}e_{jl}$ for $1 \le i < j < k < l \le n$. The expression for the invariant corresponding to W is therefore (using equation (2.44))

$$\begin{aligned} I_{(2^2)}(\psi) &= \langle \psi, P_W \psi \rangle \\ &= \sum_{1 \le i < j \le n} |\psi_{ij}^2|^2 + \sum_{i=1}^{n} \sum_{\substack{1 \le j < k \le n \\ j \ne i \ne k}} 2|\psi_{ij}\psi_{ik}|^2 \\ &+ \sum_{1 \le i < j < k < l \le n} \left(|\psi_{ij}\psi_{kl} + \psi_{ik}\psi_{jl}|^2 + \frac{1}{3}|\psi_{ij}\psi_{lk} + 2\psi_{il}\psi_{jk} - \psi_{ik}\psi_{jl}|^2 \right) \end{aligned} \quad (3.61)$$

In this case we can also show that $W^\perp$ is irreducible. To this end, let us recall that for $n = 4$ there exists a degree two $SL(4, \mathbb{C})$-invariant over $\Lambda^2\mathbb{C}^4$, namely, the polynomial in the Plücker relation which is known to be a sufficient and necessary condition of separability. The subrepresentation generated by this polynomial is the representation indexed by the partition (1^4), therefore this one must appear also in the $n > 4$ case. As the dimension of this is $\binom{n}{4}$, and

$$\dim W + \binom{n}{4} = \frac{n(n-1)(n^2-n+2)}{8} = \dim S^2(\Lambda^2\mathbb{C}^n) \quad (3.62)$$

therefore $W^\perp$ is irreducible, and the unitary invariant associated to it gives a generalization of the Plücker relation. The explicit formula turns out to be simpler than the previous one:

$$I_{(1^4)}(\psi) = \langle \psi, P_{W^\perp}\psi \rangle = \frac{2}{3}\sum_{1\le i<j<k<l\le n} |\psi_{ij}\psi_{kl} + \psi_{ik}\psi_{lj} + \psi_{il}\psi_{jk}|^2 \tag{3.63}$$

3.6.2 $k = 2$, $m = 3$ case

In this case a weight vector for the highest weight is e_{12}^3. Again, $W := \langle GL(n,\mathbb{C})e_{12}^3\rangle$. We are looking for a generating set of W. We extend e_{12}^3 into a *good* set $\{e_{ij}\}_{1\le i<j\le n}$. Now we need to add the coefficient of s in

$$u_{32}(s)(e_{12}^3) = (e_{12} + se_{13})^3 = e_{12} + 3se_{12}^2e_{13} + 3s^2e_{12}e_{13}^2 + s^3e_{13}^3 \tag{3.64}$$

and one vector from each element of the orbit of the subspace generated by it: $\{e_{ij}^2e_{ik}\}_{i,j,k\in[n]}$. The next steps are:

$$u_{43}(s)(e_{12}e_{13}^2) = e_{12}(e_{13} + se_{14})^2 = \ldots + 2se_{12}e_{13}e_{14} + s^2(\ldots) \tag{3.65}$$

$$\begin{aligned} u_{m1}(s)(e_{12}e_{13}^2) &= (e_{12} + se_{m2})(e_{13} + se_{m3})^2 \\ &= \ldots + s(2e_{12}e_{13}e_{m3} + e_{m2}e_{13}^2) + s^2(\ldots) + s^3(\ldots) \end{aligned} \tag{3.66}$$

Here $m = 2$ is special, in this case the second term in the coefficient of s vanishes, hence we have to add $\{2e_{ij}e_{ik}e_{mk} + e_{mj}e_{ik}^2\}$ for any ordered pair of disjoint pairs $(\{i,j\},\{k,m\})$, and also $e_{ij}e_{jk}e_{kj}$ for $\{i,j,k\} \in \binom{[n]}{3}$. The remaining steps are

$$\begin{aligned} u_{m1}(s)e_{12}e_{13}e_{14} &= (e_{12} + se_{m2})(e_{13} + se_{m3})(e_{14} + se_{m4}) \\ &= \ldots + s(e_{m2}e_{13}e_{14} + e_{12}e_{m3}e_{14} + e_{12}e_{13}e_{m4}) + \\ &\quad + s^2(\ldots) + s^3(\ldots) \end{aligned} \tag{3.67}$$

$$\begin{aligned} u_{m1}(s)(e_{52}e_{13}e_{14} + e_{12}e_{53}e_{14} + e_{12}e_{13}e_{54}) = \ldots + s(&e_{52}e_{m3}e_{14} + e_{52}e_{13}e_{m4} \\ &+ e_{m2}e_{53}e_{14} + e_{12}e_{53}e_{m4} + \\ &+ e_{m2}e_{13}e_{54} + e_{12}e_{m3}e_{54}) + \\ &+ s^2(\ldots) \end{aligned} \tag{3.68}$$

Here $m \le 5$ does not lead to a new subspace.

It turns out that the vectors obtained so far are enough to generate W. In this case, orthogonalization turns out to be a bit lengthy, especially in the case of the six-term vectors like in equation (3.68). These span a five dimensional subspace for each six-element set of indices $i_1, \ldots, i_6$. For these the coefficients of the monomials are given as a matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 & 3 & 0 & 1 & 1 & 3 & 1 & -2 & 3 & 1 & -2 \\ 0 & 0 & 0 & 2 & 1 & -1 & 2 & 1 & -1 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 2 & 2 & 0 & 1 & 1 & 0 & 1 & 1 & -1 & -1 & 0 & -1 & -1 & 0 \\ 4 & 2 & -2 & 2 & 1 & -1 & -2 & -1 & 1 & -1 & 1 & 2 & 1 & -1 & -2 \end{pmatrix} \tag{3.69}$$

The order of the monomials is (12|34|56), (12|35|46), (12|36|45), (13|24|56), (13|25|46), (13|26|45), (14|23|56), (14|25|36), (14|26|35), (15|23|46), (15|24|36), (15|26|34), (16|23|45), (16|24|35), (16|25|34), where $(ab|cd|ef)$ is a short notation for $e_{i_a,i_b} e_{i_c,i_d} e_{i_e,i_f}$. The norms inverse squared of these vectors are

$$1, \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8} \tag{3.70}$$

respectively. The orthogonal generators coming from the remaining vectors are given in table (3.2)

form	indices	dimension	$\\| \cdot \\|^{-2}$
e_{ij}^3	$\{i, j\}$	$\binom{n}{2}$	1
$e_{ij}^2 e_{ik}$	$\{i\}, \{j\}, \{k\}$	$6\binom{n}{3}$	3
$e_{ij} e_{ik} e_{il}$	$\{i\}, \{j, k, l\}$	$4\binom{n}{4}$	6
$e_{ij} e_{ik} e_{kj}$	$\{i, j, k\}$	$\binom{n}{3}$	6
$e_{ij}^2 e_{kl} + 2e_{ij} e_{il} e_{kj}$	$\{i, j\}, \{k, l\}$	$2\binom{n}{2}\binom{n-2}{2}$	1
$-2e_{ij}^2 e_{kl} + 6e_{ij} e_{ik} e_{lj} + 2e_{ij} e_{il} e_{kj}$			$\frac{1}{8}$
$e_{ik} e_{il} e_{jm} + e_{ik} e_{jl} e_{im} + e_{jk} e_{il} e_{im}$	$\{i\}, \{j, k, l, m\}$	$15\binom{n}{5}$	2
$e_{ik} e_{il} e_{jm} + 3e_{ik} e_{ij} e_{ml} + e_{im} e_{il} e_{kj} + 3e_{im} e_{ij} e_{kl} + 2e_{im} e_{ik} e_{jl}$			$\frac{1}{4}$
$2e_{ij} e_{il} e_{mk} + e_{ik} e_{il} e_{mj} + e_{ik} e_{ij} e_{ml} + e_{im} e_{il} e_{jk} + e_{im} e_{ij} e_{kl}$			$\frac{3}{4}$

Table 3.2: Orthogonalized generators for W. Indices shown in one set are indistinguishable for counting purposes.

Using these data the value of the invariant $I_{(3,3)}$ can be calculated in a straightforward way, but the full formula is too long to be presented explicitely.

The orthogonal complement of W clearly has a highest weight of $(2^2, 1^2)$, and we could find a generator of the unique one dimensional weight space corresponding to it, and calculate the projector of its invariant subspace. Instead of this, we follow another approach. According to the plethysm

$$s_{(3)}[s_{(1^2)}] = s_{(3^2)} + s_{(2^2,1^2)} + s_{(1^6)} \tag{3.71}$$

for $n = 6$, an $SL(6, \mathbb{C})$-invariant polynomial appears. It is easy to guess how this should look like: for a state $\psi \in \Lambda^2\mathbb{C}^6$, we can construct $\psi \wedge \psi \wedge \psi$ which is an element of $\Lambda^6\mathbb{C}^6$, a one dimensional vector space on which $GL(6, \mathbb{C})$ acts by multiplication with the determinant. Therefore this element remains unchanged under $SL(6, \mathbb{C})$, and its norm squared is an $U(6, \mathbb{C})$-invariant polynomial in the coefficients of ψ and their conjugates. Our invariant corresponding to the subrepresentation indexed by the partition (1^6) must be proportional to it. Explicitely, it equals to

$$\frac{1}{11520} \left| \sum_{\pi \in S_6} \sigma(\pi) \psi_{\pi(1),\pi(2)} \psi_{\pi(3),\pi(4)} \psi_{\pi(5),\pi(6)} \right|^2 \tag{3.72}$$

Here the sum is over all the permutations, but actually there are 15 different terms, each counted $48 = 3! \cdot 2^3$ times. Alternatively, we could sum over the partitions of $[n]$ into three two-element sets.

The $n \geq 6$ case can be obtained similarly to the previous section. Taking all the six-element subsets of $[n]$ polynomials like this span an $\binom{n}{6}$ dimensional subspace which is also the dimension of the invariant subspace we are looking for. Therefore in the general case the invariant is

$$I_{(1^6)}(\psi) = \frac{1}{11520} \sum_{I \in \binom{[n]}{6}} \left| \sum_{\pi \in S_6} \sigma(\pi) \psi_{i_{\pi(1)},i_{\pi(2)}} \psi_{i_{\pi(3)},i_{\pi(4)}} \psi_{i_{\pi(5)},i_{\pi(6)}} \right|^2 \tag{3.73}$$

where $I = \{i_1, \ldots, i_6\}$.

These two invariants are linearly independent, and they sum to 1 with the one associated to the third irreducible subspace.

3.6.3 $k = 3$, $m = 2$ case

Now we turn to the first case with more than two particles. In $S^2(\Lambda^3\mathcal{H})$ the vector with highest weight is e_{123}^2. We proceed in a similar way as before:

$$u_{n3}(s)(e_{123}^2) = e_{123}^2 + 2se_{123}e_{12n} + s^2 e_{12n}^2 \tag{3.74}$$

$$u_{n2}(s)(e_{123}e_{124}) = \ldots + s(e_{123}e_{1n4} + e_{1n3}e_{124}) + s^2(\ldots) \tag{3.75}$$

$$u_{n1}(s)(e_{123}e_{154} + e_{153}e_{124}) = \\ = \ldots + s(e_{123}e_{n54} + e_{n23}e_{154} + e_{153}e_{n24} + e_{n53}e_{124}) + s^2(\ldots) \tag{3.76}$$

These vectors already form a generating set, we only need to orthogonalize this set. For a fixed subset of six indices, the vectors of the form like in (3.76) span a five dimensional subspace. Orthogonal generators for this are again given with the coefficients of the monomials as a matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 2 & 1 & 0 & 0 & 2 & 1 \\ 1 & 4 & 2 & -1 & 2 & 1 & -2 & 0 & -2 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & 3 & 1 & -1 \end{pmatrix} \tag{3.77}$$

The order of monomials is (123|456), (124|356), (125|346), (126|345), (134|256), (135|246), (136|245), (145|236), (146|235), (156|234), where $(abc|def)$ is a shorthand notation for the vector $e_{i_a,i_b,i_c}e_{i_d,i_e,i_f}$. The inverse squared norms of these vectors are

$$\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{18}, \frac{1}{9} \tag{3.78}$$

respectively. The orthogonal generators coming from the remaining vectors are given in table (3.3)

form	indices	dimension	$\|\cdot\|^{-2}$
e_{ijk}^2	$\{i,j,k\}$	$\binom{n}{3}$	1
$e_{ijk}e_{ijl}$	$\{i,j\},\{k,l\}$	$\binom{n}{2}\binom{n-2}{2}$	2
$e_{ijm}e_{ikl} + e_{ijk}e_{iml}$	$\{i\},\{j,k,l,m\}$	$2n\binom{n-1}{4}$	1
$e_{ijm}e_{ikl} + 2e_{ijl}e_{imk} - e_{ijk}e_{iml}$			$\frac{1}{3}$

Table 3.3: Orthogonalized generators for the subspace generated by the highest weight vector. Indices shown in one set are indistinguishable for counting purposes.

The value of $I_{(2^3)}$ can now be calculated. This time the orthocomplement is also irreducible, so we get one independent invariant in this case.

3.7 Fermionic SLOCC invariants

In the examples we have seen local unitary invariants with a special property: for a particular value of n, the corresponding irreducible subspace becomes one dimensional, and the subspace is pointwise fixed under the action of $SL(n,\mathbb{C})$, that is, the local unitary invariant turns out to be a SLOCC-invariant. Let us examine this case in more detail.

The irreducible polynomial representation of $SL(n,\mathbb{C})$ indexed by the partition λ is one dimensional for some n precisely when λ consists of equal parts (see section B.4 in the appendix). In this case $\lambda = (r^n)$ is a partition of nr, hence a neccesary condition for it to occur as a subrepresentation of $S^m(\Lambda^k\mathbb{C}^n)$ is that $mk = nr$, and in this case $GL(n,\mathbb{C})$ acts on it by multiplication with the rth power of the determinant. The norm squared is therefore invariant under $U(n,\mathbb{C})$.

In our notations this subspace is spanned by a polynomial w in the basis vectors $e_1,\ldots,e_n$. w is a weight vector with weight (r^n), and it generates a one dimensional $U(n,\mathbb{C})$-invariant subspace. The crucial thing is that when we increase the dimension n of the single particle state space to n', w remains a weight vector that generates an irreducible $U(n',\mathbb{C})$-invariant subspace, but it is no longer one dimensional. Therefore, the invariant corresponding to this subspace will be a generalization of the SLOCC-invarant we have begun with, but is now only a unitary invariant.

The explicit form of the resulting invariant can be obtained in general using the method outlined above: we must act on it with $u_{ij}(s)$-s and elements of $S_{n'}$. A particularly simple special case is when $r = 1$. In this case the dimension of the representation corresponding to λ is $\binom{n'}{n}$, and an orthonormal basis can be obtained by acting on w by elements of $S_{n'}$. Therefore the invariant can be obtained by calculating the value of the SLOCC-invariant with the initial index set $[n]$ replaced by every element of $\binom{[n']}{n}$, and summing their absolute values squared.

3.8 Fourth order LU-invariants of multipartite systems with identical particles

Similarly to the case of distinguishable particles, the $m = 2$ case is special in that the decomposition of the symmetric square of a fermionic state space into irreducible subrepresentations of the local unitary group involves only representations with multiplicity 1. Explicitely, we have the following isomorphism of $U(\mathcal{H})$-spaces[22]:

$$S^2(\Lambda^k\mathcal{H}) \simeq \bigoplus_{\substack{0\leq a\leq k\\ 2|a}} \mathbb{S}_{(2^{k-a},1^{2a})}\mathcal{H} \tag{3.79}$$

It is apparent that the number of linearly independent fourth order LU-invariants of a k-fermion system is $1+\lfloor\frac{k}{2}\rfloor$. We have seen that this is exactly the number of permutation-invariant LU-invariants for a system consisting of k distinguishable particles with equal single-particle state space dimensions. This means that in this special case, *every* permutation invariant LU-invariant is a truncation of a corresponding fermionic invariant.

Let us look at the examples worked out above comparing them with the invariants of eq. (2.58), indexed by a nonnegative even integer d. In the $k=m=2$ case $I_{(2^2)}$ corresponds to $d=0$, while $I_{(1^4)}$ to $d=2$, while in the $k=3$, $m=2$ case $I_{(2^3)}$ reduces to the $d=0$ and $I_{(2,1^4)}$ to $d=2$. This suggests that in general the invariant associated to the subspace $\mathbb{S}_{(2^{k-a},1^{2a})}\mathcal{H}$ reduces to I_a when viewed as an invariant of a quantum system with distinguishable constituents.

Finally, let us look at the symmetric square of the state space of a quantum system of k bosons. The isomorphism[22]

$$S^2(S^k\mathcal{H}) \simeq \bigoplus_{\substack{0\leq a\leq k\\ 2|a}} \mathbb{S}_{(2k-a,a)}\mathcal{H} \tag{3.80}$$

shows that the space of fourth order LU-invariant polynomials is $1+\lfloor\frac{k}{2}\rfloor$ in this case too, which suggests that the fourth order LU-invariants with permutation symmetry for distinguishable particles can also be obtained by restricting bosonic invariants of the same degree.

Appendix A

Multilinear algebra and Hilbert spaces

A.1 Categories and functors

Category theory is an abstract way to speak of mathematical structures and structure-preserving maps between them. As such, it appears in all branches of modern mathematics and also in mathematical physics, providing a language in which many important constructions can be discussed in a unified way.

Definition. A (locally small) category $\mathbf{C}$ consists of a class $\mathrm{Ob}(\mathbf{C})$ of objects, for any two objects a and b a set $\mathrm{Hom}_{\mathbf{C}}(a,b)$ (for $f \in \mathrm{Hom}_{\mathbf{C}}(a,b)$, we often write $f : a \to b$) and for any three objects a, b and c, a binary operation $\circ : \mathrm{Hom}_{\mathbf{C}}(a,b) \times \mathrm{Hom}_{\mathbf{C}}(b,c) \to \mathrm{Hom}_{\mathbf{C}}(a,c)$ sending (f,g) to $g \circ f$ such that

1. if $f : a \to b$, $g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$
2. for every object x there exists $\mathrm{id}_x \in \mathrm{Hom}_{\mathbf{C}}(x,x)$ such that for $f : a \to b$ the equations $f \circ \mathrm{id}_a = \mathrm{id}_b \circ f = f$ hold.

Elements of $\mathrm{Hom}_{\mathbf{C}}(a,b)$ for any two objects a and b are called morphisms.

A morphism $f : a \to a$ is called an endomorphism of a, and the semigroup of endomorphisms of a is denoted by $\mathrm{End}_{\mathbf{C}}(a)$.

A morphism $f : a \to b$ for which another morphism $f^{-1} : b \to a$ exists such that $f \circ f^{-1} = \mathrm{id}_b$ and $f^{-1} \circ f = \mathrm{id}_a$ is called an isomorphism, while f^{-1} is called the inverse of f. In this case a and b are said to be isomorphic, denoted by $a \simeq b$.

An isomorphism $f : a \to a$ is called an automorphism of a, and the group of automorphisms of a is denoted by $\mathrm{Aut}_{\mathbf{C}}(a)$.

Note that the subscripts in the definition above are usually omitted when it is clear from the context.

Sometimes it is convenient to look at morphisms of a category "backwards". This is made precise by the concept of the opposite category:

Definition. Let $\mathbf{C}$ be a category. The category $\mathbf{C}^{op}$ consists of the same objects as $\mathbf{C}$, for a, b in $\mathrm{Ob}(\mathbf{C}^{op}) = \mathrm{Ob}(\mathbf{C})$, $\mathrm{Hom}_{\mathbf{C}^{op}}(a, b) = \mathrm{Hom}_{\mathbf{C}}(b, a)$, and for a, b and c in $\mathrm{Ob}(\mathbf{C}^{op})$, $f \in \mathrm{Hom}_{\mathbf{C}^{op}}(a, b)$ and $g \in \mathrm{Hom}_{\mathbf{C}^{op}}(b, c)$, the binary operation is defined by $g \circ^{op} f = f \circ g$. $\mathbf{C}^{op}$ is called the opposite category of $\mathbf{C}$.

In many cases a category consists of sets with some extra structure and functions between them preserving this extra structure with composition as $\circ$. These are called concrete categories.

Well-known examples include the category **Set** consisting of sets as objects and functions as morphisms, the category $\mathbf{Vect}_K$ with vector spaces over a fixed field K as objects and K-linear mappings between them as morphisms and the category **Grp** of groups and group-homomorphisms.

We would also like to consider structure-preserving maps between categories:

Definition. Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. A covariant functor F from $\mathbf{C}$ to $\mathbf{D}$ is a mapping which associates to each object a in $\mathrm{Ob}(\mathbf{C})$ an object $F(a)$ in $\mathrm{Ob}(\mathbf{D})$, and to each morphism $f \in \mathrm{Hom}_{\mathbf{C}}(a, b)$ a morphism $F(f) \in \mathrm{Hom}_{\mathbf{D}}(F(a), F(b))$ such that

1. for any object $x \in \mathrm{Ob}(\mathbf{C})$, $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$
2. for two morphisms $f : a \to b$ and $g : b \to c$, $F(g \circ f) = F(g) \circ F(f)$.

A contravariant functor F from $\mathbf{C}$ to $\mathbf{D}$ is a mapping which associates to each object a in $\mathrm{Ob}(\mathbf{C})$ an object $F(a)$ in $\mathrm{Ob}(\mathbf{D})$, and to each morphism $f \in \mathrm{Hom}_{\mathbf{C}}(a, b)$ a morphism $F(f) \in \mathrm{Hom}_{\mathbf{D}}(F(b), F(a))$ such that

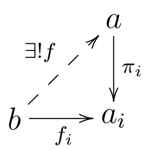
1. for any object $x \in \mathrm{Ob}(\mathbf{C})$, $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$
2. for two morphisms $f : a \to b$ and $g : b \to c$, $F(g \circ f) = F(f) \circ F(g)$.

An example of a covariant functor is the power set $P : \mathbf{Set} \to \mathbf{Set}$ sending each set to its power set and each function $f : X \to Y$ to the function $P(f) : P(X) \to P(Y)$ which maps $U \subseteq X$ to $f(U) \subseteq Y$, while an example of a contravariant functor is the dual $* : \mathbf{Vect}_K \to \mathbf{Vect}_K$, sending each vector space V to its dual V^*, and a linear map $A : V \to W$ to $A^* : W^* \to V^*$ defined by $(A^*\varphi)(v) = \varphi(Av)$.

Note that a contravariant functor $F : \mathbf{C} \to \mathbf{D}$ can also be viewed as a covariant functor $\mathbf{C}^{op} \to \mathbf{D}$.

In category theory an abstract generalization of various "product" constructions in algebra and other fields of mathematics can be given as follows:

Definition. Let $\mathbf{C}$ be a category and $(a_i)_{i\in I}$ a family of objects in $\mathbf{C}$. An object a together with a family of morphisms $\pi_i : a \to a_i$ is called the product of the family $(a_i)_{i\in I}$ if for any object $b \in \mathrm{Ob}(\mathbf{C})$ and any collection of morphisms $f_i : b \to a_i$ there exists a unique morphism $f : b \to a$ making the diagram



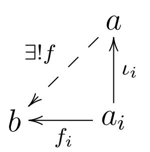
(A.1)

commute for all $i \in I$. The product of $(a_i)_{i\in I}$ is denoted by $\prod_{i\in I} a_i$.

For two objects a_1 and a_2 the product is usually written as $a_1 \times a_2$ and for $f_1 : b \to a_1$ and $f_2 : b \to a_2$ the unique morphism may be denoted by (f_1, f_2).

A dual concept is obtained by "reversing the arrows":

Definition. Let $\mathbf{C}$ be a category and $(a_i)_{i\in I}$ a family of objects in $\mathbf{C}$. An object a together with a family of morphisms $\iota_i : a_i \to a$ is called the coproduct of the family $(a_i)_{i\in I}$ if for any object $b \in \mathrm{Ob}(\mathbf{C})$ and any collection of morphisms $f_i : a_i \to b$ there exists a unique morphism $f : a \to b$ making the diagram



(A.2)

commute for all $i \in I$. The coproduct of $(a_i)_{i\in I}$ is denoted by $\coprod_{i\in I} a_i$.

Products and coproducts do not exist in every category. Examples in which they do include **Set** (in which they correspond to Cartesian products and disjoint unions, respectively), $\mathbf{Vect}_K$ (direct products and direct sums) and **Grp** (direct products and free products).

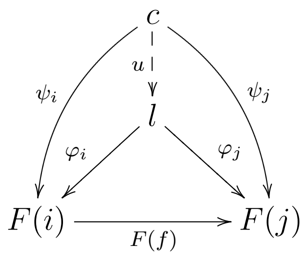
A generalization of the categorical product is obtained if the family of objects is parametrized not merely by a set, but by a partially ordered set, and we require that for indices being in relation, the corresponding objects are "glued together" along certain morphisms. Alternatively, the index set may be replaced by an arbitrary category as follows:

Definition. Let $\mathbf{C}$ and $\mathbf{I}$ be a two categories, and $F : \mathbf{I} \to \mathbf{C}$ a covariant functor, called a diagram of type $\mathbf{I}$ in $\mathbf{C}$.

A cone to F is an object $c \in \mathrm{Ob}(\mathbf{C})$ together with a morphism $\psi_i : c \to F(i)$ for each object $i \in \mathrm{Ob}(\mathbf{I})$ such that for every morphism $f : i \to j$ in $\mathbf{I}$, we have $F(f) \circ \psi_i = \psi_j$.

A limit of $F : \mathbf{I} \to \mathbf{C}$ is a cone (l, φ) to F such that for any cone (c, ψ) to F there exists a unique morphism $u : c \to l$ in $\mathbf{C}$ such that for all $i \in \mathrm{Ob}(\mathbf{I})$ we have $u \circ \phi_i = \psi_i$.

(see eq. (A.3) for a visualization of the morphisms appearing in the definition)



(A.3)

Note that when the only morphisms in $\mathbf{I}$ are the identity morphisms, then the limit reduces to the product.

One common special case is when $\mathbf{I}$ is a partially ordered set:

Definition. Let $\mathbf{I}$ be a partially ordered set considered as a small category, that is, objects of $\mathbf{I}$ are elements of a partially ordered set, $\mathrm{Hom}(i, j)$ has exactly one element if $i \leq j$, and it is empty otherwise. An inverse system in $\mathbf{C}$ is a functor $F : \mathbf{I}^{op} \to \mathbf{C}$. The limit of an inverse system F is called its inverse limit.

If I is a partially ordered set, and (a_i, f_{ij}) is an inverse system, then its inverse limit is denoted by

$$\varprojlim_{i \in I} a_i \tag{A.4}$$

Like products and coproducts, limits do not always exist in every category. When they do, however, they are unique up to isomorphism.

A.2 Vector spaces

Let K be a field, and let $\mathbf{Vect}_K$ denote the category for which $\mathrm{Ob}(\mathbf{Vect}_K)$ is the class of K-vector spaces, and for $V, W \in \mathrm{Ob}(\mathbf{Vect}_K)$, $\mathrm{Hom}_{\mathbf{Vect}_K}(V, W)$

is the set of linear maps from V to W. One can easily check that in this way we obtain a category with id_V being the identity map of V and $\circ$ being the composition of maps. An important property of this category is that $\mathrm{Hom}_{\mathbf{Vect}_K}(V, W)$ itself is also a linear space. The group $\mathrm{Aut}_{\mathbf{Vect}_K}(V)$ is often denoted by $GL(V)$.

For a collection $(V_i)_{i\in I}$ of K-vector spaces, the product is given by the vector space

$$\prod_{i\in I} V_i := \{(v_i)_{i\in I} | \forall i \in I : v_i \in V_i\} \tag{A.5}$$

and the linear maps

$$\begin{aligned} \pi_j : \prod_{i\in I} V_i &\to V_j \\ (v_i)_{i\in I} &\mapsto v_j \end{aligned} \tag{A.6}$$

Similarly, the coproduct of $\{V_i\}_{i\in I}$ is given by the vector space

$$\bigoplus_{i\in I} V_i := \coprod_{i\in I} V_i = \{(v_i)_{i\in I} | \forall i \in I : v_i \in V_i, |\{i : v_i \neq 0\}| < \infty\} \tag{A.7}$$

and the linear maps

$$\begin{aligned} \iota_j : V_j &\to \coprod_{i\in I} V_i \\ v_j &\mapsto \left(i \mapsto \begin{cases} v_j & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \right) \end{aligned} \tag{A.8}$$

Note that when $|I| < \infty$ then $\prod_{i\in I} V_i$ and $\coprod_{i\in I} V_i$ are isomorphic.

The dual gives rise to a contravariant functor $* : \mathbf{Vect}_K \to \mathbf{Vect}_K$ sending V to V^* and the linear map $A : V \to W$ to its transpose (or dual) map $A^* : W^* \to V^*$ which is defined by $(A^*\varphi)(v) = \varphi(Av)$. There is a canonical injection i from any vector space V into its bidual V^{**} given by $i(v)(\varphi) = \varphi(v)$. i is an isomorphism if and only if V is finite dimensional.

The product and the coproduct are related by the following isomorphism involving the dual space functor:

$$\left(\bigoplus_{i\in I} V_i\right)^* \simeq \prod_{i\in I} V_i^* \tag{A.9}$$

where $\{V_i\}_{i\in I}$ are arbitrary vector spaces.

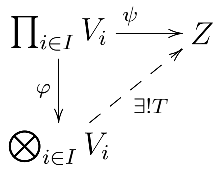
Consider a family $(V_i)_{i\in I}$ of K-vector spaces and another K-vector space Z. A map $\psi : \prod_{i\in I} V_i \to Z$ is called multilinear if for any $j \in I$ and any collection $(v_i)_{i\in I\setminus\{j\}}$ the map

$$\begin{aligned} \psi_j : V_j &\to Z \\ v_j &\mapsto \psi((v_i)_{i\in I}) \end{aligned} \tag{A.10}$$

is linear, that is, ψ is linear in each variable while the values of the other variables are fixed.

A.3 Tensor products

Definition. Given a family of vector spaces $(V_i)_{i\in I}$, a tensor product of $(V_i)_{i\in I}$is a vector space $\bigotimes_{i\in I} V_i$ together with a multilinear map $\varphi : \prod_{i\in I} V_i \to \bigotimes_{i\in I} V_i$ such that for any mulitilinear map $\psi : \prod_{i\in I} V_i \to Z$ with Z an arbitrary vector space, there exists a unique linear map $T : \bigotimes_{i\in I} V_i \to Z$ making the following diagram commute:



(A.11)

Vectors in $\operatorname{ran}\varphi$ are called elementary tensors.

One can show that the tensor product exists, and it is unique up to isomorphism. The tensor product of V and W is denoted by $V \otimes W$, and for $v \in V$ and $w \in W$ the notation $v \otimes w$ is used to denote $\varphi(v, w)$.

The tensor product of two vector spaces can be viewed as a bifunctor from $\mathbf{Vect}_K$ to itself, covariant in both variables, or in other words, a covariant functor from $\mathbf{Vect}_K \times \mathbf{Vect}_K$ to $\mathbf{Vect}_K$, sending (V_1, V_2) to $V_1 \otimes V_2$ and $(\varphi_1 : V_1 \to W_1, \varphi_2 : V_2 \to W_2)$ to $\varphi_1 \otimes \varphi_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$.

In particular, given two vector spaces V and W together with a linear map $A : V \to W$, the tensor product with an arbitrary vector space X gives rise to a linear map $A\otimes \mathrm{id}_X : V\otimes X \to W\otimes X$ defined by $(A\otimes \mathrm{id}_X)(v\otimes x) = (Av\otimes x)$ on elementary tensors and extended lineraly.

An important special case is the partial trace. We give first the definition of the trace map:

Definition. Let V be a finite dimensional vector space. Then we have the isomorphism $\mathrm{Hom}_{\mathbf{Vect}_K}(V, V) \simeq V \otimes V^*$. Consider the bilinear map $\psi : V \times$

$V^* \to K$ defined by $(v, \varphi) \mapsto \varphi(v)$. By the universal property in the definition of the tensor product, there exists a unique linear map $\mathrm{Tr}_V : V \otimes V^* \to K$, such that $\mathrm{Tr}_V(v \otimes \varphi) = \varphi(v)$ The map $\mathrm{Tr}_V : \mathrm{Hom}_{\mathbf{Vect}_K}(V, V) \to K$ is called the trace.

If $(V_i)_{i=1}^k$ is a finite family of vector spaces, then we define the jth partial trace to be $\mathrm{id}_{V_1} \otimes \cdots \otimes \mathrm{id}_{V_j-1} \otimes \mathrm{Tr}_{V_j} \otimes \mathrm{id}_{V_j+1} \otimes \cdots \otimes \mathrm{id}_{V_k}$.

A.4 Tensor products of Hilbert spaces

Hilbert spaces are complex vector spaces with an inner product such that the space is complete with respect to the metric induced by the inner product. In this case the definition of a tensor product must be modified, as the usual tensor product of two Hilbert spaces might fail to be a Hilbert space: although the usual tensor product of finitely many Hilbert spaces comes equipped with an induced inner product, the resulting space is in general not complete.

However, we can resolve this problem by defining the tensor product of finitely many Hilbert spaces to be the (topological) bidual of the usual tensor product with the induced inner product. Fortunately, in the case of finite dimensional vector spaces – which are discussed in the vast majority of quantum information theory literature – no such problem arises. In this thesis, all Hilbert spaces encountered are assumed to be finite dimensional.

Hilbert spaces together with bounded linear operators between them as morphisms form the category **Hilb**.

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$. Note that we follow the physicist's convention that an inner product is always assumed to be linear in the second variable and semilinear in the first one. For any two vectors $v, w \in \mathcal{H}$, consider the linear map

$$\begin{aligned} \langle v, \cdot w \rangle_{\mathcal{H}} : \mathrm{End}(\mathcal{H}) &\to \mathbb{C} \\ A &\mapsto \langle v, Aw \rangle_{\mathcal{H}} \end{aligned} \tag{A.12}$$

By the general construction above, for any Hilbert space $\mathcal{K}$, this gives rise to a linear map (denoted the same way)

$$\begin{aligned} \langle v, \cdot w \rangle_{\mathcal{H}} : \mathrm{End}(\mathcal{H}) \otimes \mathrm{End}(\mathcal{K}) &\to \mathbb{C} \otimes \mathrm{End}(\mathcal{K}) \\ \sum_{i \in I} A_i \otimes B_i &\mapsto \sum_{i \in I} \langle v, A_i w \rangle_{\mathcal{H}} B_i \end{aligned} \tag{A.13}$$

We also have the obvious identifications $\mathrm{End}(\mathcal{H}) \otimes \mathrm{End}(\mathcal{K}) = \mathrm{End}(\mathcal{H} \otimes \mathcal{K})$ and $\mathbb{C} \otimes \mathrm{End}(\mathcal{K}) = \mathrm{End}(\mathcal{K})$.

A.5 Graded algebras

In this section we collect some properties of a very special class of algebras, namely, $\mathbb{N}$-graded algebras. By algebra we will always mean unital associative algebra, and by graded we will mean $\mathbb{N}$-graded.

Definition. Let K be a field. A graded algebra A over K is a vector space equipped with a bilinear associative map $\cdot : A \times A \to A$ sending (a, b) to $a \cdot b$ such that A has a direct sum decomposition:

$$A = \bigoplus_{n \in \mathbb{N}} A_n \tag{A.14}$$

satisfying $A_i \cdot A_j \subseteq A_{i+j}$, and such that there exists $1 \in A$ satisfying $1 \cdot a = a \cdot 1 = a$ for all $a \in A$.

A_i is called the ith graded part of A, and elements of A_i are the homogenous elements of degree i.

An element a in $A = \bigoplus_{n \in \mathbb{N}} A_n$ can be uniquely written as $a = \sum_{n \in \mathbb{N}} a_n$ where $a_n \in A_n$. The degree of an arbitrary element a is the largest natural number d such that $a_d \neq 0$.

If A and B are graded algebras over K, then a map $\varphi : A \to B$ is said to be a homomorphism of graded algebras if it is linear and the relations $\varphi(a_1 \cdot a_2) = \varphi(a_1) \cdot \varphi(a_2)$, $\varphi(1) = 1$ and $\varphi(A_i) \subseteq B_i$ hold.

Note that the unit of a graded algebra must be in its 0th graded part. Graded algebras could be defined more generally replacing K with an arbitrary ring, $\mathbb{N}$ with an arbitrary semigroup and not requiring the existence of a unit. In this thesis, however, the above-given definition suffices.

For a field K, we denote by $\mathbb{N} - \mathbf{Alg}_K$ the category consisting of graded algebras as objects and graded algebra-homomorphisms as morphisms. Similarly, we denote by $\mathbb{N} - \mathbf{CAlg}_K$ the subcategory consisting of commutative graded algebras as objects and graded algebra-homomorphisms as morphisms.

Definition. Let A be a graded algebra over the field K. A subset $I \subseteq A$ is called an ideal, denoted by $I \triangleleft A$ if I is a (vector) subspace, $I \cdot A \subseteq A$, and $A \cdot I \subseteq A$. I is a homogenous ideal if in addition $I = \sum_{n \in \mathbb{N}} (I \cap A_n)$.

If $I \triangleleft A$ is a homogenous ideal, then the factor A/I becomes a graded algebra with the obvious multiplication and the following grading $(A/I)_n = A_n + I$. An important special case is the ideal generated by commutators: the abelianization of a graded algebra is also a graded algebra.

One often deals with infinite dimensional graded algebras, and in many cases the dimension of each graded part is finite. In this case, these dimensions can be conveniently encoded in a single object as follows:

Definition. Let $A = \bigoplus_{n \in N} A_n$ be a graded algebra over K. The Hilbert series of A is the formal power series

$$\sum_{n \in \mathbb{N}} (\dim_K A_n) t^n \tag{A.15}$$

in the indeterminate t.

As a particularly important example, we calculate the Hilbert series of the graded algebra $A = K[x^d]$ over K freely generated by the element x^d, assumed to be a degree d homogenous element as the notation suggests. A vector space basis of this algebra is easily seen to be $1, x^d, x^{2d}, \ldots$, and these are homogenous elements of degrees 0, d, $2d$, ... respectively. Therefore the Hilbert series is

$$\sum_{n \in N} (\dim_K A_n) t^n = \sum_{i=0}^{\infty} t^{d \cdot i} = \frac{1}{1 - t^d} \tag{A.16}$$

In $\mathbb{N} - \mathbf{Alg}_K$ products and coproducts always exist, and are called direct products and free products, respectively. For a family $(A_i)_{i \in I}$ of graded algebras, the product is given by

$$\prod_{i \in I} = \coprod_{n \in \mathbb{N}} (\prod_{i \in I} (A_i)_n) \tag{A.17}$$

where on the right hand side the product and the coproduct are understood in $\mathbf{Vect}_K$, and the multiplication is the obvious one. The free product is the graded algebra generated freely by the disjoint union of the A_i modulo the identities holding for their elements.

In $\mathbb{N} - \mathbf{Alg}_K$ (and many other categories of algebraic objects like abelian groups, rings, modules over a fixed ring, etc.) inverse limits can be conveniently realized as subobjects of products. Let $(I, \leq)$ be a partially ordered set, and $(A_i)_{i \in I}$ and $\varrho_{ij} : A_j \to A_i$ $(i \leq j)$ be an inverse system (where the A_i are objects and ϱ_{ij} are morphisms in the category under consideration, $\mathbb{N} - \mathbf{Alg}_K$ in this case). The inverse limit of this inverse system is

$$\varprojlim_{i \in I} A_i = \left\{ (a_i)_{i \in I} \in \prod_{i \in I} A_i \middle| \forall i \leq j : a_i = \varrho_{ij} a_j \right\} \tag{A.18}$$

Let V be a vector space over the field K. The graded algebra $T(V)$ is defined as follows. Its nth graded part is

$$T(V)_n = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}} \tag{A.19}$$

(and $T(V)_0 = K$), while multiplication is given by the canonical isomorphism

$$\underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}} \otimes \underbrace{V \otimes V \otimes \cdots \otimes V}_{m \text{ times}} \to \underbrace{V \otimes V \otimes \cdots \otimes V}_{n+m \text{ times}} \tag{A.20}$$

$T(V)$ is called the tensor algebra of V. The tensor algebra satisfies the following universal property: any linear transformation $\varphi : V \to A$ to an algebra A over K extends uniquely to a homomorphism $\varphi' : T(V) \to A$ of algebras. This shows that the construction gives a covariant functor $T : \mathbf{Vect}_K \to \mathbb{N} - \mathbf{Alg}_K$.

Denoting by I the ideal generated by commutators in $T(V)$, the abelianization $T(V)/I$ of the tensor algebra is called the symmetric algebra of V, denoted by $S(V)$. Clearly, we have a contravariant functor $S : \mathbf{Vect}_K \to \mathbb{N} - \mathbf{CAlg}_K$ which is the composition of T with abelianization. The algebra of polynomials *over* a vector space V is $S(V^*)$.

Definition. Let K be a field and A, B two commutative graded algebras over K. The graded tensor product $A \otimes B$ is the tensor product as a vector space, it is equipped with the multiplication $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 \cdot a_2) \otimes (b_1 \cdot b_2)$ and with a grading

$$A \otimes B = \bigoplus_{n \in \mathbb{N}} \underbrace{\bigoplus_{\substack{i,j \in \mathbb{N} \\ i+j=n}} A_i \otimes B_j}_{(A \otimes B)_n} \tag{A.21}$$

and becomes this way a commutative graded K-algebra.

Note that the graded tensor product is just the coproduct in $\mathbb{N} - \mathbf{CAlg}_K$.

Let $f(t)$ and $g(t)$ be the Hilbert series of the commutative graded algebras A and B. Then the Hilbert series of their tensor product is the product of their Hilbert series:

$$\begin{aligned} \sum_{n \in N} \dim_K((A \otimes B)_n) t^n &= \sum_{n \in N} \sum_{\substack{i,j \in \mathbb{N} \\ i+j=n}} \dim_K(A_i) \cdot \dim_K(B_j) t^{i+j} \\ &= \sum_{i,j \in} \dim_K(A_i) t^i \cdot \dim_K(B_j) t^j \\ &= \underbrace{\left(\sum_{i \in \mathbb{N}} \dim_K(A_i) t^i \right)}_{f(t)} \cdot \underbrace{\left(\sum_{j \in \mathbb{N}} \dim_K(B_j) t^j \right)}_{g(t)} \end{aligned} \tag{A.22}$$

In particular, the Hilbert series of a free commutative unital graded algebra A with n_d generators of degree d in an algebraically independent generating set is

$$\sum_{n \in N} \dim_K A_n t^n = \prod_{d=1}^{\infty} \frac{1}{(1 - t^d)^{n_d}} \tag{A.23}$$

as a consequence of eq. (A.16) and the previous calculation.

Appendix B

Elements of representation theory

B.1 Group representations

The theory of group representations deals with homomorphisms from a group to automorphism groups of vector spaces.

Definition. Let G be a group and V a complex vector space. A representation (or linear action) of G is a group homomorphism ϱ from G to $\mathrm{Aut}_{\mathbf{Vect}_{\mathbb{C}}}(V)$. One often writes $g \cdot v$ or even gv to denote the vector $\varrho(g)(v)$, if there is no ambiguity. V is then said to be a G-space, sometimes without explicitely mentioning the map ϱ.

Let V, W be two G-spaces, with representations $\varrho_V : G \to \mathrm{Aut}_{\mathbf{Vect}_{\mathbb{C}}}(V)$ and $\varrho_W : G \to \mathrm{Aut}_{\mathbf{Vect}_{\mathbb{C}}}(W)$, respectively. A linear map $\varphi : V \to W$ is G-equivariant if the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle \varrho_V}\downarrow & & \downarrow{\scriptstyle \varrho_W} \\ V & \xrightarrow[\varphi]{} & W \end{array} \tag{B.1}$$

More precisely, what we have just defined is a *left* action of G. One can similarly define a *right* action.

If V is a G-space, then V^* also carries a representation of G, by setting $(g \cdot \varphi)(v) = \varphi(g^{-1}v)$ for $\varphi \in V^*$ and $v \in V$, called the dual representation.

Two G-spaces V and W are said to be isomorphic if there exists a G-equivariant isomorphism in $\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{C}}}(V, W)$.

A (linear) subspace U of a G-space V is a G-subspace if for all $g \in G$ the relation $g \cdot U \subseteq U$ holds. A G-space V is irreducible if the only G-subspaces of V are $\{0\}$ and V.

If $U \leq V$ is a G-subspace, then the quotient V/U can be given the structure of a G-space, defined by the requirement that the quotient map $q : V \to V/U$ is G-equivariant.

Given two G-spaces V and W and a G-equivariant map $\varphi : V \to W$, it is easy to see that $\ker \varphi \leq V$ and $\operatorname{ran} \varphi \leq W$ are G-subspaces. Assuming that V and W are irreducible, we have Schur's lemma, stating that in this case the subspace of G-equivariant maps is one dimensional when V and W are isomorphic and consists of only the zero map when V and W are not isomorphic.

Various constructions exist which produce new representations from old ones. If G and H are groups, V is a G-space and W is a H-space, then the tensor product $V \otimes W$ comes with a $G \times H$-representation defined by $(g, h) \cdot (v \otimes w) := (g \cdot v) \otimes (h \cdot w)$ on elementary tensors and extended linearly. If V and W are irreducible, then $V \otimes W$ is also irreducible.

Consider the special case when $G = H$, that is, V and W are two G-spaces. Then $V \otimes W$ is a $G \times G$-space, or in other words, we have a group homomorphism $\varrho : G \times G \to \operatorname{Aut}_{\mathbf{Vect}_{\mathbb{C}}}(V \otimes W)$. Composing this with the diagonal morphism

$$\begin{aligned} \Delta_G : G &\to G \times G \\ g &\mapsto (g, g) \end{aligned} \tag{B.2}$$

we get a representation of G on $V \otimes W$ called the tensor product representation.

Representations of a group G as objects and G-equivariant maps between them as morphisms form a category, denoted by ${}_G\mathbf{Mod}$. In this category products and coproducts exist, and they correspond to products and coproducts of the underlying vector spaces equipped with the obvious (componentwise) action of G. As in $\mathbf{Vect}_{\mathbb{C}}$, these are also called direct products and direct sums, respectively. The dual representation gives a contravariant functor from ${}_G\mathbf{Mod}$ to itself.

Consider the free commutative group on the isomorphism classes of finite dimensional complex representations of a group G divided by the relations $[V \oplus W] = [V] + [W]$ where V and W are two such representations. The tensor product then induces a binary operation, turning this abelian group into a commutative ring called the representation ring of G, and denoted by $R(G)$. If G has the property that every finite dimensional G-space is the direct sum of irreducible ones – for example when G is finite –, then the additive group of the representation ring is the abelian group generated freely by the isomorphism classes of irreducible representations.

Note that a group G may be viewed as a category with a single object and such that every morphism is an isomorphism. In this way a group representation is simply a functor from this category to $\mathbf{Vect}_{\mathbb{C}}$. One may also consider functors to other categories, leading to other types of representations. For example, composing an ordinary (linear) representation with the symmetric algebra functor – which sends every vector space to its symmetric algebra – yields a group homomorphism from G to the automorphism group of a commutative unital associative algebra.

A particularly important special case is when we compose the representation with a Schur functor introduced section B.3. In this way every representation of a group G is mapped to another representation and G-equivariant maps are sent to G-equivariant maps. Thus, a Schur functor may also be viewed as a functor $\mathbb{S}_\lambda : {}_G\mathbf{Mod} \to {}_G\mathbf{Mod}$. In section B.3 we proceed in a slightly different way, defining Schur functors directly for representations.

B.2 Character theory of finite groups

Character theory is an indispensable tool in the study of representations of groups. For our purposes it suffices to formulate the statements for finite groups, but many results of character theory can be extended to compact topological groups as well.

Definition. Let G be a finite group, V a finite dimensional complex vector space and $\varrho : G \to \mathrm{Aut}_{\mathbf{Vect}_{\mathbb{C}}}(V)$ a complex representation of G. Then the function

$$\begin{aligned} \chi_V : G &\to \mathbb{C} \\ g &\mapsto \mathrm{Tr}(\varrho(g)) \end{aligned} \tag{B.3}$$

is called the character of the representation.

The equation

$$\begin{aligned} \chi_V(hgh^{-1}) &= \mathrm{Tr}(\varrho(hgh^{-1})) = \mathrm{Tr}(\varrho(h)\varrho(g)\varrho(h)^{-1}) \\ &= \mathrm{Tr}(\varrho(g)) = \chi_V(g) \end{aligned} \tag{B.4}$$

shows that a character is constant along conjugacy classes of G. Such functions are called class functions, and the set of class functions on G is denoted by $C(G)$. Also, isomorphic representations have the same character. It is an important result that the converse is also true, that is, the isomorphism class of a finite dimensional representation of G is uniquely determined by its character.

On the space of functions $G \to \mathbb{C}$ we have the usual inner product defined by

$$\langle f_1, f_2 \rangle_G = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g) \tag{B.5}$$

Class functions clearly form a subalgebra, and we have the following important result:

Theorem B.2.1. *Let G be a finite group,* $\mathrm{Irr}(G)$ *a maximal set of inequivalent irreducible representations, and let χ_i denote the character of the representation $i \in \mathrm{Irr}(G)$. Then $\{\chi_i\}_{i \in \mathrm{Irr}(G)}$ is an orthonormal basis in the space of class functions on G.*

In particular, the number of inequivalent irreducible representations is equal to the number of conjugacy classes in G.

Using the properties of the trace it is easy to see that for two G-spaces V and W, the following hold:

$$\begin{aligned} \chi_{V \oplus W} &= \chi_V + \chi_W \\ \chi_{V \otimes W} &= \chi_V \cdot \chi_W \\ \chi_{V^*} &= \overline{\chi_V} \end{aligned} \tag{B.6}$$

Therefore, the map $V \mapsto \chi_V$ induces a ring homomorphism from the representation ring $R(G)$ to the ring of class functions $C(G)$. In the case of finite groups, this is in fact injective.

Therefore in the case of finite groups, the theory of finite dimensional representations can be encoded in a table containing the values of irreducible characters on each conjugacy class, indicating also the sizes of the conjugacy classes. Such a table is usually referred to as a character table.

A special kind of linear representation is obtained in the following way. Let S be a set and $\alpha : G \times S \to S$ a group action, that is, a map such that

1. $\forall s \in S : \alpha(1, s) = s$
2. $\forall s \in S : \forall g, h \in G : \alpha(g, \alpha(h, s)) = \alpha(gh, s)$

and let $\mathbb{C}S$ be the vector space with basis S. Then the map $\varrho : G \to \mathrm{Aut}_{\mathbf{Vect}_{\mathbb{C}}}(\mathbb{C}S)$ defined by

$$\varrho(g) \sum_{s \in S} a_s s := \sum_{s \in S} a_s \alpha(g, s) \tag{B.7}$$

is a representation of G called a permutation representation. In this case the value of the character on a group element g is the number of elements in S fixed by g.

In the special case when $S = G$ and $\alpha : G \times G \to G$ is the group multiplication, the resulting representation is called the left regular representation. Note that $\mathbb{C}G$ can be turned into an algebra by extending the group multiplication linearly, called the group algebra.

Another important representation is given by conjugation. Let $S = G$ as above and $\alpha(g, h) := ghg^{-1}$. The corresponding representation on $\mathbb{C}G$ is the conjugation representation. We have the following result [21]:

Theorem B.2.2. *Let G be a finite group and $\mathbb{C}G$ be the group algebra equipped with the conjugation representation. Let $\mathrm{Irr}(G)$ be the set of isomorphism classes of irreducible representations of G, and let V_i denote a representation in the class $i \in \mathrm{Irr}(G)$. Then we have the isomorphism*

$$\mathbb{C}G \simeq \bigoplus_{i\in\mathrm{Irr}(G)} V_i \otimes V_i^* \tag{B.8}$$

as G-spaces.

In particular, we have that the character of the conjugation representation is

$$\sum_{i\in\mathrm{Irr}(G)} |\chi_i|^2 \tag{B.9}$$

where χ_i is the character of V_i.

B.3 Representations of the symmetric group

Let S_d be the group of bijections from $[d]$ to itself, called the symmetric group on d elements. Its group algebra $\mathbb{C}S_d$ consists of formal linear combinations of the group elements, with multiplication defined as the bilinear extension of the group multiplication.

Given a partition λ, fix a numbering of the corresponding Young diagram by the integers $[d]$ and let P_λ (Q_λ) denote the subgroup of S_d which fixes each row (column). In the group algebra we introduce two elements:

$$a_\lambda = \sum_{g\in P_\lambda} g \quad \text{and} \quad b_\lambda = \sum_{g\in Q_\lambda} sgn(g)g \tag{B.10}$$

Then $c_\lambda = a_\lambda b_\lambda$ is called a Young symmetrizer. We have the following theorem:

Theorem B.3.1. $c_\lambda^2 = n_\lambda c_\lambda$ *for some* $n_\lambda \in \mathbb{C}$ *and the image of* c_λ *by right multiplication on* $\mathbb{C}S_d$ *is an irreducible representation* V_λ *of* S_d*. Moreover, every irreducible representation of* S_d *arises in this way for a unique partition* λ*.*

In principle, this theorem enables us to calculate the characters of the irreducible representations of S_d. The number of inequivalent irreducible representations is the number of partitions of d. This is also the number of conjugacy classes in S_d, and it turns out that conjugacy classes can be indexed by partitions quite naturally as follows.

A permutation of the set $[d]$ can always be written essentially uniquely as the product of cycles. A cycle is a permutation of the form $a_1 \mapsto a_2 \mapsto \ldots \mapsto a_k \mapsto a_1$ where $\{a_i\}_{i=1}^k \subseteq [d]$ are distinct elements, and sending the other elements to themselves. As the length of a cycle does not change upon conjugation, we can conclude that a conjugacy class contains elements which are built up from disjoint cycles such that each length occurs the same number of times in each element. The collection of these numbers is called the cycle type of the group element, considering fixed elements as cycles of length 1.

Alternatively, cycle types can be mapped bijectively to partitions sending a cycle type with n_i at the ith place to the partition containing i n_i times.

Character tables of the first few symmetric groups are shown in tables B.1-B.4. In these tables the character of V_λ is denoted by χ_λ.

S_1	(1) 1
$\chi_{(1)}$	1

Table B.1: Character table of the symmetric group S_1

S_2	(1) 1	(1) (ab)
$\chi_{(2)}$	1	1
$\chi_{(1^2)}$	1	-1

Table B.2: Character table of the symmetric group S_2

Let V be a finite dimensional vector space. Then on the vector space

$$\underbrace{V \otimes V \otimes \cdots \otimes V}_{d \text{ factors}} \tag{B.11}$$

S_3	(1) 1	(3) (ab)	(2) (abc)
$\chi_{(3)}$	1	1	1
$\chi_{(2,1)}$	2	0	-1
$\chi_{(1^3)}$	1	-1	1

Table B.3: Character table of the symmetric group S_3

S_4	(1) 1	(6) (ab)	(3) $(ab)(cd)$	(8) (abc)	(6) $(abcd)$
$\chi_{(4)}$	1	1	1	1	1
$\chi_{(3,1)}$	3	1	-1	0	-1
$\chi_{(2,2)}$	2	0	2	-1	0
$\chi_{(2,1^2)}$	3	-1	-1	0	1
$\chi_{(1^4)}$	1	-1	1	1	-1

Table B.4: Character table of the symmetric group S_4

$GL(V)$ acts from the left as the tensor product of the standard actions on V, and S_d acts from the right by permuting the factors, that is, for $\pi \in S_d$, the action is defined by

$$(v_1 \otimes \cdots \otimes v_d) \cdot \pi = v_{\pi(1)} \otimes \cdots \otimes v_{\pi(d)} \tag{B.12}$$

on elementary tensors and extended linearly. It is easy to see that the two actions commute. For any partition λ of d, let c_λ be a corresponding Young symmetrizer, which acts also from the right on the vector space. Let

$$\mathbb{S}_\lambda V = \operatorname{ran}(c_\lambda|_{V\otimes\cdots\otimes V}) \tag{B.13}$$

denote the image of c_λ acting on $V \otimes \cdots \otimes V$ from the right.

Now let W be a $GL(V)$-space, and $\varphi \in \operatorname{Hom}_{GL(V)\mathbf{Mod}}(V, W)$. then φ determines a $GL(V)$-equivariant map

$$\varphi \otimes \cdots \otimes \varphi : V \otimes \cdots \otimes V \to W \otimes \cdots \otimes W \tag{B.14}$$

which is easily seen to be also S_d-equivariant when S_d acts on both spaces from the right as above. It follows that

$$\varphi \otimes \cdots \otimes \varphi|_{\operatorname{ran}(c_\lambda|_{V\otimes\cdots\otimes V})} : \operatorname{ran}(c_\lambda|_{V\otimes\cdots\otimes V}) \to \operatorname{ran}(c_\lambda|_{W\otimes\cdots\otimes W}) \tag{B.15}$$

is a $GL(V)$-equivariant map from $\mathbb{S}_\lambda V \to \mathbb{S}_\lambda W$. this map will be denoted by $\mathbb{S}_\lambda(\varphi)$.

In this way what we have obtained is a covariant functor from ${}_{GL(V)}\mathbf{Mod}$ to itself, called the Schur functor corresponding to λ. Note also that we can view V to be a G-space for any group G instead of $GL(V)$, if we are given a representation $\varrho : G \to GL(V)$. Then the same construction leads us to a functor $\mathbb{S}_\lambda : {}_G\mathbf{Mod} \to {}_G\mathbf{Mod}$. In particular, for $G = \{1\}$, a G-space is simply a vector space, and hence we have $\mathbb{S}_\lambda : \mathbf{Vect}_\mathbb{C} \to \mathbf{Vect}_\mathbb{C}$.

B.4 Rational representations of the unitary and the general linear groups

Let V be a finite dimensional complex vector space. The general linear group of V is $GL(V) := \mathrm{Aut}_{\mathbf{Vect}_\mathbb{C}}(V)$. Suppose that V is also equipped with an inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$. The subgroup $U(V) := \{g \in GL(V) | \forall v, w \in V : \langle g \cdot v, g \cdot w\rangle = \langle v, w\rangle\}$ preserving the inner product is called the unitary group of $(V, \langle\cdot,\cdot\rangle)$. Note that V is automatically a $GL(V)$-space and a $U(V)$-space with the (appropriate restriction of the) identity homomorphism $\mathrm{id}_{\mathrm{Aut}_{\mathbf{Vect}_\mathbb{C}}(V)}$ as representation.

We are interested in locally finite dimensional rational representations of these groups, that is, representations $\varrho : GL(V) \to \mathrm{Aut}_{\mathbf{Vect}_\mathbb{C}}(W)$ where $W = \bigoplus_{i\in I} W_i$ is the direct sum of finite dimensional $GL(V)$-spaces given by rational maps $\varrho_i : GL(V) \to \mathrm{Aut}_{\mathbf{Vect}_\mathbb{C}}(W_i)$. (A rational map $\phi : GL(V) \to \mathrm{Aut}_{\mathbf{Vect}_\mathbb{C}}(W_i) \subset \mathrm{End}_{\mathbf{Vect}_\mathbb{C}}(W_i)$ may be imagined as an element of the module $\mathbb{C}[\mathrm{End}_{\mathbf{Vect}_\mathbb{C}}(V)](\det) \otimes \mathrm{End}_{\mathbf{Vect}_\mathbb{C}}(W_i)$.) We have the following:

Theorem B.4.1. *Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space with $n = \dim V < \infty$. Then any locally finite dimensional rational representation of $GL(V)$ ($U(V)$) is isomorphic to the direct sum of irreducible ones. Irreducible rational representations of $GL(V)$ ($U(V)$) are of the form*

$$\mathbb{S}_\lambda V \otimes D_k \tag{B.16}$$

where D_k is the k-fold tensor product of $\Lambda^n V$ with itself when $k \geq 0$ and $D_{-k} = D_k^*$, and λ is an arbitrary partition with at most $n-1$ parts.

Moreover, $\mathbb{S}_\lambda V \otimes D_k \simeq \mathbb{S}_{\lambda'} V \otimes D_{k'}$ iff $\lambda = \lambda'$ and $k = k'$.

Note that D_k is a one dimensional vector space of $GL(V)$ ($U(V)$), on which $GL(V)$ ($U(V)$) acts by multipication with the kth power of the determinant.

Alternatively, we may index finite dimensional irreducible representations of the two groups with nonincreasing integer sequences $\lambda_1 \geq \ldots \geq \lambda_n$, corresponding to the representation

$$\mathbb{S}_{(\lambda_1+k,\cdots,\lambda_n+k)} V \otimes D_{-k} \tag{B.17}$$

for any sufficiently large k. This equation may also be used to extend the definitions of Schur functors to nonincreasing integer sequences (sometimes called generalized partitions) instead of partitions, and then $\mathbb{S}_{(\lambda_1+k,\cdots,\lambda_n+k)}V \otimes D_{-k} \simeq \mathbb{S}_{(\lambda_1,\cdots,\lambda_n)}V$ for any value of k. The representation $\mathbb{S}_{(\lambda_1,\cdots,\lambda_n)}V$ is polynomial iff λ is a partition.

Theorem B.4.1 shows that there is a deep connection between the representation theory of $GL(V)$ and that of $U(V)$. In particular, we have that any locally finite dimensional rational representation of $U(V)$ may be extended to a representation of $GL(V)$. But as $U(V)$ is a compact matrix Lie group, *every* finite dimensional representation of $U(V)$ is rational, and *every* unitary representation of $U(V)$ has a locally finite dimensional dense $U(V)$-subspace [40].

We would also like to identify one dimensional representations of $GL(V)$, which are exactly those which are trivial when viewed as representations of $SL(V) := \{g \in GL(V) | \det g = 1\}$. We have the dimension formula for $\lambda = (\lambda_1, \ldots, \lambda_n)$:

$$\dim \mathbb{S}_\lambda V = \prod_{1\le i<j\le n} \frac{\lambda_i - \lambda_j + j - i}{j-i} = \prod_{1\le i<j\le n} \left(1 + \frac{\lambda_i - \lambda_j}{j-i}\right) \qquad \text{(B.18)}$$

As each factor in the product is at least 1, the product can only be 1 if $\lambda_i - \lambda_j = 0$ for all $1 \le i < j \le n$, that is, $\lambda = (k, k, \ldots, k)$. where $k \in \mathbb{Z}$ and this representation is polynomial iff $k \ge 0$.

We have seen that polynomial representations of $GL(V)$ and representations of the symmetric groups S_d are intimately connected. We will make use of various isomorphisms which show some other aspects of this connection. For $\nu \vdash d$,

$$\mathbb{S}_\nu(V \otimes W) \simeq \bigoplus_{\lambda,\mu\vdash d} C_{\lambda\mu\nu}(\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu W) \qquad \text{(B.19)}$$

as $GL(V) \times GL(W)$-spaces, for $\mu, \nu \vdash d$,

$$V_\lambda \otimes V_\mu \simeq \bigoplus_{\nu\vdash d} C_{\lambda\mu\nu} V_\nu \qquad \text{(B.20)}$$

as S_d-spaces. Note that as the irreducible representations of S_d are self-dual, $C_{\lambda\mu\nu}$ is also the multiplicity of the trivial representation in $V_\lambda \otimes V_\mu \otimes V_\nu$, and in particular, is invariant under permutations of its three indices.

For our purposes it is particularly important to understand better how Schur functors interact with tensor products. We have the following generalization of eqs. (B.19-B.20):

Proposition B.4.2. *Let $\nu \vdash d$ and $V_1, \dots, V_k$ be finite dimensional vector spaces. Then*

$$\mathbb{S}_\nu(V_1 \otimes \cdots \otimes V_k) \simeq \bigoplus_{\lambda_1,\dots,\lambda_k \vdash d} C_{\nu\lambda_1\dots\lambda_k}(\mathbb{S}_{\lambda_1}V_1 \otimes \cdots \otimes \mathbb{S}_{\lambda_k}V_k) \tag{B.21}$$

as $GL(V_1) \times \cdots \times GL(V_k)$-spaces where the coefficient $C_{\nu\lambda_1\dots\lambda_k}$ is defined by

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_k} \simeq \bigoplus_{\nu \vdash d} C_{\nu\lambda_1\dots\lambda_k} V_\nu \tag{B.22}$$

Proof. We prove by induction, using eqs. (B.19-B.20) in the induction step. For $k = 1$ (and $k = 2$) the statement is clearly true. Suppose that eq. (B.21) holds for $k - 1$ instead of k. Then we have that

$$\begin{aligned}
&\mathbb{S}_\nu((V_1 \otimes \cdots \otimes V_{k-1}) \otimes V_k) \simeq \\
&\qquad \simeq \bigoplus_{\mu,\lambda_k \vdash d} C_{\mu\lambda_k\nu}(\mathbb{S}_\mu(V_1 \otimes \cdots \otimes V_{k-1}) \otimes \mathbb{S}_{\lambda_k}V_k) \\
&\qquad \simeq \bigoplus_{\mu,\lambda_k \vdash d} C_{\mu\lambda_k\nu}(\bigoplus_{\lambda_1,\dots,\lambda_{k-1} \vdash d} C_{\mu\lambda_1\dots\lambda_{k-1}}(\mathbb{S}_{\lambda_1}V_1 \otimes \cdots \otimes \mathbb{S}_{\lambda_{k-1}}V_{k-1}) \otimes \mathbb{S}_{\lambda_k}V_k) \\
&\qquad \simeq \bigoplus_{\lambda_1,\dots,\lambda_k \vdash d} \left(\sum_{\mu \vdash d} C_{\mu\lambda_k\nu} C_{\mu\lambda_1\dots\lambda_{k-1}} \right) (\mathbb{S}_{\lambda_1}V_1 \otimes \cdots \otimes \mathbb{S}_{\lambda_k}V_k)
\end{aligned} \tag{B.23}$$

But

$$\begin{aligned}
V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{k-1}} \otimes V_{\lambda_k} &\simeq \bigoplus_{\mu \vdash d} C_{\mu\lambda_1\dots\lambda_{k-1}} V_\mu \otimes V_{\lambda_k} \\
&\simeq \bigoplus_{\mu,\nu \vdash d} C_{\mu\lambda_1\dots\lambda_{k-1}} C_{\mu\lambda_k\nu} V_\nu \\
&\simeq \bigoplus_{\nu \vdash d} \underbrace{\left(\sum_{\mu \vdash d} C_{\mu\lambda_1\dots\lambda_{k-1}} C_{\mu\lambda_k\nu} \right)}_{C_{\nu\lambda_1\dots\lambda_k}} V_\nu
\end{aligned} \tag{B.24}$$

so the coefficient in the equation above is $C_{\nu\lambda_1\dots\lambda_k}$. □

Appendix C

The axioms of quantum mechanics

C.1 States and observables

The presentation of quantum mechanical states and observables in this section is based on ref.[46]. To every quantum system we associate a complex separable Hilbert space, that is, a complex vector space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that $\mathcal{H}$ is complete with respect to the metric induced by the inner product, and such that there exists a dense countable subset. In addition, there is a distinguished sub-C^*-algebra $\mathcal{A} \leq \mathrm{End}(\mathcal{H})$ of the C^*-algebra of bounded linear operators of $\mathcal{H}$.

Self-adjoint elements of $\mathcal{A}$ are called observables and they correspond to physical quantities which can be measured. A state is a positive element ϱ of $\mathcal{A}$ with $\mathrm{Tr}(\varrho) = 1$. A state is called pure if it is a rank 1 projector. Note that $\mathcal{A}$ may be a proper subset of $\mathrm{End}(\mathcal{H})$ meaning that not every rank 1 projector (or unit vector in $\mathcal{H}$) describes a physically realizable state. In general the representation of $\mathcal{A}$ on $\mathcal{H}$ is reducible, and there are self-adjoint elements in the commutant of $\mathcal{A}$ which are not proportional to the identity, called superselection operators. Subrepresentations of $\mathcal{H}$ are called superselection sectors.

One common way in which nontrivial superselection operators arise is when a group G acts on $\mathcal{H}$ and $\mathcal{A}$ in such a way that the action of $\mathcal{A}$ on $\mathcal{H}$ (which is a linear map $\mathcal{A} \otimes \mathcal{H} \to \mathcal{H}$) is G-equivariant. If this representation of G on $\mathcal{H}$ is not irreducible, then $\mathcal{H}$ decomposes as a direct sum of subspaces, each of which is the tensor product of an irreducible representation of G and a representation of $\mathcal{A}$.

If we measure the value of an observable A when the quantum system is in the state described by ϱ, then the expectation value of the outcome of the measurement is $\mathrm{Tr}(\varrho A)$. Note that $A \mapsto \mathrm{Tr}(\varrho A)$ is a positive linear functional on $\mathcal{A}$.

If we are given k quantum systems with state spaces $\mathcal{H}_1, \ldots, \mathcal{H}_k$ then the Hilbert space associated to the composite system is their tensor product $\mathcal{H} := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$. One can think of this as a consequence of linearity: loosely speaking, given an observable $A \in \mathrm{End}(\mathcal{H})$ of the whole system, its expectation value must depend linearly on the state of each subsystem, but also linearly on the state of the composite system. But the "most general" vector space which has the property that a multilinear map from $\mathcal{H}_1 \times \cdots \times \mathcal{H}_k$ factors through it is the tensor product.

An important case is when among the subsystems we can find identical ones. In this case the corresponding Hilbert spaces are isomorphic via a distinguished isomorphism, and hence can be identified. Then the appropriate permutation group acts on the Hilbert space of the composite system by permuting the factors in the tensor product, and observables are invariant under this action, giving rise to superselection sectors. In the simplest case when $\mathcal{H}_1 = \ldots = \mathcal{H}_k$, we readily see that S_k acts on the tensor product and on $\mathrm{End}(\mathcal{H})$ and the action of the observables is equivariant. Then the images of Young symmetrizers are subrepresentations of the algebra of observables. In particular, we can restrict ourselves to any subrepresentation, the partition (k) corresponds to bosons, and in this case the relevant Hilbert space is

$$\mathbb{S}_{(k)}\mathcal{H}_1 = S^m(\mathcal{H}_1) \tag{C.1}$$

while the partition (1^k) corresponds to fermions, with the relevant part of the Hilbert space being

$$\mathbb{S}_{(1^k)}\mathcal{H}_1 = \Lambda^m(\mathcal{H}_1) \tag{C.2}$$

C.2 Time evolution with restricted interactions

Let $\mathcal{H}$ be the state space of a quantum system interacting with its environment, the state space of which is $\mathcal{H}_{ENV}$. Suppose that we are given a distinguished C^*-algebra $\mathcal{A} \leq \mathrm{End}(\mathcal{H})$, and the Hamiltonian governing the unitary evolution of the composite system is of the form

$$H = \sum_{i \in I} A_i \otimes B_i \tag{C.3}$$

where $\forall i \in I : A_i \in \mathcal{A}$ and $B_i \in \mathrm{End}(\mathcal{H}_{ENV})$, that is, observables in $\mathcal{A}$ correspond to physical quantities through which the quantum system is coupled to its environment.

If the inital state of the joint system is $\varrho \otimes \psi\psi*$ where $\varrho \in \mathrm{End}(\mathcal{H})$ and $\psi \in \mathcal{H}_{ENV}$ is a unit vector, then after a time interval t has elapsed, the state of the quantum system is

$$\sum_{j \in J} E_j \varrho E_j^* \tag{C.4}$$

where $E_j = \langle e_j, e^{\frac{i}{\hbar}tH}\psi\rangle_{\mathcal{H}_{ENV}} \in \mathrm{End}(\mathcal{H})$.

We claim that $\forall j \in J : E_j \in \mathcal{A}$. Firstly, for a polynomial $p(x) = \sum_{k=0}^{m} a_k x^k$ and $v, w \in \mathcal{H}_{ENV}$, we have that

$$\begin{aligned} &\langle v, p(\sum_{i \in I} A_i \otimes B_i) w\rangle_{\mathcal{H}_{ENV}} \\ &= \sum_{k=0}^{m} a_k \sum_{i_1,\ldots,i_k \in I} \langle v, (A_{i_1} A_{i_2} \cdots A_{i_k} \otimes B_{i_1} B_{i_2} \cdots B_{i_k}) w\rangle_{\mathcal{H}_{ENV}} \\ &= \sum_{k=0}^{m} a_k \sum_{i_1,\ldots,i_k \in I} \langle v, B_{i_1} B_{i_2} \cdots B_{i_k} w\rangle_{\mathcal{H}_{ENV}} A_{i_1} A_{i_2} \cdots A_{i_k} \in \mathcal{A} \end{aligned} \tag{C.5}$$

Secondly, any continuous function on the spectrum of H can be uniformly approximated by polynomials, from which the claim follows.

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