Mass of the bottom quark from Upsilon(1S) at NNLO: an update

Mass of the bottom quark from Upsilon(1S) at NNNLO: an update

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Abstract. We update our perturbative determination of $\overline{\rm MS}$ mass $\overline{m}_b(\overline{m}_b)$, by including the recently obtained four-loop coefficient in the relation between the pole and $\overline{\rm MS}$ mass. First the renormalon subtracted (RS or RS') mass is determined from the known mass of the $\Upsilon(1S)$ meson, where we use the renormalon residue N_m obtained from the asymptotic behavior of the coefficient of the 3-loop static singlet potential. $\overline{\rm MS}$ mass is then obtained using the 4-loop renormalon-free relation between the RS (RS') and $\overline{\rm MS}$ mass. We argue that the effects of the charm quark mass are accounted for by effectively using $N_f = 3$ in the mass relations. The extracted value is $\overline{m}_b(\overline{m}_b) = 4222(40)$ MeV, where the uncertainty is dominated by the renormalization scale dependence.

1. Introduction

The ($\overline{\rm MS}$) mass of the bottom (b) quark, $\overline{m}_b \equiv \overline{m}_b(\overline{m}_b)$, is an important quantity in particle physics, free of renormalon ambiguities, and appears in many physical observables. Since it is relatively high, ~ 4 GeV, perturbative QCD methods are suitable for its extraction. The mass of the ground state of the $b\bar{b}$ quarkonium, $\Upsilon(1S)$, is one of the best quantities for such an extraction, $M^{\rm (th)}_{\Upsilon(1S)} = 2m_b + E_{\Upsilon(1S)} = 9.460$ GeV, where m_b is the pole mass of the bottom quark, and $E_{\Upsilon(1S)}$ is the binding energy. We use the available perturbative expansions of $2m_b/\overline{m}_b$ and of $E_{\Upsilon(1S)}/\overline{m}_b$ in powers of QCD coupling $a(\mu) \equiv \alpha_s(\mu)/\pi$ and thus extract the value of $\overline{m}_b$. In the extraction, we use the fact that the leading infrared (IR) renormalon ambiguity of $2m_b$ cancels out with that of $E_{\Upsilon(1S)}$ [1, 2, 3].

These proceedings are a brief review of our previous work [4], which we update by including in the analysis the recently calculated [5] four-loop coefficient of the relation between the pole mass and the $\overline{\rm MS}$ mass. Here we outline: (1) The correct treatment of charm quark mass effects in the perturbation expansion of $m_b/\overline{m}_b$; (2) Asymptotic expressions for the coefficients in the perturbation expansion of the ratio $m_b/\overline{m}_b$ and of the static singlet potential $V(r)$, and the extraction of the renormalon residue N_m; (3) The construction of the (modified) renormalon-subtracted mass $m_{b,{\rm RS}^{(\prime)}}$ (using N_m), and the renormalon-free relation between $m_{b,{\rm RS}^{(\prime)}}$ and $\overline{m}_b$; (4) Renormalon-free perturbation expansion for $M^{\rm (th)}_{\Upsilon(1S)}$ in terms of $m_{b,{\rm RS}^{(\prime)}}$, and extraction, from $M^{\rm (th)}_{\Upsilon(1S)} = 9.460$ GeV, of the values of $m_{b,{\rm RS}^{(\prime)}}$ ($\Rightarrow \overline{m}_b$).

2. Charm mass effects in the bottom pole mass

The pole mass m_b and the $\overline{\rm MS}$ mass $\overline{m}_b$ are related:

$$m_b = \overline{m}_b \left(1 + S(N_f)\right) + \delta m_c^{(+)} \ , \tag{1}$$

where $$S(N_f) = \frac{4}{3} a_+(\mu) \left[1 + r_1^{(+)}(\mu) a_+(\mu) + r_2^{(+)}(\mu) a_+^2(\mu) + r_3^{(+)}(\mu) a_+^3(\mu) + \mathcal{O}(a_+^4)\right] \tag{2}$$

and the evaluation is usually performed in QCD with $N_f = N_l + 1 = 4$ active flavors: $r_j^{(+)}(\mu) \equiv r_j(\mu; N_f)$, $a_+(\mu) = a(\mu; N_f)$. The coefficients $R_0 = 4/3$ and r_j $(j = 1, 2)$ were obtained in Refs. [6], [7], [8, 9], respectively. Recently, numerical values of the 4-loop coefficient r_3 were obtained [5], and we incorporate them here in the form given in [10].

These coefficients have a specific dependence on the renormalization scale μ, dictated by μ-independence of $S(N_f)$

$$r_1(\mu; N_f) = r_1(N_f) + \beta_0 L_m(\mu) \ , \text{ etc.} \tag{3}$$

where $L_m(\mu) = \ln(\mu^2/\overline{m}_b^2)$, and we maintain, for simplicity, the notation $r_j \equiv r_j(\overline{m}_b)$. We will use the notations $\beta_0 = (1/4)(11 - 2N_f/3)$ and $\beta_1 = c_1\beta_0 = (102 - 38N_f/3)/16$ for the first two coefficients of the RGE of $a(\mu)$

$$\frac{da(Q)}{d \ln Q^2} = -\beta_0 a^2(Q) \left(1 + c_1 a(Q) + c_2 a^2(Q) + c_3 a^3(Q) + \cdots \right) \ . \tag{4}$$

Finite-mass charm quark effects are incorporated in

$$\delta m_c^{(+)} = \delta m_{(c,+)}^{(1)} a_+^2(\overline{m}_b) + \delta m_{(c,+)}^{(2)} a_+^3(\overline{m}_b) + \mathcal{O}(a_+^4) \, , \tag{5}$$

which vanishes in the $m_c \to 0$ limit. We have

$$\delta m_{(c,+)}^{(1)} = \frac{4}{3} \overline{m}_b \Delta[\overline{m}_c/\overline{m}_b] = 1.9058 \text{ MeV [7]}, \quad \delta m_{(c,+)}^{(2)} = 48.6793 \text{ MeV [11]}, \tag{6}$$

$$\Rightarrow \quad \delta m_{(c,+)}^{(1)} a_+^2(\overline{m}_b) = 9.3 \text{ MeV}, \qquad \delta m_{(c,+)}^{(2)} a_+^3(\overline{m}_b) = 18.1 \text{ MeV}, \tag{7}$$

so $\delta m_c^{(+)}$ is badly divergent. Why? At loop order n, the natural scale of the loop integral for m_b is $m_b e^{-n}$ [12], which for n large enough is: $m_b e^{-n} < m_c$. Therefore, for large n (> 2) charm quark appears as very heavy (decoupled), leading to the effective number of flavors being $N_l = 3$ and not $N_f = N_l + 1 = 4$. Therefore, it is convenient to rewrite the relation between the pole and the $\overline{\rm MS}$ mass in terms of $a_-(\mu) = a(\mu; N_l)$ and $r_j^{(-)}(\mu) \equiv r_j(\mu; N_l)$ $[N_l = 3]$

$$m_b = \overline{m}_b \left(1 + S(N_l)\right) + \delta m_c \ , \tag{8}$$

where $$S(N_l) = \frac{4}{3} a_-(\mu) \left[1 + r_1^{(-)}(\mu) a_-(\mu) + r_2^{(-)}(\mu) a_-^2(\mu) + r_3^{(-)}(\mu) a_-^3(\mu) + \mathcal{O}(a_-^4)\right], \tag{9}$$

and $r_j^{(-)}(\overline{m}_b) = 7.74,\ 87.2,\ 1265.3 \pm 16.1$, for $j = 1, 2, 3$. The effects of the decoupling of S $(N_f \mapsto N_l = 3)$ are absorbed in the new δm_c

$$\delta m_c = \left[\delta m_{(c,+)}^{(1)} + \delta m_{(c,\text{dec.})}^{(1)}\right] a_-^2(\overline{m}_b) + \left[\delta m_{(c,+)}^{(2)} + \delta m_{(c,dec.)}^{(2)}\right] a_-^3(\overline{m}_b) + \mathcal{O}(a_-^4) \ , \tag{10}$$

where $\delta m^{(j)}_{(c,\text{dec.})}$ are generated by this decoupling and read

$$\delta m^{(1)}_{(c,\text{dec.})} = \frac{2}{9}\overline{m}_b\left(\ln\left(\frac{\overline{m}_b^2}{\overline{m}_c^2}\right) - \frac{71}{32} - \frac{\pi^2}{4}\right) \tag{11}$$

and $\delta m^{(2)}_{(c,\text{dec.})}$ can be found in Ref. [4].

Numerical evaluation gives for $\left[\delta m^{(1)}_{(c,+)} + \delta m^{(1)}_{(c,\text{dec.})}\right] a_-^2(\overline{m}_b) = -1.6$ MeV and $\left[\delta m^{(2)}_{(c,+)} + \delta m^{(2)}_{(c,\text{dec.})}\right] a_-^3(\overline{m}_b) = -0.3$ MeV. This means that the previous divergent series (in $\text{QCD}_{N_f=4}$) $\delta m_c^{(+)} = (9.3 + 18.1 + \ldots)$ MeV [Eq. (7)] now tranforms (in $\text{QCD}_{N_l=3}$) to

$$\delta m_c = (-1.6 - 0.3 + \ldots)\ \text{MeV}. \tag{12}$$

The series for δm_c in $\text{QCD}_{N_l=3}$ formulation is convergent, strong cancellation takes place between $\delta m^{(j)}_{(c,+)}$ and $\delta m^{(j)}_{(c,\text{dec.})}$, as expected.

3. Leading renormalon of the pole mass

The asymptotic behaviour of r_N is determined by the leading IR renormalon:

$$\frac{4}{3}r_N^{\text{asym}}(\mu) \simeq \pi N_m \frac{\mu}{\overline{m}_b}(2\beta_0)^N \frac{\Gamma(\nu+N+1)}{\Gamma(\nu+1)}\left[1 + \sum_{s=1}^{3}\frac{\nu\cdots(\nu-s+1)}{(N+\nu)\cdots(N+\nu-s+1)}\widetilde{c}_s + \mathcal{O}\left(N^{-4}\right)\right]. \tag{13}$$

$$\frac{4}{3}r_N(\mu) = \pi N_m \frac{\mu}{\overline{m}_b}(2\beta_0)^N \frac{\Gamma(\nu+N+1)}{\Gamma(\nu+1)}\left[1 + \sum_{s\geq 0}\frac{\nu\cdots(\nu-s+1)}{(N+\nu)\cdots(N+\nu-s+1)}\widetilde{c}_s\right] + h_N(\mu), \tag{14}$$

where h_N is dominated by subleading renormalons, and the coefficients $\widetilde{c}_s$ $(s = 1, 2, 3)$ are given in [13, 14, 15, 4] ($\widetilde{c}_0 = 1$ by convention).

Determining the pole mass from $\Upsilon(1S)$ mass has large uncertainties due to the pole mass renormalon ambiguity $\delta m_b \sim \Lambda_{\text{QCD}}$ [13]. In order to avoid this problem, we work with the renormalon-subtracted (RS) bottom mass $m_{b,\text{RS}}$ instead [14]. Then, $\overline{m}_b$ is obtained from its stable (renormalon-free) relation with the m_{RS} mass.

The use of m_{RS} in the theoretical evaluation of the $\Upsilon(1S)$ mass is convenient because it has no leading IR renormalon ambiguity, and the renormalon cancellation in the quarkonium mass $M_{\Upsilon(1S)} = 2m_b + E_{\Upsilon(1S)}$ is implemented automatically and explicitly.

4. Determination of the renormalon residue N_m and N_V

The asymptotic behavior of the coefficients $v_N(\mu)$ of the static singlet potential,

$$V(r) = -\frac{4\pi}{3}\frac{1}{r}a_-(\mu)\left[1 + v_1(\mu)a_-(\mu) + v_2 a_-(\mu)^2 + v_3 a_-(\mu)^3 + \ldots\right], \tag{15}$$

can be determined in complete analogy with those of r_N

$$-\frac{4}{3}v_N(\mu) = N_V \mu r (2\beta_0)^N \sum_{s\geq 0}\widetilde{c}_s \frac{\Gamma(\nu+N+1-s)}{\Gamma(\nu+1-s)} + d_N(\mu) \quad \Rightarrow \tag{16}$$

$$-\frac{4}{3}v_N^{\text{asym}}(\mu) \approx N_V \mu r (2\beta_0)^N \frac{\Gamma(\nu+N+1)}{\Gamma(\nu+1)}\left[1 + \sum_{s=1}^{3}\frac{\nu\cdots(\nu-s+1)}{(N+\nu)\cdots(N+\nu-s+1)}\widetilde{c}_s\right], \tag{17}$$

where in Eq. (17) $d_N = 0$ was taken. We can determine the "strength", N_V, of the leading IR renormalon by approximating the asymptotic $v_N^{\rm asym}(\mu)$ with the exact $v_N(\mu)$ ($N = 0, 1, 2, 3$): $v_N^{\rm asym}(\mu) \approx v_N(\mu) \;\Rightarrow$

$$N_V \approx -\frac{4}{3} v_N(\mu) \Bigg/ \left\{ \mu r (2\beta_0)^N \frac{\Gamma(\nu + N + 1)}{\Gamma(\nu + 1)} \left[1 + \sum_{s=1}^{3} \frac{\nu \cdots (\nu - s + 1)}{(N + \nu) \cdots (N + \nu - s + 1)} \widetilde{c}_s \right] \right\}. \quad (18)$$

The result for N_V should be the best for the highest available N ($N = 3$) and should also have reduced spurious μ-dependence. At present, the v_j are known up to N^{3}LO (v_3) [16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

In the sum $2m_b + V(r)$ the leading IR renormalon gets cancelled. N_V is then related with N_m by the renormalon cancellation of the sum $2m_b + V(r)$: $2N_m + N_V = 0$. Determining N_m via N_V gives us the value that we use [4]

$$N_m = -N_V/2 = 0.56255(260) \; (N_l = 3). \quad (19)$$

5. Renormalon-subtracted (RS, RS') mass of bottom

The RS mass is defined by subtracting the leading IR renormalon singularity from the pole mass [14]:

$$m_{b,\rm RS}(\nu_f) = m_b - N_m \pi \nu_f \sum_{N=0}^{\infty} a_-^{N+1}(\nu_f)(2\beta_0)^N \sum_{s \geq 0} \widetilde{c}_s \frac{\Gamma(\nu + N + 1 - s)}{\Gamma(\nu + 1 - s)} \quad (20)$$

Equation (20) is still formal. In practice, one rewrites m in terms of $\overline{m}$ using Eqs. (8)-(9)

$$m_b = \overline{m}_b (1 + (4/3) a_-(\nu_f) + \ldots), \quad (21)$$

and reexpands the perturbation series in Eq. (20) around the same coupling $a_-(\mu)$, at fixed but otherwise arbitrary scale μ:

$$m_{b,\rm RS}(\nu_f) = \overline{m}_b \left[1 + \sum_{N=0}^{\infty} h_N(\nu_f) a_-^{N+1}(\nu_f) \right] \Rightarrow m_{b,\rm RS}(\nu_f) = \overline{m}_b \left[1 + \sum_{N=0}^{\infty} \widetilde{h}_N(\nu_f; \mu) a_-^{N+1}(\mu) \right], \quad (22)$$

where $h_N(\nu_f)$ is determined from Eq. (14) (with $\mu = \nu_f$ and with the sum truncated at $\widetilde{c}_3$) for $N = 0, 1, 2, 3$. For $N \geq 4$ we take $h_N(\overline{m}_b) = 0$. The coefficients $\widetilde{h}_N(\nu_f; \mu)$ in Eq. (22) are obtained by expanding $a_-(\nu_f)$ in the expansion in powers of $a_-(\mu)$. Note that $m_{b,\rm RS}(\nu_f)$ will only marginally depend on μ when we truncate the infinite sum in Eq. (22). On the other hand, the coefficients h_N are functions of ν_f, μ, and $\overline{m}_b$, and are much smaller than $r_N(\mu)$.

A variant of the RS mass is the modified renormalon-subtracted (RS') mass $m_{b,\rm RS'}$, where subtractions start at $\sim a^2$ [14]. Specifically in this case, Eqs. (20) and (22) are repeated, with the replacements $m_{b,\rm RS}(\nu_f) \mapsto m_{b,\rm RS'}(\nu_f)$ and $\sum_{N=0}^{\infty} \mapsto \sum_{N=1}^{\infty}$.

6. Bottom mass from heavy quarkonium

The perturbation expansion of $M_{\Upsilon(1S)}^{(th)}$ is presently known up to $\mathcal{O}(m_b a^5)$ [19, 20, 21, 22]:

$$\begin{aligned} M_{\Upsilon(1S)}^{(th)} &= 2m_b - \frac{4\pi^2}{9} m_b a_-^2(\mu) \Bigg\{ 1 + a_-(\mu) \left[K_{1,0} + K_{1,1} L_p(\mu) \right] + a_-^2(\mu) \sum_{j=0}^{2} K_{2,j} L_p(\mu)^j \\ &\quad + a_-^3(\mu) \Big[K_{3,0,0} + K_{3,0,1} \ln a_-(\mu) + \sum_{j=1}^{3} K_{3,j} L_p(\mu)^j \Big] + \mathcal{O}(a_-^4) \Bigg\}, \end{aligned} \quad (23)$$

μ is the renormalization scale, $L_p(\mu) = \ln(\mu/\mu_b)$ where $\mu_b = (4\pi/3)m_b a_-(\mu)$. $K_{i,j}(N_f)$ and $K_{3,0,j}$ are given, e.g., in [4]. We then rewrite m_b in terms of $m_{b,\mathrm{RS}}$ to implement the leading IR renormalon cancellation. This gives

$$\frac{M^{(th)}_{\Upsilon(1S)}}{m_{b,\mathrm{RS}}(\nu_f)} = 2 + \left[2\pi N_m b a \mathcal{K}_0 - \frac{4\pi^2}{9}a^2\right] + \left[2\pi N_m b a^2 (\mathcal{K}_1 + z_1\mathcal{K}_0) - \frac{4\pi^2}{9}a^3 (K_{1,0} + K_{1,1}L_{\mathrm{RS}})\right]$$
$$+ \left[2\pi N_m b a^3 (\mathcal{K}_2 + 2z_1\mathcal{K}_1 + z_2\mathcal{K}_0) - \frac{4\pi^2}{9}\left(a^4\sum_{j=0}^{2} K_{2,j}L^j_{\mathrm{RS}} + b a^3 \pi N_m \mathcal{K}_0\right)\right] + \mathcal{O}(ba^4, a^5). \quad (24)$$

The terms $\mathcal{O}(ba^4, a^5)$ have a similar structure and were written in [4]. The notations are

$$a \equiv a_-(\mu) = a(\mu, N_f = 3)\ ; \quad b \equiv b(\nu_f) = \nu_f/m_{b,\mathrm{RS}}(\nu_f)\ , \quad N_m = N_m(N_l = 3)\ , \quad (25a)$$

$$L_{\mathrm{RS}} \equiv L_{\mathrm{RS}}(\mu) = \ln\left(\frac{\mu}{(4\pi/3)m_{b,\mathrm{RS}}(\nu_f)a_-(\mu)}\right),\ \mathcal{K}_N = (2\beta_0)^N \sum_{s=0}^{3} \widetilde{c}_s \frac{\Gamma(\nu + N + 1 - s)}{\Gamma(\nu + 1 - s)}\ . \quad (25b)$$

In the expression (24) for $M_{\Upsilon(1S)}$, the terms of the same order $(\nu_f/m_{b,\mathrm{RS}})a^n$ and a^{n+1} were combined in common brackets $[\ldots]$, in order to account for the renormalon cancellation.

If using the RS' mass in our approach instead, the above expressions are valid without changes, except that $m_{b,\mathrm{RS}} \mapsto m_{b,\mathrm{RS'}}$ and $\mathcal{K}_0 \mapsto 0$ (and: $h_0(\mu) \mapsto 4/3$).

We note that we take $N_l = 3$ active flavours, as the charm quark mass effects in the binding energy $E_{\Upsilon(1S)}$ are negligible [26].

We extract the bottom masses from the condition $M^{(th)}_{\Upsilon(1S)} = M^{(exp)}_{\Upsilon(1S)} (= 9.460\ \mathrm{GeV})$. The error estimates are made assuming $\mu = 2.5^{+1.5}_{-1.0}$ GeV [we varied μ in Eq. (24) but not in Eq. (22)], $\nu_f = 2 \pm 1$ GeV, $\alpha_s(M_z) = 0.1184(7)$ (and decoupling at $\overline{m}_b = 4.2$ GeV and at $\overline{m}_c = 1.27$ GeV), $N_m = 0.56255(260)$, and $(4/3)r_3(\overline{m}_b; N_l) = 1687.1 \pm 21.5$ [10].

In RS and RS' approaches we extract, in MeV, respectively

$$m_{b,\mathrm{RS}}(2\mathrm{GeV}) = 4\,437^{-11}_{+43}(\mu)^{-3}_{+5}(\nu_f)^{-2}_{+2}(\alpha_s)^{-41}_{+41}(N_m)^{-0}_{+0}(r_3); \quad (26a)$$
$$\Rightarrow \quad \overline{m}_b = 4\,217^{-10}_{+39}(\mu)^{-3}_{+5}(\nu_f)^{-5}_{+5}(\alpha_s)^{-1}_{+1}(N_m)^{-4}_{+4}(r_3). \quad (26b)$$
$$m_{b,\mathrm{RS'}}(2\ \mathrm{GeV}) = 4\,761^{-16}_{+41}(\mu)^{-3}_{+5}(\nu_f)^{+4}_{-3}(\alpha_s)^{-26}_{+26}(N_m)^{-0}_{+0}(r_3); \quad (26c)$$
$$\Rightarrow \quad \overline{m}_b = 4\,223^{-14}_{+36}(\mu)^{-2}_{+4}(\nu_f)^{-4}_{+4}(\alpha_s)^{-1}_{+1}(N_m)^{-4}_{+4}(r_3). \quad (26d)$$

The uncertainties in $\overline{m}_b$ are dominated by the variation of the renormalization scale μ.

The renormalon cancellations are reflected numerically in Eq. (24) [we take $\mu = 2.5$ GeV]:

$$\mathrm{RS} : M_{\Upsilon(1S)} = (8874 + 431 + 167 + 18 - 30)\ \mathrm{MeV}\,, \quad (27a)$$
$$\mathrm{RS'} : M_{\Upsilon(1S)} = (9521 - 150 + 112 + 8 - 31)\ \mathrm{MeV}\,, \quad (27b)$$

The convergence is good; except for the last (NNNLO) term $\mathcal{O}(a^5, ba^4)$, where the factorization scale dependence becomes stronger, which may signal the importance of ultrasoft effects.

The relations between RS (RS') mass and $\overline{\mathrm{MS}}$ mass are reasonably convergent:

$$m_{b,\mathrm{RS}}(2\ \mathrm{GeV}) = (4217 + 191 + 36 + 12 - 19)\ \mathrm{MeV}\ , \quad (28a)$$
$$m_{b,\mathrm{RS'}}(2\ \mathrm{GeV}) = (4223 + 478 + 60 + 18 - 17)\ \mathrm{MeV}\ , \quad (28b)$$

where the expansion parameter is taken to be $a(2.5\ \mathrm{GeV})$. A bigger value for the renormalization scale, closer to the bottom quark mass, makes the last term smaller.

Until now we have approximated $\delta m_c = 0$ in Eq. (20). However, $\delta m_c \approx -2$ MeV, Eq.(12). Hence, we have to add 2 MeV to the values of $\bar{m}_b$ obtained in Eqs. (26b) and (26b) (in Ref. [4] it was incorrectly subtracted), leading to the final average of the RS and RS' extractions

$$\overline{m}_b = 4222(40) \text{ MeV} . \tag{29}$$

where we have rounded the $\pm$ variation of each parameter to the maximum and added them in quadrature.

7. Conclusions

(i) We presented strong numerical indications that the charm quark decouples in the relation between m_b and $\overline{m}_b$ ($\Rightarrow N_l = 3$).

(ii) An improved determination of the residue of the leading renormalon for the bottom pole mass (and static potential with $N_l = 3$) was performed: $N_m = 0.56255(260)$.

(iii) Use of the 3-loop ($\sim a^5\overline{m}_b$) corection to the $\Upsilon(1S)$ binding energy, and 4-loop relation between m_b and $\overline{m}_b$, allowed us to perform extraction of $m_{b,\mathrm{RS}^{(\prime)}}$ and $\overline{m}_b$ to NNNLO, with the resulting values Eq. (29). The uncertainties are dominated by the variation of the renormalization scale.

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