On Hyperbolic Monopoles

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Declaration

The contents of Chapters 1 to 10 are expository and the remaining chapters of the thesis from Chapter 11 onwards are my original contributions. Chapter 14 is a collaboration with Michael Murray.

Letter

> But, at night, under the full moon, they (mathematicians) dream, they float among the stars and wonder at the miracle of the heavens. They are inspired. Without dreams there is no art, no mathematics, no life. - Sir Michael Atiyah.

One begins the undertaking that is doctoral study with the illusion that one will produce a Great Work. One finishes stripped of those illusions. So it was with me. Completing this thesis was but a first step. Though I leave this chapter of my life, the gauge theory of magnetic monopoles will forever remain my first step.

My goals at the outset were rather more lofty. I wished to examine some ideas on the Kapustin-Witten theory. As is common with ambitious expeditions, the pathfinding party gets lost. I was shaken by the vastness, the sheer depth of the theory that I would need to feel at ease at before even beginning on the actual conquest.

Paul, my thesis supervisor, encouraged me to keep my ambitions but I felt panic at my lack of progress. I shelved several months of study to divert my attention to a humble component of Kapustin-Witten theory: magnetic monopoles. I sought some concrete calculations and saw the equivariant Atiyah-Singer index calculations which occur in magnetic monopole gauge theory as my chance.

The going was tough and the theory behind my new direction was no less deep. The literature surrounding magnetic monopole gauge theory goes back to the 1980s, at the height of the Golden Age of the *Oxford School*, work for which Atiyah and Donaldson received Fields Medals.

Over and over, I was advised to study Atiyah. But I found his writing impenetrable. I turned to other sources who would explain the material better to mere mortals, surely. In the end, I returned to Atiyah and everything was so elegant, so *clear*. Volume 5 of Atiyah′s Collected Works became the book that I carried around with me, ate and slept next to. It was the breadcrumb trail out of the maze that I had finally found.

With the above experiences in mind, I wrote this thesis to be a roadmap to the reader, to leave an opening in the hedge behind me for those that come after.

Nevertheless, the Italian Lectures of Atiyah and Differential Forms in Algebraic Topology by Bott and Tu are still indispensable and explain what I do not have the space for.

If despite all this, the reader finds themselves forlorn, they should remember that the fruits of mathematical inquisition is not only the subject matter but the struggle itself and all the skills of analysis, problem solving, persistence and imagination which come with it.

Good luck and have fun.
Joe.

Acknowledgements

Dedicated to my beloved parents, Chee Keong Chan and Wai Fong Soo.

My first thanks goes to my PhD supervisor Paul Norbury without whom this thesis could not have come to be. I thank him for guiding me on my doctoral journey and I owe my understanding of complex geometry to him. My education in mathematics has been made all the richer by masterful mentors Arun Ram, Norman Do, Ben Burton and Nora Ganter. I also thank James Barnard for sitting on my supervisory panel.

I thank the Australian Department of Education and Training of Australia for awarding me the Australian Postgraduate Award which supported me for three and a half years of my doctoral programme. I thank the University of Melbourne, especially the Department of Mathematics.

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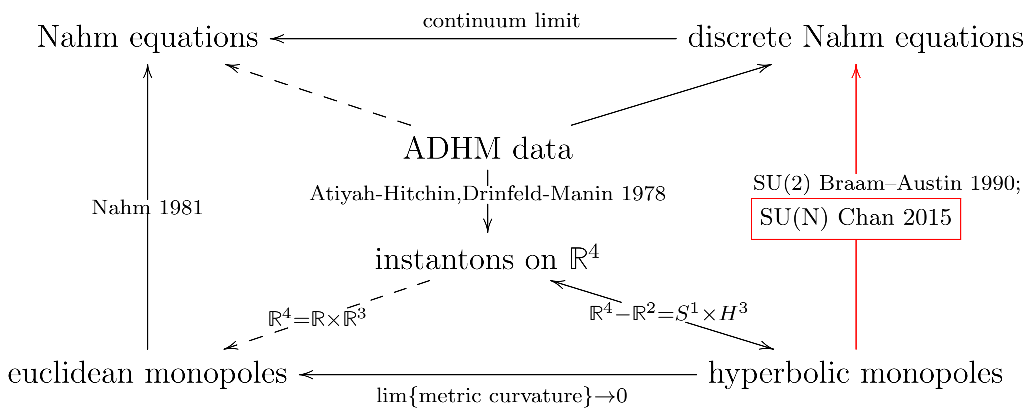
CHAPTER 1

Introduction

This thesis, On Hyperbolic Monopoles, completes the following circle of ideas:



The algebro-geometric techniques of hyperbolic monopoles are intimately tied to the techniques developed in the study of its gauge-theoretical predecessors - instantons and euclidean monopoles. Hence, this thesis consists of three parts which may be read independently of each other. The trinity of euclidean monopoles, instantons and hyperbolic monopoles are unified by the use of twistorial, homological and inverse scattering methods in their study.

Instantons, euclidean monopoles and hyperbolic monopoles are solutions to the following systems of partial differential equations:

$$F_A = \star_{\mathbb{R}^4} F_A$$

$$F_A = \star_{\mathbb{R}^3} D_A \Phi$$

$$F_A = \star_{H^3} D_A \Phi.$$

Since $F_A = D_A A$ and in three dimensions, the fourth component of A is Φ, these three equations are really the same system of non-linear PDEs written for euclidean 4-space $\mathbb{R}^4$, euclidean 3-space $\mathbb{R}^3$ and hyperbolic 3-space H^3. In all three cases, they arise as a minimal action condition.

Our story begins with the mathematical treatment of instantons. Yang-Mills(-Higgs) theory gained popularity in the field of particle physics in the 1960s but a mathematical treatment of Yang-Mills instantons only occured in 1978 when Atiyah and Hitchin, and working independently of the former two, Drinfeld and Manin, wrote the landmark paper Construction of Instantons [Ati+78]. The Atiyah-Drinfeld-Hitchin-Manin (ADHM) paper showed how matrices with some constraints, the so-called ADHM matrix data, could be used to explicitly construct instantons. In the following year, Atiyah spoke at a series of lectures at the Scuola Normale Superiore in Pisa. The notes from his “Italian lectures” [Ati79] have become an indispensable exposition and reference for the geometry of instantons.

In the prior year, Atiyah and Ward [AW77] had already considered the relevance of algebraic geometry to the construction of instantons. The Atiyah-Ward ansatz from this paper was applied by Ward [War81], Prasad and Rossi [PR81], and Corrigan and Goddard [CG81] to treat euclidean monopoles. They considered monopoles in $\mathbb{R}^3$ as “time-translation invariant” instantons (but with infinitie action).

In 1981, Nahm [Nah83] found that replacing the ADHM matrices with differential operators in the ADHM construction of instantons yielded a matrix-valued, non-linear system of ordinary differential equations on an interval called Nahm equations. The Nahm equations are in correspondence with and can be used to construct monopoles in $\mathbb{R}^3$.

The following year, in Monopoles and Geodesics [Hit82], Hitchin studied euclidean monopoles via its mini-twistor space $\mathcal{T}$ - the space of geodesics in $\mathbb{R}^3$. Hitchin showed that the data of a monopole could be used to construct a holomorphic vector bundle $\tilde{E}$ on the twistor space. The boundary conditions for a euclidean monopole imply that there exists special line sub-bundles L^+, L^-. The line bundles determine an algebraic curve $S \subset \mathcal{T}$, the spectral curve of the monopole.

In 1984, following the work of Nahm, Donaldson [Don84b] classified the euclidean monopoles. He showed that the solutions of the Nahm equations are in bijection with the based rational maps $\mathbb{P}^1 \to \mathbb{P}^1$. In the same year, Donaldson wrote Instantons and Geometric Invariant Theory [Don84a] where it was shown that the ADHM construction could be essentially carried out on $\mathbb{P}^2$ by entirely complex methods.

Also in 1984, Atiyah [Ati84a; Ati84b] began the mathematical study of hyperbolic monopoles by considering them as circle-invariant instantons and showing that they too were in bijection with rational maps.

Murray published Non-abelian Magnetic Monopoles [Mur83; Mur84], a generalisation of Hitchin's Monopoles and Geodesics to monopoles with classical Lie groups $\mathrm{SU}(N)$, $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$, as their structure group. This was followed up by algebro-geometric papers Hurtubise-Murray [HM89] and Hurtubise [Hur89], studying the extension of all the previous results of Hitchin and Donaldson to the classical Lie groups.

The next breakthrough came in 1990 when Braam–Austin [BA90] showed that hyperbolic monopoles gave rise to a discrete analogue of the Nahm equations and the *fundamental* difference that, unlike their euclidean counterparts, hyperbolic monopoles were entirely determined by their values on the asymptotic boundary sphere of hyperbolic 3-space H^3. The work only limits itself to hyperbolic monopoles of mass $\mathbb{Z} + \frac{1}{2}$. This boundary data consists of a holomorphic map $\mathbb{P}^1 \to \mathbb{P}^k$.

In 1996, Murray-Singer [MS96] study, from a twistorial point of view, the spectral curves of $\mathrm{SU}(N)$ hyperbolic monopoles with no restriction on the mass. However, because they did not treat hyperbolic monopoles as circle-invariant instantons, they had to posit boundary conditions and it is not clear if they are equivalent to the boundary conditions automatically imposed in the standard treatment.

In 2000, Murray-Singer [MS00] showed that the discrete Nahm equations were completely integrable: that they come with spectral curves and the time evolution of the discrete Nahm equations can be described as a walk in the Jacobian of the spectral curve. The formula of the spectral curve here is in terms of the solutions of the discrete Nahm equations. Three years later, Murray, Norbury and Singer [MNS03] show that for charge k and any mass, there exists a “holomorphic sphere" $\mathbb{P}^1 \to \mathbb{P}^k$ which determines the monopole and defines a spectral curve for the monopole.

The study of hyperbolic monopoles then turned to amassing a collection of symmetric monopoles[MS14; NR07; BHS15]. There are attempts, starting with Braam–Austin [BA90] to look for some model of low energy scattering of monopoles by finding geodesics in the moduli space of monopoles. Unlike in the euclidean case as studied by Atiyah and Hitchin [AH85; Ati+85; AH14], the L^2 metric on the hyperbolic moduli space is not finite. Attempts with a different metric, such as the metric on the $\mathrm{U}(1)$ connections on the boundary S^2 which determine the monopole, have not yielded notable success. There has been some progress in the charge $k = 1, 2$ cases [Hit93; BCS15].

My contribution to this long progression of ideas, as detailed in this thesis, is a generalisation of the $\mathrm{SU}(2)$ results of Braam–Austin by different methods (closer to those of Atiyah's work on hyperbolic monopoles) to the $\mathrm{SU}(N)$ case [Cha15] and then further to the other classical groups $\mathrm{Sp}(N)$ and $\mathrm{SO}(N)$ cases. With Murray, I also write down the spectral curve in terms of solutions of the discrete Nahm equations. These generalisations take the same form as the euclidean counterparts of Murray, Hurtubise-Murray and Murray-Singer.

Catalogue of Results

Let $k_1, \dots, k_N \in \mathbb{Z}$ satisfy $\sum_{i=1}^{N} k_i = 0$. Let $p_1, \dots, p_N \in \mathbb{Z}$ be ordered $p_1 < \dots < p_N$ (maximal symmetry breaking) and satisfy $\sum_{i=1}^{N} p_i = 0$. ($p_1, \dots, p_N \in \frac{1}{2} + \mathbb{Z}$ is allowed for the case of $SU(2N)$.)

The main theorem of this thesis, found in chapter 12 is, Theorem 38 p. 88,

Main Theorem. *There is an equivalence between*

(1) *framed $SU(N)$ monopoles (A, ϕ) on hyperbolic space H^3 of mass $(p_i, \dots, p_{N-1})$ and charge $(k_1, \dots, k_{N-1})$, and*
(2) *solutions of the $(N-1)$-interval discrete Nahm equations of type $(p_1, \dots, p_{N-1}; k_1, \dots, k_{N-1})$.*

The latter, a solution to the $(N-1)$-interval discrete Nahm equations are block matrices $(\{\beta_i\}, \{\gamma_i\}, \{a_{p_j}\}, \{b_{p_j}\})$ which satisfy,

$$\begin{cases} \beta_{i+\frac{1}{2}}\gamma_{i+1} - \gamma_{i+1}\beta_{i+\frac{3}{2}} + b_{i+1}a_{i+1} = 0 & \text{for } i+1 = p_j,\ 2 \le j \le N-1 \\ \beta_{i+\frac{1}{2}}\gamma_{i+1} - \gamma_{i+1}\beta_{i+\frac{3}{2}} = 0 & \text{otherwise} \end{cases} \tag{1.1}$$

$$\begin{cases} \left[\beta_{i+\frac{1}{2}}, \beta^*_{i+\frac{1}{2}}\right] + \gamma_{i+1}\gamma^*_{i+1} - \gamma^*_i\gamma_i - a^*_i a_i = 0 & \text{when } i = -p_j,\ 1 \le j \le N-1 \\ \left[\beta_{i+\frac{1}{2}}, \beta^*_{i+\frac{1}{2}}\right] + \gamma_{i+1}\gamma^*_{i+1} - \gamma^*_i\gamma_i + b_{i+1}b^*_{i+1} = 0 & \text{when } \begin{matrix} i+1 = p_j,\ 2 \le j \le N-1 \\ i = p_N - 1 \end{matrix} \\ \left[\beta_{i+\frac{1}{2}}, \beta^*_{i+\frac{1}{2}}\right] + \gamma_{i+1}\gamma^*_{i+1} - \gamma^*_i\gamma_i = 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

The ADHM construction uses matrix data to construct holomorphic bundles over $\mathbb{CP}^3$ as the cohomology of a *monad.* The Main Theorem hinges on a technical result Proposition 36 p.83, proven in Chapter 11, regarding the $\mathbb{C}^\times$-equivariant monad associated to a monopole with mass numbers $(p_1, \dots, p_{N-1})$ and charge numbers $(k_1, \dots, k_{N-1})$,

Proposition. *Let there be a $\mathbb{C}^\times$-action on $\mathbb{P}^3$, with $c \in \mathbb{C}^\times$,*

$$[x : y : z : w] \mapsto [c^{-1/2}x : c^{1/2}y : c^{-1/2}z : c^{1/2}w].$$

Let E be a $\mathbb{C}^\times$-equivariant holomorphic vector bundle on $\mathbb{P}^3$ corresponding to a monopole with mass $(p_1, \dots, p_N)$, and charge $(k_1, \dots, k_N)$.

Then the decomposition of the monad for E,

$$\underline{H} \xrightarrow{A_X} \underline{K} \xrightarrow{B_X} \underline{L}$$

restricted to $\mathbb{P}^1_+$*, into weight* p *components* $\mathbb{C}_p$ *with respect to the* $\mathbb{C}^\times$*-action is*

$$H = \mathbb{C}^{k_1}_{p_1} \oplus \ldots \oplus \mathbb{C}^{k_1}_{p_2-1} \oplus \mathbb{C}^{k_1+k_2}_{p_2} \oplus \mathbb{C}^{k_1+k_2}_{p_2+1} \oplus \ldots \oplus \mathbb{C}^{-k_N}_{p_N-1},$$

$$K = \mathbb{C}^{k_1+1}_{p_1} \oplus \mathbb{C}^{2k_1}_{p_1+1} \oplus \ldots \oplus \mathbb{C}^{2k_1}_{p_2-1} \oplus \mathbb{C}^{2(k_1+k_2)+1}_{p_2} \oplus \mathbb{C}^{2(k_1+k_2)}_{p_2+1} \oplus \ldots \oplus \mathbb{C}^{2(k_1+\ldots+k_{N-1})}_{p_N-1} \oplus \mathbb{C}^{-k_N+1}_{p_N},$$

$$L = \mathbb{C}^{k_1}_{p_1+1} \oplus \ldots \oplus \mathbb{C}^{k_1}_{p_2} \oplus \mathbb{C}^{k_1+k_2}_{p_2+1} \oplus \mathbb{C}^{k_1+k_2}_{p_2+2} \oplus \ldots \oplus \mathbb{C}^{-k_N}_{p_N}.$$

The ADHM matrices $(\alpha_1, \alpha_2, a, b)$ which preserve this weight space decomposition have zeros everywhere except for some block matrices $(\{\beta_i\}, \{\gamma_i\}, \{a_{p_j}\}, \{b_{p_j}\})$ which satisfy the discrete Nahm equations.

A corollary of the Main Theorem is a generalisation of the Braam–Austin boundary values result found in Chapter 13. (Theorem 48, p. 93)

THEOREM. *Let* (A, Φ) *be a framed* $SU(N)$ *hyperbolic monopole of charge* $(k_1, \ldots, k_N)$ *and mass* $(p_1, \ldots, p_N)$*. Then*

(1) *the* $(N-1)$ *tuple of* $U(1)$ *connections* $(A_1, \ldots, A_{N-1})$ *on* S^2_∞ *determines the connection* A *(up to gauge transformations);*

(2) *there exists for* $i = 1, \ldots, N-1$*, holomorphic maps*

$$F_i : \mathbb{P}^1 \to Fl(k_1 + \ldots + k_i, k_1 + \ldots + k_i + 1, 2k_1 + \ldots + 2k_{i-1} + k_i + 1)$$

into the manifold of two term partial flags for which each A_i *is the pullback of the unitary invariant connection on the "hyperplane bundle"* $\mathcal{O}(1,-1)$ *of the* i*-th flag manifold; and*

(3) *the map* $A \mapsto (A_1, \ldots, A_{N-1})$ *is an immersion of the moduli space of* $SU(N)$ *framed hyperbolic monopoles in the moduli of* $(N-1)$ *tuples of* $U(1)$ *connections on* S^2*.*

Another corollary is an explicit formula for the rational map of an SU(N) hyperbolic monopole, found in Chapter 13. (Proposition 44, p.92)

PROPOSITION. *Let* $(\{\gamma_i\}, \{\beta_i\}, \{a_{-p_j}\}, \{b_{-p_{j+1}}\})$ *be a solution of the* $(N-1)$*-interval discrete Nahm equations of type* $(p_1, \ldots, p_{N-1}; k_1, \ldots, k_{N-1})$*. Then the solution can be put into the form* $(\{\beta_{[-p_i]}\}, \{\gamma_{[-p_i]}\}, \{a_{[-p_i]}\}, \{b_{[-p_{i+1}]}\})$ *and the rational map*

$$f : \mathbb{P}^1 \to Fl_{full}(N)$$

$$x \mapsto (V_1, \ldots, V_{N-1}), \dim V_i = i,$$

into the manifold of full flags in $\mathbb{C}^N$ can be written as the maps $(r_1(x), \ldots, r_{N-1}(x))$

$$\begin{aligned} r_{N-1}(x) &= (-h)^{p_{N-1}-p_N} a_{[p_{N-1}]} \left(x - \beta_{[p_{N-1}]}\right)^{-1} b_{[p_N]} r_N(x) \\ &\vdots \\ r_j(x) &= \sum_{i=j+1}^{N} (-h)^{p_j - p_i} a_{[p_j]} \left(x - \beta_{[p_j]}\right)^{-1} b_{[p_i]}^{k_1+\ldots+k_j} r_i(x) \\ &\vdots \\ r_1(x) &= \sum_{i=2}^{N} (-h)^{p_1 - p_i} a_{[p_1]} \left(x - \beta_{[p_1]}\right)^{-1} b_{[p_i]}^{k_1} r_i(x) \end{aligned}$$

where for each $x \in \mathbb{P}^1$, $r_{N-1}(x)$ specifies an $(N-1)$-dimensional linear subspace in $\mathbb{C}^N$ and each successive $r_i(x)$ specifies an i-dimensional linear subspace inside the $(i+1)$-dimensional linear subspace specified by $r_{i+1}(x)$. The superscript $k_1 + \ldots + k_j$ indicates that only the first $k_1 + \ldots + k_j$ entries of the vector are involved.

The proof requires the following lemma.

LEMMA. *On $\mathbb{P}^2 - \mathbb{P}^1_-$, there exists unique holomorphic sub-bundles $L_1^+ \subset L_2^+ \subset \ldots \subset L_{N-1}^+$ of E which is preserved by the $\mathbb{C}^\times$-action and each L_i^+ restricted to $\mathbb{P}^1_+$ coincides with the last i-th factors.*

The spectral curve can be calculated quite simply when one is in possession of discrete Nahm data of a hyperbolic monopole.

THEOREM. *The spectral curve S is the variety of points $(p, q) \in Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ satisfying $\det B_{\hat{q}} A_p = 0$.*

The natural generalisation is to ask for discrete Nahm equations of $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$ hyperbolic monopoles. The Theorem 56 p.105 from Chapter 15 summarises this result.

THEOREM. *Let $(\{\gamma_i\}, \{\beta_i\}, \{a_{p_j}\}, \{b_{p_{j+1}}\})$ be a solution of the $(N-1)$-interval discrete Nahm equations of type $(p_1, \ldots, p_{N-1}; k_1, \ldots, k_{N-1})$. If $p_i = p_{N+1-i}$ and $k_i = -k - N + 1 - i$ then*

(1) *if N is even, $\beta_i = \beta_{-i}^T$, $\gamma_i = \gamma_{-i}^T$ and for $i > 0$, $b_i = -a_{-i}^T$ and $b_{-i} = a_i^T$ then this is a solution of the $Sp(N/2)$ discrete Nahm equations;*

(2) *if κ is even, $\beta_i = \beta_{-i}^T$, $\gamma_i = \gamma_{-i}^T$ except for $i = 0$ where $\gamma_0 = -\gamma_0^T$ and for $i > 0$, $b_i = -a_{-i}^T$ and $b_{-i} = a_i^T$ then this is a solution of the $SO(N)$ discrete Nahm equations.*

How to read this thesis

This thesis was written such that the three parts could be read independently. There are references to the other parts since the three are very tightly related and the context for some of the results arise from the other two parts.

All of my original contributions are in Part 3, on hyperbolic monopoles. However, I have laboured to make the other two parts accessible introductions to the mathematical theories of euclidean monopoles and instantons.

The mathematical prerequisites are mostly the differential geometric theories of connections and fibre bundles. There are good books which treat these and my favourite is the excellent tome by Bott and Tu [BT82]. Other choices include Milnor-Stasheff [MS74] (my PhD supervisor's rite of passage book), Lee [Lee03], Chern-Chen-Lam [LCC00], Kobayashi-Nomizu [KN63] (very complete) and (the verbose) Spivak [Spi70]. Atiyah's Italian Lectures [Ati79] or Donaldson-Kronheimer [DK90] are also good places to start for the more mature reader. I have an appendix on differential geometry and Lie groups where I remind the reader of some definitions and discuss the subtle concept of connections.

Part 1

Euclidean Monopoles

CHAPTER 2

SU(2) monopoles

The theory of the euclidean SU(2) magnetic monopole is the simplest and most well-trod theory of non-abelian monopoles. The equivalences between the various objects equivalent to monopoles can be seen in Figure 2.1.

(2.1)

$$
\begin{tikzcd}
\{\text{spectral curves } S\} \arrow[r, leftrightarrow] & \{\text{solutions to the Nahm equations } (T_1, T_2, T_3)\} \arrow[dl] \arrow[d, leftrightarrow] \\
\{\text{magnetic monopoles } (A, \Phi)\} \arrow[u] \arrow[r] & \{\text{rational maps}\}
\end{tikzcd}
$$

In this chapter, I show how a spectral curve and a rational map can be produced from the data of an SU(2) magnetic monopole. In the following chapter, I discuss the Nahm equations. Finally, the case of SU(N) will be dealt with in Monopoles of the Classical Groups. The Sp(N) and SO(N) monopoles are special cases of the SU(N) monopole arising from embedding Sp(N) and SO(N) into SU($2N$).

Definition 1. Let P be the trivial principal SU(2) bundle on $\mathbb{R}^3$ with its euclidean metric and the Hodge star dual $\star_{\mathbb{R}^3}$. A euclidean SU(2) monopole (A, Φ) of charge $k \in \mathbb{Z}$, is a gauge equivalence-class of

- a connection A on P with curvature F_A, and
- a section Φ of the adjoint bundle ad $P = P \times_{\mathrm{Ad}} \mathfrak{su}(2)$ (a Higgs field),

with asymptotic conditions equivalent to F_A being finite with respect to the L_2 norm,

$$\|\Phi\| = \left(-\frac{1}{2}\mathrm{Tr}(\Phi^2)\right) = 1 - \frac{k}{r} + O(r^{-2}) \tag{2.2}$$

$$\frac{\partial\|\Phi\|}{\partial\Omega} = \left((\frac{\partial\|\Phi\|}{\partial\theta}) + \sin^2\theta(\frac{\partial\|\Phi\|}{\partial\phi})\right)^{1/2} = O(r^{-2}) \tag{2.3}$$

$$\|\nabla\Phi\| = O(r^{-2}) \tag{2.4}$$

and satisfying the Bogolmonyi equations

$$F_A = \star_{\mathbb{R}^3} D_A \Phi. \tag{2.5}$$

Given a gauge transformation $g : \mathbb{R}^3 \to \mathrm{SU}(2)$ preserving the above asymptotic conditions, there is an equivalence relation on (A, Φ)

$$
\begin{aligned}
A &\sim g^{-1}Ag + g^{-1}dg, \\
\Phi &\sim g^{-1}\Phi g.
\end{aligned}
$$

Example 2 (Prasad-Sommerfield monopole)**.** The Prasad-Sommerfield monopole centred at the origin is a $k = 1$ monopole with Higgs field $\Phi : \mathbb{R}^3 \to \mathfrak{su}(2)$,

$$
\Phi = \frac{\sum_i x_i e_i}{r} \begin{bmatrix} \coth(r) - \dfrac{1}{r} & 0 \\ 0 & -\coth(r) + \dfrac{1}{r} \end{bmatrix}
$$

in spherical coordinates (r, ϕ, θ) and where the e_i is an orthonormal basis of $\mathfrak{su}(2)$. From Φ, it is possible to solve for A with the Bogolmonyi equations (2.5).

Generally, ordinary differential equations (ODEs) are easier to study than partial differential equations (PDEs). The strategy which was adopted by Hitchin in his seminal paper Monopoles and Geodesics [Hit82] was to associate a vector bundle to the geodesic and then to lift it to a holomorphic vector bundle on the "mini-twistor" space. This kind of technique was used earlier as a line of attack on instantons. The holomorphic vector bundle arises as the solutions to an ODE problem and it is in this sense that the PDE has been reduced to an ODE. More details will follow after I define the mini-twistor space.

Mini-twistor space

The mini-twistor space $\mathcal{T}$ (mini because it is the lower-dimensional analogue of the twistor space in the study of instantons which predates the use of this technique in the study of monopoles) models the space of (oriented) geodesics of $\mathbb{R}^3$ and the correspondence looks like

$$
\{\text{geodesics in } \mathbb{R}^3\} \rightsquigarrow \{\text{points of } \mathcal{T}\},
$$

$$
\begin{tikzcd}
 & S^2 \times \mathbb{R}^3 \arrow[dl, "p"'] \arrow[dr, "p_2"] & \\
\mathcal{T} & & \mathbb{R}^3.
\end{tikzcd}
$$

Geodesics in $\mathbb{R}^3$ are simply the straight lines,

$$
\gamma = \{x \in \mathbb{R}^3 \mid x = v + tu,\ v \in \mathbb{R}^3, u \in S^2, t \in \mathbb{R}, u \cdot v = 0\}.
$$

The reader is invited to view the geodesics of $\mathbb{R}^3$ as a choice of a vector $v \in \mathbb{R}^3$ and a unit vector $u \in S^2 \subset \mathbb{R}^3$ orthogonal to v (this requires a metric). The mini-twistor space $\mathcal{T}$ is isomorphic to

$$\mathcal{T} \simeq \{(u, v) \in S^2 \times \mathbb{R}^3 \mid u \cdot v = 0\} \simeq TS^2,$$

the tangent space of the two sphere $S^2 = \mathbb{P}^1$.

The mini-twistor space $\mathcal{T}$ can be constructed as the quotient of $S^2 \times \mathbb{R}^3$ by the "flow"

$$(u, v) \sim (u, v - tu).$$

If the quantity $(u \cdot v)$ is a time parameter t for the flow then the representative at time 0 is exactly the choice of v which satisfies $u \cdot v = 0$.

There is a complex structure J on $\mathcal{T}$ which acts on $(\dot{u}, \dot{v}) \in T\mathcal{T}$ by

$$(\dot{u}, \dot{v}) \mapsto J \cdot (\dot{u}, \dot{v}) = (\dot{u} \times u, \dot{v} \times u),$$

and a real structure τ which acts as the (negative) antipodal map on S^2,

$$(u, v) \mapsto (-u, v).$$

The real structure is the symmetry transformation arising from the reversal of the orientations of the geodesics of $\mathbb{R}^3$.

Given an $\mathrm{SU}(2)$ monopole (A, Φ) for a principal bundle $P \to \mathbb{R}^3$, we say that the vector bundle E is associated to P if $E = P \times_\rho \mathbb{C}^2$ where $\rho : \mathrm{SU}(2) \to \mathrm{GL}(\mathbb{C}^2)$ is a representation of $\mathrm{SU}(2)$ (here, the defining representation) so for any $[x, v] \in E$,

$$(x, v) \sim (xg^{-1}, \rho(g)v).$$

Now, we will "reduce" the PDE to an ODE. For any geodesic $\gamma \in \mathcal{T}$ defined by $(u, v) \in S^2 \times \mathbb{R}^3$, we can write down an ODE (Hitchin's equation, not to be confused with Hitchin's equation for Higgs pairs) for the sections $s \in \Gamma(E|_\gamma)$ of E restricted to $\gamma \subset \mathbb{R}^3$,

$$(\nabla_t^A - i\Phi)s = 0, \tag{2.6}$$

where t is a parameter on γ. The solutions of (2.6) define a vector bundle $\tilde{E} \to \mathcal{T}$ (the "lift" of E to the mini-twistor space) with fibre at $(u, v) \in \mathcal{T}$,

$$\tilde{E}_{(u,v)} = \{s \in \Gamma(\gamma, E|_\gamma) \mid (\nabla_t^A - i\Phi)s = 0\}.$$

The crucial property of $\tilde{E}$ is that it is holomorphic so the tools of complex geometry can be brought to bear on our problem - in particular, spectral curves can be associated to monopoles. I will spend the rest of the section providing enough technical detail to convince the reader of the non-trivial claim that $\tilde{E}$ is holomorphic.

The first of the Bogolmonyi equations (2.5) can be written as an integrability condition for (2.6),

$$[\partial_{\bar{z}}^A, \nabla_t^A - i\Phi] = 0.$$

This integrability condition implies that the $\bar{\partial}$-operator which we will define on $\tilde{E}$,

$$\bar{\partial} : \Gamma(\tilde{E}) \to \Gamma\left(\tilde{E} \otimes \bigwedge^{0,1}\right)$$

satisfying the Leibniz rule

$$\bar{\partial}(fs) = f\bar{\partial}s + s \otimes \bar{\partial}f \text{ for } f \in \Gamma(\mathcal{T}),$$

also satisfies $\bar{\partial}^2 = 0$. The Newlander-Nirenberg theorem says that if the operator satisfies $\bar{\partial}^2 = 0$ then $\tilde{E}$ is holomorphic.

THEOREM 3 (Hitchin [Hit82]). *If (A, Φ) is a magnetic monopole in $\mathbb{R}^3$ then $\tilde{E}$ is naturally a holomorphic vector bundle on the mini-twistor space $\mathcal{T}$ such that*

(1) *$\tilde{E}$ is trivial on real sections of $\mathcal{T} \to \mathbb{P}^1$,*
(2) *$\tilde{E}$ has a symplectic structure, and*
(3) *$\tilde{E}$ has a quaternionic structure, ie. an anti-holomorphic linear map*

$$\sigma : \tilde{E}_z \to \tilde{E}_{\tau z}$$

such that $\sigma^2 = -1$.

Now, I will define the $\bar{\partial}$-operator. Remember that $\mathcal{T}$ was considered as a quotient of $S^2 \times \mathbb{R}^3$ by the "flow" relation,

$$(u, v) \sim (u, x - tu).$$

We would like to define the $\bar{\partial}$-operator on sections over $S^2 \times \mathbb{R}$ instead of on sections over $\mathcal{T}$.

The bundle $E \to \mathbb{R}^3$ and connection ∇^A can be lifted to $S^2 \times \mathbb{R}^3$ by pulling back along the second projection map to get $p_2^* E \to S^2 \times \mathbb{R}^3$ and $p_2^* \nabla^A$. A section $\hat{s} \in \Gamma(S^2 \times \mathbb{R}^3, p_2^* E)$ which satisfies $(p_2^* \nabla^A - i\Phi)\hat{s} = 0$ projects to a section $s \in \Gamma(\mathcal{T}, \tilde{E})$. Under the exponential map

$$\exp : S^2 \times \mathbb{R} \to \mathbb{R}^3$$

given by $(u, t) \mapsto ut$, $\hat{s}$ arises from a section $\hat{s} \in \Gamma(S^2 \times \mathbb{R}, p_2^* E)$.

The operator $\bar{\partial}$ is defined by

$$(\bar{\partial}s)^\wedge = \nabla^{0,1}\hat{s} = (\nabla_{\partial x} + i\nabla_{\partial y})\hat{s}\, d\bar{z}, \tag{2.7}$$

where $x + iy = z \in S^2$.

The first property on $\bar{\partial}$ is satisfied due to the Bogolmonyi equations. The second property arises because

$$(\bar{\partial}^2 \hat{s})^\wedge = F^{0,2}\hat{s},$$

is a holomorphic (0,2) 2-form. This means that given a basis (u_1, u_2) for the cotangent bundle of $\mathcal{T}$, $F^{0,2}$ only has non-zero coefficient in the $\bar{u}_1 \wedge \bar{u}_2$ term which is zero since it is pulled back along p_2 so is flat in the fibre direction. Thus, both conditions on $\bar{\partial}$ are satisfied and $\tilde{E}$ is holomorphic.

The line sub-bundles of $\tilde{E}$

Let $E = \mathbb{R}^3 \times \mathbb{C} \to \mathbb{R}^3$, ∇ be a flat connection and $\Phi = i \in \mathbb{C}$. Then $\tilde{E} = L$, the line bundle with fibre at $z \in \mathcal{T}$ given by

$$L_z = \{s \in \Gamma(\gamma_z) \mid \frac{ds}{dt} + s = 0\}.$$

The solutions to the ODE on each geodesic γ_z can simply be written down, $s = Ae^{-t}$ for some constant A. To glue these solutions together to get a global solution on all of $\mathcal{T}$, allow the "constant" A_z to vary smoothly with respect to γ_z.

For coordinates $(u, x_0) \in \mathcal{T}$ represented by some $(u, x) \in S^2 \times \mathbb{R}^3$,

$$A_z e^{-t} = e^{-u\cdot x_0} e^{-t} = e^{-u\cdot(x_0+tu)} = e^{-u\cdot x} =: \hat{l}(u, x).$$

We have just written down a section $\hat{l} \in \Gamma(S^2 \times \mathbb{R}^3, p^*L)$ which descends to a global section $l \in \Gamma(\mathcal{T}, L)$.

If we choose coordinates $(a, b) \in \mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3$, and treat $\mathcal{T}$ as a fibration $\mathcal{T} \to \mathbb{P}^1$ then we have coordinates (η, ζ) for $\mathcal{T}$ where ζ is a complex coordinate for $\mathbb{P}^1$ and

$$\eta = a - 2b\zeta - \bar{a}\zeta^2.$$

Stereographic projection of $u \in S^2$ into $\mathbb{R}^3$ allows us to write down $u\cdot x$ in coordinates. Then the $\bar{\partial}$-operator on a section l can be written explicitly as

$$\bar{\partial} l = \frac{-l\eta\, d\bar{\zeta}}{(1 + \zeta\bar{\zeta})^2}.$$

The line bundle L can now be explicitly constructed. There are two charts on $\mathcal{T}$ in analogy with $\mathbb{P}^1$:

$$U_0 = \{(\eta, \zeta) \in \mathcal{T} \mid \zeta \neq 0\}$$

with L trivialisation

$$l \exp\left(\frac{-\eta}{\zeta(1 + \zeta\bar{\zeta})}\right)$$

and

$$U_1 = \{(\eta, \zeta) \in \mathcal{T} \mid \zeta \neq \infty\}$$

with L trivialisation

$$l \exp\left(\frac{-\eta}{\zeta(1+\zeta\bar{\zeta})} + \frac{\eta}{\zeta}\right).$$

On the intersection $U_0 \cap U_1$, the transition function is

$$\exp(-\frac{\eta}{\zeta}).$$

For a monopole (A, Φ) on $\mathbb{R}^3$, the (Hitchin [Hit82]) boundary conditions on the 2-sphere S^2_∞ with coordinates (θ, ϕ) as the radial coordinate r in $\mathbb{R}^3$ approaches infinity, were shown by Jaffe and Taubes to be implied by the finite energy of the monopole [JT80].

$$\begin{aligned}
&(1)\ \ \|\Phi\| = \sqrt{-\tfrac{1}{2}\mathrm{Tr}\Phi^2} = 1 - \frac{k}{r} + \mathcal{O}(r^{-2}),\\
&(2)\ \ \frac{\partial\|\Phi\|}{\partial\Omega} = \left[\left(\frac{\partial\|\Phi\|}{\partial\theta}\right)^2 + \sin^2\theta\left(\frac{\partial\|\Phi\|}{\partial\phi}\right)^2\right]^{\frac{1}{2}} = \mathcal{O}(r^{-2}),\\
&(3)\ \ \|\nabla\Phi\| = \mathcal{O}(r^{-2}).
\end{aligned}$$

These conditions imply that at the asymptotic S^2_∞, the vector bundle E splits into a sum of line bundles $M \oplus M^*$ where M has first chern class $c_1(M) = \pm k$. We call k the charge of the monopole.

Theorem 4 (Hitchin [Hit82]). *Let (A, Φ) be an $SU(2)$ monopole on $\mathbb{R}^3$ of charge k satisfying the (Hitchin) boundary conditions 1-3.*

Let $\tilde{E} \to \mathcal{T}$ be defined by $\tilde{E}_z = \{s \in \Gamma(\gamma_z, E|_{\gamma_z}) \mid (\nabla^A - i\Phi)s = 0\}$.

Let L^+ be the line sub-bundle of $\tilde{E}$ defined by

$$L^+_z = \{s \in \tilde{E}_z \mid s \sim \mathcal{O}(t^k e^{-t}) \text{ as } t \to +\infty\}.$$

Then $L^+ \simeq L(-k) \simeq L \otimes \mathcal{O}_{\mathbb{P}^1}(-k)$ as holomorphic bundles and $\tilde{E}$ is an extension

$$0 \to L(-k) \to \tilde{E} \to L^*(k) \to 0.$$

ie. an exact sequence.

Before discussing the proof idea, I would like to say what the theorem implies more explicitly. Being able to express a holomorphic vector bundle $\tilde{E}$ as an extension implies that it has a transition function on $U_0 \cap U_1$ of the form

$$\begin{bmatrix} \zeta^{-k}e^{\eta/\zeta} & f(\eta,\zeta) \\ & \zeta^k e^{-\eta/\zeta} \end{bmatrix}$$

where $f(\eta, \zeta)$ is a representative of a cohomology class in the sheaf cohomology $H^1(\mathcal{T}, L^2(-2k))$ which classifies all such extensions.

As an aside, isomorphism classes of extensions B of C by A, exact sequences,

$$0 \to A \to B \to C \to 0$$

are in correspondence with the elements of $\mathrm{Ext}^1(C, A)$. This is defined as, for $C = L^*(k)$ and $A = L(-k)$,

$$\begin{aligned}
\mathrm{Ext}^1(L^*(k), L(-k)) &= R^1\mathrm{Hom}(-, L(-k))(L^*(k)) \\
&= \begin{array}{c} H^1(0 \to \mathrm{Hom}(P^0, L(-k)) \xrightarrow{\partial_0} \mathrm{Hom}(P^1, L(-k)) \\ \xrightarrow{\partial_1} \mathrm{Hom}(P^2, L(-k)) \to \ldots) \end{array} \\
&= \ker \partial_1 / \mathrm{im}\partial_0
\end{aligned}$$

where

$$\ldots \to P^1 \to P^0 \to L^*(k)$$

is a projective resolution of $L^*(k)$. On the other hand, $H^1(\mathcal{T}, L^2(-2k))$ is $R^1\Gamma(\mathcal{T}, L^2(-2k))$. Tensoring with $L(-k)$ gives us the relationship between the two sheaf cohomology groups.

The crucial idea behind the theorem is the existence of the holomorphic sub-line bundle L^+ consisting of the solutions which decay in the appropriate sense at (positive) infinity on oriented geodesics. For more detail, refer to Hitchin's Monopoles and Geodesics [Hit82].

Once a gauge that is covariantly constant in the radial direction has been chosen, Hitchin's equation for a geodesic γ can be written as,

$$\frac{dx}{dt} - \begin{bmatrix} -1 + \frac{k}{t} & 0 \\ 0 & 1 - \frac{k}{t} \end{bmatrix} x + C(t)x = 0,$$

where $\|C(t)\| = \mathcal{O}(t^{-2})$. The boundary conditions (2.2) are needed to do this.

Then, ODE theory (see Coddington-Levinson [CL55]) tells us that there exists solutions x_0, x_1 such that as $t \to +\infty$,

$$x_0(t)t^{-k}e^t \to e_0, \quad x_1(t)t^k e^{-t} \to e_1,$$

where e_0, e_1 are eigenvectors of $\lim\limits_{t\to\infty} \Phi$ and give us a trivialisation of E in a neighbourhood around γ. The solution x_0 defines a 1-dimensional subspace $L_z^+ \subset \tilde{E}_z$ of solutions which decay as $t \to +\infty$.

The line sub-bundle L^+ is holomorphic. This follows from the vanishing of the quantity $< \bar{\partial} s, s >$ where $<, >$ is the symplectic form of $\tilde{E}$. The same arguments show that there exists a holomorphic line sub-bundle L^- of the solutions which vanish as $t \to -\infty$.

It is possible to figure out which holomorphic line bundle on $\mathcal{T}$ L^+ is. Let the holomorphic isomorphism $\alpha : L^+ \otimes L^* \underset{\sim}{\to} \pi^* M$ where M is the positive factor of the splitting of E on S^2_∞, be defined on local sections s of $L^+ \otimes L^*$ in terms of sections $\hat{s}$ on the space $S^2 \times \mathbb{R}^3$,

$$\alpha(s) = \lim_{t \to +\infty} t^{-k} \hat{s}(z, t).$$

On $\mathbb{P}^1$, any line bundle is isomorphic to $\mathcal{O}(\pm k)$ so $L^+ \otimes L^* \simeq \pi^* \mathcal{O}(\pm k)$. The sign is negative since $\tilde{E}$ is holomorphically trivial on any real section of $\mathcal{T} \to \mathbb{P}^1$. Thus,

$$L^+ \simeq L(-k).$$

The spectral curve of a monopole

The spectral curve is an algebraic curve whose points parametrise the geodesics of $\mathbb{R}^3$ on which there exist solutions to the Hitchin equation $(\nabla^A - i\Phi)s = 0$ which decay at both limits of the geodesic ($st^{-k}e^t \to$ const as $t \to +\infty$ and $st^k e^{-t} \to$ const as $t \to -\infty$).

The bundle $\tilde{E}$ can be expressed as extensions [Hit82]

$$0 \to L^+ \to \tilde{E} \to (L^+)^* \to 0$$

$$0 \to L^- \to \tilde{E} \to (L^-)^* \to 0,$$

where L^+ and L^- are the line sub-bundles of solutions decaying at the $+\infty$ and $-\infty$ of geodesics respectively. The real structure induces an antiholomorphic isomorphism $\sigma : L^+ \xrightarrow{\sim} L^-$.

The upshot of the previous section was that $L^+ \simeq L(-k)$ where L is the line sub-bundle of solutions to the trivial U(1) Hitchin equations constructed in the last section. The same line of reasoning also gives $L^- \simeq L^*(-k)$. Then the short exact sequences can be written,

$$0 \to L(-k) \to \tilde{E} \to L^*(k) \to 0$$

$$0 \to L^*(-k) \to \tilde{E} \to L(k) \to 0.$$

The projection of L^- in $\tilde{E}$ onto $(L^+)^*$ defines a degree $2k$ section

$$\psi \in H^0(\mathcal{T}, (L^+ \otimes L^-)^*) \simeq H^0(\mathcal{T}, \mathcal{O}(2k)).$$

The points z of $\mathcal{T}$ for which the line subspaces L^+_z and L^-_z coincide in the fibre $\tilde{E}_z$ are the geodesics with decaying solutions at both ends. Put another way, the subspace L^-_z should not intersect with the complement $(L^+_z)^*$ of L^+_z in E_z over points z of the spectral curve. Thus, the spectral curve S of an SU(2) monopole on $\mathbb{R}^3$ of charge k can be defined as $\ker \psi$.

The spectral curve has some notable properties as stated in the following proposition of Hitchin.

Proposition 5 (Hitchin [Hit83]). *Let S be the spectral curve of an $SU(2)$ monopole on $\mathbb{R}^3$ of charge k. Then*

(1) *S is compact;*
(2) *S is defined by an equation $p(\eta, \zeta) = \eta^k + a_1(\zeta)\eta^{k-1} + \ldots + a_k(\zeta) = 0$ where $\deg a_i = 2i$.*
(3) *The line bundle L^2 is holomorphically trivial on S.*
(4) *S is preserved by the real structure τ of $T\mathbb{P}^1$.*
(5) *S has no multiple components.*
(6) *$L(k-1)$ is real.*
(7) *$H^0(S, L^z(k-2)) = 0$ for $z \in (-1, 1)$.*

The final condition is the condition for the spectral curve to correspond to a non-singular monopole.

An important example is the spectral curve of a charge k axially-symmetric monopole when $k = 2n+1$,

$$\psi = \eta \prod_{l=1}^{n} (\eta^2 + l^2\pi^2\zeta^2)$$

and when $k = 2n+2$,

$$\psi = \prod_{l=0}^{n} [\eta^2 + (l + \frac{1}{2})^2\pi^2\zeta^2].$$

CHAPTER 3

The Nahm equations

The Euler top system of equations [Gol65]

$$\frac{d\omega_1}{dt} = \omega_2\omega_3 \tag{3.1}$$

$$\frac{d\omega_2}{dt} = \omega_3\omega_1 \tag{3.2}$$

$$\frac{d\omega_3}{dt} = \omega_1\omega_2 \tag{3.3}$$

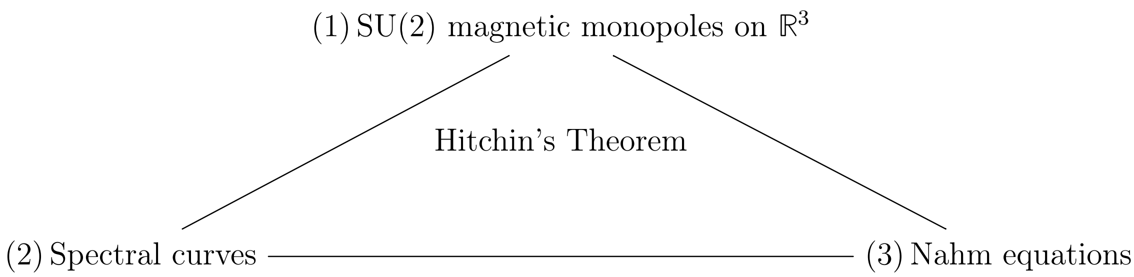
for a spinning top with no external torque, where $\omega_1, \omega_2, \omega_3$ are the angular velocities along the three axes is a historical example of an integrable system. It is said to be integrable because the angular momentum and energy are conserved, allowing the angular velocities to be written in terms of the conserved quantities and Jacobi elliptic functions.

Another way to write the Euler top equations is

$$\frac{dT_i}{dt} = \frac{1}{2}\sum_{j,k} \varepsilon_{ijk}[T_j, T_k]$$

where $T_i = \omega_i \tau_i$ and $\tau_i = i\sigma_i$ are the generators of SU(2) given by the Pauli matrices σ_i. The Nahm equations can be viewed as a generalisation of the Euler top where T_i are skew-Hermitian $k \times k$ matrices and $t \in (-1, 1)$.

Hitchin's theorem summarises the previous chapter and introduces Nahm's equations in terms of the previously discussed objects.



THEOREM 6 (Hitchin [Hit83]). *There is an equivalence between*

(1) *$SU(2)$ magnetic monopoles (A, Φ) on $\mathbb{R}^3$ consisting of the $\mathfrak{su}(2)$-valued connection A and conjugacy class of scalar field Φ satisfying the Bogolmonyi*

equations $\star F_A = D_A \Phi_A$ *and the boundary conditions as* $r \to \infty$ *(see previous chapter for details):*

(a) $\|\Phi\| = 1 - \frac{k}{r} + \mathcal{O}(r^{-2})$,

(b) $\frac{\partial \|\Phi\|}{\partial \Omega} = \mathcal{O}(r^{-2})$,

(c) $\|D_A \Phi\| = \mathcal{O}(r^{-2})$.

(2) *Spectral curves* S *consisting of a compact algebraic curve in* $T\mathbb{P}^1$ *in the linear system of divisors* $\mathcal{O}(2k)$ *with the conditions,*

(a) S *is connected;*

(b) S *is real with respect to a reality involution on* $T\mathbb{P}^1$*;*

(c) L^2 *is trivial on* S *and* $L(k-1)$ *is real;*

(d) $H^0(S, L^z(k-2)) = 0$ *for* $z \in (-1, 1)$*;*

where L *is the line bundle on* $T\mathbb{P}^1$ *defined by* $\exp(s\pi^*\omega)$ *where* $\pi : T\mathbb{P}^1 \to \mathbb{P}^1$ *is the projection,* $s \in \pi^* T\mathbb{P}^1$ *and* $\omega \in H^1(\mathbb{P}^1, T_*\mathbb{P}^1)$.

(3) *Solutions of the Nahm equations consisting of a gauge equivalence class of matrix-valued functions* $T_0, T_1, T_2, T_3 : [-1, 1] \to Mat_{k\times k}$*,*

$$\left(\frac{dT_i}{dz} + [T_0, T_i]\right) + \frac{1}{2}\sum_{j,k} \epsilon_{ijk}[T_j, T_k] = 0,$$

satisfying for $i, j, k = 1, 2, 3$*,*

(a) $T_i^* = -T_i$*;*

(b) $T_i(1-z) = T_i(z-1)^T$*;*

(c) T_i *are meromorphic with simple poles only at* ± 1*;*

(d) *the residues of the* T_i *at the poles define an irreducible representation of* $SU(2)$.

where the gauge action is through $u : (-1, 1) \to U(k)$*,*

$$u \cdot T_i = u T_i u^{-1} \tag{3.4}$$

$$u \cdot T_0 = u T_0 u^{-1} - \frac{du}{dz} u^{-1}. \tag{3.5}$$

The T_0 part of a solution of the Nahm equations can be gauged away and was introduced by Donaldson [Don84b]. It is important to include the T_0 to make the Nahm equations more "physical". Let me explain what I mean by this.

The Higgs field Φ can be united with the gauge connection A if there is an extra time dimension $t \in \mathbb{R}$ such that

$$A_\square = \Phi dt + A$$

is a gauge connection on $\mathbb{R}^4$ with anti-self-dual curvature 2-form (see the part on instantons) and which is t-invariant. Analogous to the Fourier Transform is the Nahm transform which exchanges time t with momenta p_i (the conjugate variables to space) and space coordinates x_i with energy z (the conjugate variable to time). The field is now a momentum-invariant gauge field

$$\hat{A}_{\square} = T_0 dz + \sum_{i=1}^{3} T_i dp_i,$$

depending on $(z, p_1, p_2, p_3) \in \mathbb{R}^4$.

From the Nahm matrices, an infinite-dimensional version of the ADHM operator seen in Part 2 can be constructed. The bounded linear operator $\Delta : W \to V$ between Sobolev spaces defined in terms of a solution of the Nahm equations is

$$\Delta = \frac{d}{dz} + \sum_{j=0}^{3} (T_j - I_k x_j)\, \sigma_j,$$

where I_k is the $k \times k$ identity matrix and $\sigma_0, \ldots, \sigma_3$ are the Pauli matrices as a basis of the quaternions, making V a quaternioninc vector space. Then an orthonormal basis of solutions v_1, v_2 of

$$\Delta v = 0$$

allows us to construct

$$A_{\mu\nu} = \int_{-1}^{1} (v_\mu^* dv_\nu)\, dz, \quad \Phi_{\mu\nu} = i \int_{-1}^{1} v_\mu^* z v_\nu\, dz.$$

There is another form of the Nahm equations which is more suited to their analysis [Don84b]. Let

$$\sigma = \frac{1}{2}(T_0 + iT_1), \quad \tau = \frac{1}{2}(T_2 + iT_3).$$

The Nahm equations are,

$$\frac{1}{2}\frac{d\tau}{dz} = [\sigma, \tau], \quad \frac{1}{2}\frac{d\tau^*}{dz} = [\sigma^*, \tau^*];$$

and

$$\frac{1}{2}\frac{d(\sigma + \sigma^*)}{dz} = [\sigma, \sigma^*] + [\tau, \tau^*].$$

They can be interpreted as moment maps

$$\mu := \frac{d\tau}{dz} - [\sigma, \tau]; \tag{3.6}$$

$$\mu_{\mathbb{R}} := \frac{d(\sigma + \sigma^*)}{dz} - [\sigma, \sigma^*] + [\tau, \tau^*]. \tag{3.7}$$

The moduli space of (gauge-equivalent) solutions of the Nahm equations can then be seen as the quotient

$$\mathcal{M}_{\text{Nahm}} = \{[-1,1] \to \text{Mat}_k(\mathbb{C})\} \cap \{\mu = 0 = \mu_{\mathbb{R}}\}/\text{U}(k).$$

Let $A = 2\tau + 4\sigma^*\zeta + 2\tau^*\zeta^2$ and $A_+ = 2\sigma^* + 2\tau\zeta = \frac{dA}{d\zeta}$. As foreshadowed, the Lax form of the Bogolmonyi equations [Hit83] are the Nahm equations for A, A_+,

$$\frac{dA}{dz} = [A, A_+].$$

As in the case of other integrable systems, the equation for the spectral curve can be written

$$\det(\eta 1 + A(\zeta)) = 0.$$

If the residues of (σ, τ) are (a, b) at -1 then the complex Nahm equation implies that

$$b = [a, b].$$

So if v is the eigenvector of a with the smallest eigenvalue λ, then

$$abv = (\lambda + 1)bv$$

and $\{v, bv, \ldots, b^{k-1}v\}$ is a basis for $\mathbb{C}^k$. The moduli space of solutions to the Nahm equations can be enlarged by a unit vector v "phase factor" to get the moduli space

$$\tilde{\mathcal{M}}_{\text{Nahm}} = \mathcal{M}_{\text{Nahm}} \times S^{k-1}$$

of Nahm complexes (σ, τ, v). This vector v acts as the framing or data of the monopole and stabilises the moduli space.

The moduli space of monopoles is in bijection with the moduli space of something far more familiar - the moduli space of rational maps, $p(x)/q(x)$ where $p, q \in \mathbb{C}[x]$ with a basing/framing condition that $p(\infty)/q(\infty)$ for a point $\infty \in \mathbb{P}^1$ be fixed. As an example, all charge $k = 1$ monopoles in $\mathbb{R}^3$ correspond to, for $f_1, f_2 \in \mathbb{C}$, rational maps of the form,

$$f(z) = \frac{f_1}{f_2 - z}.$$

Proposition 7 (Donaldson [Don84b]). *There is an equivalence between*

(1) *Nahm complexes $[\sigma, \tau, v]$ for a charge k monopole, and*
(2) *based rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree k.*

Let $B = \sigma(0)$. Since $\{v, bv, \ldots, b^{k-1}v\}$ is a basis for $\mathbb{C}^k$, the residue of σ at -1, a can be diagonalised. This means that we can solve for a $\mathbb{C}^k$-valued function u such that $\alpha u = \frac{d}{dz}u$. What we have done is parallel transported our frame vector v along the interval $(-1, 1)$. Then let $w = u(0)$.

Now we have an equivalence class $[B, w]$ consisting of a complex symmetric $k \times k$ matrix and a column k-vector with the condition that $\{w, Bw, \ldots, B^{k-1}w\}$ be a basis for $\mathbb{C}^k$ (w is a cyclic vector for B). The equivalence $[B, w]$ is in bijection with the data (σ, τ, v) and we can write the rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ with ζ a coordinate on the domain,

$$f(\zeta) = w^T (I\zeta - B)^{-1} w.$$

The physical interpretation of the rational map is in terms of a Rutherford type scattering experiment where test particles are fired through the monopole field from some point on the sphere, ∞. If the monopole is surrounded by a spherical detector then a graph of how many of the test particles are transmitted through the monopole field, hence not scattered, is exactly a function on $\mathbb{P}^1$ whose value is "normalised" with regards to the value at ∞.

The target $\mathbb{P}^1$ is actually the flag variety $\mathrm{SU}(2)/T$. At each $z \in \mathbb{P}^1$, the choice of an element of $\mathrm{SU}(2)/T$,

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} \exp(i\theta) & 0 \\ 0 & \exp(-i\theta) \end{pmatrix}$$

fixes some line L_z in $E_z \simeq \mathbb{C}^2$ and the choice of all such flags $L_z \subset E_z$ is $\mathbb{P}^1$.

The interpretation of the rational map as a map into a flag variety can be related to the scattering experiment interpretation in the following way. The choice of a point ∞ on $\mathbb{P}^1$ is equivalent to choosing some special direction in $\mathbb{R}^3$ hence a choice $\mathbb{R}^3 \simeq \mathbb{R} \times \mathbb{C}$. The rational map is then a section of the projectivised bundle $\mathbb{P}(\tilde{E})$ restricted to all the geodesics in the direction of $\{\infty\}$, parametrised by $\mathbb{C}$. A section of $\mathbb{P}(\tilde{E})$ defines a line sub-bundle L^- of $\tilde{E}$. This is precisely the sub-bundle of solutions to the equation $(\nabla_t^A - i\Phi)s = 0$ from the previous chapter which decay at $t \to -\infty$. The poles of the rational map indicate where L^+ and L^- coincide. This is a finite number of points.

One advantage of having access to the rational map isomorphism is in the counting of the dimensions of the moduli space of framed monopoles. The conventional count involves an Atiyah-Singer index theorem type argument. However, counting the number of degrees of freedom arising from the coefficients of polynomials is easier.

The rational map also provides intuition since higher charge monopoles correpond to higher degree rational maps. The monopole field is concentrated at the poles of the factorised rational map

$$f(z) = \sum_{i=1}^{k} \frac{a_i}{b_i - z}.$$

Charge 1 monopoles have a single pole at which the monopole field energy density function has a global maximum. When the poles are sufficiently far apart, the charge k monopole field can be approximated by a superposition of charge 1 monopoles with a pole at each of the k poles.

CHAPTER 4

Monopoles of the classical groups

The results of the previous two chapters extend to the classical groups $\mathrm{SU}(N), \mathrm{SO}(N)$ and $\mathrm{Sp}(N)$. At the time of writing, a paper posted on arχiv the pre-print server, discussed an example of a G_2 monopole [SZ15]. Much of the work on this extension to other classical groups is due to Murray [Mur84], Hurtubise [Hur89] and Hurtubise-Murray [HM90].

Definition 8. Let P be the trivial principal $\mathrm{SU}(N)$ bundle on $\mathbb{R}^3$. A euclidean $\mathrm{SU}(N)$ monopole (A, Φ) is a gauge equivalence class of

- a connection A on P with curvature F_A, and
- a section Φ of the adjoint bundle $\operatorname{ad} P = P \times_{\mathrm{Ad}} \mathfrak{su}(N)$

with asymptotic conditions (due to Murray [Mur84], existence of monopoles satisfying them due to Taubes [Tau81])

$$\begin{aligned}\Phi &= i\mathrm{diag}(p_1, \ldots, p_N) - \frac{i}{2r}\mathrm{diag}(k_1, \ldots, k_N) + O(r^-2)\\ \frac{\partial\|\Phi\|}{\partial\Omega} &= \left((\frac{\partial\|\Phi\|}{\partial\theta}) + \sin^2\theta(\frac{\partial\|\Phi\|}{\partial\phi}) \right)^{1/2} = O(r^{-2})\\ \|\nabla\Phi\| &= O(r^{-2})\end{aligned}$$

where $p_1, \ldots, p_N \in \mathbb{R}$, $p_1 > \ldots > p_N$ and $k_1, \ldots, k_N \in \mathbb{Z}$; and satisfying the Bogolmonyi equations

$$F_A = \star_{\mathbb{R}^3} D_A \Phi$$

where $\star_{\mathbb{R}^3}$ is the Hodge dual with respect to the standard metric of $\mathbb{R}^3$.

Locally, the connection A is a 1-form

$$A = A_x dx + A_y dy + A_z dz$$

where $A_x, \ldots, A_z$ are $N \times N$ traceless anti-hermitian matrices of complex functions and for any fixed $(x, y, z) \in \mathbb{R}^3$, $A_x, \ldots, A_z \in \mathfrak{su}(N)$. The Higgs field Φ is locally an $\mathfrak{su}(N)$-valued function. The constant terms $p_1, \ldots, p_{N-1}$ of the asymptotic eigenvalues of Higgs bundle Φ are called the mass numbers of the monopole. The next terms $k_1, \ldots, k_{N-1}$ are called the charge numbers. Note that the condition that Φ

be valued in $\mathfrak{su}(N)$ imposes the conditions

$$p_1 + \ldots + p_N = 0, \quad k_1 + \ldots + k_N = 0$$

which fixes the value of p_N and k_N.

I will only discuss the case when $p_1, \ldots, p_{N-1}$ are distinct, the so-called *maximal symmetry breaking* case. (Spectral curves and associated data still determine the monopole in the non-maximal symmetry breaking case [Mur84] so one could hope that the other results follow but not much else is known [HM90].) For the SU(2) case, it is possible to scale the only mass number to unity and that is why p does not appear in the asymptotic conditions in the previous chapters.

Since A and Φ are a SU(N) connection and a section of the adjoint bundle of P respectively, the reader is reminded that they are invariant under the action of the gauge group, the group of functions $g : \mathbb{R}^3 \to \mathfrak{su}(N)$ on (A, Φ),

$$\begin{aligned} A &\mapsto g^{-1}Ag + g^{-1}dg \\ \Phi &\mapsto g^{-1}\Phi g. \end{aligned}$$

The Murray [Mur84] and Hurtubise-Murray [HM90] extension of Hitchin's equations are:

THEOREM 9 (Murray, Hurtubise-Murray). *There is a natural correspondence between:*

(1) *$SU(N)$ magnetic monopoles (A, Φ) on $\mathbb{R}^3$ with mass numbers $p_1, \ldots, p_{N-1} \in \mathbb{R}$, $p_1 > \ldots > p_N$ and charge numbers $k_1, \ldots, k_{N-1} \in \mathbb{Z}$.*

(2) *$U(k_1) \oplus \ldots \oplus U(k_1 + \ldots + k_{N-1})$-conjugacy classes of analytic functions*

$$T_j^i = \begin{cases} T_1^i : (p_1, p_2) \to \mathfrak{u}(k_1) & \text{if } j = 1 \\ \vdots & \vdots \\ T_1^i : (p_{N-1}, p_N) \to \mathfrak{u}(k_1 + \ldots + k_{N-1}) & \text{if } j = N-1 \end{cases}$$

which are solutions to the Nahm equations

$$\frac{dT_j^i}{dz} + \frac{1}{2}\sum_{m,n} \epsilon_{imn}[T_j^m, T_j^n] = 0, \tag{4.1}$$

and satisfy boundary conditions at p_j, with $t = z - p_j$, and with the selector function

$$\xi(j) = \begin{cases} j & \text{if } k_j > k_{j-1} \\ j-1 & \text{if } k_j < k_{j-1} \end{cases}$$

$$\xi^*(j) = \begin{cases} j-1 & \textit{if } k_j > k_{j-1} \\ j & \textit{if } k_j < k_{j-1} \end{cases}$$

there exists finite nonzero limits

$$C^i_j = \lim_{t \to 0^{-sign(k_j)}} T^i_{\xi^*(j)}(t),$$

$T^i_{\xi^*(j)}(t)$ *is analytic at* $t = 0$ *and* $T^i_{\xi(j)}(t)$ *can be conjugated by a unitary* $(k_1 + \ldots + k_j) \times (k_1 + \ldots + k_j)$ *matrix such that*

$$T^i_{\xi(j)} \simeq \left(\begin{array}{c|c} C^i_j + O(t) & O\left(t^{\frac{1}{2}(|k_j|-1)}\right) \\ \hline O\left(t^{\frac{1}{2}(|k_j|-1)}\right) & \frac{r^i_j}{t} + O(1) \end{array} \right) \tag{4.2}$$

where r^i_j *are* $k_j \times k_j$ *matrices defining an irreducible representation of* $\mathfrak{su}(2)$. *In the case when* $k_{j-1} = k_j$*, the upper and lower limits* C^-_i, C^+_i *defined earlier define matrices*

$$A^{\pm}(\zeta) = (C^{\pm}_2 + iC^{\pm}_3) + (2iC^{\pm}_1)\zeta + (C^{\pm}_2 - iC^{\pm}_3)\zeta^2, \tag{4.3}$$

which need to satisfy rank$(A^+(\zeta) - A^-(\zeta)) \leq 1$.

(3) *Spectral curves* $S_1, \ldots, S_{N-1} \subset T\mathbb{P}^1$ *which are compact curves curves* $S_j \in |\mathcal{O}(2k_1 + \ldots + 2k_j)|$ *satisfying,*
   (a) *invariant under* $\tau : (\eta, \zeta) \mapsto (-\bar{\zeta}^{-2}\bar{\eta}, -\bar{\zeta}^{-1})$ *(τ-real);*
   (b) *intersect as a disjoint union* $S_{j-1} \cap S_j = S_{j-1,j} \sqcup S_{j,j-1}$*;*
   (c) *over* S_j,

$$\mathcal{O} \simeq L^{p_j}(k_{j+1} + k_j + 2k_{j-1} + \ldots + 2k_1)[-S_{j,j+1} - S_{j,j-1}];$$

   (d)

$$H^0\left(S_j, L^{(m_j - z)}(k_j + 2k_{j-1} + \ldots + 2k_1 - 2)[-S_{j,j-1}]\right) = 0$$

   holds for $z \in (p_j, p_{j+1})$ *and for* $z = m_{\xi(j+1)}$*;*
   (e) $\tau(S_{j,j+1}) = S_{j+1,j}$*;*
   (f) $(-1)^{k_j + 2k_{j-1} + \ldots + k_1 + 1}(\psi_p \psi^*_p)/(g_{j-1}g_{j+1}) > 0$.

where the square brackets means restriction to the divisor.

The proof of this theorem is technical and beyond the scope of this chapter. I will point out features which differ from the SU(2) case. The most notable difference is the generalisation of the mass and charge data $p_1, \ldots, p_{N-1}, k_1, \ldots, k_{N-1}$.

The boundary condition applies to an asymptotically large sphere S^2_∞. On this sphere, the Higgs field Φ is map $\Phi_\infty : S^2 \to \mathfrak{su}(N)$ with image lying in an orbit of the adjoint action $\Phi \mapsto g^{-1}\Phi g$ of $\mathrm{SU}(N)$. The homotopy class $[\Phi_\infty] \in \pi_2(\mathrm{SU}(N)/T)$

where T is a maximal torus, is a topological invariant given by $N-1$ integers,

$$\pi_2(\mathrm{SU}(N)/T) \simeq \pi_1(T) \simeq \mathbb{Z}^{N-1}$$

coming from the long exact sequence of $T \to \mathrm{SU}(N) \to \mathrm{SU}(N)/T$.

As in the SU(2) case, the bundle E associated to the monopole splits into line bundle factors over S^2_∞ determined by pairs p_j, k_j, $L^{p_j}(-2k_j)$. Outside of S^2_∞, the bundle E has two flags of sub-bundles

$$E_1^\pm \subset \ldots \subset E_{N-1}^\pm \subset E$$

where $E_j^+/E_{j-1}^+ = L^{p_j}(-k_j)$ and $E_j^-/E_{j-1}^- = L^{p_{N-j+1}}(k_{N-j+1})$.

The spectral curves are defined as the points in $T\mathbb{P}^1$ for which the sub-bundles in the opposite flags E_*^+ and E_*^- intersect with at most codimension $N-1$. To illustrate, here are some examples:

- SU(2)

$$L_1^+ = L_1^-$$

- SU(3)

$$L_1^+ \subset L_2^- \frac{L_1^+ = L_1^-}{L_2^+ = L_2^-} L_1^- \subset L_2^+$$

- SU(4)

$$L_1^+ \subset L_3^- \frac{L_2^+ \subset L_3^-}{L_1^+ \subset L_2^-} L_2^+ \overset{1}{\cap} L_2^- \frac{L_1^- \subset L_2^+}{L_2^- \subset L_3^+} L_1^- \subset L_3^+$$

The conditions at the vertices indicate the condition for a point to be in the curve S_j and the conditions above and below the edges are conditions for points to be in either the component $S_{j,j+1}$ or $S_{j+1,j}$. The number above the intersection indicates the dimension of the intersection of the sub-bundles. For SU(6), there are components of the intersections of spectral curves whose condition is of the form

$$L_3^+ \overset{2}{\cap} L_4^-.$$

Observe that the splitting of the intersection of "adjacent" spectral curves into two disjoint components arises naturally from this definition of the spectral curve.

There is a second definition of the spectral curve defined from solutions of the Nahm equations

$$\det(\eta \mathbf{1} - A(t,\zeta)) = 0$$

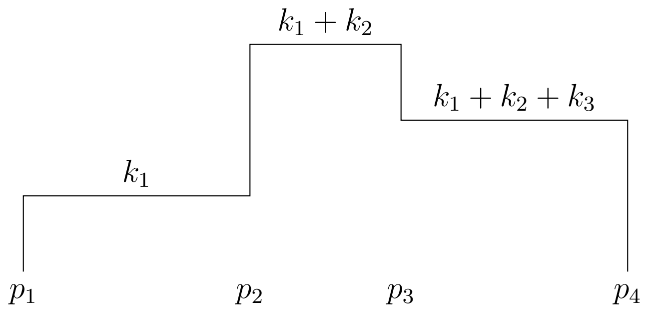
where

$$A(t,\zeta) = (T_1(t) + iT_2(t)) + 2iT_3(t)\zeta + (T_1(t) - iT_2(t))\zeta^2.$$

Each interval defines a new spectral curve.

The condition (3c) tells us which line bundle is the sheaf of regular functions on S_j. The condition (3d) is a vanishing cohomology condition which guarantees that the monopole associated to the spectral curve is non-singular, in analogy with the SU(N) case.

The Nahm equations for SU(N) are interesting because the matrix dimensions jump by k_j at the point p_j. An example of the graph of matrix dimensions versus size is



The Nahm equations for Sp(N) and SO(N) can be found from the SU(N) Nahm equations according to the table (reproduced from Hurtubise-Murray [HM89] but with a different convention for charge numbers):

G	G-charges	in SU(N)	mass numbers	SU(N)-charges
Sp(n)	$l_1, \ldots, l_n$	$N = 2n$	$p_i = -p_{2n+1-i}$ $i = 1, \ldots, n$	$k_i = l_i = -k_{2n-i+1}$, $i = 1, \ldots, n$.
SO$(2n)$	$l_1, \ldots, l_{n-2}$, l_+, l_-	$N = 2n$	$p_i = -p_{2n-i+1}$ $i = 1, \ldots, n$	$k_i = l_i = -k_{2k-i}$, $i = 1, \ldots, n-2$; $\pm k_{n\pm 1} = l_+ + l_-$ $+ l_1 + \ldots + l_{n-2}$, $k_n = l_+ - l_-$.
SO$(2n+1)$	$l_1, \ldots, l_n$	$N = 2n+1$	$p_i = -p_{2n-i+2}$ $i = 1, \ldots, n+1$	$k_i = l_i = -k_{2n-i+2}$ $i = 1, \ldots, n-1$; $-k_{n+2} = k_n$ $= 2l_n + l_{n-1} + \ldots + l_1$, $k_{n+1} = 0$.

TABLE 1.

These identifications on the mass and charge numbers effectively cause the Nahm solutions to be symmetrical about the origin of the line of z. The U$(k_1 + \ldots + k_i)$

gauge action on the Nahm matrices need to respect this "folding" of the Nahm equations. The gauge action matrices C_j need to obey

$$T_i^T(-z) = C_j T_i(z) C_j^{-1},$$

and

$$C_{N-j+1} = \begin{cases} -C_j^T & \text{if } G = \mathrm{SO} \\ C_j^T & \text{if } G = \mathrm{Sp}. \end{cases}$$

These identifications cause the unitary gauge action to become either a symplectic or orthogonal gauge action. One interesting feature of this is that the gauge action is symplectic when $G = \mathrm{SO}$ and orthogonal when $G = \mathrm{Sp}$. The gauge action in the Nahm equations is dual to the gauge action on the monopoles and this duality is known as reciprocity. For a more detailed discussion, see chapters 6 and 15.

The folding of the Nahm equations is suggestive of the intervals of the Nahm equations being in bijection with the vertices of the corresponding Dynkin diagram. (Thanks to a reviewer who pointed out that this is a conjecture by Atiyah. See the conclusion of [HM88].) Dynkin diagrams for Lie algebras of type C can be folded from type A and type B can be folded from type D. We see that the analogy is incomplete since Nahm equations with G whose Lie algebra is of type B,C and D are being acquired from type A here.

Part 2

Instantons

CHAPTER 5

Instantons

The word instanton comes from quantum mechanical tunneling. Before defining *gauge-theoretic* instantons, I will provide motivation by sketching out the case of the *quantum mechanical* instanton [Wei12].

For quantum mechanics on a line, a particle can be represented by a wave-function $\psi : \mathbb{R}_x \times \mathbb{R}_t \to \mathbb{R}$. Given a (time-independent) potential $V : \mathbb{R}_x \to \mathbb{R}$ and constants $m, E \in \mathbb{R}$ representing the mass and energy of the system, the wave-function obeys the Schrödinger equation,

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{2m(V(x) - E)}{\hbar^2} \psi.$$

If V is the potential barrier

$$V(x) = \begin{cases} V & x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

with $V \gg E$, then a solution is the wave $\psi = e^{-ikx}$ transmitted through the potential barrier with

$$k = \frac{1}{\hbar} \int_a^b \sqrt{2m(E - V(x))} dx.$$

When outside the well $x \notin [a, b]$, this is an imaginary quantity so the wave decays.

This setup is the stationary point of the action

$$S = \int_{t_0}^{t_f} \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - V(x) \, dt$$

with canonical quantisation $\dfrac{dx}{dt} \to \dfrac{\partial}{\partial x}$.

After reparametrisation (Wick rotation in physics parlance), the tunneling exponent k can be obtained as the minimum of the equivalent euclidean action,

$$S_E = \int_{\tau_i}^{\tau_f} \frac{1}{2} m \left(\frac{dx}{d\tau} \right)^2 + V(x) \, d\tau.$$

Thus, *quantum mechanical* instantons can be realised as a quantum mechanical tunneling amplitude. The gauge field theory analogy is the minimisation of an action involving a curvature 2-form which is equivalent to the condition that the 2-form be anti-self-dual.

On four-dimensional real euclidean space $\mathbb{R}^4$, let $P \to \mathbb{R}^4$ be the trivial principal $\mathrm{SU}(N)$-bundle $P \simeq \mathrm{SU}(N) \times \mathbb{R}^4$. Let A be a connection on P, $D_A = d + A$ be its covariant derivative and $F_A = dA + A \wedge A$ be the curvature 2-form of A.

Instantons on $\mathbb{R}^4$ are connections A whose curvature 2-form F_A minimise the euclidean action defined with the Hodge star $\star$ of the Euclidean metric,

$$S = \int \mathrm{Tr}\, F_A \wedge \star F_A.$$

By considering the self-dual F^+ and anti-self-dual F^- parts of an arbitrary curvature form F, the action is

$$S = \int \|F^+\|^2 + \|F^-\|^2,$$

which takes on its minimal value when either part vanishes.

Thus let us define an $\mathrm{SU}(N)$ instanton on $\mathbb{R}^4$ to be a connection A satisfying the condition that its curvature is anti-self dual (with respect to the Hodge star dual $\star$),

$$\star F_A = -F_A,$$

which extends to a connection on the (non-trivial) principal $\mathrm{SU}(N)$-bundle $P \to S^4$ on the conformal compactification S^4 of $\mathbb{R}^4$.

By Chern-Weil theory, the first Pontryagin class p_1 (the only non-vanishing one for a four-dimensional manifold) of the principal $\mathrm{SU}(N)$-bundle P is

$$p_1 = \frac{1}{8\pi^2} \mathrm{Tr}\, F_A \wedge F_A.$$

The integral of this form on S^4 is an integer κ which is variously known as the Pontryagin index, topological charge or instanton number. Since the first chern class c_1 of an instanton vanishes, $p_1 = c_2$, the second Chern class.

Alternatively, consider a very large 3-sphere S^3 in $\mathbb{R}^4$ "approaching infinity". The condition that A extends to a connection on S^4 implies that on this 3-sphere S^3_∞, the connection is gauge equivalent to $g^{-1}(x)\, dg(x)$ for some gauge transformation g (A is said to be pure gauge on this sphere). The gauge transformation

$$g : S^3_\infty \to \mathrm{SU}(N),$$

taking values in the compact, simply-connected group $\mathrm{SU}(N)$ is a representative of an element of the third homotopy group $\pi_3(\mathrm{SU}(N))$. Hence,

$$\pi_3(\mathrm{SU}(3)) \simeq \pi_2(T) \simeq \mathbb{Z}.$$

This integral topological invariant is called the instanton charge.

We can augment the data of an instanton with a framing condition. Fix a distinguished point ∞ on S^4 (the point at infinity for $\mathbb{R}^4$). An instanton is framed if only the gauge transformations g which satisfy $g(\infty) = 1$ are allowed to act. This effectlively fixes the value of A at ∞, imparting an extra $\dim G$ degrees of freedom.

At this point it would be good to provide some examples of instantons. The following is due to Belavin et. al. [Bel+75]. Let the $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ be written as a quaternion $x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H}$. The charge $\kappa = 1$ SU(2) instanton can be written with $x \in \mathbb{H}$ as

$$A = \frac{1}{2} \frac{\bar{x} dx - d\bar{x}\, x}{1 + |x|^2}.$$

Its self-dual curvature form is

$$F = \frac{d\bar{x} \wedge dx}{(1 + |x|^2)^2}.$$

Note that this curvature 2-form resembles the Kähler form of the Fubini-Study on $\mathbb{P}^1$ or the ball metric on S^2 wherre $x \in \mathbb{C}$. (However, the structure group for the field would be U(1).)

It is known that a solution ρ of the Laplace equation

$$\nabla \cdot \nabla \rho = 0,$$

yields the instanton

$$\boldsymbol{A} = \frac{i}{2} \sigma \nabla \log \rho$$

with

$$\sigma = \begin{bmatrix} 0 & \sigma_1 & \sigma_2 & \sigma_3 \\ -\sigma_1 & 0 & \sigma_3 & -\sigma_2 \\ -\sigma_2 & -\sigma_3 & 0 & \sigma_1 \\ -\sigma_3 & \sigma_2 & -\sigma_1 & 0 \end{bmatrix}.$$

’t Hooft and Jackiw-Nohl-Rebbi [JNR77] provide a family of solutions (JNR ansatz) of charge κ depending on parameters $\lambda_j \in \mathbb{R}_{>0}$ and $y_j \in \mathbb{R}^4$:

$$\rho = \sum_{j=0}^{\kappa} \frac{\lambda_j^2}{|x - y_j|^2}.$$

When the $j = 0$ term is 1, this is known as the ’t Hooft ansatz.

The JNR ansatz can be explained as the approximation of an instanton as a collection of greatly separated, weakly interacting point particles. The y_j data is the locations of these point particles and the λ_j is the phase data. Each point particle contributes unit charge and one unit charge is consumed by the framing. In this interpretation, the 't Hooft ansatz posits that one of the point particles sits at infinity.

The JNR ansatz has $5\kappa+4$ real parameters (the expression ρ can be scaled). We can ask if this is the total number of instantons. The counting of instantons can be done by an Atiyah-Singer index theorem calculation [AHS78; AS68b; AS68a].

Let $S^+ \oplus S^- \simeq \mathbb{C}^2 \oplus \mathbb{C}^2$ be the oppositely-oriented components of the spin representation of $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$. Let $\gamma : \mathbb{R}^4 \to \mathrm{Hom}_{\mathbb{C}}(S^-, S^+)$ be the map sending the standard basis to the Pauli matrices. Define the Dirac operator $D_A : \Gamma(E \otimes S^+) \to \Gamma(E \otimes S^-)$ by

$$D_A = -\sum_{i=0}^{3} \gamma_i^* \left(\frac{\partial}{\partial x_i} + A_i \right). \tag{5.1}$$

This map is equivalent to the complex

$$\Omega^0(\mathfrak{su}(N)) \to \Omega^1(\mathfrak{su}(N)) \to \Omega^2_-(\mathfrak{su}(N)),$$

of $\mathfrak{su}(N)$-valued forms (the negative means anti-self-dual). By the irreducibility of the connection and positive curvature of S^4, only the first cohomology H^1 of this complex does not vanish. This implies that the index

$$\mathrm{ind}(D_A) = \dim\ker(D_A) - \dim\mathrm{coker}(D_A)$$

is exactly $\dim\ker(D_A) = \dim H^1 =: h^1$.

The Atiyah-Singer index theorem says that

$$\begin{aligned}
\mathrm{ind}(D_A) &= \mathrm{ch}(E)\hat{\mathcal{A}}(S^4)[S^4] \\
&= (\dim E + p_1(E)[1 - \frac{1}{24}p_1(S^4)] \\
&= p_1(E) + \dim(G)(\mathrm{ch}(S^-)\hat{\mathcal{A}}(S^4)) \\
&= p_1(E) + \dim(G)(\mathrm{ind}\, D : \Gamma(S^+ \times S^- \to S^- \otimes S^-)) \\
&= p_1(E) - \frac{1}{2}\dim(G)(\chi(S^4) - \mathrm{signature}(S^4)) \\
&= p_1(E) - \dim(G).
\end{aligned}$$

This formula applies to any of the simple Lie algebras (for large κ since when κ is small compared to N, there are some additional symmetries) [AHS78].

G	$p_1(\mathfrak{g}) - \dim(G)$	
$\mathrm{SU}(N)$	$4N\kappa - N^2 + 1$	$\kappa \geq N/2$
$\mathrm{Spin}(N)$	$4(N-2)\kappa - \frac{1}{2}N(N-1)$	$\kappa \geq N/4$
$\mathrm{Sp}(N)$	$4(N+1)\kappa - N(2N+1)$	$\kappa \geq N$
G_2	$16\kappa - 14$	$\kappa \geq 2$
F_4	$36\kappa - 52$	$\kappa \geq 3$
E_6	$48\kappa - 78$	$\kappa \geq 3$
E_7	$72\kappa - 133$	$\kappa \geq 3$
E_8	$120\kappa - 248$	$\kappa \geq 3$

Note that these are the dimensions of the moduli spaces of unframed instantons. The framing increases the dimension of the moduli space by $\dim(G)$.

CHAPTER 6

The ADHM construction

The Atiyah-Drinfeld-Hitchin-Manin (ADHM) transform [Ati+78] turns the anti-self-dual equations

$$F_A = - \star F_A,$$

written as non-linear partial differential equations for a connection A,

$$\epsilon_{ijkl} \left(\partial_i A_j - \partial_j A_i + \partial_k A_l - \partial_l A_k + [A_i, A_j] + [A_k, A_l] \right) = 0$$

where ϵ_{ijkl} vanishes if i, j, k, l are not all distinct and is otherwise the sign of the permutation $(ijkl)$, into non-linear *algebraic* matrix equations, called the *ADHM equations*,

$$[\alpha_1, \alpha_2] + ba = 0 \tag{6.1}$$

$$[\alpha_1, \alpha_1^*] + [\alpha_2, \alpha_2^*] + bb^* - a^* a = 0. \tag{6.2}$$

A solution $(\alpha_1, \alpha_2, a, b)$ to the ADHM equations is an equivalence class of complex valued

(1) $\kappa \times \kappa$ matrices α_1, α_2,
(2) an $N \times \kappa$ matrix a, and
(3) a $\kappa \times N$ matrix b,

which satisfy (6.1) and (6.2) and for each $g \in \mathrm{U}(N)$, the equivalence relations

$$\begin{aligned} \alpha_i &\mapsto g \alpha_i g^{-1} \\ a &\mapsto \lambda a g^{-1} \\ b &\mapsto g b \lambda^{-1}. \end{aligned}$$

Given a solution of the ADHM equations, one can construct a holomorphic vector bundle E on $\mathbb{P}^3$ trivial over a $\mathbb{P}^1$. The holomorphic vector bundle E has fibres given by the cohomology

$$E_X = \ker B_X / \mathrm{im} A_X$$

of the sequence of vector spaces

$$H \overset{A_X}{\to} K \overset{B_X}{\to} L,$$

where K is $2\kappa + N$-dimensional and H, L are κ-dimensional.

The linear maps A_X and B_X where $X = [x : y : z : w] \in \mathbb{P}^3$, assembled from a solution $(\alpha_1, \alpha_2, a, b)$ of the ADHM equations as follows,

$$A_X = \begin{bmatrix} x + z\alpha_1 - w\alpha_2^* \\ y + z\alpha_2 + w\alpha_1^* \\ za + wb^* \end{bmatrix};$$

$$B_X = \begin{bmatrix} -y - z\alpha_2 - w\alpha_1^* & x + z\alpha_1 - w\alpha_2^* & zb - wa^* \end{bmatrix}.$$

Thus, a solution of the ADHM equations allows one to construct a holomorphic vector bundle over $\mathbb{P}^3$ trivial over a $\mathbb{P}^1$ with unitary structure. The rest of this chapter discusses how such a holomorphic vector bundle constructs an instanton field by algebro-geometric methods [Ati79]. The first section of the following chapter provides more detail on the construction of the vector bundle from $(\alpha_1, \alpha_2, a, b)$.

We begin by treating S^4 as the quaternionic projective line $\mathbb{HP}^1$. There exists a fibration

$$\mathbb{P}^3 \to S^4 \simeq \mathbb{HP}^1,$$

$$[x : y : z : w] \mapsto [x + yj : z + wj] \tag{6.3}$$

called the *twistor fibration* with fibres $\mathbb{P}^1$. The complex projective 3-space $\mathbb{P}^3$ parametrising geodesics is called the twistor space of $\mathbb{R}^4$.

The action of $j \in \mathbb{H}$ on $\mathbb{H}^2$ by left multiplication

$$(x + yj, z + wj) \mapsto (\bar{x}j - \bar{y}, \bar{z}j - \bar{w})$$

induces the reality map $\sigma : \mathbb{P}^3 \to \mathbb{P}^3$ given by

$$[x : y : z : w] \mapsto [\bar{y} : -\bar{x} : \bar{w} : -\bar{z}].$$

Multiplication by j fixes $\mathbb{HP}^1$ but σ acts as the antipodal map on the fibres $\mathbb{P}^1$ of the fibration. These fibres are called *real lines*.

The relevant theorem is then:

THEOREM 10 (Ward [War77]; Atiyah-Ward [Ati79]). *There is a correspondence between*

1. *irreducible anti-self-dual $SU(N)$ connections over S^4, and*
2. *holomorphic vector bundles on $\mathbb{P}^3$ with fibre $\mathbb{C}^N$ trivial on real lines with a positive real form.*

Let $\tilde{E}$ be a holomorphic vector bundle on $\mathbb{P}^3$. An antilinear isomorphism $\tilde{\sigma} : \tilde{E} \to \tilde{E}^*$ covering σ such that

$$(u, \tilde{\sigma}v) = \overline{(v, \tilde{\sigma}u)}, \quad v \in \tilde{E}_X,\ u \in \tilde{E}_{\sigma X},$$

is said to be a real form. Since the bundle we are considering is trivial over any real line, the holomorphic sections restricted to a real line is isomorphic to $\mathbb{C}^N$ and the real form induces a (non-degenerate) hermitian form on the restricted holomorphic sections. In the pushforward E of $\tilde{E}$ along the twistor fibration, the holomorphic sections restricted to a real line integrate to the fibre of E over a point. Hence the real form induces a hermitian form on the fibres of E.

I would like to discuss this theorem by seeing what happens to the data of a $\mathrm{SU}(N)$-invariant connection on S^4 as it is encoded into the holomorphic structure of a vector bundle on the twistor space S^4.

Lemma 11. *[Ati79]*

(1) *For a vector bundle E on S^4 equipped with a unitary metric, a 2-form is anti-self-dual if and only if it is a (1,1)-form with respect to all complex structures compatible with the unitary metric (and orientation).*

(2) *A (1,1) 2-form on S^4 lifts to a (1,1) 2-form on $\mathbb{P}^3$.*

Choosing some coordinates on $\mathbb{R}^4 \simeq \mathbb{C}^2$, a 2-form F has the form

$$F = F_{12} dz_1 \wedge dz_2 + \sum_{\substack{\rho=1 \\ \sigma=1}}^{2} F_{\rho\sigma} dz_\rho \wedge d\bar{z}_\sigma + F_{\bar{1}\bar{2}} d\bar{z}_1 \wedge d\bar{z}_2,$$

and we say that the three parts are type (2,0), (1,1) and (0,2) respectively with respect to the complex structure. Under $\star$, three of the basis elements are self-dual and three are anti-self-dual. It can be checked that all of the anti-self-dual 2-forms are all type (1,1).

Furthermore, this anti-self-dual subspace is irreducible under the action of unitary matrices ($\mathrm{U}(2)$ to be precise) so it is invariant under different choices of complex structure which are compatible with the unitary metric. Thus the anti-self-duality has been encoded in the type of the 2-form.

The second part of the lemma follows from noticing that lifting a (1,1) 2-form along the twistor fibration produces a horizontal 2-form on the lifted bundle $\tilde{E}$ (that is, $F_{\rho\sigma} = 0$ if either ρ or σ is in a fibre-direction). This is due to the remarkable existence of such a fibration. The two parts of the lemma combine to give us the following proposition.

Proposition 12. *[Ati79] A vector bundle E on S^4 with unitary structure and connection has anti-self-dual curvature if and only if the lifted bundle $\tilde{E}$ and connection on $\mathbb{P}^3$ has curvature form in the type $(1,1)$ 2-forms $\Omega^{1,1}$.*

If a vector bundle $\tilde{E} \to \mathbb{P}^3$ is both holomorphic and unitary, there is a unique connection which is also both holomorphic and unitary. To be precise, a unitary gauge is a choice of basis for $\tilde{E}_X$ at each $X \in \mathbb{R}^3$ which is orthonormal with respect to the metric and varies continuously with X. A holomorphic gauge is a choice of basis which varies holomorphically with respect to X. Geometrically, these can be thought of as a continuous section of a principal $\mathrm{U}(N)$-bundle and a holomorphic section of a principal $\mathrm{GL}(N, \mathbb{C})$-bundle respectively (as frame bundles). Any gauge which is both unitary and holomorphic is constant.

Proposition 13. *[Ati79] Let E be a holomorphic vector bundle with a unitary structure. There exists a unique connection A such that*

(1) *in every unitary gauge, $A^* = -A$,*
(2) *in every holomorphic gauge, $A = \sum A_\mu dz_\mu$ (it is of type (1,0)).*

The curvature 2-form F of A is of type (1,1).

Conversely, a unitary vector bundle with a connection with a (1,1) curvature form defines the unique holomorphic structure for which the above is true.

Let $A = \sum A_\mu dz_\mu$ in a holomorphic gauge. A gauge change to a unitary gauge results in

$$\tilde{A} = \sum g^{-1} A_\mu g dz_\mu + g^{-1} dg.$$

The term $g^{-1}dg$ has an antiholomorphic part $d\bar{z}$ since it is not a holomorphic gauge. This is the only source of the (0,1) part of the connection in the unitary gauge. Since conjugation under the unitary structure exchanges the holomorphic and anti-holomorphic parts,

$$\sum g^{-1} A_\mu g dz_\mu = -\left(g^{-1}dg\right)^*,$$

the (1,0) part is uniquely determined.

Hence, a holomorphic vector bundle $\tilde{E} \to \mathbb{P}^3$ associated to an principal $\mathrm{SU}(N)$-bundle uniquely determines an instanton on $\mathbb{R}^4$. If the data includes a trivialisation $\tilde{E}|_{l_\infty} \simeq \mathbb{C}^N \times l_\infty$ over the line $l_\infty \subset \mathbb{P}^3$ over $\infty \in S^4$ then the instanton is framed.

An explicit expression for the connection

The ADHM Horrocks formalism requires a skew form so it constructs $\mathrm{Sp}(n)$ instantons [Ati79].

Let V, W be vector spaces with $\dim V = 2\kappa + 2n$ and $\dim W = \kappa$. Let there be a non-degenerate skew form ω on V making it into a quaternionic vector space. Then let

$$\Delta : W \otimes \mathbb{C}^4 \to V$$

be any linear map such that for nonzero $X \in \mathbb{C}^4$, $\Delta(X) : W \to V$ is injective and the image of $\Delta(X)$ is isotropic with respect to the skew form ω on V. This last condition is

$$\Delta(X)(W) \subset \Delta(X)(W)^{\perp}$$

where

$$v(X)(W)^{\perp} = \{v \in V \mid \text{if } w \in \Delta(X)(W) \text{ then } \omega(v, w) = 0\}$$

is the symplectic complement.

The fibres

$$E_X = \Delta(X)(W)^{\perp}/\Delta(X)(W)$$

vary holomorphically with X and determine a holomorphic vector bundle E with skew-form inherited from V. E pushes forwards along the twistor fibration $\mathbb{P}^3 \to \mathbb{HP}^1$ (6.3) to the bundle of an instanton.

Complex conjugation of scalars induces an anti-holomorphic involution on W whose fixed point set is the real-valued bundle $W_{\mathbb{R}}$. Then A is equivalently a quaternionic-linear map

$$\Delta : W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{H}^2 \to V. \tag{6.4}$$

In the physics literature, the ADHM construction is commonly carried out by choosing bases for V and $W_{\mathbb{R}}$ such that $\Delta(X) = \Delta(x + yj, z + wj) : W_{\mathbb{R}} \to V$ can be written in terms of quaternionic matrices L and M [Ati+78; Wei12],

$$\Delta(x + yj, z + wj) = \begin{bmatrix} L \\ M \end{bmatrix} - \left(\frac{x + yj}{z + wj}\right) \begin{bmatrix} 0_{n\times\kappa} \\ I_{\kappa\times\kappa} \end{bmatrix}.$$

The dimensions of the quaternionic matrices L and M are $n\times\kappa$ and $\kappa\times\kappa$ respectively. The first term on the right side of the equality is often written $\hat{M}$ and is called the ADHM matrix.

In this formalism, the above conditions take the form of a non-linear reality condition,

$$\hat{M}^*\hat{M} = R_0,$$

for some real nonsingular $\kappa \times \kappa$ matrix R_0, and an invertibility condition that

$$R(X) = \Delta(X)^*\Delta(X)$$

be real-valued and $\det R(q) \neq 0$ for all $q \in \mathbb{H}$.

The connection A can be produced by solving the equation

$$\Psi^*\Delta = 0$$

where $\Psi(X)$ is an $(n+\kappa)\times n$ matrix obeying $\Psi^*\Psi = 1$. Then,

$$A = \Psi^* d\Psi.$$

The insight here is that Ψ is an inclusion $\Psi : \mathbb{R}^n \to \mathbb{H}^{n+\kappa}$ from which we get a projection operator $P = \Psi\Psi^*$ onto the fibres of the rank n bundle E in $\mathbb{H}^{n+k}$. The covariant derivative is the projection of the usual derivative

$$\nabla_\mu f = P\partial_\mu f$$

and if we interpret Ψ as a choice of gauge then

$$\nabla(\Psi f) = \Psi\Psi^* d(\Psi f) = \Psi(df + \Psi^* d\Psi f).$$

Thus the connection A in the usual expression for the covariant derivative $\nabla = d + A$ is given by $A = \Psi^* d\Psi$. Note that the different choices of Ψ give the same connection A in different gauges.

Note that this gauge choice can be changed by the action of the automorphism groups $\mathrm{Sp}(n+\kappa)$ for V and $\mathrm{O}(\kappa)$ for W. The action of $\mathrm{Sp}(n)$ induces an action on the connection A but the $\mathrm{O}(\kappa)$ action does not. This is called reciprocity by Corrigan and Goddard [CG84].

CHAPTER 7

Holomorphic bundles over $\mathbb{P}^2$

Donaldson applied Geometric Invariant Theory (GIT) to prove the following theorem:

THEOREM 14 (Donaldson [Don84a]).

$$\tilde{M}(G,\kappa) \simeq \mathcal{O}\tilde{M}(G^{\mathbb{C}},\kappa).$$

The left side $\tilde{M}(G,\kappa)$ of the isomorphism is the moduli of space of framed G instantons of charge κ. This means pairs (A,Θ) consisting of an anti-self-dual connections A and an isomorphism $\Theta : P_\infty \to G$ of the fibre of P at some fixed point of S^4 which we label ∞.

Let $\mathbb{P}^2$ be the subvariety defined by $\{[x:y:z:w] \in \mathbb{P}^3 \mid w = 0\}$. $\mathcal{O}\tilde{M}(G^{\mathbb{C}},\kappa)$ is the moduli of pairs (E,θ) consisting of a holomorphic bundle E on $\mathbb{P}^2$ associated to a principal G-bundle P, of second chern class κ, trivial on a line l_∞ at infinity, and a trivialisation θ there.

Hulek and Barth have a construction of the holomorphic bundles parametrised by $\mathcal{O}\tilde{M}(G^{\mathbb{C}},\kappa)$ [OSS80]. Donaldson has a particularly nice form for this construction which realises the holomorphic bundle E of the ADHM construction as the cohomology of a *monad* on $\mathbb{P}^2$ with the *complexified* group $G^{\mathbb{C}}$.

With $\mathcal{O}(1)$ being the Hopf bundle over $\mathbb{P}^2$, let

(1) H,K,L be κ,$2\kappa+N$,κ dimensional vector spaces over $\mathbb{C}$ respectively; and
(2) $\underline{H} = H \otimes \mathcal{O}(-1)$, $\underline{K} = K \otimes \mathcal{O}$, $\underline{L} = L \otimes \mathcal{O}(1)$.

A *monad* over $\mathbb{P}^2$ is the following pair of families of linear maps A_X,B_X for each $[x:y:z] = X \in \mathbb{P}^2$ depending linearly on X,

$$\underline{H} \overset{A_X}{\to} \underline{K} \overset{B_X}{\to} \underline{L}\ .$$

The map A_X needs to be injective, the map B_X needs to be surjective and $B_X A_X \equiv 0_\kappa$.

Since the maps A_X,B_X vary holomorphically with $X \in \mathbb{P}^2$, the holomorphic bundle E can be defined fibre-wise by the cohomology

$$E_X = \ker B_X / \mathrm{im}\ A_X$$

of the monad. This construction is unique up to an action of $\mathrm{GL}_{HKL} = \mathrm{GL}(H) \times \mathrm{GL}(K) \times \mathrm{GL}(L)$.

Following Donaldson, the conditions on A_X and B_X can be used to write them in a more useful form. By the linearity of A_X and B_X, we can write them in terms of constant matrices A_x, A_y, A_Z and B_x, B_y, B_z,

$$A_X = A_x x + A_y y + A_z z, \qquad B_X = B_x x + B_y y + B_z z.$$

Triviality of the bundle on the line $\{z = 0\}$, and $B_X A_X \equiv 0_\kappa$ implies that $B_x A_y$ is an isomorphism, and

$$B_x A_y = -B_y A_x, \qquad B_y A_y = 0, \qquad B_x A_x = 0.$$

Wield GL_{HKL} symmetry (ie. row reduce) to set $A_x^T = (\mathbb{1}_\kappa, 0_\kappa, 0_N) = -B_y$ and $A_y^T = (0_\kappa, \mathbb{1}_\kappa, 0_N) = B_x$ to satisfy these conditions.

Let α_1, α_2 be $\kappa \times \kappa$ matrices, a be an $N \times \kappa$ matrix, and b be a $\kappa \times N$ matrix. Now A_X and B_X are of the form,

$$A_X = \begin{bmatrix} x + z\alpha_1 \\ y + z\alpha_2 \\ za \end{bmatrix};$$

$$B_X = \begin{bmatrix} -y - z\alpha_2 & x + z\alpha_1 & zb \end{bmatrix};$$

and to satisfy $B_X A_X \equiv 0_\kappa$, it has to satisfy the complex ADHM equation

$$[\alpha_1, \alpha_2] + ba = 0. \tag{7.1}$$

The $G^{\mathbb{C}}$ action on the monad is

$$\alpha_i \mapsto g\alpha_i g^{-1}$$
$$a \mapsto \lambda a g^{-1}$$
$$b \mapsto g b \lambda^{-1}.$$

For the fibre $\mathbb{P}^1_\infty = \{[x : y : 0]\}$ over infinity, the trivialisation data θ is

$$A_X = \begin{bmatrix} xI_\kappa \\ yI_\kappa \\ 0_{N\times\kappa} \end{bmatrix},$$

$$B_X = \begin{bmatrix} -yI_\kappa & xI_\kappa & 0_{\kappa\times N} \end{bmatrix}.$$

An analogous monad over $\mathbb{P}^3$ (it is understood that over $\mathbb{P}^3$, $\mathcal{O}$ means $\mathcal{O}_{\mathbb{P}^3}$) can be defined with maps A_X and B_X over $\mathbb{P}^3$,

$$A_X = \begin{bmatrix} x + z\alpha_1 - w\alpha_2^* \\ y + z\alpha_2 + w\alpha_1^* \\ za + wb^* \end{bmatrix};$$

$$B_X = \begin{bmatrix} -y - z\alpha_2 - w\alpha_1^* & x + z\alpha_1 - w\alpha_2^* & zb - wa^* \end{bmatrix}.$$

If the monad maps satisfy the *real* ADHM equation

$$[\alpha_1, \alpha_1^*] + [\alpha_2, \alpha_2^*] + bb^* - a^*a = 0, \tag{7.2}$$

and the $G^{\mathbb{C}}$ action is restricted to a G action to preserve it then this monad can be shown to produce the corresponding real bundle on $\mathbb{P}^3$.

The real ADHM equation is the vanishing of a moment map $\mu : \mathcal{O}\tilde{M} \to \mathfrak{g}^*$ in Donaldson's application of Geometry Invariant Theory (GIT) [Don84a].

The quaternion action on $\mathbb{C}^4$ is generated by

$$I(x, y, z, w) = (ix, iy, iz, iw), \quad J(x, y, z, w) = (\bar{y}, -\bar{x}, \bar{w}, -\bar{z}),$$

and the action of J induces a map $J^* : E \to E$. The subspace of holomorphic bundles E fixed under the action of the quaternions, that is, those for which there exists an isomorphism $\overline{J^*(E)} \simeq E^*$ is a GIT quotient equivalent to $\mu^{-1}(0)$.

In the case of $\mathrm{SU}(N)$ instantons, $G^{\mathbb{C}} = \mathrm{GL}(N, \mathbb{C})$ and $G = \mathrm{U}(N)$.

To see why this version of the ADHM construction is equivalent to the one presented in the previous chapter, note that the isomorphism on E on the level of the monad induces a Hermitian metric on $\underline{K}$ and an isomorphism $\bar{H}^* \equiv L$. Thus the monad can be rewritten as a map

$$\underline{L} \oplus \underline{L} \to \underline{K}.$$

Compare this with equation (6.4).

Hence Donaldson's theorem says that given *ADHM data* $(\alpha_1, \alpha_2, a, b)$ (to be precise, the $G^{\mathbb{C}}$ equivalence class satisfying the complex ADHM equation), there is a way of constructing instantons and holomorphic bundles on $\mathbb{P}^2$ trivial on a line from the ADHM data.

The ADHM quiver and perverse instantons

In this section, I will divert off the main path to discuss a generalisation of ADHM data. The following is a brief mention of work by Nakajima [Nak99] that ties in with current research on quiver varieties. The modern context for this work lies in the theory of perverse coherent sheaves and the geometric Langlands conjecture.

Definition 15. Let V, W be vector spaces over $\mathbb{C}$ of dimension κ and N respectively. ADHM data $(\alpha_1, \alpha_2, a, b)$ are maps $\alpha_1, \alpha_2 \in \mathrm{Hom}(V, V)$, $a \in \mathrm{Hom}(V, W)$ and $b \in \mathrm{Hom}(W, V)$ satisfying

$$[\alpha_1, \alpha_2] + ba = 0.$$

The previous usage of ADHM data is now referred to as regular ADHM data.

Definition 16. ADHM data $(\alpha_1, \alpha_2, a, b)$ is

(1) Stable if there is no subspace $S \subsetneq V$ such that $\alpha_1 S, \alpha_2 S, aW \subseteq S$;
(2) Costable if there is no non-trivial subspace $S \subseteq V$ such that $\alpha_1 S, \alpha_2 S \subseteq S$ and $S \subseteq \ker b$;
(3) Regular if it is stable and costable.

Let X be ADHM data. The stabilising subspace $\Sigma_X \subseteq V$ of X is defined as $\bigcap S$ for all $S \subseteq V$ satisfying $\alpha_1 S, \alpha_2 S, aS \subseteq S$. Σ_X is the smallest subspace of V for which X is stable.

There are correspondences

$$\begin{array}{ccc}
\{\text{ADHM data}\}/\mathrm{GL}(N) & \longrightarrow & \{\text{framed perverse coherent sheaves on } \mathbb{P}^2\} \\
\cup\!\mid & & \cup\!\mid \\
\{\text{stable ADHM data}\}/\mathrm{GL}(N) & \longrightarrow & \{\text{framed torsion-free coherent sheaves on } \mathbb{P}^2\} \\
\cup\!\mid & & \cup\!\mid \\
\{\text{regular ADHM data}\}/\mathrm{GL}(N) & \longrightarrow & \{\text{framed holomorphic vector bundles on } \mathbb{P}^2\}
\end{array}$$

Definition 17. A coherent sheaf E on $\mathbb{P}^2$ is torsion free if for $X \in \mathbb{P}^2$, the stalk E_X is a torsion free $\mathcal{O}_{\mathbb{P}^2,X}$-module, that is, if $f \in \mathcal{O}_{\mathbb{P}^2,X}$ and $a \in E_X$ then $fa = 0$ implies that $a = 0$ or $f = 0$.

Remember that a holomorphic vector bundle is a locally free sheaf. Torsion free (coherent) sheaves on $\mathbb{P}^2$ are locally free except on a set of dimension zero. Equivalently, torsion free sheaves are rank r holomorphic vector bundles except at a set of points where they fail to be rank r.

The usual definition of perverse coherent sheaves requires discussion of the t-structures of Kashiwara [Kas04]. However, the exposition can be vastly simplified if a theorem of Jardim-Martins [JM11] is taken as a definition.

Definition 18. A perverse coherent sheaf E^* on $\mathbb{P}^2$ trivial at infinity is a complex $E^{-i} \to E^{-i+1} \to \ldots \to E^{j-1} \to E^j$ of coherent sheaves on $\mathbb{P}^2$ in the bounded derived category (ie. $i, j \in \mathbb{Z}$ are finite) satisfying

(1) $H^i(E^*)) = 0$ for $i \neq 0, 1$;
(2) $H^0(E^*)$ is a torsion free sheaf, trivial at infinity (isomorphic to a free sheaf/ trivial holomorphic vector bundle at l_∞); and
(3) $H^1(E^*))$ is a torsion sheaf with support outside of l_∞.

Remember that our holomorphic bundle E is $H^0(E^*)$ of a monad

$$E^{-1} \to E^0 \to E^1$$

and the non-degeneracy or regularity conditions guarantee that $H^1(E^{-1}) = 0$. In fact, we see that the monad in the ADHM construction is an example of a perverse coherent sheaf on $\mathbb{P}^2$ trivial at infinity.

Definition 19. The ADHM quiver is the following directed graph

$$\begin{tikzcd} \bullet v \arrow[loop left] \arrow[loop right] \arrow[d, bend right] \\ \bullet w \arrow[u, bend right] \end{tikzcd}$$

The upshot which takes us into deeper waters, is the category of perverse instanton sheaves equivalent to ADHM data is the category of representations of the ADHM quiver [JM11].

Part 3

Hyperbolic Monopoles

Into the hyperbolic

Hyperbolic 3-space can be realised as the upper half space

$$H^3 = \left\{(x, y, r) \in \mathbb{R}^3 \mid r > 0\right\}$$

with the negative curvature metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dr^2}{r^2}.$$

The treatment of hyperbolic monopoles differs from the treatment of euclidean monopoles because hyperbolic monopoles can be studied as circle-invariant instantons. This limits us to only examining monopoles with *integral mass*. The main consequence is that hyperbolic monopoles are uniquely determined up to gauge equivalence by their restriction to an asymptotically large sphere.

The challenge in the study of hyperbolic monopoles is that the L^2 norm on the monopole moduli space is not finite and no hyperKähler metric has been found for the moduli space. This issue is outside the scope of this thesis but receives airtime in the following references: [BA90; BCS15; Hit93].

CHAPTER 8

Circle-invariant instantons

Let P' be a principal $\mathrm{SU}(N)$ bundle on hyperbolic 3-space. A magnetic monopole (A, Φ) on hyperbolic 3-space H^3 consists of an $\mathrm{SU}(N)$ connection A on P' and a section Φ of the adjoint bundle ad P' which satisfy the Bogolmonyi equations with Hodge star dual $\star_{H^3}$ taken with respect to the hyperbolic metric,

$$F_A = \star_{H^3} D_A \Phi$$

whose energy is finite.

However, the analysis for boundary conditions which are implied by finite energy has not been thoroughly treated. An alternative definition of hyperbolic monopoles as S^1-invariant instantons was suggested by Atiyah [Ati84a; Ati84b]. This sidesteps the analytical difficulties. From here onwards, this is the definition of hyperbolic monopole which will be used. The limitation of this is that for the remainder of the thesis, unless otherwise specified, hyperbolic monopoles will only have *integral mass*.

For completeness, let me say what an instanton is and list some properties. For more detail, refer to Chapter 5.

Definition 20. Let P be the (trivial) principal $\mathrm{SU}(N)$ bundle on $\mathbb{R}^4$. An instanton $A_\square$ is a gauge equivalence class of connection 1-forms on P whose curvature form $F_{A_\square}$ is anti-self dual,ie.

$$F_{A_\square} = - \star F_{A_\square}$$

and extends to a connection 1-form on the extension of P to a bundle on S^4, the conformal compactification of $\mathbb{R}^4$.

The latter condition on $A_\square$ is the clean way of saying that $A_\square$ decays asymptotically. Another common statement of the decay condition is that there exists a gauge transformation g on a sphere S^3 of large radius such that $A \sim g^{-1}dg$ there.

The bundle P extended to $\{\infty\} \in S^4$ is now no longer necessarily trivial. The degree κ of $A_\square|_{S^3} : S^3 \to \mathrm{SU}(N)$ (which can be considered as an element of the third homotopy group $\pi_3(\mathrm{SU}(N))$) is a topological invariant of P which we call the *instanton charge*.

An instanton is *framed at* $\{\infty\} \in S^4$ if the gauge equivalence is given by the subset of gauge transformations g where $g(\infty) = 1$.

Let (x_1, x_2, x_3, x_4) be a coordinate basis for $\mathbb{R}^4$. The circle S^1 action on $\mathbb{R}^4$ which we will consider is a rotation in the x_3x_4-plane $\mathbb{R}^2_{34} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = 0 = x_2\}$. Let (r, θ), $r > 0$, $\theta \in [0, 2\pi)$ be polar coordinates for the x_3x_4-plane (sans the origin). This S^1 action is then

$$(x_1, x_2, r, \theta) \mapsto (x_1, x_2, r, \theta + \theta')$$

for $\theta' \in S^1$. The x_1x_2-plane $\mathbb{R}^2_{12} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_3 = 0 = x_4\}$ is the fixed-point set of the S^1-action, thought of as the axis of rotation.

The euclidean metric in these coordinates is

$$ds^2 = r^2 \left(\frac{dx_1^2 + dx_2^2 + dr^2}{r^2} + d\theta^2 \right).$$

Without the axis of rotation, $\mathbb{R}^4$ is foliated by upper half spaces and the euclidean metric restricted to each upper half space is conformally equivalent to the Poincaré hyperbolic metric. Conformally,

$$\mathbb{R}^4 - \mathbb{R}^2 \simeq S^1 \times H^3.$$

An S^1-invariant instanton $A_\square$ on $\mathbb{R}^4$ is an instanton A on a principal $\mathrm{SU}(N)$-bundle P and a lifting of the S^1-action to an action on P which preserves A. In particular, for framed instantons, the homomorphism $\alpha : S^1 \to \mathrm{SU}(N)$ determined by the lifting at ∞ is invariant (Atiyah calls it the *type*). Hence, it makes sense to define hyperbolic monopoles as S^1-invariant instantons.

Definition 21. A magnetic monopole (A, Φ) on hyperbolic 3-space H^3 is an S^1-invariant framed instanton on $\mathbb{R}^4$.

Over $\mathbb{R}^2_{12}$, the *prescribed* lifting $\rho : S^1 \to \mathrm{Aut}P$ of the S^1-action from $\mathbb{R}^4$ to P induces a representation of S^1 on the fibres of $P|_{\mathbb{R}^2_{12}}$. The gauge transformations g compatible with the lifted action are the S^1-equivariant gauge transformations

$$g(x_1, x_2, r, \theta + \theta') = \rho(\theta') g(x_1, x_2, r, \theta).$$

For framed instantons, the compatible gauge transformations are S^1-invariant, ie. $g = g(x_1, x_2, r)$.

From here on, I will make use of the decay condition to work over S^4 instead of $\mathbb{R}^4$. Thus, the principal $\mathrm{SU}(N)$ bundle P and connection 1-form $A_\square$ will be treated as objects over S^4. The 4-sphere S^4 will be regarded as $\mathbb{HP}^1$, the projective quaternion line $(\mathbb{H}^2 - 0)/\mathbb{H}^\times$.

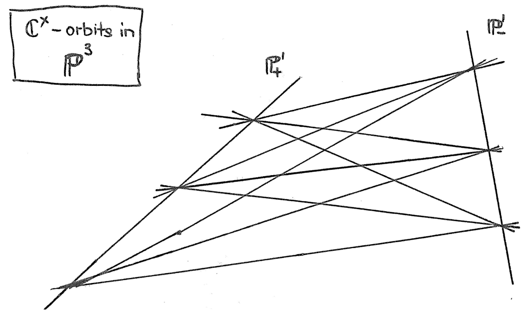


FIGURE 1. The decomposition of $\mathbb{P}^3$ by the $\mathbb{C}^\times$-action into fixed lines and $\mathbb{C}^\times$ orbits.

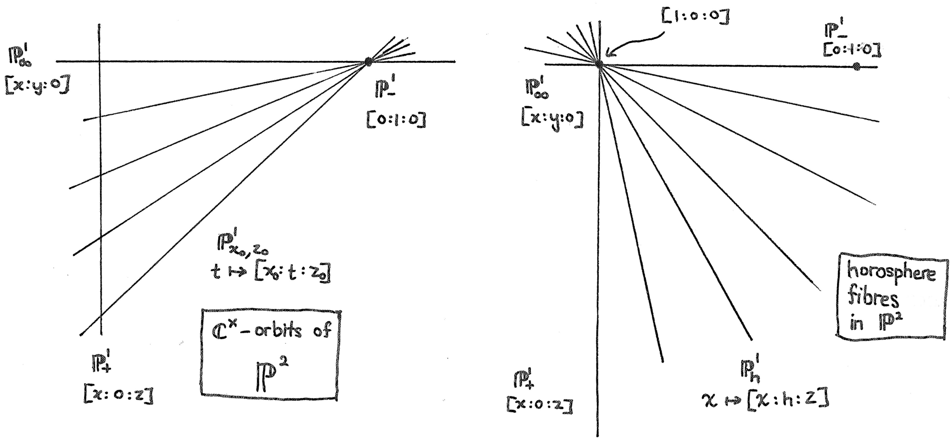


FIGURE 2. The $\mathbb{C}^\times$ orbits of $\mathbb{P}^2$ and the fibres of horospheres intersecting $\{\infty\} \in \partial H^3$.

The data of an instanton is equivalent to the rank N complex vector bundle $\mathcal{E}$ associated to P with anti-self-dual curvature. Applying the twistor technique, lift $\mathcal{E}$ to a bundle E on $\mathbb{P}^3$ along the twistor fibration

$$\mathbb{P}^3 \to \mathbb{HP}^1 \cong S^4$$

$$[x : y : z : w] \mapsto [x + yj : z + wj].$$

The bundle E is a rank N holomorphic vector bundle.

The S^1-action on S^4 lifts to an action of $\mathbb{C}^\times$ on $\mathbb{P}^3$

$$c \mapsto [c^{-1/2}x : c^{1/2}y : c^{-1/2}z : c^{1/2}w].$$

This action has two fixed lines (see Figure 1)

$$\mathbb{P}^1_+ = \{[x : 0 : z : 0]\},\ \mathbb{P}^1_- = \{[0 : y : 0 : w]\}$$

with opposite orientations. The orbits of $\mathbb{C}^\times$ intersect both fixed lines. The real structure on $\mathbb{P}^3$ is the anti-holomorphic linear map

$$\sigma : \mathbb{P}^3 \to \mathbb{P}^3$$

$$[x : y : z : w] \mapsto [-\bar{y} : \bar{x} : -\bar{w} : \bar{z}]$$

which exchanges the two fixed lines. The real structure σ induces a positive real form on E, an anti-linear isomorphism $p : E \to E^*$ such that for $v \in E_X$, $u \in E_{\sigma(X)}$,

$$(u, pv) = \overline{(v, pu)}.$$

An important subspace of $\mathbb{P}^3$ is the projective plane $\mathbb{P}^2$ defined by $w = 0$ (see Figure 2). This $\mathbb{P}^2$ contains the fixed line $\mathbb{P}^1_+$, the fixed point $P_- = [0 : 1 : 0]$ as well as the line $\mathbb{P}^1_\infty$, the twistor fibre over $\infty \in S^4$. The $\mathbb{C}^\times$-action on $\mathbb{P}^2$ is, for $c \in \mathbb{C}^\times$,

$$c \mapsto [x : cy : z].$$

Proposition 22. *A framed $SU(N)$ instanton is S^1-invariant and hence a hyperbolic monopole (A, Φ) if and only if the associated holomorphic vector bundle $E \to \mathbb{P}^3$ is $\mathbb{C}^\times$-equivariant.*

There are two important families of lines in $\mathbb{P}^2$. For every point of $\mathbb{P}^1_+$, there is a $\mathbb{C}^\times$-orbit in $\mathbb{P}^2$ intersecting that point and P_-. The lines which meet $[1 : 0 : 0] = \mathbb{P}^1_+ \cap \mathbb{P}^1_\infty$ map to horospheres in H^3 and so they will be called the horosphere lines. This latter family of lines makes $\mathbb{P}^2$ akin to the “complex manifold version” of H^3.

Over $\mathbb{P}^1_+$, the $\mathbb{C}^\times$-action induced by the $\mathbb{C}^\times$-action on $\mathbb{P}^3$, on $E|_{\mathbb{P}^1_+}$ is a representation on each fibre. By Schur’s Lemma, this representation can be written, for $c \in \mathbb{C}^\times$, and for some $p_1, \ldots, p_N \in \mathbb{Z}$, and $\sum_{i=1}^N p_i = 0$ as

$$c \mapsto \mathrm{diag}(c^{p_1}, c^{p_2}, \ldots, c^{p_N}).$$

Thus $E|_{\mathbb{P}^1_+}$ splits into line bundles.

Let $\mathcal{L}_p \cong \mathbb{C}$ be the representation of $\mathbb{C}^\times$ of weight p, $p \in \frac{1}{2}\mathbb{Z}$, ie. for $c \in \mathbb{C}^\times$, $x \in \mathcal{L}_p$, $c \cdot x = c^p x$.

Since the action is algebraic, the line bundles are algebraic or equivalently, holomorphic. By Birkhoff–Grothendieck [OSS80], this splitting is unique up to permutation of the summand thus for some $k_1, \ldots, k_N \in \mathbb{Z}$,

$$E|_{\mathbb{P}^1_+} = \mathcal{O}(k_1) \otimes \mathcal{L}_{p_1} \oplus \ldots \oplus \mathcal{O}(k_{N-1}) \otimes \mathcal{L}_{p_{N-1}} \oplus \mathcal{O}(k_N) \otimes \mathcal{L}_{p_N}$$

where $\sum_{i=1}^N p_i = 0$ and $\sum_{i=1}^N k_i = 0$. The numbers $k_1, \ldots, k_N$ are the 1st chern numbers of the line bundles.

Over $\mathbb{P}^1_-$, the same arguments imply that

$$E|_{\mathbb{P}^1_-} \cong \mathcal{O}(k_1) \otimes \mathcal{L}_{p_1} \oplus \ldots \oplus \mathcal{O}(k_N) \otimes \mathcal{L}_{p_N}.$$

Definition 23. Let (A, Φ) be a $\mathrm{SU}(N)$ hyperbolic monopole with $\mathbb{C}^\times$-equivariant holomorphic vector bundle $E \to \mathbb{P}^3$. Then the mass numbers $p_1, \ldots, p_N \in \mathbb{Z}$ (or $\frac{1}{2} + \mathbb{Z}$ for N even) and charge numbers $k_1, \ldots, k_N \in \mathbb{Z}$ of (A, Φ) are the numbers appearing in the $\mathbb{C}^\times$-equivariant splitting of E restricted to $\mathbb{P}^1_+$ (or $\mathbb{P}^1_-$).

I work exclusively with the case of *maximal symmetry breaking*, the condition that $p_1, \ldots, p_N$ are distinct. For convenience, I will order them $p_1 < \ldots < p_N$.

The rational map

The simplest handle on the moduli space of monopoles is the rational map. In parallel with euclidean monopoles, there is an isomorphism of moduli spaces.

THEOREM 24 (Atiyah [Ati84a]). *For any classical group G and any homomorphism $\alpha : S^1 \to G$ with centraliser $G(\alpha)$ in G, there is a natural isomorphism between*

(1) *the parameter space of framed S^1-invariant charge κ instantons of type α, and*
(2) *the parameter space of based holomorphic maps $f : \mathbb{P}^1 \to G/G(\alpha)$ of degree κ.*

Note that the degree of the rational map is κ which should be a polynomial in the mass and charge numbers. Generically, α is integral and $G(\alpha)$ is a maximal torus T of G so

$$G/G(\alpha) \cong G/T$$

is a flag manifold.

This theorem then says that for each hyperbolic monopole, we have a map $\mathbb{P}^1 \to G/T$. The data of α is the mass data in the form of winding numbers.

In the $\mathrm{SU}(2)$ case, $\kappa = 2kp$ and the only generic homomorphisms $\alpha : \mathrm{U}(1) \to T \subset \mathrm{SU}(2)$ are classified by a positive integer. Thus the $\mathrm{SU}(2)$ rational map is actually a degree κ rational function $\mathbb{P}^1 \to \mathbb{P}^1$. Since the rational maps are based, the numerator is of a lower degree than the denominator.

The physical interpretation for the rational map is the scattering of some test particle by the monopole field. From the distinguished point $\{\infty\}$ where the monopole is framed, the test particles travel along all the geodesics beginning at $\{\infty\}$. The rational map returns a fraction for each point on S^2 representing the other end of the geodesic. This fraction can be interpreted as the ratio of test particles transmitted through the monopole field to the test particles reflected by the field.

This interpretation is not so straightforward for higher rank G. One way to generalise the interpretation is to replace transmitted and reflected in the SU(2) case with whether or not the test particle has undergone a phase change. In the case of a rank r group G, there are r charge types and the i-th number records the relative number of test particles which have had i of their phases changed.

Atiyah has a procedure for a concrete realisation of the SU(2) rational map which Braam–Austin make use of to write the rational map in terms of solutions of the discrete Nahm equations. I will use this procedure for my treatment of rational maps and so I will state two lemmas from Atiyah's work here without proof:

Lemma 25 ([Ati84a]). *Let E be a rank r holomorphic vector bundle over $\mathbb{C}^2$ with a $\mathbb{C}^\times$-action covering the $\mathbb{C}^\times$-action on $\mathbb{C}^2$. Then E and $\mathbb{C}^2 \times E_0$ are isomorphic as $\mathbb{C}^\times$-representations (E_0 is the fibre of E over 0).*

Lemma 26 ([Ati84a]). *The previous lemma holds for $\mathbb{C}^1$ instead of $\mathbb{C}^2$ and the $\mathbb{C}^\times$-automorphisms of $E = \mathbb{C} \times E_0$, for a point $z \in \mathbb{C}$ are*

$$\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} z^{p_1 - p_2} & \cdots & a_{1r} z^{p_{r-1}-p_r} \\ & a_{22} & & \vdots \\ & & \ddots & a_{r-1,r} z^{p_{r-1}-p_r} \\ & & & a_{rr} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}, \quad a_{11}, \dots, a_{rr} \neq 0.$$

Returning to theorem 24, the proof proceeds by making the links

$$\begin{array}{c} \{\mathbb{P}^1 \to \Omega G\} \\ \big\downarrow \wr \\ \{\text{holomorphic } G^c \text{ bundles on } \mathbb{P}^1 \times \mathbb{P}^1\} \\ \big\downarrow \wr \\ \{\text{holomorphic } G^c \text{ bundles on } \mathbb{P}^2\}. \end{array}$$

Here G^c is the complexification of G. For example, $\mathrm{SU}(N)$ complexifies to $\mathrm{SL}(N, \mathbb{C})$. In the $\mathbb{P}^2$ ADHM construction of Donaldson, the $\mathrm{U}(N)$-action complexified to a $\mathrm{GL}(N, \mathbb{C})$-action.

Lemma 27 ([Ati84a]). *Let $(\mathbb{P}^2, x)$ be the complex projective plane with a choice of basepoint. Then there is an equivalence between*

(1) *based holomorphic maps $\mathbb{P}^1 \to \Omega G$, and*

(2) *framed holomorphic G^c-bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ trivial on $\mathbb{P}^1 \vee \mathbb{P}^1$.*

Lemma 28 ([Ati84a]). *There is an identification between the parameters spaces of framed holomorphic G^c bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ trivial over $\mathbb{P}^1 \vee \mathbb{P}^1$ and $\mathbb{P}^2$ trivial over $\mathbb{P}^1$.*

Consider a holomorphic vector bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ as a holomorphic family of holomorphic vector bundles over the second $\mathbb{P}^1$ parametrised by points of the first $\mathbb{P}^1$. These vector bundles can be constructed using maps $h : S^1 \to G^c$ as transition functions to glue together trivial vector bundles on the northern and southern hemispheres of $\mathbb{P}^1$. This is the data provided by the a holomorphic map $\mathbb{P}^1 \to \Omega G$. The blow up and blow downs preserve the trivialisation of a trivial vector bundle.

There is a birational equivalence between $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. Look at the union of the axes $\mathbb{P}^1 \vee \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Blow up at the intersection of the axes and then blow down the two now disjoint $\mathbb{P}^1$ axes.

This birational equivalence can be written explicitly.

$$([x : y], [z : w]) \mapsto [xw : zy : yw]\,, \quad [X : Y : Z] \mapsto ([X : Z], [Y : Z]).$$

Observe that the axes $\{y = 0\}$ and $\{w = 0\}$ of $\mathbb{P}^1 \times \mathbb{P}^1$ each map to a point $[1 : 0 : 0]$ and $[0 : 1 : 0]$ respectively on the $\mathbb{P}^1 \subset \mathbb{P}^2$. Conversely, that $\{Z = 0\} = \mathbb{P}^1 \subset \mathbb{P}^2$ maps to the intersection of the axes. Thus the holomorphic map $\mathbb{P}^1 \to \Omega G$ constructs a holomorphic vector bundle on $\mathbb{P}^2$ trivial over $\mathbb{P}^1_\infty$.

Proof of Theorem 24. The preceeding lemmas along with Donaldson's GIT arguments [Don84a] give a diffeomorphism between

(1) the moduli of charge κ, G framed instantons on $\mathbb{R}^4$ and
(2) the parameter space of based holomorphic maps $\mathbb{P}^1 \to \Omega G$ of charge κ.

It remains to show that the instantons are S^1-invariant of type α if and only if the maps reduce to maps $\mathbb{P}^1 \to G/G(\alpha)$ for $\alpha : S^1 \to G$.

Let $f \in \Omega G$ be a based loop, $f : S^1 \to G$ such that $f(1) = 1$. There is a natural action of S^1 on ΩG which rotates the loop. For $c \in S^1$ and $f \in \Omega G$,

$$(cf)(\theta) = f(c\theta)f(c)^{-1}.$$

If f is invariant under the action of S^1 then

$$(cf)(\theta) = f(\theta) = f(c\theta)f(c)^{-1}$$

so $f(c)f(\theta) = f(c\theta)$ and f is a homomorphism.

The S^1-action on the physical space S^4 lifts to an action on the principal G-bundle P. Over the point ∞ of the fixed S^2 (or any point on the fixed S^2), the

S^1-action lifts to a representation on the fibre P_∞. Equivalently, we get a homomorphism $\alpha : S^1 \to G$. This homomorphism determines the lifting since it is invariant under automorphisms of P which preserve the connection.

Since the image of $\mathbb{P}^1$ in ΩG lies in a conjugacy class and $\infty \in \mathbb{P}^1$ maps to α, $\mathbb{P}^1 \to \Omega G$ reduces to $\mathbb{P}^1 \to \Gamma^\alpha$, the moduli space of conjugacy classes of α. This is isomorphic to $G/G(\alpha)$ (translated by α to be precise) where $G(\alpha) = g \in G \mid g\alpha = \alpha g$ is the centraliser of the image of α as a subgroup. $\square$

Mini-twistor space and SU(2) spectral curves

Monopole solutions to the Yang-Mills-Higgs system of equations are integrable systems. This implies that they have a Lax form and a spectral curve. The discrete Nahm equations are the Lax form of hyperbolic monopoles. Here, I will say what the spectral curve is and what its significance is in term of inverse scattering.

Definition 29. The mini-twistor space Q of H^3 is the quotient

$$Q = \frac{\mathbb{P}^3 - (\mathbb{P}^1_+ \cup \mathbb{P}^1_-)}{\mathbb{C}^\times}.$$

Geometrically, Q is the moduli of oriented geodesics of H^3, considered as the ball model. The oriented geodesics of H^3 are uniquely determined by a starting point $\hat{z} = -1/\bar{z}$ and an end point w on an asymptotic sphere S^2_∞. Considering, S^2_∞ as $\mathbb{P}^1$, an oriented geodesic can be uniquely specified by a pair $(z, w) \in \mathbb{P}^1 \times \mathbb{P}^1$. Of course, the anti-diagonal pairs $(\hat{z}, z)$ specify the same point for the start and the end so they don't specify geodesics. Thus I will make the identification

$$Q \cong \mathbb{P}^1_+ \times \mathbb{P}^1_- - \bar{\Delta}$$

where $\bar{\Delta} = \{(\hat{z}, z) \in \mathbb{P}^1 \times \mathbb{P}^1\}$.

The anti-holomorphic linear involution

$$\sigma : Q \to Q$$

$$(z, w) \mapsto (\hat{w}, \hat{z})$$

is the the real structure induced by the real structure of $\mathbb{P}^3$ and is a reversal of the orientations of the geodesics in H^3.

On each oriented geodesic $\gamma \subset H^3$ with parameter t, a hyperbolic monopole (A, Φ) defines a scattering equation

$$(\nabla^A_t - i\Phi)s = 0.$$

Since (A, Φ) satisfy the Bogolmonyi equations, $[\nabla^A_{\bar{z}}, \nabla^A_t - i\Phi] = 0$ making solutions of the scattering holomorphic. Let s_+, s_- be the solutions which decay to the order

$\sim \exp(\text{const } t)$ in the $+\infty, -\infty$ directions respectively. Then the spectral curve S is defined as

$$S = \{\gamma \in Q \mid s_+|_\gamma = s_-|_\gamma\}.$$

The spectral curve is the image of a $\mathbb{P}^1$ in Q. It is uniquely determined by the monopole and furthermore, is enough to reconstruct the vector bundle and connection of the monopole. In Chapter 14, I will discuss spectral curves for $\mathrm{SU}(N)$ hyperbolic monopoles.

From a geometric point of view, there are two line sub-bundles L^+, L^- of the S^1-invariant holomorphic vector bundle $E \to \mathbb{P}^3$. The spectral curve S is the image in Q of the lines $\mathbb{P}^1$ over which L^+ coincides with L^-. More precisely,

$$0 \to L^+ \to E \to (L^-)^* \to 0$$

is exact. For more details, look at Section 2 as well as [Ati84b; Nor04; MNS03].

The S^1-invariant ADHM construction

Summarising chapters 6 and 7, there is an equivalence:

$$\left\{\text{Framed charge } \kappa \text{ instantons on } \mathbb{R}^4\right\} \simeq \left\{ \begin{array}{l} \text{Holomorphic vector bundles on } \mathbb{P}^3 \\ \text{trivial over } \mathbb{P}^1_\infty \text{ with } c_1 = 0, c_2 = \kappa \end{array} \right\}.$$

The holomorphic vector bundle is constructed as the cohomology at the middle position of the sequence of holomorphic vector bundles

$$\underline{H} \otimes \mathcal{O}(-1) \overset{A_X}{\to} \underline{K} \otimes \mathcal{O} \overset{B_X}{\to} \underline{L} \otimes \mathcal{O}(1) \tag{8.1}$$

where

- $H, L \simeq \mathbb{C}^\kappa$ are κ-dimensional vector spaces.
- $K \simeq \mathbb{C}^{2\kappa+n}$ is a $(2\kappa + n)$-dimensional vector space.
- $A_X = \begin{bmatrix} x + \alpha_1 z - \alpha_2^* w \\ y + \alpha_2 z + \alpha_1^* w \\ az + b^* w \end{bmatrix}$
- $B_X = \begin{bmatrix} -y - \alpha_2 z - \alpha_1^* w & x + \alpha_1 z - \alpha_2^* w & bz - a^* w \end{bmatrix}$ and
- α_1, α_2 are $\kappa \times \kappa$ complex matrices, a is an $n \times \kappa$ complex matrix and b is a $\kappa \times n$ complex matrix satisfying the ADHM equations

$$[\alpha_1, \alpha_2] + ba = 0$$

$$[\alpha_1, \alpha_1^*] + [\alpha_2, \alpha_2^*] + bb^* - a^* a = 0.$$

For a more detailed exposition of the monad ADHM construction, refer to chapters 6 and 7.

A tuple $(\alpha_1, \alpha_2, a, b)$ satisfying the ADHM equations is known as an ADHM datum. The action of the automorphism groups of the vector spaces H, K, L of the monad induces an action on the ADHM datum. For $g \in \mathrm{U}(\kappa)$ and $\lambda \in \mathrm{SU}(N)$,

$$\begin{aligned} \alpha_1 &\mapsto g\alpha_1 g^{-1} \\ \alpha_2 &\mapsto g\alpha_2 g^{-1} \\ a &\mapsto \lambda a g^{-1} \\ b &\mapsto g b \lambda^{-1}. \end{aligned}$$

The goal is to find additional conditions for an ADHM datum $(\alpha_1, \alpha_2, a, b)$ to produce an S^1-invariant instanton and thus be the datum of a hyperbolic magnetic monopole. To that end, there is a Proposition in the PhD thesis of Norbury [Nor94] which provides exactly this condition. By a theorem of Donaldson, holomorphic bundles on $\mathbb{P}^2$ trivial on $\mathbb{P}^1_\infty$ are in correpondence with our bundles on $\mathbb{P}^3$ and the monad ADHM construction is the same except that $w = 0$, $U(\kappa)$ is complexified to $\mathrm{GL}(\kappa, \mathbb{C})$ and S^1 is complexified to $\mathbb{C}^\times$. Then the proposition takes the following form:

Proposition 30 (Norbury)**.** *[Nor94] A holomorphic $\mathbb{C}^N$-vector bundle E on $\mathbb{P}^2$ trivial on a line is $\mathbb{C}^\times$-invariant if and only if there exists a homomorphism $P : \mathbb{C}^\times \to GL(\kappa, \mathbb{C})$ with $c \mapsto P_c$ such that the ADHM data $(\alpha_1, \alpha_2, a, b)$ of the monad associated to E satisfies*

(1) $\alpha_1 = P_c \alpha_1 P_c^{-1}$
(2) $\alpha_2 = c P_c \alpha_2 P_c^{-1}$
(3) $a = \lambda a P_c^{-1}$
(4) $b = c P_c b \lambda^{-1}$.

Proof. First note that a monad that is $\mathbb{C}^\times$-equivariant will always have a $\mathbb{C}^\times$-invariant holomorphic vector bundle E for its cohomology. For the converse, consider the $\mathbb{C}^\times$-action on $\mathbb{P}^2$ given by $c \cdot (x, y, z) = (x, cy, z)$ as in our setting. The linear dependence of the monad maps A,B, implies that an action is induced on the space of monads. For example,

$$A_X = \begin{bmatrix} x + z\alpha_1 \\ y + z\alpha_2 \\ za \end{bmatrix} \mapsto \begin{bmatrix} x + z\alpha_1 \\ cy + z\alpha_2 \\ za \end{bmatrix}.$$

Since the holomorphic vector bundle E in question is $\mathbb{C}^\times$-invariant, it is a fixed point of the $\mathbb{C}^\times$-action induced on vector bundles. Thus the $\mathbb{C}^\times$ orbit of a monad associated to E must all have E as their cohomology. By the uniqueness (up to a

GL_{HKL}-action) of the monad associated to a holomorphic vector bundle, the monad must be a fixed point of the $\mathbb{C}^\times$-action. The $\mathbb{C}^\times$-action acts only through GL_{HKL}.

There needs to be an element (σ, ρ, σ') of GL_{HKL} for which the maps A_X and B_X satisfy $\rho(c)A_{(x,y,z)} = A_{(x,cy,z)}\sigma(c)$ and $\sigma'(c)B_{(x,y,z)} = B_{(x,cy,z)}\rho(c)$. We can ask that the choice of basis made for K be preserved which means that $\rho(c)$ should split into blocks on the diagonal, $\mathrm{diag}\,(\rho_1, \rho_2, \rho_3) \in \mathrm{GL}(\kappa, \mathbb{C}) \times \mathrm{GL}(\kappa, \mathbb{C}) \times \mathrm{GL}(N, \mathbb{C})$.

The condition $A_{(x,cy,z)} = \rho(c)A_{(x,y,z)}\sigma^{-1}(c)$ in this basis is

$$\begin{bmatrix} x + z\alpha_1 \\ y + z\alpha_2 \\ za \end{bmatrix} \mapsto \begin{bmatrix} x + z\alpha_1 \\ cy + z\alpha_2 \\ za \end{bmatrix} = \mathrm{diag}\,(\rho_1, \rho_2, \rho_3) \begin{bmatrix} x + z\alpha_1 \\ y + z\alpha_2 \\ za \end{bmatrix} \sigma^{-1}.$$

Note that $x = \rho_1 x \sigma^{-1}$ implies that $\rho_1 = \sigma$ and $cy = \rho_2 y \sigma^{-1}$ implies that $\rho_2 = c\sigma$.

Likewise, $B_{(x,cy,z)} = \sigma'(c)B_{(x,y,z)}\rho^{-1}(c)$ in the chosen basis reads as

$$\begin{bmatrix} -cy - z\alpha_2 & x + z\alpha_1 & zb \end{bmatrix} = \sigma' \begin{bmatrix} -y - z\alpha_2 & x + z\alpha_1 & zb \end{bmatrix} \mathrm{diag}\left(\rho_1^{-1}, \rho_2^{-1}, \rho_3^{-1}\right).$$

From the first two blocks, $-cy = -\sigma' y \rho_1^{-1}$ implies that $c\rho_1 = \sigma'$ and $x = \sigma' x \rho_2^{-1}$ implies that $\rho_2 = \sigma'$.

Together, this means $\sigma = P_c = \rho_1$ and $\sigma' = cP_c = \rho_2$ for some $P_c \in \mathrm{GL}(\kappa, \mathbb{C})$. Recall that the last N basis elements of K provide the framing so ρ_3 needs to be the representation λ_c. Thus, the conditions (1)-(4) of the theorem are exactly the conditions for the $\mathbb{C}^\times$-equivariance of A_X and B_X. $\square$

Thus we see that in the case of a circle invariant instanton, the $\mathbb{C}^\times$-action on the monad's bundles is multiplication by

$$c \mapsto \mathrm{diag}\,(P_c, \mathrm{diag}\,(P_c, cP_c, \lambda_c)\,, cP_c) \in \mathrm{GL}(H) \times \mathrm{GL}(K) \times \mathrm{GL}(L).$$

The homomorphism P_c is a representation of $\mathbb{C}^\times$ so we can diagonalise it. This means that H, K and L can be decomposed into weight spaces for the $\mathbb{C}^\times$-action. The ADHM data α_1, α_2, a, b must then preserve these weight spaces.

Although the original proposition is for the $\mathbb{P}^2$ case, the statement and proof for $\mathbb{P}^3$ is the same but with S^1 instead of $\mathbb{C}^\times$ and $\mathrm{U}(\kappa)$ instead of $\mathrm{GL}(\kappa, \mathbb{C})$.

CHAPTER 9

Discrete Nahm equations and boundary values

Braam and Austin [BA90] were the first to study the hyperbolic analogue of the Nahm equations. Their main theorem is:

THEOREM 31 (Braam–Austin [BA90]). *Let k and $2p$ be integers. There is a 1-1 correspondence between*

(1) *the moduli space of charge k, mass p framed monopoles $(A, \Phi, \theta : E|_\infty \xrightarrow{\sim} \mathbb{C}^2)$, and*
(2) *solutions of the discrete Nahm equations consisting of matrices*

$$\beta_j \in \mathfrak{gl}(k, \mathbb{C}) \qquad j = -p + \frac{1}{2}, \ldots, p - \frac{1}{2}; \tag{9.1}$$

$$\gamma_j \in \mathfrak{gl}(k, \mathbb{C}) \qquad j = -p + 1, \ldots, p - 1; \tag{9.2}$$

$$v \in \mathbb{C}^k; \tag{9.3}$$

satisfying the discrete Nahm equations

$$\beta_{j-\frac{1}{2}}\gamma_j - \gamma_j\beta_{j+\frac{1}{2}} = 0 \qquad -p+1 \le j \le p-1; \tag{9.4}$$

$$[\beta_j, \beta_j^*] + \gamma_{j+\frac{1}{2}}\gamma_{\frac{1}{2}}^* - \gamma_{j-\frac{1}{2}}^*\gamma_{j-\frac{1}{2}} = 0 \qquad -p + \frac{3}{2} \le j \le p - \frac{3}{2}; \tag{9.5}$$

$$[\beta_{p-\frac{1}{2}}, \beta_{p-\frac{1}{2}}^*] + v^T\bar{v} - \gamma_{p-1}^*\gamma_{p-1} = 0; \tag{9.6}$$

$$\gamma_j = \gamma_{-j}^T \qquad p+1 \le j \le p-1; \tag{9.7}$$

$$\beta_j = \beta_{-j}^T \qquad -p + \frac{1}{2} \le j \le p - \frac{1}{2}; \tag{9.8}$$

and modulo the action of the map

$$g = (g_{-p+\frac{1}{2}}, \ldots, g_{p-\frac{1}{2}}) : (-p, p) \cap \left(\mathbb{Z} + \frac{1}{2}\right) \to U(k),$$

$$\beta_j \mapsto g_j\beta_j g_j^{-1}; \tag{9.9}$$

$$\gamma_j \mapsto g_{j-\frac{1}{2}}\gamma_j g_{j+\frac{1}{2}}^{-1}; \tag{9.10}$$

$$v \mapsto v g_{-p+\frac{1}{2}}^{-1}. \tag{9.11}$$

Note that I use a different labelling from Braam–Austin. Braam–Austin's theorem is actually for an Sp(1) hyperbolic monopole, not SU(2) which does not come with the final two discrete Nahm equations and has two vectors a, b instead of just v. However, the isomorphism of the groups imply that they are the same equations. Hence, Braam–Austin reveals a "hidden" symmetry of the SU(2) discrete Nahm equations.

The discrete Nahm equations can be interpreted as a discrete evolution equation along the interval-lattice. In analogy with the Nahm equations, this can be interpreted as gauge field theory on a lattice. Lattice field theories have gained attention of late in the study of quantum chromodynamics (QCD/ strong sector).

Here is how Braam and Austin proved their theorem. Hitchin describes the vector spaces $\underline{K}$ and $\underline{L}$ in the ADHM construction as the solutions of Dirac operators,

$$\underline{K} = \ker \left\{ D_A^* : \Gamma(S^4, E \otimes S_- \otimes S_-) \to \Gamma(S^4, E \otimes S_+ \otimes S_-) \right\} \tag{9.12}$$

$$\underline{L} = \ker \left\{ D_A^* : \Gamma(S^4, E \otimes S_-) \to \Gamma(S^4, E \otimes S_+) \right\}. \tag{9.13}$$

Here, S_+ and S_- are the spin bundles on S^4. The fibres of $S_\pm$ are isomorphic to $\mathbb{C}^2$ and are representations of $\mathrm{Spin}(3) \simeq \mathrm{SU}(2)$. D_A^* is the adjoint Dirac operator $S_- \to S_+$ with coefficients in $E \otimes S_-$ for $\underline{K}$ and E for L. Since $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$, both E and S_- carry quaternionic structures so $\underline{K}$ is quaternionic and $\underline{L}$ is real.

Braam–Austin assume that $p \in \mathbb{Z} + \frac{1}{2}$ and use the equivariant Atiyah-Singer index theorem to compute the S^1 characters of $\underline{K}$ and $\underline{L}$ as representations of S^1. Since S^1 is abelian, this is a sequence of integers. The theorem actually computes the *difference* between the kernel and cokernel of the Dirac operator but the cokernel vanishes due to the positive scalar curvature of S^4. They found that,

$$\underline{K} = \mathbb{C}^{k+1}_{-p} \oplus \mathbb{C}^{2k}_{-p+1} \oplus \ldots \oplus \mathbb{C}^{2k}_{p-1} \oplus \mathbb{C}^{k+1}_{p},$$

and

$$\underline{L} = \left(\mathbb{R}^{2k}_{p-\frac{1}{2}} \oplus \mathbb{R}^{2k}_{p-\frac{3}{2}} \oplus \ldots \oplus \mathbb{R}^{k}_{0} \right) \otimes_{\mathbb{R}} \mathbb{C}.$$

The S^1-characters tell us that there is a basis in which the ADHM matrices are sparse with block entries on the diagonal and off-diagonal. The skew-form on $\underline{K}$ is defined component-wise on $K_{-j} \oplus K_j$,

$$\left(\begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} v' \\ w' \end{pmatrix} \right) = -v^T w' + w^T v'.$$

The isotropy condition of the ADHM construction with respect to this skew form then becomes the discrete Nahm equations.

In my generalisation of the discrete Nahm equations to groups other than SU(2), I use a different method to calculate the S^1-characters of $\underline{K}$ and $\underline{L}$. I will use many of the same arguments used by Braam and Austin in the rest of their paper so I will only briefly outline them here and provide more detail when I discuss the generalised case.

Analogous to work by Donaldson [Don84b], Braam–Austin then reduce the solutions of the discrete Nahm equations to a symmetric matrix β and a vector v. This is done by passing to the Donaldson form of the ADHM construction via GIT [Don84a]. The real discrete Nahm (9.5) is now a moment map condition and the gauge freedom is $\mathrm{GL}(k, \mathbb{C})$ instead of $\mathrm{U}(k)$. The extra freedom of $\mathrm{GL}(k, \mathbb{C})$ can be used to row reduce the γ matrices into the identity matrices. I will use the lemmas which are used for this purpose later in my work on the rational map.

Braam–Austin show that the rational map of an $\mathrm{SU}(2)$ hyperbolic monopole is given by

$$r = -\lambda^{-p} v(z - \beta)^{-1} v^T.$$

Braam and Austin also point out that the end-points of the discrete Nahm equations on the interval-lattice define line bundles on $\mathbb{P}^k$ by defining a linear map between complex spaces, for example,

$$\begin{pmatrix} \beta_{-p+\frac{1}{2}} - z \\ v \end{pmatrix} : \mathbb{C}^k_{-p+\frac{1}{2}} \to \mathbb{C}^{k+1}_{-p}$$

This is a (small) monad on S^2 but it is also a map $S^2 \to \mathbb{P}^k$ where $z \in S^2$. The cohomology (which reduces to the quotient) of this map defines a line subspace of $\mathbb{C}^{k+1}$ for each $z \in S^2$. Since this map varies holomorphically with z, we have a sub-line bundle on $\mathbb{P}^k$ which we pullback to a line bundle on S^2. The canonical derivative of $\mathbb{C}^{k+1}$ passes to a non-trivial connection A_∞ in the quotient which can be pulled back to S^2.

Conversely, given a connection for a line bundle on S^2_∞ induced by the monopole connection A, there is a small monad unique up to gauge transforms defined by A_∞. The small monad can be used as boundary conditions for the discrete Nahm equations. Thus $\mathrm{SU}(2)$ hyperbolic monopoles are holographic, that is, the boundary connection induced by the monopole uniquely determines the monopole.

Theorem 32 (Braam–Austin [BA90]).

(1) *The connection A_∞ on the factor L^* of $E|_{S^2} = L \oplus L^*$ determines the monopole (A, Φ) up to gauge transformations.*
(2) *The connection A_∞ is the pullback of the $U(k+1)$-invariant connection on $\mathcal{O}(1)$ along a holomorphic map $S^2 \to \mathbb{P}^k$.*
(3) *The map $A \mapsto A_\infty$ is an immersion of the monopole moduli space in the space of $U(1)$ connections on S^2.*

CHAPTER 10

Symmetric examples and ansätze

There has been a history of highly symmetric examples of hyperbolic monopole solutions, notably Murray-Norbury-Singer[MNS03] and Norbury-Romao[NR07]. I will discuss the case of Bolognesi-Cockburn-Sutcliffe's [BCS15] clever embedding of the JNR ansatz into the ADHM construction which leads to a family of solutions suggestive of low energy scattering of hyperbolic monopoles.

Bolognesi et al. [BCS15] show how a subset of the JNR data (and as a special case, 't Hooft ansatz) can be embedded into the ADHM matrices as circle-invariant solutions (see Part 2 for the full JNR ansatz).

Definition 33. Instanton charge κ *circle-invariant JNR data* $\{\gamma_i, \lambda_i^2\}_{i\in\{0,\dots,\kappa\}}$ consists of $\kappa+1$ complex constants $\gamma_i \in \mathbb{C}$ and real weights $\lambda_i^2 \in \mathbb{R}_{>0}$. *Circle-invariant 't Hooft data* $\{\gamma_i, \lambda_i^2\}_{i\in\{1,\dots,\kappa\}}$ is JNR data with $\gamma_0 = \infty$ and $\lambda_0^2 = \infty$.

The circle-invariant JNR ansatz defines a harmonic function

$$\psi(x_1 + ix_2, r) = \sum_{i=0}^{\kappa} \frac{\lambda_i^2}{|x_1 + ix_2 - \gamma_i| + r^2}$$

with poles $\gamma_0, \dots, \gamma_\kappa$ on $r = 0$, the conformal boundary of H^3. A formula for the connection is

$$A_\mu = \frac{i}{2}\varepsilon_{\mu\nu\rho}\sigma_\rho\partial_{\mu\nu}\log\psi$$

with $\Phi = A_0$.

Note that the moduli space of hyperbolic monopoles arising from JNR data is of dimension $3\kappa+2$ (the scaling of ψ by a constant leaves the monopole unchanged, reducing the number of parameters by one). The full hyperbolic monopole moduli space is of dimension $4\kappa-1$. Hence the JNR moduli space is only the full hyperbolic monopole space for charge $\kappa \leq 3$.

't Hooft data $\{\gamma_1, \dots, \gamma_\kappa, \lambda_1^2, \dots, \lambda_\kappa^2\}$ can be written as an ADHM matrix

$$\begin{bmatrix} \lambda_1 & \cdots & \lambda_\kappa \\ \gamma_1 & & \\ & \ddots & \\ & & \gamma_\kappa \end{bmatrix}$$

which satisfies the discrete Nahm equations of charge k, mass p such that $2kp = \kappa$.

The general JNR data produces an ADHM matrix

$$S\Gamma V$$

with

$$\Gamma = \begin{bmatrix} \lambda_1\gamma_0 & \cdots & \lambda_\kappa\gamma_0 \\ \lambda_0\gamma_1 & & \\ & \ddots & \\ & & \lambda_0\gamma_\kappa \end{bmatrix}$$

and matrices $S \in \mathrm{O}(\kappa+1)$, $V \in \mathrm{GL}(\kappa, \mathbb{R})$ satisfying

$$S \begin{bmatrix} \lambda_1 & \cdots & \lambda_\kappa \\ \lambda_0 & & \\ & \ddots & \\ & & \lambda_0 \end{bmatrix} V = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}.$$

The matrices S and V can be found from iterating the rules:

$$V_{ij} = \begin{cases} 0 & \text{if } i > j \\ p_i/(\lambda_0 p_{i-1}) & \text{if } i = j \\ -\lambda_i\lambda_j p_j p_{j-1}/\lambda_0 & \text{if } i < j \end{cases} \tag{10.1}$$

$$S_{i1} = \lambda_0\lambda_{i-1}p_{i-1}p_{i-2} \qquad \text{for } i = 1, \ldots, \kappa+1 \tag{10.2}$$

$$S_{1j} = -\lambda_{j-1}p_N \qquad \text{for } j = 2, \ldots, \kappa+1 \tag{10.3}$$

$$S_{ij} = \lambda_0 V_{j-1,i-1} \qquad \text{for } i, j = 2, \ldots, \kappa+1 \tag{10.4}$$

Here, $p_i = (\sum_{j=0}^{i} \lambda_j^2)^{-1/2}$ for $i = 0, \ldots, \kappa$, $p_{-1} = p_\kappa$ and $\lambda_{-1} = \lambda_0$.

For the sake of convenience, here are the matrices for $\kappa = 1, 2, 3$.

$\kappa = 1$:

$$V = (\lambda_0^2 + \lambda_1^2)^{-1/2}, \quad S = V \begin{pmatrix} \lambda_0 & -\lambda_1 \\ \lambda_1 & \lambda_0 \end{pmatrix}.$$

$\kappa = 2$:

$$V = \begin{pmatrix} p_1 & -\lambda_1\lambda_2 p_0 p_1 p_2 \\ 0 & \lambda_0^{-1}p_1^{-1}p_2 \end{pmatrix} \quad S = \begin{pmatrix} \lambda_0 p_2 & -\lambda_1 p_2 & -\lambda_2 p_2 \\ \lambda_1 p_1 & \lambda_0 p_1 & 0 \\ \lambda_0\lambda_2 p_1 p_2 & -\lambda_1\lambda_2 p_1 p_2 & p_1^{-1}p_2 \end{pmatrix}.$$

$\kappa = 3:$

$$V = \begin{pmatrix} p_1 & -\lambda_0^{-1}\lambda_1\lambda_2 p_1 p_2 & -\lambda_0^{-1}\lambda_1\lambda_3 p_2 p_3 \\ 0 & p_2\lambda_0^{-1}p_1^{-1} & -\lambda_0^{-1}\lambda_2\lambda_3 p_2 p_3 \\ 0 & 0 & p_3\lambda_0^{-1}p_2^{-1} \end{pmatrix}$$

$$S = \begin{pmatrix} \lambda_0 p_3 & -\lambda_1 p_3 & -\lambda_2 p_3 & -\lambda_3 p_3 \\ \lambda_1 p_1 & \lambda_0 p_1 & 0 & 0 \\ \lambda_0\lambda_2 p_1 p_2 & -\lambda_1\lambda_2 p_1 p_2 & p_2 p_1^{-1} & 0 \\ \lambda_0\lambda_3 p_2 p_3 & -\lambda_1\lambda_3 p_2 p_3 & -\lambda_2\lambda_3 p_2 p_3 & p_3 p_2^{-1} \end{pmatrix}.$$

In terms of the JNR data, the spectral curve is given by

$$\sum_{j=0}^{\kappa} \lambda_j^2 \prod_{\substack{k=0,\\ k\neq j}}^{\kappa} (\zeta - \gamma_k)(1 + \eta\bar{\gamma}_k) = 0. \tag{10.5}$$

By evaluating the spectral curve at $(\eta, \zeta) = (0, z)$, we get the denominator of the rational map. Using the additional requirement that the points be invariant under the circle-symmetry, the full rational map is

$$R = \frac{\sum\limits_{\substack{i=0\\ j=i+1}}^{\kappa} \lambda_i^2\lambda_j^2(\gamma_i - \gamma_j)^2 \prod\limits_{\substack{k=0\\ k\neq i,j}}^{\kappa} (z - \gamma_k)}{\sum\limits_{j=0}^{\kappa} \lambda_j^2 \prod\limits_{\substack{k=0\\ k\neq j}}^{\kappa} (\zeta - \gamma_k)}.$$

Choosing $\gamma_j = \omega^j = \exp(2\pi i/(\kappa+1))^j$ gives the axially symmetric spectral curve

$$\sum_{i=0}^{\kappa} (-1)^i \eta^i \zeta^{\kappa - i} = 0,$$

and rational map

$$R = \frac{1}{z^\kappa}.$$

The position of the monopole can be moved to $(0, 0, \xi)$ by shifting $\omega \mapsto \frac{1+\xi}{1-\xi}\omega$. The discrete Nahm equations for axially-symmetric hyperbolic monopoles have an interpretation as discrete Hitchin equations [War15].

Bolognesi-Cockburn-Sutcliffe [BCS15] discuss a host of symmetric examples. I will mention the charge 2 monopole with D_2 symmetry. For $a \in (-1, 1)$, choose

$$\gamma_0 = 1, \gamma_1 = \frac{a-1}{2} + \frac{1}{2}\sqrt{3 + 2a - a^2}, \gamma_2 = \gamma_1^{-1}, \lambda_1^2 = \lambda_2^2 = 1, \lambda_0^2 = \frac{1+a}{1-a}.$$

In terms of the harmonic function ψ, we can find an expression for the energy density of the monopole,

$$\nabla^2|\Phi|^2 = \nabla^2 \frac{r^2}{4\psi^2}\left[(\frac{\partial\psi}{\partial x_1})^2 + (\frac{\psi}{r} + \frac{\partial\psi}{\partial r})^2 + (\frac{\partial\psi}{\partial x_1})^2\right].$$

The energy density isosurfaces of the charge 2 D_2-symmetric monopole resembles two particles colliding in the $x-y$ plane, combining into a torus in the $x-y$ plane and then separating along the z-axis. This is suggestive of a quantum mechanical scattering.

CHAPTER 11

A calculation by localisation

In this chapter, I will detail the calculations that I will need to prove proposition 34. The reader may wish to read the following chapter before returning to this one.

Magnetic monopoles in hyperbolic space are defined to be circle-invariant instantons [Ati84a; Ati84b]. Instantons are equivalent to instanton bundles, holomorphic vector bundles on the twistor space of $\mathbb{R}^4$, $\mathbb{P}^3$ satisfying some conditions [Ati79]. In the ADHM construction (See Part II of this theses for details), these instanton bundles are constructed as the homology of a sequence of three simpler holomorphic vector bundles on $\mathbb{P}^3$, called a monad [Don84a]. If the monad is circle-equivariant then the instanton bundle will be circle-equivariant. Thus the instanton produced will be circle-invariant and hence a hyperbolic monopole.

Over the fixed-point set of the circle action on $\mathbb{P}^3$, the circle action acts trivially on the point in the base of the fibration but it induces a representation of the circle group on the fibres of both the instanton bundle and the vector bundles of the monad. Since this fixed-point set is the disjoint union of two-spheres $\mathbb{P}^1_+ \sqcup \mathbb{P}^1_-$, all bundles on the fixed-point set split into a sum of line bundles. The condition that the monad be circle-equivariant will be shown to be the same as the condition that over the fixed-point set, the vector bundles of the monad decompose into a sum of weight spaces of the induced circle-action and that the maps between them preserve this decomposition.

The requirement that the maps of the monad preserve the weight-decomposition of the vector bundles of the monad over the fixed-point set, reduces the ADHM equations into the $(N-1)$-interval discrete Nahm equations. A solution of the ADHM equations over the fixed-point set determines the solution over all of the twistor space $\mathbb{P}^3$.

The proof of the weight decomposition of the monad vector bundles proceeds by computing the equivariant chern characters of monad vector bundles and of the instanton bundle localised to the fixed-point set $\mathbb{P}^1_+$. The circle action that is induced on the monad by the action on the base constrain the equivariant chern characters of the monad enough that they can be determined by comparison with the equivariant chern characters of the instanton bundle. The chern characters of the instanton bundle is determined by the charge and mass data of the hyperbolic monopole.

The upshot is, the following calculations will be carried out:

(1) The equvariant chern character of the circle-equivariant instanton bundle is equal to the alternating sum of the equivariant chern characters of the monad.
(2) The calculation of the equivariant chern character of the circle-invariant instanton bundle.
(3) The calculation of the equivariant chern characters of the monad vector bundles.
(4) A comparison of the equivariant chern characters of the circle-invariant instanton bundle and monad vector bundles.

Calculation 4 will prove the following proposition.

Proposition 34. *Let there be a $\mathbb{C}^\times$-action on $\mathbb{P}^3$,*

$$[x : y : z : w] \mapsto [c^{-1/2}x : c^{1/2}y : c^{-1/2}z : c^{1/2}w].$$

Let E be a $\mathbb{C}^\times$-equivariant holomorphic vector bundle on $\mathbb{P}^3$ corresponding to a monopole with mass numbers $p_1, \ldots, p_N \in \mathbb{Z}$ (or $\frac{1}{2}+\mathbb{Z}$ if N is even) ordered $p_1 < \ldots < p_N$, and charge numbers $k_1, \ldots, k_N \in \mathbb{Z}$, with $\sum_{i=1}^N p_i = 0$ and $\sum_{i=1}^N k_i = 0$.

Then the decomposition of the monad for E,

$$\underline{H} \overset{A_X}{\to} \underline{K} \overset{B_X}{\to} \underline{L}$$

restricted to $\mathbb{P}^1_+$, into weight p components $\mathbb{C}_p$ with respect to the $\mathbb{C}^\times$-action is

$$H = \mathbb{C}_{p_1}^{k_1} \oplus \ldots \oplus \mathbb{C}_{p_2-1}^{k_1} \oplus \mathbb{C}_{p_2}^{k_1+k_2} \oplus \mathbb{C}_{p_2+1}^{k_1+k_2} \oplus \ldots \oplus \mathbb{C}_{p_N-1}^{-k_N},$$

$$K = \mathbb{C}_{p_1}^{k_1+1} \oplus \mathbb{C}_{p_1+1}^{2k_1} \oplus \ldots \oplus \mathbb{C}_{p_2-1}^{2k_1} \oplus \mathbb{C}_{p_2}^{2(k_1+k_2)+1} \oplus \mathbb{C}_{p_2+1}^{2(k_1+k_2)} \oplus \ldots \oplus \mathbb{C}_{p_N-1}^{2(k_1+\ldots+k_{N-1})} \oplus \mathbb{C}_{p_N}^{-k_N+1},$$

$$L = \mathbb{C}_{p_1+1}^{k_1} \oplus \ldots \oplus \mathbb{C}_{p_2}^{k_1} \oplus \mathbb{C}_{p_2+1}^{k_1+k_2} \oplus \mathbb{C}_{p_2+2}^{k_1+k_2} \oplus \ldots \oplus \mathbb{C}_{p_N}^{-k_N}.$$

At the end of the chapter, I will also prove that a formula for the instanton charge in terms of the mass and charge data of the hyperbolic monopole arising from the circle-equivariant instanton bundle and the newly-computed monad weight decomposition are in agreement.

The equivariant chern character of instanton bundles and monads

The starting point of the calculation is the following display (which can be found in Ch.2 §3 [OSS80]) for a monad

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \underline{H} \arrow[r] \arrow[d, equal] & \ker B_X \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{H} \arrow[r, "A_X"] & \underline{K} \arrow[r] \arrow[d, "B_X"'] & \operatorname{coker} A_X \arrow[r] \arrow[d] & 0 \\
 & & \underline{L} \arrow[r, equal] \arrow[d] & \underline{L} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{11.1}$$

where the rows and columns are all exact.

The equivariant Chern character of $\mathbb{P}^1$, here denoted ch, is a map $K_{\mathbb{C}^\times}(\mathbb{P}^1) \to H^*_{\mathbb{C}^\times}(\mathbb{P}^1)$, from the equivariant K-theory to the equivariant cohomology of a space $\mathbb{P}^1$. By the additivity of the Chern character, the right vertical and bottom horizontal exact sequences of the display gives us the following

$$\mathrm{ch}(\text{coker } A_X) = \mathrm{ch}(E) + \mathrm{ch}(\underline{L})$$

$$\mathrm{ch}(\underline{K}) = \mathrm{ch}(\underline{H}) + \mathrm{ch}(\text{coker } A_X).$$

Putting them together yields

$$\mathrm{ch}(E) = \mathrm{ch}(\underline{K}) - \mathrm{ch}(\underline{H}) - \mathrm{ch}(\underline{L}). \tag{11.2}$$

The upshot is that $\mathbb{C}^\times$-invariance is a strong enough condition that, if we know the equivariant Chern character of a $\mathbb{C}^\times$-invariant holomorphic bundle E associated to a hyperbolic monopole, we can compute the equivariant Chern character of the monad vector spaces H, K and L over $\mathbb{P}^1_+$ of the $\mathbb{C}^\times$-equivariant monad which produces E, and hence their $\mathbb{C}^\times$ weight decomposition. Concretely, this data is encoded in the exponents of the matrix P_c and will induce a decomposition of the ADHM matrices.

Since the bundle E is trivial over $\mathbb{P}^1_+$, we have a representation of $\mathbb{C}^\times$ on the fibres which allows us to compute the equivariant Chern character of $E|_{\mathbb{P}^1_+}$. Over any $\mathbb{P}^1$, all holomorphic vector bundles split into line bundles by the Birkoff-Grothendieck splitting principle [OSS80]. The strategy is to localise to $\mathbb{P}^1_+$, split all the relevant bundles and compute the exponents of P_c. Since the ADHM matrices are constant, any conditions on them over any line hold globally.

The equivariant chern character of circle-equivariant instanton bundles

For SU(2), Atiyah showed that over $\mathbb{P}^1_+$, $E = \mathcal{O}(k) \otimes \mathcal{L}^{-p} \oplus \mathcal{O}(-k) \otimes \mathcal{L}^p$ where $\mathcal{L}^p$ is the trivial line bundle with the c^p representation of $\mathbb{C}^\times$ [Ati84b]. This follows from a result of equivariant K-theory that over a fixed point set M,

$$K_{\mathbb{C}^\times}(M) = K(M) \otimes R(\mathbb{C}^\times)$$

where $R(\mathbb{C}^\times) = \mathbb{Z}[u]$ is the ring of characters of the representations of $\mathbb{C}^\times$ [Seg68; AS68a].

Over $\mathbb{P}^1_+$, the $\mathbb{C}^\times$-action on E is a representation and can be diagonalised

$$c \mapsto \lambda(c) = \operatorname{diag}\begin{pmatrix} c^{p_1} & \dots & c^{p_N} \end{pmatrix},$$

since the irreducible algebraic representations of $\mathbb{C}^\times$ are 1-dimensional with weights $p_1, \dots, p_N \in \mathbb{Z}$ and this splits E into a sum of line bundles. Since the action is algebraic, the line bundles are algebraic or equivalently, holomorphic. By Birkhoff–Grothendieck [OSS80], this splitting is unique up to permutation of the summand thus for some $k_1, \dots, k_N \in \mathbb{Z}$,

$$E|_{\mathbb{P}^1_+} = \mathcal{O}(k_1) \otimes \mathcal{L}^{p_1} \oplus \dots \oplus \mathcal{O}(k_{N-1}) \otimes \mathcal{L}^{p_{N-1}} \oplus \mathcal{O}(k_N) \otimes \mathcal{L}^{p_N}$$

where $\sum_{i=1}^N p_i = 0$ and $\sum_{i=1}^N k_i = 0$.

Using results in [Ati84b; AB84], we calculate the equivariant first Chern class and the total Chern class of E. The equivariant first Chern class of a line bundle of the form $\mathcal{O}(k) \otimes \mathcal{L}^p$ is

$$c_1^{eq} = kx + pu$$

where x is the second degree generator of the usual $H^2(\mathbb{P}^1)$ and u is the second degree generator of $R(\mathbb{C}^\times)$.

This is enough to calculate the equivariant Chern character

$$\operatorname{ch}(E) = e^{k_1 x + p_1 u} + \dots + e^{k_N x + p_N u}$$

and since $H^*(\mathbb{P}^1) = \mathbb{Z}[x]/\langle x^2 \rangle$, the following series expansion with respect to x is exact

$$\begin{aligned} \operatorname{ch}(E) = {} & e^{p_1 u} + \dots + e^{p_N u} \\ & + x\left(k_1 e^{p_1 u} + \dots + k_N e^{p_N u}\right). \end{aligned} \tag{11.3}$$

The equivariant total Chern class of E is given by

$$\prod_{i=1}^N (1 + k_i x + p_i u) \qquad \operatorname{mod} x^2.$$

The localisation formula from Atiyah and Bott (p.5-9) [AB84] tells us that the second Chern class c_2 (remember that $c_1(E) = 0$) can be found by looking at the coefficient of x and dividing it by u. This is the *positive* integer

$$c_2(E) = -\left[2\sum_{i=1}^{N-1} k_i p_i + \sum_{1\le i<j\le N-1} (k_i p_j + k_j p_i)\right] \tag{11.4}$$

which reduces to $2kp$ as expected for the SU(2) case $p_1 = -p$ which is known.

The main calculation

Since the x-terms in the Chern character of E only have terms up to $e^{p_1 u}$ and $e^{p_N u}$, the lowest weight of P_c and highest weight of cP_c are c^{p_1} and c^{p_N} respectively. This is required because for the x-terms, the lowest weight term of $\underline{H}$ and the highest weight term of $\underline{L}$ do not cancel with any other terms on the right side of (11.2) and therefore must exactly match x-terms of $\mathrm{ch}(E)$.

The homomorphism P_c has the form

$$\mathrm{diag}\ \Big(\underbrace{c^{p_1} \quad \dots \quad c^{p_1}}_{\longleftarrow\ \chi_{p_1}\ \longrightarrow} \quad \underbrace{c^{p_1+1} \quad \dots \quad c^{p_1+1}}_{\longleftarrow\ \chi_{p_1+1}\ \longrightarrow} \quad \dots \quad \underbrace{c^{p_N-1} \quad \dots \quad c^{p_N-1}}_{\longleftarrow\ \chi_{p_N-1}\ \longrightarrow} \Big)$$

and the $p_N - p_1$ numbers $\chi_{p_1}, \dots, \chi_{p_N-1}$ are what we need to calculate.

The vector bundles $\underline{H}$,$\underline{K}$ and $\underline{L}$ decompose as follows:

$$\underline{H} = \bigoplus_{i=p_1}^{p_N-1} \left(\mathcal{O}(-1)\otimes\mathcal{L}^i\right)^{\oplus\chi_i}$$

$$\underline{K} = \bigoplus_{i=p_1}^{p_N-1} \left(\mathcal{L}^i\right)^{\oplus\chi_i} \oplus \bigoplus_{i=p_1}^{p_N-1} \left(\mathcal{L}^{i+1}\right)^{\oplus\chi_i} \oplus \left(\mathcal{L}^{p_1}\oplus\dots\oplus\mathcal{L}^{p_N}\right)$$

$$\underline{L} = \bigoplus_{i=p_1}^{p_N-1} \left(\mathcal{O}(1)\otimes\mathcal{L}^{i+1}\right)^{\oplus\chi_i}.$$

Note that $\underline{K}$ has been arranged into the parts on which the $\mathbb{C}^\times$-action is via P_c, cP_c and λ respectively.

The corresponding equivariant Chern characters are:

$$\begin{aligned}\mathrm{ch}(\underline{H}) &= \sum_{i=p_1}^{p_N-1} \chi_i e^{-x+iu} \\ &= \sum_{i=p_1}^{p_N-1} \chi_i e^{iu} - x\left(\sum_{i=p_1}^{p_N-1} \chi_i e^{iu}\right)\end{aligned}$$

$$
\begin{aligned}
\mathrm{ch}(\underline{K}) &= \sum_{i=p_1}^{p_N-1} \chi_i e^{iu} + \sum_{i=p_1}^{p_N-1} \chi_i e^{(i+1)u} + (e^{p_1 u} + \ldots + e^{p_N u}) \\
&= \chi_{p_1} e^{p_1 u} + \sum_{i=p_1+1}^{p_N-1} (\chi_{i-1} + \chi_i) e^{iu} + \chi_{p_{N-1}} e^{p_N u} + (e^{p_1 u} + \ldots + e^{p_N u})
\end{aligned}
\tag{11.5}
$$

$$
\begin{aligned}
\mathrm{ch}(\underline{L}) &= \sum_{i=p_1}^{p_N-1} \chi_i e^{x+(i+1)u} \\
&= \sum_{i=p_1}^{p_N-1} \chi_i e^{(i+1)u} + x \left(\sum_{i=p_1}^{p_N-1} \chi_i e^{(i+1)u} \right).
\end{aligned}
$$

We proceed by comparing coefficients of xe^{ju} in $\mathrm{ch}(E)$ and $\mathrm{ch}\underline{K} - \mathrm{ch}\underline{H} - \mathrm{ch}\underline{L}$. The x-terms are enough to determine the unknowns $\chi_{p_1}, \ldots, \chi_{p_N-1}$.

$$xe^{p_1 u} : k_1 = \chi_{p_1}$$

$$xe^{p_N u} : k_N = -\chi_{p_{N-1}}$$

$$xe^{p_i u}, \text{ for } 1 < i \leq N-1 : k_i = \chi_{p_i} - \chi_{p_i - 1}$$

and all the other x-terms require that $\chi_j = \chi_{j-1}$ when $j \neq p_i$ for any of the $1 \leq i \leq N$.

The interesting 1-terms (constant terms, to be clear) are the ones of the form $e^{p_i u}$. The rightmost terms of (11.5) supply the 1-terms of $\mathrm{ch}(E)$. We expected to see this because in the monad, $\underline{K}$ carries the trivialisation/framing data of E in its last N basis elements. The rest of the 1-terms $\mathrm{ch}(\underline{K})$ cancel with the 1-terms of $\mathrm{ch}(\underline{H})$ and $\mathrm{ch}(\underline{L})$ to show that they are consistent with the constraints set by the x-terms.

In the case of SU(3), the weights run from p_1 to p_2 with coefficients $\chi_i = k_1$ and then from p_2 to $-p_1 - p_2$ with multiplicities $\chi_i = k_1 + k_2$. At p_2, the multiplicity jumps from $\chi_{p_2-1} = k_1$ to $\chi_{p_2} = k_1 + k_2$. This is illustrated by the following diagram (which should be viewed as an interval - the domain of an evolution equation)

$$\underset{p_1}{\cdot} \quad \frac{p_2 - p_1}{k_1} \quad \underset{p_2}{\cdot} \quad \frac{-2p_2 - p_1}{k_1 + k_2} \quad \underset{p_3}{\cdot}$$

where the quantity above the line is the number of distinct weights with corresponding coefficient being the quantity under the line. The dimensions of P_c (as a square matrix) are given by

$$(p_2 - p_1)\, k_1 - (2p_2 + p_1)\, (k_1 + k_2) = -(2p_1 k_1 + 2p_2 k_2 + p_1 k_2 + p_2 k_1)$$

which is exactly the formula 11.4 for the second Chern class $c_2(E)$ from the previous subsection.

In general, we have

$$\underset{p_1}{\bullet}\xrightarrow[k_1]{p_2-p_1}\underset{p_2}{\bullet}\quad\cdots\quad\underset{p_{N-2}}{\bullet}\xrightarrow[k_1+\ldots+k_{N-2}]{p_{N-1}-p_{N-2}}\underset{p_{N-1}}{\bullet}\xrightarrow[k_1+\ldots+k_{N-1}]{p_N-p_{N-1}}\underset{p_N}{\bullet}$$

and this gives us the dimensions of P_c

$$\kappa=\sum_{i=1}^{N-1}\left[(p_{i+1}-p_i)\sum_{j=1}^{i}k_j\right]. \tag{11.6}$$

In [Nor94], Norbury proved the SU(2) case of the following proposition by a different method.

Proposition 35. *Let $N\in\mathbb{N}_{\geq 2}$. Then the formulae for κ in the dimensions $\kappa\times\kappa$ of P_c and the second Chern number $\kappa=c_2(E)$ compute the same quantity.*

Proof. We proceed by induction. In the case of $N=n$, denote κ by κ_n and $c_2(E)$ by χ_n. When $N=2$, $\kappa_2=-2kp=\chi_2$. The $N=3$ case was dealt with in the previous page.

In the case of $N=n$, the expression for κ in (11.6) can be rewritten using $\sum_{i=1}^{n}p_i=0$ to eliminate p_n. The result is

$$\kappa_n=\sum_{i=1}^{n-2}(p_{i+1}-p_i)\sum_{j=1}^{i}k_j\ -\ \left(p_{n-1}+\sum_{i=1}^{n-1}p_i\right)\sum_{j=1}^{n-1}k_j.$$

For the inductive step, we assume that the proposition holds for the case of $N=n-1$, that is, $\kappa_{n-1}=\chi_{n-1}$. Then if the differences $\kappa_n-\kappa_{n-1}$ and $\chi_n-\chi_{n-1}$ are equal, the proof is done.

$$\begin{aligned}
&\kappa_n-\kappa_{n-1}\\
&=\sum_{i=1}^{n-2}(p_{i+1}-p_i)\sum_{j=1}^{i}k_j\ -\ \left(p_{n-1}+\sum_{i=1}^{n-1}p_i\right)\sum_{j=1}^{n-1}k_j-\sum_{i=1}^{n-1}(p_{i+1}-p_i)\sum_{j=1}^{i}k_j\\
&=-p_{n-1}\sum_{j=1}^{n-1}k_j-\left(\sum_{j=1}^{n-1}p_j\right)k_{n-1}\\
&=-2k_{n-1}p_{n-1}-k_{n-1}\sum_{j=1}^{n-2}p_j-\left(\sum_{j=1}^{n-2}k_j\right)p_{n-1}\\
&=\chi_n-\chi_{n-1}
\end{aligned}$$

which is exactly the extra terms of $c_2(E)$ in (11.6) in going from $N-1$ to N. □

CHAPTER 12

The generalised discrete Nahm equations

The preceding section proves that,

Proposition 36. *Let there be a $\mathbb{C}^\times$-action on $\mathbb{P}^3$,*

$$[x : y : z : w] \mapsto [c^{-1/2}x : c^{1/2}y : c^{-1/2}z : c^{1/2}w].$$

Let E be a $\mathbb{C}^\times$-equivariant holomorphic vector bundle on $\mathbb{P}^3$ corresponding to a monopole with mass numbers $p_1, \ldots, p_N \in \mathbb{Z}$ (or $\frac{1}{2}+\mathbb{Z}$ if N is even) ordered $p_1 < \ldots < p_N$, and charge numbers $k_1, \ldots, k_N \in \mathbb{Z}$, with $\sum_{i=1}^N p_i = 0$ and $\sum_{i=1}^N k_i = 0$.

Then the decomposition of the monad for E,

$$\underline{H} \overset{A_X}{\to} \underline{K} \overset{B_X}{\to} \underline{L}$$

restricted to $\mathbb{P}^1_+$, into weight p components $\mathbb{C}_p$ with respect to the $\mathbb{C}^\times$-action is

$$H = \mathbb{C}^{k_1}_{p_1} \oplus \ldots \oplus \mathbb{C}^{k_1}_{p_2-1} \oplus \mathbb{C}^{k_1+k_2}_{p_2} \oplus \mathbb{C}^{k_1+k_2}_{p_2+1} \oplus \ldots \oplus \mathbb{C}^{-k_N}_{p_N-1},$$

$$K = \mathbb{C}^{k_1+1}_{p_1} \oplus \mathbb{C}^{2k_1}_{p_1+1} \oplus \ldots \oplus \mathbb{C}^{2k_1}_{p_2-1} \oplus \mathbb{C}^{2(k_1+k_2)+1}_{p_2} \oplus \mathbb{C}^{2(k_1+k_2)}_{p_2+1} \oplus \ldots \oplus \mathbb{C}^{2(k_1+\ldots+k_{N-1})}_{p_N-1} \oplus \mathbb{C}^{-k_N+1}_{p_N},$$

$$L = \mathbb{C}^{k_1}_{p_1+1} \oplus \ldots \oplus \mathbb{C}^{k_1}_{p_2} \oplus \mathbb{C}^{k_1+k_2}_{p_2+1} \oplus \mathbb{C}^{k_1+k_2}_{p_2+2} \oplus \ldots \oplus \mathbb{C}^{-k_N}_{p_N}.$$

Note that anti-self-dual instantons have negative instanton charge $-\kappa$ so $\kappa > 0$ which constrains the allowed mass and charge numbers of a hyperbolic monopole.

The Proposition 30 implies that the ADHM data $(\alpha_1, \alpha_2, a, b)$ for a magnetic monopole commutes with the $\mathbb{C}^\times$-action. By Schur's Lemma, the monad maps only between components of the same weight. Now I will describe the form of the ADHM data $(\alpha_1, \alpha_2, a, b)$ which preserve the above weight decomposition.

I remind the reader that when linear transformations $V \to W$ are represented as matrices and when they preserve some decompositions $V = \bigoplus V_i$, $W = \bigoplus W_j$ of V, W, the matrix is equivalent to a matrix with zeros everywhere except for block matrices. These block matrices are the linear transformations between components V_i and W_j. In this case, the components are labelled by weights of the $\mathbb{C}^\times$-action. I have found it convenient to label the block matrices with subscripts denoting the weight i of the weight spaces $\mathbb{C}_i$ between which they map.

The matrix α_1 is a sparse matrix with square blocks $\left\{\beta_{i+1/2}\right\}$, $p_1 \le i \le p_N - 1$ running down the diagonal of the indicated size. The matrix dimensions increase

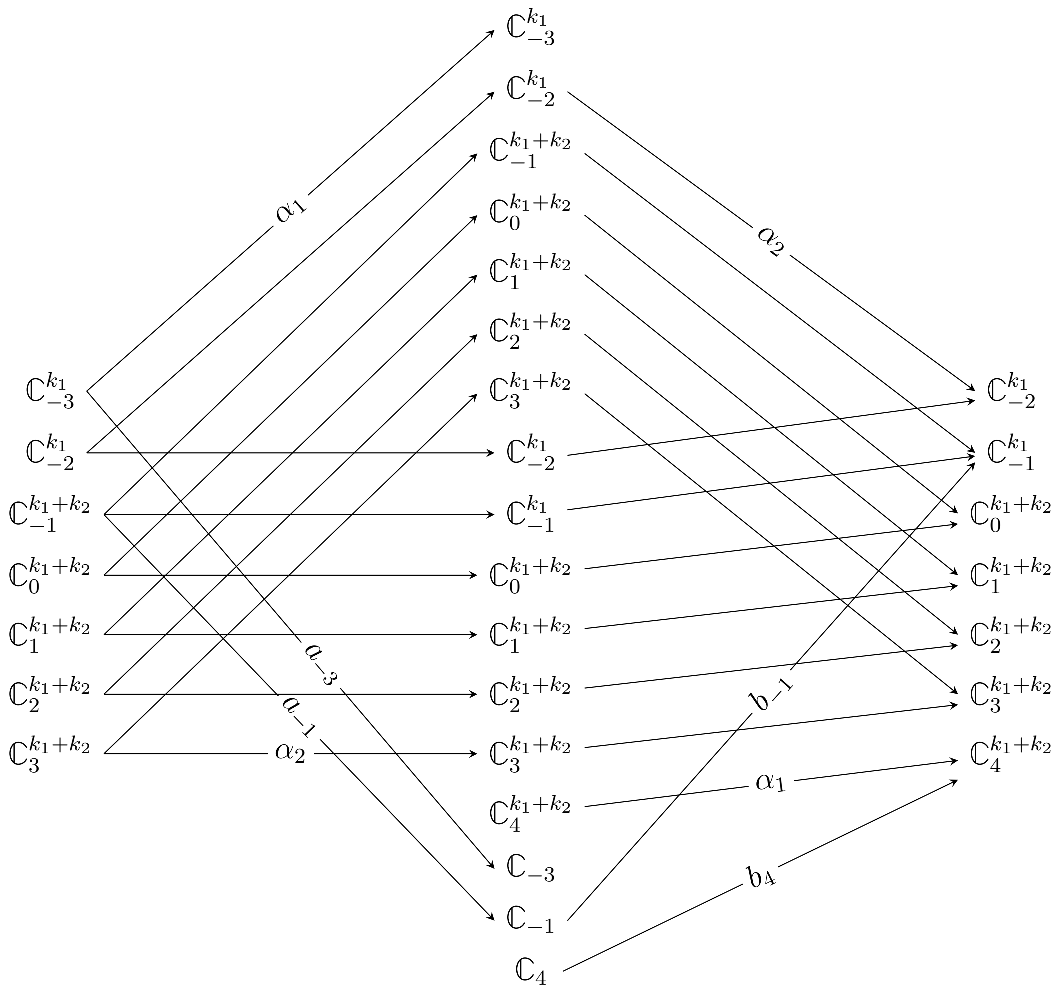


FIGURE 3. The weight decomposition of the monad of an SU(3) hyperbolic monopole with $p_1 = -3$ and $p_2 = -1$ (hence $\kappa = 7k_1 + 5k_2$).

from $(k_1 + \ldots + k_{j-1}) \times (k_1 + \ldots + k_{j-1})$ to $(k_1 + \ldots + k_j) \times (k_1 + \ldots + k_j)$ at each $i = p_j$, $2 \leq j \leq N-1$. Notice the cheeky $\frac{1}{2}$ in the subscript. This is because each $\beta_{i+\frac{1}{2}}$ block does double duty as maps $\mathbb{C}^k_i \to \mathbb{C}^k_i$ and as $\mathbb{C}^k_{i+1} \to \mathbb{C}^k_{i+1}$.

The sparse matrix α_2 has (square except at transitions) blocks $\{\gamma_i\}$, $p_1+1 \leq i \leq p_N-1$ along the super-diagonal. At $i = p_j$, $2 \leq j \leq N-1$, the diagonal block of zeros increases in dimensions from $(k_1 + \ldots + k_{j-1}) \times (k_1 + \ldots + k_{j-1})$ to $(k_1 + \ldots + k_j) \times (k_1 + \ldots + k_j)$. The matrix γ_{p_j} sitting in the transition is a *rectangular* matrix of dimensions $(k_1 + \ldots + k_{j-1}) \times (k_1 + \ldots + k_j)$. The next matrix γ_{p_j+1} returns to being a square block, now of dimensions $(k_1 + \ldots + k_j) \times (k_1 + \ldots + k_j)$.

The $N \times \kappa$ matrix a is divided by P_c into columns labelled by weight space. The non-zero entries are row vectors $\{a_1, \ldots, a_{N-1}\}$ in the columns with weight p_i,

$$
\alpha_1 = \begin{array}{c}\left(\begin{array}{ccccccccc}
\beta_{p_1+\frac{1}{2}} & & & & & & & & \\
& \beta_{p_1+\frac{3}{2}} & & & & & & & \\
& & \ddots & & & & & & \\
& & & \beta_{p_i-\frac{1}{2}} & & & & & \\
& & & & \beta_{p_i+\frac{1}{2}} & & & & \\
& & & & & \beta_{p_i+\frac{3}{2}} & & & \\
& & & & & & \ddots & & \\
& & & & & & & \beta_{p_N-\frac{3}{2}} & \\
& & & & & & & & \beta_{p_N-\frac{1}{2}}
\end{array}\right)\end{array}
\begin{array}{l}
\} \; k_1 \\
\} \; k_1 \\
\\
\} \; k_1+\ldots+k_{i-1} \\
\} \; k_1+\ldots+k_i \\
\} \; k_1+\ldots+k_i \\
\\
\} \; -k_N \\
\} \; -k_N
\end{array}
$$

$$
\alpha_2 = \left(\begin{array}{ccccccccc}
0_{k_1} & \gamma_{p_1+1} & & & & & & & \\
& 0_{k_1} & \ddots & & & & & & \\
& & \ddots & \gamma_{p_i-1} & & & & & \\
& & & 0 & \gamma_{p_i} & & & & \\
& & & & 0_{k_1+\ldots+k_i} & \gamma_{p_i+1} & & & \\
& & & & & 0_{k_1+\ldots+k_i} & \gamma_{p_i+2} & & \\
& & & & & & \ddots & \ddots & \\
& & & & & & & 0_{-k_N} & \gamma_{p_N} \\
& & & & & & & & 0_{-k_N}
\end{array}\right)
\begin{array}{l}
\} \; k_1 \\
\\
\\
\} \; k_1+\ldots+k_{i-1} \\
\} \; k_1+\ldots+k_i \\
\} \; k_1+\ldots+k_i \\
\\
\} \; -k_N \\
\} \; -k_N
\end{array}
$$

$1 \leq i \leq N-1$ and i-th rows of length $k_1+\ldots+k_i$. The last weight space of the domain of a correponding to the last $-k_N$ columns has weight $p_N - 1$.

$$a = \begin{array}{c} \left[\begin{array}{ccccccc} a_{p_1} & & & & & & \\ & a_{p_2} & & & & & \\ & & \ddots & & & & \\ & & & a_{p_i} & & & \\ & & & & \ddots & & \\ & & & & & a_{p_{N-1}} & \\ & & & & & & 0_{p_N} \end{array}\right] \\ \begin{array}{ccccccc} \underbrace{\quad}_{k_1} & \underbrace{\quad}_{k_1+k_2} & & \underbrace{\quad}_{k_1+\ldots+k_i} & & \underbrace{\quad}_{-k_N} & \end{array} \end{array}$$

$$b = \begin{array}{cl} \left[\begin{array}{ccccccc} 0_{k_1} & & & & & & \\ & b_{p_2} & & & & & \\ & & \ddots & & & & \\ & & & b_{p_i} & & & \\ & & & & \ddots & & \\ & & & & & b_{p_{N-1}} & \\ & & & & & & b_{p_N} \end{array}\right] & \begin{array}{l} \\ \}\ k_1 \\ \\ \}\ k_1+\ldots+k_{i-1} \\ \\ \}\ k_1+\ldots+k_{N-2} \\ \}\ -k_N \end{array} \end{array}$$

The $\kappa \times N$ matrix b is divided into rows labelled by weight space. The non-zero entries are column vectors $\{b_2, \ldots, b_N\}$ in the rows with weight p_i, $2 \leq i \leq N-1$ and p_N, and i-th columns of length $k_1 + \ldots + k_{i-1}$. Note that the first weight space of the image of b corresponding to the first k_1 rows has weight $p_1 + 1$.

The complex equation (6.2) is now a series of equations in terms of the blocks $\left\{\beta_{i+1/2}\right\}_{p_1 \leq i \leq p_N - 1}$ and $\{\gamma_j\}_{p_1+1 \leq j \leq p_N - 1}$,

$$\begin{cases} \beta_{i+\frac{1}{2}}\gamma_{i+1} - \gamma_{i+1}\beta_{i+\frac{3}{2}} + b_{i+1}a_{i+1} = 0 & \text{for } i+1 = p_j,\ 2 \leq j \leq N-1 \\ \beta_{i+\frac{1}{2}}\gamma_{i+1} - \gamma_{i+1}\beta_{i+\frac{3}{2}} = 0 & \text{otherwise} \end{cases} \tag{12.1}$$

which we call the complex discrete Nahm equations.

The real ADHM equation becomes the real discrete Nahm equations

(12.2)

$$\begin{cases} \left[\beta_{i+\frac{1}{2}}, \beta^*_{i+\frac{1}{2}}\right] + \gamma_{i+1}\gamma^*_{i+1} - \gamma^*_i\gamma_i - a^*_i a_i = 0 & \text{when } i = p_j,\ 1 \le j \le N-1 \\ \left[\beta_{i+\frac{1}{2}}, \beta^*_{i+\frac{1}{2}}\right] + \gamma_{i+1}\gamma^*_{i+1} - \gamma^*_i\gamma_i + b_{i+1}b^*_{i+1} = 0 & \text{when } i+1 = p_j,\ 2 \le j \le N \\ \left[\beta_{i+\frac{1}{2}}, \beta^*_{i+\frac{1}{2}}\right] + \gamma_{i+1}\gamma^*_{i+1} - \gamma^*_i\gamma_i = 0 & \text{otherwise} \end{cases}$$

where $\gamma_{p_1} = 0 = \gamma_{p_N}$ so the first real equation is

$$\left[\beta_{p_1+\frac{1}{2}}, \beta^*_{p_1+\frac{1}{2}}\right] + \gamma_{p_1+1}\gamma^*_{p_1+1} - a^*_{p_1}a_{p_1} = 0$$

and the last one is

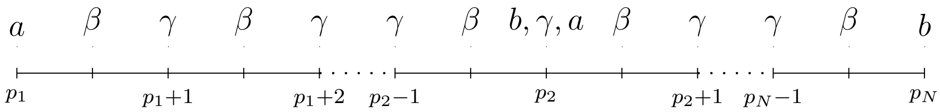
$$\left[\beta_{p_N-\frac{1}{2}}, \beta^*_{p_N-\frac{1}{2}}\right] + b_{p_N+\frac{1}{2}}b^*_{p_N+\frac{1}{2}} - \gamma^*_{p_N-1}\gamma_{p_N-1} = 0.$$

Definition 37. Let N be a natural number. Let $p_1, \ldots, p_N \in \mathbb{Z}$ be distinct and satisfy $\sum_{i=1}^N p_i = 0$. Let $k_1, \ldots, k_N \in \mathbb{Z}$ satisfy $\sum_{i=1}^N k_i = 0$. A solution of the $(N-1)$-interval discrete Nahm equations of type $(p_1, \ldots, p_{N-1}; k_1, \ldots, k_{N-1})$ is an equivalence class of

(1) matrices

$$\left(\{\beta_{j+\frac{1}{2}}\}, \{\gamma_j\}, \{a_{p_i}\}, \{b_{p_i}\}\right)$$

labeled by integral points on an interval $j \in [p_1, p_N] \cap \mathbb{Z}$ as shown



with dimensions $(k_1 + \ldots + k_i) \times (k_1 + \ldots + k_i)$ at half integer points on an interval (p_i, p_{i+1}) and at a boundary point p_i between intervals, the matrices a_{p_i} , γ_{p_i} and b_{p_i} have dimensions $1 \times (k_1 + \ldots + k_i)$, $(k_1 + \ldots + k_{i-1}) \times (k_1 + \ldots + k_i)$ and $(k_1 + \ldots + k_{i-1}) \times 1$ respectively;

(2) satisfying the $(N-1)$-interval discrete Nahm equations; and

(3) with the equivalence relations (gauge transformations)

$$\beta_{j+\frac{1}{2}} \sim g_{j+\frac{1}{2}}\beta_{j+\frac{1}{2}}g^{-1}_{j+\frac{1}{2}}$$

$$\gamma_j \sim g_{j-\frac{1}{2}}\gamma_j g_{j+\frac{1}{2}}$$

$$a_{p_i} \sim \lambda_{p_i}a_{p_i}g^{-1}_{p_i+\frac{1}{2}}$$

$$b_{p_i} \sim g_{p_i-\frac{1}{2}}b_{p_i}\lambda^{-1}_{p_i}$$

where $g_j \in \mathrm{U}(k_1 + \ldots + k_i)$ when $j \in (p_i, p_{i+1}) \cap \mathbb{Z}$.

Thus we have proven our main theorem:

THEOREM 38 (MAIN THEOREM). *Let N be a natural number. Let $p_1, \ldots, p_N \in \mathbb{Z}$ be distinct with $\sum_{i=1}^{N} p_i = 0$. Let $k_1, \ldots, k_N \in \mathbb{Z}$ with $\sum_{i=1}^{N} k_i = 0$. There is an equivalence between*

(1) *framed $SU(N)$ monopoles (A, ϕ) on hyperbolic space H^3 of mass $(p_1, \ldots, p_N)$ and charge $(k_1, \ldots, k_N)$ (up to gauge equivalence), and*
(2) *solutions of the $(N-1)$-interval discrete Nahm equations of type $(p_1, \ldots, p_N; k_1, \ldots, k_N)$.*

CHAPTER 13

The rational map and boundary values revisited

In this chapter, I will discuss two fruits of the labour of generalising the discrete Nahm equations from SU(2) to SU(N). First, an explicit expression for the rational map in terms of a solution of the discrete Nahm equations will be produced. Motivation for having such an explicit expression is the work of Braverman et al. [BDF16] who place a representation-theoretic coordinate system on the moduli space of rational maps (there, they are called Zastava - Croatian for flag). It would be interesting to study the coordinate system induced on the solutions of the discrete Nahm equations by the coordinate system on the moduli space of rational maps.

The rational map

Atiyah [Ati84a] showed (where Atiyah's k_i is $k_1 + \ldots + k_i$ in this work) that:

THEOREM 39 (Atiyah). *For a compact classical group G, the moduli space of circle-invariant instantons or equivalently, hyperbolic monopoles of mass $(p_1, \ldots, p_N)$ with the $p_i \in \mathbb{Z}$ distinct and charge $\boldsymbol{k} = (k_1, \ldots, k_N)$ is naturally isomorphic to the space of degree $\boldsymbol{k}$ based rational maps*

$$f : \mathbb{P}^1 \to G/T$$

where T is a maximal torus.

(Atiyah proved a more general theorem for any homomorphism $\alpha : S^1 \to G$ and based holomorphic maps $f : \mathbb{P}^1 \to G/G(\alpha)$. For a monopole with maximal symmetry breaking, α is integral and generic so $G(\alpha)$ is a maximal torus.)

When $G = \mathrm{SU}(N)$, $G/T = \mathrm{Fl}_{\mathrm{full}}(N) = \left\{0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \ldots \subset \mathbb{C}^N\right\}$, the manifold of full flags in N-dimensional space. For magnetic monopoles, we have the following corollary.

Corollary 40. *There is an isomorphism between the moduli space of framed $SU(N)$ magnetic monopoles on H^3 and the moduli of degree $(k_1, k_1 + k_2, \ldots, k_1 + \ldots + k_{N-1})$ rational maps such that $f(\infty) = \mathbf{0}$,*

$$f : \mathbb{P}^1 \to \mathrm{Fl}_{\mathrm{full}}(N).$$

Along the lines of Braam and Austin [BA90], I will derive an explicit formula for the rational map of a hyperbolic monopole in terms of its discrete Nahm boundary data. To do this, restrict the bundle to the projective plane $\mathbb{P}^2 = \{[x : y : z : 0] \in \mathbb{P}^3\}$. Over this $\mathbb{P}^2$, the solutions of the discrete Nahm equations have a $\mathrm{GL}(\boldsymbol{k}, \mathbb{C})$ freedom. We first require two lemmas of Braam and Austin whose conditions are satisfied in our case.

Lemma 41 (Braam–Austin 4.2 [BA90]). *If $(\{\gamma_i\}, \{\beta_i\}, \{a_{p_j}\}, \{b_{p_{j+1}}\})$ lies in a stable orbit then the γ_i are all injective.*

By the injectivity of the γ_i and using the $\mathrm{GL}(\boldsymbol{k}, \mathbb{C})$ action,

$$g_{i-\frac{1}{2}}\gamma_i g_{i+\frac{1}{2}}^{-1} = \mathrm{I}$$

we set each γ_i, except when $i = p_j$, to the identity matrix. Let square brackets in the subscript indicate that this is the matrix after the $\mathrm{GL}(\boldsymbol{k}, \mathbb{C})$ action has been applied. Then in each interval $p_i \leq j < p_i + 1$, the $\beta_{[j]}$ are all equal to a constant matrix $\beta_{[p_i]}$.

Lemma 42 (Braam–Austin 4.3 [BA90]). *The data $(\{\beta_{[p_i]}\}, \{\gamma_{[p_i]}\}, \{a_{[p_i]}\}, \{b_{[p_{i+1}]}\})$ defines a monad satisfying the ADHM equations if and only if $\{\beta^l_{[p_i]}a_{[p_i]}\}$ for $l = 0, \ldots, k_1 + \ldots + k_i$ span $\mathbb{C}^{k_1+\ldots+k_i}$.*

The procedure is as follows. Choose a "horosphere line" $\mathbb{P}^1_h$ in $\mathbb{P}^2$ with coordinates say $x \mapsto [x : h : -1]$. The trivialisation of E over $\mathbb{P}^1_\infty$ is also a trivialisation of the monad in the sense that in the vector space K of the monad, over $\mathbb{P}^1_\infty$, $(\mathbf{0}, \mathbf{0}, r) \in K$, $r \in \mathbb{C}^N$ are representatives of the global sections of $E|_{\mathbb{P}^1_\infty}$. Extended to $\mathbb{P}^1_h$, this trivialisation (written as cosets) is

$$\begin{bmatrix} -(h-\alpha_2)^{-1} b \\ 0_{\kappa\times N} \\ I_N \end{bmatrix} r + \begin{bmatrix} (h-\alpha_2)^{-1}(x-\alpha_1) \\ I_\kappa \\ 0_{N\times\kappa} \end{bmatrix} Y \in K$$

where $Y \in \mathbb{C}^\kappa$. Note that the second term is the image of A.

Consider the splitting of E over $\mathbb{P}^1_+$,

$$E = \mathcal{O}(k_1) \otimes \mathcal{L}^{p_1} \oplus \ldots \oplus \mathcal{O}(k_r) \otimes \mathcal{L}^{p_r} \oplus \ldots \oplus \mathcal{O}(k_N) \otimes \mathcal{L}^{p_N}.$$

Atiyah showed that in the $\mathrm{SU}(2)$ case, the last factor extends by flowing along the $\mathbb{C}^\times$-action to a sub-line-bundle over $\mathbb{P}^3 - \mathbb{P}^1_-$. The sum of the last two factors extends to a sub-plane-bundle and the sum of the last three extends to a rank 3 sub-bundle of E, etc.

Remember I denote the intersection of $\mathbb{P}^2 = \{w = 0\}$ and $\mathbb{P}^1_-$ by X_-.

Lemma 43. *On $\mathbb{P}^2 - \{\mathbb{P}^1_+ \cap X_-\}$, there exists unique holomorphic sub-bundles $L_1^+ \subset L_2^+ \subset \ldots \subset L_{N-1}^+$ of E which are preserved by the $\mathbb{C}^\times$-action, and each L_i^+ extended to $\mathbb{P}^1_+$ coincides with the last i factors.*

Proof. The bundle E restricted to a $\mathbb{C}^\times$-orbit has the following $\mathbb{C}^\times$-action:

$$c \cdot (z; u_1, \ldots, u_N) = (cz; c^{p_1} u_1, \ldots, c^{p_N} u_N)\,.$$

Near $\mathbb{P}^1_+$, the global holomorphic sections of the form $(0, 0, \ldots, 0, u_N(z))$ are preserved by the $\mathbb{C}^\times$-action since multiplication by $c \in \mathbb{C}^\times$ cannot change zero into a non-zero number. Since the space of such sections is one dimensional, they give us a sub-line bundle L_1^+ of E. The sections have weight p_N and so must coincide with the last factor in the splitting of E over $\mathbb{P}^1_+$ in the limit $c \to 0$.

Similarly for $1 < i < N$, the global holomorphic sections

$$(0, \ldots, 0, u_i(z), u_{i+1}(z), \ldots, u_N(z)),$$

are preserved by the $\mathbb{C}^\times$-action and have weights $(p_i, \ldots, p_N)$. The set of them is $(N - i + 1)$-dimensional so they define a rank $(N - i + 1)$ sub-bundle L^+_{N-i+1} of E.

By induction, a section of the form $(0, \ldots, 0, u_i(z), \ldots, u_N(z))$ is also a section of the sub-bundle given by sections of the form $(0, \ldots, u_{i-1}(z), \ldots, u_N(z))$ so $L^+_{N-i+1} \subset L^+_{N-i}$ and thus the sub-bundles are a chain ordered by subset.

These are the only sections preserved by the $\mathbb{C}^\times$-action which extend to $\mathbb{P}^1_+$ since the $\mathbb{C}^\times$-action is transitive on the non-zero entries of sections. Hence the holomorphic sub-bundles $L_1^+ \subset \ldots \subset L_{N-1}^+$ preserved by the $\mathbb{C}^\times$-action thus defined are unique.

From the discussion on the previous page, the trivialisation of E on the line $\mathbb{P}^1_\infty$ extends to a trivialisation on a neighbourhood of $\mathbb{P}^1_\infty$. The line bundles L_i^+ can be seen in the explicit expression to extend to the whole $\mathbb{P}^2 - \{\mathbb{P}^1_+ \cap X_-\}$. ☐

The rational map f is defined by sending each point x of $\mathbb{P}^1_+$ to the fibre of the restriction of $L_1^+ \subset \ldots \subset L_{N-1}^+ \subset E$ to the orbit of $\mathbb{C}^\times$ whose limit is x. The chain of sub-bundles over the $\mathbb{C}^\times$-orbit is trivialised by taking the intersection of the $\mathbb{C}^\times$-orbit with the chosen horosphere line $\mathbb{P}^1_h$ as the unit point and then the rest of the isomorphism is constructed by flowing along the $\mathbb{C}^\times$-orbit using the $\mathbb{C}^\times$-action. Canonically,

$$\left(L_1^+, \ldots, L_{N-1}^+\right)|_{\mathbb{C}^\times} \cong (\mathbb{C}^1, \ldots, \mathbb{C}^{N-1}) \times \mathbb{C}^\times$$

so that $f(z)$ is an element of the manifold of full flags $\mathrm{Fl}_{\mathrm{full}}(N)$.

Since E has a canonical trivialisation over $\mathbb{P}^1_h$, we can find equations for the rational map. On the level of the monad, the rank i sub-bundle is produced exactly when the $p_1, \ldots, p_{N-i}$ weight spaces are in the kernel of A_X. This happens when the expression for each p_i weight space in the monad trivialisation is equal to the negative of some element of the image of A_X.

Using Lemma 41 to linearly transform $\{\gamma_{[j]}\}_{j\neq p_i}$ into identity matrices, we can invert $(h-\alpha_2)$. Writing $r=(r_1,\ldots,r_N)$, we define the algebraic equations of a flag of subspaces by recursion. The condition that the p_1 weight space be in the kernel of A_X is equivalent to solving the equations

$$(-h)^{p_{N-1}-p_N}b_{[p_N]}r_N+(x-\beta_{[p_{N-1}+\frac{1}{2}]})w_{p_{N-1}}=0$$
$$r_{N-1}+a_{[p_{N-1}]}w_{p_{N-1}}=0.$$

Solving for r_{N-1} in terms of r_N, this is

$$r_{N-1}=(-h)^{p_{N-1}-p_N}a_{[p_{N-1}]}\left(x-\beta_{[p_{N-1}]}\right)^{-1}b_{[p_N]}r_N$$

which defines a line in a plane for any $x\in\mathbb{P}^1$.

Proceeding in the same way for the other weight spaces, we have:

Proposition 44. *Let $(\{\gamma_i\},\{\beta_i\},\{a_{p_j}\},\{b_{p_{j+1}}\})$ be a solution of the $(N-1)$-interval discrete Nahm equations of type $(p_1,\ldots,p_{N-1};k_1,\ldots,k_{N-1})$. Then the solution can be put into the form $(\{\beta_{[p_i]}\},\{\gamma_{[p_i]}\},\{a_{[p_i]}\},\{b_{[p_{i+1}]}\})$ and the rational map,*

$$f:\mathbb{P}^1\to \mathrm{Fl}_{\mathrm{full}}(N)$$
$$x\mapsto (V_1,\ldots,V_{N-1}),\quad dim\ V_i=i,$$

into the manifold of full flags in $\mathbb{C}^N$ can be written as the maps $(r_1(x),\ldots,r_{N-1}(x))$,

$$\begin{aligned}
r_{N-1}(x) &= (-h)^{p_{N-1}-p_N}a_{[p_{N-1}]}\left(x-\beta_{[p_{N-1}]}\right)^{-1}b_{[p_N]}r_N(x)\\
&\vdots\\
r_j(x) &= \sum_{i=j+1}^{N}(-h)^{p_j-p_i}a_{[p_j]}\left(x-\beta_{[p_j]}\right)^{-1}b_{[p_i]}^{k_1+\ldots+k_j}r_i(x)\\
&\vdots\\
r_1(x) &= \sum_{i=2}^{N}(-h)^{p_1-p_i}a_{[p_1]}\left(x-\beta_{[p_1]}\right)^{-1}b_{[p_i]}^{k_1}r_i(x)
\end{aligned}$$

where for each $x\in\mathbb{P}^1$, $r_{N-1}(x)$ specifies an $(N-1)$-dimensional linear subspace in $\mathbb{C}^N$ and each successive $r_i(x)$ specifies an i-dimensional linear subspace inside the $(i+1)$-dimensional linear subspace specified by $r_{i+1}(x)$. The superscript $k_1+\ldots+k_j$ indicates that only the first $k_1+\ldots+k_j$ entries of the vector are involved.

Note that when $N = 2$, the equation of the rational map is of the form

$$r(x) = \frac{r_2(x)}{r_1(x)} = (-h)^{2p} v(x - \beta)^{-1} v^t$$

which is the rational map found by Atiyah for SU(2) hyperbolic monopoles [Ati84a; Ati84b].

The Boundary Value of a Monopole

On the conformal sphere at infinity, S^2_∞, the holomorphic vector bundle $\mathcal{E}$ splits into holomorphic line bundles $\mathcal{O}(k_1) \oplus \ldots \oplus \mathcal{O}(k_N)$ and the gauge field A restricted to S^2_∞, induces a a U(1) connection A_i on each factor $\mathcal{O}(k_i)$. We define the $(N-1)$-tuple $(A_1, \ldots, A_{N-1})$ to be the boundary value or connections at infinity.

We shall prove the following generalisation of Braam–Austin's theorem [BA90] regarding the boundary values of SU(2) hyperbolic monopoles.

THEOREM 45. *Let (A, Φ) be a framed $SU(2)$ hyperbolic monopole. Then*

(1) *the $(N-1)$ tuple of $U(1)$ connections $(A_1, \ldots, A_{N-1})$ on S^2_∞ determines the connection A (up to gauge transformations);*
(2) *there exists for $i = 1, \ldots, N-1$, holomorphic maps*

$$F_i : \mathbb{P}^1 \to Fl(k_1 + \ldots + k_i, k_1 + \ldots + k_i + 1, 2k_1 + \ldots + 2k_{i-1} + k_i + 1)$$

into the manifold of two term partial flags for which each A_i is the pullback of the unitary invariant connection on the "hyperplane bundle" $\mathcal{O}(1, -1)$ of the i-th flag manifold; and

(3) *the map $A \mapsto (A_1, \ldots, A_{N-1})$ is an immersion of the moduli space of $SU(N)$ framed hyperbolic monopoles in the space of $(N-1)$ tuples of $U(1)$ connections on S^2.*

PROOF. From Proposition 34, we have a decomposition of the monad $H \to K \to L$ restricted to $\mathbb{P}^1_+$ (which by abuse of notation, I conflate with S^2_∞ since any connections on $\mathbb{P}^1_+$ descend to connections on S^2_∞ along the twistor transform) into weight spaces. By considering the maps A_x and B_x restricted to a weight subspace, we get what is called a small monad. By dimensional considerations, the cohomology of a generic small monad $(p_i < j < p_{i+1})$

$$\begin{tikzcd}
& \mathbb{C}_j^{k_1+\ldots+k_i} \arrow[r, "\gamma_j"] & \mathbb{C}_j^{k_1+\ldots+k_{i-1}} \\
\mathbb{C}_j^{k_1+\ldots+k_i} \arrow[ur, "\beta_{j+\frac{1}{2}}"] \arrow[r, "\gamma_j"'] & \mathbb{C}_j^{k_1+\ldots+k_{i-1}} \arrow[ur, "\beta_{j-\frac{1}{2}}"'] &
\end{tikzcd}$$

is trivial except for the weight spaces $p_1, \ldots, p_N$ which take the form

$$\begin{tikzcd}
 & & \mathbb{C}_{p_i}^{k_1+\ldots+k_i} \arrow[drr, "\gamma_{p_i}"] & & \\
\mathbb{C}_{p_i}^{k_1+\ldots+k_i} \arrow[urr, "\beta_{p_i+\frac{1}{2}}"] \arrow[r, "\gamma_{p_i}"] \arrow[drr, "a_{p_i}"'] & \mathbb{C}_{p_i}^{k_1+\ldots+k_{i-1}} \arrow[rrr, "\beta_{p_i-\frac{1}{2}}"] & & & \mathbb{C}_{p_i}^{k_1+\ldots+k_{i-1}} \\
 & & \mathbb{C}_{p_i} \arrow[urr, "b_{p_i}"'] & &
\end{tikzcd}$$

The cohomology of these small monads are holomorphic line bundles defined fibre-wise

$$\mathcal{L}_{p_i}(x) = \ker(\mathbb{C}^{2k_1+\ldots+2k_{i-1}+k_i+1} \to \mathbb{C}^{k_1+\ldots+k_{i-1}})/A_x(\mathbb{C}^{k_1+\ldots+k_i})$$

which are exactly the line bundles in the splitting of $\mathcal{E}$.

Furthermore, there is a natural interpretation of the maps A_x and B_x, restricted to each weight space of weight p_i as a pair of maps,

$$B_x^t : \mathbb{C}^{k_1+\ldots+k_{i-1}} \to \mathbb{C}^{2k_1+\ldots+2k_{i-1}+k_i+1}$$

$$A_x : \mathbb{C}^{k_1+\ldots+k_i} \to B_x^t(\mathbb{C}^{k_1+\ldots+k_{i-1}})^{\perp} \cong \mathbb{C}^{k_1+\ldots+k_i+1} \subset \mathbb{C}^{2k_1+\ldots+2k_{i-1}+k_i+1}$$

defining a map $F_i = (A_x(H_{p_i}), B_x^t(L_{p_i})^{\perp})$ into the two term partial flag manifold $\mathrm{Fl}(k_1+\ldots+k_i, k_1+\ldots+k_i+1, 2k_1+\ldots+2k_{i-1}+k_i+1)$. Then each line bundle L_{p_i} and its $\mathrm{U}(1)$ connection is the pullback of the invariant line bundle and (limiting) connection over the two term partial flag manifold. This proves (2) of the theorem.

The map F_i thus defined is an embedding of $\mathbb{P}^1$ into the partial flag manifold, for the ADHM equations guarantee that the monad is non-degenerate [Cal53], and so im F_i has no self-intersections and its derivative is non-zero. Compose F_i with the Plücker embedding and then the Segre embedding to get

$$F_i^{\mathbb{P}} : \mathbb{P}^1 \hookrightarrow \mathbb{P}^{\mathfrak{k}(i)}$$

where

$$\mathfrak{k}(i) = \begin{pmatrix} 2k_1+\ldots+2k_{i-1}+k_i+1 \\ k_1+\ldots+k_i \end{pmatrix} \begin{pmatrix} 2k_1+\ldots+2k_{i-1}+k_i+1 \\ k_1+\ldots+k_i+1 \end{pmatrix} - 1.$$

The pullback of the $\mathrm{U}(\mathfrak{k}(i)+1)$ invariant connection A_i by the embedding $F_i^{\mathbb{P}}$ induces a Kähler form F_{A_i} (the curvature form of A_i) on $\mathbb{P}^1$. The work of Calabi

[Cal53] tells us that any such embedding $F_i^{\mathbb{P}}$ is locally rigid, that is, the embedding is determined by the Kähler form up to the isometry group of the target space.

Hence the boundary values $(A_1, \ldots, A_{N-1})$ descend by the twistor transform to $\mathrm{U}(1)$ connections on S^2 and determine the small monad for the weight spaces corresponding to the weights $p_1, \ldots, p_{N-1}$. These small monads provide boundary values for the $(N-1)$-interval discrete Nahm equations and their propagation uniquely specifies a complete solution up to gauge transformations. Thus the boundary values on S^2_∞ or equivalently $\mathbb{P}^1_+$ uniquely determine the monopole.

On the moduli space of $\mathrm{SU}(N)$ framed hyperbolic monopoles, the boundary values $(A_1, \ldots, A_{N-1})$ are local coordinates. Thus $A \mapsto (A_1, \ldots, A_{N-1})$ is a local immersion of the moduli of monopoles into the moduli of $(N-1)$-tuples of $\mathrm{U}(1)$ connections on S^2. $\square$

CHAPTER 14

Spectral curves

This chapter deals with the analysis of the spectral curve of an $\mathrm{SU}(N)$ hyperbolic monopole in relation to the discrete Nahm equations. Notably, the discrete Nahm equations provide an explicit expression for the equation for the spectral curve. The spectral curve condition was suggested to me by Michael Murray who first wrote it down for the $\mathrm{SU}(2)$ case many years back. Paul Norbury′s thesis and calculations of the $\mathbb{C}^\times$-action on lines were instrumental in the analysis of this chapter.

The spectral curve for an $\mathrm{SU}(2)$ hyperbolic monopole was previously defined by Atiyah [Ati84b]. For the higher rank case, Murray–Singer studied the spectral curve for hyperbolic monopoles defined with different boundary conditions to the ones used in this thesis [MS96]. To be precise, it would do well to have here a definition of spectral curves for $\mathrm{SU}(N)$ hyperbolic monopoles of integral mass, defined as circle-invariant instantons. The following definition is analogous to the definition used by Atiyah and is in my opinion conceptually clearer than the more analytic definition used in Murray–Singer.

Remember that the mini-twistor space Q parametrising oriented geodesics of H^3 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with the antidiagonal

$$\bar{\Delta} = \{(\hat{\zeta}, \zeta) \in \mathbb{P}^1 \times \mathbb{P}^1\}$$

removed. Given an oriented geodesic in H^3 with limits on the sphere at infinity represented in $\mathbb{P}^3$ by $\zeta, \eta \in \mathbb{P}^1$ respectively, the corresponding point in Q is $(\hat{\zeta}, \eta)$ where $\hat{\zeta}$ is $-1/\bar{\zeta}$.

Definition 46. Define projection maps $\pi_\pm : \mathbb{P}^3 - \mathbb{P}^1_\mp \to \mathbb{P}^1_\pm$. Let $E \to \mathbb{P}^3$ be the holomorphic vector bundle of a framed $\mathrm{SU}(N)$ hyperbolic monopole with mass $p_1 < \ldots < p_N$ and charge $k_1, \ldots, k_N$. Define sub-bundles $E_i^\pm$ of E by

$$E_i^+ = \pi_+^* \left(\bigoplus_{j=N-i+1}^{N} \mathcal{O}(k_j) \otimes \mathcal{L}_{p_j} \right) \quad \text{and} \quad E_i^- = \pi_-^* \left(\bigoplus_{j=1}^{i} \mathcal{O}(k_j) \otimes \mathcal{L}_{p_j} \right).$$

The *spectral curve* of the hyperbolic monopole is the curve in Q defined by

$$S = \left\{ \gamma \in Q \mid \operatorname{codim}_E(E_i^+|_{\gamma^\times} \oplus E_{N-i}^-|_{\gamma^\times}) \geq 1, \text{ for some } i \in 1, \ldots, N-1 \right\}.$$

The codimension 1 condition for each value of i defines a component S_i of the spectral curve.

I will take the liberty of referring to the closure of a $\mathbb{C}^\times$-orbit in $\mathbb{P}^3$ as an *equivariant line*. Every equivariant line intersects the fixed point sets $\mathbb{P}^1_\pm$ in two points, p and q so that we can write $\mathbb{P}^1_{pq}$ for an equivariant line. If an equivariant line $\mathbb{P}^1_{pq}$ corresponds to a point in the spectral curve then we call it a *spectral line*. The spectral curve parametrises the spectral lines.

Remember that the holomorphic bundle E is the cohomology of a monad (Chapter 6)

$$H \stackrel{A_X}{\to} K \stackrel{B_X}{\to} L.$$

The condition for an equivariant line $\mathbb{P}^1_{pq}$ to be a spectral line is

$$\det B_{\hat{q}} A_p = 0.$$

A way to make sense of this is to define a bilinear form $< \quad >: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{C}$ by $< p, q >= \det B_{\hat{q}} A_p$. This bilinear form invokes the ideas in Norbury's boundary algebras work [Nor04] as well as the work of Murray–Norbury–Singer [MNS03]. The spectral curve could have been defined by the vanishing of this bilinear form. We will come back to this bilinear form at the end of the chapter.

One additional point is the existence of a holomorphic line bundle $\mathcal{L}_i \to S_i$ on the (components of the) spectral curve for SU(2) hyperbolic monopoles [MS00]. The map $B \circ A : H \otimes Q \to L$ is itself a monad whose cohomology (or kernel in this case) defines a holomorphic line bundle over the spectral lines. I conjecture that this holomorphic line bundle will push down along $\mathbb{P}^3 \to Q$ to a holomorphic line bundle on the spectral curve $S \subset Q$ analogous to the one described by Murray–Singer. The benefit of this approach is that it ties the codimension 1 condition in with the existence of the line bundle geometrically and naturally.

Before I prove that the above condition is the equation of the spectral curve, let me show you the explicit form of the spectral curve equation in terms of discrete Nahm data $\{\beta_j, \gamma_j, a_{p_i}, b_{p_i}\}$ implied by this condition.

Points $p, \hat{q}$ on the fixed point sets $\mathbb{P}^1_\pm$ can be represented in $\mathbb{P}^3$ by $[1 : 0 : \zeta : 0]$ and $[0 : \lambda : 0 : \eta]$ respectively for $\zeta, \lambda, \eta \in \mathbb{C}$. The component spectral curve S_i is given by the formula, where $p_i < j < p_{i+1}$,

$$\det((\lambda + \eta \beta^*_{j+\frac{1}{2}})(1 + \zeta \beta_{j+\frac{1}{2}}) + \eta \zeta \gamma^*_j \gamma_j) = 0.$$

What happens at the points $p_1, \ldots, p_N$? Let us look at the illustrative example of SU(3) for Nahm data with p_1, p_2, p_3 and k_1, k_2, k_3. The monad maps are

$$A_p = \begin{bmatrix} 1 + \zeta\alpha_1 \\ \zeta\alpha_2 \\ \zeta a \end{bmatrix}, \quad B_{\hat{q}} = \begin{bmatrix} -\lambda - \eta\alpha_1^* & -\eta\alpha_2^* & -\eta a^* \end{bmatrix},$$

where

$$\alpha_1 = \begin{bmatrix} \beta_{p_1+\frac{1}{2}} & & & & \\ & \beta_{p_1+\frac{3}{2}} & & & \\ & & \ddots & & \\ & & & \beta_{p_N-\frac{3}{2}} & \\ & & & & \beta_{p_N-\frac{1}{2}} \end{bmatrix},$$

$$\alpha_2 = \begin{bmatrix} 0_{k_1} & \gamma_{p_1+1} & & & & & \\ & \ddots & \ddots & & & & \\ & & 0_{k_1} & \gamma_{p_2} & & & \\ & & & 0_{k_1+k_2} & \ddots & & \\ & & & & \ddots & \gamma_{p_N-1} & \\ & & & & & 0_{k_1+k_2} & \end{bmatrix},$$

$$a = \begin{bmatrix} a_{p_1} & & & & \\ & & & a_{p_2} & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$b = \begin{bmatrix} 0 & & \\ 0 & & \\ & b_{p_1} & \\ 0 & & \\ 0 & & b_{p_2} \end{bmatrix}.$$

Now we compute $\det B_{\hat{q}} A_p$ and find that the determinant reduces to a product of the determinants of the independent non-zero blocks on the diagonal of the $\kappa \times \kappa$

matrix $B_{\hat{q}}A_p$. The equivariant line $\mathbb{P}^1_{pq}$ is spectral line if one of the following is true:

$$\begin{aligned}
\det((\lambda+\eta\beta^*_{p_1+\frac{1}{2}})(1+\zeta\beta_{p_1+\frac{1}{2}})+\eta\zeta a^*_{p_1}a_{p_1}) &= 0 \qquad \leftarrow\\
\det((\lambda+\eta\beta^*_{p_1+\frac{3}{2}})(1+\zeta\beta_{p_1+\frac{3}{2}})+\eta\zeta\gamma^*_{p_1+1}\gamma_{p_1+1}) &= 0\\
\det((\lambda+\eta\beta^*_{p_1+\frac{5}{2}})(1+\zeta\beta_{p_1+\frac{5}{2}})+\eta\zeta\gamma^*_{p_1+2}\gamma_{p_1+2}) &= 0\\
&\vdots\\
\det((\lambda+\eta\beta^*_{p_2-\frac{1}{2}})(1+\eta\beta_{p_2-12})+\eta\zeta\gamma^*_{p_2-1}\gamma_{p_2-1}) &= 0\\
\det((\lambda+\eta\beta^*_{p_2+\frac{1}{2}})(1+\eta\beta_{p_2+12})+\eta\zeta(\gamma^*_{p_2}\gamma_{p_2}+a^*_{p_2}a_{p_2}) &= 0 \qquad \leftarrow\\
\det((\lambda+\eta\beta^*_{p_2+\frac{3}{2}})(1+\eta\beta_{p_2+32})+\eta\zeta\gamma^*_{p_2+1}\gamma_{p_2+1}) &= 0\\
&\vdots\\
\det((\lambda+\eta\beta^*_{p_3-\frac{1}{2}})(1+\eta\beta_{p_3-12})+\eta\zeta\gamma^*_{p_3-1}\gamma_{p_3-1}) &= 0.
\end{aligned}$$

Notice the Nahm equation boundary cases as indicated by arrows. Notice also that the b matrices play no role.

Reversing the orientation of the geodesics gives us the backward equations of the form

$$\det((1+\zeta\beta_{j+\frac{1}{2}})(\lambda+\eta\beta^*_{j+\frac{1}{2}})+\zeta\eta\gamma_j\gamma^*_j)=0$$

which involve only the b matrices and not the a matrices. For the reality condition to be satisfied, every forward equation must be equivalent to a backward equation.

From Ward's formalism [War99], the "discrete Lax pair" for the discrete Nahm equations are

$$W_+ = \gamma^* + \eta\beta + \zeta^{-1}\eta$$

and

$$W_- = -\eta^2\gamma + \eta\beta^* + 1.$$

Together,

$$[W_+, W_-] = \eta^2([\gamma,\gamma^*]+[\beta,\beta^*]) + \eta([\gamma^*,\beta^*]+\eta^3([\gamma,\beta])$$

which is recognised as the discrete Nahm equations.

Now I will bring rigour to the discussion.

Lemma 47. *Let there be a $\mathbb{C}^\times$-action on $\mathbb{P}^1$ with fixed points P_+, P_-. Let $\mathcal{O}(p)\to\mathbb{P}^1$ be a holomorphic line bundle with a lifting of the $\mathbb{C}^\times$-action. Then the weights p_+, p_- of the fibres of $\mathcal{O}(p)$ as an equivariant bundle, over P_+, P_- respectively as $\mathbb{C}^\times$-representations satisfy $p_+ - p_- = p$.*

Proof. Let $\{U_+, U_-\}$ be charts covering $\mathbb{P}^1$. In the local coordinate (z, v) of E over U_+, the $\mathbb{C}^\times$-action is

$$(z,v)\mapsto(cz, c^{p_+}v).$$

In the U_- chart, the point (z, v) is $(z, v') = (z, z^{-p}v)$. Then by compatibility of the action and the transition function, the $\mathbb{C}^\times$-action on E over U_- must be

$$(z, v) \mapsto (cz, c^{p_+}v) \simeq (cz, c^{p_+}(cz)^{-p}v) = (cz, c^{p_+-p}(z^{-p}v)) = (cz, c^{p_-}v').$$

Thus the representation of $\mathbb{C}^\times$ over P_- must satisfy $p_+ - p_- = p$. $\square$

Proposition 48. *Let $l = \mathbb{P}^1_{P_+,P_-}$ be an equivariant line. l is a spectral line if and only if there exists an equivariant line subbundle of $E|_l$ isomorphic to $\mathcal{O}(p_1 - p_2)$ where p_1, p_2 are mass numbers of the hyperbolic monopole.*

PROOF. Let F be a rank $N-1$ equivariant sub-bundle of the rank N equivariant holomorphic vector bundle $E \to \mathbb{P}^1$, defined by the union of E_i^+ and E_{N-i}^- for some $1 \leq i \leq N$. The quotient bundle E/F is an equivariant holomorphic line bundle L over $\mathbb{P}^1$. Thus L is $\mathcal{O}(p)$ for some $p \in \mathbb{Z}$. Let $p_\pm$ be the weights of $\mathbb{C}^\times$ of $(E/F)|_{P_\pm}$ at $P_\pm$ respectively. By Lemma 47, $p = p_+ - p_-$. $\square$

Proposition 49 (Norbury, [Nor94] p.54)**.** *If $B_q A_p : \mathbb{C}^\kappa \to \mathbb{C}^\kappa$ is invertible then $E|_{\overline{pq}}$ is trivial.*

A *jumping line* is a line $\mathbb{P}^1$ for which E is non-trivial. The proposition then says that when $\mathbb{P}^1_{pq}$ is a jumping line for E, $\det B_{\hat{q}} A_p = 0$. Conversely,

Proposition 50. $\dim \ker B_q A_p = n$ *if and only if for some partition $n_1, \ldots, n_k$ of n by non-negative integers, and $N - k$ negative integers $n_{k+1}, \ldots, n_N$ such that $\sum_{i=1}^N n_i = 0$,*

$$E|_{\overline{pq}} \simeq \mathcal{O}(n_1) \oplus \ldots \oplus \mathcal{O}(n_N).$$

PROOF. The line bundle $\mathcal{O}(m)$ of $\mathbb{P}^1$ contributes $m+1$ global holomorphic sections for $m \geq 0$ and none for negative m. The bundle $\sum_{i=1}^N \mathcal{O}(n_i)$ as in the proposition has $n+k$ global holomorphic sections. Furthermore, $\sum_{i=1}^N \mathcal{O}(n_i)$ tensored with $\mathcal{O}(-1)$ has n holomorphic global sections. Since all holomorphic vector bundles over $\mathbb{P}^1$ split into a sum of $\mathcal{O}(m)$ line bundles, a holomorphic vector bundle over $\mathbb{P}^1$ with n global holomorphic sections vanishing at a divisor p is of the form $\sum_{i=1}^N \mathcal{O}(n_i)$ for some positive k.

It remains to show that when $\dim \ker B_q A_p = n$, the holomorphic vector bundle $E|_{\mathbb{P}^1}$ has n global holomorphic sections which vanish at p. The rest of the proof follows the proof of Proposition 7 in the Norbury thesis [Nor94] p.55 and requires the lemma that is to follow. $\square$

Lemma 51 (Norbury [Nor94] p.55)**.** *A global holomorphic section of $E|_{\overline{pq}}$ is given by $v \in K$ independent of the coordinate $z \in \overline{pq}$.*

The holomorphic vector bundle E of the instanton trivial on a line will be generically trivial when restricted to lines $\mathbb{P}^1$ in $\mathbb{P}^3$.

Proposition 52. *The spectral lines of the hyperbolic monopole are exactly the equivariant lines of the bundle $E \to \mathbb{P}^3$ which are jumping lines.*

PROOF. Consider, without loss of generality, a spectral line where $E_i^+ \subset E_{N-i}^-$. Over $\mathbb{P}^1_+$, the $\mathbb{C}^\times$-action has at most weight p_i on E_i^+. Likewise, over $\mathbb{P}^1_-$, the $\mathbb{C}^\times$-action has at least weight p_{i+1} on E_i^-. The definition of a spectral line implies that E_i^+ and E_{N-i}^- intersect in a subspace of at least dimension 1 over $\mathbb{C}^\times$. Since E_i^+ and E_{N-i}^- are both holomorphic vector bundles, when they intersect over a whole $\mathbb{C}^\times$, their intersection is a holomorphic line sub-bundle of E. Furthermore, when restricted to $P_\pm$, this line bundle has weights $p_+ \le p_i$ and $p_- \ge p_{i+1}$ respectively. By Lemma 47, this line bundle is $\mathcal{O}(p_+ - p_-)$.

Suppose that an equivariant line l is a jumping line. Then there exists at least one $\mathcal{O}(p)$ with $p \neq 0$ in its unique decomposition into holomorphic line bundles. Since at the endpoints, $P_\pm$, the holomorphic vector bundle E is a representation of $\mathbb{C}^\times$, the line bundle must coincide with weight spaces there. Therefore, $p = p_+ - p_-$ and l is a spectral line.

Conversely, suppose that an equivariant line l is not a jumping line. That is, $E|_l$ is a trivial equivariant vector bundle over l. Then E is determined by the fibre over a single point and has no non-trivial holomorphic line sub-bundles. Therefore, l is not a spectral line for E.

□

It follows that:

THEOREM 53. *The spectral curve S is the variety of points $(p, q) \in Q$ satisfying* $\det B_{\hat{q}} A_p = 0$.

I end the chapter with two conjectures.

Conjecture 54. *The bilinear form $< p, q >= \det B_{\hat{q}} A_p$ coincides with the two point function of Norbury [Nor04].*

Conjecture 55. *Let Φ be the polynomial equation defining the spectral curve S. Then the irreducible factors of Φ restricted to the anti-diagonal $\bar{\Delta}$ are Hermitian metrics whose associated 1-forms are the boundary $U(1)$ fields which together determine the monopole.*

CHAPTER 15

The classical Lie groups

In this thesis, hyperbolic monopoles were mostly treated for $\mathrm{SU}(N)$. However, any classical group G can play the role of $\mathrm{SU}(N)$. In this chapter, I will describe the procedure for finding solutions of the discrete Nahm equations for the groups $G = \mathrm{Sp}(N), \mathrm{SO}(N)$ as an invariant subset of the solutions of the $\mathrm{SO}(N)$ $(N-1)$-interval discrete Nahm equations. I will then write down the resulting conditions. I attempted to investigate the case of G_2 for which Shnir and Zhilin recently produced an example in the euclidean setting [SZ15], but I am not confident of my ansatz so I have not included it here.

As far as I can find, Corrigan-Goddard [CG84] and Nekrasov-Shadchin [NS04] treat ADHM for the classical groups via a the concept of "reciprocity" where to produce an instanton for a group G, a hyper-Kähler quotient of the ADHM data by a "reciprocal" group is taken. It is unclear to me what reciprocity means geometrically.

A more geometric take, in terms of monads, is suggested by both Atiyah [Ati79] and Donaldson [Don84a] but neither treat it explicitly.

I will use the latter approach and use the more explicit former papers as a guide for the correctness of the process. The original method *cannot* be applied verbatim in the S^1-equivariant setting. Instead, a modification needs to be done to preserve the S^1-equivariance.

Returning to the monad 8.1, with H, K, L vector spaces and $\mathcal{O}(k)$ the k-th tensor product of the Hopf or hyperplane bundle over $\mathbb{P}^3$,

$$H \otimes \mathcal{O}(-1) \overset{A_X}{\to} K \otimes \mathcal{O} \overset{B_X}{\to} L \otimes \mathcal{O}(1),$$

the main idea is to equip the vector bundles with the appropriate bilinear or sesquilinear forms and then insist that the monad maps A, B respect the form.

A non-degenerate bilinear form on a vector space V

$$V \times V \to \mathbb{F}$$

over the field $\mathbb{F}$ is equivalent to an $\mathbb{F}$-linear isomorphism

$$V \to V^*.$$

Hence, this is equivalent to requiring that the monad maps A, B be invariant under the linear maps defined by the chosen forms on the monad vector bundles.

The point is which forms to put on which vector bundles for which G. Atiyah [Ati79] discusses this for the symplectic ADHM construction but we are starting with a unitary ADHM. Nekrasov-Shadchin [NS04] start with the unitary ADHM of Donaldson as well so their prescription is the correct one to use.

Note that the solutions of the discrete Nahm equations for $\mathrm{SO}(N)$ are a subset of the solutions of the $\mathrm{SU}(N)$ discrete Nahm equations with the condition that κ be *even*. $\mathrm{Sp}(N)$ is produced from $\mathrm{SU}(2N)$ but with no additional condition on κ. The former "folds" the κ and the latter "folds" the N.

The choice of basis for the monad in Donaldson's ADHM construction for $\mathrm{SU}(N)$ amounts to

$$K = V \oplus V \oplus W, \qquad L = V = H$$

where V is a κ-dimensional vector space and W is an N-dimensional vector space. We will now write explicit matrices for the forms on V, W in bases which respect both the unitary structure and the weight decomposition with respect to the S^1-action.

For orthogonal G, the forms on V and W are the sparse matrices

$$\mathbb{J} = \left[\begin{array}{c|c} & \begin{matrix} & & \mathbb{1} \\ & \iddots & \\ \mathbb{1} & & \end{matrix} \\ \hline \begin{matrix} & & -\mathbb{1} \\ & \iddots & \\ -\mathbb{1} & & \end{matrix} & \end{array}\right], \qquad \mathbb{K} = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \tag{15.1}$$

where the non-zero blocks in $\mathbb{J}$ corresponding to weight modules p_i are identity matrices $\mathbb{1}$ of rank $k_1 + \ldots + k_j$ if it is of weight $\in [p_j, p_j + 1)$.

For symplectic G, the forms on V and W are the sparse matrices

$$\mathbb{J} = \begin{bmatrix} & & \mathbb{1} \\ & \iddots & \\ \mathbb{1} & & \end{bmatrix}, \qquad \mathbb{K} = \left[\begin{array}{c|c} & \begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix} \\ \hline \begin{matrix} & & -1 \\ & \iddots & \\ -1 & & \end{matrix} & \end{array}\right] \tag{15.2}$$

where the non-zero blocks in $\mathbb{J}$ corresponding to weight modules p_i are identity matrices $\mathbb{1}$ of rank $k_1 + \ldots + k_j$ if it is of weight $p_i \in [p_j, p_j + 1)$.

The condition that the monad maps respect the bilinear forms is equivalent to

$$\alpha_1 = (\mathbb{J}\alpha_1\mathbb{J}^{-1})^T \tag{15.3}$$

$$\alpha_2 = (\mathbb{J}\alpha_2\mathbb{J}^{-1})^T \tag{15.4}$$

$$b = (\mathbb{K}a\mathbb{J}^{-1})^T. \tag{15.5}$$

For $G = \mathrm{SO}(N) \subset \mathrm{SU}(N)$, this implies that for $i \in [p_1, p_N]$ and $j \in \{p_1, \ldots, p_N\} \cap \{> 0\}$,

$$\beta_i = \beta_{-i}^T \tag{15.6}$$

$$\gamma_i = \gamma_{-i}^T \tag{15.7}$$

$$\gamma_0 = -\gamma_0^T \tag{15.8}$$

$$b_{-j} = a_j^T \tag{15.9}$$

$$b_j = -a_{-j}^T. \tag{15.10}$$

For $G = \mathrm{Sp}(N) \subset \mathrm{SU}(2N)$, this implies that for $i \in [p_1, p_{2N}]$ and $j \in \{p_1, \ldots, p_{2N}\} \cap \{> 0\}$,

$$\beta_i = \beta_{-i}^T \tag{15.11}$$

$$\gamma_i = \gamma_{-i}^T \tag{15.12}$$

$$\gamma_0 = \gamma_0^T \text{ or } \beta_0 = \beta_0^T \tag{15.13}$$

$$b_{-j} = a_j^T \tag{15.14}$$

$$b_j = -a_{-j}^T. \tag{15.15}$$

To summarise,

THEOREM 56. *Let* $(\{\gamma_i\}, \{\beta_i\}, \{a_{p_j}\}, \{b_{p_{j+1}}\})$ *be a solution of the* $(N-1)$*-interval discrete Nahm equations of type* $(p_1, \ldots, p_{N-1}; k_1, \ldots, k_{N-1})$*. If* $p_i = p_{N+1-i}$ *and* $k_i = -k - N + 1 - i$ *then*

(1) *if* N *is even,* $\beta_i = \beta_{-i}^T$*,* $\gamma_i = \gamma_{-i}^T$ *and for* $i > 0$*,* $b_i = -a_{-i}^T$ *and* $b_{-i} = a_i^T$ *then this is a solution of the Sp*$(N/2)$ *discrete Nahm equations;*
(2) *if* κ *is even,* $\beta_i = \beta_{-i}^T$*,* $\gamma_i = \gamma_{-i}^T$ *except for* $i = 0$ *where* $\gamma_0 = -\gamma_0^T$ *and for* $i > 0$*,* $b_i = -a_{-i}^T$ *and* $b_{-i} = a_i^T$ *then this is a solution of the SO*(N) *discrete Nahm equations.*

Note that in the case of $\mathrm{Sp}(1)$ and $p = \frac{1}{2} + \mathbb{Z}$, the first two conditions provide us with the extra two discrete Nahm equations seen in Braam–Austin [BA90] and the final two reduce a, b to a single vector v. Since $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$, all $\mathrm{SU}(2)$ monopoles are Sp(1) monopoles. This reveals a hidden symmetry in the SU(2) discrete Nahm equations.

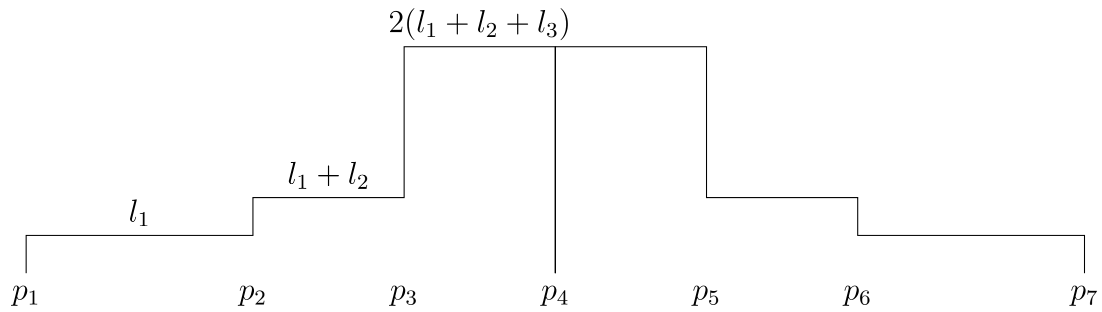
It remains to match up the mass and charge numbers of the new Sp(N) and SO(N) discrete Nahm equations with the mass and charge numbers from the old SU(N) discrete Nahm equations. The algebraic geometry and representation theory for the hyperbolic setting is identical to that of the euclidean setting as treated by Hurtubise-Murray [HM89]. Refer to Table 1 from Part 1.

In the euclidean case [HM89], the symmetry condition was arrived by a different means and was of the form

$$T_j(z) = C(z)T_j^T(z)C^{-1}(z).$$

In the euclidean case, these were smooth functions whereas since they are discretely indexed in the hyperbolic case, it is possible to row reduce them with the gauge freedom. Hence the two cases are morally the same but with the hyperbolic case being simpler.

It will be instructive to see the graph of matrix dimensions against interval number, sometimes known as a skyline diagram. Here is the diagram for SO(7) as a symmetrical SU(7) discrete Nahm equation.



CHAPTER 16

Conclusion

This thesis juxtaposes monopoles in euclidean space $\mathbb{R}^3$ with monopoles in hyperbolic space H^3. The main achievement here is the development of the discrete Nahm equations for all classical groups analogous to the Nahm equations for the classical groups. The rational map and spectral curve of a hyperbolic monopole was also discussed.

However, a significant difference between the euclidean and hyperbolic cases is the result that the asymptotic field of a hyperbolic monopole uniquely determines the monopole up to gauge freedom.

The moduli space of euclidean monopoles has a hyperKähler metric. No such hyperKähler metric has been found on the moduli space of hyperbolic monopoles. One usage of such a hyperKähler metric has been to study the dynamics of monopoles where the geodesics of the metric are assumed to be the scattering behaviour of monopoles in the low energy regime.

It has been conjectured that a metric on the space of U(1) connections and hence the asymptotic fields of hyperbolic monopoles could play the same role that the hyperKähler metric does in the euclidean case. Some evidence in favour of this can be found in the study of symmetric monopoles. This is a first possible avenue of future research that the thesis opens up.

As is often the case with ventures such as a PhD research project, there are loose ends which lie just outside the scope of the project which would have been achieved if there were more time or funding.

One such loose end are the conjectures at the end of the chapter on spectral curves in Part 3. Another example is the task of writing down an example of symmetric SU(4) discrete Nahm data from JNR data [BCS15] and the spectral curve equation that comes along with it.

Finally, I am confident that an example of G_2 discrete Nahm data can be guessed from studying the example of Shnir and Zhilin [SZ15].

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