RICCI YANG-MILLS FLOW

by

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Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Mathematics
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2007

ABSTRACT

(Mathematics)

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Abstract

Let (M^n, g) be a Riemannian manifold. Say $\mathcal{K} \to E \to M$ is a principal $\mathcal{K}$-bundle with connection A. We define a natural evolution equation for the pair (g, A) combining the Ricci flow for g and the Yang-Mills flow for A which we dub Ricci Yang-Mills flow. We show that these equations are, up to diffeomorphism equivalence, the gradient flow equations for a Riemannian functional on M. Associated to this energy functional is an entropy functional which is monotonically increasing in areas close to a developing singularity. This entropy functional is used to prove a non-collapsing theorem for certain solutions to Ricci Yang-Mills flow.

We show that these equations, after an appropriate change of gauge, are equivalent to a strictly parabolic system, and hence prove general unique short-time existence of solutions. Furthermore we prove derivative estimates of Bernstein-Shi type. These can be used to find a complete obstruction to long-time existence, as well as to prove a compactness theorem for Ricci Yang Mills flow solutions.

Our main result is a fairly general long-time existence and convergence theorem for volume-normalized solutions to Ricci Yang-Mills flow. The limiting pair (g, A) satisfies equations coupling the Einstein and Yang-Mills conditions on g and A respectively. Roughly these conditions are that the associated curvature F_A must be large, and satisfy a certain "stability" condition determined by a quadratic action of F_A on symmetric two-tensors.

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Chapter 1

Introduction

1.1 Background

"Does every smooth compact manifold admit a best metric?" This basic question was first posed by Rene Thom to Marcel Berger [2] in the early 1960's . Of course the question above is not precise mathematically. Indeed, part of the question involves finding a proper definition of "best." The guiding example for answering this question has always been the classical uniformization theorem for surfaces. In particular this example tells us that the definition of "best metrics" should use the Riemannian curvature tensor and furthermore be invariant under the action of the diffeomorphism group of the underlying manifold. Secondly we would hope that constant curvature metrics, should they exist, fall into this class of metrics. Third is that we should not necessarily expect our "best" metrics to be unique, indeed a finite-dimensional moduli space of "best metrics" is acceptable.

While a huge amount of work has been done in answering this question, here we will only give a very brief history of some partial answers. The first natural question to ask is whether Einstein metrics, that is, metrics satisfying

$$\mathrm{rc} = \lambda g \tag{1.1}$$

where rc is the Ricci tensor of the Riemannian metric g and $\lambda \in \mathbb{R}$, fulfill the requirements of the Thom question. Indeed for three-dimensional manifolds this condition is equivalent to the metric having constant curvature, making it a very natural candidate for the definition of "best metric." However because of this very fact it is immediately clear that not every three-manifold can admit such a metric, for example the manifold $S^2 \times S^1$. The constant curvature metrics give but three classes (depending on the sign of the curvature) of natural metrics one could expect on a three-manifold. In his famous Geometrization Conjecture Thurston proposed [30] that given a three-manifold, one could decompose it into pieces using certain surgeries, and that each resulting piece admitted one of eight canonical "geometries." The recent proof of this conjecture [11] [21] [25] [26] [27] [31] is the crowning achievement of decades of work in geometric analysis, and we will describe it later.

For manifolds with dimension greater than three, Einstein metrics need not have constant curvature, and one naive guess would be that every such manifold admits an Einstein metric. However, the Hitchin-Thorpe inequality [19] [29] says that a four-manifold admits an Einstein metric only if

$$(2\chi + 3\,|\tau|)\,(M) \geq 0$$

where $\chi(M)$ denotes the Euler characteristic and $\tau(M)$ the signature of M. This inequality is a consequence of the generalized Gauss-Bonnet theorem [1] and the Hirzebruch signature theorem [18]. Thus other classes of metrics must be considered for four-manifolds. In particular Berger [2] proposed metrics minimizing a global

L^p norm of curvature as potential "best metrics". For scaling reasons it is most sensible to consider metrics minimizing the $L^{n/2}$-norm of curvature on a Riemannian n-manifold. While in general these metrics are not well understood, in dimension 4 much progress has been made. Einstein metrics fall into this class, but other natural classes of metrics do as well. Claude Lebrun found simply connected 4-manifolds which admit no such minimizing metric, and also showed that the existence of such metrics depends strictly on the diffeotype of the underlying topological manifold [23]. We close by noting that the question of whether every manifold of dimension five or greater admits an Einstein metric is still completely open.

We note that in all of the examples above the notion of "best metric" has *only* involved the Riemannian curvature tensor. Here we propose a definition of "best metric" which uses an auxiliary term. In particular we will introduce the self-duality equations into the definition of "best metric." Let (M^4, g) be a Riemannian four-manifold, and let $S^1 \rightarrow E \rightarrow M$ denote a circle bundle over M. Let A be a connection on this bundle, and let $F = dA$ denote the curvature of this connection. Consider the system of equations

$$\begin{gathered}\mathrm{rc}_{ij} - \frac{1}{2} g^{kl} F_{ik} F_{jl} = \lambda g_{ij} \\ d^* F = 0\end{gathered} \tag{1.2}$$

where d^* is the L^2-adjoint of the exterior derivative d. We first point out that given any Einstein metric g, we can get a solution of (1.2) by taking the trivial bundle $S^1 \times M$ with the flat connection. In chapter 2 we will give a simple example of a manifold which admits no Einstein metric but does admit a solution to (1.2). These equations arise naturally from adding the energy $\int |F|^2 \, dV$, considered as a functional in both the metric and the two-form F, to the scalar curvature Lagrangian which gives the Einstein equation.

The equation $d^*F = 0$ is known as the Yang-Mills equation for A. These equations were introduced by physicists in an attempt to understand electromagnetism and the nuclear strong interaction. An important subclass of Yang-Mills connections on principal bundles over four-manifolds are the self-dual and anti-self-dual connections. Indeed such connections are intimately linked with the topology of four-manifolds, and they have had a tremendous impact on our understanding of four-manifold geometry and topology. We hasten to add that the tensor $g^{kl}F_{ik}F_{jl}$ is itself scalar, i.e. its traceless piece vanishes, if and only if the connection A is self-dual or anti-self dual with respect to g (see lemma A.4). Since in every case described above the existence of best metrics is intimately linked to the topology and smooth structure of the underlying manifold, it is natural to combine the Riemannian curvature with the equations of self-duality to find "best metrics," and indeed equation (1.2) does this in a very simple way.

While (1.2) is clearly a natural equation, this alone does not justify its study. To describe our reasoning for introducing these equations, let us give a very broad overview of the proof of the Geometrization Conjecture. In 1983 Hamilton [14] introduced the Ricci flow equation

$$\begin{aligned} \frac{d}{dt} g_{ij} &= -2\,\mathrm{rc}_{ij} \\ g(0) &= g_0 \end{aligned} \tag{1.3}$$

and showed that if g_0 has positive Ricci curvature then the solution to (1.3), after dilating to fix the volume of the time-dependent metric, exists for all time and converges to a spherical space form. In light of this result, it was thought that given *any* metric on *any* three-manifold, it might be possible to completely describe the behaviour of the solution to (1.3). It was conjectured that only certain fairly simple singularities would occur, and furthermore that by performing "surgeries" one could remove these singularities and then continue the Ricci flow [16]. Ideally all of the limiting objects would admit rigid geometries, and thus we could understand their topological structure. This picture was validated in the spectacular work of Perelman [11] [21] [25] [26] [27] [31] resulting in a proof of the Geometrization Conjecture, and in particular the Poincare Conjecture.

This amazing result shows how strong geometric evolution equations can be in understanding the geometry and topology of manifolds. In fact, the Ricci flow equation can be used to give a proof of the uniformization theorem for surfaces [10], and also to show the existence of Kahler-Einstein metrics on Kahler manifolds with zero or negative first Chern class [13]. Also there is the recent result of Böhm and Wilking [4] showing that any manifold admitting a metric of strictly positive curvature operator admits a metric of constant positive sectional curvature. Despite these successes, analysing the behaviour of solutions to the Ricci flow on manifolds of dimension 4 or higher remains extremely difficult. One could argue that the main reason for this difficulty is simply the complexity of the Riemannian curvature tensor in these dimensions. Thus, using the notation from above, we propose to study the following system of equations:

$$\begin{aligned} \frac{d}{dt} g_{ij} &= -2\,\mathrm{rc}_{ij} + g^{kl} F_{ik} F_{jl} \\ \frac{d}{dt} A &= -d^* F \end{aligned} \tag{1.4}$$

which we dub *Ricci Yang-Mills flow*, and ask the following question which motivates the rest of this thesis:

Main Question: *To what extent can the bundle curvature F_A be used to control the behaviour of solutions to Ricci Yang-Mills flow, and thus produce metrics and connections on manifolds which satisfy (1.2)?*

Indeed we will study a version of equation (1.4) associated to a general principal bundle. Given that the bundle curvature F is typically much simpler than the full Riemannian curvature tensor, this question justifies introducing F into the already complicated Ricci flow equation. This perspective is also our original motivation for introducing the system (1.2).

1.2 Statements of Results

Here we will state the main results of this thesis and briefly describe their significance. In what follows a *bundle* metric is a natural metric on a principal bundle defined in chapter 2. The Ricci Yang-Mills flow can be defined in terms of these metrics, which we will denote below as G. Also, the curvature of this metric G will be denoted by Rm.

Proposition 6.1 (Short-time Existence): *Given (g, A) as above there exists $\epsilon > 0$ so that a unique solution to Ricci Yang-Mills flow exists on $[0, \epsilon)$ with initial condition*

(g, A).

One expects this short-time existence result from the corresponding results for the Ricci and Yang-Mills flow. Since the coupling introduces lower-order terms into the individual Ricci and Yang-Mills flows, the proof is a straightforward modification of the proof of short-time existence of Ricci flow using a diffeomorphism gauge-fixing procedure. Given this result, one can compute the evolution of the curvatures. On a formal level these equations are very close to the evolution equations for Ricci flow. Thus one can prove many further analytical properties of Ricci Yang-Mills flow.

Theorem 6.2 (Bernstein-Shi Derivative Estimates): *Let $(E, G(t))$ be a solution to RYM-flow on a compact principal bundle. For each $\alpha > 0$ and every $m \in \mathbb{N}$ there exists a constant C_m depending only on m, n and $\max\{\alpha, 1\}$ such that if*

$$|\mathrm{Rm}| \leq K \text{ for all } x \in M^n \text{ and } t \in \left[0, \frac{\alpha}{K}\right]$$

then

$$|\nabla^m \mathrm{Rm}| \leq \frac{C_m K}{t^{m/2}} \text{ for all } x \in M^n \text{ and } t \in \left(0, \frac{\alpha}{K}\right]$$

Proposition 6.6 (Long Time Existence Obstruction): *Let G_0 be a bundle metric on $E \to M$ a principal bundle. Then Ricci Yang-Mills flow (and its normalizations) has a unique solution $G(t)$ on a maximal time interval $0 \leq t < T \leq \infty$. Moreover, if $T < \infty$ then*

$$\lim_{t \nearrow T} \left(\sup_{x \in E} |\mathrm{Rm}(x, t)| \right) = \infty$$

Theorem 6.10 (Compactness Theorem): *Let*

$$\{E_i, G_i(t), p_i, F_i : i \in \mathbb{N}\}$$

be a sequence of complete solutions to Ricci Yang-Mills flow existing for $t \in (\alpha, \omega)$ *where* $-\infty \leq \alpha < 0 < \omega \leq \infty$*. Each solution has a fixed origin* $p_i \in M_i$ *and a frame* F_i *at* p_i *which is orthonormal with respect to* $G_i(0)$*. Suppose there exists* $K < \infty$ *such that*

$$\sup_{E_i \times (\alpha,\omega)} |\mathrm{Rm}| \leq K$$

and $\delta > 0$ *such that:*

$$\mathrm{inj}_{G_i(0)}(E_i) \geq \delta \text{ for all } i \in \mathbb{N}$$

Then there exists a subsequence which converges in the pointed category to a complete solution

$$\{E_\infty, G_\infty(t), p_\infty, F_\infty\}$$

to Ricci Yang-Mills flow on (α, ω) *with the same bounds on curvature and injectivity radius.*

This compactness result will be a useful tool in producing models for singular solutions to Ricci Yang-Mills flow. Typically one blows up a forming singularity on the scale of curvature and attempts to derive a limit space using the compactness theorem. One must have the crucial injectivity radius estimate to apply theorem 6.10. For the Ricci flow, Perelman [25] showed a noncollapsing result which provides such an estimate. Using similar techniques we are able to show a noncollapsing estimate for a low energy singularity of Ricci Yang-Mills flow. Low energy singularities are defined in definition 3.9, and the definition for a sequence to be locally collapsing is definition 3.13.

Theorem 3.14 (No Local Collapsing of Low Energy Solutions): *Let* $(M, g(t), A(t))$

be a low energy solution to Ricci Yang-Mills flow on $[0,T)$*,* $T<\infty$*. Then* $(M,g(t),A(t))$ *is not locally collapsing at* T*.*

We also show that the Ricci Yang-Mills flow is the gradient flow for a particular energy functional. This result can be used to show

Proposition 3.7 (Steady Breathers are Gradient Solitons) *If* M *is closed any steady breather on* M *is a steady gradient soliton.*

We also prove a formula satisfied by gradient Ricci Yang-Mills solitons, which are defined in definition 3.15.

Proposition 3.17 (Formula for Gradient Ricci Yang-Mills Solitons): *A Gradient Ricci Yang-Mills solition satisfies*

$$\nabla\left(R-\frac{3}{4}\left|F\right|^2+\left|\nabla f\right|^2+2\lambda f\right)=0$$

Next we have a result which shows an interesting pinching property of Ricci Yang-Mills flow on a $U(1)$-bundle. Let $\eta_{ij}=F_i^kF_{kj}$. As we mentioned above, lemma A.4 shows that the pinching of the eigenvalues η is related to the (anti-)self-duality of F. This next proposition shows that if one has a curvature bound and a bound on ${}^A\nabla F$, which we refer to as the ϵ-low-order estimate and is clarified in chapter 4, and furthermore F is symplectic initially, then one expects exponential decay of $\left|\overset{\circ}{\eta}\right|$, but only up to a certain point depending on the curvature bound. Indeed one does not expect that F will become exactly self-dual in a convergent limit typically because this implies a decoupling of the Einstein and Yang-Mills equations.

Proposition 4.7 (Pinching of η**):** *Suppose* $(g(t),A(t))$ *is a solution to Ricci Yang-Mills flow on a* $U(1)$*-bundle over a four-manifold which exists on* $[0,T)$*. Suppose that* *on this time interval the ϵ-low-order estimate holds and moreover*

$$\min_{M\times[0,T)} |F|^2 \geq \zeta \fint |F|^2$$

There exists universal $C > 0$ (not depending on any of the constants/objects above) so that $\left|\overset{\circ}{\eta}\right|^2 / |F|^4$ is bounded above by the solution to the ODE

$$\frac{d}{dt}\phi(t) = C\left(\epsilon + \frac{\epsilon^2}{\zeta}\right)[F^{\wedge 2}] - \zeta[F^{\wedge 2}]\phi\,(1 - 4\phi)$$

$$\phi(0) = \sup_{(M,g_0,A_0)} \frac{\left|\overset{\circ}{\eta}\right|^2}{|F|^4}$$

where $[F^{\wedge 2}] = \left|\int F \wedge F\right|$.

Next we have our main analytic result, which describes conditions under which we can prove convergence of the volume-normalized Ricci Yang-Mills flow. Roughly the statement says that if one has a principle bundle over M^4 satisfying a particular curvature condition (μ-stability, see definition 7.5)) at each point, and moreover this bundle curvature is very large compared to the base curvature and the tensor ${}^A\nabla F$, then the flow will exist for all time and converge. The curvature of the standard $SU(2)$-instanton on S^4 satisfies the (pointwise) condition of μ-stability. Indeed the round instanton on S^4 gives a global example of a connection satisfying the stability condition. Other global examples include connections arising from Riemannian metrics on manifolds which are half-conformally flat. This result, and how to use it, is described in more depth in chapter 7. In the statement below $\eta_{ij} = {}^\theta F_{ki}\ {}_\theta F_j^k$.

Theorem 7.9 (Main Convergence Result): *Let $E \to (M^4, g)$ be a principal bundle. For fixed $\mu > 0$, $B > 0, \Omega > 0$ there exists a large $N > 0$ depending on μ,*

B, Ω and the base metric g with the following property: if A is a μ-stable connection on E which satisfies

$$\left|\mathring{\eta}\right|_{C^0} + \left|\ {}^g\nabla\mathring{\eta}\right|_{C^0} + \left|\ {}^g\nabla^2\mathring{\eta}\right|_{C^0} \leq \Omega$$

and

$$\frac{1}{B}\max_M |F|^2 \leq |F|^2(x) \leq B\min_M |F|^2 \text{ for all } x \in M$$

and

$$\min_M |F|^2 > N^2$$

and furthermore

$$\left|\ {}^A\nabla F\right|_{C^0} < B\,|F|_{C^0(M_0)}$$

then the volume normalized Ricci Yang-Mills flow with initial condition (g, A) exists for all time and converges to an Einstein-Yang-Mills metric.

Chapter 2

Definition of Ricci Yang-Mills Flow

2.1 Metrics on Principal Bundles

Let $\mathcal{K}$ be a compact Lie group with Lie algebra $\mathfrak{k}$. Let (M^n, g) be a Riemannian manifold, and say E is a principal $\mathcal{K}$-bundle on M with connection $\mathcal{A}$. Fix $U \subset M$ a local coordinate patch. Over U the bundle E is trivial, so we have a local cross-section $s : U \to E$. Let $A := s^*\mathcal{A}$ be the local representation of the connection with respect to this section. Let x^i be coordinate functions on U, and let v_θ denote a basis for the Lie algebra $\mathfrak{k}$. Using this we can write

$$A = A_i dx^i \tag{2.1}$$

where each $A_i = A_i^\theta v_\theta$ is a $\mathfrak{k}$-valued function on M.

Let $\overline{g}$ denote a smooth family of bi-invariant metrics on $\mathcal{K}$ parametrized by the base manifold M. Let v^θ denote a left-invariant coframe on $\mathcal{K}$. Using the local product structure of E over U, both dx^i and v^θ are defined locally on E. Following the usual Kaluza-Klein ansatz, we can write down the following metric

$$G = g_{ij} dx^i dx^j + \overline{g}_{\theta\rho}(v^\theta + A_k^\theta dx^k)(v^\rho + A_l^\rho dx^l) \tag{2.2}$$

It follows from [3] 9.15 that this definition in fact gives a well-defined global metric on E. We will call the metric determined by this data $G = G(g, A, \overline{g})$. Also, we will now go ahead and make an important simplifying assumption. We will assume that the metric $\overline{g}$ is given by the Killing form for $\mathfrak{k}$ over each point. In particular then the functions $\overline{g}_{\theta\rho} v^\theta v^\rho$ are constant.

Definition: 2.1. We call a metric satisfying the above properties a *bundle* metric.

We now introduce some more convenient notation. Let us choose the local basis for one-forms on E given by $\omega^i = dx^i, \omega^\theta = v^\theta + A_i^\theta dx^i$. Then the dual basis for the tangent space is given by e_θ the dual to v^θ taken with respect to the metric $\overline{g}$, and $e_i = \frac{\partial}{\partial x^i} - A_i^\theta e_\theta$. In this basis the metric is:

$$G = \begin{pmatrix} g_{ij} & 0 \\ 0 & \overline{g}_{\theta\rho} \end{pmatrix}$$

Recall the formula for the curvature of A

$$F = dA + A \wedge A \tag{2.3}$$

Define $[e_i, e_j] = C_{ij}^\alpha e_\alpha$. Note the basic computation

$$\begin{aligned} [e_i, e_j] &= \left[\frac{\partial}{\partial x^i} - A_i^\mu e_\mu, \frac{\partial}{\partial x^j} - A_j^\nu e_\nu\right] \\ &= A_{i,j}^\mu e_\mu - A_{j,i}^\nu e_\nu + A_i^\mu A_j^\nu [e_\mu, e_\nu] \\ &= \left(A_{i,j}^\theta - A_{j,i}^\theta + (A \wedge A)_{ij}^\theta\right) e_\theta \\ &= F_{ij}^\theta e_\theta \end{aligned}$$

Recall the formula for the connection tensor of a Riemannian metric G:

$$\Gamma_{ij}^k G_{kl} = \frac{1}{2}\left(e_i(G_{jl}) + e_j(G_{il}) - e_l(G_{ij}) - C_{ilj} - C_{jil} + C_{lji}\right)$$

Using this formula the following lemma is immediate

Lemma: 2.2. *In the above frame a bundle metric satisfies*

$$[e_i, e_j] = {}^{\theta}F_{ij}\, e_\theta, [e_i, e_\theta] = -A_i^\mu\, [e_\mu, e_\theta]$$

$$\Gamma_{ij}^k = {}^{g}\Gamma_{ij}^k, \Gamma_{\theta\rho}^\mu = {}^{\overline{g}}\Gamma_{\theta\rho}^\mu$$

$$\Gamma_{ij}^\theta = \frac{1}{2}{}^{\theta}F_{ij}\,, \Gamma_{i\theta}^k = \frac{1}{2}\ {}_{\theta}F_i^k$$

$$\Gamma_{i\rho}^\theta = \overline{g}^{\theta\mu} A_i^\nu C_{\nu\mu\rho}$$

$$\Gamma_{\rho i}^\theta = \Gamma_{\theta\rho}^k = 0$$

where ${}^{g}\Gamma$ *and* ${}^{\overline{g}}\Gamma$ *are the Christoffel symbols of* g *and* $\overline{g}$ *respectively.*

Note that in the term ${}^{\theta}F_i^k$ above we have raised the *first* index of F, and this will be our convention throughout. Also note that the bundle index in ${}_{\theta}F_i^k$ has been lowered with respect to the metric $\overline{g}$. We now want to compute the curvature of this metric. We will denote the curvature tensor for G by R, the curvature tensor for g by ${}^{g}r$, and the curvature tensor for $\overline{g}$ by ${}^{\overline{g}}r$. In general covariant derivatives will be taken with respect to the Levi-Civita connection associated to G. We use the notation ${}^{g}\nabla$ to refer to the Levi-Civita connection of g, ${}^{\overline{g}}\nabla$ the connection or $\overline{g}$ and ${}^{A}\nabla$ is the connection on $\mathfrak{k}$-valued differential forms.

Lemma: 2.3. *The curvature of a bundle metric G is*

$$R^l_{ijk} = {}^g r^l_{ijk} + \frac{1}{4}({}^\theta F_{jk} \ {}_\theta F^l_i - {}^\theta F_{ik} \ {}_\theta F^l_j - 2 \ {}^\theta F_{ij} \ {}_\theta F^l_k) \tag{2.4}$$

$$R^l_{\theta jk} = -\frac{1}{2} \ {}^A\nabla_j \ {}_\theta F^l_k \tag{2.5}$$

$$R^l_{\theta j\rho} = \frac{1}{4} \ {}_\rho F^m_j \ {}_\theta F^l_m \tag{2.6}$$

$$R^\rho_{ij\theta} = \frac{1}{4}\left({}^\rho F_{im} \ {}_\theta F^m_j - {}^\rho F_{jm} \ {}_\theta F^m_i\right) - \overline{g}^{\rho\mu} F^\nu_{ij} C_{\nu\mu\theta} \tag{2.7}$$

$$R^l_{\theta\rho\mu} = 0 \tag{2.8}$$

$$R^\nu_{\theta\rho\mu} = {}^{\overline{g}} r^\nu_{\theta\rho\mu} \tag{2.9}$$

$$R_{jk} = R^\beta_{\beta jk} = {}^g r_{jk} - \frac{1}{2}({}_\theta F_{ij} \ {}^\theta F^i_k) \tag{2.10}$$

$$R_{j\theta} = \frac{1}{2} d^*{}_\theta F_j \tag{2.11}$$

$$R_{\theta\rho} = {}^{\overline{g}} r_{\theta\rho} + \frac{1}{4}\left\langle {}_\theta F, {}_\rho F\right\rangle_g \tag{2.12}$$

Proof. This is an immediate consequence of lemma 2.2 and the formula for curvature, given in proposition A.7. □

Next we will define the objects which will be stationary for Ricci Yang-Mills flow. First we need some definitions. The operator π will denote projection of a vector field onto its vertical components, and can be written

Definition: 2.4.

$$\pi(\cdot) = G\left(\frac{\partial}{\partial y^\mu}, \cdot\right) \overline{g}^{\mu\nu} \frac{\partial}{\partial y^\nu} \tag{2.13}$$

This projection operator allows us to define the following projections of symmetric two-tensors on E:

Definition: 2.5. Given $W \in S^2T^*E$ define:

$$W^H = W - \pi W - W\pi + \pi W \pi$$

$$W^C = \pi W + W\pi - 2\pi W \pi$$

$$W^V = \pi W \pi$$

where juxtaposition of π means to compose the lowered index with π, i.e. $\pi W(X, Y) = W(\pi(X), Y)$. The idea is that W^H corresponds to the piece of W with two horizontal (base manifold) components, with similar interpretations for the other two projections.

Definition: 2.6. A metric $G(g, A, \overline{g})$ is Einstein-Yang-Mills if

$$\overset{\circ}{\mathrm{Rc}}{}^H = 0$$

$$\mathrm{Rc}^C = 0$$

where $\overset{\circ}{\mathrm{Rc}}{}^H$ refers to the traceless piece of Rc^H. The first condition is a natural analogue of the usual Einstein condition, while the second is the condition that the connection A is Yang-Mills. We deliberately do not require a condition on the vertical component of the Ricci tensor.

2.2 Definition of Evolution Equation

A natural question to ask at this point is: "why not simply study the Ricci flow of a metric of the form given in (2.2)". Recall (see the Main Question of the introduction) that we would like to use the curvature of a principal bundle to produce a canonical (in this case Einstein Yang-Mills) metric on a manifold. More specifically, we would like to construct a flow of metrics and see the extent to which the bundle curvature F can be used to control the behaviour of this flow. Examining the Ricci curvature in lemma 2.3, it is clear that if F is very large compared to the base curvatures ${}^{g}r$ and ${}^{\overline{g}}r$, then the metric G has *no chance* of being Einstein. This is because F contributes a negative-semidefinite tensor to the horizontal component of the Ricci tensor, while it contributes a positive semi-definite tensor in the vertical component. This difference of sign prevents such a metric from being Einstein. Thus attempting to use F to control the Ricci flow of a metric as given in (2.2) seems unlikely since Einstein metrics are the most natural objects for Ricci flow to converge to, and indeed the only *smooth* objects Ricci flow will converge to. This is one reason for introducing the notion of an Einstein Yang-Mills metric (definition 2.6). Now we must consider the question of finding a natural evolution equation which would produce EYM-metrics as critical objects.

If we are going to flow to such a metric, it would be natural to renormalize the volume of the base manifold. A cursory examination of the variation of the Ricci tensor shows that this prevents the evolution of the Ricci tensor from resembling a heat equation, unless the divergence of the pullback of g with respect to the total Levi-Civita connection vanishes. It is not hard to see using lemma 2.2 that this divergence vanishes in the case of a bundle metric. Thus we are led to *fixing* $\overline{g}$ to be the Killing form along the flow, so that it stays constant in space along the flow, allowing the divergence of the pullback of g to vanish along the flow. In the homogeneous examples of the next section we will see another strong reason for wanting to fix $\overline{g}$, that is, to ignore the vertical component of the Ricci flow. In short, allowing $\overline{g}$ to flow will rapidly decrease the effect of F on the flow, a situation we would like to avoid. Thus we are led to defining the following evolution equation.

Definition: 2.7. Given $G(t)$ a family of bundle metrics of the form (2.2), we say that $G(t)$ is a solution to *Ricci Yang-Mills flow* if

$$\dot{G} = -2\left(\mathrm{Rc}^H + \mathrm{Rc}^C\right) \tag{2.14}$$

We will sometimes abbreviate this by calling it a solution to *RYM flow*.

The next lemma shows our justification for calling this equation Ricci Yang-Mills flow. In particular we see that the equation induces evolution equations on g and A whose leading order terms are the same as the Ricci flow and Yang-Mills flow respectively.

Lemma: 2.8. *The Ricci Yang-Mills flow of a bundle metric is equivalent to*

$$\begin{aligned}
\dot{g}_{jk} &= -2r_{jk} + \eta_{jk} \\
\dot{A}_j^\theta &= -d^* F_j^\theta \\
\dot{\overline{g}}_{\theta\rho} &= 0
\end{aligned} \tag{2.15}$$

where

$$\eta_{jk} = {}_\theta F_j^l \,{}^\theta F_{lk} \tag{2.16}$$

Proof. Let us differentiate expression 2.2 in time.

$$\dot{G}_{\alpha\beta} dx^\alpha dx^\beta = \frac{d}{dt}(g_{ij}) dx^i dx^j + \overline{g}_{\theta\rho} \frac{d}{dt}(A_j^\rho) dx^j \hat{\otimes} dz^\theta + \frac{d}{dt}(\overline{g}_{\theta\rho}) dz^\theta dz^\rho$$

Now the individual evolution equations follows from (2.14) and lemma 2.3. $\square$

We will also study the following volume-normalized version of Ricci Yang-Mills flow.

Definition: 2.9. Given $G(t)$ a family of bundle metrics, we say that $G(t)$ is a solution to *volume renormalized Ricci Yang-Mills flow* if

$$\dot{G} = -2\left(\mathrm{Rc}^H + \mathrm{Rc}^C\right) + \frac{2}{n}\left(\fint r - \frac{n}{4(n-2)}\left|F\right|^2 - \fint \frac{4-n}{4(n-2)}\left|F\right|^2\right) G^H \qquad (2.17)$$

Let us make three observations about this definition. First, we have added a term of the form $|F|^2 G^H$ to the evolution of G. Recall that for Ricci flow one cannot simply add a conformal term $\frac{2}{n} r g$ to fix the volume, as the resulting equation is not parabolic. In our case the term $|F|^2$ is zeroth-order in the metric and so this will pose no issues as far as parabolicity is concerned. Secondly, we note that we did not simply add the appropriate multiple of $|F|^2$ to completely remove the trace of η in the definition of RYM flow. This constant was chosen carefully so that a particular Bianchi-type identity will hold for the tensor on the right hand side of (2.17). Finally, we note that in dimension 4 only, we *have* removed the trace of η completely.

2.3 A Homogeneous Example

Next we will work through a specific homogeneous example which highlights the qualitative behaviour differences between Ricci flow and Ricci Yang-Mills flow. Also we will see the qualitative differences between the Ricci flow of the natural metric on a principal bundle and the Ricci Yang-Mills flow.

Example: 2.10. Let N be a surface of genus $g \geq 2$ and let g^N_{can} denote a metric of constant curvature -1. Now let $M^4 = S^2 \times N$ equipped with the Riemmanian metric $g = g^{S^2}_{can} \oplus g^N_{can}$. More generally, for real numbers A and B, define ${}^{AB}g = A g^{S^2}_{can} \oplus B g^N_{can}$. Let US^2, UN denote the unit tangent bundles of S^2 and N respectively. These circle bundles have curvature forms equal to the usual volume forms (denoted $\omega_{can}^{S^2}$ and ω_{can}^{N}) on their respective base manifolds. Let $E = \pi_1^* US^2 \otimes \pi_2^* UN$. The curvature of this bundle is then $\pi_1^* \omega_{can}^{S^2} \oplus \pi_2^* \omega_{can}^{N}$. We will examine the Ricci flow of the base metric and the Ricci Yang-Mills flow of the associated line bundle.

First of all, let us note that as a product of surfaces M has signature zero [24]. Also, the Euler characteristic of this manifold is easily computed as the product of the Euler characteristics which is $4 - 4g$. Thus we easily see that M cannot admit an Einstein metric by the Hitchin-Thorpe inequality [19], [29] which states that if a four-manifold M admits an Einstein metric then

$$2\chi + 3\left|\tau\right| \geq 0 \tag{2.18}$$

As we will see, this manifold does in fact admit an EYM metric, and that for any initial value for A and B the volume-normalized RYM flow converges to this metric. First of all, let us see how unnormalized Ricci flow acts on the base metric. An easy computation shows that at any point of M we may write

$$^{AB}\operatorname{Rc} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad \frac{R}{2}\ ^{AB}g = \begin{pmatrix} \frac{(B-A)}{B} & \\ & \frac{(B-A)}{A} \end{pmatrix}$$

and so the equation $\dot{g} = -2\operatorname{rc}$ reduces to the system of ODEs:

$$\dot{A} = -2$$

$$\dot{B} = 2$$

with solution $A(t) = A_0 - 2t, B(t) = B_0 + 2t$, which exists on the time interval $[0, \frac{A_0}{2})$. It is easy to see that even under the volume normalized equation the sphere contracts to a point and the higher genus surface blows up, and no natural convergence occurs.

Indeed in surgery arguments on four-manifolds [6], [17] singularities modeled on this space must be explicitly avoided.

Now let us examine Ricci Yang-Mills flow on E. The metric is totally homogeneous in space and $d^* F = 0$, so in particular the connection on E will not vary. An easy computation shows

$$F_{ij}F^i_k = \begin{pmatrix} \frac{1}{A} & & & \\ & \frac{1}{A} & & \\ & & \frac{1}{B} & \\ & & & \frac{1}{B} \end{pmatrix}, \qquad \left|F\right|^2 = \frac{2(A^2+B^2)}{A^2B^2}$$

Thus Ricci Yang-Mills flow corresponds to the system of ODEs

$$\dot{A} = -2 + \frac{1}{A}$$

$$\dot{B} = 2 + \frac{1}{B}$$

It is easy to see that this flow will exist for all time for any initial choice of A and B. Let us now examine the volume-renormalized flow. This amounts to adding the term $\left(\frac{1}{2}R - \frac{1}{4}\left|F\right|^2\right) g$ to the evolution of g. Then we have $AB = 1$ for all time, thus we are reduced to the single ODE

$$\dot{A} = \ -1 + \frac{1}{2A} - A^2 - \frac{1}{2}A^3$$

This clearly converges for any initial choice of A. It is important to note here that the condition of F being symplectic is very important in this example, and certainly no convergence if one chooses F to be just the volume form of one of the factors, as then it will not influence the other factor.

Finally, let us point out an important aspect of the *Ricci flow* of the natural metric on the bundle E. Specifically, this is the Ricci flow of the metric

$$G = \left(A g_{can}^{S^2} \oplus B g_{can}^{N}\right) + C\left(dy + A_i dx^i\right)^2 \tag{2.19}$$

An easy computation using lemma 2.3 shows that the Ricci flow of this metric corresponds to the system of ODE

$$\dot{A} = -2 + \frac{C}{A} \tag{2.20}$$

$$\dot{B} = 2 + \frac{C}{B} \tag{2.21}$$

$$\dot{C} = -\frac{A^2 + B^2}{A^2 B^2} C \tag{2.22}$$

In particular we see that allowing the vertical metric (in this case represented by the constant C) to vary causes the size of F to be rapidly decreased, and so the qualitative behaviour of this flow in the limit is essentially the same as that of the Ricci flow of the base metric, giving no natural smooth limit space.

Chapter 3

Gradient Properties of Ricci Yang-Mills Flow

In the study of geometric evolution equations monotonic quantities have always played an important role. Often the system is given as the gradient flow of an energy functional (see for instance the seminal paper of Eells and Sampson [9]) in which case the monotone energy bound plays an important role in understanding the long-time behaviour of solutions. Also, it was the remarkable monotonicity of the Hawking mass [12] shown by Geroch that inspired Huisken and Ilmanen to construct weak solutions to inverse mean curvature flow which they used to prove the Riemannian Penrose Inequality for connected horizons [20]. In another striking example Bray constructed a conformal flow of metrics which proved the full Riemannian Penrose Inequality [5], and monotonicity formulas play a crucial role in the proof.

Monotonic quantities for solutions to Ricci flow have only recently been found. As was done for Ricci flow [32] we have partially motivated the introduction of the Ricci Yang-Mills flow using an energy functional, but as for Ricci flow the resulting equations are not a-priori gradient equations. In his breakthrough paper Perelman [25] found that Ricci flow was indeed a gradient flow by explicitly introducing the action of the diffeomorphism group on the equations. He furthermore found an entropy functional associated to this energy and a very important related quantity called reduced volume, both of which are monotonic along solutions to Ricci flow. Here we follow those ideas and define an energy functional which explicitly includes the action of the diffeomorphism group on the Ricci Yang-Mills flow equations. This is used to show that steady breather solutions are steady gradient solitons. Next we define an entropy functional which is monotonic if one is approaching a certain kind of singularity. This entropy can be used to prove a non-collapsing result for type I singularities. Ruling out collapsing-type behaviour is essential in Perelman's analysis of singularities of the Ricci flow on three-manifolds [27]. Though our result is not yet strong enough for deeper applications (say surgery arguments) it is an important first step in understanding singular behaviour of solutions to Ricci Yang-Mills flow. We will close this chapter by making a few remarks about Ricci Yang-Mills solitons, which are critical points for the energy functional.

3.1 Energy Functional

Let (M, g) be a Riemannian manifold and let $E \to M$ denote a principal $\mathcal{K}$-bundle over M with connection A. In this chapter ∇ will always refer to the Levi-Civita connection of g. Consider the functional

$$\mathcal{F}(g, A, f) = \int_M \left(R - \frac{1}{4} |F|^2 + |\nabla f|^2 \right) e^{-f} dV \tag{3.1}$$

where R is the scalar curvature of the base metric, $f \in C^\infty(M)$ and dV denotes the volume form of g. In what follows every object, say g for instance, will implicitly be part of a one-parameter family $g(s)$. We define

$$\delta g = \frac{d}{ds} g(s)_{|s=0}$$

and similarly use the notation δ to refer to the first variation at 0 of other quantities with respect to the parameter s.

Lemma: 3.1. *Let $\delta g_{ij} = v_{ij}, \delta A_i = \alpha_i, \delta f = h$. Then*

$$\begin{aligned}\delta\mathcal{F}(v,\alpha,h) = &\int_M e^{-f}\big[-v_{ij}\left(\mathrm{rc}_{ij} - \frac{1}{2}\eta_{ij} + \nabla_i\nabla_j f\right) - \alpha_j\left(d^*F_j - \nabla^i f F_{ij}\right) \\ &+ (v/2 - h)\left(2\Delta f - |\nabla f|^2 + R - \frac{1}{4}|F|^2\right)\big]dV\end{aligned} \tag{3.2}$$

where $v = g^{ij}v_{ij}$.

Proof. We first note that $\delta F = d\delta A = d\alpha$. Now we directly compute

$$\begin{aligned}\delta\mathcal{F}(v,\alpha,h) = &\int_M e^{-f}\big[-\Delta v + \nabla_i\nabla_j v_{ij} - \mathrm{rc}_{ij}\, v_{ij} \\ &+\frac{1}{2}\eta_{ij}v_{ij} - \frac{1}{2}\langle d\alpha, F\rangle - v_{ij}\nabla_i f\nabla_j f + 2\langle\nabla f, \nabla h\rangle \\ &+\left(R - \frac{1}{4}|F|^2 + |\nabla f|^2\right)(v/2 - h)\big]dV \\ = &\int_M e^{-f}\big[-v_{ij}\left(\mathrm{rc}_{ij} - \frac{1}{2}\eta_{ij} + \nabla_i\nabla_j f\right) \\ &-\alpha_j\left(d^*F_j - \nabla^i f F_{ij}\right) \\ &+ (v/2 - h)\left(2\Delta f - |\nabla f|^2 + R - \frac{1}{4}|F|^2\right)\big]dV\end{aligned} \tag{3.3}$$

where we performed the integration by parts

$$
\begin{aligned}
-\frac{1}{2}\int \langle d\alpha, F\rangle e^{-f} dV = & -\frac{1}{2}\int \left\langle \nabla_j\alpha_i - \nabla_i\alpha_j, e^{-f}F_{ij}\right\rangle dV \\
= & \frac{1}{2}\int \left(\left\langle \alpha_i, \nabla_j F_{ij} - \nabla^j f F_{ij}\right\rangle - \left\langle \alpha_j, \nabla_i F_{ij} - \nabla^i f F_{ij}\right\rangle\right) e^{-f} dV \\
= & -\int \alpha_j \left(d^* F_j - \nabla^i f F_{ij}\right) e^{-f} dV
\end{aligned}
$$

$\square$

It is clear from the above lemma that the gradient flow equations of $\mathcal{F}$ are

$$
\begin{aligned}
\frac{d}{dt} g_{ij} = & -2\operatorname{rc}_{ij} + \eta_{ij} - 2\nabla_i\nabla_j f \\
\frac{d}{dt} A_i = & -d^* F + \nabla^i f F_{ij} \\
\frac{d}{dt} f = & -\Delta f - R + \frac{1}{2}|F|^2
\end{aligned}
\tag{3.4}
$$

Note that for this evolution system we have $v/2 - h \equiv 0$. What this means geometrically is that the measure $e^{-f}dV$ remains fixed along a solution to (3.4). The following corollary is immediate.

Corollary: 3.2. *Given $(g(t), A(t), f(t))$ a solution to (3.4) the functional $\mathcal{F}$ is monotonically increasing in t. In particular*

$$
\frac{d}{dt}\mathcal{F} = \int_M \left(2\left|\operatorname{rc} - \frac{1}{2}\eta_{ij} + \nabla_i\nabla_j f\right|^2 + \left|d^*F - \nabla^i f F_{ij}\right|^2\right) e^{-f} dV \geq 0 \tag{3.5}
$$

We now describe how equation (3.4) does in fact correspond to Ricci Yang-Mills flow modified by an appropriate diffeomorphism. Recall that given g and A we can define a metric on E based on the Kaluza-Klein ansatz in (2.2). Also recall that locally we have the frame $e_i = \frac{\partial}{\partial x^i} - A_i \frac{\partial}{\partial y}$ and $e_\theta = \frac{\partial}{\partial y}$. Let $\overline{\nabla}$ denote the Levi-Civita connection of $G(g, A)$.

Lemma: 3.3. *Let G be a bundle metric and fix a function $f \in C^\infty(M)$ and let $\overline{f} = \pi^*(f) \in C^\infty(E)$. Let $\overline{\nabla}$ denote the Levi-Civita connection of G. Then*

$$\begin{aligned} \overline{\nabla}_i \overline{\nabla}_j \overline{f} &= \nabla_i \nabla_j f \\ \overline{\nabla}_i \overline{\nabla}_\theta \overline{f} &= \overline{\nabla}_\theta \overline{\nabla}_i \overline{f} = -\frac{1}{2} \nabla^k f \;{}_\theta F_{ki} \\ \overline{\nabla}_\theta \overline{\nabla}_\mu \overline{f} &= 0 \end{aligned} \tag{3.6}$$

Now if ϕ_t is the one-parameter family of diffeomorphisms of M generated by the time-dependent vector field $W(t)$, i.e.

$$\frac{d}{dt}\phi_t(p) = W(\phi_t(p), t)$$

$$\phi_0 = \mathrm{Id}_M$$

Then a straightforward calculation shows that

$$\frac{d}{dt}\left(\phi_t^* g(t)\right) = \phi_t^* \left(\frac{d}{dt} g(t)\right) + \phi_t^* \left(L_{W(t)} g(t)\right) \tag{3.7}$$

Using lemma 3.3 and the formula $\mathcal{L}_{\nabla f} G_{\alpha\beta} = \overline{\nabla}_\alpha \overline{\nabla}_\beta f + \overline{\nabla}_\beta \overline{\nabla}_\alpha f$ we see that (3.4) is diffeomorphism equivalent to

$$\begin{aligned} \frac{d}{dt} g_{ij} &= -2\,\mathrm{rc}_{ij} + \eta_{ij} \\ \frac{d}{dt} A_i &= -d^* F \\ \frac{d}{dt} f &= -\Delta f + |\nabla f|^2 - R + \frac{1}{2}|F|^2 \end{aligned} \tag{3.8}$$

in the sense that we may pull back $G(t)$ by the family of diffeomorphisms generated by ∇f to remove the terms $\nabla^2 f$ and $\nabla f * F$ from the evolutions of g and A respectively.

Thus we see that we have explicitly introduced a diffeomorphism parameter by means of the function f. However, we would like to remove it so as to get a monotonic quantity which depends only on the given metric and connection. Following the ideas of Perelman [25] we consider the quantity

$$\lambda(g, A) := \inf_{\int_M e^{-f} dV = 1} \mathcal{F}(g, A, f) \tag{3.9}$$

An easy calculation shows that λ is the lowest eigenvalue of the Schrödinger operator

$$-4\Delta + R - \frac{1}{4} |F|^2 \tag{3.10}$$

Proposition: 3.4. *The quantity $\lambda(g, A)$ is monotonically increasing along a solution to Ricci Yang-Mills flow.*

Proof. Let $u(t)$ be a family eigenvectors for lowest eigenvector of the operators

$$-4\Delta + R - \frac{1}{4} |F|^2 \tag{3.11}$$

Normalize u so that

$$\int_M u(t)^2 dV(t) \equiv 1 \tag{3.12}$$

These functions u are in fact positive, so that we may define the family of functions

$$e^{-f(t)/2} = u(t) \tag{3.13}$$

The eigenvalue equation for u is equivalent to the equation

$$2\Delta f - |\nabla f|^2 + R - \frac{1}{4} |F|^2 = \lambda \tag{3.14}$$

It is clear that for each time t, $f(t)$ minimizes the functional $\mathcal{F}$. Using (3.14) we note that the last term in (3.2) is

$$\begin{aligned}
&\int_M (v/2-h)\left(2\Delta f - |\nabla f|^2 + R - \frac{1}{4}|F|^2\right)e^{-f}dV \\
&= \lambda \int_M (v/2-h)\, e^{-f} dV \\
&= \lambda \frac{d}{dt}\int_M e^{-f} dV \\
&= 0
\end{aligned}$$

where the last line follows using (3.12). Using this calculation and our observations about diffeomorphism invariance above, it is clear that

$$\frac{d\lambda}{dt} = \int_M \left(2\left|\mathrm{rc} - \frac{1}{2}\eta_{ij} + \nabla_i\nabla_j f\right|^2 + \left|d^*F - \nabla^i f F_{ij}\right|^2\right)e^{-f}dV \geq 0 \tag{3.15}$$

□

This proposition can be used to rule out certain kinds of behaviour which can be problematic in studying singularities. In particular we can show that steady breathers must be steady gradient solitons.

Definition: 3.5. A solution $(g(t), A(t))$ to Ricci Yang-Mills flow is called a *breather* if there exists $t_1 < t_2$ and $\alpha > 0$ so that $\alpha G(t_1)$ and $G(t_2)$ differ only by a diffeomorphism, where $G(t)$ is the associated global metric on E defined in definition 2.2. In the cases $\alpha = 1, \alpha < 1, \alpha > 1$ we say that the breather is *steady, shrinking*, or *expanding*, respectively.

Definition: 3.6. We say that (g, A) is a *gradient Ricci-Yang-Mills soliton* if there exists $f \in C^\infty(M)$ so that

$$\mathrm{rc} - \frac{1}{2}\eta + \nabla^2 f + \lambda g = 0 \tag{3.16}$$

$$d^* F = \nabla f \neg F \tag{3.17}$$

the soliton is *steady, shrinking*, or *expanding* if $\lambda = 0, \lambda < 0, \lambda > 0$ respectively.

Proposition: 3.7. *If M is closed any steady breather on M is a steady gradient soliton.*

Proof. We note by (3.15) that λ is monotonically increasing, and fixed if and only if we are on a steady gradient soliton. The proposition follows. □

3.2 Entropy Functional

Perelman made great strides in understanding finite-time singularities of the Ricci flow by introducing an entropy functional. We would like to understand the behaviour of finite-time singularities of Ricci Yang-Mills flow. Suppose that $(g(t), A(t))$ is a solution to Ricci Yang-Mills flow which exists on a maximal time interval of the form $[0, T)$ where $T < \infty$. Below we introduce a new entropy functional designed to understand the behaviour of g and A as $t \nearrow T$. This functional is not always monotonic along Ricci Yang-Mills flow, but as we will show it is monotonic for certain singularities. Though not essential for the calculations, in what follows below the parameter τ should be thought of as "backwards time," specifically $\tau = T - t$. Let

$$\mathcal{W}(g, A, f, \tau) = \int_M \left[\tau \left(|\nabla f|^2 + R + \frac{1}{4} |F|^2 \right) + f - n \right] (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \tag{3.18}$$

We hasten to point out that the sign of the term $|F|^2$ has changed from the definition of the energy functional $\mathcal{F}$. Although the reason for doing this will be borne out in the calculations below, we can give a nice explanation for the difficulties which arise. Recall that the Ricci Yang-Mills flow can be though of as the Ricci flow of a bundle metric which keeps the metric on the fiber fixed. Thus if one simply looks at the entropy given by the scalar curvature of the corresponding bundle metric (which would have the opposite sign on the term $|F|^2$), which would be the Perelman entropy, an extra term arises in the evolution corresponding to the part of the volume form which has been fixed. Unfortunately this term has the wrong (negative) sign to get a monotonic quantity. We now proceed with the calculations.

Lemma: 3.8. *Let $v_{ij} = \delta g_{ij}, \delta A_i = \alpha_i, \delta f = h$ and $\delta\tau = \sigma$. Then*

$$
\begin{aligned}
\delta\mathcal{W}(v,\alpha,h,\sigma) = \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \Bigg[\sigma\left(|\nabla f|^2 + R + \frac{1}{4}|F|^2 \right) \\
-\tau v_{ij}\left(\mathrm{rc}_{ij} + \frac{1}{2}\eta_{ij} + \nabla_i\nabla_j f \right) - \tau\alpha_j\left(d^*F_j - \nabla^i f F_{ij} \right) + h \\
+ \left[\tau\left(2\Delta f - |\nabla f|^2 + R - \frac{1}{4}|F|^2 \right) + f - n \right] \left(\frac{v}{2} - h - \frac{n\sigma}{2\tau} \right) \Bigg]
\end{aligned}
\tag{3.19}
$$

Proof. First of all we note the variational formula

$$
\delta\left((4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \right) = \left(\frac{v}{2} - h - \frac{n\sigma}{2\tau} \right) (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV
$$

Now using (3.3) we see

$$\delta\mathcal{W} = \int_M \left[\sigma\left(|\nabla f|^2 + R + \frac{1}{4}|F|^2\right) + \tau\left(\frac{v}{2} - h\right)\left(2\Delta f - 2|\nabla f|^2\right)\right.$$

$$-\tau v_{ij}\left(\mathrm{rc}_{ij} + \frac{1}{2}\eta_{ij} + \nabla_i\nabla_j f\right) - \tau\alpha_j\left(d^*F_j - \nabla^i f F_{ij}\right) + h$$

$$\left. + \left[\tau\left(|\nabla f|^2 + R + \frac{1}{4}|F|^2\right) + f - n\right]\left(\frac{v}{2} - h - \frac{n\sigma}{2\tau}\right)\right]$$

$$\cdot (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV$$

Now since

$$\int \left(\Delta f - |\nabla f|^2\right) e^{-f} dV = 0$$

we conclude

$$\delta\mathcal{W} = \int_M \left[\sigma\left(|\nabla f|^2 + R + \frac{1}{4}|F|^2\right) - \tau v_{ij}\left(\mathrm{rc}_{ij} + \frac{1}{2}\eta_{ij} + \nabla_i\nabla_j f\right)\right.$$

$$-\tau\alpha_j\left(d^*F_j - \nabla^i f F_{ij}\right) + h$$

$$\left. + \left[\tau\left(2\Delta f - |\nabla f|^2 + R - \frac{1}{4}|F|^2\right) + f - n\right]\left(\frac{v}{2} - h - \frac{n\sigma}{2\tau}\right)\right]$$

$$\cdot (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV$$

as required. □

Consider the following system of equations.

$$\begin{aligned}
\frac{d}{dt}g_{ij} &= -2\left(\mathrm{rc}_{ij} - \frac{1}{2}\eta_{ij} + \nabla_i\nabla_j f\right) \\
\frac{d}{dt}A_i &= -d^*F + \nabla^i f F_{ij} \\
\frac{d}{dt}f &= -\Delta f - R + \frac{1}{2}|F|^2 + \frac{n}{2\tau} \\
\frac{d}{dt}\tau &= -1
\end{aligned} \tag{3.20}$$

Note that for this system we have the equation

$$\frac{v}{2} - h - \frac{n\sigma}{2\tau} = 0 \tag{3.21}$$

which is to say the measure $(4\pi\tau)^{-\frac{n}{2}} e^{-f} dV$ is fixed. Using (3.19) we compute that for a solution to (3.20) we have

$$\begin{aligned}
\frac{d\mathcal{W}}{dt} = & \int_M \Bigg[\left(-\frac{1}{4}|F|^2 - |\nabla f|^2 - R\right) + 2\tau \left|\mathrm{rc}_{ij} + \nabla_i \nabla_j f\right|^2 - \frac{1}{2}\tau |\eta|^2 \\
& + \tau \left|d^* F_j - \nabla^i f F_{ij}\right|^2 - \Delta f - R + \frac{1}{2}|F|^2 + \frac{n}{2\tau}\Bigg] (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \\
= & \int_M \Bigg[-2(R + \Delta f) + \frac{n}{2\tau} + 2\tau \left|\mathrm{rc}_{ij} + \nabla_i \nabla_j f\right|^2 \\
& + \tau \left|d^* F_j - \nabla^i f F_{ij}\right|^2 + \frac{1}{4}|F|^2 - \frac{1}{2}\tau |\eta|^2\Bigg] (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \\
= & \int_M \left[2\tau \left|\mathrm{rc}_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}\right|^2 + \tau \left|d^* F_j - \nabla^i f F_{ij}\right|^2 + \frac{1}{4}|F|^2 - \frac{1}{2}\tau |\eta|^2\right] \\
& \cdot (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV
\end{aligned} \tag{3.22}$$

As above, this system is diffeomorphism equivalent to

$$\begin{aligned}
\frac{d}{dt} g_{ij} = & -2\,\mathrm{rc}_{ij} + \eta_{ij} \\
\frac{d}{dt} A_i = & -d^* F \\
\frac{d}{dt} f = & -\Delta f + |\nabla f|^2 - R + \frac{1}{2}|F|^2 + \frac{n}{2\tau} \\
\frac{d}{dt} \tau = & -1
\end{aligned} \tag{3.23}$$

Thus we see that $\mathcal{W}$ is monotonically increasing under Ricci Yang-Mills flow whenever $\frac{1}{4}|F|^2 - \frac{1}{2}\tau |\eta|^2 > 0$. In fact only the weaker integral condition is needed. This may appear to be a strong condition, but note that the two terms in fact have the same parabolic scaling factor.

3.3 Noncollapsing of Low-Energy Singularities

Definition: 3.9. Let $(M, g(t), A(t))$ be a solution to RYM-flow which exists on a maximal time interval $T < \infty$. $(M, g(t), A(t))$ is a *low-energy* solution if

$$\lim_{t \to T} (T-t) \left| F \right|^2_{C^0(M_t)} = 0 \tag{3.24}$$

We point out that the only known examples of finite-time singularities to Ricci Yang-Mills flow are all low-energy. Specifically, if one modifies example 2.10 to the case where F is the pullback of the volume form on the higher genus surface, then the singularity encountered is low-energy. This hypothesis codifies the sense in which we don't expect singularities in areas where F is nonvanishing.

Corollary: 3.10. *Let $(M, g(t), A(t))$ be a low energy solution to RYM flow on $[0, T)$. Then there exists $t_0 < T$ such that for all $t_0 \le t < T$, we have*

$$\frac{d\mathcal{W}}{dt} \ge 0 \tag{3.25}$$

Proof. By assumption equation (3.24) holds, so choose t_0 so that for all $t_0 \le t < T$ we have $(T-t) \left| F \right|^2_{C^0(M_t)} \le 1$. Then we have using lemma A.5

$$\begin{aligned}
\frac{1}{4} \left| F \right|^2 - \frac{1}{2}(T-t) \left| \eta \right|^2 &\ge \frac{1}{4} \left| F \right|^2 - \frac{1}{4}(T-t) \left| F \right|^4 \\
&\ge \frac{1}{4} \left| F \right|^2 \left(1 - (T-t) \left| F \right|^2 \right) \\
&\ge \frac{1}{4} \left| F \right|^2 \left(1 - (T-t) \left| F \right|^2_{C^0(M_t)} \right) \\
&\ge 0
\end{aligned}$$

Using this in (3.22) the result clearly follows. □

Definition: 3.11. Given (M, g, A) at $\tau \in \mathbb{R}$ let

$$\mu(g, A, \tau) := \inf_f \{\mathcal{W}(g, A, f, \tau) | \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1\} \tag{3.26}$$

Corollary: 3.12. *Let $(M, g(t), A(t))$ be a low-energy solution to RYM flow on $[0, T)$. Then there exists $t_0 < T$ such that for all $t_0 \leq t < T$ we have*

$$\frac{d\mu}{dt} \geq 0 \tag{3.27}$$

Proof. The proof is identical to that of proposition 3.4, again exploiting the diffeomorphism invariance of the functional $\mathcal{W}$. □

Definition: 3.13. A solution to RYM flow defined on $[0, T)$ is said to be *locally collapsing at T* if there exists a sequence of times $t_k \to T$ and a sequence of metric balls $B_k := B_{g(t_k)}(p_k, r_k)$ such that $\frac{r_k^2}{t_k}$ is bounded, $|\mathrm{rm}|_{C^0(B_k)} \leq r_k^{-2}$ and $\lim_{k\to\infty} r_k^{-n} \operatorname{vol}(B_k) = 0$.

Theorem: 3.14. *Let $(M, g(t), A(t))$ be a low energy solution to RYM flow on $[0, T)$, $T < \infty$. Then $(M, g(t), A(t))$ is not locally collapsing at T.*

Proof. Using the monotonicity of μ the proof is essentially identical to the proof of noncollapsing for finite time singularities of Ricci flow. We follow the proof in [22]. If the solution were locally collapsing, we can find functions f_k localized around the singularity so that $\mathcal{W}(g(t_k), A(t_k), f_k, r_k^2) \to -\infty$ as $k \to \infty$. Thus $\mu(g(t_k), r_k^2) \to -\infty$. By the (eventual) monotonicity of μ, it then follows that $\mu(g(0), A(0), t_k + r_k^2) \to -\infty$ as $k \to \infty$. Since r_k^2 and t_k are both bounded, this contradicts continuity of $\mu(g(0), A(0), \tau)$ as a function of τ.

So, given a low energy solution, by translating time assume without loss of generality by corollary 3.12 that μ is monotonically increasing on $[0,T)$. Suppose B_k is a sequence of locally collapsing balls as in definition 3.13. Let us change variables in $\mathcal{W}$ and set $\Phi = e^{-\frac{f}{2}}$. This gives

$$\mathcal{W}(g,A,f,\tau) = (4\pi\tau)^{-\frac{n}{2}} \int_M \left[4\tau \left|\nabla\Phi\right|^2 + \left(\tau R + \frac{1}{4}\tau \left|F\right|^2 - 2\ln\Phi - n\right)\Phi^2\right] dV \tag{3.28}$$

Set

$$\Phi_k(x) = e^{-\frac{c_k}{2}}\phi(\mathrm{dist}_{g(t_k)}(x,p_k)/r_k) \tag{3.29}$$

where $\phi : [0,\infty) \to [0,1]$ is a monotonically nonincreasing functions such that

$$\begin{cases} \phi(s) = 1 & 0 \le s \le \frac{1}{2} \\ \phi(s) = 0 & s \ge 1 \\ |\phi'(s)| \le 10 & \frac{1}{2} \le s \le 1 \end{cases}$$

The constant c_k in (3.29) is determined by the condition in (3.26), so

$$e^{c_k} = \int_M (4\pi r_k^2)^{-\frac{n}{2}}\phi^2(\mathrm{dist}_{g(t_k)}(x,p_k)/r_k)dV \le (4\pi r_k^2)^{-\frac{n}{2}}\,\mathrm{vol}(B_k) \tag{3.30}$$

Thus by the assumption of local collapsing, $c_k \to -\infty$. By assumption we have the bound

$$|F|^2_{C^0(r_k^2)} \le \frac{C}{r_k^2} \tag{3.31}$$

along a low energy solution. Let $A_k(s)$ denote the area of the distance sphere $S(p_k, r_k s)$ around p_k. For a given function v let

$$\overline{v}_k(s) = r_k^2 A_k(s)^{-1}\int_{S(p_k,r_ks)} v\,d\Sigma \tag{3.32}$$

It is clear by the assumed bound on curvature and (3.31) that both $\overline{R}_k(s)$ and $\overline{|F|^2}_k(s)$ are bounded. Thus we can compute the integral in (3.28) radially to conclude

$$\begin{aligned}
\mathcal{W}(g(t_k), A(t_k), f_k, r_k^2) &= \frac{\int_0^1 \left[4\phi'(s)^2 + \left(\overline{R}_k(s) + \frac{1}{4}\overline{|F|^2}_k(s) + c_k - 2\ln\phi(s) - n\right)\phi^2(s)\right] A_k(s)ds}{\int_0^1 \phi^2(s)A_k(s)ds} \\
&\leq C + c_k + \frac{\int_0^1 \left[4\phi'(s)^2 - 2\ln\phi(s)\phi^2(s)\right] A_k(s)ds}{\int_0^1 \phi^2(s)A_k(s)ds} \\
&\leq C + c_k + C\frac{\operatorname{vol}(B(p_k, r_k)) - \operatorname{vol}(B(p_k, r_k/2))}{\operatorname{vol}(B(p_k, r_k/2))} \\
&\leq C + c_k
\end{aligned} \tag{3.33}$$

where the last line follows from the bound on curvature and the Bishop-Gromov inequality. So we have shown that $\mathcal{W}(g(t_k), A(t_k), f_k, r_k^2) \leq C + c_k$ so that

$$\mu(g(t_k), A(t_k), f_k, r_k^2) \to -\infty$$

as required. □

3.4 Gradient Ricci-Yang-Mills Solitons

In this section we will prove two basic formulas satisfied by gradient Ricci-Yang-Mills solitons, which as noted above are the stationary (up to diffeomorphism equivalence) solutions to Ricci Yang-Mills flow.

Definition: 3.15. Given (M, g) a Riemannian manifold and $E \to M$ a principal $\mathcal{K}$-bundle over M with connection A, we say that (g, A) is a gradient Ricci-Yang-Mills soliton if there exists $f \in C^\infty(M)$ and $\lambda \in \mathbb{R}$ so that

$$\operatorname{rc} - \frac{1}{2}\eta + \nabla^2 f + \lambda g = 0 \tag{3.34}$$

$$d^*F = \nabla^i f F_{ij} \tag{3.35}$$

We call the solition *expanding*, *stationary*, or *shrinking* if $\lambda > 0, \lambda = 0$ or $\lambda < 0$ respectively.

Lemma: 3.16. *A gradient RYM-soliton satisfies*

$$R - \frac{1}{2}|F|^2 + \Delta f + n\lambda = 0 \tag{3.36}$$

$$\langle d^* F, \nabla f \rangle = 0 \tag{3.37}$$

Proof. The first equation is simply the trace of (3.34). To get the second equation simply take the inner of product of (3.35) with ∇f. The second equation also follows by taking the divergence of (3.35). □

Proposition: 3.17. *A gradient RYM-solition satisfies*

$$\nabla \left(R - \frac{3}{4}|F|^2 + |\nabla f|^2 + 2\lambda f \right) = 0 \tag{3.38}$$

Proof. Taking the divergence of equation (3.34), applying the Bianchi identity and commuting derivatives gives

$$\begin{aligned} 0 &= \nabla_j \left(\mathrm{rc}_{ij} - \frac{1}{2}\eta_{ij} + \nabla_i \nabla_j f + \lambda g_{ij} \right) \\ &= \frac{1}{2}\nabla_i R - \frac{1}{2}(\operatorname{div} \eta)_i + \nabla_i \Delta f - R_{jkik} \nabla_k f \\ &= \frac{1}{2}\nabla_i R - \frac{1}{2}(\operatorname{div} \eta)_i + \nabla_i \Delta f + \mathrm{rc}_{ik} \nabla_k f \end{aligned} \tag{3.39}$$

Now from (5.25) we have

$$\operatorname{div} \eta_i = \frac{1}{4}\nabla_i |F|^2 - (d^*_\theta F)_k \ {}^\theta F_i^k \tag{3.40}$$

Also, taking the gradient of equation (3.36) we know

$$\nabla_i \Delta f = -\nabla_i R + \frac{1}{2}\nabla_i |F|^2 \tag{3.41}$$

Plugging (3.40) and (3.41) into (3.39) gives

$$0 = -\frac{1}{2}\nabla_i R + \frac{3}{8}\nabla_i |F|^2 + \frac{1}{2}(d^*F)_k F_i^k + \mathrm{rc}_{ik}\nabla_k f \tag{3.42}$$

Now, applying (3.34) we see that

$$\begin{aligned} \mathrm{rc}_{ik}\nabla^k f = & \left(\frac{1}{2}\eta - \nabla^2 f - \lambda g\right)_{ik}\nabla^k f \\ = & \frac{1}{2}\eta_{ik}\nabla^k f - \frac{1}{2}\nabla_i |\nabla f|^2 - \lambda\nabla_i f \end{aligned} \tag{3.43}$$

Also, using (3.35) we have

$$\begin{aligned} \frac{1}{2}(d_\theta^* F)_k \ {}^\theta F_i^k = & \frac{1}{2}\left(\nabla^l f \ {}_\theta F_{lk}\right) \ {}^\theta F_i^k \\ = & -\frac{1}{2}\eta_{ik}\nabla^k f \end{aligned} \tag{3.44}$$

Thus, plugging (3.43) and (3.44) into (3.42) we get

$$0 = \nabla_i\left(-\frac{1}{2}R + \frac{3}{8}|F|^2 - \frac{1}{2}|\nabla f|^2 - \lambda f\right)$$

and the result follows immediately. □

Chapter 4

Evolution of F

Here we will study the evolution of the bundle curvature F in the case of a $U(1)$ bundle over M. If we ignore the effects of the curvature of g, it is natural to expect that if M has dimension 4 then F should get closer to being self-dual, that is we expect to be able to bound $\left|\mathring{\eta}\right|$ (see lemma A.4) and moreover to show that under certain conditions it decays. Note that this is a stronger statement than one expects from Yang-Mills flow alone, where any Yang-Mills (not necessarily self-dual) connection is critical for Yang-Mills flow. We bear this expectation out, and indeed it is the term η appearing in the evolution of g which provides the required decay.

4.1 Evolution Equations

Fix E a $U(1)$ bundle over M. Then the curvature form F is just a two-form on M, and so we can use the usual Bochner formula and compute the evolution of quantities related to F under Ricci Yang-Mills flow. We will actually study the following more general system of equations

$$\begin{aligned}\dot{g}_{jk} &= -2\operatorname{rc}_{jk} + \eta_{jk} \\ \dot{F} &= \Delta_d F\end{aligned} \tag{4.1}$$

which is induced by Ricci Yang-Mills flow in this case. In the propositions that follow we compute the evolution of curvature quantities related to F along a solution to (4.1). Note that throughout this section ∇ will always refer to the Levi-Civita connection of g.

Proposition: 4.1. *A solution to (4.1) satisfies*

$$\begin{aligned}\frac{d}{dt}\eta &= \Delta\eta_{ij} - 2\nabla_l F_{ki}\nabla^l F_j^k - \eta^{kl}F_{ik}F_{jl} \\ &\quad + g^{kl}r_{ikmn}F^{mn}F_{jl} + g^{kl}r_{jlmn}F^{mn}F_{ik} - \operatorname{rc}_i^m \eta_{mj} - \operatorname{rc}_j^m \eta_{mi}\end{aligned} \tag{4.2}$$

Proof. Using the Bochner formula of proposition A.2 and (4.1) we calculate

$$\begin{aligned}\frac{d}{dt}\eta_{ij} &= \frac{d}{dt}g^{kl}F_{ik}F_{jl} \\ &= \dot{g}^{kl}F_{ik}F_{jl} + g^{kl}\left(\Delta_d F_{ik}\right)F_{jl} + g^{kl}F_{ik}(\Delta_d F_{jl}) \\ &= (2\operatorname{rc} - \eta)_{mn}g^{mk}g^{nl}F_{ik}F_{jl} + g^{kl}\left(\Delta_d F_{ik}\right)F_{jl} + g^{kl}F_{ik}(\Delta_d F_{jl}) \\ &= (2\operatorname{rc} - \eta)^{kl} F_{ik}F_{jl} \\ &\quad + g^{kl}(\Delta F_{ik} + r_{ikmn}F^{mn} - g^{mn}\operatorname{rc}_{im}F_{nk} - g^{mn}\operatorname{rc}_{km}F_{in})F_{jl} \\ &\quad + g^{kl}F_{ik}\left(\Delta F_{jl} + r_{jlmn}F^{mn} - g^{mn}\operatorname{rc}_{jm}F_{nl} - g^{mn}\operatorname{rc}_{lm}F_{jn}\right) \\ &= \Delta\eta_{ij} - 2\nabla_l F_{ki}\nabla^l F_j^k - \eta^{kl}F_{ik}F_{jl} \\ &\quad + g^{kl}r_{ikmn}F^{mn}F_{jl} + g^{kl}r_{jlmn}F^{mn}F_{ik} - \operatorname{rc}_i^m \eta_{mj} - \operatorname{rc}_j^m \eta_{mi}\end{aligned}$$

□

Corollary: 4.2. *A solution to (4.1) satisfies*

$$\frac{d}{dt}|F|^2 = \Delta |F|^2 - 2|\nabla F|^2 + 2\operatorname{rm}(F,F) - 2|\eta|^2 \tag{4.3}$$

Proof. Using the above proposition we compute

$$\begin{aligned}
\frac{d}{dt}|F|^2 &= \frac{d}{dt} g^{ij}\eta_{ij} \\
&= (2\operatorname{rc} - \eta)^{ij}\eta_{ij} + g^{ij}\left(\Delta\eta_{ij} - 2(\nabla F\cdot\nabla F)_{ij} - \eta^{kl}F_{ik}F_{jl}\right) \\
&\quad + g^{ij}\left(2g^{kl}r_{ikmn}F^{mn}F_{jl} - 2\operatorname{rc}_i^m \eta_{mj}\right) \\
&= \Delta|F|^2 - 2|\nabla F|^2 + 2\operatorname{rm}(F,F) - 2|\eta|^2
\end{aligned}$$

□

Proposition: 4.3. *A solution to (4.1) satisfies*

$$\frac{d}{dt}\mathring{\eta} = \Delta\mathring{\eta} - 2(\nabla F\cdot\nabla F)^{\circ} + \operatorname{rm} * F^{*2} - \mathring{\eta}^{kl}F_{ik}F_{jl} + \frac{2}{n}\left|\mathring{\eta}\right|^2 g - \frac{2}{n}|F|^2\mathring{\eta} \tag{4.4}$$

Proof. Using the above propositions we compute

$$\begin{aligned}
\frac{d}{dt}\mathring{\eta} &= \frac{d}{dt}\left(\eta - \frac{1}{n}|F|^2 g\right) \\
&= \Delta\eta - 2\nabla F\cdot\nabla F + \operatorname{rm} * F^{*2} - \eta^{kl}F_{ik}F_{jl} \\
&\quad - \frac{1}{n}\left(\Delta|F|^2 - 2|\nabla F|^2 - 2|\eta|^2\right)g - \frac{1}{n}|F|^2\eta \\
&= \Delta\mathring{\eta} - (\nabla F\cdot\nabla F)^{\circ} + \operatorname{rm} * F_p^{*2} - \eta^{kl}F_{ik}^pF_{jl}^p + \frac{2}{n}|\eta|^2 g - \frac{1}{n}|F|^2\eta
\end{aligned} \tag{4.5}$$

But now the we can simplify the F^4 curvature term as

$$\begin{aligned}
-\eta^{kl}F_{ik}F_{jl} \ &+\frac{2}{n}\left|\eta\right|^2 g-\frac{1}{n}\left|F\right|^2\eta \\
&= -\left(\mathring{\eta}+\frac{1}{n}\left|F\right|^2 g\right)^{kl}F_{ik}F_{jl} \\
&\quad +\frac{2}{n}\left(\left|\mathring{\eta}\right|^2+\frac{1}{n}\left|F\right|^4\right)g-\frac{1}{n}\left|F\right|^2\left(\mathring{\eta}+\frac{1}{n}\left|F\right|^2 g\right) \\
&= -\mathring{\eta}^{kl}F_{ik}F_{jl}-\frac{1}{n}\left|F\right|^2\eta_{ij}+\frac{2}{n}\left|\mathring{\eta}\right|^2 g+\frac{1}{n^2}\left|F\right|^4 g-\frac{1}{n}\left|F\right|^2\mathring{\eta} \\
&= -\mathring{\eta}^{kl}F_{ik}F_{jl}+\frac{2}{n}\left|\mathring{\eta}\right|^2 g-\frac{2}{n}\left|F\right|^2\mathring{\eta}
\end{aligned}$$

Plugging this into (4.5) gives

$$\frac{d}{dt}\mathring{\eta}=\Delta\mathring{\eta}-2(\nabla F\cdot\nabla F)^\circ+\mathrm{rm}*F^{*2}-\mathring{\eta}^{kl}F_{ik}F_{jl}+\frac{2}{n}\left|\mathring{\eta}\right|^2 g-\frac{2}{n}\left|F\right|^2\mathring{\eta}$$

□

Corollary: 4.4. *A solution to (4.1) satisfies*

$$\begin{aligned}
\frac{d}{dt}\left|\mathring{\eta}\right|^2 &= \Delta\left|\mathring{\eta}\right|^2-2\left|\nabla\mathring{\eta}\right|^2-2\left\langle(\nabla F\cdot\nabla F)^\circ,\mathring{\eta}\right\rangle \\
&\quad -2\left\langle\mathring{\eta},\mathring{\eta}^2\right\rangle-2\left\langle\mathring{\eta}^{kl}F_{ik}F_{jl},\mathring{\eta}\right\rangle-\frac{6}{n}\left|F\right|^2\left|\mathring{\eta}\right|^2 \\
&\quad +\mathrm{rm}*F^{*4}
\end{aligned} \tag{4.6}$$

Proof. Using the above proposition we compute

$$\begin{aligned}
\frac{d}{dt}\left|\mathring{\eta}\right|^2 &= \frac{d}{dt}\left(g^{ij}g^{kl}\mathring{\eta}_{ik}\mathring{\eta}_{jl}\right) \\
&= -2\dot{g}^{ij}g^{kl}\mathring{\eta}_{ik}\mathring{\eta}_{jl} + 2\left\langle \frac{d}{dt}\mathring{\eta}, \mathring{\eta}\right\rangle \\
&= -2\left\langle \eta, \mathring{\eta}^2\right\rangle + 2\left\langle \Delta\mathring{\eta} - (\nabla F\cdot\nabla F)^\circ, \mathring{\eta}\right\rangle + \mathrm{rm} * F^{*4} \\
&\quad -2\left\langle \mathring{\eta}^{kl}F_{ik}F_{jl} - \frac{2}{n}\left|\mathring{\eta}\right|^2 g + \frac{2}{n}\left|F\right|^2\mathring{\eta}, \mathring{\eta}\right\rangle \\
&= \Delta\left|\mathring{\eta}\right|^2 - 2\left|\nabla\mathring{\eta}\right|^2 - 2\left\langle (\nabla F\cdot\nabla F)^\circ, \mathring{\eta}\right\rangle + \mathrm{rm} * F^{*4} \\
&\quad -2\left\langle \eta, \mathring{\eta}^2\right\rangle - 2\left\langle \mathring{\eta}^{kl}F_{ik}F_{jl} + \frac{2}{n}\left|F\right|^2\mathring{\eta}, \mathring{\eta}\right\rangle \\
&= \Delta\left|\mathring{\eta}\right|^2 - 2\left|\nabla\mathring{\eta}\right|^2 - 2\left\langle (\nabla F\cdot\nabla F)^\circ, \mathring{\eta}\right\rangle + \mathrm{rm} * F^{*4} \\
&\quad -2\left\langle \mathring{\eta}, \mathring{\eta}^2\right\rangle - 2\left\langle \mathring{\eta}^{kl}F_{ik}F_{jl}, \mathring{\eta}\right\rangle - \frac{6}{n}\left|F\right|^2\left|\mathring{\eta}\right|^2
\end{aligned}$$

□

The next proposition applies in the case of a general $\mathcal{K}$-bundle.

Proposition: 4.5. *Suppose $(g(t), A(t))$ is a solution to Ricci Yang-Mills flow on $E \to M$ a principal $\mathcal{K}$-bundle. Then*

$$\frac{d}{dt}\left|F\right|^2 = \Delta\left|F\right|^2 - 2\left|{}^A\nabla F\right|^2 + \mathrm{rm} * F^{*2} + F^{*3} - 2\left|\eta\right|^2 \tag{4.7}$$

Moreover, a solution to volume-normalized Ricci Yang-Mills flow satisfies

$$\frac{d}{dt}\left|F\right|^2 = \Delta\left|F\right|^2 - 2\left|{}^A\nabla F\right|^2 + \mathrm{rm} * F^{*2} + F^{*3} - 2\left|\mathring{\eta}\right|^2 \tag{4.8}$$

Proof. The proof is identical to that of proposition 4.2, except that one must use the Bochner formula for the Laplacian D_A, which is given in proposition A.3. Thus the answer is the same except for the introduction of the term F^{*3}. □

4.2 Pinching Behaviour

In this section we use the maximum principle to conclude statements about the pinching behaviour of F and its related quantities. Restricting to $n = 4$ will be important for these calculations. Let's begin by remarking on the pointwise pinching of F, i.e. $\left|\overset{\circ}{\eta}\right|^2$. We form the scale invariant quantity

$$q(x,t) = \frac{\left|\overset{\circ}{\eta}\right|^2}{|F|^4} \tag{4.9}$$

and compute its evolution. Note that

$$\left|\overset{\circ}{\eta}\right|^2 = \left|\eta - \frac{1}{4}|F|^2 g\right|^2 = |\eta|^2 - \frac{1}{4}|F|^4$$

Also, since η is positive definite we have the inequality $|\eta|^2 \leq |F|^4$. However, because η only has two distinct eigenvalues, in fact $|\eta|^2 \leq \frac{1}{2}|F|^4$ (see lemma A.5). Thus $q \leq \frac{1}{4}$ with equality if and only if F is not symplectic. Indeed, assuming that η is strictly positive definite to begin with, i.e. assuming F is pointwise symplectic initially we have

$$q \leq \frac{1}{4} - \epsilon_0 \tag{4.10}$$

everywhere. This ϵ_0 represents the initial pinching of η. The hypothesis described below and the proposition that both follow are discussed after the proof.

Definition: 4.6. A solution to Ricci Yang-Mills flow satisfies the ϵ-low order estimate on $[0,T)$ if the following inequalities hold on $[0,T)$:

$$\begin{aligned} |\nabla F|_g &\leq \epsilon[F^{\wedge 2}] \\ |\mathrm{rm}|_g &\leq \epsilon[F^{\wedge 2}] \end{aligned} \tag{4.11}$$

where $[F^{\wedge 2}] = \left|\int F \wedge F\right|$. Either of these terms is referred to as a term of *low order.*

Proposition: 4.7. *Suppose $(g(t), A(t))$ is a solution to Ricci Yang-Mills flow on a $U(1)$-bundle over a four-manifold which exists on $[0,T)$. Suppose that on this time interval the ϵ-low-order estimate holds and moreover*

$$\min_{M\times[0,T)} |F|^2 \geq \zeta \fint |F|^2 \tag{4.12}$$

There exists universal $C > 0$ (not depending on any of the constants/objects above) so that $\left|\mathring{\eta}\right|^2 / |F|^4$ is bounded above by the solution to the ODE

$$\begin{aligned} \frac{d}{dt}\phi(t) &= C\left(\epsilon + \frac{\epsilon^2}{\zeta}\right)[F^{\wedge 2}] - \zeta[F^{\wedge 2}]\phi\left(1 - 4\phi\right) \\ \phi(0) &= \sup_{(M,g_0,A_0)} \frac{\left|\mathring{\eta}\right|^2}{|F|^4} \end{aligned} \tag{4.13}$$

where $[F^{\wedge 2}] = \left|\int F \wedge F\right|$.

Proof. Before we begin we will examine some of the quantities in the evolution of $\left|\mathring{\eta}\right|^2$ specialized to dimension 4. After a change of basis we may assume at a fixed point x that $g(x) = I$ and

$$\eta = \begin{pmatrix} \lambda_1^2 I_2 & 0 \\ 0 & \lambda_2^2 I_2 \end{pmatrix}, \qquad \mathring{\eta} = \frac{1}{2}\begin{pmatrix} (\lambda_1^2 - \lambda_2^2) I_2 & 0 \\ 0 & (\lambda_2^2 - \lambda_1^2) I_2 \end{pmatrix}$$

where $\lambda_1 \leq \lambda_2$. Then:

$$\begin{aligned} -2\left\langle \mathring{\eta}, \mathring{\eta}^2 \right\rangle &= -\frac{1}{4}\left((\lambda_1^2 - \lambda_2^2)^3 + (\lambda_2^2 - \lambda_1^2)^3\right) \\ &= 0 \end{aligned}$$

Also,

$$-2\left\langle \overset{\circ}{\eta}^{kl} F_{ik}F_{jl}, \overset{\circ}{\eta}\right\rangle = 2\,\mathrm{tr}(F\overset{\circ}{\eta})^2$$

$$= -\left(\lambda_1^2+\lambda_2^2\right)\left(\lambda_1^2-\lambda_2^2\right)^2$$

$$= -\frac{1}{2}\left|F\right|^2\left|\overset{\circ}{\eta}\right|^2$$

Now let $L = \frac{\partial}{\partial t} - \Delta$. Using lemma A.10 with $f = \left|\overset{\circ}{\eta}\right|^2, g = \left|F\right|^2, \alpha = 1$ and $\beta = 2$ we see that:

$$L(q) = \frac{1}{\left|F\right|^4}L(\left|\overset{\circ}{\eta}\right|^2) - 2\frac{\left|\overset{\circ}{\eta}\right|^2}{\left|F\right|^6}L(\left|F\right|^2)$$

$$-6\frac{\left|\overset{\circ}{\eta}\right|^2}{\left|F\right|^8}\left|\nabla\left|F\right|^2\right|^2 + 4\frac{1}{\left|F\right|^6}\left\langle\nabla\left|\overset{\circ}{\eta}\right|^2, \nabla\left|F\right|^2\right\rangle$$

$$= \frac{1}{\left|F\right|^4}\left(-2\left|\nabla\overset{\circ}{\eta}\right|^2 - 2\left\langle(\nabla F\overset{\circ}{\cdot}\nabla F), \overset{\circ}{\eta}\right\rangle + \mathrm{rm}*F^{*4} - 2\left|F\right|^2\left|\overset{\circ}{\eta}\right|^2\right)$$

$$-2\frac{\left|\overset{\circ}{\eta}\right|^2}{\left|F\right|^6}\left(-\left|\nabla F\right|^2 + \mathrm{rm}*F^{*2} - 2\left|\eta\right|^2\right)$$

$$-6\frac{\left|\overset{\circ}{\eta}\right|^2}{\left|F\right|^8}\left|\nabla\left|F\right|^2\right|^2 + 4\frac{1}{\left|F\right|^6}\left\langle\nabla\left|\overset{\circ}{\eta}\right|^2, \nabla\left|F\right|^2\right\rangle$$

The highest order (in F) terms are easy to collect, and they are:

$$q\left(\frac{-2\left|F\right|^4 + 4\left|\eta\right|^2}{\left|F\right|^2}\right) = q\left(\frac{-\left|F\right|^4 + 4\left|\overset{\circ}{\eta}\right|^2}{\left|F\right|^2}\right)$$

$$= -\left|F\right|^2 q\left(1-4q\right)$$

Finally by (4.12) and lemma A.6 we have

$$-\left|F\right|^2 q\left(1-4q\right) \le -\zeta[F^{\wedge 2}]q\left(1-4q\right) \tag{4.14}$$

Using the ϵ-low-order estimate and (4.12) also gives the bounds

$$C\frac{|\nabla F|^2}{|F|^2} \leq C\frac{\epsilon^2}{\zeta}[F^{\wedge 2}] \tag{4.15}$$

$$\frac{\left|\overset{\circ}{\eta}\right|^2}{|F|^6}\,\mathrm{rm} * F^{*2} \leq C\epsilon[F^{\wedge 2}] \tag{4.16}$$

Thus we may apply the maximum principle (lemma A.1) to conclude that q is bounded above by the solution to the ODE

$$\frac{d\phi}{dt} = C\left(\epsilon + \frac{\epsilon^2}{\zeta}\right)[F^{\wedge 2}] - \zeta[F^{\wedge 2}]\phi\,(1 - 4\phi)$$

□

Let us make a few remarks on this result. Since F arises as the curvature of a $U(1)$-bundle we can actually examine the family of equations parametrized by the twisting power of this bundle, which we call p. We refer to the curvature of this bundle as F_p and in fact $F_p = pF$. Above when we spoke of "ignoring the effects of the curvature of g" what we meant was to think of picking a very large value for p so that the bundle curvature is much larger in norm than the curvature of g. We note that if p is chosen very large, then we may take $\epsilon = \frac{1}{p}$ in the definition of ϵ-low-order, so that the $-\zeta[F_p^{\wedge 2}]\phi\,(1 - 4\phi)$ term will dominate the behaviour of the ODE component of the evolution of $\left|\overset{\circ}{\eta}\right|^2 / |F|^4$. In particular we will have exponential decay of this quantity. We note also that the form of the term $\phi(1 - 4\phi)$ is to be expected, since the condition $q = \frac{1}{4}$ is equivalent to F not having full rank. Thus if for instance F had a nontrivial kernel everywhere, we would not expect the rank of F to suddenly jump.

Chapter 5

General Curvature Evolution Equations

In this section we will show formulas for the evolution of curvature for a solution to Ricci Yang-Mills flow. In the first section below we will compute variation formulae for symmetric tensors. Then in the second section we will compute evolution equations for solutions to Ricci Yang-Mills flow. In the second section we compute evolution equations for an interesting partially normalized version of RYM flow. Finally in the last section we compute evolution equations for the full volume-normalized RYM flow.

5.1 Variation Formulae

In this section we will compute variation formulae for tensors in a moving frame. The results are the same as the usual formulae for the evolution of curvature quantities in coordinates.

Lemma: 5.1. *If* $\dot{G} = W$ *then*

$$\begin{aligned} \dot{g} &= W^H \\ \dot{A} &= W^C \\ \dot{F} &= dW^C \end{aligned} \tag{5.1}$$

Proof. This is immediate, as in lemma 2.8. □

Lemma: 5.2. *If* $\dot{G} = W$, *then*

$$\dot{\Gamma}^{\delta}_{\alpha\beta} = \frac{1}{2} G^{\delta\epsilon} \left(\nabla_\alpha W_{\beta\epsilon} + \nabla_\beta W_{\alpha\epsilon} - \nabla_\epsilon W_{\alpha\beta} \right) \tag{5.2}$$

Proof. We start with the computation

$$\begin{aligned} \frac{\partial}{\partial s}\Gamma^{\delta}_{\alpha\beta} = \frac{\partial}{\partial s}\frac{1}{2}G^{\delta\epsilon}\left(e_\alpha G_{\beta\epsilon} + e_\beta G_{\alpha\epsilon} - e_\epsilon G_{\alpha\beta} - C_{\alpha\epsilon\beta} - C_{\beta\alpha\epsilon} + C_{\epsilon\beta\alpha}\right) \\ = \; -\frac{1}{2}G^{\delta\nu}\dot{G}_{\nu\mu}G^{\mu\epsilon}\left(e_\alpha G_{\beta\epsilon} + e_\beta G_{\alpha\epsilon} - e_\epsilon G_{\alpha\beta} - C_{\alpha\epsilon\beta} - C_{\beta\alpha\epsilon} + C_{\epsilon\beta\alpha}\right) \\ + \frac{1}{2}G^{\delta\epsilon}\left(e_\alpha W_{\beta\epsilon} + e_\beta W_{\alpha\epsilon} - e_\epsilon W_{\alpha\beta}\right) \\ - \frac{1}{2}G^{\delta\epsilon}\left(\dot{C}_{\alpha\epsilon\beta} + \dot{C}_{\beta\alpha\epsilon} - \dot{C}_{\epsilon\beta\alpha}\right) \\ = \; -G^{\delta\nu}W_{\nu\mu}\Gamma^{\mu}_{\alpha\beta} + \frac{1}{2}G^{\delta\epsilon}\left(e_\alpha W_{\beta\epsilon} + e_\beta W_{\alpha\epsilon} - e_\epsilon W_{\alpha\beta}\right) \\ - \frac{1}{2}G^{\delta\epsilon}\left(\dot{C}_{\alpha\epsilon\beta} + \dot{C}_{\beta\alpha\epsilon} - \dot{C}_{\epsilon\beta\alpha}\right) \end{aligned} \tag{5.3}$$

Now we compute

$$\frac{1}{2}G^{\delta\epsilon}\left(e_\alpha W_{\beta\epsilon} + e_\beta W_{\alpha\epsilon} - e_\epsilon W_{\alpha\beta}\right)$$

$$\begin{aligned}
&= \frac{1}{2}G^{\delta\epsilon}\big(\nabla_\alpha W_{\beta\epsilon} + \Gamma^\mu_{\alpha\beta}W_{\mu\epsilon} + \Gamma^\mu_{\alpha\epsilon}W_{\beta\mu} + \nabla_\beta W_{\alpha\epsilon} + \Gamma^\mu_{\beta\alpha}W_{\mu\epsilon} + \Gamma^\mu_{\beta\epsilon}W_{\alpha\mu} \\
&\qquad - \nabla_\epsilon W_{\alpha\beta} - \Gamma^\mu_{\epsilon\alpha}W_{\mu\beta} - \Gamma^\mu_{\epsilon\beta}W_{\alpha\mu}\big) \qquad (5.4) \\
&= \frac{1}{2}G^{\delta\epsilon}\left(\nabla_\alpha W_{\beta\epsilon} + \nabla_\beta W_{\alpha\epsilon} - \nabla_\epsilon W_{\alpha\beta}\right) \\
&\qquad + \frac{1}{2}G^{\delta\epsilon}\left(\left(\Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\beta\alpha}\right)W_{\mu\epsilon} + \left(\Gamma^\mu_{\alpha\epsilon} - \Gamma^\mu_{\epsilon\alpha}\right)W_{\mu\beta} + \left(\Gamma^\mu_{\beta\epsilon} - \Gamma^\mu_{\epsilon\beta}\right)W_{\mu\alpha}\right)
\end{aligned}$$

Also, since our frame is independent of the variation, we clearly have that $\dot{C}^\delta_{\alpha\beta} = 0$. Thus

$$\begin{aligned}
-\frac{1}{2}G^{\delta\epsilon}\left(\dot{C}_{\alpha\epsilon\beta} + \dot{C}_{\beta\alpha\epsilon} - \dot{C}_{\epsilon\beta\alpha}\right) &= -\frac{1}{2}G^{\delta\epsilon}\frac{\partial}{\partial s}\left(C^\nu_{\alpha\epsilon}G_{\nu\beta} + C^\nu_{\beta\alpha}G_{\nu\epsilon} - C^\nu_{\epsilon\beta}G_{\nu\alpha}\right) \\
&= -\frac{1}{2}G^{\delta\epsilon}\left(C^\nu_{\alpha\epsilon}W_{\nu\beta} + C^\nu_{\beta\alpha}W_{\nu\epsilon} - C^\nu_{\epsilon\beta}W_{\nu\alpha}\right)
\end{aligned} \qquad (5.5)$$

Plugging 5.4 and 5.5 into 5.3 and using that $\Gamma^\epsilon_{\alpha\beta} - \Gamma^\epsilon_{\beta\alpha} = C^\epsilon_{\alpha\beta}$ the result follows. □

Proposition: 5.3. *If $\dot{G} = W$, then*

$$\frac{\partial}{\partial s}\operatorname{Rc}_{\alpha\beta} = -\frac{1}{2}\left(\Delta_L W_{\alpha\beta} - G^{\mu\epsilon}\nabla_\alpha\nabla_\mu W_{\epsilon\beta} - G^{\mu\epsilon}\nabla_\beta\nabla_\mu W_{\epsilon\alpha} + \nabla_\alpha\nabla_\beta \operatorname{tr} W\right) \qquad (5.6)$$

where

$$\left(\Delta_L W\right)_{\alpha\beta} = \Delta W_{\alpha\beta} + 2R(W)_{\alpha\beta} - (\operatorname{Rc}\cdot W)_{\alpha\beta} - (W\cdot\operatorname{Rc})_{\alpha\beta} \qquad (5.7)$$

is the Lichneorwicz Laplacian.

Proof. We begin with the equation

$$\operatorname{Rc}_{\alpha\beta} = R^\mu_{\mu\alpha\beta} = e_\mu\Gamma^\mu_{\alpha\beta} - e_\alpha\Gamma^\mu_{\mu\beta} + \Gamma^\nu_{\alpha\beta}\Gamma^\mu_{\mu\nu} - \Gamma^\nu_{\mu\beta}\Gamma^\mu_{\alpha\nu} - C^\nu_{\mu\alpha}\Gamma^\mu_{\nu\beta}$$

Taking the variation of this equation gives

$$
\begin{aligned}
\frac{\partial}{\partial s}\operatorname{Rc}_{\alpha\beta} &= e_{\mu}\dot{\Gamma}^{\mu}_{\alpha\beta} - e_{\alpha}\dot{\Gamma}^{\mu}_{\mu\beta} \\
&\qquad + \dot{\Gamma}^{\nu}_{\alpha\beta}\Gamma^{\mu}_{\mu\nu} + \Gamma^{\nu}_{\alpha\beta}\dot{\Gamma}^{\mu}_{\mu\nu} - \dot{\Gamma}^{\nu}_{\mu\beta}\Gamma^{\mu}_{\alpha\nu} - \Gamma^{\nu}_{\mu\beta}\dot{\Gamma}^{\mu}_{\alpha\nu} - C^{\nu}_{\mu\alpha}\dot{\Gamma}^{\mu}_{\nu\beta}
\end{aligned}
\tag{5.8}
$$

Now we compute

$$
\begin{aligned}
e_{\mu}\dot{\Gamma}^{\mu}_{\alpha\beta} &= \nabla_{\mu}\dot{\Gamma}^{\mu}_{\alpha\beta} + \Gamma^{\nu}_{\mu\alpha}\dot{\Gamma}^{\mu}_{\nu\beta} + \Gamma^{\nu}_{\mu\beta}\dot{\Gamma}^{\mu}_{\alpha\nu} - \Gamma^{\mu}_{\mu\nu}\dot{\Gamma}^{\nu}_{\alpha\beta} \\
-e_{\alpha}\dot{\Gamma}^{\mu}_{\mu\beta} &= -\nabla_{\alpha}\dot{\Gamma}^{\mu}_{\mu\beta} - \Gamma^{\nu}_{\alpha\mu}\dot{\Gamma}^{\mu}_{\nu\beta} - \Gamma^{\nu}_{\alpha\beta}\dot{\Gamma}^{\mu}_{\mu\nu} + \Gamma^{\mu}_{\alpha\nu}\dot{\Gamma}^{\nu}_{\mu\beta} \\
&= -\nabla_{\alpha}\dot{\Gamma}^{\mu}_{\mu\beta} - \Gamma^{\nu}_{\alpha\beta}\dot{\Gamma}^{\mu}_{\mu\nu}
\end{aligned}
$$

Plugging this into (5.8) gives

$$
\begin{aligned}
\frac{\partial}{\partial s}\operatorname{Rc}_{\alpha\beta} &= \nabla_{\mu}\dot{\Gamma}^{\mu}_{\alpha\beta} - \nabla_{\alpha}\dot{\Gamma}^{\mu}_{\mu\beta} + \dot{\Gamma}^{\nu}_{\mu\beta}\left(\Gamma^{\mu}_{\nu\alpha} - \Gamma^{\mu}_{\alpha\nu} - C^{\mu}_{\nu\alpha}\right) \\
&= \nabla_{\mu}\dot{\Gamma}^{\mu}_{\alpha\beta} - \nabla_{\alpha}\dot{\Gamma}^{\mu}_{\mu\beta}
\end{aligned}
\tag{5.9}
$$

where the last line follows because $\Gamma^{\mu}_{\nu\alpha} - \Gamma^{\mu}_{\alpha\nu} - C^{\mu}_{\nu\alpha} = 0$. We now plug 5.2 into 5.9. This gives

$$
\begin{aligned}
\frac{\partial}{\partial s}\operatorname{Rc}_{\alpha\beta} &= \frac{1}{2}G^{\mu\epsilon}\nabla_{\mu}\left(\nabla_{\alpha}W_{\beta\epsilon} + \nabla_{\beta}W_{\alpha\epsilon} - \nabla_{\epsilon}W_{\alpha\beta}\right) \\
&\qquad - \frac{1}{2}G^{\mu\epsilon}\nabla_{\alpha}\left(\nabla_{\mu}W_{\beta\epsilon} + \nabla_{\beta}W_{\mu\epsilon} - \nabla_{\epsilon}W_{\mu\beta}\right)
\end{aligned}
\tag{5.10}
$$

First we note that

$$
-\frac{1}{2}G^{\mu\epsilon}\nabla_{\alpha}\left(\nabla_{\mu}W_{\beta\epsilon} - \nabla_{\epsilon}W_{\mu\beta}\right) = 0 \tag{5.11}
$$

We commute the derivatives to compute

$$
\frac{1}{2}G^{\mu\epsilon}\nabla_{\mu}\nabla_{\alpha}W_{\beta\epsilon} = \frac{1}{2}G^{\mu\epsilon}\left(\nabla_{\alpha}\nabla_{\mu}W_{\epsilon\beta}\right) + \frac{1}{2}G^{\mu\epsilon}G^{\nu\rho}\left(R_{\mu\alpha\beta\rho}W_{\nu\epsilon} + R_{\mu\alpha\epsilon\rho}W_{\beta\nu}\right) \tag{5.12}
$$

and similarly

$$
\frac{1}{2}G^{\mu\epsilon}\nabla_{\mu}\nabla_{\beta}W_{\alpha\epsilon} = \frac{1}{2}G^{\mu\epsilon}\left(\nabla_{\beta}\nabla_{\mu}W_{\epsilon\alpha}\right) + \frac{1}{2}G^{\mu\epsilon}G^{\nu\rho}\left(R_{\mu\beta\alpha\rho}W_{\nu\epsilon} + R_{\mu\beta\epsilon\rho}W_{\alpha\nu}\right) \tag{5.13}
$$

Note that in the two formulas above the second derivatives represent *total* second covariant derivatives, so that their commutators alone give curvature. Plugging 5.11 - 5.13 into 5.10 gives

$$\begin{aligned}\frac{\partial}{\partial s}\operatorname{Rc}_{\alpha\beta} = & -\frac{1}{2}\left(\Delta W_{\alpha\beta} - G^{\mu\epsilon}\nabla_\alpha\nabla_\mu W_{\epsilon\beta} - G^{\mu\epsilon}\nabla_\beta\nabla_\mu W_{\epsilon\alpha} + \nabla_\alpha\nabla_\beta \operatorname{tr} W\right) \\ & + \frac{1}{2}G^{\mu\epsilon}G^{\nu\rho}\left(R_{\mu\alpha\beta\rho}W_{\nu\epsilon} + R_{\mu\alpha\epsilon\rho}W_{\beta\nu} + R_{\mu\beta\alpha\rho}W_{\nu\epsilon} + R_{\mu\beta\epsilon\rho}W_{\alpha\nu}\right)\end{aligned} \tag{5.14}$$

The Laplacian term and the curvature terms clearly combine to give

$$-\frac{1}{2}\Delta_L W_{\alpha\beta} = -\frac{1}{2}\left(\Delta W_{\alpha\beta} + 2R(W)_{\alpha\beta} - (\operatorname{Rc}\cdot W)_{\alpha\beta} - (W\cdot\operatorname{Rc})_{\alpha\beta}\right) \tag{5.15}$$

And the result follows. $\square$

Proposition: 5.4. *If $\dot{G} = W$ and $Y(t)$ is a family of symmetric two-tensors such that $\dot{Y} = Z$, then*

$$\frac{d}{dt}\left(Y^H + Y^C\right) = \left(Z^H + Z^C\right) + \pi^*\left(\left\langle W^C, Y^C\right\rangle_g\right) \tag{5.16}$$

Proof. First of all, we write $Y^H + Y^C = Y - Y^V$. Then in our frame this reads

$$\left(Y^H + Y^C\right)_{\alpha\beta} = Y_{\alpha\beta} - \left(G^V\right)_\alpha^\mu Y_{\mu\nu}\left(G^V\right)_\beta^\nu \tag{5.17}$$

We would like to take the variation of this equation. First of all we take the variation

$$\begin{aligned}\frac{d}{dt}\left(G^V\right)_\alpha^\beta &= \frac{d}{dt}\left(G^V\right)_{\alpha\delta}G^{\delta\beta} \\ &= \left(W^V\right)_\alpha^\beta - \left(G^V\right)_{\alpha\delta}G^{\delta\mu}W_{\mu\nu}G^{\nu\beta} \\ &= \left(W^V\right)_\alpha^\beta - \left(G^V\right)_\alpha^\mu W_\mu^\beta\end{aligned} \tag{5.18}$$

Thus we see

$$\begin{aligned}\frac{d}{dt}\left(Y^H+Y^C\right)_{\alpha\beta} = Z_{\alpha\beta} - \left(G^V\right)_\alpha^\mu Z_{\mu\nu}\left(G^V\right)_\beta^\nu \\ &- \left(\left(W^V\right)_\alpha^\mu - \left(G^V\right)_\alpha^\delta W_\delta^\mu\right) Y_{\mu\nu}\left(G^V\right)_\beta^\nu \\ &- \left(G^V\right)_\alpha^\mu Y_{\mu\nu}\left(\left(W^V\right)_\beta^\nu - \left(G^V\right)_\beta^\delta W_\delta^\nu\right) \\ &= \left(Z^H+Z^C\right)_{\alpha\beta} + \left(W^C\right)_\alpha^\mu Y_{\mu\nu}\left(G^V\right)_\beta^\nu + \left(G^V\right)_\alpha^\mu Y_{\mu\nu}\left(W^C\right)_\beta^\nu \\ &= \left(Z^H+Z^C\right)_{\alpha\beta} + \pi^*\left(\left\langle W_\alpha^C, Y_\beta^C\right\rangle_g\right)\end{aligned}$$

$\square$

Proposition: 5.5. *If* $\dot{G} = W$ *then*

$$\begin{aligned}\frac{\partial}{\partial s}\left(d^*F\right)_i^\theta = & \left(d^*dW^C\right)_i^\theta - \left(W^H\right)^{jk}\ {}^A\nabla_j F_{ki}^\theta \\ & - \frac{1}{2}g^{jk}g^{mn}\left({}^g\nabla_j W_{kn}^H + {}^g\nabla_k W_{jn}^H - {}^g\nabla_n W_{jk}^H\right)F_{mi}^\theta \\ & - \frac{1}{2}g^{jk}g^{mn}\left({}^g\nabla_j W_{in}^H - {}^g\nabla_n W_{ji}^H\right)F_{km}^\theta\end{aligned} \tag{5.19}$$

Proof. Let τ denote the Christoffel symbol of g. Using normal coordinates for g we may assume that τ vanishes at a chosen point. Using lemma 5.1 and proposition A.8

we compute that

$$
\begin{aligned}
\frac{\partial}{\partial s}(d^*F)_i^\theta &= \frac{\partial}{\partial s}\left(g^{jk}\ {}^A\nabla_j F_{ki}^\theta\right) \\
&= \frac{\partial}{\partial s}\left(g^{jk}\left(\partial_j F_{ki}^\theta - \tau_{jk}^m F_{mi}^\theta - \tau_{ji}^m F_{km}^\theta\right)\right) \\
&= -g^{jp}\dot{g}_{pq}g^{qk}\ {}^A\nabla_j F_{ki}^\theta + g^{jk}\left(\partial_j \dot{F}_{ki}^\theta - \dot{\tau}_{jk}^m F_{mi}^\theta - \dot{\tau}_{ji}^m F_{km}^\theta\right) \\
&= -\left(W^H\right)^{jk}\ {}^A\nabla_j F_{ki}^\theta + g^{jk}\left(d^*dW^C\right)_i^\theta \\
&\quad - \frac{1}{2}g^{jk}g^{mn}\left({}^g\nabla_j W_{kn}^H + {}^g\nabla_k W_{jn}^H - {}^g\nabla_n W_{jk}^H\right)F_{mi}^\theta \\
&\quad - \frac{1}{2}g^{jk}g^{mn}\left({}^g\nabla_j W_{in}^H + {}^g\nabla_i W_{jn}^H - {}^g\nabla_n W_{ji}^H\right)F_{km}^\theta
\end{aligned}
\tag{5.20}
$$

It is clear that $g^{jk}g^{mn}\,{}^g\nabla_i W_{jn}\,{}_\theta F_{km} = 0$. Plugging this into (5.20) gives the result. □

5.2 Curvature Evolution Equations

5.2.1 Bianchi-type Identities

In this subsection we will compute certain curvature identities which are related to the usual Bianchi identities for the Riemannian curvature tensor. Just to simplify notation, given a metric G of the form (2.2) let

$$N := \mathrm{Rc}^H + \mathrm{Rc}^C \tag{5.21}$$

Lemma: 5.6. *Let G be a bundle metric. Then*

$$
\begin{aligned}
(\operatorname{div} N)_i &= \frac{1}{2}\nabla_i r - \frac{1}{8}\nabla_i |F|^2 \\
(\operatorname{div} N)_\theta &= 0
\end{aligned}
\tag{5.22}
$$

Proof. We note using lemma 2.3 that for a bundle metric we have $\mathrm{Rc}^H = \mathrm{rc} - \frac{1}{2}\eta$ and $\mathrm{Rc}^C = \frac{1}{2}d^*F$. Using this and lemma 2.2 we compute

$$\begin{aligned}
(\operatorname{div} N)_i &= G^{\alpha\beta}\left(\partial_\alpha N_{\beta i} - \Gamma^\nu_{\alpha\beta}N_{\nu i} - \Gamma^\nu_{\alpha i}N_{\beta\nu}\right) \\
&= ({}^g\operatorname{div} N^H)_i - \frac{1}{2}\,{}^\theta F^n_i N_{\theta n} - \frac{1}{2}g^{ab}\,{}_\theta F_{ai}N^\theta_b \\
&= ({}^g\operatorname{div} N^H)_i - \frac{1}{2}\,{}^\theta F^k_i (d^*F)_{k\theta}
\end{aligned} \tag{5.23}$$

Now, using the usual contracted differential Bianchi identity for the Ricci tensor of g we see

$$\begin{aligned}
({}^g\operatorname{div} N^H)_i &= \left({}^g\operatorname{div}\left(\mathrm{rc} - \frac{1}{2}\eta\right)\right)_i \\
&= \frac{1}{2}\nabla_i r - \frac{1}{2}\left({}^g\operatorname{div}\eta\right)_i
\end{aligned} \tag{5.24}$$

Finally we compute, using that $dF = 0$,

$$\begin{aligned}
\left({}^g\operatorname{div}\eta\right)_i &= {}^g\nabla^m\left(g^{kl}\,{}_\theta F_{ki}\,{}^\theta F_{lm}\right) \\
&= g^{kl}\left({}^A\nabla^m F^\theta_{ki}\right){}_\theta F_{lm} - {}^\theta F^k_i (d^*F)_{k\theta} \\
&= -g^{kl}g^{mn}\left({}^A\nabla_i F^\theta_{mk} + {}^A\nabla_k F^\theta_{im}\right){}_\theta F_{ln} - {}^\theta F^k_i\left(d^*F\right)_{k\theta} \\
&= \frac{1}{2}\nabla_i\left|F\right|^2 - g^{mn}\left({}^A\nabla^k F^\theta_{mi}\right){}_\theta F_{nk} - {}^\theta F^k_i\left(d^*F\right)_{k\theta} \\
&= \frac{1}{4}\nabla_i\left|F\right|^2 - {}^\theta F^k_i\left(d^*F\right)_{k\theta}
\end{aligned} \tag{5.25}$$

where the last line follows by rearranging the fourth and second lines. Plugging (5.25) into (5.24) and then plugging the result into (5.23) gives the first claim. As for the second, we compute

$$\begin{aligned}(\operatorname{div} N)_\theta &= G^{\alpha\beta}\left(\partial_\alpha N_{\beta\theta} - \Gamma^\nu_{\alpha\beta} N_{\nu\theta} - \Gamma^\nu_{\alpha\theta} N_{\beta\nu}\right) \\ &= \frac{1}{2}\left(d^*(d^*F)\right) - \frac{1}{2}\operatorname{tr}\left({}_\theta F \cdot N^H\right) \\ &= 0\end{aligned} \tag{5.26}$$

since $(d^* d^* F) = 0$ and the trace of the product of a symmetric and a skew-symmetric matrix is zero. □

Lemma: 5.7. *Let G be a bundle metric. Then*

$$\nabla_\alpha (\operatorname{div} N)_\beta = \nabla_\alpha \nabla_\beta \left(\frac{1}{2} r - \frac{1}{8}|F|^2\right) \tag{5.27}$$

Proof. We compute using (5.22)

$$\begin{aligned}\nabla_i (\operatorname{div} N)_j &= \left(\partial_i \operatorname{div} N_j - \Gamma^\alpha_{ij} \operatorname{div} N_\alpha\right) \\ &= \nabla_i \left(\frac{1}{2}\nabla_j r - \frac{1}{8}\nabla_j |F|^2\right) \\ &= \frac{1}{2}\nabla_i \nabla_j r - \frac{1}{8}\nabla_i \nabla_j |F|^2\end{aligned} \tag{5.28}$$

Next we compute

$$\begin{aligned}\nabla_i (\operatorname{div} N)_\theta &= \left(\partial_i \operatorname{div} N_\theta - \Gamma^\alpha_{i\theta} \operatorname{div} N_\alpha\right) \\ &= -\frac{1}{2}\, {}_\theta F^k_i \left(\operatorname{div} N_k\right) \\ &= -\frac{1}{2}\, {}_\theta F^k_i \left(\frac{1}{2}\nabla_k r - \frac{1}{8}\nabla_k |F|^2\right)\end{aligned} \tag{5.29}$$

and similarly

$$\begin{aligned}\nabla_\theta (\operatorname{div} N)_j &= \left(\partial_\theta \operatorname{div} N_j - \Gamma^\alpha_{\theta j} \operatorname{div} N_\alpha\right) \\ &= -\frac{1}{2}\, {}_\theta F^k_j \left(\operatorname{div} N_k\right) \\ &= -\frac{1}{2}\, {}_\theta F^k_j \left(\frac{1}{2}\nabla_k r - \frac{1}{8}\nabla_k |F|^2\right)\end{aligned} \tag{5.30}$$

Finally we have

$$\begin{aligned}\nabla_\theta \left(\operatorname{div} N\right)_\rho &= \left(\partial_\theta \operatorname{div} N_\rho - \Gamma^\alpha_{\theta\rho} \operatorname{div} N_\alpha\right) \\ &= 0\end{aligned} \tag{5.31}$$

Comparing these results with lemma 3.3 gives the result. □

5.2.2 Curvature Evolution for RYM Flow

Before stating these evolution equations let us introduce an important piece of notation. If A and B are two tensors on a Riemannian manifold, we denote by $A * B$ any quantity obtained from $A \otimes B$ by one or more of the following operations: summation over pairs of matching upper and lower indices, contraction on upper indices with respect to the metric, contraction on lower indices with respect to the metric inverse, and multiplication by constants depending only on the dimension of the total space and the ranks of A and B.

Proposition: 5.8. *A solution to Ricci Yang-Mills flow satisfies*

$$\frac{d}{dt} \operatorname{Rm} = \Delta \operatorname{Rm} + F * \nabla \operatorname{Rm} + \operatorname{Rm}^{*2} \tag{5.32}$$

Proof. Recall ([7] lemma 7.4) that in the case of the usual Ricci flow we have

$$\frac{d}{dt} \operatorname{Rm} = \Delta \operatorname{Rm} + \operatorname{Rm}^{*2}$$

Now, in our case we have $\dot{G} = -2N = -2 \operatorname{Rc} + 2 \operatorname{Rc}^V$. Thus to compute the evolution for the curvature tensor we can start with the evolution given by usual Ricci flow and compute the evolution of curvature given by $\dot{G} = 2 \operatorname{Rc}^V = \frac{1}{2} \pi^* \left(\langle F, F \rangle_g \right)$. We know that in general the evolution of curvature is given by second derivatives of the evolution of G. Thus we see that for $\dot{G} = \frac{1}{2}\pi^* \left(\langle F, F\rangle_g\right)$ we have

$$\begin{aligned}\frac{d}{dt}\operatorname{Rm} &= \nabla^2 (F * F) \\ &= \left(\nabla^2 F\right) * F + (\nabla F)^{*2}\end{aligned} \tag{5.33}$$

Now using lemma 2.3 it is clear that we formally have $\nabla F = \operatorname{Rm}$, thus this term may be written $F * \nabla \operatorname{Rm} + \operatorname{Rm}^{*2}$ and the result follows. □

Proposition: 5.9. *A solution to Ricci Yang-Mills flow satisfies*

$$\frac{d}{dt}\nabla^k \operatorname{Rm} = \Delta\nabla^k \operatorname{Rm} + F * \nabla^{k+1}\operatorname{Rm} + \sum_{j=0}^{k} \nabla^j \operatorname{Rm} * \nabla^{k-j}\operatorname{Rm} \tag{5.34}$$

Proof. First consider the case $k = 1$. Using lemma 5.2 and proposition 5.8 we see

$$\begin{aligned}\frac{d}{dt}\nabla \operatorname{Rm} &= \frac{d}{dt}(\partial \operatorname{Rm} + \Gamma * \operatorname{Rm}) \\ &= \partial\left(\Delta \operatorname{Rm} + F * \nabla \operatorname{Rm} + \operatorname{Rm}^{*2}\right) \\ &\quad + \nabla \operatorname{Rm} * \operatorname{Rm} + F * \left(\Delta \operatorname{Rm} + F * \nabla \operatorname{Rm} + \operatorname{Rm}^{*2}\right) \\ &= \nabla\Delta \operatorname{Rm} + F * \nabla^2 \operatorname{Rm} + \nabla \operatorname{Rm} * \operatorname{Rm} + F * \operatorname{Rm}^{*2}\end{aligned} \tag{5.35}$$

Using the formula

$$\nabla\Delta A = \Delta\nabla A + \operatorname{Rm} * \nabla A + \nabla \operatorname{Rm} * A \tag{5.36}$$

gives the result for $k=1$. As for the general case we compute as above

$$\begin{aligned}
\frac{d}{dt}\nabla^{k+1}\operatorname{Rm} &= \frac{d}{dt}\left(\partial+\Gamma\right)*\cdots*\left(\partial+\Gamma\right)*\operatorname{Rm} \\
&= \left(\nabla\operatorname{Rm}\right)*\left(\partial+\Gamma\right)*\ldots\left(\partial+\Gamma\right)*\operatorname{Rm} \\
&\quad+\left(\partial+\Gamma\right)*\left(\nabla\operatorname{Rm}\right)*\left(\partial+\Gamma\right)*\cdots*\left(\partial+\Gamma\right)*\operatorname{Rm} \\
&\quad+\cdots+\left(\partial+\Gamma\right)*\cdots*\left(\partial+\Gamma\right)*\left(\nabla\operatorname{Rm}\right)*\operatorname{Rm} \\
&\quad+\nabla^k\left(\frac{d}{dt}\operatorname{Rm}\right) \qquad (5.37)\\
&= \nabla\operatorname{Rm}*\nabla^{k-1}\operatorname{Rm}+\nabla^2\operatorname{Rm}*\nabla^{k-2}\operatorname{Rm}+\cdots+\nabla^k\operatorname{Rm}*\operatorname{Rm} \\
&\quad+\nabla^k\left(\Delta\operatorname{Rm}+F*\nabla\operatorname{Rm}+\operatorname{Rm}^{*2}\right) \\
&= \nabla^k\Delta\operatorname{Rm}+\nabla^k\left(F*\nabla\operatorname{Rm}\right)+\sum_{j=0}^{k}\nabla^j\operatorname{Rm}*\nabla^{k-j}\operatorname{Rm}
\end{aligned}$$

Iterating (5.36) it is easy to see that

$$\nabla^k\Delta\operatorname{Rm}=\Delta\nabla^k\operatorname{Rm}+\sum_{j=0}^{k}\nabla^j\operatorname{Rm}*\nabla^{k-j}\operatorname{Rm} \qquad (5.38)$$

Also, we have using lemma 2.3 that $\nabla F=\operatorname{Rm}$ formally, thus

$$\begin{aligned}
\nabla^k\left(F*\nabla\operatorname{Rm}\right) &= \sum_{j=0}^{k}\left(\nabla^jF*\nabla^{k+1-j}\operatorname{Rm}\right) \\
&= F*\nabla^{k+1}\operatorname{Rm}+\sum_{j=1}^{k}\left(\nabla^jF*\nabla^{k+1-j}\operatorname{Rm}\right) \qquad (5.39)\\
&= F*\nabla^{k+1}\operatorname{Rm}+\sum_{j=0}^{k}\left(\nabla^j\operatorname{Rm}*\nabla^{k-j}\operatorname{Rm}\right)
\end{aligned}$$

Plugging (5.39) and (5.38) into (5.37) gives the result. $\square$

Proposition: 5.10. *A solution to Ricci Yang-Mills flow satisfies*

$$\dot{\mathrm{Rc}}_{\alpha\beta} = \Delta_L N_{\alpha\beta} - \frac{1}{4}\nabla_\alpha \nabla_\beta |F|^2 \tag{5.40}$$

Proof. Plugging $W = -2N$ into (5.6) gives

$$\frac{d}{dt}\mathrm{Rc}_{\alpha\beta} = \Delta_L N_{\alpha\beta} - G^{\mu\epsilon}\nabla_\alpha\nabla_\mu N_{\epsilon\beta} - G^{\mu\epsilon}\nabla_\beta\nabla_\mu N_{\epsilon\alpha} + \nabla_\alpha\nabla_\beta \operatorname{tr} N \tag{5.41}$$

Now using lemma 5.7 we compute

$$\begin{aligned}
-G^{\mu\epsilon}&\nabla_\alpha\nabla_\mu N_{\epsilon\beta} - G^{\mu\epsilon}\nabla_\beta\nabla_\mu N_{\epsilon\alpha} + \nabla_\alpha\nabla_\beta \operatorname{tr} N \\
&= -\nabla_\alpha (\operatorname{div} N)_\beta - \nabla_\beta (\operatorname{div} N)_\alpha + \nabla_\alpha\nabla_\beta \operatorname{tr} N \\
&= -\nabla_\alpha\nabla_\beta\left(r - \frac{1}{4}|F|^2\right) + \nabla_\alpha\nabla_\beta\left(r - \frac{1}{2}|F|^2\right) \\
&= -\frac{1}{4}\nabla_\alpha\nabla_\beta |F|^2
\end{aligned} \tag{5.42}$$

□

5.2.3 Curvature Evolution for a Renormalized Equation

In this section we will consider a particular renormalization of Ricci Yang-Mills flow. The term $-\frac{1}{4}\nabla_\alpha\nabla_\beta |F|^2$ in (5.40) suggests that we add a multiple of $|F|^2 G^H$ to the evolution of G directly to cancel this term and give a more natural heat-type equation. Since this term is lower order in the metric, it is clear that this poses no short-time existence problems, as opposed to trying to add the scalar curvature to the usual Ricci flow. So, consider the equation

$$\dot{G} = -2S := -2\left(\mathrm{Rc}^H + \mathrm{Rc}^C + \frac{1}{4(n-2)}|F|^2 G^H\right) \tag{5.43}$$

The factor of $\frac{1}{4(n-2)}$ is chosen with perfect hindsight.

Proposition: 5.11. *A solution to equation (5.43) satisfies*

$$\frac{d}{dt}\operatorname{Rc}_{\alpha\beta} = \Delta_L S_{\alpha\beta} \tag{5.44}$$

Proof. Using proposition 5.6 with $W = -\frac{1}{2(n-2)}|F|^2 G^H$ and proposition 5.10 shows that a solution to (5.43) satisfies

$$\begin{aligned}\dot{\operatorname{Rc}}_{\alpha\beta} &= \Delta_L N_{\alpha\beta} - \frac{1}{4}\nabla_\alpha\nabla_\beta |F|^2 + \frac{1}{4(n-2)}\Delta_L\left(|F|^2 G^H\right) \\ &\quad - \frac{1}{4(n-2)}G^{\mu\epsilon}\nabla_\alpha\nabla_\mu\left(|F|^2 G^H\right)_{\epsilon\beta} - \frac{1}{4(n-2)}G^{\mu\epsilon}\nabla_\beta\nabla_\mu\left(|F|^2 G^H\right)_{\epsilon\alpha} \\ &\quad + \frac{n}{4(n-2)}\nabla_\alpha\nabla_\beta|F|^2 \\ &= \Delta_L S_{\alpha\beta}\end{aligned} \tag{5.45}$$

where in the last line we have used that $\operatorname{div} G^H = 0$ □

Next we would like to compute the evolution of S itself. To do this we will of course need to first compute the evolution of $|F|^2$. So, we have

Proposition: 5.12. *A solution to equation (5.43) satisfies*

$$\frac{d}{dt}|F|^2 = 4\left\langle S^H, \eta\right\rangle + 8\left\langle {}^A\nabla S^C, F\right\rangle \tag{5.46}$$

Proof. First of all given the equation $\dot{A} = -2S^C$ and $F^\theta_{ij} = A^\theta_{i,j} - A^\theta_{j,i} + (A\wedge A)^\theta_{ij}$ we clearly see that $\dot{F}^\theta_{ij} = -2\left({}^A\nabla_j\left(S^C\right)^\theta_i - {}^A\nabla_i\left(S^C\right)^\theta_j\right)$. Thus we compute

$$\begin{aligned}\frac{d}{dt}|F|^2 &= \frac{d}{dt}\left(\overline{g}^{\theta\rho}g^{ij}g^{kl}\ {}_\theta F_{ik}\ {}_\rho F_{jl}\right) \\ &= -2g^{im}\dot{g}_{mn}g^{nj}g^{kl}\ {}^\theta F_{ik}\ {}_\theta F_{jl} - 4\left\langle {}^A\nabla_j\left(S^C\right)^\theta_i - {}^A\nabla_i\left(S^C\right)^\theta_j, F^\theta_{ij}\right\rangle \\ &= 4\left\langle S^H,\eta\right\rangle + 8\left\langle {}^A\nabla S^C, F\right\rangle\end{aligned}$$

□

Proposition: 5.13. *A solution to equation (5.43) satisfies*

$$\begin{aligned}\frac{d}{dt}S = \ & (\Delta_L S)^H + (\Delta_L S)^C - \frac{1}{2(n-2)}|F|^2 S^H \\ & + \left(\frac{1}{n-2}\left\langle S^H, \eta\right\rangle + \frac{2}{n-2}\left\langle {}^A\nabla S^C, F\right\rangle\right) G^H \\ & - \pi^*\left(\left\langle S^C, S^C\right\rangle_g\right)\end{aligned} \tag{5.47}$$

Proof. Let $A = \operatorname{Rc} + \frac{1}{4(n-2)}|F|^2 G^H$. We combine propositions 5.11 and 5.12 to compute

$$\begin{aligned}\frac{d}{dt}A = \Delta_L S - \frac{1}{2(n-2)}|F|^2 S^H \\ + \left(\frac{1}{n-2}\left\langle S^H, \eta\right\rangle + \frac{2}{n-2}\left\langle {}^A\nabla S^C, F\right\rangle\right) G^H\end{aligned} \tag{5.48}$$

Using that $S = A^H + A^C$ we can apply proposition 5.4 to get the result. ∎

Proposition: 5.14. *A solution to equation (5.43) satisfies*

$$\frac{d}{dt}r = \Delta r + 2\langle S, \operatorname{rc}\rangle + 2\left|S^C\right|^2 + \frac{4-n}{4(n-2)}\Delta|F|^2 + 2\left\langle {}^A\nabla S^C, F\right\rangle \tag{5.49}$$

Proof. We use the equation $r = \operatorname{tr}\operatorname{Rc} + \frac{1}{4}|F|^2$ and compute

$$\begin{aligned}\frac{d}{dt}r &= \frac{d}{dt}\left(G^{\alpha\beta}\operatorname{Rc}_{\alpha\beta} + \frac{1}{4}|F|^2\right) \\ &= -G^{\alpha\mu}\dot{G}_{\mu\nu}G^{\nu\beta}\operatorname{Rc}_{\alpha\beta} + G^{\alpha\beta}\Delta_L S_{\alpha\beta} + \left\langle S^H, \eta\right\rangle \\ &\quad + 2\left\langle {}^A\nabla S^C, F\right\rangle \\ &= \langle S, 2\operatorname{Rc} + \eta\rangle + G^{\alpha\beta}\Delta_L S_{\alpha\beta} + 2\left\langle {}^A\nabla S^C, F\right\rangle \\ &= 2\left\langle S^H, \operatorname{rc}\right\rangle + 2\left|S^C\right|^2 + G^{\alpha\beta}\Delta_L S_{\alpha\beta} + 2\left\langle {}^A\nabla S^C, F\right\rangle\end{aligned} \tag{5.50}$$

Now we simplify

$$
\begin{aligned}
G^{\alpha\beta}\Delta S_{\alpha\beta} &= \Delta \operatorname{tr} S \\
&= \Delta\left(r + \left(\frac{n}{4(n-2)} - \frac{1}{2}\right)|F|^2\right) \\
&= \Delta r + \frac{4-n}{4(n-2)}\Delta |F|^2
\end{aligned}
\tag{5.51}
$$

Also, it is a general fact that $\operatorname{tr}\mathcal{R}(S)$, the trace of the curvature term appearing in the Lichnerowicz Laplacian, vanishes (see proposition A.8). Thus

$$
\frac{d}{dt}r = \Delta r + 2\langle S, \mathrm{rc}\rangle + 2\left|S^C\right|^2 + \frac{4-n}{4(n-2)}\Delta|F|^2 + 2\left\langle {}^A\nabla S^C, F\right\rangle \tag{5.52}
$$

□

Proposition: 5.15. *A solution to (5.43) satisfies*

$$
\begin{aligned}
\frac{d}{dt}d^*F_i^\theta = {}& \Delta_d d^*F_i^\theta + 2\left(S^H\right)^{jk}\ {}^A\nabla_j F_{ki}^\theta + 2g^{jk}g^{lm}\ {}^g\nabla_j S_{li}^H F_{km}^\theta \\
&+ \left(F^\theta \cdot\ {}_\rho F \cdot d^*F^\rho\right)_i
\end{aligned}
\tag{5.53}
$$

Proof. We apply proposition 5.5 with $W = -2S$. This gives

$$
\begin{aligned}
\frac{d}{dt}d^*F_i^\theta = {}& -d^*dd^*F_i^\theta + 2\left(S^H\right)^{jk}\ {}^A\nabla_j F_{ki}^\theta \\
&+ g^{jk}g^{mn}\left({}^g\nabla_j S_{kn}^H + {}^g\nabla_k S_{jn}^H - {}^g\nabla_n S_{jk}^H\right)F_{mi}^\theta \\
&+ g^{jk}g^{mn}\left({}^g\nabla_j S_{in}^H - {}^g\nabla_n S_{ji}^H\right)F_{km}^\theta
\end{aligned}
\tag{5.54}
$$

First of all, since $d^*d^*F = 0$ we have that $-d^*dd^*F_i = \Delta_d d^*F_i$. Next we simplify using the Bianchi identity and the proof of lemma 5.6

$$g^{jk}g^{mn}\left({}^{g}\nabla_j S^H_{kn} + {}^{g}\nabla_k S^H_{jn} - {}^{g}\nabla_n S^H_{jk}\right) {}_{\theta}F_{mi}$$

$$= g^{mn}\left(-\left({}^{g}\operatorname{div}\eta\right)_n + \frac{1}{4}\nabla_n |F|^2\right) F^{\theta}_{mi}$$

$$= g^{mn}\left({}_{\rho}F^k_n (d^*F)^{\rho}_k\right) F^{\theta}_{mi}$$

$$= (F^{\theta}\cdot\ {}_{\rho}F\cdot\ d^*F^{\rho})_i$$

Finally using the skew-symmetry of F we simplify

$$g^{jk}g^{mn}\left({}^{g}\nabla_j S^H_{in} - {}^{g}\nabla_n S^H_{ji}\right) F^{\theta}_{km} = 2g^{jk}g^{lm}\ {}^{g}\nabla_j S^H_{li} F^{\theta}_{km}$$

□

5.2.4 Curvature Evolution for Volume Renormalized RYM-flow

In this subsection we will consider volume renormalized RYM flow. Noting that $\operatorname{tr} S = r + \frac{4-n}{4(n-2)}|F|^2$ we define $V = \mathrm{Rc}^H + \mathrm{Rc}^C + \frac{1}{4(n-2)}|F|^2 G^H - \frac{1}{n}\left(\fint r + \fint \frac{4-n}{4(n-2)}|F|^2\right) G^H$, and then we can write volume renormalized RYM flow as

$$\frac{d}{dt}G = -2V = -2\left(S - \frac{1}{n}\left(\fint r + \fint \frac{4-n}{4(n-2)}|F|^2\right) G^H\right) \tag{5.55}$$

In computing the evolution equations of various tensors we will compute the effect of the renormalization term and use the results of the previous section.

Proposition: 5.16. *A solution to volume renormalized RYM flow satisfies*

$$\frac{d}{dt}\mathrm{Rc}_{\alpha\beta} = \Delta_L V_{\alpha\beta} \tag{5.56}$$

Proof. Use (5.6) with $W = \frac{2}{n}\left(\fint r + \fint \frac{4-n}{4(n-2)}\right) G^H$ (and the fact that the divergence of G^H vanishes) and proposition 5.11 to see that under volume renormalized RYM flow we have

$$\frac{d}{dt}\operatorname{Rc}_{\alpha\beta} = \Delta_L S_{\alpha\beta} - \Delta_L \left(\frac{1}{n}\left(\fint r + \fint \frac{4-n}{4(n-2)}\right) G^H\right)$$

$$= \Delta_L V$$

□

Proposition: 5.17. *A solution to volume renormalized RYM flow satisfies*

$$\frac{d}{dt}|F|^2 = 4\left\langle V^H, \eta\right\rangle + 8\left\langle {}^A\nabla V^C, F\right\rangle \tag{5.57}$$

Proof. It is not hard to see that given $\frac{d}{dt}G = cG^H$ one has $\frac{d}{dt}F = 0$, and hence $\frac{d}{dt}|F|^2 = -2c|F|^2$. Thus using proposition 5.12 and the fact that $S^C = V^C$ we have

$$\begin{aligned}\frac{d}{dt}|F|^2 &= 4\left\langle S^H, \eta\right\rangle + 8\left\langle {}^A\nabla S^C, F\right\rangle - \frac{4}{n}\left(\fint r + \fint \frac{4-n}{4(n-2)}|F|^2\right)|F|^2 \\ &= 4\left\langle V^H, \eta\right\rangle + 8\left\langle {}^A\nabla V^C, F\right\rangle\end{aligned} \tag{5.58}$$

□

Proposition: 5.18. *A solution to volume renormalized RYM flow satisfies*

$$\frac{d}{dt}r = \Delta r + 2\langle V, \operatorname{rc}\rangle + 2\left|V^C\right|^2 + \frac{4-n}{4(n-2)}\Delta|F|^2 + 2\left\langle {}^A\nabla V^C, F\right\rangle \tag{5.59}$$

Proof. It is clear that for $\dot{G} = cG^H$ we have $\dot{r} = -cr$. Thus using proposition 5.14 we compute

$$\begin{aligned}\frac{d}{dt}r &= \Delta r + 2\langle S, \operatorname{rc}\rangle + 2\left|S^C\right|^2 + \frac{4-n}{4(n-2)}\Delta|F|^2 + 2\left\langle {}^A\nabla S^C, F\right\rangle \\ &\quad - \frac{2}{n}\left(\fint r + \fint \frac{4-n}{4(n-2)}|F|^2\right) r \\ &= \Delta r + 2\langle V, \operatorname{rc}\rangle + 2\left|V^C\right|^2 + \frac{4-n}{4(n-2)}\Delta|F|^2 + 2\left\langle {}^A\nabla V^C, F\right\rangle\end{aligned}$$

□

Before the next proposition we recall that for a function $f \in C^\infty(M)$ we define

$$⨍ f := \left(\int f dV \right) / \operatorname{vol} M \tag{5.60}$$

Proposition: 5.19. *A solution to volume renormalized RYM flow with $n = 4$ satisfies*

$$\frac{d}{dt} ⨍ r = ⨍ 2 \langle V, \mathrm{rc} \rangle - r\delta r \tag{5.61}$$

Proof. Recall that the volume stays fixed under volume renormalized RYM flow. Without loss of generality assume $\operatorname{vol}(M_t) \equiv 1$ for all t. Using proposition 5.18 and plugging in $n = 4$ we see that

$$\begin{aligned}
\frac{d}{dt} ⨍ r &= \frac{d}{dt} \left(\int r dV \right) / (\operatorname{vol} M) \\
&= \frac{d}{dt} \left(\int r dV \right) \\
&= \int \left(\frac{d}{dt} r \right) dV - \int r\delta r dV \\
&= \int 2 \langle V, \mathrm{rc} \rangle + 2 \left| V^C \right|^2 + 2 \left\langle {}^g\nabla V^C, F \right\rangle - r\delta r \\
&= \int 2 \langle V, \mathrm{rc} \rangle - r\delta r
\end{aligned}$$

where the last line follows because the term $2 \left| V^C \right|^2 + 2 \left\langle {}^g\nabla V^C, F \right\rangle$ vanishes upon integrating by parts. The result follows. $\square$

Corollary: 5.20. *A solution to volume renormalized RYM flow with $n = 4$ satisfies*

$$\begin{aligned}
\frac{d}{dt} (\delta r)^2 = \Delta (\delta r)^2 - 2 |\nabla r|^2 + 4 \langle V, \mathrm{rc} \rangle \delta r + 4 \left| V^C \right|^2 \delta r + 4 \left\langle {}^A\nabla V^C, F \right\rangle \delta r \\
- 4 \left(⨍ \langle V, \mathrm{rc} \rangle \right) \delta r + 2 \left(⨍ r\delta r \right) \delta r
\end{aligned} \tag{5.62}$$

Proof. This is immediate using propositions 5.18 and 5.19. □

Proposition: 5.21. *A solution to volume renormalized RYM flow satisfies*

$$\begin{aligned}\frac{d}{dt}S = {} & (\Delta_L V)^H + (\Delta_L V)^C - \frac{1}{2(n-2)}|F|^2 V^H \\ & + \left(\frac{1}{n-2}\left\langle V^H, \eta\right\rangle + \frac{2}{n-2}\left\langle {}^A\nabla V^C, F\right\rangle\right) G^H \\ & - \pi^*\left(\left\langle V^C, V^C\right\rangle_g\right)\end{aligned} \tag{5.63}$$

Proof. Let $Z = \mathrm{Rc} + \frac{1}{4(n-2)}|F|^2 G^H$. Combine propositions 5.16 and 5.17 to compute

$$\begin{aligned}\frac{d}{dt}Z_{\alpha\beta} = {} & \Delta_L V_{\alpha\beta} - \frac{1}{2(n-2)}|F|^2 V^H \\ & + \left(\frac{1}{n-2}\left\langle V^H, \eta\right\rangle + \frac{2}{n-2}\left\langle {}^A\nabla V^C, F\right\rangle\right) G^H\end{aligned} \tag{5.64}$$

Now using that $S = Z^H + Z^C$ we apply proposition 5.4 to get the result. □

Proposition: 5.22. *A solution to volume renormalized RYM flow satisfies*

$$\begin{aligned}\frac{d}{dt}d^*F_i^\theta = {} & \Delta_d d^*F_i^\theta + 2\left(V^H\right)^{jk}\ {}^A\nabla_j F_{ki}^\theta + 2g^{jk}g^{lm}\ {}^g\nabla_j V_{li}^H F_{km}^\theta \\ & + \left({}_\theta F \cdot {}^\rho F \cdot d^*{}_\rho F\right)_i\end{aligned} \tag{5.65}$$

Proof. Applying proposition 5.5 with $W = cG^H$ shows that under this variation one has $\frac{d}{dt}d^*F_i^\theta = -cd^*F_i^\theta$. Thus combining this with proposition 5.15 we have

$$\begin{aligned}\frac{d}{dt}d^*F_i^\theta = {} & \Delta_d d^*F_i^\theta + 2\left(S^H\right)^{jk}\ {}^A\nabla_j F_{ki}^\theta + 2g^{jk}g^{lm}\ {}^g\nabla_j S_{li}^H F_{km}^\theta \\ & + \left(F^\theta \cdot {}_\rho F \cdot d^*F^\rho\right)_i \\ & - \frac{2}{n}\left(\fint r + \fint \frac{4-n}{4(n-2)}|F|^2\right) d^*F_i^\theta \\ = {} & \Delta_d d^*F_i^\theta + 2\left(V^H\right)^{jk}\ {}^A\nabla_j F_{ki}^\theta + 2g^{jk}g^{lm}\ {}^g\nabla_j V_{li}^H F_{km}^\theta \\ & + \left(F^\theta \cdot {}_\rho F \cdot d^*F^\rho\right)_i\end{aligned}$$

□

Proposition: 5.23. *A solution to volume normalized RYM flow satisfies*

$$\frac{d}{dt}|V|^2 = \Delta|V|^2 - 2|\nabla V|^2 + \operatorname{Rm} * V^{*2} \tag{5.66}$$

$$\frac{d}{dt}|\nabla V|^2 = \Delta|\nabla V|^2 - 2\left|\nabla^2 V\right|^2 + \operatorname{Rm} * \nabla V^{*2} + \nabla \operatorname{Rm} * V * \nabla V \tag{5.67}$$

$$\frac{d}{dt}\left|\nabla^2 V\right|^2 = \Delta\left|\nabla^2 V\right| - 2\left|\nabla^3 V\right|^2 + \operatorname{Rm} * \nabla^2 V^{*2} \tag{5.68}$$

$$+ \nabla \operatorname{Rm} * \nabla V * \nabla^2 V + \nabla^2 \operatorname{Rm} * V * \nabla^2 V \tag{5.69}$$

Proof. Using propositions 5.17, 5.18, 5.21 and the basic integration by parts

$$\int_M \left\langle {}^A\nabla V^C, F\right\rangle = -\int_M \left\langle V^C, d^* F\right\rangle = V * \operatorname{Rm}$$

we see

$$\frac{d}{dt}V = \Delta V + \operatorname{Rm} * V \tag{5.70}$$

from which the first claim follows immediately. We can also use this to compute

$$\begin{aligned} \frac{d}{dt}\nabla V &= \frac{d}{dt}(\partial V + \Gamma * V) \\ &= \partial\frac{d}{dt}V + \nabla \operatorname{Rm} * V \\ &= \nabla(\Delta V + \operatorname{Rm} * V) + \nabla \operatorname{Rm} * V \\ &= \Delta \nabla V + \nabla \operatorname{Rm} * V + \operatorname{Rm} * \nabla V \end{aligned}$$

from which the second result follows immediately. The third is entirely similar. □

Chapter 6

Analytic Properties of Ricci Yang Mills Flow

In this chapter we will discuss many analytic properties of Ricci Yang-Mills flow. Based on the calculations of the previous chapter and our intuition from the study of Ricci flow, we expect that the Ricci Yang-Mills flow should behave like a nonlinear parabolic equation. In the first section we will bear this out and show that for any initial condition the flow always exists for a short time. Next we prove a lower bound for the existence time based on an initial curvature bound. Also we will find the obstruction to long-time existence of a solution to Ricci Yang-Mills flow.

In the second section we prove certain decay estimates for the derivatives of the curvature along a solution to Ricci Yang-Mills flow. These are natural heat kernel estimates, and the corresponding estimates for the Ricci flow are called *Bernstein-Bando-Shi* estimates. Our proof of these will closely follow the corresponding proof for Ricci flow found in [7]. The estimates will play a crucial role in the proof of our main convergence theorem. Also using these estimates we will prove a compactness theorem for solutions to Ricci Yang-Mills flow.

6.1 Existence Properties

Proposition: 6.1. *Given G_0 a smooth bundle metric, there exists $\epsilon > 0$ so that a unique solution to Ricci Yang-Mills flow exists on $[0, \epsilon)$ with initial condition G_0.*

Proof. To prove this theorem we will use the interpretation of Ricci Yang-Mills flow as a coupled system of equations, i.e. the viewpoint of lemma 2.8. In particular we must solve the equations

$$\begin{aligned}
\frac{d}{dt} g_{ij} &= -2 \operatorname{rc}_{ij} + \eta_{ij} \\
\frac{d}{dt} A &= -d_A^* F \\
g(0) &= g_0 \\
A(0) &= A_0
\end{aligned} \tag{6.1}$$

for arbitrary initial g_0, A_0. Recall that short-time existence for both Ricci flow and Yang-Mills flow is proved by using a gauge-fixing procedure. Here we will combine both of these gauge-fixing procedures.

First we must compute the linearization of the operator $-2 \operatorname{rc} + \eta$. As far as parabolicity is concerned, we only need to consider the rc term as it will contain the highest derivatives of the variation of g. In particular, we recall the following formula from [7] pg. 79

$$-2[D(\operatorname{rc})(h)]_{jk} = \Delta h_{jk} + g^{pq} \left(\nabla_j \nabla_q h_{pq} - \nabla_j \nabla_q h_{pk} - \nabla_k \nabla_q h_{pj} \right) + \mathcal{R}(h)$$

where now we have grouped all of the lower order terms together into $\mathcal{R}(h)$. Fix $\widetilde{\nabla}$ a torsion-free connection on M. For instance we could take $\widetilde{\nabla} = {}^g\nabla(0)$. Now define the vector field

$$W^k = g^{pq}\left(\Gamma^k_{pq} - \widetilde{\Gamma}^k_{pq}\right)$$

and consider the differential operator

$$P(g) := \mathcal{L}_W g$$

The calculation in ([7] pg. 80) shows that

$$\sigma[D(-2\operatorname{rc} + P)](\zeta)(h) = |\zeta|^2 h \tag{6.2}$$

Now let us set up the gauge-fixing procedure for the Yang-Mills flow. We note that above we found that by adding a certain Lie derivative of the metric to the Ricci flow equation (which can be accounted for by a diffeomorphism flow), it was equivalent to a parabolic equation. We will follow the exact same procedure in showing that the Yang-Mills flow is equivalent to a parabolic system, now using gauge transformations of the principal bundle. In particular, paramaterize a family of connections $A(t)$ as

$$A(t) = A_0 + a_t \tag{6.3}$$

where $a_t \in \Omega^1(\mathfrak{g})$. Consider the function

$$\beta_t = d_A^* a_t \in \Omega^0(\mathfrak{g}) \tag{6.4}$$

Then there is the computation

$$\begin{aligned} -\left(d_A^* F_A + d\beta\right) &= -\left(d_A^* d_A A + d_A d_A^* a\right) \\ &= -\Delta_A a \end{aligned} \tag{6.5}$$

which is a strictly elliptic operator acting on the one-form a. Motivated by this discussion we define the following *gauge-fixed flow*

$$
\begin{aligned}
\frac{d}{dt} g_{ij} = & -2\,\mathrm{rc}_{ij} + \eta_{ij} + \mathcal{L}_W g \\
\frac{d}{dt} A = & -d^*_{A,\phi^*(g)} F_A + d\beta \\
g(0) = & \; g_0 \\
A(0) = & \; A_0
\end{aligned}
\tag{6.6}
$$

where W and β satisfy

$$
\begin{aligned}
W^k = & \; g^{pq}\left(\Gamma^k_{pq} - \widetilde{\Gamma}^k_{pq}\right) \\
\beta = & \; d^*_A\left(A(t) - A_0\right)
\end{aligned}
\tag{6.7}
$$

and $\phi_t : M \to M$ is the unique one-parameter family of diffeomorphisms satisfying

$$
\begin{aligned}
\frac{\partial}{\partial t}\phi_t = & -W(t) \\
\phi_0 = & \; \mathrm{Id}_M
\end{aligned}
\tag{6.8}
$$

What is immediately clear from equations (6.2) and (6.5) is that (6.6) is a strictly parabolic system of equations, and as such has unique short-time existence of solutions on compact manifolds. Thus there exists $\epsilon > 0$ so that the solution $(g(t), A(t))$ to (6.6) exists on $[0, \epsilon)$. Let $\overline{g}(t) = \phi^*_t g(t)$. Analagous to the diffeomorphism gauge ϕ, let $u_t : E \to E$ to be the unique one-parameter family of gauge transformations so that $\overline{A}(t) = u_t(A(t))$ satisfies

$$
\begin{aligned}
d^*_{\overline{A}}\left(\frac{\partial}{\partial t}\overline{A}(t)\right) = & \; 0 \\
u_0 = & \; \mathrm{Id}_E
\end{aligned}
\tag{6.9}
$$

This is the Coloumb gauge condition for the time-varying connection $B(t)$, and its existence is proved in [8]. Indeed it is equivalent to solving an ODE over M.

We claim that the pair $(\overline{g}, \overline{A})$ is a solution to RYM flow. First we will show that g satisfies the right equation. Since the crux of this calculation is the removal of the Lie derivative term in the gauge-fixed flow, we briefly introduce the shorthand $H = -2\operatorname{rc} + \eta$. We may now calculate

$$\begin{aligned}
\frac{\partial}{\partial t}\overline{g}(t) &= \frac{\partial}{\partial t}\left(\phi_t^* g(t)\right) \\
&= \phi_t^*\left(\frac{\partial}{\partial t} g(t)\right) + \frac{\partial}{\partial s}|_{s=0}\left(\phi_{t+s}^* g(t)\right) \\
&= \phi_t^*\left(H(g(t)) + L_{W(t)} g(t)\right) \\
&\quad + \frac{\partial}{\partial s}|_{s=0}\left[\left(\phi_t^{-1}\circ\phi_{t+s}\right)^* \phi_t^* g(t)\right] \\
&= H\left(\phi_t^* g(t)\right) + \phi_t^*\left(L_{W(t)} g(t)\right) - L_{\left[\left(\phi_t^{-1}\right)_* W(t)\right]}\phi_t^* g(t) \\
&= H(\overline{g}(t))
\end{aligned} \tag{6.10}$$

Note that since η is a well-defined tensor on the base manifold M, this formula holds independent of any gauge-transformation we may apply to the time-varying connection A. Next we calculate

$$\begin{aligned}
\frac{\partial}{\partial t}\overline{A}(t) &= \frac{\partial}{\partial t} u_t A(t) \\
&= -\left(d_{\overline{A}}^* F_{\overline{A}} + d_{\overline{A}}\left(u\beta u^{-1} + \frac{\partial u}{\partial t} u^{-1}\right)\right)
\end{aligned} \tag{6.11}$$

Using the Coloumb gauge condition, taking $d_{\overline{A}}^*$ of the equation above yields 0, and since $d_{\overline{A}}^* d_{\overline{A}}^* F_{\overline{A}} = 0$, this means that we have $d_{\overline{A}}^* d_{\overline{A}}\left(u\beta u^{-1} + \frac{\partial u}{\partial t} u^{-1}\right) = 0$, which implies that $d_{\overline{A}}\left(u\beta u^{-1} + \frac{\partial u}{\partial t} u^{-1}\right) = 0$ since M is compact. Thus we see that indeed $\overline{A}(t)$ satisfies the required equation, and so we have shown short-time existence of solutions to RYM flow.

Now we must show uniqueness. To do this we will use the important fact that the diffeomorphisms ϕ_t satisfy the harmonic map heat flow with respect to the pulled-back metric $\phi^* g(t)$ ([7] pg. 89), i.e.

$$\begin{aligned} \frac{d}{dt}\phi_t &= \Delta_{\overline{g}(t),\widetilde{g}}\phi_t \\ \phi_0 &= \mathrm{Id}_M \end{aligned} \tag{6.12}$$

Note then that if we *push-forward* a solution to Ricci Yang-Mills flow by this family of diffeomorphisms, then the metric satisfies its part of the gauge-fixed flow.

Next we point out that the gauge transformation of the bundle obeys a parabolic equation as well. In particular, given a solution $A(t)$ of Yang-Mills flow, let $u(t)$ be the unique solution to the parabolic equation

$$\begin{aligned} \frac{d}{dt}u &= \Delta_{A(t)}u \\ u_0 &= \mathrm{Id}_E \end{aligned} \tag{6.13}$$

Then the family of connections $u_t^{-1}\overline{A}(t)$ satisfy the Yang-Mills component of the gauge-fixed flow. Using these facts we can prove uniqueness of solutions as follows. Suppose $(g_i(t), A_i(t))$ are two solutions to Ricci Yang-Mills flow with the same initial condition g_0, A_0. One can construct the gauge transformations as above $\phi_i(t), u_i(t)$ with respect to these two solutions. As noted above, the resulting pairs $(\overline{g}_i(t), \overline{A}_i(t)) = \left((\phi_i(t))_* g_i(t), u_i^{-1}(t)A_i(t)\right)$ are both solutions to the gauge-fixed flow with the same initial conditions. Since this is a strictly parabolic system, these solutions are unique. But given these solutions, we may reinterpret $\phi_i(t)$ and $u_i(t)$ as above as solutions to ODE with respect to the (same) pair $(\overline{A}(t), \overline{g}(t))$. Thus $\phi_1(t) = \phi_2(t)$ and $u_1(t) = u_2(t)$ for all time and so the original solutions to Ricci Yang-Mills flow are equal for all time. $\square$

6.2 Derivative Estimates and Existence Obstructions

In this section we will derive derivative estimates for the curvature tensor for solutions to RYM-flow and use them to prove a compactness theorem for such solutions. Our estimates are the basic analogue of the Bernstein-Shi estimates for Ricci flow, and the method we use to prove them is directly analagous to the technique used in [7]. We also point out that although we have stated the results for solutions to RYM flow, they in fact hold for the different renormalizations of RYM flow as well. To see this one simply notes that the proofs below only use that the curvature and its derivatives obey evolution equations with certain orders of nonlinearity. For a solution to a renormalization of RYM flow the curvature obeys an evolution equation with the same nonlinearity as RYM flow, and thus the proof will apply in these cases as well.

Theorem: 6.2. *Let $(E, G(t))$ be a solution to RYM-flow on a compact principal bundle E. For each $\alpha > 0$ and every $m \in \mathbb{N}$ there exists a constant C_m depending only on m, n and $\max\{\alpha, 1\}$ such that if*

$$|\mathrm{Rm}| \leq K \text{ for all } x \in M^n \text{ and } t \in \left[0, \frac{\alpha}{K}\right]$$

then

$$|\nabla^m \mathrm{Rm}| \leq \frac{C_m K}{t^{m/2}} \text{ for all } x \in M^n \text{ and } t \in \left(0, \frac{\alpha}{K}\right]$$

Remark:. We note that one does not actually require that E is compact, instead we merely require that the maximum principle with respect to the time varying metrics $G(t)$ must hold on E. This is true for all of the results in this section.

Proof. We first recall using lemma 2.3 that both ${}^{A}\nabla F$ and F^{*2} are bounded by $|\mathrm{Rm}|$, so that the assumption $|\mathrm{Rm}| \leq K$ implies $\left| {}^{A}\nabla F \right| \leq K$ and $|F|^2 \leq K$. Thus an easy consequence of proposition 5.8 is

$$\begin{aligned} \frac{d}{dt} |\mathrm{Rm}|^2 &= \Delta |\mathrm{Rm}|^2 - 2 |\nabla \mathrm{Rm}|^2 + F * \nabla \mathrm{Rm} * \mathrm{Rm} + \mathrm{Rm}^{*3} \\ &\leq \Delta |\mathrm{Rm}|^2 - 2 |\nabla \mathrm{Rm}|^2 + C |\nabla \mathrm{Rm}| |F| |\mathrm{Rm}| + C' |\mathrm{Rm}|^3 \\ &\leq \Delta |\mathrm{Rm}|^2 - |\nabla \mathrm{Rm}|^2 + C'' |\mathrm{Rm}|^3 \end{aligned} \tag{6.14}$$

So let us consider the case $m = 1$. Proposition 5.9 gives

$$\begin{aligned} \frac{d}{dt} |\nabla \mathrm{Rm}|^2 &= \Delta |\nabla \mathrm{Rm}|^2 - 2 \left|\nabla^2 \mathrm{Rm}\right|^2 + F * \nabla^2 \mathrm{Rm} * \nabla \mathrm{Rm} \\ &\quad + \mathrm{Rm} * \nabla \mathrm{Rm}^{*2} + F * \mathrm{Rm} * \mathrm{Rm} * \nabla \mathrm{Rm} \\ &\leq \Delta |\nabla \mathrm{Rm}|^2 - \left|\nabla^2 \mathrm{Rm}\right|^2 + c_1 |\mathrm{Rm}| |\nabla \mathrm{Rm}|^2 + c_2 |\mathrm{Rm}|^4 \end{aligned} \tag{6.15}$$

We now define

$$Z(x,t) := t |\nabla \mathrm{Rm}|^2 + \beta |\mathrm{Rm}|^2 \tag{6.16}$$

where β is a constant which will be determined below. Using 6.14 and 6.15 we get the estimate

$$\frac{d}{dt} Z \leq \Delta Z + (1 + c_1 t |\mathrm{Rm}| - \beta) |\nabla \mathrm{Rm}|^2 + c_2 t |\mathrm{Rm}|^4 + C_3 \beta |\mathrm{Rm}|^3$$

Now, using that $|\mathrm{Rm}| \leq K$ for all $t \in [0, \alpha/K]$ we have

$$\frac{d}{dt} Z \leq \Delta Z + (1 + c_1 \alpha - \beta) |\nabla \mathrm{Rm}|^2 + (c_2 \alpha + C'' \beta) K^3$$

Choose $\beta \geq (1 + c_1 \alpha)$, and we then have

$$\frac{d}{dt} Z \leq \Delta Z + c_4 (\beta + \alpha) K^3$$

Thus using the parabolic maximum principle we conclude

$$\sup_{x\in M} Z(x,t) \le \beta K^2 + c_4\left(\alpha+\beta\right)K^3 t \le \left(1 + c_4(\alpha+\beta)\alpha\right)K^2 \le C_1^2 K^2$$

where C_1 depends only on n and $\max\{\alpha, 1\}$. Thus

$$|\nabla \operatorname{Rm}| \le \sqrt{\frac{Z}{t}} \le \frac{C_1 K}{t^{1/2}} \qquad \text{for } 0 < t \le \frac{\alpha}{K}$$

Now we proceed by induction, assuming we have the required estimate on $|\nabla^j \operatorname{Rm}|$ for all $1 \le j < m$. We first make a basic estimate on the evolution of $\left|\nabla^k \operatorname{Rm}\right|^2, 1 \le k \le m$ using proposition 5.9.

$$\begin{aligned}
\frac{d}{dt}\left|\nabla^k \operatorname{Rm}\right|^2 &= \Delta\left|\nabla^k \operatorname{Rm}\right|^2 - 2\left|\nabla^{k+1}\operatorname{Rm}\right|^2 + F * \nabla^{k+1}\operatorname{Rm} * \nabla^k \operatorname{Rm} \\
&\quad + \sum_{j=0}^{k} \nabla^j \operatorname{Rm} * \nabla^{k-j}\operatorname{Rm} * \nabla^k \operatorname{Rm} \\
&\le \Delta\left|\nabla^k \operatorname{Rm}\right|^2 - \left|\nabla^{k+1}\operatorname{Rm}\right|^2 \\
&\quad + C\sum_{j=0}^{k} \left|\nabla^j \operatorname{Rm}\right|\left|\nabla^{k-j}\operatorname{Rm}\right|\left|\nabla^k \operatorname{Rm}\right| \qquad (6.17)\\
&\le \Delta\left|\nabla^k \operatorname{Rm}\right|^2 - \left|\nabla^{k+1}\operatorname{Rm}\right|^2 \\
&\quad + C\left(K\left|\nabla^k \operatorname{Rm}\right|^2 + \frac{K^2}{t^{k/2}}\left|\nabla^k \operatorname{Rm}\right|\right) \\
&\le \Delta\left|\nabla^k \operatorname{Rm}\right|^2 - \left|\nabla^{k+1}\operatorname{Rm}\right|^2 + CK\left(\left|\nabla^k \operatorname{Rm}\right|^2 + \frac{K^2}{t^k}\right)
\end{aligned}$$

where in the second to last line we used the inductive hypothesis and the estimate $|\operatorname{Rm}| \le K$. This estimate is exactly of the form given in [7] line 7.5 pg. 229. The inductive proof now follows precisely as there using this estimate. $\square$

Corollary: 6.3. *Let $(E, G(t))$ be a solution to RYM flow on a compact principal bundle E. If there exist $\beta > 0$ and $K > 0$ so that*

$$|\mathrm{Rm}(x,t)| \leq K \qquad \text{for all } x \in M \text{ and } t \in [0,T] \tag{6.18}$$

where $T > \beta/K$ then for each $m \in \mathbb{N}$ there exists a constant C_m depending only on m and $\min\{\beta, 1\}$ so that

$$|\nabla^m \mathrm{Rm}(x,t)| \leq C_m K^{1+m/2} \text{ for all } x \in M^n \text{ and } t \in \left[\frac{\min\{\beta,1\}}{K}, T\right]$$

Proof. Let $\beta' = \min\{\beta, 1\}$. Fix a time $t_0 \in [\beta'/K, T]$. Translate time so that the flow starts at $t_0 - \beta'/K$. In fact, by changing notation we may assume that $t_0 = \beta'/K$. Note that the curvature is still bounded by K up to time t_0, so applying theorem 6.2 with $\alpha = \beta'$ we get constants C_m depending only on m so that

$$|\nabla^m \mathrm{Rm}(x,t)| \leq \frac{C_m K}{t^{m/2}} \tag{6.19}$$

for all $x \in M$ and $t \in (0, \beta'/K]$. So, for $t \in \left[\frac{\beta'}{2K}, \frac{\beta'}{K}\right]$ we have

$$t^{m/2} \geq \beta'^{m/2} 2^{-m/2} K^{-m/2} \tag{6.20}$$

so that in particular we find

$$|\nabla^m \mathrm{Rm}(x,t_0)| \leq \left(\frac{2^{m/2} C_m}{\beta'^{m/2}}\right) K^{1+m/2} \text{ for all } x \in M \tag{6.21}$$

and since $t_0 \in [\beta'/K, T]$ was arbitrary the result follows. □

Proposition: 6.4. *Let $(E, G(t)$ be a solution to RYM flow on a compact manifold, and fix a background metric $\overline{G}$ and connection $\overline{\nabla}$. If there exists $K > 0$ so that*

$$|\mathrm{Rm}(x,t)|_G \leq K \text{ for all } x \in E \text{ and } t \in [0,T) \tag{6.22}$$

then for every $m \in \mathbb{N}$ there exists a constant C_m depending on m, K, T, G_0 and $(\overline{G}, \overline{\nabla})$ so that

$$\left|\overline{\nabla}^m G(x,t)\right|_{\overline{G}} \leq C_m \text{ for all } x \in E \text{ and } t \in [0,T) \tag{6.23}$$

Proof. Since E is compact and the metric $\overline{G}$ and connection $\overline{\nabla}$ are fixed it suffices to show the bound

$$|\partial^m G(x,t)| \leq C_m \tag{6.24}$$

in a coordinate chart. First we need the following lemma ([7] lemma 6.49).

Lemma: 6.5. *Let M^n be a closed manifold. For $0 \leq t < T \leq \infty$ let $g(t)$ be a smooth one-parameter family of metrics on M^n. If there exists a constant $C < \infty$ so that*

$$\int_0^T \left| \frac{d}{dt} g(x,t) \right|_{g(t)} dt \leq C \tag{6.25}$$

for all $x \in M$ then

$$e^{-C} g(x,0) \leq g(x,t) \leq e^C g(x,0) \tag{6.26}$$

for all $x \in M$ and $t \in [0,T)$. Furthermore, as $t \nearrow T$ the metrics $g(t)$ converge uniformly to a continuous metric $g(T)$ such that for all $x \in M$

$$e^{-C} g(x,0) \leq g(T) \leq e^C g(x,0) \tag{6.27}$$

Using this and our assumption of bounded curvature it is clear that we have uniform pointwise upper and lower bounds for the metric $G(t)$ on $[0,T)$. To estimate the first derivatives of the metric we write

$$\begin{aligned} \left| \frac{d}{dt} \left(\frac{\partial}{\partial x^i} G_{jk} \right) \right| &= |\partial \operatorname{Rm}| \\ &= |\nabla \operatorname{Rm} + \Gamma * \operatorname{Rm}| \\ &\leq |\nabla \operatorname{Rm}| + CK\,|\Gamma| \end{aligned} \tag{6.28}$$

Using our assumed curvature bound. Next, using corollary 6.3 we see that the covariant derivatives of the curvature are bounded (here the constant depends on T). Thus we may bound

$$\left| \frac{d}{dt} \Gamma \right| = |\nabla \operatorname{Rm}| \leq C \tag{6.29}$$

so that we can conclude that Γ is bounded on our finite time interval. Using these facts in (6.28) clearly gives a bound on the time derivative of $\frac{\partial}{\partial x^i} G_{jk}$ and thus we can conclude a bound on this derivative on $[0, T)$. All of the higher-order derivative bounds are similar, and the result follows. $\square$

Proposition: 6.6. *Let G_0 be a bundle metric on $E \to M$ a principal bundle. Then Ricci Yang-Mills flow (and its normalizations) has a unique solution $G(t)$ on a maximal time interval $0 \leq t < T \leq \infty$. Moreover, if $T < \infty$ then*

$$\lim_{t \nearrow T} \left(\sup_{x \in E} |\mathrm{Rm}(x,t)| \right) = \infty \tag{6.30}$$

Proof. First we will show that

$$\lim_{t \nearrow T} \sup \sup_{x \in E} |\mathrm{Rm}(x,t)| = \infty \tag{6.31}$$

First by proposition 6.1 we know that the solution exists for a short time. Suppose there exists $K < \infty$ so that $\sup_{t \in [0,T)} \sup_{x \in E} |\mathrm{Rm}(x,t)| \leq K$. Using lemma 6.5 we have uniform convergence to a continuous metric $G(T)$. Using proposition 6.4 we have uniform bounds on all of the coordinate derivatives of G. This allows us to conclude that $G(t) \to G(T)$ in any C^m norm. Since $G(T)$ is now smooth, we can apply proposition 6.1 to conclude that a unique solution to Ricci Yang-Mills flow exists with initial condition $G(T)$, which contradicts maximality of T.

Now we must show that in fact not just the lim sup but in fact the supremum of curvature goes to infinity. Suppose not, then there exists $K_0 < \infty$ and a sequence of times $t_i \nearrow T$ so that $\sup_{x \in E} |\mathrm{Rm}(x, t_i)| \leq K_0$. In the next proposition we will show the existence of a universal constant c so that

$$\sup_{x \in E} |\mathrm{Rm}(x,t)| \leq 2 \sup_{x \in E} |\mathrm{Rm}(x, t_i)| \leq 2K_0 \tag{6.32}$$

for any time t satisfying $t_i \leq t \leq \{T, t_i + \frac{c}{K_0}\}$. Since $t_i \nearrow T$ it is clear that for i large enough $t_i + \frac{c}{K_0} \geq T$, so that

$$\sup_{t_i \leq t < T} \sup_{x \in E} |\mathrm{Rm}(x,t)| \leq 2K_0 \tag{6.33}$$

which contradicts the claim above, and so the proposition follows. □

Proposition: 6.7. *There exists $c > 0$ depending only on the dimension of the total space E so that if $(E, G(t)), t \in [0,T]$ is a solution to Ricci Yang-Mills flow on a compact manifold and*

$$M(t) := \sup_{x \in E} |\mathrm{Rm}(x,t)|_{G(x,t)}$$

then

$$M(t) \leq 2M(0) \text{ for all } 0 \leq t \leq \min\left\{T, \frac{c}{M(0)}\right\}$$

Proof. A simple consequence of proposition 5.8 is that

$$\begin{aligned}\frac{d}{dt}|\mathrm{Rm}| &= \Delta |\mathrm{Rm}|^2 - 2|\nabla \mathrm{Rm}|^2 + F * \nabla \mathrm{Rm} * \mathrm{Rm} + \mathrm{Rm}^{*3} \\ &\leq \Delta |\mathrm{Rm}|^2 - |\nabla \mathrm{Rm}|^2 + C|\mathrm{Rm}|^3\end{aligned}$$

where the constant C depends only on dimension. Thus by the maximum principle $M(t)$ satisfies

$$\frac{dM}{dt} \leq \frac{CM^3}{2M} = \frac{C}{2}M^2$$

This implies that

$$M(t) \leq \frac{1}{\frac{1}{M(0)} - \frac{C}{2}t} \tag{6.34}$$

Thus $M(t) \leq 2M(0)$ as long as $0 \leq t \leq \min\{T, c/M(0)\}$. □

6.3 Compactness of Ricci Yang-Mills Flow Solutions

In this section we will prove a compactness result for solutions to Ricci Yang-Mills flow. Our proof is a straightforward modification of the case of regular Ricci flow. We use our Bernstein-Bando-Shi type estimates from theorem 6.2 to reduce to the case where one has a uniform bound on curvatures and their derivatives, and then apply a convergence result from [15]. We start with the basic definitions of evolving Riemannian manifolds and a convergent sequence of such.

Definition: 6.8. Given M a smooth manifold and $g(t)$ a smooth 1-parameter family of complete metrics on M, $p \in M$ and F a local frame centered at p, we call the grouping $(M, g(t), p, F)$ an *evolving complete marked Riemannian manifold.* If the metric does not vary this is simply a *complete marked Riemannian manifold.*

Definition: 6.9. We say that a sequence $M_i = \{M_i, g_i(t), p_i, F_i\}$ of evolving complete marked Riemannian manifolds converges to the evolving complete marked Riemannian manifold $M = \{M, g(t), p, F\}$ if there exists a sequence of open sets U_i in M containing p and a sequence of diffeomorphisms ϕ_i mapping U_i to $V_i \subset M_i$ and mapping p to p_i and F to F_i such that any compact set in M eventually lies in all U_i and the pullbacks $\phi^*(g_i)$ converge to g uniformly on compact sets in $M \times (\alpha, \omega)$ together with all of their derivatives.

Theorem: 6.10. Compactness Theorem *Let*

$$\{E_i, G_i(t), p_i, F_i : i \in \mathbb{N}\}$$

be a sequence of connected complete solutions to Ricci Yang-Mills flow existing for $t \in (\alpha, \omega)$ *where* $-\infty \leq \alpha < 0 < \omega \leq \infty$*. Each solution has a fixed origin* $p_i \in M_i$

and a frame F_i at p_i which is orthonormal with respect to $G_i(0)$. Suppose there exists $K < \infty$ such that

$$\sup_{E_i \times (\alpha,\omega)} |\mathrm{Rm}| \leq K$$

and $\delta > 0$ such that:

$$\mathrm{inj}_{G_i(0)}(p_i) \geq \delta \textit{ for all } i \in \mathbb{N}$$

Then there exists a subsequence which converges in the pointed category to a complete solution

$$\{E_\infty, G_\infty(t), p_\infty, F_\infty\}$$

to Ricci Yang-Mills flow on (α, ω) with the same bounds on curvature and injectivity radius.

Proof. First of all by a diagonalization argument it suffices to prove the case where $\alpha > -\infty$ and $\omega < \infty$. So, for fixed $\epsilon > 0$, using our assumption of uniformly bounded curvature, theorem 6.2 gives us uniform C^0 bounds on covariant derivatives of Rm of arbitrarily high order on $(\alpha + \epsilon, \omega)$. So if we can prove the theorem for sequences with uniform C^k bounds on curvature then we can do another diagonalization as $\epsilon \to 0$ to conclude the theorem. Thus we make this assumption.

Fix a sequence (E_i, G_i, p_i, F_i) of complete marked solutions to Ricci Yang-Mills flow satisfying the hypotheses of the theorem as well as uniform C^k bounds on curvature. We will use the following theorem ([15] Theorem 2.3).

Theorem: 6.11. *Given any sequence $M_i = (M_i, g_i, p_i, F_i)$ of complete marked Riemannian manifolds such that there exist constants $C_m, m \geq 0$ independent of i so that*

$$|{}^{g_i}\nabla_m \mathrm{rm}(g_i)| \leq C_m$$

and $\delta > 0$ independent of i so that

$$\text{inj}_{g_i(t)}(p_i) \geq \delta$$

there exists a convergent subsequence.

In particular, this shows that the slices at time 0 contain a subsequence which converges to a metric G on a manifold (E, p). Recall that this convergence means that there exist a sequence of open sets $\{U_i\}$ in M containing p and a sequence of maps $F_i : U_i \to V_i \subset M_i$ such that $\widetilde{G}_i := F_i^*(g_i)$ converges uniformly in C^∞ to G. Note that these pullbacks $\widetilde{G}_i$ still have uniform bounds on all derivatives of curvatures and are defined on (α, ω), whereas G is only defined at time 0. We now want to show that we have uniform bounds on the covariant derivatives of $\widetilde{G}_i$ taken with respect to the metric G to conclude the existence of a convergent subsequence on this new manifold. We sketch the proof here for solutions to Ricci Yang-Mills flow, as it is identical to the proof of lemma 2.4 in [15].

Lemma: 6.12. *Let (E, G) be a principal bundle with bundle metric, K a compact subset of E and G_i a collection of solutions Ricci Yang-Mills flow defined on neighborhoods of $K \times [a, b]$ where $0 \in [a, b]$. Let ∇ denote the covariant derivative with respect to G and ∇_i denote the covariant derivative with respect to G_i.*

Suppose that the metrics G_i are uniformly equivalent to G at $t = 0$ on K and that the covariant derivatives of G_i with respect to G are uniformly bounded at $t = 0$ on K. Finally assume that the covariant derivatives of the curvatures of Rm_i with respect to G_i are uniformly bounded on $K \times [a, b]$. Then the metrics G_i are uniformly equivalent to G on $K \times [a, b]$, and the covariant derivatives of G_i with respect to G are uniformly bounded on $K \times [a, b]$.

Proof. We have assumed the bound

$$cG(V,V) \leq G_i(V,V) \leq CG(V,V)$$

at time $t = 0$. Using the equation

$$\frac{d}{dt} G_i(V,V) = -2\left(\mathrm{Rc}_i^H + \mathrm{Rc}_i^C\right)(V,V)$$

and the bound

$$\left|\mathrm{Rc}_i(V,V)\right| \leq C\left|G_i(V,V)\right|$$

we get

$$\left|\frac{d}{dt}\ln G_i(V,V)\right| \leq C$$

which allows us to extend the bound on $\ln G_i(V,V)$ at $t = 0$ to the finite time interval $[a,b]$. The derivative bounds are the same, simply bounding the time derivative using our assumption of uniform bounds on all derivatives of curvature. See [15] pg. 550 for more detail. □

Applying this lemma to the sequence of metrics $\widetilde{G}_i$ above it is clear that the metric G is defined on $[a,b]$ and is a solution to Ricci Yang-Mills flow. In particular it is clear that the infinitesimal isometries of the $\mathcal{K}$-action are preserved under this convergence process, so that the limiting space still retains the structure of a principal $\mathcal{K}$-bundle. □

Chapter 7

Convergence Theorem

7.1 Estimating the Volume-Normalized Equation for $n = 4$

Fix $n = 4$. Consider the tensor $T = \mathrm{Rc}^H + \mathrm{Rc}^C - \frac{1}{4} r G^H + \frac{1}{8} |F|^2 G^H$. Note that in this dimension $V = T + \frac{1}{4}\delta r$. We begin by computing the evolution of $|T|^2$. We will denote the 1-form that represents T^C, i.e. $\frac{1}{2} d^* F$, by ω. This is to avoid any confusion about taking the inner products of one forms or symmetric matrices. In particular note $\left|T^C\right|^2 = 2|\omega|^2$. We will make use the following convenient shorthand notation.

Definition: 7.1. In the calculations below, the letter Q will refer to any universal linear expression using the tensors rm and ${}^A\nabla F$. We will refer to both of these as *terms of low order*

We will often use the fact that

$$\overset{\circ}{\eta} = 2\left(\overset{\circ}{\mathrm{rc}} - T\right) = Q - 2T$$

Proposition: 7.2. *A solution to volume renormalized RYM flow with $n = 4$ satisfies*

$$\begin{aligned}\frac{d}{dt}\left|d^*F\right|^2 = \Delta\left|d^*F\right|^2 - 2\left|\ {}^A\nabla d^*F\right|^2 \\ + 4\left\langle\ {}^g\nabla_j T_{li}\ {}_\theta F^{jl}, d^*F_i^\theta\right\rangle + \frac{4}{n}\left\langle\ {}^g\nabla_j r\ {}_\theta F_i^j, d^*F_i^\theta\right\rangle \\ + 2\left(\ {}_\theta F\cdot\ {}^\rho F\cdot d^*{}_\rho F\right)_i d^*F_i^\theta + Q * T^{*2} + V * T^{*2}\end{aligned} \tag{7.1}$$

Proof. We start with the calculation

$$\begin{aligned}\frac{d}{dt}\left|d^*F\right|^2 &= \frac{d}{dt} g^{ij} d^*F_i d^*F_j \\ &= -g^{im}\dot{g}_{mn}g^{nj}d^*F_i d^*F_j + 2\left\langle\frac{d}{dt}d^*F, d^*F\right\rangle \\ &= V * T^{*2} + 2\left\langle\frac{d}{dt}d^*F, d^*F\right\rangle\end{aligned}$$

Now using proposition 5.22 with $n = 4$ and the Bochner formula we see

$$\begin{aligned}2\left\langle\frac{d}{dt}d^*F, d^*F\right\rangle &= 2\left\langle\Delta_d d^*F_i, d^*F_i\right\rangle + 4\left\langle\ {}^g\nabla_j V_{li}\ {}_\theta F^{jl}, d^*F_i^\theta\right\rangle \\ &\quad + 2\left(\ {}_\theta F\cdot\ {}_\rho F\cdot d^*F^\rho\right)_i d^*F_i^\theta + Q * T^{*2} \\ &= 2\left\langle\Delta d^*F, d^*F\right\rangle + 4\left\langle\ {}^g\nabla_j V_{li}\ {}_\theta F^{jl}, d^*F_i^\theta\right\rangle \\ &\quad + 2\left(\ {}_\theta F\cdot\ {}_\rho F\cdot d^*F^\rho\right)_i d^*F_i^\theta + Q * T^{*2} \\ &= \Delta\left|d^*F\right|^2 - 2\left|\ {}^A\nabla d^*F\right|^2 \\ &\quad + 4\left\langle\ {}^g\nabla_j T_{li}\ {}_\theta F^{jl}, d^*F_i^\theta\right\rangle + \frac{4}{n}\left\langle\ {}^g\nabla_j r\ {}_\theta F_i^j, d^*F_i^\theta\right\rangle \\ &\quad + 2\left(\ {}_\theta F\cdot\ {}^\rho F\cdot d^*F^\rho\right)_i d^*F_i^\theta + Q * T^{*2}\end{aligned}$$

Plugging this into the above calculation gives the result. □

Proposition: 7.3. *A solution to volume-normalized Ricci Yang-Mills flow with $n = 4$ satisfies*

$$\begin{aligned}\frac{d}{dt}|T|^2 = {} & \Delta|T|^2 - 2|\nabla T|^2 + 3\operatorname{tr}\left(FT^H FT^H\right) \\ & + 4\left\langle {}^{\theta}F \cdot {}^{\rho}F \cdot \omega_\rho, \omega_\theta \right\rangle - 2\left\langle {}^{\theta}F \cdot {}^{\rho}F \cdot \omega_\theta, \omega_\rho \right\rangle \\ & + \frac{1}{2}|F|^2|\omega|^2 - \left\langle {}^{\theta}F, {}^{\rho}F \right\rangle \langle \omega_\theta, \omega_\rho \rangle \\ & - \left\langle g^{lm}\left({}^{g}\nabla_l r\right) F^{\theta}_{im}, \omega_{i\theta} \right\rangle + \delta r * \mathrm{rm} * T + Q * T^{*2} + T^{*3}\end{aligned} \tag{7.2}$$

Proof. We start with the calculation

$$\begin{aligned}\frac{d}{dt}|T|^2 &= \frac{d}{dt} G^{\alpha\delta} G^{\beta\epsilon} T_{\alpha\beta} T_{\delta\epsilon} \\ &= -2G^{\alpha\mu}\dot{G}_{\mu\nu}G^{\nu\delta}G^{\beta\epsilon}T_{\alpha\beta}T_{\delta\epsilon} + 2\left\langle \dot{T}, T \right\rangle \\ &= 4V^{\alpha\delta}G^{\beta\epsilon}T_{\alpha\beta}T_{\delta\epsilon} + 2\left\langle \dot{T}, T \right\rangle \\ &= T^{*3} + Q * T^{*2} + 2\left\langle \dot{T}, T \right\rangle\end{aligned} \tag{7.3}$$

Now we calculate using the fact that $T = S - \frac{1}{4}rG^H$. Thus using proposition 5.21 and the fact that T is traceless we see that

$$\begin{aligned}2\left\langle \dot{T}, T \right\rangle &= 2\left\langle \frac{d}{dt}\left(S - \frac{1}{4}rG^H\right), T \right\rangle \\ &= 2\left\langle \Delta_L V - \frac{1}{4}|F|^2 V^H + \frac{1}{2}rV^H, T \right\rangle \\ &= 2\langle \Delta_L V, T \rangle - \frac{1}{2}|F|^2 \left|T^H\right|^2 \\ &\quad + Q * T^{*2}\end{aligned} \tag{7.4}$$

First we simplify

$$
\begin{aligned}
2\left\langle \Delta_L V, T\right\rangle &= 2\left\langle \Delta V + \mathcal{R}(V), T\right\rangle \\
&= 2\left\langle \Delta\left(T + \frac{1}{4}(\delta r)G^H\right), T\right\rangle \\
&\quad + 2\left\langle \mathcal{R}\left(T + \frac{1}{4}(\delta r)G^H\right), T\right\rangle \qquad (7.5)\\
&= \Delta\left|T\right|^2 - 2\left|\nabla T\right|^2 + \frac{1}{2}\left\langle \Delta(\delta r G^H), T\right\rangle \\
&\quad + 2\left\langle \mathcal{R}(T), T\right\rangle + \frac{1}{2}\left\langle \mathcal{R}((\delta r)G^H), T\right\rangle
\end{aligned}
$$

So, plugging 7.4 - 7.5 into 7.3 gives

$$
\begin{aligned}
\frac{d}{dt}\left|T\right|^2 = \Delta\left|T\right|^2 - 2\left|\nabla T\right|^2 + 2\left\langle \mathcal{R}(T), T\right\rangle - \frac{1}{2}\left|F\right|^2\left|T^H\right|^2 \\
+ \frac{1}{2}\left\langle \Delta\left(\delta r G^H\right), T\right\rangle + \frac{1}{2}\left\langle \mathcal{R}\left(\delta r G^H\right), T\right\rangle \qquad (7.6)\\
+ Q * T^{*2} + T^{*3}
\end{aligned}
$$

Now, using theorem B.3 we compute that

$$
\begin{aligned}
\frac{1}{2}\left\langle \Delta\left(\delta r G^H\right), T\right\rangle &= \frac{1}{2}\left\langle {}^g\Delta\left(\delta r G^H\right) - \frac{1}{2}(\delta r)\eta, T^H\right\rangle \\
&\quad + \left\langle 2\delta r(\omega_\theta) - {}^g\nabla_l\left(\delta r G^H\right)_{in}\ {}_\theta F^{nl}, \omega_\theta\right\rangle \\
&= -\frac{1}{4}\delta r\left\langle \mathring{\eta}, T^H\right\rangle - \left\langle g^{lm}({}^g\nabla_l r)F^\theta_{im}, \omega_{i\theta}\right\rangle + Q * T^{*2} \\
&= -\left\langle g^{lm}({}^g\nabla_l r)F^\theta_{im}, \omega_{i\theta}\right\rangle + \delta r * \mathrm{rm} * T + Q * T^{*2}
\end{aligned}
$$

Similarly using lemma 2.3 we have

$$
\begin{aligned}
\frac{1}{2}\left\langle \mathcal{R}((\delta r)G^H), T\right\rangle &= -\frac{1}{4}\delta r\left\langle \mathring{\eta}, T^H\right\rangle \\
&= \delta r * \mathrm{rm} * T + Q * T^{*2}
\end{aligned}
$$

Next we break $2\langle \mathcal{R}(T), T\rangle$ into its parts:

$$2\langle \mathcal{R}(T), T\rangle = 2\langle 2R(T) - \mathrm{Rc}\cdot T - T\cdot \mathrm{Rc}, T\rangle$$

Now using lemma 2.3 we compute

$$\begin{aligned} 4\langle R(T), T\rangle &= 4R_{ijkl}T_{il}T_{jk} + 4R_{i\theta k\rho}T_{i\rho}T_{\theta k} + 4R_{\theta j\rho l}T_{\theta l}T_{j\rho} \\ &\quad + 4R_{ij\theta\rho}T_{i\rho}T_{j\theta} + 4R_{\theta\rho kl}T_{\theta l}T_{\rho k} \\ &= 3\,\mathrm{tr}\left({}^{\theta}F\cdot T^H \cdot {}_{\theta}F\cdot T^H\right) + 4\,{}^{\theta}F_m^k\,{}^{\rho}F_i^m\omega_{i\rho}\omega_{k\theta} \\ &\quad - 2\,{}^{\theta}F_{im}\,{}^{\rho}F_j^m\omega_{j\theta}\omega_{i\rho} + Q * T^{*2} \\ &= 3\,\mathrm{tr}(FT^HFT^H) + 4\left\langle {}^{\theta}F\cdot {}^{\rho}F\cdot\omega_{\rho}, \omega_{\theta}\right\rangle \\ &\quad - 2\left\langle {}^{\theta}F\cdot {}^{\rho}F\cdot\omega_{\theta}, \omega_{\rho}\right\rangle + Q * T^{*2} \end{aligned}$$

Next using lemma 2.3 we simplify

$$\begin{aligned} -2\langle \mathrm{Rc}\cdot T, T\rangle = &- 2\,\mathrm{Rc}_i^j\,T_{jk}T_{ik} - 2\,\mathrm{Rc}_i^j\,T_{j\theta}T_{\theta}^i \\ &- 2\,\mathrm{Rc}_{i\theta}\,T_{\theta j}T_{ij} - 2\,\mathrm{Rc}_{\theta i}\,T_{ij}T_{\theta}^j - 2\,\mathrm{Rc}_{\theta\rho}\,T_i^{\theta}T_{\rho i} \\ = &\left\langle \eta\cdot T^H, T^H\right\rangle + \eta\cdot\omega^{\theta}\cdot\omega_{\theta} - \frac{1}{2}\left\langle {}^{\theta}F, {}^{\rho}F\right\rangle\langle\omega_{\theta},\omega_{\rho}\rangle + Q * T^{*2} + T^{*3} \\ = &\frac{1}{4}|F|^2\left|T^H\right|^2 + \frac{1}{4}|F|^2|\omega|^2 - \frac{1}{2}\left\langle {}^{\theta}F, {}^{\rho}F\right\rangle\langle\omega_{\theta},\omega_{\rho}\rangle + Q * T^{*2} + T^{*3} \end{aligned}$$

The inner product $-2\langle T\cdot \mathrm{Rc}, T\rangle$ is the same since all the matrices involved are symmetric. Plugging the above simplifications into 7.6 gives the result. $\square$

Because of the presence of the $\nabla r\cdot F\cdot\omega$ term in the evolution of T it is necessary to study the evolution of the following slightly modified quantity

$$Z_c := |T|^2 - c\,|\omega|^2 \tag{7.7}$$

where $c < 2$. Clearly these quantities bound $|T|^2$ up to a universal constant depending on c.

Proposition: 7.4. *A solution to volume-renormalized Ricci Yang-Mills flow with $n = 4$ satisfies*

$$\frac{d}{dt} Z_c \leq \Delta Z_c + L_0(T^H) + L_1(T^C) + \delta r * \mathrm{rm} * T + Q * T^{*2} + T^{*3} \tag{7.8}$$

where

$$\begin{aligned} L_0(T^H) = & \operatorname{tr}\left({}^{\theta}F \cdot {}_{\theta}F \cdot T^H \cdot T^H + 2\, {}^{\theta}F \cdot T^H \cdot {}_{\theta}F \cdot T^H \right) + \left(\frac{c}{4(2-c)} \right) |F|^2 \left| T^H \right|^2 \\ L_1(\omega) = & -\left\langle {}^{\theta}F, {}^{\rho}F \right\rangle \langle \omega_\theta, \omega_\rho \rangle + \frac{c^2 - 4c}{4} \left| {}^{\theta}F_{ij}\omega_{\theta k} \right|^2 \\ & + 3\left(-1 + c - \frac{1}{4}c^2 \right) \left\langle {}^{\theta}F \cdot {}^{\rho}F \cdot \omega_\rho, \omega_\theta \right\rangle \\ & + \left(-1 + c - \frac{c^2}{4} \right) \left\langle {}^{\rho}F \cdot {}^{\theta}F \cdot \omega_\rho, \omega_\theta \right\rangle \end{aligned} \tag{7.9}$$

Proof. To compute the evolution of Z_c, first let us reinterpret the result of proposition 7.2 using $\omega = \frac{1}{2} d^* F$. It reads

$$\begin{aligned} \frac{d}{dt} |\omega|^2 = & \Delta |\omega|^2 - 2 \left| {}^{A}\nabla \omega \right|^2 \\ & + 2 \left\langle {}^{g}\nabla_j T_{li}\, {}_{\theta}F^{jl}, \omega_{\theta i} \right\rangle + \frac{1}{2} \left\langle {}^{g}\nabla_j r\, {}_{\theta}F_i^j, \omega_{\theta i} \right\rangle \\ & + 2 \left\langle {}^{\theta}F \cdot {}^{\rho}F \cdot \omega_\rho, \omega_\theta \right\rangle + Q * T^{*2} + T^{*3} \end{aligned} \tag{7.10}$$

Combining this with the result of proposition 7.3 gives

$$
\begin{aligned}
\frac{d}{dt}Z &= \Delta\left|T\right|^2 - 2\left|\nabla T\right|^2 - c\Delta\left|\omega\right|^2 + 2c\left|\ {}^{A}\nabla\omega\right|^2 \\
&\quad + 3\,\mathrm{tr}\left(FT^H FT^H\right) + (4-2c)\left\langle\ {}^{\theta}F\cdot\ {}^{\rho}F\cdot\omega_\rho,\omega_\theta\right\rangle \\
&\quad - 2\left\langle\ {}^{\theta}F\cdot\ {}^{\rho}F\cdot\omega_\theta,\omega_\rho\right\rangle + \frac{1}{2}\left|F\right|^2\left|\omega\right|^2 - \left\langle\ {}^{\theta}F,\ {}^{\rho}F\right\rangle\left\langle\omega_\theta,\omega_\rho\right\rangle \\
&\quad + \left(\frac{c}{2}-1\right)\left\langle g^{lm}\left(\ {}^{g}\nabla_l r\right)\ {}_{\theta}F_{im},\omega_{i\theta}\right\rangle - 2c\left\langle\ {}^{g}\nabla_j T_{li}\ {}_{\theta}F^{jl},\omega_{i\theta}\right\rangle \\
&\quad + Q*T^{*2} + V*T^{*2} \\
&= \Delta Z - 2\left|\nabla T\right|^2 + 2c\left|\ {}^{A}\nabla\omega\right|^2 + 3\,\mathrm{tr}\left(FT^H FT^H\right) \\
&\quad + (4-2c)\left\langle\ {}^{\theta}F\cdot\ {}^{\rho}F\cdot\omega_\rho,\omega_\theta\right\rangle - 2\left\langle\ {}^{\theta}F\cdot\ {}^{\rho}F\cdot\omega_\theta,\omega_\rho\right\rangle \\
&\quad + \frac{1}{2}\left|F\right|^2\left|\omega\right|^2 - \left\langle\ {}^{\theta}F,\ {}^{\rho}F\right\rangle\left\langle\omega_\theta,\omega_\rho\right\rangle \\
&\quad + \left(\frac{c}{2}-1\right)\left\langle g^{lm}\ {}^{g}\nabla_l r\ {}_{\theta}F_{im},\omega_{\theta i}\right\rangle - 2c\left\langle\ {}^{g}\nabla_j T_{li}\ {}_{\theta}F^{jl},\omega_{i\theta}\right\rangle \\
&\quad + Q*T^{*2} + T^{*3}
\end{aligned}
\tag{7.11}
$$

First we note using lemma B.1 that

$$
3\,\mathrm{tr}\left(\ {}^{\theta}FT^H\ {}_{\theta}FT^H\right) - 2\left|\nabla_\theta T_{ij}\right|^2 = \mathrm{tr}\left(\ {}^{\theta}F\ {}_{\theta}FT^H T^H + 2\ {}^{\theta}FT^H\ {}_{\theta}FT^H\right) \tag{7.12}
$$

Now using parts of $-2\left|\nabla T\right|^2$ we make the bound

$$
\begin{aligned}
2c\left|\ {}^{A}\nabla\omega\right|^2 &- 2\left|\nabla_i T_{\theta j}\right|^2 - 2\left|\nabla_i T_{j\theta}\right|^2 \\
&= 2c\left|\ {}^{g}\nabla\omega\right|^2 - 4\left|\ {}^{A}\nabla_i\omega_{\theta j} + \frac{1}{2}\left(\ {}_{\theta}FT^H\right)_{ik}\right|^2 \\
&= 2c\left|\ {}^{A}\nabla\omega\right|^2 - 4\left|\ {}^{A}\nabla\omega\right|^2 - 4\left\langle\ {}^{A}\nabla\omega_\theta,\ {}_{\theta}FT^H\right\rangle - \operatorname{Tr}\left|F\cdot T^H\right|^2 \\
&\leq (4\epsilon + 2c - 4)\left|\ {}^{A}\nabla\omega\right|^2 + \frac{1}{\epsilon}\operatorname{Tr}\left|F\cdot T^H\right|^2 - \operatorname{Tr}\left|F\cdot T^H\right|^2 \\
&\leq \left(\frac{2c}{4-2c}\right)\operatorname{Tr}\left|F\cdot T^H\right|^2 \\
&= \left(\frac{c}{4(2-c)}\right)\left|F\right|^2\left|T^H\right|^2 + Q * T^{*2} + T^{*3}
\end{aligned}
\tag{7.13}
$$

for $\epsilon = \frac{4-2c}{4}$. Next we will rewrite the the term $-2c\left\langle\ {}^{g}\nabla_j T_{li}\ {}_{\theta}F^{jl}, \omega_\theta\right\rangle$ using part of $-2\left|\nabla T\right|^2$

$$
\begin{aligned}
-2&\left|\ {}^{g}\nabla_i T^H_{jk} - \frac{1}{2}\left(\ {}^{\theta}F_{ij}\omega_{\theta k} +\ {}^{\rho}F_{ik}\omega_{\rho j}\right)\right|^2 - 2c\left\langle\ {}^{g}\nabla_j T_{li}\ {}_{\theta}F^{jl}, \omega_{\theta i}\right\rangle \\
&= -2\left|\ {}^{g}\nabla_i T_{jk}\right|^2 + 2\left\langle\ {}^{g}\nabla_i T_{jk},\ {}^{\theta}F_{ij}\omega_{\theta k} +\ {}^{\rho}F_{ik}\omega_{\rho j}\right\rangle - \frac{1}{2}\left|\ {}^{\theta}F_{ij}\omega_{\theta k} +\ {}^{\rho}F_{ik}\omega_{\rho j}\right|^2 \\
&\quad - 2c\left\langle\ {}^{g}\nabla_j T_{li}\ {}_{\theta}F^{jl}, \omega_{\theta i}\right\rangle \\
&= -2\left|\ {}^{g}\nabla_i T_{jk} - \frac{2-c}{4}\left(\ {}^{\theta}F_{ij}\omega_{\theta k} +\ {}^{\rho}F_{ik}\omega_{\rho j}\right)\right|^2 + \frac{c^2-4c}{8}\left|\ {}^{\theta}F_{ij}\omega_{\theta k} +\ {}^{\rho}F_{ik}\omega_{\rho j}\right|^2
\end{aligned}
\tag{7.14}
$$

Now we want to bound the $\nabla r\cdot F\cdot\omega$ term using the first term above. In particular we will only use the trace component of that inner product, and use the equation

${}^{g}\operatorname{div} T^H = \frac{1}{4}\nabla r + {}^{\theta}F_{lk}\omega_\theta^l$ to estimate

$$
\begin{aligned}
&-2\left| {}^{g}\nabla_i T_{jk} - \frac{2-c}{4}\left({}^{\theta}F_{ij}\omega_{\theta k} + {}^{\rho}F_{ik}\omega_{\rho j}\right)\right|^2 + \left(\frac{c}{2}-1\right)\left\langle {}^{g}\nabla_l r,\ {}^{\theta}F_l^i\omega_{\theta i}\right\rangle \\
&\leq\ -2\left|\frac{1}{n+1}\left[\left({}^{g}\operatorname{div} T^H\right)_k g_{ij} + \left({}^{g}\operatorname{div} T^H\right)_j g_{ik}\right] - \frac{2-c}{4(n+1)}\left({}^{\theta}F_{lk}\omega_\theta^l g_{ij} + {}^{\rho}F_{lj}\omega_\rho^l g_{ik}\right)\right|^2 \\
&\quad + \left(\frac{c}{2}-1\right)\left\langle {}^{g}\nabla_l r,\ {}^{\theta}F_l^i\omega_{\theta i}\right\rangle \\
&=\ -2\left|\frac{1}{4(n+1)}\left[\nabla_k r g_{ij} + \nabla_j r g_{ik}\right] + \frac{2+c}{4(n+1)}\left({}^{\theta}F_{lk}\omega_\theta^l g_{ij} + {}^{\rho}F_{lj}\omega_\rho^l g_{ik}\right)\right|^2 \\
&\quad + \left(\frac{c}{2}-1\right)\left\langle {}^{g}\nabla_l r,\ {}^{\theta}F_l^i\omega_{\theta i}\right\rangle \\
&=\ -\frac{1}{20}\left|\nabla r\right|^2 - \frac{2+c}{2(n+1)}\left\langle {}^{g}\nabla_l r,\ {}^{\theta}F_l^i\omega_{\theta i}\right\rangle - \frac{(2+c)^2}{8(n+1)^2}\left| {}^{\theta}F_{lk}\omega_\theta^l g_{ij} + {}^{\rho}F_{lj}\omega_\rho^l g_{ik}\right|^2 \\
&\quad + \left(\frac{c}{2}-1\right)\left\langle {}^{g}\nabla_l r,\ {}^{\theta}F_l^i\omega_{\theta i}\right\rangle \\
&=\ -\frac{1}{20}\left|\nabla r\right|^2 + \frac{4c-12}{10}\left\langle {}^{g}\nabla_l r,\ {}^{\theta}F_l^i\omega_{\theta i}\right\rangle - \frac{(2+c)^2}{200}\left| {}^{\theta}F_{lk}\omega_\theta^l g_{ij} + {}^{\rho}F_{lj}\omega_\rho^l g_{ik}\right|^2 \\
&\leq\ -\frac{(12-4c)^2}{20}\left\langle {}^{\theta}F\cdot\ {}^{\rho}F\cdot\ \omega_\rho,\omega_\theta\right\rangle - \frac{(2+c)^2}{200}\left| {}^{\theta}F_{lk}\omega_\theta^l g_{ij} + {}^{\rho}F_{lj}\omega_\rho^l g_{ik}\right|^2 \\
&=\ \left(-7+5c-\frac{3}{4}c^2\right)\left\langle {}^{\theta}F\cdot\ {}^{\rho}F\cdot\ \omega_\rho,\omega_\theta\right\rangle
\end{aligned}
\tag{7.15}
$$

Finally let us extract and the rest of the terms from $-2\left|\nabla T\right|^2$ using lemma B.1. First we have

$$
\begin{aligned}
-2\left|\nabla_i T_{\theta\rho}\right|^2 &=\ -\frac{1}{2}\left|\left({}_{\rho}F\cdot\omega_\theta + {}_{\theta}F\cdot\omega_\rho\right)\right|^2 \\
&= {}^{\theta}F\cdot\ {}_{\theta}F\cdot\omega_\rho\cdot\omega_\rho + \left\langle {}^{\theta}F\cdot\ {}^{\rho}F\cdot\omega_\theta,\omega_\rho\right\rangle \qquad (7.16)\\
&=\ -\frac{1}{4}\left|F\right|^2\left|\omega\right|^2 + \left\langle {}^{\theta}F\cdot\ {}^{\rho}F\cdot\omega_\theta,\omega_\rho\right\rangle + Q * T^{*2} + T^{*3}
\end{aligned}
$$

Next we have

$$-2\left|\nabla_\theta T_{\rho j}\right|^2 - 2\left|\nabla_\theta T_{j\rho}\right|^2 = -\left|\ {}_\theta F\cdot\omega_\rho\right|^2$$

$$= -\frac{1}{4}\left|F\right|^2\left|\omega\right|^2 + Q * T^{*2} + T^{*3} \tag{7.17}$$

Finally we simplify

$$\frac{c^2-4c}{8}\left|\ {}^\theta F_{ij}\omega_{\theta k} + \ {}^\rho F_{ik}\omega_{\rho j}\right|^2 = \frac{c^2-4c}{4}\left(\left|\ {}^\theta F_{ij}\omega_{\theta k}\right|^2 - \left\langle\ {}^\rho F\cdot\ {}^\theta F\cdot\omega_\rho,\omega_\theta\right\rangle\right) \tag{7.18}$$

Combining the above estimates allows us to conclude

$$\frac{d}{dt}Z \leq \Delta Z + L_0(T^H) + L_1(\omega) + \delta r * \mathrm{rm} * T + Q * T^{*2} + V * T^{*2}$$

where

$$L_0(T^H) = \operatorname{tr}\left(\ {}^\theta F\cdot\ {}_\theta F\cdot T^H\cdot T^H + 2\ {}^\theta F\cdot T^H\cdot\ {}_\theta F\cdot T^H\right) + \left(\frac{c}{4(2-c)}\right)\left|F\right|^2\left|T^H\right|^2$$

$$L_1(\omega) = -\left\langle\ {}^\theta F,\ {}^\rho F\right\rangle\langle\omega_\theta,\omega_\rho\rangle + \frac{c^2-4c}{4}\left|\ {}^\theta F_{ij}\omega_{\theta k}\right|^2$$

$$+3\left(-1+c-\frac{1}{4}c^2\right)\left\langle\ {}^\theta F\cdot\ {}^\rho F\cdot\omega_\rho,\omega_\theta\right\rangle$$

$$+\left(-1+c-\frac{c^2}{4}\right)\left\langle\ {}^\rho F\cdot\ {}^\theta F\cdot\omega_\rho,\omega_\theta\right\rangle$$

□

Definition: 7.5. We say that a connection A is *μ-stable* if there exists $c<2$ so that for any $W\in\operatorname{Sym}^2 T^*M$ and $\omega\in\bigwedge^1(\mathfrak{g})$ we have

$$L_0(W) \leq -\mu\left|F\right|^2\left|W\right|^2$$

$$L_1(\omega) \leq -\mu\left|F\right|^2\left|\omega\right|^2 \tag{7.19}$$

Proposition: 7.6. *The standard anti-self-dual $SU(2)$ instanton is stable.*

Proof. The stability condition is a condition that must be satisfied by the curvature of the bundle at each point. Recall that the curvature of the standard $SU(2)$ instanton at a point is given by

$$F = \lambda\left(dx^{12} - dx^{34}\right) i + \lambda\left(dx^{13} + dx^{24}\right) j + \lambda\left(dx^{14} - dx^{23}\right) k$$

where $\lambda \in \mathbb{R}_{\geq 0}$ and i, j, k are the standard basis for the imaginary quaternions. Without loss of generality we may assume that the metric is given by the identity matrix at our fixed point. Let W be any symmetric two-tensor, i.e.

$$W = (w_{ij})$$

A straightforward computation shows that

$$\begin{aligned} &\operatorname{tr}\left({}^{\theta}F \cdot {}_{\theta}F \cdot W^H \cdot W^H + 2\, {}^{\theta}F \cdot W^H \cdot {}_{\theta}F \cdot W^H\right) \\ &= -2\lambda^2 \sum_{i \neq j} w_{ij}^2 - 4\lambda^2 \sum_{i\neq j} w_{ii} w_{jj} - 3\lambda^2 \sum_i w_{ii}^2 \end{aligned} \tag{7.20}$$

But now we can simplify

$$\begin{aligned} -4\lambda^2 \sum_{i\neq j} w_{ii} w_{jj} &= -2\lambda^2 \sum_i \left(w_{ii} \sum_{j \neq i} w_{jj} \right) \\ &= 2\lambda^2 \sum_i w_{ii}^2 - 2\lambda^2 \left(\operatorname{tr} W\right)^2 \\ &\leq 2\lambda^2 \sum_i w_{ii}^2 \end{aligned}$$

Using this gives

$$\begin{aligned} &\operatorname{tr}\left({}^{\theta}F \cdot {}_{\theta}F \cdot W^H \cdot W^H + 2\, {}^{\theta}F \cdot W^H \cdot {}_{\theta}F \cdot W^H\right) \\ &\leq -2\lambda^2 \sum_{i\neq j} w_{ij}^2 - \lambda^2 \sum_i w_{ii}^2 \\ &\leq -\lambda^2 \left|W\right|^2 \\ &\leq -\frac{1}{12} \left|F\right|^2 \left|W\right|^2 \end{aligned} \tag{7.21}$$

Thus we see that for any choice of $c < \frac{1}{2}$ the first condition of stability will hold. Now we turn to the second condition. Fix ω an arbitrary $\mathfrak{su}(2)$-valued one-form. For concreteness fix $c = \frac{1}{2}$. Noting that $\{F^{\theta}, F^{\rho}\} = 0$ for $\theta \neq \rho$ we see

$$\begin{aligned}
-\frac{27}{16}&\left\langle {}^{\theta}F \cdot {}^{\rho}F \cdot \omega_{\rho}, \omega_{\theta} \right\rangle - \frac{9}{16}\left\langle {}^{\rho}F \cdot {}^{\theta}F \cdot \omega_{\rho}, \omega_{\theta} \right\rangle \\
&= -\frac{36}{16}\left\langle {}^{\theta}F \cdot {}^{\theta}F \cdot \omega_{\theta}, \omega_{\theta} \right\rangle - \frac{9}{8}\sum_{\theta \neq \rho}\left\langle {}^{\theta}F \cdot {}^{\rho}F \cdot \omega_{\rho}, \omega_{\theta} \right\rangle \\
&\leq \frac{9}{16}\left| {}^{\theta}F \right|^2 \left|\omega_{\theta}\right|^2 + \frac{9}{16}\sum_{\theta \neq \rho}\left|F_{\theta} \cdot \omega_{\theta}\right|^2 + \left|F_{\rho} \cdot \omega_{\rho}\right|^2 \\
&\leq \frac{9}{16}\left|F_{\theta}\right|^2 \left|\omega_{\theta}\right|^2 + \frac{9}{4}\sum_{\theta}\left|F_{\theta} \cdot \omega_{\theta}\right|^2 \\
&= \frac{18}{16}\left|F_{\theta}\right|^2 \left|\omega_{\theta}\right|^2
\end{aligned}$$

Also we compute

$$-\left\langle {}^{\theta}F, {}^{\rho}F \right\rangle \langle \omega_{\theta}, \omega_{\rho} \rangle = -\left|F_{\theta}\right|^2 \left|\omega_{\theta}\right|^2$$

and

$$-\frac{7}{16}\left| {}^{\theta}F_{ij}\omega_{\theta k} \right|^2 = -\frac{7}{16}\left|F_{\theta}\right|^2 \left|\omega_{\theta}\right|^2$$

and hence it is clear that the second condition of stability holds. Thus for any choice of c slightly less than $\frac{1}{2}$ both conditions of stability will hold, and the result follows. □

7.2 Proof of Main Convergence Theorem

In this section we will prove the main convergence result. Our proof is loosely modeled on Ye's result [32] on convergence of Ricci flow assuming certain "stability conditions." Roughly, Ye assumes L^2-stability of the Einstein operator, which is the curvature operator appearing in the evolution of the traceless Ricci tensor. This assumption implies exponential L^2-decay of the traceless Ricci tensor for a short time. By applying the Moser weak maximum principle, which requires certain delicate assumptions on the volume and diameter, he bootstraps this decay using Bernstein-Shi estimates to conclude exponential decay of the C^0 norm of the second covariant derivative of the traceless Ricci tensor. Decay of this form can be integrated over an arbitrarily large time interval to show a bound on the curvature for all time, and also continued decay for all time. Though most of the estimates are straightforward, the overall proof is fairly delicate, due mostly to the fact that only a weak stability is assumed, and so the Moser weak maximum principle must be used.

Let us describe the hypotheses of theorem 7.9. We have stated the hypotheses to emulate the "high power of a line bundle" case as much as possible. In particular, if one had a line bundle $L \to M$ satisfying μ-stability, then a sufficiently high power of the bundle (in other words dropping a sufficiently large constant in front of the bundle curvature) would satisfy the hypotheses of the theorem. We have repackaged this idea and phrased it as a sequence of connections with arbitrarily large minimum of curvature, uniform control over $\max |F|^2 / \min |F|^2$, and also a uniform bound $\left| {}^{A}\nabla F \right| \leq C \left| F \right|$.

Now we will describe the proof of theorem 7.9. The assumption of large, stable bundle curvature F amounts to assuming very fast (on the order of $|F|^2$) C^0-decay of the tensor T. Due to a particular technical problem we will first prove a much weaker convergence result which in addition to the hypotheses of theorem 7.9 also assumes that $||T||_{L^2}$ is initially very small, small even depending on the size of the bundle curvature. We prove this theorem because the proof illustrates the techniques of proving the main theorem in a more straightforward way.

The proof has two main steps. Roughly, if the base curvature rm stays small enough with respect to the bundle curvature, then we should have decay of T and convergence of the Ricci Yang-Mills flow. We first show such a bound on the guaranteed short interval of existence of length $\left[0, \frac{1}{|F|^2}_{C^0(M_0)}\right)$. These bounds are very straightforward ODE estimates. After a time interval of this length, one has the Bernstein-Shi type derivative estimates available to us. Thus we can exploit the L^2-decay of T and use these estimates and the Sobolev inequality to conclude C^0 decay of $|\nabla^2 T|$. The resulting bound on rm is not good enough without the hypothesis of very small $||T||_{L^2}$ initially (depending on F).

So, to get around this problem for the main proof we prove a better short-time existence theorem for the volume-normalized Ricci Yang-Mills flow. Essentially we show bounds on rm on an interval of the form $\left[0, \frac{1}{|F|^{2-\delta}}\right)$ for some small but universal (independent of P) $\delta > 0$. The basic reason that this is possible is because the bundle curvature $|F|^2$ actually has a lower-order nonlinearity than one expects from Ricci flow alone. In particular, the highest order term in the evolution $\left(\left|\overset{\circ}{\eta}\right|^2\right)$ appears with the negative sign, so that as far as growth is concerned, the nonlinearity is of a lower order. Once one has this a-priori bound on $|F|^2$ getting bounds on everything else is straightforward. So, using this short-time existence theorem and the exponential C^0 decay of T, the L^2 norm at the time $\frac{1}{|F|^{2-\delta}}$ is arbitrarily small, even with respect to powers of P. At this point one can cite the weaker convergence result to conclude convergence, but we carry out the estimates anyways.

Theorem: 7.7. *Let $E \to (M^4, g)$ be a principal bundle. For fixed $\mu > 0$, $\Omega > 0$ and $B > 0$ there exists a large $N = N(\mu, \Omega, B, g) > 0$ depending on μ, Ω, B and the base metric g with the following property: if A is a μ-stable connection on E which satisfies*

$$\left|\mathring{\eta}\right|_{C^0} + \left|\,{}^g\nabla\mathring{\eta}\right|_{C^0} + \left|\,{}^g\nabla^2\mathring{\eta}\right|_{C^0} \leq \Omega \tag{7.22}$$

and

$$\frac{1}{B}\max_M |F|^2 < |F|^2(x) < B\min_M |F|^2 \text{ for all } x \in M \tag{7.23}$$

and

$$\min_M |F|^2 > N^2 \tag{7.24}$$

and furthermore

$$\left|\,{}^A\nabla F\right|_{C^0} < B\,|F|_{C^0(M_0)} \tag{7.25}$$

and finally

$$||T||_{L^2(M_0)} < |F|^{-7}_{C^0(M_0)} \tag{7.26}$$

then the volume normalized Ricci Yang-Mills flow with initial condition $G(g, A)$ exists for all time and converges to an Einstein-Yang-Mills metric.

Proof. Let

$$P := \min_M |F| > N$$

First we describe a set of conditions on the metric which guarantee decay of T. Our ultimate goal is to show that for certain choices of the constants above, these conditions hold for all time. In particular we say that a volume normalized Ricci Yang-Mills flow satisfies condition $\alpha(\mu, B, C, N)$ on $[0, \tau)$ if for every $t \in [0, \tau)$ we have

1. The connection $A(t)$ is μ-stable
2. The bounds of (7.23) and (7.24) hold for our given B and N.
3. $|\mathrm{rm}|_{C^0(M_t)} < C, |\nabla \mathrm{rm}|_{C^0(M_t)} < C \left|F\right|_{C^0(M_t)}$
4. $\left| {}^A\nabla F\right|_{C^0(M_t)} < C \left|F\right|_{C^0(M_t)}$
5. $|\nabla T|_{C^0(M_t)} < C P^{3/2} \left|T\right|_{C^0(M_t)}$
6. $\frac{1}{2} g(t) \leq g(0) \leq 2g(t)$

The hypotheses of the theorem amount to assuming condition $\alpha(\mu, B, C, N)$ for certain choices of the constants. We will show that for a constant C chosen with respect to the initial data and N large enough condition $\alpha := \alpha\left(\mu/2, 2B, C, \frac{N}{2}\right)$ holds for all time.

We now institute a very important notational convention. In the calculations below, there will be many implicit constants. In most cases, these constants will be universal (depending only on the dimension), although in some cases the constants will depend on the initial metric, or possibly the constants μ and B. What is most important though is that these constants will be *independent of* N. Note that the hypotheses of the theorem state that μ and B are both independent of N as well. This will allow us to freely choose N large with respect to given constants at various points of the theorem. Notationally, we will refer to *any* such constant with the letter

C. Thus C, and sometimes lowercase c, will denote different constants on different lines, but must be *independent of* N. Alternatively, one could think that C is the supremum of all universal constants encountered in the proof thus far.

Lemma: 7.8. *There exists a constant c so that if N is sufficiently large then the solution to volume normalized RYM flow exists and moreover condition α holds on $\left[0, \frac{c}{P^2}\right)$.*

Proof. First we want to choose N so large that the derivatives of the curvature are dominated by the bundle curvature connection terms. In particular, first choose N so large that

$$\begin{aligned} \left|\mathrm{Rm}\right|_{C^0(M_0)} &\leq CP^2 \\ \left|\nabla \mathrm{Rm}\right|_{C^0(M_0)} &\leq CP^3 \\ \left|\nabla^2 \mathrm{Rm}\right|_{C^0(M_0)} &\leq CP^4 \end{aligned} \tag{7.27}$$

for a constant C. This is possible because of hypothesis (7.25). We first show that for a larger choice of C the bounds of (7.27) are preserved on an interval of the form $\left[0, \frac{c_1}{P^2}\right)$, with a different choice of C. An immediate corollary of proposition 5.8 is

$$\begin{aligned} \frac{d}{dt}\left|\mathrm{Rm}\right|^2 &= \Delta\left|\mathrm{Rm}\right|^2 - 2\left|\nabla \mathrm{Rm}\right|^2 + F * \nabla \mathrm{Rm} * \mathrm{Rm} + \mathrm{Rm}^{*3} \\ &\leq \Delta\left|\mathrm{Rm}\right|^2 - \left|\nabla \mathrm{Rm}\right|^2 + \mathrm{Rm}^{*3} \\ &\leq \Delta\left|\mathrm{Rm}\right|^2 + C\left|\mathrm{Rm}\right|^3 \end{aligned} \tag{7.28}$$

using the Cauchy-Schwarz inequality. Applying the maximum principle, we see that $\left|\mathrm{Rm}\right|_{C^0(M_t)}$ is bounded above by the solution to the ODE

$$\frac{d}{dt}\phi = C\phi^2 \tag{7.29}$$

which has solution

$$\phi(t) = \frac{1}{\frac{1}{\phi(0)} - Ct} \tag{7.30}$$

so that as long as $t \leq \frac{1}{2C|\mathrm{Rm}|_{C^0(M_0)}}$ we have

$$|\mathrm{Rm}|_{C^0(M_t)} \leq 2\,|\mathrm{Rm}|_{C^0(M_0)} \leq 2CP^2 \tag{7.31}$$

Recall that this bound on curvature is sufficient to conclude existence of the flow up to this point. We want to use this bound to show a bound on $|\nabla \mathrm{Rm}|_{C^0(M_t)}$ on this time interval. A consequence of proposition 5.9 is

$$\begin{aligned} \frac{d}{dt}\,|\nabla \mathrm{Rm}|^2 &= \Delta\,|\nabla \mathrm{Rm}|^2 - 2\left|\nabla^2 \mathrm{Rm}\right|^2 + F * \nabla^2 \mathrm{Rm} * \nabla \mathrm{Rm} + \mathrm{Rm} * \nabla \mathrm{Rm}^{*2} \\ &= \Delta\,|\nabla \mathrm{Rm}|^2 - \left|\nabla^2 \mathrm{Rm}\right|^2 + C\,|\mathrm{Rm}|\,|\nabla \mathrm{Rm}|^2 \\ &\leq \Delta\,|\nabla \mathrm{Rm}|^2 + CBP^2\,|\nabla \mathrm{Rm}|^2 \end{aligned} \tag{7.32}$$

Again applying the maximum principle it is straightforward to conclude

$$|\nabla \mathrm{Rm}|_{C^0(M_t)} \leq |\nabla \mathrm{Rm}|_{C^0(M_0)}\, e^{CBP^2 t}$$

and so by restricting to the time interval $\frac{\ln 2}{CBP^2}$ we clearly conclude

$$|\nabla \mathrm{Rm}|_{C^0(M_t)} \leq 2CP^3 \tag{7.33}$$

A completely analogous argument shows that

$$\left|\nabla^2 \mathrm{Rm}\right|_{C^0(M_t)} \leq CP^4 \tag{7.34}$$

and so indeed the bounds of (7.27) hold on this interval.

So, we need to use these bounds to show condition α on a possibly shorter time interval $\left[0, \frac{c}{P^2}\right)$. Using the bound of (7.22) and arguing as in line (7.27), there exists a constant C so that if N is chosen large enough then

$$\begin{aligned} |V|_{C^0(M_0)} &\leq C \\ |\nabla V|_{C^0(M_0)} &\leq CP \\ \left|\nabla^2 V\right|_{C^0(M_0)} &\leq CP^2 \\ \left|\nabla^3 V\right|_{C^0(M_0)} &\leq CP^3 \end{aligned} \tag{7.35}$$

We would like to show that there exists c and C independent of N so that the bounds of (7.35) hold on $\left[0, \frac{c}{P^2}\right)$ for a different choice of C. From proposition 5.67 we see that

$$\frac{d}{dt} |V|^2 \leq \Delta |V|^2 + C |\mathrm{Rm}| |V|^2 \tag{7.36}$$

so that by applying the curvature bound above on the given time interval we conclude by the maximum principle that $|V|_{C^0(M_t)}$ is bounded above by the solution to the ODE

$$\frac{d}{dt}\phi = CP^2\phi \tag{7.37}$$

which has the solution $\phi(t) = \phi(0)e^{CP^2 t}$. Clearly for $t \in \left[0, \frac{\ln 2}{CP^2}\right)$ the required bound will hold. Similarly using proposition 5.66 we see that

$$\begin{aligned} \frac{d}{dt} |\nabla V|^2 &\leq \Delta |\nabla V|^2 + C |\mathrm{Rm}| |\nabla V|^2 + C |\nabla \mathrm{Rm}| |V| |\nabla V| \\ &\leq \Delta |\nabla V|^2 + CP^2 |\nabla V|^2 + CP^4 |V|^2 \end{aligned} \tag{7.38}$$

where we have used the bounds on curvature and its derivatives from (7.27) and the Cauchy-Schwarz inequality. Using the bound above for V we can conclude using the maximum principle that $|\nabla V|^2_{C^0(M_t)}$ is bounded above by the solution to the ODE

$$\frac{d}{dt}\phi = CP^2\phi + CP^4 \tag{7.39}$$

which is bounded above by

$$P^2 e^{CP^2 t}\left(1 - e^{-CP^2 t} + C\right) \tag{7.40}$$

so that $|\nabla V|^2_{C^0(M_t)} \leq CP^2$ on $\left[0, \frac{c}{P^2}\right)$ for c small. The bounds on $\nabla^2 V$ and $\nabla^3 V$ are identical.

Finally we are in a position to show that condition α holds on a time interval of the form $\left[0, \frac{c}{P^2}\right)$. Note that using the variation formula for curvature, we can write

$$\frac{d}{dt}\,\mathrm{rm} = \mathcal{L}_0(\,{}^g\nabla^2 V) \tag{7.41}$$

where here $\mathcal{L}_0(\,{}^g\nabla^2 V)$ refers to a universal linear polynomial in ${}^g\nabla^2 V$. Using our bounds above we get that

$$\begin{aligned} \frac{d}{dt}\,|\mathrm{rm}| &\leq C\left|\nabla^2 V\right| + C\,|V|\,|\mathrm{rm}| \\ &\leq CP^2 + C\,|\mathrm{rm}| \end{aligned} \tag{7.42}$$

It is clear that on the interval $\left[0, \frac{c}{P^2}\right)$ we can conclude

$$\begin{aligned} |\mathrm{rm}|_{C^0(M_t)} &\leq e^{Ct}\left(|\mathrm{rm}|_{C^0(M_0)} + P^2\left(1 - e^{Ct}\right)\right) \\ &\leq 2\left(1 + \frac{C}{P^2}\right) \\ &\leq 4 \end{aligned} \tag{7.43}$$

for c chosen small with respect to C and N chosen large. Also we have the following variation formula:

$$\begin{aligned}\frac{d}{dt}\ {}^{A}\nabla F &= \frac{d}{dt}\left(\partial F + \ {}^{g}\Gamma * F\right)\\ &= \ {}^{A}\nabla\left(\frac{d}{dt}F\right) + F * \left(\frac{d}{dt}\ {}^{g}\Gamma\right)\\ &= \mathcal{L}_1\left(\nabla^2\left(\frac{d}{dt}A\right)\right) + \mathcal{L}_2\left(F * \nabla\left(\frac{d}{dt}g\right)\right)\\ &= \mathcal{L}_1\left(\nabla^2 V\right) + \mathcal{L}_2\left(F * \nabla V\right)\end{aligned}$$

where again the $\mathcal{L}_i$ refer to universal linear polynomials. Thus we can conclude

$$\frac{d}{dt}\left|\ {}^{A}\nabla F\right| \leq C\left(\left|\nabla^2 V\right| + P\left|\nabla V\right| + \left|V\right|\left|\ {}^{A}\nabla F\right|\right) \tag{7.44}$$

so that again on the interval $\left[0, \frac{c_3}{P^2}\right)$ we can conclude

$$\left|\ {}^{A}\nabla F\right|_{C^0(M_t)} \leq \left|\ {}^{A}\nabla F\right|_{C^0(M_0)} + cC \tag{7.45}$$

One can bound the time derivative of F in an entirely similar way. This will give upper and lower bounds, and thus in particular give conditions $\alpha.1$ and $\alpha.2$. Thus we have shown conditions $\alpha.1 - \alpha.4$. We note that concluding condition $\alpha.5$ is exactly the same as the estimate of $\left|\nabla V\right|_{C^0(M_t)}$ proved above. For the C^0 base metric bound $\alpha.6$, we first note that for a fixed tangent vector v we have

$$\frac{d}{dt}\left|v\right|^2 = \dot{g}(v,v) = -2V(v,v) \tag{7.46}$$

thus

$$\left|\frac{d}{dt}\left|v\right|^2\right| \leq 2\left|V\right| \leq C \tag{7.47}$$

We can then use the estimate

$$\begin{aligned}\left|\log\left(\frac{g_t(v,v)}{g_0(v,v)}\right)\right| &= \left|\int_0^t \frac{\partial}{\partial s}\log g_s(v,v)ds\right| \\ &= \left|\int_0^t \frac{\frac{\partial}{\partial s}g_s(v,v)}{g_s(v,v)}ds\right| \\ &\leq \int_0^t \left|\frac{\partial}{\partial s}g_s\left(\frac{v}{|v|},\frac{v}{|v|}\right)\right| ds \\ &\leq \int_0^t \left|\frac{\partial}{\partial s}g_s\right|_{g_s} ds \\ &\leq C\end{aligned}$$

and exponentiate to give the bounds

$$e^{-C/P^2}g_t \leq g_0 \leq e^{C/P^2}g_t \tag{7.48}$$

from which condition $\alpha.6$ immediately follows. $\square$

Now we turn to the second part of the proof, which is concluding existence and convergence on the infinite time interval $\left[\frac{c}{P^2},\infty\right)$. Our condition α is open, so we assume that it holds on an open time interval $[0,\tau)$ where $\tau \geq \frac{c}{P^2}$ and prove that in fact it holds at τ. This will show that the interval is both open and closed which will give long time existence and convergence. To do this we must use the exponential decay of T, the Sobolev inequality and integral interpolation estimates.

First let us describe the decay estimate for $|T|_{C^0(M_t)}$. We will assume $c=0$ in the definition of μ-stability for notational convenience. Our main problem is the term $\delta r * \mathrm{rm} * T$. At a given time t, fix a point $p \in M$ such that $\delta r = 0$. Using the formula $\operatorname{div} T = \frac{1}{4}\nabla r$ we can write

$$\delta r(q) = \int_p^q \nabla r = 4\int_p^q \operatorname{div} T$$

which gives the estimate

$$|\delta r|_{C^0(M_t)} \leq C \int_p^q |\nabla T| \leq C \operatorname{diam}(M_t) |\nabla T|_{C^0(M_t)} \tag{7.49}$$

Using condition α it is clear now that

$$\begin{aligned} \delta r * \mathrm{rm} * T &\leq C |\delta r|_{C^0(M_t)} |\mathrm{rm}|_{C^0(M_t)} |T|_{C^0(M_t)} \\ &\leq C \operatorname{diam}(M_t) |\nabla T|_{C^0(M_t)} |\mathrm{rm}|_{C^0(M_t)} |T|_{C^0(M_t)} \\ &\leq C P^{3/2} |T|^2_{C^0(M_t)} \end{aligned} \tag{7.50}$$

Using this estimate, it is clear that for N chosen very large we will have

$$\begin{aligned} \frac{d}{dt} |T|^2 &\leq \Delta |T|^2 - \frac{\mu}{8B} P^2 |T|^2 + C |T|^2 + C P^{3/2} |T|^2_{C^0(M_t)} \\ &\leq \Delta |T|^2 - \frac{\mu}{16B} P^2 |T|^2 + C P^{3/2} |T|^2_{C^0(M_t)} \end{aligned} \tag{7.51}$$

as long as condition α holds. By the maximum principle we conclude

$$|T|_{C^0(M_t)} \leq e^{-\frac{\mu}{32B} P^2 t} |T|_{C^0(M_0)} \tag{7.52}$$

Now we point out how to conclude condition $\alpha.5$ at time τ. It is a Bernstein-Shi type estimate, and we will prove it in exactly that manner. First of all we have the evolution equations

$$\begin{aligned} \frac{d}{dt} |\nabla T|^2 = \Delta |\nabla T|^2 - 2 \left|\nabla^2 T\right|^2 + \nabla \mathrm{Rm} * T * \nabla T \\ + \mathrm{Rm} * \nabla T^{*2} + \delta r * \nabla \mathrm{rm} * \nabla T \end{aligned} \tag{7.53}$$

and

$$\frac{d}{dt} |T|^2 = \Delta |T|^2 - 2 |\nabla T|^2 + \mathrm{Rm} * T^{*2} + \delta r * \mathrm{rm} * T \tag{7.54}$$

Define $s = t - \left(\tau - \frac{c}{P^2}\right)$. Let

$$H(x,s) = s |\nabla T|^2 + \beta |T|^2 \tag{7.55}$$

where β is a universal constant to be determined. Note then that $s \leq \frac{c}{P^2}$ on the time interval of interest. We can then compute

$$\frac{\partial}{\partial s} H \leq \Delta H + (1 - 2\beta) |\nabla T|^2 + s \nabla \operatorname{Rm} * T * \nabla T$$
$$+ s \operatorname{Rm} * \nabla T^{*2} + s \delta r * \nabla \operatorname{rm} * \nabla T + \operatorname{Rm} * T^{*2} + \delta r * \operatorname{rm} * T$$

We note that at any time after $\frac{c}{P^2}$ we can assume uniform bounds on the covariant derivatives of curvature using our Bernstein-Shi estimates (corollary 6.3). In particular we may assume

$$\left|\nabla^k \operatorname{Rm}\right|_{C^0(M_t)} \leq C_k P^{2+k} \tag{7.56}$$

Using these bounds, choosing β large and applying condition α we can estimate the equation further. We will use the bound of (7.50), which uses condition $\alpha.5$ up to time τ. This is an extra subtlety, so in the calculations below $\overline{C}$ will denote the constant of condition $\alpha.5$. Thus we have

$$\begin{aligned}
\frac{\partial}{\partial s} H &\leq \Delta H + (1 - 2\beta + C) |\nabla T|^2 + CP |T| |\nabla T| \\
&\quad + \frac{C}{P} |\delta r| |\nabla T| + CP^2 |T|^2 + C |\delta r| |T| \\
&\leq \Delta H + (1 - 2\beta + C) |\nabla T|^2 + C\overline{C}^2 P^2 |T|^2_{C^0(M_s)} + C\overline{C} P^{3/2} |T|^2_{C^0(M_s)} \\
&\leq \Delta H + C\overline{C}^2 P^2 |T|^2_{C^0(M_s)}
\end{aligned} \tag{7.57}$$

Applying the maximum principle and using the decay of T we can conclude

$$\begin{aligned}
H\left(s\left(\frac{c}{P^2}\right),x\right) &\leq H(0,x)+C\overline{C}^2P^2\int_0^{\frac{c}{P^2}}|T|^2_{C^0(M_{s(t)})}\,dt \\
&\leq \beta\,|T|^2_{C^0(M_{s(0)})}+C\overline{C}^2\,|T|^2_{C^0(M_{s(0)})}\,P^2\int_0^{\frac{c}{P^2}}e^{-\frac{\mu}{32B}P^2t}dt \\
&\leq C\left(1+\overline{C}^2\right)|T|^2_{C^0(M_{s(0)})} \\
&\leq C\left(1+\overline{C}^2\right)|T|^2_{C^0\left(M_{s\left(\frac{c}{P^2}\right)}\right)}
\end{aligned}$$

This allows us to conclude

$$|\nabla T|^2_{C^0(M_\tau)} \leq C\left(1+\overline{C}^2\right)P^2\,|T|^2_{C^0(M_\tau)} \leq \overline{C}P^3\,|T|^2_{C^0(M_\tau)} \tag{7.58}$$

for N chosen large.

We now describe how to use the decay estimate to conclude the rest of condition α at time τ. First of all, condition $\alpha.6$ gives a uniform bound on the Sobolev constant. Since we are on a four-manifold we can thus conclude in particular that

$$|f|_{C^0(M_t)} \leq C_S(M_t)||f||_{H^2_3(M_t)} \leq C||f||_{H^2_3(M_t)} \tag{7.59}$$

We can get bounds on these integrals using interpolation inequalities. Specifically we have

$$\begin{aligned}
\int_{M_t}\left|\nabla^kT\right|^2 &= \int_{M_t}\left\langle T,\nabla^{2k}T\right\rangle \\
&\leq \left(\int_{M_t}|T|^2\right)^{\frac{1}{2}}\left(\int_{M_t}\left|\nabla^{2k}T\right|^2\right)^{\frac{1}{2}} \\
&\leq CP^{2k+2}||T||_{L^2(M_0)}e^{-\frac{\mu}{32B}P^2t}
\end{aligned} \tag{7.60}$$

Thus using the Sobolev inequality we see that

$$\begin{aligned}
|\nabla T|_{C^0(M_t)} &\leq CP^5||T||_{L^2(M_0)}e^{-\frac{\mu}{64B}P^2t} \\
\left|\nabla^2 T\right|_{C^0(M_t)} &\leq CP^6||T||_{L^2(M_0)}e^{-\frac{\mu}{64B}P^2t}
\end{aligned} \tag{7.61}$$

for a universal constant C. Note also that the argument of (7.49) now gives the bound

$$|V|_{C^0(M_t)} \leq CP^5 ||T||_{L^2(M_0)} e^{-\frac{\mu}{64B}P^2 t} \tag{7.62}$$

These bounds are sufficient to conclude the rest of condition α by repeating the arguments of (7.44) - (7.48). In particular we can conclude

$$\begin{aligned} \frac{d}{dt}|\mathrm{rm}| &\leq C\left|\nabla^2 T\right|_{C^0(M_t)} + C|V|_{C^0(M_t)}|\mathrm{rm}| \\ &\leq \frac{C}{P}e^{-\frac{\mu}{64B}P^2 t} + \frac{C}{P^2}e^{-\frac{\mu}{64B}P^2 t}|\mathrm{rm}| \end{aligned} \tag{7.63}$$

where in the last line we used our assumption on $||T||_{L^2(M_0)}$. We can integrate this bound to any time τ to conclude

$$|\mathrm{rm}|_{C^0(M_\tau)} \leq |\mathrm{rm}|_{C^0(M_0)} + \frac{C}{P} \leq C \tag{7.64}$$

for N chosen large. Indeed, the proofs of all of the bounds for condition α are identical to those performed in lines (7.44) - (7.48). Thus we can conclude condition α at time τ and the result follows. $\square$

Theorem: 7.9. *Let $E \to (M^4, g)$ be a principal bundle. For fixed $\mu > 0$, $B > 0, \Omega > 0$ there exists a large $N > 0$ depending on μ, B, Ω and the base metric g with the following property: if A is a μ-stable connection on E which satisfies*

$$\left|\mathring{\eta}\right|_{C^0} + \left|{}^g\nabla\mathring{\eta}\right|_{C^0} + \left|{}^g\nabla^2\mathring{\eta}\right|_{C^0} \leq \Omega \tag{7.65}$$

and

$$\frac{1}{B}\max_M |F|^2 \leq |F|^2(x) \leq B\min_M |F|^2 \text{ for all } x \in M \tag{7.66}$$

and

$$\min_M |F|^2 > N^2 \tag{7.67}$$

and furthermore

$$\left| {}^{A}\nabla F \right|_{C^0} < B \left| F \right|_{C^0(M_0)} \tag{7.68}$$

then the volume normalized Ricci Yang-Mills flow with initial condition $G(g, A)$ exists for all time and converges to an Einstein-Yang-Mills metric.

Proof. Let us define a new condition α. We say that a solution to volume-renormalized Ricci Yang-Mills flow satisfies condition $\alpha(\epsilon) = \alpha(\epsilon, \mu, B, C, N)$ on $[0, \tau)$ if for every $t \in [0, \tau)$ we have

1. The connection $A(t)$ is μ-stable
2. The bounds of (7.66) and (7.67) hold for our given B.
3. $|\mathrm{rm}|_{C^0(M_t)} < P^{\epsilon}, |\nabla \mathrm{rm}| < P^{1+\epsilon}$
4. $\left| {}^{A}\nabla F \right|_{C^0(M_t)} < C \left| F \right|_{C^0(M_t)}$
5. $|\nabla T|_{C^0(M_t)} < C P^{3/2} |T|_{C^0(M_t)}$
6. $\frac{1}{2} g(t) \leq g(0) \leq 2g(t)$

We would now like to show condition $\alpha(\epsilon)$ on a time interval of the form $\left[0, \frac{1}{P^{2-\delta}}\right)$ for some small $\delta > 0$. To see why this is a reasonable thing to try to do, we first point out that bundle curvature is a-priori bounded on this interval. This is because the evolution of $|F|^2$ has a nonlinearity which is of a lower-order than what is expected by Ricci flow. In particular, from proposition 4.5 we have

$$\frac{d}{dt} |F|^2 \leq \Delta |F|^2 + C |\mathrm{rm}| |F|^2 + C |F|^3 \tag{7.69}$$

Thus, assuming condition $\alpha(\epsilon)$ on the time interval $[0, T)$, we have by the maximum principle that $|F|_{C^0(M_t)}$ is bounded above by the solution to the ODE

$$\begin{aligned} &\frac{d}{dt}\phi = CP^{\epsilon}\phi + C\phi^2 \\ &\phi(0) = BP \end{aligned} \tag{7.70}$$

which in turn is bounded above by the solution to the ODE

$$\begin{aligned} &\frac{d}{dt}\phi = C\phi^2 \\ &\phi(0) = BP \end{aligned} \tag{7.71}$$

which has solution

$$\phi(t) = \frac{1}{\frac{1}{BP} - Ct} \tag{7.72}$$

So that if $t \leq \frac{1}{2CBP}$ then $|F|_{C^0(M_t)} \leq 2\,|F|_{C^0(M_0)}$. This is noteworthy because as mentioned above the "doubling time" for curvature under Ricci flow normally is $\frac{1}{|\mathrm{Rm}|_{C^0(M_0)}}$, which in this case would be $\frac{1}{P^2}$. We will now show how to use this a-priori bound on curvature to show condition $\alpha(\epsilon)$ on the time interval $\left[0, \frac{1}{P^{2-\delta}}\right)$. Our main goal is an a-priori bound on $|V|_{C^0(M_t)}$.

First we point out that condition $\alpha(\epsilon)$ is sufficient to show decay of $|T|_{C^0(M_t)}$. In particular, for N chosen large enough we can conclude line (7.52). Next we need an a-priori bound on δr. Fix $\gamma << \epsilon$. Let

$$H(x,t) = (\delta r)^2 + P^{\gamma}\,|T|^2$$

We note using propositions 5.20 and 7.3 and the bound on $|T|_{C^0(M_t)}$ that

$$
\begin{aligned}
\frac{d}{dt}H \leq \Delta H &+ 4\langle V, \mathrm{rc}\rangle \delta r + 2\delta r \left|T^C\right|^2 + 4\left\langle {}^A\nabla T^C, F\right\rangle \delta r \\
&- 4\left(\fint \langle V, \mathrm{rc}\rangle\right)\delta r + 2\left(\fint r\delta r\right)\delta r \\
&- 2P^\gamma |\nabla T|^2 + P^\gamma \operatorname{Rm} * T^{*2} + P^\gamma \delta r * \mathrm{rm} * T
\end{aligned}
\tag{7.73}
$$

Now we have the bound

$$
\begin{aligned}
\langle V, \mathrm{rc}\rangle \delta r = &\left\langle T + \frac{1}{4}\delta r G^H, \mathrm{rc}\right\rangle \delta r \\
&\leq |\mathrm{rm}|^2_{C^0(M_t)} |T|_{C^0(M_t)} + \frac{1}{4} r (\delta r)^2 \\
&\leq CP^{2\epsilon} + P^\epsilon (\delta r)^2 \\
&\leq CP^{3\epsilon}
\end{aligned}
\tag{7.74}
$$

and similarly

$$
\begin{aligned}
-4\left(\fint \langle V, \mathrm{rc}\rangle\right)\delta r + 2\left(\fint r\delta r\right)\delta r &\leq |T| |\mathrm{rm}|^2 + |\mathrm{rm}|^3 \\
&\leq CP^{3\epsilon}
\end{aligned}
\tag{7.75}
$$

And we also bound

$$
4\left\langle {}^A\nabla T^C, F\right\rangle \delta r \leq P^\gamma |\nabla T|^2 + CP^{2-\gamma} (\delta r)^2 \tag{7.76}
$$

We also have the simple bounds

$$
P^\gamma \delta r * \mathrm{rm} * T \leq CP^{\gamma + 2\epsilon} \leq CP^{3\epsilon} \tag{7.77}
$$

and

$$
P^\gamma \operatorname{Rm} * T^{*2} \leq CP^{2+\gamma} \tag{7.78}
$$

Plugging these bounds into 7.73 gives

$$\frac{d}{dt}H \leq \Delta H + CP^{2+\gamma} + CP^{2-\gamma}H \tag{7.79}$$

Applying the maximum principle, H is bounded above by the solution to the ODE

$$\frac{d}{dt}\phi = CP^{2+\gamma} + CP^{2-\gamma}\phi \tag{7.80}$$

with $\phi(0)$ bounded independent of P. On the interval $\left[0, \frac{1}{P^{2-\delta}}\right)$ for $\delta < \gamma$ this is bounded above by

$$H \leq P^{\gamma} < P^{\epsilon/4} \tag{7.81}$$

Thus we may conclude

$$|V|_{C^0(M_t)} < P^{\epsilon/4} \tag{7.82}$$

On $\left[0, \frac{1}{P^{2-\delta}}\right)$. We would like to show the estimates

$$\begin{aligned} |\nabla V|_{C^0(M_t)} &\leq P^{1+\epsilon/4} \\ \left|\nabla^2 V\right|_{C^0(M_t)} &\leq P^{2+\epsilon/4} \\ \left|\nabla^3 V\right|_{C^0(M_t)} &\leq P^{3+\epsilon/4} \end{aligned} \tag{7.83}$$

on this time interval as well. We have already shown this estimate on the interval $\left[0, \frac{c}{P^2}\right)$ in the proof of lemma 7.8. After this time interval we have the Bernstein-Shi estimates available to us. In particular, using our total bound on curvature, corollary (6.3) gives

$$\left|\nabla^k \operatorname{Rm}\right| \leq C_k P^{2+k} \tag{7.84}$$

Given a time $t_0 > \frac{c}{P^2}$, let $\bar{t} := t - \left(t_0 - \frac{c}{P^2}\right)$. Now let

$$H(x,t) = \bar{t}\,|\nabla V|^2 + \beta\,|V|^2 \tag{7.85}$$

where β is a universal constant to be determined in the calculation below. We would like to get an estimate for $H(x, t_0)$, corresponding to $\bar{t} = \frac{c}{P^2}$. In particular on the interval of interest we have $\bar{t} \leq \frac{c}{P^2}$. Using proposition 5.67 we compute

$$\begin{aligned}
\frac{d}{dt} H \leq \Delta H &+ \left(1 + \bar{t} C B P^2 - 2\beta\right) |\nabla V|^2 + \bar{t} C |\nabla \operatorname{Rm}| |V| |\nabla V| \\
&+ C |\operatorname{Rm}| |V|^2 \\
&\leq \Delta H + (1 + CB - 2\beta) |\nabla V|^2 + CP |V| |\nabla V| + CP^2 |V|^2 \\
&\leq \Delta H + (1 + C - 2\beta) |\nabla V|^2 + CP^2 |V|^2 \\
&\leq \Delta H + CP^2 |V|^2 \\
&\leq \Delta H + CP^{2+\epsilon/2}
\end{aligned} \tag{7.86}$$

Applying the maximum principle as usual and integrating over the interval $\left(t_0 - \frac{c}{P^2}, t_0\right]$ gives the bound

$$|\nabla V|_{C^0(M_{t_0})} \leq CP^{1+\epsilon/4} \tag{7.87}$$

for $t_0 \in \left[0, \frac{1}{P^{2-\delta}}\right)$ Again, the Bernstein-Shi type estimates for the second and third derivatives are entirely similar, giving the bounds

$$\left|\nabla^2 V\right|_{C^0(M_t)} \leq CP^{2+\epsilon/4} \tag{7.88}$$

$$\left|\nabla^3 V\right|_{C^0(M_t)} \leq CP^{3+\epsilon/4} \tag{7.89}$$

for $t \in \left[0, \frac{1}{P^{2-\delta}}\right)$. As we have demonstrated in lines (7.43) - (7.48) this bound is sufficient to conclude for instance

$$\begin{aligned}
|\mathrm{rm}|_{C^0(M_t)} \leq\ & |\mathrm{rm}|_{C^0(M_0)} + CP^{\delta+\epsilon/4} \\
& < P^{\epsilon/2}
\end{aligned} \tag{7.90}$$

for $\delta < \frac{\epsilon}{4}$ and P chosen large. Concluding the rest of condition $\alpha(\epsilon)$ is is the same as in the proof of theorem 7.7 using the bound on $|\nabla^2 V|_{C^0(M_t)}$.

Now that we have shown condition $\alpha(\epsilon)$ on the time interval $\left[0, \frac{1}{P^{2-\delta}}\right]$, the rest of the proof is simple. We note that as a consequence of the decay of T we can conclude that at time $t_0 = \frac{1}{P^{2-\delta}}$ we have

$$||T||_{L^2(M_{t_0})} \leq ||T||_{L^2(M_0)} e^{-\frac{\mu}{16B}P^\delta} \tag{7.91}$$

Thus for N chosen large we have $||T||_{L^2(M_{t_0})}$ arbitrarily small, even smaller than any polynomial in positive powers of P. That is, in particular we can conclude for N large

$$||T||_{L^2(M_{t_0})} \leq P^{-7} \tag{7.92}$$

so that for instance we could apply theorem 7.7 to conclude convergence after this point. Proving convergence directly at this point is easy enough though. In particular using the Sobolev inequality as in lines (7.60) - (7.61) we can conclude for any time t

$$\left|\nabla^2 V\right|_{C^0(M_t)} \leq P^6 ||T||_{L^2(M_0)} e^{-\frac{\mu}{16B}P^2 t} \tag{7.93}$$

Thus for $t > \frac{1}{P^{2-\delta}}$ we have

$$\begin{aligned} |\mathrm{rm}|_{C^0(M_t)} = \ & |\mathrm{rm}|_{C^0\left(M_{\frac{1}{P^{2-\delta}}}\right)} + CP^6 \int_{\frac{1}{P^{2-\delta}}}^{t} e^{-\frac{\mu}{32BP^2 t}} \\ & \leq P^{\epsilon/2} + CP^4 e^{-\frac{\mu}{32B}P^\delta} \\ & < P^\epsilon \end{aligned} \tag{7.94}$$

for N chosen large. Concluding the rest of condition $\alpha(\epsilon)$ is the same as in lines (7.44) - (7.48), and the result follows. $\square$

Chapter 8

Summary

Let us briefly recap the results. We have justified the definition of a new geometric evolution equation. We have shown many analytic properties which would would expect from the study of Ricci flow, namely short-time existence, Bernstein-Shi derivative estimates, and compactness of solutions under certain natural hypotheses. Furthermore using the ideas of Perelman we showed that Ricci Yang-Mills flow is the gradient flow for the minimum eigenvalue of a particular Schrödinger operator, and found an entropy-like quantity which is monotonic near a type I singularity. One clear area for improvement of the results is to prove noncollapsing of general finite-time singularities, not just type I singularities. This may be possible by adding a correction term which accounts for the change in volume form as discussed in chapter 3. Finally, we succeeded in showing the existence of a canonical metric on a four-manifold in the presence of a particular kind of connection on a principal bundle.

The next obvious step in light of theorem 7.9 is to attempt to construct stable connections on four-manifolds. The theorem of Taubes [28] constructs anti-self-dual

$SU(2)$ connections by gluing together S^4 instantons (which have stable curvature) over different points in the manifold. Thus if one could make the curvature arbitrarily large everywhere and also control ${}^g\nabla F$ and $\max |F|^2 / \min |F|^2$ the resulting connection satisfy the hypotheses of theorem 7.9. This is a clear direction for future research.

Another direction would be to understand the circle-bundle case better. Here the most natural assumption is that F is symplectic initially. As pointed out in chapter 4, if F is large compared to the base curvature then one gets a pinching estimate for $\mathring{\eta}$, meaning that F is becoming closer to self-dual or anti-self-dual. The following conjecture seems reasonable.

Conjecture: 8.1. *Let $L \to M$ be a line bundle over a Riemannian four-manifold (M, g) with connection A. Suppose F_A is symplectic. Then there exists N large depending on g so that the volume-normalized Ricci Yang-Mills flow on $L^{\otimes p} \to M$ with connection A_p exists for all time and converges to an Einstein Yang-Mills metric.*

We hasten to point out that the techniques used in theorem 7.9 have no chance of proving this conjecture since the curvature of a line bundle is *never* stable. One possibility is to first look at the unnormalized Ricci Yang-Mills flow equation and prove an a-priori C^0 lower bound on the metric. See example (2.10) for the intuition behind this. One would then have to bootstrap this into higher-order regularity of the metric, and it is not a-priori clear how to do this. Another possibility is to do a blowup analysis at a potential singularity of the flow and attempt to classify ancient solutions of Ricci Yang-Mills flow. Our noncollapsing result for type I solutions would be a useful tool in such an analysis. We note that for both of these ideas it doesn't even seem as though a large bundle curvature is necessary. Thus we close with the following further conjecture.

Conjecture: 8.2. *Let $L \to M$ be a line bundle over a Riemannian four-manifold (M, g) with connection A. Suppose F_A is symplectic. Then the solution to volume-normalized Ricci Yang-Mills flow on L with connection A exists for all time and converges to an Einstein Yang-Mills metric.*

Also, a more general singularity analysis is called for. There may be quantities analogous to Perelman's reduced length and volume which are monotonic along a solution to RYM flow. If so these would undoubtedly be useful in classifying singularities. Guided by example 2.10, we make the following specific conjecture.

Conjecture: 8.3. *Every solution to RYM flow has low energy.*

If this conjecture is true, using theorem 3.14 we get as a corollary is that every solution to RYM flow is noncollapsed.

Finally, we point out that the general idea behind the main question could be applied to many different geometric situations. First of all one could change the extra "field" that one is introducing into the Einstein equation. Specifically one could introduce a three-form on a six manifold, or even coupling the Einstein equation to the Seiberg-Witten equations. Both of these situations arise naturally in physics. Also, one could consider coupling an external field to other elliptic equations which arise naturally in Riemannian geometry, say the harmonic map equation. It is possible that one could introduce terms which make the resulting parabolic equation easy to control, and thus produce a map satisfying a coupled elliptic system.

Appendix A

Riemannian Geometry Results

Lemma: A.1. *Let $u : M^n \times [0,T) \to \mathbb{R}$ be a C^2 subsolution to*

$$\frac{\partial u}{\partial t} = \Delta_{g(t)} u + \langle X, \nabla u \rangle + F(u)$$

on a closed manifold. Suppose there exists $C \in \mathbb{R}$ such that $u(x,0) \leq C$ for all $x \in M^n$. Let $\phi(t)$ be the solution to the initial value problem

$$\frac{d\phi}{dt} = F(\phi)$$

$$\phi(0) = C$$

Then

$$u(x,t) \leq \phi(t)$$

for all $x \in M^n$ and $t \in [0,T)$ such that $\phi(t)$ exists.

Proposition: A.2. *If α is a 2-form on a Riemannian manifold (M,g) then*

$$\Delta_d \alpha = (\Delta \alpha_{ij} + g^{kp} g^{lq} R_{ijkl} \alpha_{pq} - g^{kl} R_{ik} \alpha_{lj} - g^{kl} R_{jk} \alpha_{il}) dx^i \wedge dx^j \tag{A.1}$$

Proposition: A.3. *Let $E \to M$ a principal $\mathcal{K}$-bundle with connection A. Let*

$$\Delta_D := D_A D_A^* + D_A^* D_A \tag{A.2}$$

acting on $\mathfrak{k}$-valued differential forms. Then for ω a given $\mathfrak{k}$-valued differential form the following general formula holds:

$$\Delta_D \omega = \Delta\omega + F_A * \omega + \mathrm{rm} * \omega \tag{A.3}$$

Proof. This can be found in [8]. $\square$

Lemma: A.4. *Let ω be a two-form on a Riemannian four-manifold (M, g). Then ω is self-dual or anti-self-dual if and only if $\mathring{\eta} = 0$.*

Proof. Recall that $\eta_{ij} = g^{kl}\omega_{ik}\omega_{jl}$. Fix a given point $x \in M$ and choose coordinates such that $g(x) = \mathrm{Id}$ and also ω has been skew-diagonalized. Specifically this means

$$g = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad \omega = \begin{pmatrix} 0 & \lambda & & \\ -\lambda & 0 & & \\ & & 0 & \mu \\ & & -\mu & 0 \end{pmatrix} \tag{A.4}$$

where $\lambda, \mu \in \mathbb{R}$. Thus we may easily compute

$$\mathring{\eta} = \begin{pmatrix} \frac{1}{2}\left(\lambda^2 - \mu^2\right) & & & \\ & \frac{1}{2}\left(\lambda^2 - \mu^2\right) & & \\ & & \frac{1}{2}\left(\mu^2 - \lambda^2\right) & \\ & & & \frac{1}{2}\left(\mu^2 - \lambda^2\right) \end{pmatrix} \tag{A.5}$$

It is then clear that $\mathring{\eta} = 0$ at x if and only if $\lambda = \pm\mu$, i.e. if ω is self-dual or anti-self-dual. $\square$

Lemma: A.5. *Let ω be a two-form on a Riemannian four-manifold (M, g). Then*

$$|\eta|^2 \leq \frac{1}{2}|\omega|^4 \tag{A.6}$$

with equality if and only if ω has a non-trivial kernel.

Proof. Recall $\eta_{ij} = g^{kl}\omega_{ik}\omega_{jl}$. Fix a given point $x \in M$ and choose coordinates such that $g(x) = \mathrm{Id}$ and also ω has been skew-diagonalized as in the above lemma. Then an easy computation using (A.4) gives

$$\begin{aligned} |\eta|^2 &= 2\lambda^4 + 2\mu^4 \\ &\leq 2\left(\lambda^2 + \mu^2\right)^2 \\ &= \frac{1}{2}|\omega|^4 \end{aligned}$$

and moreover it is clear that equality holds in the second line if and only if one of λ or μ is zero, which corresponds exactly to when ω has a non-trivial kernel. □

Lemma: A.6. *Let ω an n-form on a $2n$-manifold. Then*

$$\int_M |\omega|^2 \geq \left|[\omega^{\wedge 2}]\right| \tag{A.7}$$

Proof. Using that the operator $*$ is an isometry, we see

$$\begin{aligned} \int \omega \wedge \omega &= (-1)^n \int \langle \omega, *\omega \rangle \\ &\leq \int |\omega|\,|*\omega| \\ &= \int_M |\omega|^2 \end{aligned}$$

One can bound $-\int \omega \wedge \omega$ similarly, so the result follows. □

Proposition: A.7. *In a general frame the curvature tensor of a Riemannian metric is given by*

$$R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{jk}^m \Gamma_{im}^l - \Gamma_{ik}^m \Gamma_{jm}^l - C_{ij}^m \Gamma_{mk}^l \tag{A.8}$$

Proposition: A.8. *In any coordinate system, if $\dot{g} = h$, then:*

$$\dot{\Gamma}_{ij}^{k} = \frac{1}{2} g^{kl} \left(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij} \right) \tag{A.9}$$

$$\dot{R}_{jk} = -\frac{1}{2} \left[\Delta_L h_{jk} + \nabla_j \nabla_k (\operatorname{tr} h) - \nabla_j \nabla_i h_{ik} - \nabla_k \nabla_i h_{ij} \right] \tag{A.10}$$

$$\dot{R} = -\Delta(\operatorname{tr} h) + \operatorname{div} \operatorname{div} h - \langle h, Rc \rangle \tag{A.11}$$

where Δ_L is the Lichnerowicz laplacian, given by:

$$\Delta_L A_{ij} = \Delta A_{ij} + 2 g^{kp} g^{lq} R_{ikjl} A_{pq} - g^{kl} R_{ik} A_{lj} - g^{kl} R_{jk} A_{il} \tag{A.12}$$

For our convenience we will write

$$\Delta_L A = \Delta A + \mathcal{R}(A) \tag{A.13}$$

where $\mathcal{R}$ is defined by the equality.

Lemma: A.9. *Let $L = \left(\frac{d}{dt} - \Delta \right)$. Then:*

$$L(fg) = (Lf)g + f(Lg) - 2 \nabla f \cdot \nabla g$$

Lemma: A.10. *If f a non-negative function and g a positive function then:*

$$\begin{aligned} \left(\frac{d}{dt} - \Delta \right) \left(\frac{f^\alpha}{g^\beta} \right) = \alpha \frac{f^{\alpha-1}}{g^\beta} \left(\frac{d}{dt} - \Delta \right) f - \beta \frac{f^\alpha}{g^{\beta+1}} \left(\frac{d}{dt} - \Delta \right) g \\ - \alpha(\alpha - 1) \frac{f^{\alpha-2}}{g^\beta} \left| \nabla f \right|^2 - \beta(\beta+1) \frac{f^\alpha}{g^{\beta+2}} \left| \nabla g \right|^2 \\ + 2\alpha\beta \frac{f^{\alpha-1}}{g^{\beta+1}} \left\langle \nabla f, \nabla g \right\rangle \end{aligned}$$

Appendix B

Decomposition of the Laplacian

In the lemma below the letters α, β and γ refer to general indices, the letters i, j and k refer to base indices and μ, ν refer to bundle indices.

Lemma: B.1. *Given a bundle metric we have*

$$
{}^{G}\nabla_{\alpha}\pi^{*}(s)_{\beta\gamma} = \begin{cases} {}^{g}\nabla_{i}s_{jk} & \alpha = i, \beta = j, \gamma = k \\ \frac{1}{2}\left({}_{\mu}F \cdot s - s \cdot {}_{\mu}F\right)_{jk} & \alpha = \mu \\ \frac{1}{2}\left({}_{\mu}F \cdot s\right)_{ik} & \beta = \mu \\ \frac{1}{2}\left({}_{\mu}F \cdot s\right)_{ij} & \gamma = \mu \\ 0 & \textit{otherwise} \end{cases}
$$

$$
{}^{G}\nabla_{\alpha}\pi^{*}(\omega)_{\beta\gamma} = \begin{cases} -\frac{1}{2}({}^{\theta}F_{ij}\omega_{k\theta} + {}^{\rho}F_{ik}\omega_{j\rho}) & \alpha = i, \beta = j, \gamma = k \\ 0 & \alpha = \mu \\ {}^{A}\nabla_{i}\omega_{\mu k} & \beta = \mu \\ {}^{A}\nabla_{i}\omega_{j\mu} & \gamma = \mu \\ \frac{1}{2}\left({}_{\mu}F \cdot \omega_{\nu} + {}_{\nu}F \cdot \omega_{\mu}\right)_{i} & \beta = \mu, \gamma = \nu \\ \frac{1}{2}\left({}_{\mu}F \cdot \omega_{\nu}\right)_{k} & \alpha = \mu, \beta = \nu \\ \frac{1}{2}\left({}_{\mu}F \cdot \omega_{\nu}\right)_{j} & \alpha = \mu, \gamma = \nu \\ 0 & \textit{otherwise} \end{cases}
$$

$$
{}^{G}\nabla_\alpha \pi^*(f)_{\beta\gamma} = \begin{cases} 0 & \alpha = i, \beta = j, \gamma = k \\ 0 & \alpha = \mu \\ -\frac{1}{2}{}^{\nu}F_{ik}\, f_{\nu\mu} & \beta = \mu \\ -\frac{1}{2}{}^{\nu}F_{ij}\, f_{\nu\mu} & \gamma = \mu \\ 0 & \alpha = \mu, \beta = \nu \\ {}^{g}\nabla_i f_{\mu\nu} & \beta = \mu, \gamma = \nu \\ 0 & otherwise \end{cases}
$$

Proof. In general for $W \in S^2T^*E$ we have

$$
\nabla_\alpha W_{\beta\gamma} = \partial_\alpha W_{\beta\gamma} - \Gamma^{\nu}_{\alpha\beta} W_{\nu\gamma} - \Gamma^{\nu}_{\alpha\gamma} W_{\nu\beta}
$$

The equations above now all follow from lemma 2.2. □

Lemma: B.2. *Given W a symmetric two-tensor on a Riemmanian manifold,*

$$
\begin{aligned}
\nabla_k \nabla_m W_{ij} = W_{ij,km} &- W_{lj}\Gamma^{l}_{mi,k} - W_{il}\Gamma^{l}_{mj,k} \\
&- W_{lj,k}\Gamma^{l}_{mi} - W_{il,k}\Gamma^{l}_{mj} - W_{nj,m}\Gamma^{n}_{ki} \\
&- W_{in,m}\Gamma^{n}_{kj} - W_{ij,n}\Gamma^{n}_{km} \\
&+ W_{lj}\Gamma^{l}_{mn}\Gamma^{n}_{ki} + W_{ln}\Gamma^{l}_{mi}\Gamma^{n}_{kj} + W_{lj}\Gamma^{l}_{ni}\Gamma^{n}_{km} \\
&+ W_{nl}\Gamma^{l}_{mj}\Gamma^{n}_{ki} + W_{il}\Gamma^{l}_{mn}\Gamma^{n}_{kj} + W_{il}\Gamma^{l}_{nj}\Gamma^{n}_{km}
\end{aligned}
$$

Theorem: B.3. *Given a metric of type 2.2 with ${}^{g}\nabla\overline{g} \equiv 0$ we have*

$$
{}^{G}\Delta\pi^*(h)_{ij} = {}^{g}\Delta h_{ij} + \frac{1}{2}\left({}^{\theta}F \cdot {}_{\theta}F \cdot h + h \cdot {}^{\theta}F \cdot {}_{\theta}F\right)_{ij} - \frac{1}{2}\left({}^{\theta}F \cdot h \cdot {}_{\theta}F\right)_{ij}
$$

$$
{}^{G}\Delta\pi^*(h)_{i\theta} = \frac{1}{2}(h \cdot d^*F)_{i\theta} - {}^{g}\nabla_l h_{in}\, {}_{\theta}F^{nl}
$$

$$
{}^{G}\Delta\pi^*(h)_{\theta\rho} = -\frac{1}{2}\left\langle {}_{\theta}F \cdot {}_{\rho}F, h\right\rangle
$$

Proof. Set $W = \pi^*(h)$. We recall the result of lemma B.2:

$$
\begin{aligned}
{}^G\Delta W_{\alpha\beta} = G^{\delta\epsilon}(&W_{\alpha\beta,\delta\epsilon} - W_{\nu\beta}\Gamma^{\nu}_{\delta\alpha,\epsilon} - W_{\alpha\nu}\Gamma^{\nu}_{\delta\beta,\epsilon} \\
&- W_{\nu\beta,\epsilon}\Gamma^{\nu}_{\delta\alpha} - W_{\alpha\nu,\epsilon}\Gamma^{\nu}_{\delta\beta} - W_{\mu\beta,\delta}\Gamma^{\mu}_{\epsilon\alpha} \\
&- W_{\alpha\mu,\delta}\Gamma^{\mu}_{\epsilon\beta} - W_{\alpha\beta,\mu}\Gamma^{\mu}_{\epsilon\delta} \\
&+ W_{\nu\beta}\Gamma^{\nu}_{\delta\mu}\Gamma^{\mu}_{\epsilon\alpha} + W_{\nu\mu}\Gamma^{\nu}_{\delta\alpha}\Gamma^{\mu}_{\epsilon\beta} + W_{\nu\beta}\Gamma^{\nu}_{\mu\alpha}\Gamma^{\mu}_{\epsilon\delta} \\
&+ W_{\mu\nu}\Gamma^{\nu}_{\delta\beta}\Gamma^{\mu}_{\epsilon\alpha} + W_{\alpha\nu}\Gamma^{\nu}_{\delta\mu}\Gamma^{\mu}_{\epsilon\beta} + W_{\alpha\nu}\Gamma^{\nu}_{\mu\beta}\Gamma^{\mu}_{\epsilon\delta})
\end{aligned}
$$

Thus we have:

$$
\begin{aligned}
{}^G\Delta W_{ij} = G^{\delta\epsilon}(&W_{ij,\delta\epsilon} - W_{\nu j}\Gamma^{\nu}_{\delta i,\epsilon} - W_{i\nu}\Gamma^{\nu}_{\delta j,\epsilon} \\
&- W_{\nu j,\epsilon}\Gamma^{\nu}_{\delta i} - W_{i\nu,\epsilon}\Gamma^{\nu}_{\delta j} - W_{\gamma j,\delta}\Gamma^{\gamma}_{\epsilon i} \\
&- W_{i\gamma,\delta}\Gamma^{\gamma}_{\epsilon j} - W_{ij,\gamma}\Gamma^{\gamma}_{\epsilon\delta} \\
&+ W_{\nu j}\Gamma^{\nu}_{\delta\gamma}\Gamma^{\gamma}_{\epsilon i} + W_{\nu\gamma}\Gamma^{\nu}_{\delta i}\Gamma^{\gamma}_{\epsilon j} + W_{\nu j}\Gamma^{\nu}_{\gamma i}\Gamma^{\gamma}_{\epsilon\delta} \\
&+ W_{\gamma\nu}\Gamma^{\nu}_{\delta j}\Gamma^{\gamma}_{\epsilon i} + W_{i\nu}\Gamma^{\nu}_{\delta\gamma}\Gamma^{\gamma}_{\epsilon j} + W_{i\nu}\Gamma^{\nu}_{\gamma j}\Gamma^{\gamma}_{\epsilon\delta})
\end{aligned}
\tag{B.1}
$$

Now at this point we simply note that the Christoffel symbol terms that don't involve any μ components simply comprise ${}^g\Delta k$. Now we will go term by term and compute what the rest of the operator is, only considering things which have not already appeared in ${}^g\Delta k$. So, in particular we see using lemma 2.2 that the terms of the form $W * \partial\Gamma$ and $\partial W * \Gamma$ all vanish. Next we compute

$$
\begin{aligned}
G^{\delta\epsilon}W_{\eta j}\Gamma^{\eta}_{\delta\gamma}\Gamma^{\gamma}_{\epsilon i} &= \frac{1}{4}G^{\delta\epsilon}\left(h_{lj}\ {}_{\theta}F^{l}_{\delta}\ {}^{\theta}F_{\epsilon i}\right) + \frac{1}{4}h_{lj}\ {}_{\theta}F^{l}_{\gamma}\ {}^{\theta}F^{\gamma}_{i} \\
&= \frac{1}{2}\left({}^{\theta}F \cdot\ {}_{\theta}F \cdot h\right)_{ij}
\end{aligned}
$$

and similarly

$$G^{\delta\epsilon}W_{\eta\mu}\Gamma^{\eta}_{\delta i}\Gamma^{\mu}_{\epsilon j} = \frac{1}{4}h_{nm}\ {}_{\mu}F^{n}_{i}\ {}^{\mu}F^{m}_{j}$$

$$= \ -\frac{1}{4}(\ {}^{\theta}F \cdot h \cdot_{\theta} F)_{ij}$$

Next we have

$$G^{\delta\epsilon}W_{\nu j}\Gamma^{\nu}_{\mu i}\Gamma^{\mu}_{\epsilon\delta} = \frac{1}{4}G^{\delta\epsilon}h_{kj}\ {}_{\theta}F^{k}_{i}\ {}^{\theta}F_{\delta\epsilon}$$

$$= 0$$

and

$$G^{\delta\epsilon}W_{\mu\eta}\Gamma^{\eta}_{\delta j}\Gamma^{\mu}_{\epsilon i} = \frac{1}{4}h_{mn}\ {}_{\theta}F^{n}_{j}\ {}^{\theta}F^{m}_{i}$$

$$= \ -\frac{1}{4}(\ {}^{\theta}F \cdot h \cdot_{\theta} F)_{ij}$$

Next

$$G^{\delta\epsilon}W_{i\eta}\Gamma^{\eta}_{\delta\mu}\Gamma^{\mu}_{\epsilon j} = \frac{1}{4}G^{\delta\epsilon}h_{in}\ {}_{\theta}F^{n}_{\delta}\ {}^{\theta}F_{\epsilon j} + \frac{1}{4}h_{in}\ {}_{\theta}F^{n}_{m}\ {}^{\theta}F^{m}_{j}$$

$$= \frac{1}{2}\left(h \cdot^{\theta} F \cdot_{\theta} F\right)_{ij}$$

And finally

$$G^{\delta\epsilon}W_{i\eta}\Gamma^{\eta}_{\mu j}\Gamma^{\mu}_{\epsilon\delta} = \frac{1}{4}G^{\delta\epsilon}h_{in}\ {}_{\theta}F^{n}_{j}\ {}^{\theta}F_{\epsilon\delta}$$

$$= 0$$

Collecting the above calculations gives

$${}^{G}\Delta W_{ij} = \ {}^{g}\Delta h_{ij} + \frac{1}{2}(\ {}^{\theta}F \cdot_{\theta} F \cdot h + h \cdot^{\theta} F \cdot_{\theta} F)_{ij}$$

Now we must compute ${}^G\Delta W_{i\theta}$. Again specializing lemma B.2 gives

$$\begin{aligned}
{}^G\Delta W_{i\theta} = G^{\delta\epsilon}(&W_{i\theta,\delta\epsilon} - W_{\nu\theta}\Gamma^{\nu}_{\delta i,\epsilon} - W_{i\nu}\Gamma^{\nu}_{\delta\theta,\epsilon} \\
&- W_{\nu\theta,\epsilon}\Gamma^{\nu}_{\delta i} - W_{i\nu,\epsilon}\Gamma^{\nu}_{\delta\theta} - W_{\mu\theta,\delta}\Gamma^{\mu}_{\epsilon i} \\
&- W_{i\mu,\delta}\Gamma^{\mu}_{\epsilon\theta} - W_{i\theta,\mu}\Gamma^{\mu}_{\epsilon\delta} \\
&+ W_{\nu\theta}\Gamma^{\nu}_{\delta\mu}\Gamma^{\mu}_{\epsilon i} + W_{\nu\mu}\Gamma^{\nu}_{\delta i}\Gamma^{\mu}_{\epsilon\theta} + W_{\nu\theta}\Gamma^{\nu}_{\mu i}\Gamma^{\mu}_{\epsilon\delta} \\
&+ W_{\mu\nu}\Gamma^{\nu}_{\delta\theta}\Gamma^{\mu}_{\epsilon i} + W_{i\nu}\Gamma^{\nu}_{\delta\mu}\Gamma^{\mu}_{\epsilon\theta} + W_{i\nu}\Gamma^{\nu}_{\mu\theta}\Gamma^{\mu}_{\epsilon\delta})
\end{aligned} \tag{B.2}$$

First of all we note that any term involving $W_{i\theta}$ or its derivatives vanishes. Next we compute

$$\begin{aligned}
-G^{\delta\epsilon}W_{i\eta}\Gamma^{\eta}_{\delta\theta,\epsilon} &= -\frac{1}{2}G^{\delta\epsilon}h_{in\ \theta}F^{n}_{\delta,\epsilon} \\
&= \frac{1}{2}h^{n}_{i}d^{*}{}_{\theta}F_{n}
\end{aligned}$$

and

$$\begin{aligned}
-G^{\delta\epsilon}\left(W_{i\eta,\epsilon}\Gamma^{\eta}_{\delta\theta} + W_{i\mu,\delta}\Gamma^{\mu}_{\epsilon\theta}\right) &= -G^{\delta\epsilon}h_{in,\epsilon\ \theta}F^{n}_{\epsilon} \\
&= -\ {}^{g}\nabla_{l}h_{in\ \theta}F^{nl}
\end{aligned}$$

Next we note that

$$G^{\delta\epsilon}(W_{\eta\mu}\Gamma^{\eta}_{\delta i}\Gamma^{\mu}_{\epsilon\theta} + W_{\mu\eta}\Gamma^{\eta}_{\delta\theta} + W_{i\eta}\Gamma^{\eta}_{\delta\mu}\Gamma^{\mu}_{\epsilon\theta}) = 0$$

Thus we have

$${}^G\Delta W_{i\theta} = \frac{1}{2}(h\cdot d^{*}{}_{\theta}F)_{i} -\ {}^{g}\nabla_{l}h_{in\ \theta}F^{nl}$$

To compute the vertical component, we again specialize lemma B.2 to get

$$\begin{aligned}
{}^{G}\Delta W_{\theta\rho} = G^{\delta\epsilon}(&W_{\theta\rho,\delta\epsilon} - W_{\nu\rho}\Gamma^{\nu}_{\delta\theta,\epsilon} - W_{\theta\nu}\Gamma^{\nu}_{\delta\rho,\epsilon} \\
&- W_{\nu\rho,\epsilon}\Gamma^{\nu}_{\delta\theta} - W_{\theta\nu,\epsilon}\Gamma^{\nu}_{\delta\rho} - W_{\mu\rho,\delta}\Gamma^{\mu}_{\epsilon\theta} \\
&- W_{\theta\mu,\delta}\Gamma^{\mu}_{\epsilon\rho} - W_{\theta\rho,\mu}\Gamma^{\mu}_{\epsilon\delta} \\
&+ W_{\nu\rho}\Gamma^{\nu}_{\delta\mu}\Gamma^{\mu}_{\epsilon\theta} + W_{\nu\mu}\Gamma^{\nu}_{\delta\theta}\Gamma^{\mu}_{\epsilon\rho} + W_{\nu\rho}\Gamma^{\nu}_{\mu\theta}\Gamma^{\mu}_{\epsilon\delta} \\
&+ W_{\mu\nu}\Gamma^{\nu}_{\delta\rho}\Gamma^{\mu}_{\epsilon\theta} + W_{\theta\nu}\Gamma^{\nu}_{\delta\mu}\Gamma^{\mu}_{\epsilon\rho} + W_{\theta\nu}\Gamma^{\nu}_{\mu\rho}\Gamma^{\mu}_{\epsilon\delta})
\end{aligned} \tag{B.3}$$

Again any term involving $W_{i\theta}, W_{\theta\theta}$ or any of their coordinate derivatives vanishes. Indeed the only nonzero terms are the following:

$$\begin{aligned}
2G^{\delta\epsilon}W_{\nu\mu}\Gamma^{\nu}_{\delta\theta}\Gamma^{\mu}_{\epsilon\rho} &= \frac{1}{2}G^{\delta\epsilon}h_{mn}\ {}_{\theta}F^{n}_{\delta}\ {}_{\rho}F^{m}_{\epsilon} \\
&= -\frac{1}{2}\langle\ {}_{\theta}F \cdot\ {}_{\rho}F, h\rangle
\end{aligned}$$

□

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Biography

Jeffrey Streets was born on July 2, 1980 in Hinsdale, Illinois. He received his BA in Mathematics from University of Chicago in May 2002, and his MA in Mathematics from Duke University in December 2003. In the fall of 2006 he was awarded an NSF Visiting Student Research Collaborator grant and spent the fall semester at Princeton University. Also in January 2007 he was awarded an NSF Postdoctoral Fellowship to work at Princeton University.