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Hydrodynamique et intrication dans la correspondance AdS/CFT

Hydrodynamics and entanglement in the AdS/CFT correspondence

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Résumé

Nous présentons dans cette thèse deux applications de la correspondance AdS/CFT. La première est l'analyse des propriétés de transport des modes fermioniques dans les théories des champs fortement couplées. Nous étudions en particulier les propriétés de la constante de diffusion du phonino dans $\mathcal{N} = 4$ SYM à densité nulle ou finie. Nous trouvons que la constante de diffusion dépend du potentiel chimique et par conséquent qu'elle n'a pas une propriété d'universalité similaire à celle de la viscosité de cisaillement. La deuxième application traite du comportement de l'entropie d'intrication dans des théories qui contiennent des degrés de liberté massifs. Pour identifier les contributions de la masse à l'entropie d'intrication nous l'évaluons dans un système comprenant des branes de saveur et nous identifions certains des termes dépendants de la masse. Nous trouvons que le coefficient du terme logarithmique est différent de celui calculé dans la théorie des champs libre, un résultat qui est qualitativement en accord avec de résultats holographique antérieurs. De plus nous calculons d'autres termes, prédits dans la théorie des champs, mais qui n'ont pas été identifiés auparavant dans un système holographique.

Mots clés : Correspondance AdS/CFT, hydrodynamique, supergravité, entropie d'intrication, théorie des cordes, théorie conforme des champs

Abstract

In this thesis we present two applications of the AdS/CFT correspondence. The first one is the analysis of transport properties of fermionic modes in strongly coupled field theories. In particular we study the properties of the diffusion constant of the phonino in $\mathcal{N} = 4$ SYM at zero and finite density. We find that the diffusion constant depends on the chemical potential and therefore we conclude that it does not have a universality property analogous to the one of the shear viscosity. The second application deals with the behavior of entanglement entropy in theories with massive degrees of freedom. To identify the mass contributions to entanglement entropy we compute it in a setup with backreacted massive flavor branes and identify some of the mass dependent terms. We find that the logarithmic term has a different coefficient from the one computed in free field theory, a result that qualitatively agrees with previous holographic results. Furthermore we identify some other terms predicted in the field theory but not found before in a holographic setting.

Keywords : AdS/CFT correspondence, hydrodynamics, supergravity, entanglement entropy, string theory, conformal field theory

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Contents

Introduction

Since its original proposal by Maldacena [Maldacena, 1998] in 1997 the AdS/CFT correspondence has been applied to address questions in strongly coupled gauge theories (see [Casalderrey-Solana et al., 2011] for a recent review) and in condensed matter systems (see e.g. [Hartnoll, 2009]). The AdS/CFT correspondence relates a classical supergravity theory to a strongly coupled conformal field theory; in particular classical type IIB supergravity is conjectured to be dual to strongly coupled $\mathcal{N} = 4$ SYM theory. A dictionary has been established so that we can calculate observables on the gravity side and then "translate" them on the field theory side. In this thesis we will be interested in two problems: transport properties of fermonic modes in strongly coupled theories and the behavior of entanglement entropy in systems containing massive degrees of freedom. The work presented in this thesis was published in [Kontoudi and Policastro, 2012] and [Kontoudi and Policastro, 2013].

Using the AdS/CFT correspondence, a lot of work has been done in computing transport coefficients of strongly coupled field theories from the supergravity dual in the hydrodynamic limit (see [Policastro et al., 2001, Policastro et al., 2002a, Policastro et al., 2002b] or [Casalderrey-Solana et al., 2011] for a recent review). In this limit a field theory can be described by means of an effective theory, hydrodynamics, which contains some unspecified parameters, the transport coefficients. The value of these coefficients depends on the microscopic details of the theory and can be computed using the AdS/CFT correspondence. One of the most striking results of these investigations has been the discovery that the ratio of the shear viscosity to the entropy density is a universal quantity [Buchel and Liu, 2004], that has the same value of $1/4\pi$ in every theory that admits a description in terms of Einstein gravity coupled to matter fields. The universality is spoiled if the Lagrangian has higher-derivative terms, which means generically, that it holds only in the limit of infinitely strong coupling of the dual field theory [Cremonini, 2011]. The underlying reason for the universality is the uniqueness of Einstein gravity, or in other words the universal coupling of gravity to all matter fields; as a consequence, it can be shown that in any planar black hole background, the transverse graviton (i.e. with polarization parallel to the directions in which the horizon extends) satisfies the same equation of motion. The shear viscosity is related to the absorption of this mode by the black hole horizon; in [Das et al., 1997] it was proven that the absorption cross-section from a black hole is universal, something which underlies the universality of the shear viscosity. It is worth mentioning that the absorption rate was computed in [Das and Mathur, 1996] for D1-D5 branes and in [Klebanov, 1997] for D3, M2 and M5 branes and the comparison to the absorption rate of black holes was one of the many hints to the AdS/CFT correspondence . This result was put in a more general framework by [Iqbal and Liu, 2009b], who showed that for some transport coefficients, the relevant correlator has no radial flow, or it is independent of the radial direction in the low-energy limit; this happens whenever the fluctuating mode that couples to the operator is a massless scalar. In that case, the transport coefficient that is in principle computed on the boundary can also be computed on the horizon, and so typically it has less dependence on all the details of the model; but it is not enough to imply universality as there can still be a residual dependence of the "effective coupling" at the horizon.

Most of the work on the subject has been devoted to bosonic correlators, with relatively little attention paid to the fermions. In fact, it is natural to wonder whether the constraints imposed by supersymmetry may be sufficient to imply universality for the transport of supersymmetry charges. Trying to answer this question we computed in [Kontoudi and Policastro, 2012] the diffusion constant of the phonino, an excitation related to fluctuations of the supercharge density, in a finite density setup. Before doing the computation at finite density we developed a more efficient way of calculating the diffusion constant using a Kubo formula. This formula relates the diffusion constant to the retarded correlator of the transverse component of the gravitino which decouples from the rest of the fields. Using this method we confirmed the result found in [Policastro, 2009] by computing the retarded correlator in a black brane background.

The background we used for the finite density calculation is the so called STU black hole. It is an asymptotically AdS charged black hole which was originally found as a solution of $\mathcal{N}=2$, $d=5$ supergravity. This solution can be embedded in type IIB supergravity and is a dual of $\mathcal{N}=4$ SYM with three charges belonging to a subgroup of the SU(4) R-symmetry group. We examined two different configurations of the three charges (one charge turned on or three equal charges) and obtained qualitatively similar results. The diffusion constant depends on the charges and its value approaches the zero density result for $T/\mu \gg 1$ as expected. Therefore our analysis showed that the diffusion constant is not universal.

Entanglement is one of the phenomena characterizing quantum systems. We can define an entropy associated to it, the entanglement entropy, which is a measure of the entanglement between two subsystems of the system of interest. Typically we divide the quantum system in two subsystems which correspond to two different regions of space. One of the most distinctive properties of entanglement entropy is that it provides non-local information on the system; its behavior can be used for instance to identify topological order or the presence of Fermi surface. The behavior of entanglement entropy has been analyzed in conformal field theories [Calabrese and Cardy, 2004] where the correlation length is infinite. It is natural then to ask what is the effect of a finite correlation length on entanglement entropy. The introduction of a finite correlation length can be realized by the introduction of a massive excitation on the theory; corrections arising from massive fields have been presented in [Hertzberg and Wilczek, 2011] and [Hertzberg, 2013].

There exists a way to compute entanglement entropy in the framework of the AdS/CFT correspondence [Nishioka et al., 2009] and since the correspondence provides us with the strong coupling behavior of the field theory it is interesting to see how entanglement entropy depends on the coupling. We concentrated on the behavior of terms induced by massive degrees of freedom. In [Kontoudi and Policastro, 2013] we used this holographic prescription to compute the entanglement entropy for $\mathcal{N}=4$ $U(N)$ SYM coupled to N_f massive hypermultiplets; this is the theory that lives at the intersection of N_c D3 and N_f D7 branes [Karch and Katz, 2002]; in the regime $N_f \ll N_c$ the theory has a dual description in terms of probe D7 branes in AdS_5. In order to see the contribution of the flavor fields to entanglement entropy we need to go beyond the probe (quenched) approximation and include the backreaction of the D7 branes (although at leading order it would also be possible to do the calculation by making use of the formula proposed in [Chang and Karch, 2013, Jensen and O'Bannon, 2013]). The backreacted solutions are known perturbatively in $\epsilon = N_f/N_c$ [Bigazzi et al., 2009a]. Our analysis showed that the results at strong coupling differ from the weak coupling results calculated in field theory in accordance with other holographic computations [Lewkowycz et al., 2013, Bea et al., 2013].

This thesis is organized as follows: In the first chapter we present the basic elements of the AdS/CFT correspondence such as superconformal field theories, string theory and supergravity. Then we state the correspondence and focus on some computational tools that we will use subsequently for the computations. Finally we present the generalizations of the correspondence to theories with finite temperature, finite density and theories with flavor degrees of freedom giving particular attention to the backreaction of flavors. In the second chapter we treat the subject of holographic hydrodynamics; we review hydrodynamics as an effective theory, we report some of the most important results of hydrodynamics in the AdS/CFT correspondence and then we present the computation of the phonino diffusion constant which was published by the author in [Kontoudi and Policastro, 2012]. In the third chapter we deal with entanglement entropy; after reviewing the concept of entanglement entropy we present field theory computations for theories with massive fields. Then we present the holographic prescription for the computation of entanglement entropy and finally we present the author's work [Kontoudi and Policastro, 2013] on the entanglement entropy of theories with massive flavors. In the last chapter we present the conclusions of our work and describe the perspectives of future work on the subject.

Chapter 1

Elements of the AdS/CFT correspondence

This chapter aims at introducing the AdS/CFT correspondence starting from a brief description of the two sides (conformal field theories and string theory) and then moving on to describe basic computational tools and generalizations that are useful for the problems that we will treat later.

1.1 Superconformal field theories

A superconformal field theory is a theory that is both conformal and supersymmetric. The first ingredient of a conformal field theory is scale invariance, which explains why they describe systems at the critical point of phase transitions. We will see however that the symmetries of conformal field theories are more involved than just scale invariance [1]. Moreover in two dimensions the algebra of conformal symmetries becomes infinite dimensional and therefore the theory has an infinite number of conserved charges which makes it integrable. A complete introduction to the subject can be found in [Francesco et al., 1997].
Scale invariance means that the theory is invariant under a scale transformation given by:

$$x^\mu \to \lambda x^\mu \,, \tag{1.1}$$

where $\mu = 0, ..., 4$. Scale transformations, also called dilatations, belong to a larger group of transformations: the conformal group. Its algebra is given by:

$$\begin{aligned}
[L_{\mu\nu}, P_\rho] &= -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} - \eta_{\mu\rho}L_{\nu\sigma} + \eta_{\nu\sigma}L_{\rho\mu} - \eta_{\mu\rho}L_{\rho\nu}) \\
[D, P_\mu] &= -iP_\mu \\
[L_{\mu\nu}, K_\rho] &= -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \\
[D, K_\mu] &= iK_\mu \\
[P_\mu, K_\nu] &= 2iL_{\mu\nu} - 2i\eta_{\mu\nu}D \,,
\end{aligned} \tag{1.2}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, $L_{\mu\nu}$ is the generator of Lorentz transformations, P_μ is the generator of translations, D is the generator of scaling transformations and K_μ the

1. In fact, for 4-dimensional theories, if we impose unitarity in addition to scale invariance the theory will also be conformally invariant in most cases [Dymarsky et al., 2013].

generator of special conformal transformations which are given by:

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2a^\mu x_\mu + a^2 x^2} \,. \tag{1.3}$$

The algebra (1.2) is isomorphic to the algebra of the group $SO(4,2)$. All of the conformal transformations leave the metric invariant up to an overall multiplicative factor:

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}(x) \,. \tag{1.4}$$

Fields in conformal field theories are characterized by their conformal dimension Δ. The operators of the lowest dimension in a given representation of the conformal algebra are called primary operators. The conformal dimensions are given by the eigenvalues $-i\Delta$ of the scaling operator D. Conformal invariance imposes strong restrictions to the form of the correlation functions; for a scalar field of dimension Δ for instance, the two-point function is determined up to an overall constant factor

$$\langle \phi(x)\phi(0)\rangle \propto \frac{1}{x^{2\Delta}} \,. \tag{1.5}$$

The two-point function vanishes for fields of different scaling dimensions. For higher spin fields $\mathcal{O}^I(x)$ the two-point function is given by:

$$\langle \mathcal{O}^I(x)\mathcal{O}^J(0)\rangle \propto \frac{D^I_K\,(I(x))\,g^{KJ}}{x^{2\Delta}} \,, \tag{1.6}$$

where $D^I_K(R)$ form a representation of $SO(3,1)$, g^{IJ} is the invariant tensor of the representation and $I(x)$ is the inversion operator which in the vector representation reads:

$$I^\nu_\mu = \delta^\nu_\mu - 2\frac{x_\mu x^\nu}{x^2} \,. \tag{1.7}$$

Conformal symmetry also restricts the form of 3-point functions up to an overall function. An account on the tensor structure of correlation functions in conformal field theories in general dimension can be found in [Osborn and Petkou, 1994].

Another ingredient we need to construct superconformal field theories in supersymmetry: a symmetry relating bosons to fermions (standard references are [Wess and Bagger, 1992] and [Gates et al., 1983]), its generators Q^a_α and $\bar{Q}_{\dot{\alpha}a}$ are respectively left and right Weyl spinors. The greek indices are spinor indices and $a = 1, ..., \mathcal{N}$ where $\mathcal{N}$ is the number of independed supersymmetries. The supersymmetry algebra is given by

$$\{Q^a_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}b}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^a_b \qquad \{Q^a_\alpha, Q^b_\beta\} = 2\epsilon_{\alpha\beta} Z^{ab} \,, \tag{1.8}$$

where $\sigma^\mu_{\alpha\dot{\beta}}$ are the Pauli matrices. The Z^{ab} are called central charges and vanish in the case $\mathcal{N} = 1$. For $\mathcal{N} > 1$ the supersymmetry generators can be rotated into one another under the group $SU(\mathcal{N})$ without affecting the algebra; this symmetry of the algebra is known as R-symmetry.

Superconformal symmetry is a symmetry incorporating both conformal symmetry and supersymmetry. It is realized by supplementing the algebras (1.2) and (1.8) with the superconformal symmetry generators $S_{\alpha a}$ and $\bar{S}^a_{\dot{\alpha}}$. The part of the algebra that combines the superconformal generators with the conformal and supersymmetry algebras is given by

$$\begin{aligned} \{S_{\alpha a}, \bar{S}^b_{\dot{\beta}}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} K_\mu \delta^b_a \,, \\ \{Q^a_\alpha, S_{\beta b}\} &= \epsilon_{\alpha\beta}(\delta^a_b D + T^a_b) + \frac{1}{2}\delta^a_b L_{\mu\nu}\sigma^{\mu\nu}_{\alpha\beta} \,, \end{aligned} \tag{1.9}$$

where T^a_b is the generator of R-symmetries and is in the adjoint representation of $SU(\mathcal{N})$. The generators Q^a_α and $\bar{S}^a_{\dot{\alpha}}$ transform in the fundamental of the R-symmetry group whereas $\bar{Q}_{\dot{\alpha}a}$ and $S_{\alpha a}$ transform in the conjugate representaion.

1.1.1 $\mathcal{N} = 4$ SYM

Let's concentrate now on a particular superconformal field theory, $\mathcal{N} = 4$ Super Yang-Mills (SYM). The Lagrangian of the theory is given by [Grimm et al., 1978] :

$$\begin{aligned} \mathcal{L} = \mathrm{tr} \Bigg\{ &-\frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - \sum_i D_\mu X^i D^\mu X^i \\ &+ \sum_{a,b,i} g C^{ab}_i \lambda_a [X^i, \lambda_b] + \sum_{a,b,i} g \bar{C}_{iab} \bar{\lambda}^a [X^i, \bar{\lambda}^b] + \frac{g^2}{2} \sum_{i,j} [X^i, X^j]^2 \Bigg\} \end{aligned} \tag{1.10}$$

where $F_{\mu\nu}$ is the field strength associated to the gauge field A_μ, λ^a_α, $a = 1, ..., 4$ are left Weyl spinors and X^i, $i = 1, ..., 6$ are real scalars. Under the $SU(4)_R$ symmetry, A_μ is a singlet, λ^a_α is a **4** and the scalars X^i are a rank 2 anti-symmetric **6**. All the fields are in the adjoint representation of the gauge group and form the gauge multiplet of the supersymmetry algebra. The constants C_{iab} are related to Clifford Dirac matrices for $SU(4)_R$.

The Lagrangian is invariant under the $\mathcal{N} = 4$ supersymmetry transformations given by

$$\begin{aligned} \delta X^i &= [Q^a_\alpha, X^i] = C^{iab}\lambda_{\alpha b} \\ \delta\lambda_b &= \{Q^a_\alpha, \lambda_{\beta b}\} = F^+_{\mu\nu}(\sigma^{\mu\nu})^\alpha_\beta \delta^a_b + [X^i, X^j]\epsilon_{\alpha\beta}(C_{ij})^a_b \\ \delta\bar{\lambda}^b_{\dot{\beta}} &= \{Q^a_\alpha, \bar{\lambda}^b_{\dot{\beta}}\} = C^{ab}_i \bar{\sigma}^\mu_{\alpha\dot{\beta}} D_\mu X^i \\ \delta A_\mu &= [Q^a_\alpha, A_\mu] = (\sigma_\mu)^{\dot{\beta}}_\alpha \bar{\lambda}^a_{\dot{\beta}} \end{aligned} \tag{1.11}$$

As we already mentioned this theory is conformal and we can check this by assigning the usual mass dimensions to the fields and couplings:

$$[A_\mu] = [X^i] = 1 \qquad [\lambda_a] = 3/2 \qquad [g] = [\theta_I] = 0 \,.$$

All the terms in the Lagrangian have dimension four and so the theory is scale invariant on the classical level. Moreover the renormalization group β-function of the theory

vanishes and so the theory is also scale invariant at the quantum level.
The $\mathcal{N}=4$ SYM theory is invariant under the full superconformal group which in this case is $SU(2,2|4)$ containing the conformal group in 4 dimensions $SO(2,4)\sim SU(2,2)$, the R-symmetry group $SO(6)\sim SU(4)$ and their supersymmetric extensions. We can organize the field content of the theory under representations of this group. We can build up a representation starting from a *superconformal primary* operator and acting on it with the supersymmetry generator Q^a_α. A superconformal primary operator $\mathcal{O}$ has the lowest dimension in a representation and it is defined to be a non vanishing operator such that:

$$[S,\mathcal{O}]_\pm = 0 \qquad \mathcal{O}\neq 0\,. \tag{1.12}$$

The simplest form of gauge invariant superconformal primary operators are single trace operators of the form:

$$\mathrm{str}\left(X^{i_1}X^{i_2}...X^{i_n}\right) \tag{1.13}$$

where i_j, $j=1,...,n$ are indices in the fundamental of $SO(6)_R\sim SU(4)_R$ and str denotes the symmetrized trace over the gauge algebra. Operators of this form play an important role in the AdS/CFT correspondence.

1.2 String theory and supergravity

In this section we give a very brief account of string theory, supergravity and their relation. String theory is a very wide subject and we choose here to focus only on the topics relevant for what is treated in the following; standard references are [Polchinski, 1998] and [Green et al., 1988] and the more modern accounts [Becker et al., 2007] and [Kiritsis, 2011]. The starting point of string theory is a classical string propagating in space time; there are open and closed strings and as they propagate they span a surface in spacetime: the worldsheet. There are many realisations of string theory but we will concentrate here on type IIB string theory which is one of the realizations of superstring theory, incorporating both bosons and fermions. We can think about strings from two points of view: from the spacetime point of view it contains vectors, such as the coordinates of the string X^μ; from the worldsheet point of view these spacetime vectors correspond to bosonic or fermionic fields living on the worldsheet. Upon quantization of the theory we obtain a spectrum of excitations corresponding to particles. The masses of the excitations turn out to be proportional to the length of the strings.
The bosonic part of the type IIB string theory worldsheet action, describing closed string excitations is given by:

$$S_x = \frac{1}{4\pi\alpha'}\int\sqrt{\gamma}\left\{\left[\gamma^{ab}G_{\mu\nu}(X)+\epsilon^{\mu\nu}B_{\mu\nu}(X)\right]\partial_a X^\mu\partial_b X^\nu+\alpha' R^{(2)}_\gamma\Phi(X)\right\} \tag{1.14}$$

where $G_{\mu\nu}$ is the metric, $B_{\mu\nu}$ is an antisymmetric rank 2 tensor, Φ a scalar field, the dilaton, γ_{ab} is the induced metric on the worldsheet, $R^{(2)}_\gamma$ the associated curvature and α' is the square of the Planck length. Closed string interactions are organized in powers of the string coupling constant g_s which is associated to the expectation value of the

dilaton ϕ as $g_s = e^{\phi}$.
The worldsheet action in (**??**) defines a two dimensional non-linear sigma model which could in principle be defined for any choice of the background fields $G_{\mu\nu}, B_{\mu\nu}$ and Φ. In order for this action to define a consistent string theory however, we need to impose worksheet conformal invariance which in turn implies scale invariance of the the action at the quantum level. From the two-dimensional point of view, the background fields can be regarded as an infinite number of coupling constants. We can therefore compute the renormalization group β-functions for the background fields [2] and require that they vanish to impose scale invariance. The β-functions are computed order by order in α' and the leading contribution reads [Callan et al., 1985]:

$$\begin{aligned}\beta^G_{\mu\nu} &= \frac{1}{2}R_{\mu\nu} - \frac{1}{8}H_{\mu\rho\sigma}H^{\rho\sigma}_{\nu} + \partial_\mu\Phi\partial_\nu\Phi + \mathcal{O}(\alpha') \\ \beta^B_{\mu\nu} &= -\frac{1}{2}D_\rho H^\rho_{\mu\nu} + \partial_\rho H^\rho_{\mu\nu} + \mathcal{O}(\alpha') \\ \beta^\Phi &= \frac{1}{6}(D-10) + \mathcal{O}(\alpha')\,,\end{aligned} \tag{1.15}$$

where $H_{\mu\rho\sigma}$ is the field strength of $B_{\mu\nu}$. The last equation simply tells us that superstring theory lives in 10 space-time dimensions. The equations $\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = 0$ turn out be the equations of motion arising from the type IIB *supergravity* action. Given the fact that the β-functions were computed to first order in α', which is the loop expansion parameter of the non-linear sigma model we started from, it is fair to say that supergravity is a low energy approximation of string theory. Moreover string theory can provide us with higher order corrections in α' to the supergravity equations of motion, which for dimensional reasons will also be higher derivative corrections in x^μ.

We are particularly interested in the $\mathcal{N} = 2$, $D = 10$ type IIB supergravity since it will constitute one of the basic elements of the AdS/CFT correspondence. Note however that there are many supergravity theories which are constructed independently of string theory; a rather detailed account can be found in [Van Nieuwenhuizen, 1981] or in the more recent [Freedman and Van Proeyen, 2012]. The massless fields contained in the theory are: two left-handed Mayorana-Weyl gravitini, two right-handed Mayorana-Weyl dilatinos, the metric $g_{\mu\nu}$, the two form $B_{\mu\nu}$, the dilaton Φ and the form fields [3] C_0, C_2, C_4. The bosonic part of the action is given by

$$\begin{aligned}S =& \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}\left[e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right)\right. \\ &\left.-\frac{1}{2}|F_1|^2 - \frac{1}{2}|\tilde{H}_3|^2 - \frac{1}{4}|\tilde{F}_5|^2\right] - \frac{1}{4\kappa_{10}^2}\int\, C_4\wedge H_3\wedge F_3\,.\end{aligned} \tag{1.16}$$

We define $F_{n+1} = dC_n$, $H_3 = dB_2$, $\tilde{F}_3 = F_3 - C_0H_3$ and $\tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3$. In addition to the equations of motion that follow from the action we must impose the

2. Actually we should rather be talking about β functionals since we have a β function in each space-time point

3. We denote form fields by their rank.

self-duality condition on F_5, i.e. $F_5 = *F_5$. The 10 dimensional gravitational constant κ_{10} is related to the Newton constant in 10 dimensions G_N^{10} by $\kappa_{10}^2 = 8\pi G_N^{(10)}$. Notice that the curvature term in the action is not of the usual form of the Einstein–Hilbert action for gravity; this is because the action is written in the string frame. To recover the familiar form of the action we may write it in the Einstein frame by rescaling the metric as $G^E_{\mu\nu} = e^{-\Phi/2}G_{\mu\nu}$. We reported here the bosonic part of the action and we will be working most of the time with bosonic solutions; for such solutions the fermionic fields are consistently set to zero, i.e. their variation under supersymmetry transformations must also vanish. The solutions are usually not invariant under all of the supersymmetry transformations of the theory and the requirement that the variation of the fermonic fields vanishes specifies the amount of supersymmetry preserved by the solutions.
An important class of supergravity solutions are p-branes which are solutions containing a flat $(p+1)$-dimensional hypersurface with Poincaré invariance group $\mathbb{R}^{p+1} \times SO(1,p)$. The transverse space is then $(9-p)$-dimensional and we can always find maximally symmetric solutions with $SO(9-p)$ symmetry group. Denoting the coordinates parallel to the brane by x^μ, $\mu = 0, 1, ..., p$ and the coordinates perpendicular to the brane by $y^a = x^{p+a}$, $a = 1, ..., 9-p$ the p-brane solution in the string frame can be expressed in terms of a single function as follows:

$$ds^2 = H(\vec{y})^{-1/2}dx^\mu dx_\mu + H(\vec{y})^{1/2}d\vec{y}^2 \tag{1.17}$$

$$e^\Phi = H(\vec{y})^{(3-p)/4} \tag{1.18}$$

$$F_{p+2} = -d(H^{-1}) \wedge \epsilon_{1,p} \tag{1.19}$$

where $\epsilon_{1,p}$ is the volume form parallel to the branes and the function H is a harmonic function given by:

$$H(y) = 1 + \frac{R^{7-p}}{y^{7-p}} \tag{1.20}$$

These solutions are supersymmetric (they only break half of the supersymmetries of the theory) and are called *extremal* solutions.
The solutions that break supersymmetry are called *non-extremal* and we give here a particular example of such non-extremal p-brane solutions [Horowitz and Strominger, 1991]:

$$ds^2_{10} = \frac{-f(r)dt^2 + dx_p^2}{\sqrt{H(r)}} + \sqrt{H(r)}\left(\frac{dr^2}{f(r)} + r^2 d\Omega^2_{8-p}\right) \tag{1.21}$$

$$H(r) = 1 + \frac{R^{7-p}}{r^{7-p}} \qquad f(r) = 1 - \frac{r_0^{7-p}}{r^{7-p}} \tag{1.22}$$

where r_0 is the horizon radius. This solution is a black brane generalization of the extremal solution.

So far we have been looking at closed strings and the corresponding low energy effective action, we now turn to open strings. In order to determine the motion of an open string in space-time we need to impose boundary conditions on the endpoints of the string; we can impose either Dirichlet or Neumann boundary conditions. We can choose to impose mixed boundary conditions in the following way:

$$\partial_\perp X^\mu = 0 \qquad X^{p+a} = 0\,, \tag{1.23}$$

where $\mu = 0, 1, ..., p$ and $a = 1, ..., 9 - p$. These boundary conditions define a $(p+1)$-dimensional defect where the endpoints of open strings are attached: a Dp-brane. An introduction to the subject of Dp-branes in string theory can be found in [Bachas, 1998]. The endpoints of open strings can couple to gauge fields and propagate in a background created by the closed string modes $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ. We can consider the non-linear sigma model describing this effect and compute the renormalizarion group β-functions just as we did for the closed strings. The equations that arise from the requirement of scale invariance, i.e. the vanishing of the beta functions, are equivalent to the equations of motion of the Dirac-Born-Infeld (DBI) action:

$$S_{\rm DBI} = -T_{Dp} \int d^{p+1}\xi \; e^{-\Phi} \sqrt{-\det(\gamma_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta} + \hat{B}_{\alpha\beta})} \tag{1.24}$$

where ξ are the coordinate on the Dp-brane, $\gamma_{\alpha\beta}$ is the induced metric on the Dp-brane, $\hat{B}_{\alpha\beta}$ is the pullback of the 2-form $B_{\mu\nu}$ and $F_{\alpha\beta}$ is the field strength of the gauge field living on the Dp-brane. The tension of the brane is related to the string coupling g_s and the Planck length ℓ_P by

$$T_{Dp} = \frac{1}{(2\pi)^p g_s \ell_P^{p+1}} \tag{1.25}$$

Minimizing this action in a given closed string background we can find the way the branes are embedded in spacetime.
In the simple case where the background metric is flat and the $B_{\mu\nu}$ field is absent it is easy to see that for small field strengths $F_{\alpha\beta} \ll 1/\alpha'$ we recover Maxwell's action. An expansion in powers of α' yields:

$$S = -T_{Dp} \int d^{p+1}\xi \left(1 + \frac{(2\pi\alpha')^2}{4} F_{\alpha\beta}F^{\alpha\beta}\right)\,. \tag{1.26}$$

1.3 Statement of the correspondence

We saw in the previous section that the low energy description of closed strings admits p-brane solutions and that in open string theory there are extended objects where open strings end, the Dp-branes; the AdS/CFT correspondence emerges from the identification of these objects in a certain limit and configuration. We report here the most relevant points for the rest of the presentation; a more complete description can be found in [D'Hoker and Freedman, 2002] and [Aharony et al., 2000].

Let's start by considering N_c coincident 3-branes; then the metric given in (1.17) takes the form:

$$ds^2 = \left(1 + \frac{R^4}{y^4}\right)^{-1/2} dx_\mu dx^\mu + \left(1 + \frac{R^4}{y^4}\right)^{1/2} \left(dy^2 + y^2 d\Omega_5^2\right) \tag{1.27}$$

with R given by

$$R^4 = 4\pi g_s N_c (\alpha')^2 \,. \tag{1.28}$$

In the limit $y \gg R$ the metric becomes the flat ten dimensional metric. On the other hand, when we take the limit $y \ll R$, called the near horizon limit because the metric has a horizon at $y = 0$, the metric appears to be singular. This singularity is not essential and we can define a coordinate

$$z \equiv R^2/y \tag{1.29}$$

which allows us to bring the metric in the following form for large z :

$$ds^2 = R^2 \left(\frac{1}{z^2} dx_\mu dx^\mu + \frac{dz^2}{z^2} \right) + R^2 d\Omega_5^2 \,. \tag{1.30}$$

The first term of this metric is the AdS space and the second one is the five-sphere S^5 both having radius R. The AdS space is a solution of Einstein gravity with negative cosmological constant.
We want now to keep only the low energy excitations arising in this spacetime. An observer at $y \to \infty$ in the background (1.27) measures the energy of a local excitation located at some constant radius $E(y = \text{const})$ as

$$E(y \to \infty) = \left(1 + \frac{R^4}{y^4} \right)^{-1/4} E(y = \text{const}) \,. \tag{1.31}$$

This implies that the energy varies with the radius and in particular that excitations near the horizon appear to have low energy even if locally they have large energy. Excitations that travel through the bulk of the space must have very small energy in order to be excited in the low energy limit. In the full theory, excitations in the bulk and on the p-branes interact, but in the low energy approximation the two sectors decouple. So to summarize we have two regions, the flat space and the $AdS_5 \times S^5$ near horizon region which are decoupled. The consecutive approximations we did starting from closed string theory are depicted in the following diagram:

$$\begin{matrix}\text{closed string}\\ \text{theory}\end{matrix} \xrightarrow[\text{energy}]{\text{low}} \begin{matrix}\text{p-brane}\\ \text{supergravity}\\ \text{solutions}\end{matrix} \xrightarrow[\text{excitations}]{\text{low energy}} \boxed{\begin{matrix}\text{supergravity}\\ \text{in}\\ AdS_5 \times S^5\end{matrix}} + \begin{matrix}\text{supergravity}\\ \text{in flat space}\end{matrix}$$

Let's move on now to consider N_c coincident $D3$-branes in string theory. When there is just one brane in the theory we can imagine that all the strings are attached to this brane as shown in fig.1.2 (a). The strings can have arbitrarily small length and therefore the excitations described by them are massless; this gives rise to a $U(1)$ gauge field living on the D-brane. When we have N_c branes, there are strings stretching between two different branes (fig.1.2 (b)) and we have massive excitations; the gauge fields living on the branes form a $U(1)^{N_c}$ gauge group. Finally if the branes are coincident (fig.1.2 (c)) the gauge group gets enhanced to $SU(N_c)$ and the excitations become massless.
The low energy limit of this configuration is described by an action of the form:

$$S = S_{\text{bulk}} + S_{\text{DBI}} + S_{\text{interaction}} \tag{1.32}$$

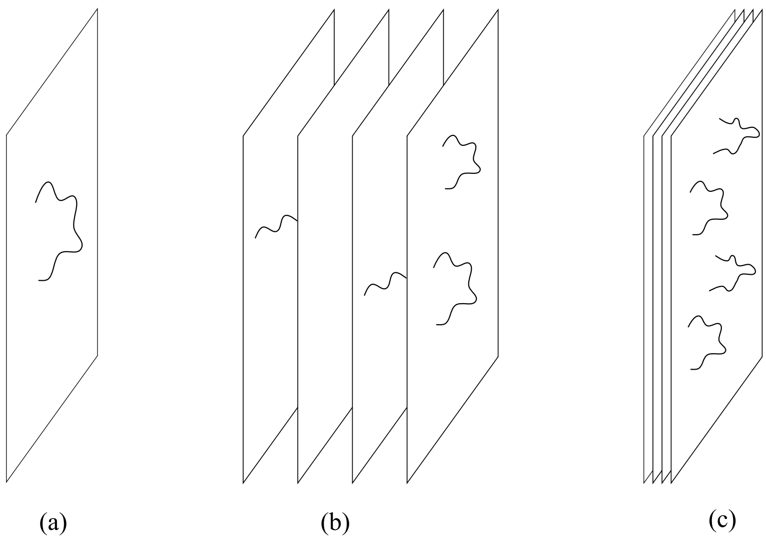

Figure 1.1: A schematic representation of D-branes: (a) a single D-brane, (b) a stack of separated D-branes, (c) a stack of coincident D-branes.

where S_{bulk} is the action describing closed strings propagating in the bulk and the DBI action is a supersymmetric non-abelian version of the action (1.24) with gauge group $SU(N_c)$. Integrating out massive modes and imposing a cutoff $\sim 1/\alpha'$ we will be left with the following ingredients: open strings propagating in flat spacetime described by a supergravity action of the form (1.16) in the bulk and $\mathcal{N}=4$ SYM coming from the small α' the DBI action. In analogy to what we saw in the previous section concerning the Maxwell action for the simplified version of the DBI action, the small α' approximation of the generalized supersymmetric DBI action appearing here gives rise to the $\mathcal{N}=4$ SYM theory with gauge group $SU(N_c)$. Finally the interactions between the bulk and the branes are negligible in the massless limit. We summarize the approximations we made in the following diagram:

D-branes in open string theory $\xrightarrow[\text{energy}]{\text{low}}$ DBI action $\xrightarrow[\text{modes}]{\text{massless}}$ $\boxed{\mathcal{N}=4 \text{ SYM on D3-branes}}$ + supergravity in flat space

We saw that starting from the two different viewpoints of string theory, open and closed strings, and considering the low-energy massless excitations we recovered two different theories each of them containing two decoupled sectors. Comparing the two results we see that supergravity in flat space appears in both so it is tempting to conjecture that also the other sectors are dual to each other. In fact, Maldacena conjectured that they are dual not only in the low energy limit but in the full quantum theory at all energies. The AdS/CFT correspondence [Maldacena, 1998] states that the two following theories are dual:

– Type IIB superstring theory in $AdS_5 \times S^5$, where both AdS_5 and S^5 have the same radius R, $N_c = \int_{S^5} F_5^+$ and the string coupling is g_s.

– $\mathcal{N} = 4$ SYM theory in 4 dimensions, with gauge group $SU(N_c)$ and Yang-Mills coupling g_{YM}.

The parameters of the two theories are identified as follows:

$$g_s = g_{YM}^2 \qquad R^4 = 4\pi g_s N_c \alpha'^2. \tag{1.33}$$

This is the statement of the correspondence in its strong form. Solving full string theory in curved backgrounds is a very difficult problem and in practice we use some versions of the correspondence that are easier to deal with.
A second form of the correspondence is established by taking the so called t'Hooft limit:

$$\lambda \equiv g_{YM}^2 N_c = \text{fixed}; \qquad N_c \to \infty. \tag{1.34}$$

In this limit the field theory $\mathcal{N} = 4$ SYM is expanded in $1/N_c$ and represents a topological expansion of the theory's Feynman diagrams ['t Hooft, 1974]. The $1/N_c$ expansion is conjectured to correspond to an expansion in g_s in the string theory.
Finally the weakest form of the correspondence concerns the large t'Hooft coupling limit. In this limit the field theory can be expanded in powers of $\lambda^{-1/2}$ corresponding to an α' expansion in the string theory side. Thus, to the leading order, this weaker form of the correspondence states that type IIB supergravity is dual to the large λ limit of the $\mathcal{N} = 4$ SYM theory. In what follows we will deal only with this weaker form of the correspondence.

Some evidence for the correspondence is provided by the fact that the symmetries of the two dual theories match; we can identify the corresponding representations of the symmetry groups and formulate an operator - field correspondence between operators in the field theory and fields in supergravity. Recall that $\mathcal{N} = 4$ has $SO(2, 4)$ conformal symmetry group and $SU(4) \sim SO(6)$ R-symmetry group. These symmetry groups correspond to the isometries of the AdS_5 space $SO(2, 4)$ and of the five-sphere $SO(6)$ on the supergravity side. The correspondence also extends to the supersymmetric extensions of the bosonic symmetry algebras. The fields $\Phi(z, x, y)$ on the supergravity side are propagating in the background (1.30). We decompose the fields in a series on S^5

$$\Phi(z, x, y) = \sum_{\Delta} \Phi_\Delta(z, x) Y_\Delta(y)\,, \tag{1.35}$$

where Y_Δ is a series of spherical harmonics on S^5. The fields are labelled by Δ just like operators in the field theory are labelled by their conformal dimension. The compactification on the five-sphere gives rise to a contribution to the mass of the fields which depends on the spin of the field. For various spin we find the following relations between the scaling dimension and the mass of the fields:

$$\begin{array}{ll} \text{scalars} & m^2 = \Delta(\Delta - 4) \\ \text{spin } 1/2,\ 3/2 & |m| = \Delta - 2 \\ \text{p-form} & m^2 = (\Delta - p)(\Delta + p - 4) \\ \text{spin } 2 & m^2 = \Delta(\Delta - 4). \end{array} \tag{1.36}$$

In general a gauge invariant single trace operator $\mathcal{O}(x)$ of dimension Δ is dual to a supergravity field that lives in the bulk of AdS. The fields Φ_Δ live in the 5-dimensional AdS space but the operators on the field theory live in 4 spacetime dimensions. In fact the field theory operators are considered to be living in the 4-dimensional boundary of AdS space at $z = 0$. Because of the fact that AdS/CFT correspondence provides a duality between theories that live in a different number of dimensions it is also called holographic correspondence. Furthermore it is believed that a general CFT in d dimensions is dual to an asymptotically AdS gravity theory in $d+1$ dimensions.

1.3.1 Computing correlators

To fix the ideas on the correspondence we describe now the relation between correlators on the two sides of the correspondence in Euclidean signature. To compute correlators in field theory we can deform the action by adding a source term

$$\int d^4x\, J(x)\mathcal{O}(x) \tag{1.37}$$

and compute correlation functions by differentiating with respect to the source $J(x)$. Notice that in order for this term to be dimensionless the operator $J(x)$ must be $(\Delta-4)$-dimensional. The generating functional is then given by

$$Z[J] = \left\langle \exp\left(-\int d^4x J\mathcal{O}\right)\right\rangle . \tag{1.38}$$

The correspondence then states that, on the supergravity side, the generating functional is given by:

$$Z[J] = \exp\left(-S^{\text{on-shell}}[\phi]\right) \tag{1.39}$$

where $S^{\text{on-shell}}$ is the on-shell supergravity action and the source $J(x)$ is related to one of the modes of the field dual to $\mathcal{O}(x)$ near the boundary as we will see shortly. Then we can compute the 2-point function as:

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = -\frac{\delta S^{\text{on-shell}}}{\delta J(x)\delta J(y)} . \tag{1.40}$$

Consider now a concrete example, a scalar operator of dimension Δ which is dual to a massive scalar field. The action for the scalar field will come from the supergravity action and we assume that it will have the form:

$$S = \frac{\mathcal{N}}{2}\int d^5x\,\sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right) , \tag{1.41}$$

where the normalization factor $\mathcal{N}$ depends on the specific supergravity theory. The scalar field obeys the equation $(-\Box + m^2)\phi = 0$ and by assuming a solution of the form $z^\alpha\phi(x)$ near the boundary we identify the following asymptotic behavior:

$$\phi(z,x) \approx z^\Delta A(x) + z^{4-\Delta}B(x). \tag{1.42}$$

The term in the expansion which is identified with the source depends on the dimension of the operator. For an operator of dimension $\Delta = 0$ for instance we identify the source with $A(x)$ whereas for an operator of dimension $\Delta = 3$ we identify the source with $B(x)$. In any case the subleading term on the boundary is identified with the expectation value of the operator and the leading one with the source. We will now compute the 2-point function for this last case, an operator of dimension $\Delta = 3$. We start by doing a Fourier transform of the field

$$\phi(z,x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \phi(z,k). \tag{1.43}$$

The equations of motion are solved by an ansatz of the form:

$$\phi(z,k) = f_k(z)\phi_o(k) \tag{1.44}$$

with $f_k(z) \approx z$ at $z \to 0$ and $\phi_o(z)$ is to be identified with the source term at the boundary. Furthermore we demand regularity in the interior of the space. Solving the equation for $f_k(z)$ and plugging in the action we are left only with a term on the boundary which will have the general form:

$$S^{\text{on-shell}} = -\frac{1}{2} \int\limits_{z=\epsilon} \frac{d^4k}{(2\pi)^4} \phi_o(-k) \mathcal{F}(z,k_\mu) \phi_o(k) \tag{1.45}$$

with

$$\mathcal{F}(z,k_\mu) = \mathcal{N} \sqrt{g} g^{zz} f_{-k} \partial_z f_k. \tag{1.46}$$

The on-shell action diverges near the boundary and one needs to subtract some counterterms on the regulator surface $z = \epsilon$ to make the result finite. These counterterms must be written in a covariant way on the boundary. The procedure of subtracting these counterterms is called *holographic renormalization* and it is used every time one needs to compute the on-shell action. We will present an example in the next section where we will explicitly compute the regularized gravitational on-shell action; the general procedure is presented in [Skenderis, 2002, Bianchi et al., 2002]. After regularizing the on-shell action we can take two derivatives with respect to the source ϕ_o to compute the correlator and we find:

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_E = \mathcal{F}(z,k_\mu)\Big|_{z\to 0}^{\text{finite}} \tag{1.47}$$

where we put the subscript E to remind that we are in euclidean signature. There is an equivalent rewriting of the correlator if one notices that the conjugate momentum of the scalar field Π is related to $\mathcal{F}$ by the simple relation

$$\Pi(z,k_\mu) = \sqrt{g} g^{zz} \partial_z f_k \phi_o = \frac{\mathcal{F}(z,k_\mu)\phi_o}{f_{-k}}. \tag{1.48}$$

Then if we take into account the asymptotic behavior of f_k we can express the correlator as [4]:

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_E = \lim_{z\to 0} z^{2(4-\Delta)} \frac{\Pi(z,k_\mu)}{\phi(z,k_\mu)}. \tag{1.49}$$

4. Note that this expression depends on the dimension of the field, for a field of dimension $\Delta = 0$ for instance this relation will become $\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_E = \lim_{z\to 0} \frac{\Pi(z,k_\mu)}{\phi(z,k_\mu)}$.

Moving back now to the $\Delta = 3$ operator, the equation satisfied by $f_k(z)$ can be solved exactly and the solution and near boundary expansion are given by:

$$\begin{aligned} f_k &= kz^2 K_1(kz) \\ &= z + \frac{1}{4}k^2 z^3 \left(2\log(k/2) + 2\log z + 2\gamma - 1\right) + ... \end{aligned} \tag{1.50}$$

After holographic renormalization we find for the 2-point function:

$$\langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_E = \frac{1}{2}k^2 \log \frac{k^2}{\mu^2} \tag{1.51}$$

where we introduced the scale μ. Fourier transforming back to position space we find:

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle_E = \frac{4}{\pi^2}\mathcal{R}\frac{1}{x^6} \tag{1.52}$$

where $\mathcal{R}\frac{1}{x^6}$ denotes the regularized version (for $x \to 0$) of $\frac{1}{x^6}$ in the sense of differential regularization [Freedman et al., 1992]. This is the expected behavior for a correlator of fields of dimension 3 in a conformal field theory as stated in (1.5).

1.4 Generalizations of the correspondence

The correspondence formulated in the previous chapter describes zero temperature field theory at zero density. Also all the fields of the theory are in the adjoint representation and therefore the correspondence needs to be modified in order to describe fields in the fundamental representation. In this section we discuss possible modifications of the correspondence to incorporate these new features.

1.4.1 Finite temperature

In quantum field theory we typically introduce finite temperature by doing a Wick rotation $t \to -i\tau$ and then periodically compactifying the euclidean time as $\tau \sim \tau + \beta$. The period of the euclidean time direction is the inverse temperature $\beta = 1/T$ and the theory is described by the canonical partition function

$$Z = e^{-\beta \int \mathcal{H}}, \tag{1.53}$$

with $\mathcal{H}$ being the hamiltonian density. This approach has the disadvantage of treating the time direction for temperature and is called imaginary time formulation. If we want to allow both time evolution and finite temperature we should use the so called real-time formulation.

In gravity, on the other side, it is a well known fact that a black hole is characterized by a temperature, the Hawking temperature. It is therefore natural to conjecture that a finite temperature field theory will be dual to an asymptotically AdS solution with a black hole in the interior. The fact that the geometry remains AdS near the boundary is required because of the importance of the boundary behavior of the fields for the correspondence

and of the symmetry matching as we saw in previous sections.
A geometry that we will heavily use is the non-extremal p-brane solution (1.21) for $p = 3$ and in the limit $r_o \ll R$. Taking this limit and defining a new coordinate $u = r_0^2/r^2$ we obtain the near extremal 3-brane metric in Minkowski signature:

$$ds_{10}^2 = \frac{(\pi TR)^2}{u}(-f(u)dt^2 + dx^i dx_i) + \frac{R^2}{4u^2 f(u)}du^2 + R^2 d\Omega_5^2 \tag{1.54}$$

with $f(u) = 1 - u^2$ and $T = r_0/(\pi R^2)$ is the temperature carried by the solution. Notice that this formulation corresponds to the real-time formulation of field theory since we have both time and finite temperature. The price we have to pay for this is that we can no longer use the prescription described in subsection 1.3.1 to compute the correlators; it has to be modified as we will explain later.
In the forthcoming problems we will only deal with perturbations of the 5-dimensional background fields and it is convenient to integrate the supergravity action (1.16) over the angular coordinates. After this integration the relevant part for the metric in the Einstein frame is given by

$$S = \frac{\pi^3 R^5}{2\kappa_{10}^2}\left[\int_0^1 du \int d^4x\sqrt{-g}(\mathcal{R} - 2\Lambda) + 2\int d^4x\sqrt{-h}K\right], \tag{1.55}$$

where the factor $\pi^3 R^5$ comes from the integration over the five-sphere and Λ is the cosmological constant. The second term in the metric is the Gibbons-Hawking boundary term with K being the extrinsic curvature and h the induced metric on the boundary. Now, we can drop the five-sphere term in the metric and consider only the 5-dimensional background:

$$ds_5^2 = \frac{(\pi TR)^2}{u}(-f(u)dt^2 + dx^i dx_i) + \frac{R^2}{4u^2 f(u)}du^2\,. \tag{1.56}$$

Since this metric (in Euclidean signature, where there is no time evolution) describes a system in equilibrium in the canonical ensemble we can deduce from it thermodynamic quantities. In particular the entropy will be given by the Bekenstein-Hawking entropy of the black hole

$$S_{BH} = \frac{A_{\rm h}}{4G_N^5} = \frac{2\pi A_{\rm h}}{\kappa_5^2}, \tag{1.57}$$

where $A_{\rm h}$ is the area of the horizon, which in our case is located at $u = 1$. Using this definition for the entropy we can compute the entropy density in the background (1.56) and we find

$$s = \frac{S_{BH}}{V_3} = \frac{\pi^2}{2}N_c^2 T^3\,, \tag{1.58}$$

where V_3 is the volume of the 3 spatial dimensions x^i and we used the fact that [5]

$$\kappa_5^2 = \frac{\kappa_{10}^2}{\pi^3 R^5} = \frac{4\pi^2 R^3}{N_c^2}. \tag{1.59}$$

5. The relation between gravitational constants in different dimensions arises from the normalization of the gravity action $S = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{g_{(10)}}\mathcal{R}_{(10)} = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{g_{(5)}}\mathcal{R}_{(5)}$.

Another thermodynamic quantity we can compute is the free energy F which is given by the on-shell euclidean action, just as in finite temperature field theory

$$F = TS_E^{\text{on-shell}}. \tag{1.60}$$

We start by computing the regularized on-shell euclidean action which is given by:

$$S_E = -\frac{\pi^3 R^5}{2\kappa_{10}^2}\left[\int_0^1 du \int d^4x\sqrt{g}(\mathcal{R} - 2\Lambda) + 2\int d^4x\sqrt{h}K\right]. \tag{1.61}$$

The surface term is to be computed at the boundary at $u = 0$ and given a unit vector n_μ normal to the boundary the extrinsic curvature it is given by

$$K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}\,, \tag{1.62}$$

with $h_{ab} = g_{ab} - n_a n_b$ with the indices a restrained on the $u = 0$ hypersurface. With this definition we find that the trace of the extrinsic curvature is given near the boundary by $K \approx 4/R$. Moreover the scalar curvature and the cosmological constant are given in the background (1.56) by:

$$\mathcal{R} = -\frac{20}{R^2}; \qquad \Lambda = -\frac{6}{R^2}\,. \tag{1.63}$$

Substituting these results in the Euclidean action and introducing the regulator surface $u = \epsilon$ we have:

$$S_E^{\text{on-shell}} = -\frac{\pi^3 R^3}{2\kappa_{10}^2}\left(-2\pi^4 R^3 T^4 + \frac{6\pi^4 R^3 T^4}{\epsilon^2}\right)\beta V_3\,. \tag{1.64}$$

Following the holographic renormalization procedure we can cancel the divergent term by adding to the action counterterms that can be written in an invariant form on the boundary. It turns out that the counter term action in this case is given by:

$$S_{\text{ct}} = \frac{\pi^3 R^3}{\kappa_{10}^2}\int d^4x\sqrt{h}\frac{3}{R} = \frac{\pi^3 R^3}{\kappa_{10}^2}\frac{3}{R}\left(\frac{\pi^4 R^4 T^4}{\epsilon^2} - \frac{1}{2}\pi^4 T^4 R^4\right)\beta V_3\,. \tag{1.65}$$

Adding the counter term action to the on-shell action we obtain the regularized action:

$$S_E^{\text{reg}} = S_E^{on-shell} + S_{\text{ct}} = -\frac{\pi^2 N_c^2 T^3}{8}V_3\,, \tag{1.66}$$

which according to equation (1.60) will give us the free energy density

$$f = \frac{F}{V_3} = -\frac{\pi^2}{8}N_c^2 T^4\,. \tag{1.67}$$

Notice that the entropy density (1.58) and the entropy density (1.58) satisfy the relation $s = -\partial f/\partial T$ as they should. Using the internal energy equation and our results we can compute the energy density of the system as

$$\epsilon = f + Ts = \frac{3\pi^2}{8}N_c^2 T^4\,. \tag{1.68}$$

Having at hand all of the equilibrium properties of the finite temperature background we will now examine how one can compute correlators in such background which, as we will see in the next chapter, will help us understand the transport properties of the system. In 1.3.1 we saw how to compute correlators in the imaginary-time formulation; now we will explain how the prescription for the correlators is altered in the case of imaginary-time formulation where the metric is in Minkowski signature and has a horizon. If we try to solve the equations of motion for a field propagating in a spacetime with a horizon we find that near the horizon the solution has two asymptotic behaviors, both corresponding to regular solutions. One of the solution describes an incoming wave and the other one an outgoing wave. Based on the argument that nothing can come out of the horizon of a black hole it was proposed [Son and Starinets, 2002] that the incoming wave condition should be imposed on the horizon in order to compute the retarded correlator. The condition on the boundary remains unchanged with respect to the Euclidean prescription. It is natural to think that since the space extends form the horizon to the boundary the correlator will receive some contribution from the boundary term of the action on the horizon but it was proven [Son and Starinets, 2002] that the correlator is given by the same formula as for the Euclidean case. Thus for a scalar operator we have

$$G_R(k_\mu) = \mathcal{F}(z, k_\mu)\Big|_{z\to 0}^{\text{finite}}, \tag{1.69}$$

just as in (1.47) but with the incoming condition on the horizon imposed on the scalar field solution. This prescription for the correlator does not follow from an action principle since we neglected the contribution coming from boundary terms on the horizon. It can be proven however [Iqbal and Liu, 2009a] that the real-time prescription is the same as the Euclidean by analytically continuing the correlator prescription in terms of the conjugate momentum (1.49). We find after analytic continuation in imaginary frequencies that the retarded correlator is given by

$$G_R(k_\mu) = \lim_{z\to 0} z^{2(4-\Delta)} \frac{\Pi(z, k_\mu)}{\phi_R(z, k_\mu)}, \tag{1.70}$$

where ϕ_R stands for the solution of the equations of motion with incoming wave boundary conditions on the horizon. We will shortly apply this prescription to compute the graviton and gravitino correlators in the background (1.56).

1.4.2 Finite density

We saw that we can introduce finite temperature in the correspondence by considering black brane supergravity solutions. We will now motivate the fact that adding finite density to the correspondence amounts to considering charged black brane solutions in supergravity.
A system at finite density is described by the grand canonical ensemble whose partition function is given by

$$Z = e^{-\beta \int (\mathcal{H} - \mu\rho)}, \tag{1.71}$$

with μ being the chemical potential and ρ being the charge density. Consider now the partition function of a finite temperature field theory containing a gauge field A_μ coupled to a conserved current J^μ

$$Z[A_\mu] = e^{-\beta \int (\mathcal{H} - A_\mu J^\mu)} \,. \tag{1.72}$$

If we choose only the time component of the gauge field to be non-zero we can, comparing (1.71) with (1.72), identify the time component of the gauge field with the chemical potential:

$$A_0 = \mu \,. \tag{1.73}$$

Following this analogy, we can say that if we have a gauge field in the supergravity solution its leading contribution near the boundary has to be identified with the background chemical potential.

1.4.3 Flavor degrees of freedom

As we saw in previous sections all the fields we can describe via the correspondence are in the adjoint representation. If we want to be more realistic though we have to find a way of describing fields on the fundamental representation, such as quarks. To have an idea on how to achieve this let's go back to the open string description of D-branes. We saw that open strings that are attached on both ends on the same stack of coincident branes belong to the adjoint representation. But if we have a string that is attached to two different stacks of branes then the string excitations will be in the bifundamental representation. Therefore if we add a stack of N_f branes apart from the stack of N_c D3-branes we will have an $SU(N_f) \times SU(N_c)$ gauge group. This type of constructions are called flavored because of the analogy to the flavor group in QCD.
Let us consider the case where we add D7 branes on top of the D3-brane background. This setup was considered for the first time in [Karch and Katz, 2002]. The D7 branes are extending in the following directions:

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	×	×	×	×						
D7	×	×	×	×	×	×	×	×		

The action they have to satisfy is the DBI action (1.24) but for the moment we consider the case without gauge field and B-field

$$S_{D7} = -N_f T_{D7} \int d^8x \; e^{-\Phi} \sqrt{-g_8} \tag{1.74}$$

and we added the factor N_f because we added N_f D7-branes. For convenience and for later reference we write the $AdS_5 \times S^5$ background in the form

$$ds^2_{10} = \frac{\alpha'}{R^2} e^{2\rho} (-dt^2 + dx^i dx_i) + R^2 \left[d\rho^2 + ds^2_{CP^2} + (d\tau + A_{CP^2})^2 \right] \,, \tag{1.75}$$

where we have written the metric on the five-sphere as a fibration over CP^2 and the coordinate ρ is related to the usual z coordinate by:

$$z = R^2 e^{-\rho} / \sqrt{\alpha'} \,. \tag{1.76}$$

The metric and connexion on CP^2 are given by:

$$ds^2_{CP^2} = \frac{1}{4}d\chi^2 + \frac{1}{4}\cos^2\frac{\chi}{2}(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{1}{4}\cos^2\frac{\chi}{2}\sin^2\frac{\chi}{2}(d\psi + \cos\theta d\varphi)^2$$
$$A_{CP^2} = \frac{1}{2}\cos^2\frac{\chi}{2}(d\psi + \cos\theta d\varphi)$$
$$\chi, \theta \in [0,\pi]\,, \phi, \theta \in [0, 2\pi]\,, \psi \in [0, 4\pi]$$

We choose an embedding of the D7-branes of the form $\chi = \chi(\rho)$, $\tau = 0$. Plugging this into the DBI action we get:

$$S_{D7} \propto \int d\rho\; e^{4\rho}\cos^3\frac{\chi}{2}\sqrt{1 + \frac{\chi'^2}{4}}\,, \tag{1.77}$$

where we integrated over the remaining 7 wordvolume coordinates. The embedding function must satisfy the equation of motion

$$\left(3\sin\frac{\chi}{2} + 2\chi'\cos\frac{\chi}{2}\right)\left(1 + \frac{\chi'^2}{4}\right) + \chi''\cos\frac{\chi}{2} = 0\,. \tag{1.78}$$

The solution to this equation is given by

$$\sin\frac{\chi}{2} = e^{\rho_q - \rho}\,. \tag{1.79}$$

Given that the coordinate χ ranges from 0 to π this solution implies that the branes extend from the boundary at $\rho \to \infty$ up to some finite radius ρ_q. Thus, the D7 branes are separated from the D3-branes located at $\rho \to -\infty$ and strings can be stretching between the two types of branes. Because the length of the strings is related to the mass of its excitations we deduce that the constant ρ_q is related to the mass of the flavor degrees of freedom introduced by the D7 branes. The massless embedding is recovered by taking the limit $\rho_q \to -\infty$.

From the field theory point of view the addition of D7 branes corresponds to the addition of $\mathcal{N} = 2$ hypermultiplets in the field theory. Furthermore the β-function of the theory becomes non-vanishing.

In principle the branes that we added can backreact on the background geometry. By comparing the normalization factors of the actions (1.16) and (1.74) which describe the D3-branes and D7-branes respectively we find that

$$\frac{\mathcal{L}_{D7}}{\mathcal{L}_{D3}} \sim \frac{N_f}{N_c}\lambda\,. \tag{1.80}$$

This implies that in the 't Hooft limit, with λ =fixed, $N_c \to \infty$, the backreaction of the flavors can be neglected. This means, in the field theory description, that we are neglecting quantum effects caused by the quarks. This approximation is the *probe* brane approximation. We will see in the next section that we can go beyond this approximation and compute the backreaction of the branes on the background geometry.

1.5 Theories with unquenched flavor degrees of freedom

We just saw that in the 't Hooft limit we can neglect the backreaction of the branes in the background geometry; one way of going beyond the probe approximation is to consider the Veneziano limit [Veneziano, 1976] for the field theory which is defined as:

$$N_f \to \infty \qquad \frac{N_f}{N_c} = \text{fixed}\,. \tag{1.81}$$

In this limit the $D7$-brane action is no longer negligible compared to the $D3$-brane action; this approximation is the so called unquenched approximation. We have therefore to find a solution of the full action of the coupled D3/D7 system including the supergravity action in the AdS background and the DBI action describing the flavor branes:

$$S = S_b + S_{fl} \tag{1.82}$$

with:

$$S_b = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g_{10}}\left[R - \frac{1}{2}\partial_M\Phi\partial^M\Phi - \frac{1}{2}e^{2\Phi}F_{(1)}^2 - \frac{1}{2}\frac{1}{5!}F_{(5)}^2\right] \tag{1.83}$$

$$S_{fl} = -T_7\sum_{N_f}\left(\int d^8\, xe^{\Phi}\sqrt{-g_8} - \int C_8\right)\,. \tag{1.84}$$

If the D7 branes are localized in the directions transverse to their worldvolume, the equations of motion have delta-function sources at the position of the branes and this makes them difficult to solve. In particular we will have an equation of the form

$$dF_{(1)} \sim \delta(x - x_0)\,. \tag{1.85}$$

with x_0 being the position of the D7 branes on the transverse space.
If we picture the space transverse to the branes as a disk (cf. fig 1.2) the probe limit corresponds to the case where we have a small number of branes (represented by points). Then, if we add more branes, it will be difficult to deal with the large number of localized branes. The smearing technique (see [Nunez et al., 2010] for a review) involves replacing the localized distribution of branes in the transverse space by a uniform brane density starting from a "seed" embedding and averaging using the symmetries of the internal space. This is done in practice by replacing the delta function by the so called smearing 2-form Ω_2,

$$dF_{(1)} = -\Omega_2\,. \tag{1.86}$$

This construction has been analyzed in the general case where the background is given by the product of an AdS space and a Sasaki-Einstein manifold [Benini et al., 2007b, Casero et al., 2006] and the supersymmetry preserving smearing forms have been constructed. In the background we are considering the Sasaki-Einstein manifold in the S^5 and the general result reads in our case:

$$\Omega_2 = -2Q_f J_{CP^2}\,, \tag{1.87}$$

where J_{CP^2} is the Kahler potential in the CP^2 base space and is related to the connexion by $dA_{CP^2} = 2J_{CP^2}$. In our case the D7 brane wraps an $S^3 \subset S^5$ and the relation of Q_f to the number of branes is given by:

$$Q_f = \frac{\text{vol}(S^3) g_s N_f}{4\text{vol}(S^5)} \,. \tag{1.88}$$

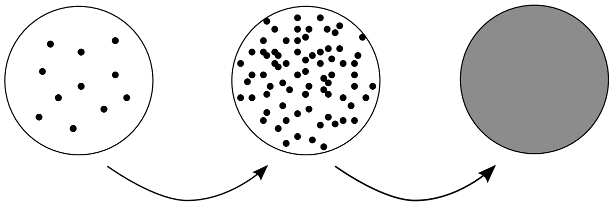

Figure 1.2: The idea of smearing: On the space transverse to the branes the branes are localized (left), as we add more branes their distribution becomes denser (middle) and we can approximate it with a homogeneous distribution over the transverse space (right).

We will concentrate now to the backreaction of massive flavors like the one we considered in 1.4.3. In this case the "seed" embedding that we use for the averaging will be (1.79). The detailed computation of the smearing form, using the $SO(6)$ symmetries of the internal space, for this particular embedding can be found in appendix C of [Bigazzi et al., 2009a]. Even after averaging, there is a memory of the breaking of the isometries of the sphere that is reflected in a squashed sphere. This motivates the following ansatz for the metric:

$$ds^2_{10} = h^{-1/2}(-dt^2 + d\vec{x}_3^2) + h^{1/2}\left[F^2 d\rho^2 + S^2 ds^2_{CP^2} + F^2(d\tau + A_{CP^2})^2\right] \tag{1.89}$$

$$\begin{gathered} ds^2_{CP^2} = \frac{1}{4}d\chi^2 + \frac{1}{4}\cos^2\frac{\chi}{2}(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{1}{4}\cos^2\frac{\chi}{2}\sin^2\frac{\chi}{2}(d\psi + \cos\theta d\varphi)^2 \\ A_{CP^2} = \frac{1}{2}\cos^2\frac{\chi}{2}(d\psi + \cos\theta d\varphi) \\ \chi, \theta \in [0,\pi]\,, \phi, \theta \in [0, 2\pi]\,, \psi \in [0, 4\pi] \end{gathered}$$

The non-trivial RR forms are given by

$$F_{(1)} = Q_f(d\tau + J_{CP^2}), \qquad F_{(5)} = Q_c(1+*)\varepsilon(S^5)\,, \tag{1.90}$$

as suggested by equation (1.87). Q_c is related to the number of D3-branes by

$$Q_c = \frac{N_c (2\pi)^4 g_s \alpha'^2}{\text{vol}(S^5)} \tag{1.91}$$

and $\varepsilon(S^5)$ is the volume form on S^5. More details can be found in [Bigazzi et al., 2009a]; we report here the part of the results that will be useful in the future.

All the fields depend only on the coordinate ρ and we can find an one dimensional effective action by plugin in the ansatz in the action and integrating out the rest of the coordinates. The equations of motion arising from this action are equivalent to the following set of equations for a zero temperature setup:

$$\begin{aligned} \partial_\rho h = -\frac{Q_c}{S^4}; \quad \partial_\rho F = F\left(3 - 2\frac{F^2}{S^2} - \frac{Q_f}{2}e^{\Phi}\cos^4\frac{\chi}{2}\right) \\ \partial_\rho S = \frac{F^2}{S}; \quad \partial_\rho \chi = -2\tan\frac{\chi}{2}; \quad \partial_\rho \Phi = Q_f e^{\Phi}\cos^4\frac{\chi}{2}\,, \end{aligned} \tag{1.92}$$

where $\chi(\rho)$ is the "seed" brane embedding and the charges Q_c and Q_f are proportional to the number of colors and flavors respectively.
If the $D7$ branes are absent, the equations are solved by the AdS metric. In the probe approximation, one sees that the branes extend along the radial direction from the boundary $\rho \to \infty$ to a finite point ρ_q, related to the mass of the flavors in the boundary theory. This feature is preserved by the smearing procedure and persists after the backreaction. The solutions are given in terms of a perturbation parameter proportional to N_f/N_c.
The solution found for $\rho > \rho_q$ is:

$$S_> = \sqrt{\alpha'}e^{\rho}\left[1 + \epsilon_*\left(\frac{1}{6} + \rho_* - \rho - \frac{1}{6}e^{6\rho_q - 6\rho} - \frac{3}{2}e^{2\rho_q - 2\rho} + \frac{3}{4}e^{4\rho_q - 4\rho} - \frac{1}{4}e^{4\rho_q - 4\rho_*} + e^{2\rho_q - 2\rho_*}\right)\right]^{1/6}$$

$$F_> = \sqrt{\alpha'}e^{\rho}\frac{\left[1 + \epsilon_*\left(\rho_* - \rho - e^{2\rho_q - 2\rho} + \frac{1}{4}e^{4\rho_q - 4\rho} + e^{2\rho_q - 2\rho_*} - \frac{1}{4}e^{4\rho_q - 4\rho_*}\right)\right]^{1/2}}{\left[1 + \epsilon_*\left(\frac{1}{6} + \rho_* - \rho - \frac{1}{6}e^{6\rho_q - 6\rho} - \frac{3}{2}e^{2\rho_q - 2\rho} + \frac{3}{4}e^{4\rho_q - 4\rho} - \frac{1}{4}e^{4\rho_q - 4\rho_*} + e^{2\rho_q - 2\rho_*}\right)\right]^{1/3}}$$

$$\Phi_> = \Phi_* - \log\left(1 + \epsilon_*\left(\rho_* - \rho - e^{2\rho_q - 2\rho} + \frac{1}{4}e^{4\rho_q - 4\rho} + e^{2\rho_q - 2\rho_*} - \frac{1}{4}e^{4\rho_q - 4\rho_*}\right)\right)\,.$$

The dilaton diverges and the metric is not asymptotically AdS when $\rho \to \infty$. The solution depends also on an arbitrary scale ρ_*, an anchoring point at which the value of the dilaton is fixed; this point should also be viewed as the effective UV cutoff of the theory. Physically, this means that because of the Landau pole the theory can not be used for arbitrarily high energy. At the end of the calculation one should be able to send $\rho_* \to \infty$.

The solution in the region where the D7 branes do not extend i.e. for $\rho < \rho_q$, reads:

$$\Phi_< = \Phi_q = \Phi_* - \log\left(1 + \epsilon_*\left(\rho_* - \rho_q - \frac{3}{4} + e^{2\rho_q - 2\rho_*} - \frac{1}{4}e^{4\rho_q - 4\rho_*}\right)\right)\,,$$

$$S_< = F_< = \sqrt{\alpha'}e^{\rho}e^{-\frac{1}{6}(\Phi_q - \Phi_*)}\,.$$

For all values of the radial coordinate we can find h by integrating the equation

$$\frac{dh}{d\rho} = -\frac{Q_c}{S^4}\,, \tag{1.93}$$

with Q_c being proportional to the number of colors N_c. The perturbation parameter is given by:

$$\epsilon_* = \frac{1}{8\pi^2}\lambda_*\frac{N_f}{N_c}\,, \tag{1.94}$$

where λ_* is the 't Hooft coupling at the ρ_* scale. For our purposes though it is preferable to express the solution in terms of a perturbation parameter fixed at the flavor mass scale given by:

$$\epsilon_q = \epsilon_* e^{\Phi_q - \Phi_*} \,. \tag{1.95}$$

Since we will be computing quantities at the scale lower than the mass of the flavors, ϵ_q is the effective expansion parameter that has to be kept small; the residual dependence on the cutoff scale leads to subleading contributions that can be suppressed sending $\rho_* \to \infty$. This observation was done in [Magana et al., 2012] in considering the dynamics of probe quarks in the unquenched flavored plasma; we verified explicitly that the same happens in our case.

Fixing the reparametrization invariance of the metric we can define a new coordinate z by imposing that h takes the form:

$$h(z) = \frac{z^4}{R^4}; \qquad R^4 \equiv \frac{1}{4} Q_c \,. \tag{1.96}$$

This form is the same as in the unflavored case and it is convenient for comparing our results with the pure AdS case. Imposing this condition and integrating equation (1.93) order by order we find an expression for $z(\rho)$. We fix the additive integration constant in h by requiring that $z \to 0$ when $\rho \to \infty$. Then we have for $\rho > \rho_q$:

$$\begin{aligned} z_>(\rho) =& \frac{e^{-\rho} R^2}{\sqrt{\alpha'}} \left[1 + \frac{\epsilon_q}{720} \left(\frac{8 e^{-6\rho} R^{12}}{\alpha'^{\,3} z_q^6} - \frac{45 e^{-4\rho} R^8}{\alpha'^2 z_q^4} + \frac{30 e^{-4\rho_*} R^8}{\alpha'^2 z_q^4} \right.\right. \\ & \left.\left. + \frac{120 e^{-2\rho} R^4}{\alpha' z_q^2} - \frac{120 e^{-2\rho_*} R^4}{\alpha' z_q^2} + 120\rho - 120\rho_* + 10 \right) \right] \,, \end{aligned} \tag{1.97}$$

where we defined $z_q = z(\rho_q)$. Now we can invert this relation to obtain $F_>(z)$ and $S_>(z)$ as expansions up to first order in ϵ_q:

$$F_>(z) = \frac{R^2}{z} + \frac{R^2 \epsilon_q}{240 z z_q^6} \left(-45 z^4 z_q^2 + 40 z^2 z_q^4 - 10 z_q^6 + 16 z^6 \right) \,,$$

$$S_>(z) = \frac{R^2}{z} + \frac{R^2 \epsilon_q}{240 z z_q^6} \left(15 z^4 z_q^2 - 20 z^2 z_q^4 + 10 z_q^6 - 4 z^6 \right) \,.$$

Imposing continuity of the function h at $\rho = \rho_q$ we obtain the following expressions for the coordinate z and for the functions $F_<(z)$ and $S_<(z)$ for $\rho < \rho_q$:

$$z_<(\rho) = \frac{e^{-\rho} R^2}{\sqrt{\alpha'}} \left[1 + \epsilon_q \left(\frac{e^{-4\rho_*} R^8}{24 \alpha^2 z_q^4} + \frac{\alpha^2 e^{4\rho} z_q^4}{240 R^8} - \frac{e^{-2\rho_*} R^4}{6 \alpha z_q^2} - \frac{1}{6} \log\left(\frac{\sqrt{\alpha'} z_q}{R^2} \right) - \frac{\rho_*}{6} + \frac{1}{8} \right) \right] \,, \tag{1.98}$$

$$F_<(z) = S_<(z) = \frac{R^2}{z} + \epsilon_q \frac{R^2 z_q^4}{720 z^5} \,.$$

These formulae provide the description of the backreacted geometry to first order in ϵ_q that we will use in chapter 3 to compute the entanglement entropy.

Chapter 2

Holographic hydrodynamics

In this chapter we deal with the subject of hydrodynamics in the context of the AdS/CFT correspondence. After reviewing briefly the basics of classical hydrodynamics and of hydrodynamics in supersymmetric field theories we present one of the most important results in this field: the universality of the shear viscosity to entropy ratio. Then we move on to discuss the diffusion constant of the phonino and its properties.

2.1 Classical and supersymmetric hydrodynamics

Hydrodynamics [Forster, 1990] is a theory describing a system at long wavelengths compared to the microscopic degrees of freedom, or in other words it describes a system where the scale L of variations away from equilibrium is much larger than the mean free path ℓ_{mfp}

$$L \gg \ell_{\text{mfp}} \, . \tag{2.1}$$

The system is in local thermal equilibrium and we can describe it in terms of the temperature $T(x)$ and the local fluid velocity $u^\mu(x)$ which both depend on the coordinates. The velocity obeys the constraint $u^\mu u_\mu = -1$. We can think of the system as being composed of patches of characteristic length L which are in local thermal equilibrium and allow the thermodynamic properties to vary across different patches. A system in equilibrium at finite temperature is characterized by the density operator

$$\rho = \frac{1}{Z} e^{\beta u_\mu P^\mu} \tag{2.2}$$

where the conserved charges P^μ are expressed in terms of the energy momentum tensor as

$$P^\mu = \int d^3x T^{\mu 0}(\vec{x}) \, . \tag{2.3}$$

Conservation of energy and momentum is imposed by requiring

$$\partial_\mu T^{\mu\nu} = 0, \tag{2.4}$$

where $T^{\mu\nu}$ is the energy momentum tensor. The conservation equation is supplemented by the so called *constitutive relations* which express the conserved charges in terms of thermodynamic quantities and their derivatives; departure from equilibrium is organized as an expansion in the thermodynamic variable derivatives. At equilibrium, i.e. at zeroth order in the derivative expansion, the energy momentum tensor is given by

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} \, , \tag{2.5}$$

where ϵ is the energy density and P is the pressure. If we also include the first order in the derivatives we allow for an additional term

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} - \sigma^{\mu\nu}\,. \tag{2.6}$$

To identify the form of this extra term it is convenient to go to the local rest frame of some point x where $u^i(x) = 0$. In this frame we can redefine the velocity and temperature in such a way that $\sigma^{00} = \sigma^{0i} = 0$. The remaining components σ_{ij} can be classified according to spacial rotations. We can therefore write the correction to the equilibrium energy momentum tensor as

$$\sigma_{ij} = \eta\left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial_k u^k\right) + \zeta\delta_{ij}\partial_k u^k\,, \tag{2.7}$$

where the first term is the traceless part of a symmetric tensor and the second one the trace part. The coefficients η and ζ are free parameters of the hydrodynamic effective theory which depend on the microscopic details and are the shear and bulk viscosities respectively. This type of coefficients are called transport coefficients. The correction to the energy-momentum tensor in a general frame in a curved background is given by

$$\sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left[\eta\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}g_{\alpha\beta}\partial_\lambda u^\lambda\right) + \zeta g_{\alpha\beta}\partial_\lambda u^\lambda\right] \tag{2.8}$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projector to the directions perpendicular to u^μ. If the theory is conformal then the trace of the energy momentum tensor should vanish and we get from (2.6) and (2.8):

$$\epsilon = 3P \qquad \zeta = 0\,. \tag{2.9}$$

If the system has some global symmetry then the corresponding current J^μ will be conserved and the thermodynamically conjugate chemical potential μ will be one of the thermodynamic quantities describing the system. In particular the density operator (**??**) will be generalized to

$$\rho = \frac{1}{Z}e^{\beta(u_\mu P^\mu - \mu N)}\,, \tag{2.10}$$

with

$$N = \int d^3x J^0(\vec{x})\,. \tag{2.11}$$

The conservation equation in this case reads

$$\partial_\mu J^\mu = 0 \tag{2.12}$$

and the constitutive relation up to first order in the derivatives is given by:

$$J^\mu = \rho u^\mu - D P^{\mu\nu}\partial_\nu\rho\,. \tag{2.13}$$

D is the charge diffusion constant and ρ the charge density.

In a supersymmetric field theory there are additional conserved quantities, the supercurrents S^{μ}_{α} and $\bar{S}^{\mu}_{\dot{\alpha}}$ which are left and right handed Weyl vector-spinors. The conservation equation is simply

$$\partial_{\mu} S^{\mu}_{\alpha} = \partial_{\mu} \bar{S}^{\mu}_{\dot{\alpha}} = 0 \, . \tag{2.14}$$

Taking into account these additional conserved charges we can introduce them in the density operator using super-chemical potentials μ^{α} and $\bar{\mu}_{\dot{\alpha}}$,

$$\rho_s = \frac{1}{Z} e^{\beta(u_{\mu} P^{\mu} + \mu^{\alpha} Q_{\alpha} + \bar{\mu}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}})} \, . \tag{2.15}$$

The supercharges are given in terms of supercharge densities by:

$$Q_{\alpha} = \int d^3 x \rho_{\alpha}(\vec{x}) \qquad \bar{Q}^{\dot{\alpha}} = \int d^3 x \bar{\rho}^{\dot{\alpha}}(\vec{x}) \, . \tag{2.16}$$

The supersymmetry transformations act on the supercurrents as

$$\begin{aligned} \{Q_{\beta}, \bar{S}^{\lambda}_{\dot{\alpha}}\} &= 2 T^{\lambda}_{\nu} \sigma^{\nu}_{\beta\dot{\alpha}} + (\text{spacetime gradients}) \, , \\ \{\bar{Q}_{\dot{\beta}}, S^{\lambda}_{\alpha}\} &= 2 T^{\lambda}_{\nu} \sigma^{\nu}_{\alpha\dot{\beta}} + (\text{spacetime gradients}) \, . \end{aligned} \tag{2.17}$$

These transformations fix the relation of the surpercharge densities and super-chemical potentials in equilibrium [Kovtun and Yaffe, 2003] once we combine them with (2.15) to compute expectation values :

$$\begin{aligned} \rho_{\alpha} &= i\beta(\epsilon\gamma^1 + (\epsilon + P) u^i \gamma_i)_{\alpha\dot{\beta}} \bar{\mu}^{\dot{\beta}} \\ \bar{\rho}_{\dot{\alpha}} &= -i\beta \mu^{\beta} (\epsilon\gamma^1 + (\epsilon + P) u^i \gamma_i)_{\alpha\dot{\beta}} \, , \end{aligned} \tag{2.18}$$

where we adopted the gamma matrix conventions described in appendix A. The constitutive relation which relates the density of charge to the spatial components of the supercurrent is up to first order in the derivative expansion:

$$\begin{aligned} S^{i}_{\alpha} &= -D_s \partial^i \rho_{\alpha} + D_{\sigma} (\gamma^{ij} \partial_j \rho)_{\alpha} - \frac{P}{\epsilon} (\gamma^i \gamma^1 \rho)_{\alpha} \\ \bar{S}^{i}_{\dot{\alpha}} &= -D_s \partial^i \bar{\rho}_{\dot{\alpha}} + D_{\sigma} (\partial_j \bar{\rho} \gamma^{ji})_{\dot{\alpha}} - \frac{P}{\epsilon} (\bar{\rho} \gamma^1 \gamma^i)_{\dot{\alpha}} \end{aligned} \tag{2.19}$$

These constitutive relations can be derived by computing the expectation value of the spatial supercurrents and combining them with the equilibrium relation (2.18). The first two term are identified as usual by classifying terms with respect to spatial rotations. D_s and D_{σ} are two new diffusion constants. If the underlying theory is superconformal then the relation $\gamma_{\mu} S^{\mu} = 0$ has to hold and this implies that $D_s = D_{\sigma}$. This diffusion constant characterizes the damping of a collective excitation, the phonino. A detailed analysis of the phonino excitation in field theory can be found in [Kratzert, 2002].

2.2 Response theory and Kubo formulas

In this section we will review linear response theory and we will derive the Kubo formulas for the viscosity and the phonino diffusion constant that will be used for computations to follow.

Linear response theory [Forster, 1990, Kadanoff and Martin, 1963] examines the response of a system to a perturbation. Suppose we have a statistical system described by the density matrix ρ and the Hamiltonian H at equilibrium. We perturb this system by adding a term to the hamiltonian

$$\tilde{H} = H + \delta H = H - J(t)\mathcal{O}(t) \tag{2.20}$$

where $J(t)$ is the source of the perturbation. The perturbed system will be described by the density matrix $\tilde{\rho} = \rho + \delta\rho$ which has to satisfy the time evolution equation

$$i\frac{\partial}{\partial t}\tilde{\rho} = [\tilde{H}, \tilde{\rho}]\,. \tag{2.21}$$

This equation is solved for small $\delta\rho$ by

$$\delta\rho(t) = i\int_{-\infty}^{t} dt'\; e^{-i(t-t')H}[\mathcal{O}, \rho]e^{i(t-t')H}J(t')\,. \tag{2.22}$$

The change in the expectation value of the operator $\mathcal{O}$ due to this perturbation can be then calculated as

$$\delta\langle\mathcal{O}(t)\rangle = \mathrm{Tr}(\delta\rho\mathcal{O}) = i\mathrm{Tr}\left(\int_{-\infty}^{t} dt'[\mathcal{O}, \rho]\mathcal{O}(t-t')J(t')\right)\,. \tag{2.23}$$

The response function $\mathcal{R}_{\mathcal{O}}(t)$ of the system is, by definition, the change in the expectation value of the operator for a delta function source $J(t) = \delta(t)$. From (2.23) we have

$$\mathcal{R}_{\mathcal{O}}(t) = i\mathrm{Tr}\left([\mathcal{O}, \rho]\mathcal{O}(t)\right) = -i\mathrm{Tr}\left(\rho[\mathcal{O}, \mathcal{O}(t)]\right)\,. \tag{2.24}$$

To generalize this result we can consider operators that have a homogeneous dependence in spacial dimensions. If we add a perturbation at some initial time y^0 then the system will respond to the perturbation only after this time. To find the response to such a perturbation we should convolute the source $J(y)$ with the response function but only after the perturbation time. For an operator that has zero initial expectation value we obtain:

$$\langle\mathcal{O}(x)\rangle = -\int dy\; G_R(x, y)J(y), \tag{2.25}$$

where

$$G_R(x, y) \equiv -i\theta(x^0 - y^0)\langle[\mathcal{O}(x), \mathcal{O}(y)]\rangle \tag{2.26}$$

is the retarded Green's function. This is the main result of response theory.

We will now use this result to derive a Kubo formula for the viscosity; Kubo formulas are in general formulas that relate transport coefficients with the low frequency limit of correlators in a theory.
The viscosity appears as we saw in the constitutive relation for the energy momentum tensor and we will see here that it is related to the retarded correlator of the transverse components of the energy momentum tensor. To fix the ideas consider a fluctuation propagating along the third spacial coordinate so that the T_{xy} component is transverse to it. The source of this fluctuation will be the metric since it is the operator that couples to the energy momentum tensor. We consider therefore a time dependent fluctuation of the form:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.27}$$

with only the $h_{xy}(t)$ component turned on. Consider now the constitutive relation (2.8) in a curved background

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} - \eta P^{\mu\alpha}P^{\nu\beta}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{3}g_{\alpha\beta}\nabla_\lambda u^\lambda\right) \tag{2.28}$$

and we take $\zeta = 0$ for simplicity. The xy component of the energy momentum tensor is then given by:

$$T_{xy} = Pg_{xy} - \eta(\nabla_x u_y + \nabla_y u_x) = Ph_{xy} + \eta\partial_t h_{xy}\,. \tag{2.29}$$

In momentum space this expression becomes

$$T_{xy}(\omega, \vec{k}) = (P - i\eta\omega)h_{xy}(\omega, \vec{k}) \tag{2.30}$$

Fourier transforming (2.25) and writing it for the energy momentum tensor we have

$$\langle T_{xy}(\omega, \vec{k})\rangle = G^R_{xy,xy}(\omega, \vec{k})h_{xy}(\omega, \vec{k})\,. \tag{2.31}$$

Comparing the two expressions we obtain the Kubo formula for the viscosity:

$$\eta = -\lim_{\omega\to 0}\lim_{k\to 0}\frac{1}{\omega}\mathrm{Im}G^R_{xy,xy}(\omega, \vec{k})\,. \tag{2.32}$$

Using the same method we can also derive a Kubo formula for the charge diffusion constant D.

Now we will derive the Kubo formula for the phonino diffusion constant using a somewhat different approach. In fact due to the fermionic nature of the supercurrents the identification of the supercurrent and its expectation value is not possible; we preferred thus to use instead the definition of the retarded correlator in order to prove the Kubo formula. Combining the constitutive relation for the supercurrents (2.19) with the continuity equation $\partial_t\rho_\alpha + \partial_i S^i_\alpha = 0$ and Fourier transforming in space yields:

$$[(\partial_t + D_s k^2)\delta^\beta_\alpha - ic_s(k^j\gamma_j\gamma^1)^\beta_\alpha]\rho_\beta(t, \vec{k}) = 0 \tag{2.33}$$

where $c_s = P/\epsilon$. The solution of this equation in the limit of small $\vec{k}$ is:

$$\rho_\alpha(t, \vec{k}) = e^{-D_s k^2 t}[\delta^\beta_\alpha\cos(kc_s t) + i(\hat{k}^j\gamma_j\gamma^1)^\beta_\alpha\sin(kc_s t)]\rho_\beta(0, \vec{k}) \tag{2.34}$$

From this solution we can deduce the following supercharge density correlators in the small $\vec{k}$ limit since $C_{\alpha\dot{\alpha}}(0,\vec{0})$ is fixed by the supersymmetry algebra.

$$\begin{aligned} C_{\alpha\dot{\alpha}}(t,\vec{k}) &= \int d^3x e^{-i\vec{k}\cdot\vec{x}} \langle \rho_\alpha(\vec{x},t)\bar{\rho}_{\dot{\alpha}}(\vec{0},0)\rangle = \\ &= -\epsilon e^{-D_s k^2 t}[(i\gamma^1)_{\alpha\dot{\alpha}}\cos(kc_s t) + (\hat{k}^j\gamma_j)_{\alpha\dot{\alpha}}\sin(kc_s t)]\,. \end{aligned} \tag{2.35}$$

We define the following correlator:

$$\tilde{C}_{\alpha\dot{\alpha}}(t,\vec{x}) = -i\theta(t)\langle \rho_\alpha(t,\vec{x})\bar{\rho}_{\dot{\alpha}}(0,\vec{0})\rangle\,. \tag{2.36}$$

We can now calculate the Fourier transform:

$$\begin{aligned} \tilde{C}_{\alpha\dot{\alpha}}(\omega,\vec{k}) &= -i\int dt \int d^3x e^{i\omega t - i\vec{k}\cdot\vec{x}}\theta(t)\langle\rho_\alpha(t,\vec{x})\bar{\rho}_{\dot{\alpha}}(0,\vec{0})\rangle \\ &= -i\int dt e^{i\omega t}\theta(t) C_{\alpha\dot{\alpha}}(t,\vec{k}) \\ &= \epsilon\left[\frac{iD_s k^2 + \omega}{(c_s k)^2 + (D_s k^2 - i\omega)^2}(i\gamma^1)_{\alpha\dot{\alpha}} + \frac{ic_s k}{(c_s k)^2 + (D_s k^2 - i\omega)^2}(\hat{k}^j\gamma_j)_{\alpha\dot{\alpha}}\right]\,. \end{aligned} \tag{2.37}$$

The retarded correlator of the supercurrents is defined as:

$$C^R_{\alpha\dot{\alpha}}(t,\vec{x}) = -i\theta(t)\langle\{\rho_\alpha(t,\vec{x}),\bar{\rho}_{\dot{\alpha}}(0,\vec{0})\}\rangle\,. \tag{2.38}$$

From the spectral decomposition of $\tilde{C}$ and C^R it follows:

$$\operatorname{Im}\tilde{C}_{\alpha\dot{\alpha}}(\omega,\vec{k}) = -\frac{\pi}{Z}\sum_{n,m} e^{-\beta E_n}\langle n|\rho_\alpha(0,\vec{k})|m\rangle\langle m|\bar{\rho}_{\dot{\alpha}}(0,\vec{0})|n\rangle\delta(\omega + E_n - E_m)$$

$$\operatorname{Im} C^R_{\alpha\dot{\alpha}}(\omega,\vec{k}) = -\frac{\pi}{Z}\sum_{n,m} e^{-\beta E_n}(1+e^{-\beta\omega})\langle n|\rho_\alpha(0,\vec{k})|m\rangle\langle m|\bar{\rho}_{\dot{\alpha}}(0,\vec{0})|n\rangle\delta(\omega + E_n - E_m)$$

Therefore we get:

$$\operatorname{Im} C^R_{\alpha\dot{\alpha}}(\omega,\vec{k}) = (1+e^{-\beta\omega})\operatorname{Im}\tilde{C}_{\alpha\dot{\alpha}}(\omega,\vec{k}) \tag{2.39}$$

Taking the small $\vec{k}$ and then the small ω limit of (2.37) and using equation (2.39) we get:

$$\epsilon D_s = \frac{1}{4}\lim_{\omega\to 0}\left[\lim_{k\to 0}\frac{\omega^2}{k^2}(i\gamma^1)^{\alpha\dot{\alpha}}\operatorname{Im} C^R_{\alpha\dot{\alpha}}\,.(\omega,\vec{k})\right] \tag{2.40}$$

Using the continuity equation we can deduce a relation between the supercharge and the supercurrent correlators, namely:

$$\omega^2 C^R_{\alpha\dot{\alpha}}(\omega,\vec{k}) = k^2\hat{k}_i\hat{k}_j G^{ij}_{\alpha\dot{\alpha}}(\omega,\vec{k}) = \frac{1}{3}k^2 G^{ii}_{\alpha\dot{\alpha}}(\omega,\vec{k}) \tag{2.41}$$

where $G^{ij}_{\alpha\dot{\alpha}}(t,\vec{x}) = -i\theta(t)\langle\{S^i_\alpha(t,\vec{x}),\bar{S}^j_{\dot{\alpha}}(0,\vec{0})\}\rangle$ and in the last equality the i indices are summed. Using this relation we can write the Kubo formula in terms of the supercurrent retarded correlator:

$$\epsilon D_s = \frac{1}{12}\lim_{\omega\to 0}\left[\lim_{k\to 0}(i\gamma^1)^{\alpha\dot{\alpha}}\operatorname{Im} G^{ii}_{\alpha\dot{\alpha}}(\omega,\vec{k})\right]\,. \tag{2.42}$$

2.3 Transport coefficients in AdS/CFT

In this section we discuss some applications of the correspondence to the calculation of transport coefficients which has been a very active subject of research during the last decade. Various transport phenomena have been studied using the correspondence such as charge diffusion and viscosity and their properties have been unraveled. In particular it has been proven that the ratio of shear viscosity and entropy density is universal, i.e. it has the same value for every system that can be described by a gravity dual. The study of hydrodynamic properties using the AdS/CFT was initiated in [Policastro et al., 2001, Policastro et al., 2002a, Policastro et al., 2002b]. Here we concentrate on the shear viscosity and in the rest of the chapter we present in detail the computation of the diffusion constant of the phonino.

We will now present following the presentation in [Policastro et al., 2002a] the computation of the correlator of the transverse component of the energy momentum tensor in the finite temperature background (1.56) whose metric we recall here [1]:

$$ds^2 = \frac{\pi^2 T^2 R^2}{u}(-f(u)dt^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{4f(u)u^2}du^2 \,, \tag{2.43}$$

where $f(u) = 1 - u^2$. The boundary is located at $u = 0$ and the horizon at $u = 1$.
We will use the supergravity action for the dual field: the metric. We can choose coordinates such that the metric perturbations propagate along the third spacial coordinate, i.e. the spacial momentum is $\vec{k} = (0, 0, q)$. Then we can classify metric perturbations according to the representations of the rotation group $O(2)$ in the (x,y) plane. The transverse perturbation h_{xy} decouples from the longitudinal ones h_{tx} and h_{ty}. Defining

$$\phi = h^x_y \qquad a_t = \phi^x_t \qquad a_z = h^t_z \,, \tag{2.44}$$

and assuming that all the other components are zero we can write the action (1.55) as follows:

$$S = \frac{N_c^2}{8\pi^2 R^3}\int d^4x du\, \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4g^2_{\text{eff}}}g^{\mu\rho}g^{\nu\sigma}f_{\mu\nu}f_{\rho\sigma}\right) . \tag{2.45}$$

We defined $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ and $g^2_{\text{eff}} = g_{xx}$. The transverse component of the metric satisfies the equation, in momentum space,

$$\phi''_k - \frac{1+u^2}{uf}\phi'_k + \frac{\omega^2 - k^2 f}{(2\pi T)^2 u f^2}\phi_k = 0 \,, \tag{2.46}$$

where primes denote derivatives with respect to u. According to 1.4.1 to solve the equation in the thermal background we need to impose incoming wave boundary conditions. The equation cannot be solved analytically; it can though be solved perturbativelly in a small frequency expansion. The solution with the right asymptotic behavior at the horizon is

$$\phi_k(u) = (1-u)^{-i\omega/(4\pi T)}F_k(u) \,, \tag{2.47}$$

1. Notice that here the radial coordinate is u and z is just the third spacial coordinate, unlike in the AdS case where z was the radial coordinate.

where the function $F_k(u)$ can be written as an expansion in frequency

$$F_k(u) = 1 - \frac{i\omega}{4\pi T}\ln\frac{1+u}{2} + \frac{\omega^2}{32\pi^2 T^2}\left[\left(\ln\frac{1+u}{2} + 8(1-q^2/\omega^2)\right)\ln\frac{1+u}{2} - 4\mathrm{Li}_2\frac{1+u}{2}\right]. \tag{2.48}$$

The relevant part of the action (2.45) in the thermal background becomes:

$$S = -\frac{\pi^2 N^2 T^4}{8}\int dud^4x\, \frac{f}{u}\phi'^2 + \ldots, \tag{2.49}$$

Then the two-point function of the stress energy tensor is given by

$$G^R_{xy,xy}(\omega,k) = \frac{N_c^2 T^2}{16}(i2\pi T\omega + q^2)\,. \tag{2.50}$$

Recall now the Kubo formula (2.32); combining with the previous result we have for the viscosity:

$$\eta = \frac{\pi}{8}N_c^2 T^3\,. \tag{2.51}$$

Using now the entropy density we found in (1.58) we find the famous result [Policastro et al., 2001]:

$$\frac{\eta}{s} = \frac{1}{4\pi}\,. \tag{2.52}$$

The same result was found in [Policastro et al., 2001] where the shear viscosity was related to the absorption cross section of low energy gravitons by the three-branes. Here we presented this result for the near-extremal 3-brane background; we will see right away that this result holds for every theory quantum theory that has a gravity dual.

2.3.1 Universality of the shear viscosity to entropy ratio

Here we present the proof of the universality [Buchel and Liu, 2004] of the shear viscosity ratio using the membrane paradigm. The membrane paradigm [Thorne et al., 1986, Parikh and Wilczek, 1998] was developed in the context of black hole physics. Consider an observer sitting just outside the horizon of a black hole. Then the effective action that will describe the physics of this observer will be given by:

$$S_{\text{eff}} = S_{\text{out}} + S_{\text{surf}}\,. \tag{2.53}$$

The S_{out} stands for the usual action outside of the horizon and the term S_{surf} is a surface term that encodes the effect of the horizon on the space outside the black hole. To illustrate this approach with an example consider a scalar field a background with metric:

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + g_{ij}dx^i dx^j\,. \tag{2.54}$$

We suppose that the metric has a horizon at $r = r_0$ where g_{tt} vanishes and g_{rr} has a pole. Furthermore we assume that the other components of the metric are finite at the horizon and that the metric exhibits translational and rotational symmetry in the spacial directions. The boundary in these coordinates is located at $r \to \infty$. Take the action of the scalar field to be given by

$$S_{\text{out}} = -\frac{1}{2}\frac{1}{q(r)} \int\limits_{r>r_0} d^5x\, \sqrt{-g}(\partial_\mu \phi)^2 \,, \tag{2.55}$$

where $q(r)$ is the effective coupling of the scalar. Varying the action provides us with a boundary term on the horizon. To cancel this term we must add the surface term

$$S_{\text{surf}} = \int\limits_{r=r_0} \sqrt{-h} \left(\frac{\Pi(r_0, x)}{\sqrt{-h}} \right) \phi(r_0, x) \,, \tag{2.56}$$

where Π is the momentum conjugate to ϕ with respect to a foliation in the r-direction

$$\Pi = -\frac{\sqrt{-g}}{q(r)} g^{rr} \partial_r \phi \,. \tag{2.57}$$

We can interpret this result by saying that the membrane at the horizon carries a scalar current

$$\Pi_{\text{mb}} \equiv \left(\frac{\Pi(r_0)}{\sqrt{-h}} \right) = -\frac{\sqrt{g^{rr}}\partial_r \phi(r_0)}{q(r)} \,. \tag{2.58}$$

Consider now a free in-falling observer hovering just outside for the horizon. For such an observer the horizon is a regular surface and therefore the scalar field on the horizon should not have a singular behavior. In other words the field ϕ should only depend on the ingoing Eddington-Finkelstein coordinate v defined as

$$dv = dt + \sqrt{\frac{g_{rr}}{g_{tt}}} dr \tag{2.59}$$

and not on the outgoing one defined as:

$$du = dt + \sqrt{\frac{g_{rr}}{g_{tt}}} dr \,. \tag{2.60}$$

The second condition, i.e. $\partial_u \phi = 0$ implies

$$\Pi_{\text{mb}} = -\frac{1}{q(r)} \sqrt{g^{tt}} \partial_t \phi(r_0) \,. \tag{2.61}$$

This condition ensures regularity of the scalar field on the horizon.

The idea of the proof of the universality using the membrane paradigm [Iqbal and Liu, 2009b] is that if we have a correlator that does not change when "moving" along the radial direction in AdS then we can compute it in the radial slice of our preference; in particular we can compute it at the horizon. Then because it remains unchanged the correlator computed on the horizon will be the same as the one computed at the boundary, i.e. the correlator of the dual field theory.

Recalling the definition of the correlator (1.70) the Kubo formula for the viscosity (2.42) can be written as:

$$\eta = \lim_{k_\mu \to 0} \lim_{r \to \infty} \frac{\Pi(r, k_\mu)}{i\omega\phi(r, k_\mu)} \,. \tag{2.62}$$

The equations of motion for ϕ in Hamiltonian form are:

$$\begin{aligned} \Pi &= -\frac{\sqrt{-g}}{q(r)} r^{rr} \partial_r \phi \\ \partial_r \Pi &= \frac{\sqrt{-g}}{q(r)} g^{rr} g^{\mu\nu} k_\mu k_\nu \phi \,. \end{aligned} \tag{2.63}$$

These equations imply in the low frequency limit

$$\partial_r \Pi = 0 \qquad \partial_r(\omega\phi) = 0 \,. \tag{2.64}$$

This means that the correlator (2.62) does not flow in the zero momentum limit and therefore we can evaluate it at the horizon. The condition for regularity at the membrane gives us the relation between the scalar field and its conjugate momentum, i.e.

$$\Pi(r_0, k_\mu) = \frac{1}{q(r_0)} \sqrt{\frac{-g}{g_{rr} g_{tt}}}\Big|_{r_0} i\omega\phi(r_0, k_\mu) = \frac{1}{q(r_0)} \frac{A_h}{V_3} i\omega\phi(r_0, k_\mu) \,, \tag{2.65}$$

where A_h is the area of the horizon and V_3 is the volume in 3 dimensions. The effective coupling of the transverse graviton is $q(r) = 1/(16\pi G_N)$ and combining with the expression of the entropy density $s = A_h/(4G_N V_3)$ we arrive again at the desired result:

$$\frac{\eta}{s} = \frac{1}{4\pi} \,. \tag{2.66}$$

We saw here that for a massless scalar field which decouples from the other fields we can express the corresponding correlator in terms of horizon data. Following this approach the universality of the shear viscosity to entropy ratio results from the universality of the effective coupling of the graviton. Another approach would be to relate the viscosity to the absorption cross section, as we mentioned earlier, and then attribute the universality of the viscosity to the universality of the absorption cross section which was established in [Das et al., 1997, Kovtun et al., 2005].

2.4 The supercurrent correlator in supergravity

A supersymmetric field theory at finite temperature has hydrodynamic excitations corresponding to fluctuations of the supercharge density (see [Lebedev and Smilga, 1989] and [Kovtun and Yaffe, 2003] for more details on hydrodynamics with supercharges). In particular there is a supersymmetric sound wave that propagates with a speed and attenuation that are different from those of ordinary sound. The attenuation rate is governed by the diffusion constant of the supercharge; this has been computed in the N=4 SYM theory for the first time in [Policastro, 2009] using holography. In this section we compute it in a different way, not using the sound attenuation but the transverse part of the correlator as explained below; along the way we present the definitions and techniques that will be used in the next section.

The retarded correlator of the supercurrents S_i^α is defined as:

$$G_{ij}^{\alpha\dot{\beta}}(k) = \int d^4x e^{-ik\cdot x} i\theta(x^0)\langle\{S_i^\alpha(x), \bar{S}_j^{\dot{\beta}}(0)\}\rangle\,. \tag{2.67}$$

Conservation of the supercurrents and superconformal invariance imply:

$$k^i G_{ij} = 0\,, \qquad \gamma^i G_{ij} = G_{ij}\gamma^j = 0\,. \tag{2.68}$$

The correlator of the supercurrents can be build out of projectors that implement these constraints. The projector on the transverse gamma-traceless part of a vector-spinor is

$$P_i^j = \delta_i^j - \frac{1}{3}\left(\gamma_i - \frac{k_i \not{k}}{k^2}\right)\gamma^j - \frac{1}{3k^2}(4k_i - \gamma_i\not{k})k^j\,. \tag{2.69}$$

The correlator can be written as $G_{ij} = P_i^k M_{kl} P_j^l$. At zero temperature, Lorentz invariance dictates the form of M to be $M_{kl} = A(k^2)\not{k}\,\eta_{kl}$.

Before doing the computation at finite temperature let us recall the results found for the correlator at zero temperature and Euclidean signature [Corley, 1999, Rashkov, 1999, Volovich, 1998]. The bulk action for the gravitino in Euclidean signature is given by:

$$S = \int d^4x\sqrt{g}(\bar{\Psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\Psi_\rho - m\bar{\Psi}_\mu\Gamma^{\mu\nu}\Psi_\nu)\,. \tag{2.70}$$

The covariant derivative acts on a spinor as:

$$D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\,. \tag{2.71}$$

The gravitino is in the AdS_5 background with metric

$$ds^2 = \frac{1}{z^2}(dx^\mu dx_\mu + dz^2) \tag{2.72}$$

and spin connection

$$\omega_\mu^{ab} = -\frac{1}{z}(\delta_\mu^a\delta_z^b - \delta_\mu^b\delta_z^a)\,. \tag{2.73}$$

The equation of motion is the Rarita-Schwinger equation

$$\Gamma^{\mu\nu\rho}D_\nu\Psi_\rho - m\Gamma^{\mu\nu}\Psi_\nu = 0\,. \tag{2.74}$$

One can decompose the field into transverse and longitudinal components with respect to the projector (2.69) and then the equations of motion of the two sectors decouple. The gravitino obeys a first order equation and imposing regularity at the interior of the space is enough to find the solution. Then expanding the solution near the boundary we see that different chirality components have different behavior near the boundary:

$$\begin{aligned}\psi_i^+(z,x^\mu) &= z^{2-m}\varphi_i(x) + ...\\ \psi_i^-(z,x^\mu) &= z^{2+m}\chi_i(x) + ...\end{aligned} \tag{2.75}$$

where $\psi_a = e_a^\mu \Psi_\mu$. Then following the standard procedure, for $m > 0$, we identify the source as the most important contribution near the boundary i.e. with $\varphi_i(x)$. The source fields φ_i must also satisfy the condition $\gamma^i\varphi_i = 0$. The solution in momentum space can be found in terms of the source fields:

$$\begin{aligned}\psi_i(z,k) =& \, z^{2-m}\left[\varphi_i - \frac{i\not{k}K_{m-1/2}(kz)}{kK_{m+1/2}(kz)}\varphi_i\right.\\ &\left. - \frac{2k_j\not{k}}{k^2}\frac{ikz\frac{K_{m-1/2}(kz)K_{m+3/2}(kz)-K^2_{m+1/2}(kz)}{K_{m+1/2}(kz)} + \frac{i\not{k}\gamma_i}{k}K_{m-1/2}(kz)}{(2m+3)K_{m+1/2}(kz) - 2kzK_{m+3/2}(kz)}\varphi_j\right]. \end{aligned} \tag{2.76}$$

Note that this solution implicitly contains a non trivial relation between χ_i and φ_i. The solution for the conjugate field $\bar{\psi}_i$ is given by a similar expression.
Given that, for the Rarita-Schwinger field, the on-shell action vanishes, the only contribution to the correlator will come from the boundary action:

$$S_{\mathrm{bdy}} = \mathcal{N}\int d^4x\sqrt{h}h^{ij}\bar{\Psi}_i\Psi^j \tag{2.77}$$

where h is the induced metric on the boundary. In order to compute the correlator we need to add counterterms to regularize the boundary action using the holographic renormalization technique. After removing the divergent terms the boundary action takes the simple form

$$S_{\mathrm{bdy}} = \mathcal{N}\int d^4k\ \delta^{ij}(\bar{\varphi}_i^\dagger(-k)\chi_j(k) + \bar{\chi}_i(-k)\varphi_j(k))\,. \tag{2.78}$$

The correlator then can be found be varying this action twice with respect to the source. The final result in momentum space is given by

$$G_{ij}(k) \propto \Pi_i^l\frac{\not{k}}{k}\left(\delta_{ln} - \frac{2(2m+1)}{2m+5}\frac{k_lk_n}{k^2}\right)\Pi_j^n \tag{2.79}$$

up to a constant multiplicative factor and $\Pi_i^j = \delta_i^j - \gamma_i\gamma^j/4$. Fourier transforming this expression we get

$$G_{ij}(x-y) \propto \Pi_i^l\frac{(x-y)_k\gamma^k}{|x-y|^{2m+5}}\left[\delta_{ln} - 2\frac{(x-y)_l(x-y)_n}{|x-y|^2}\right]\Pi_j^n\,. \tag{2.80}$$

As we saw in the previous chapter conformal symmetry severely constrains the form of two point functions and we see here an example where the holographically computed two-point function matches exactly the conformal field theory expectations. The gravitino is a vector spinor and therefore the representation appearing in equation (1.6) should be the product of the vector representation and the spinor representation with the components transverse to γ_i projected out. The inversion operator in the spinor representation is given by

$$D(I(x)) = -\frac{x^i\gamma_i}{|x|}\,. \tag{2.81}$$

Comparing (2.80) with the conformal field theory result we confirm that it has the right form for a field of dimension $m+2$, which is the dimension of a spin $3/2$ field (recall eq. (1.36)).

2.5 Supercharge diffusion at finite temperature

Let us start by considering the structure of the correlator at finite temperature, a generalized form of the projector (2.69) will be needed in this case. At finite temperature or density Lorentz invariance is broken and the correlator can depend apart from k also on the velocity of the fluid u^i. One can write another projector that is transverse both to k and u:

$$\begin{aligned} P^T_{11} &= P^T_{1i} = 0\,, \\ P^T_{jk} &= \delta_{jk} - \frac{1}{2}\left(\gamma_j - \frac{q_j \not{q}}{q^2}\right)\gamma_k - \frac{1}{2q^2}(3q_j - \gamma_j \not{q})q_k\,, \end{aligned} \tag{2.82}$$

where $k^i = (\omega - \mu, \mathbf{q})$. We define the longitudinal projector as $P^L = P - P^T$. The correlator can have three possible structures: $P^L P^L$, $P^T P^T$ and $P^L P^T + P^T P^L$. In the following we choose $\mathbf{q} = (0, 0, q)$. With this choice the mixed correlator vanishes identically. The longitudinal and the transverse part of the correlator can be written as:

$$\begin{aligned} G^L_{ij} &= (P^L)^k_i M_{km} (P^L)_{mj} \\ G^T_{ij} &= (P^T)^k_i \tilde{M}_{km} (P^T)_{mj} \end{aligned} \tag{2.83}$$

and the most general form of M and $\tilde{M}$ allowed by the symmetries is:

$$\begin{aligned} M_{km} &= \eta_{km}(a\not{k} + a'\not{u}) + u_k u_m (b\not{k} + b'\not{u})\,, \\ \tilde{M}_{km} &= \eta_{km}(\tilde{a}\not{k} + \tilde{a}'\not{u})\,. \end{aligned} \tag{2.84}$$

There are additional relations between the coefficients that come from requiring rotational invariance at zero momentum. For the following we will need these relations between the components of the correlator in the low momentum limit:

$$\begin{aligned} G^T_{22} &= G^T_{33} = -a'/2, \qquad & G^T_{44} &= 0\,, \\ G^L_{22} &= G^L_{33} = -a'/6, \qquad & G^L_{44} &= -2a'/3\,. \end{aligned} \tag{2.85}$$

The longitudinal part of the correlator was calculated in [Policastro, 2009] and the diffusion constant was extracted from the pole of the correlator. Here we will compute the transverse part of the correlator and compute the diffusion constant using the Kubo formula (2.42).

For the holographic computation we have to consider a gravitino propagating in the near-extremal 3-brane background (2.43). The action(2.70) reads, in Minkowski signature,

$$S = \int d^4x \sqrt{-g}(\bar{\Psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \Psi_\rho - m \bar{\Psi}_\mu \Gamma^{\mu\nu} \Psi_\nu) \tag{2.86}$$

while the equations of motion remain unchanged. The non-vanishing components of the spin connection are:

$$\begin{aligned} \omega^{51}_t &= \frac{\pi T(1+u^2)}{\sqrt{u}} \\ \omega^{52}_x = \omega^{53}_y = \omega^{54}_z &= \pi T \sqrt{\frac{f}{u}}\,. \end{aligned} \tag{2.87}$$

In the gauge $\Gamma^\mu\Psi_\mu = 0$ the equations for the spatial components of the field read [Policastro, 2009]:

$$\gamma^5\psi_k' + \frac{1}{2\pi T\sqrt{uf}}\left(\frac{1}{\sqrt{f}}\gamma^1\partial_t + \gamma^j\partial_j\right)\psi_k + \frac{u^2-2}{2uf}\gamma^5\psi_k - \frac{1}{2u}\gamma_k\psi_5 + \frac{mR}{2u\sqrt{f}}\psi_k = 0 \quad (2.88)$$

where $\psi_a = e_a^\mu\Psi_\mu$, $\gamma_a = e_a^\mu\Gamma_\mu$ and $\psi' = \partial\psi/\partial u$.
Define the Fourier transform as:

$$\Psi_\mu(u,x) = \frac{1}{(2\pi)^2}\int d^4k e^{ik_\nu x^\nu}\Psi_\mu(u,k) \quad (2.89)$$

with $k^\mu = 2\pi T(\omega,0,0,q)$.
The field $\eta = \gamma^2\psi_2 - \gamma^3\psi_3$ has spin 3/2 under the $O(2)$ rotational symmetry that preserves k^μ and therefore it decouples from the other components that have spin 1/2. It is easy to see that this is the field that is dual to the transverse part of the correlator. Multiplying (2.88) for $k=2$ by γ_2 and for $k=3$ by γ_3 and subtracting we arrive at the equation satisfied by the transverse component of the gravitino:

$$\eta' = -\gamma^5\left(\frac{\not{P}}{\sqrt{uf}} + \frac{u^2-2}{2uf}\gamma^5 - \frac{3}{4u\sqrt{f}}\right)\eta \quad (2.90)$$

with $\not{P} = -i\omega/\sqrt{f}\gamma^1 + iq\gamma^4$.
We cannot solve equation (2.90) analytically but we can solve it as an expansion in momenta. A useful remark is that equation (2.90) commutes with $\gamma_{23} = \mathrm{diag}(1,-1,1,-1)$ and therefore we can project on its eigenvectors. In the eigenspace $\gamma_{23} = 1$ (resp. -1) equation (2.90) becomes a matrix differential equation for (η_1,η_3) (resp. (η_2,η_4)).

We find the following solutions near the horizon :

$$\begin{aligned}&(1-u)^{-\frac{1}{4}-i\frac{\omega}{2}}(-i,1)\\&(1-u)^{-\frac{1}{4}+i\frac{\omega}{2}}(i,1)\end{aligned} \quad (2.91)$$

We have to impose incoming boundary conditions at the horizon in order to calculate the retarded correlator [Son and Starinets, 2002]. We can keep the incoming waves at the horizon by imposing the relation $\eta^+ = -i\eta^-$. For the two other components (η_2,η_4) the equations are the same after the exchange $q\to -q$. Repeating the analysis for these components we find the solution up to first order in momenta. The near boundary expansion of this solution reads:

$$\eta = \begin{pmatrix} u^{3/4}(-i3^{3/4})(q-\omega)\alpha + u^{7/4}\frac{3^{3/4}}{2\sqrt{2}}\beta(\omega,q)\\ u^{3/4}i3^{3/4}(\omega+q)\gamma + u^{7/4}\frac{3^{3/4}}{2\sqrt{2}}\delta(\omega,q)\\ u^{1/4}3^{3/4}[i\beta(\omega,q) + \sqrt{2}(q-(1-\sqrt{2}\not{L})\omega)\alpha]\\ u^{1/4}3^{3/4}[i\delta(\omega,q) - \sqrt{2}(q+(1-\sqrt{2}\not{L})\omega)\gamma]\end{pmatrix}, \quad (2.92)$$

where $\not{L} = \log(1+\sqrt{2})$. We may identify the source of the field φ with the negative chirality part $\eta^- = u^{1/4}\varphi$ and identify χ and $\tilde{\chi}$ as the coefficients of the positive chirality expansion $\eta^+ = u^{3/4}\tilde{\chi} + u^{7/4}\chi$:

$$\varphi = \begin{pmatrix} 3^{3/4}[i\beta(\omega,q)) + \sqrt{2}(q-(1-\sqrt{2}\not{L})\omega)\alpha]\\ 3^{3/4}[i\delta(\omega,q) - \sqrt{2}(q+(1-\sqrt{2}\not{L})\omega)\gamma]\end{pmatrix} \quad (2.93)$$

$$\chi = \begin{pmatrix} \frac{3^{3/4}}{2\sqrt{2}}\beta(\omega,q) \\ \frac{3^{3/4}}{2\sqrt{2}}\delta(\omega,q) \end{pmatrix} \qquad \tilde{\chi} = \begin{pmatrix} i3^{3/4}(\omega-q)\alpha \\ i3^{3/4}(\omega+q)\gamma \end{pmatrix} . \tag{2.94}$$

Multiplying ϕ by $\not{k} = -i2\pi T \,\mathrm{diag}(\omega - q, \omega + q)$ and neglecting terms that are of higher order in the momenta we can express $\tilde{\chi}$ in terms of the source:

$$\tilde{\chi} = \frac{i}{2\pi T}\not{k}\varphi \, . \tag{2.95}$$

Similarly we can verify that the following expression is correct up to higher order terms in the momenta:

$$\varphi = \mathrm{diag}(2\sqrt{2}i + 4q - 4(1-\sqrt{2}\not{L})\omega, 2\sqrt{2}i - 4q - 4(1-\sqrt{2}\not{L})\omega)\chi \tag{2.96}$$

Inverting the matrix and expanding on small momenta we can express χ in terms of the source:

$$\chi = \mathrm{diag}(-i\frac{\sqrt{2}}{4} - 4q + 4(1-\sqrt{2}\not{L})\omega, -i\frac{\sqrt{2}}{4} + 4q + 4(1-\sqrt{2}\not{L})\omega)\varphi \, . \tag{2.97}$$

The correlator will be again given by the boundary action:

$$S_{\mathrm{bdy}} = \mathcal{N}\int d^4x\sqrt{-h}h^{ij}\bar{\Psi}_i\Psi^j \, , \tag{2.98}$$

where h is the induced metric on the boundary and $\bar{\Psi} = i\Psi^\dagger\Gamma^1$. Performing the field redefinitions $\phi = \gamma^2\psi_2 + \gamma^3\psi_3$ and $\eta = \gamma^2\psi_2 - \gamma^3\psi_3$ and setting $\phi = 0$, $\psi_1 = 0$, $\psi_4 = 0$ since we are only interested in the transverse part we get:

$$S_{\mathrm{bdy}} = \frac{\mathcal{N}}{2}\int d^4x\sqrt{-h}\frac{\sqrt{u}}{\sqrt{f}\pi T R}\bar{\eta}\eta \, . \tag{2.99}$$

Fourier transforming and taking the limit $u = \epsilon \to 0$ we have:

$$S_{\mathrm{bdy}} = \frac{\mathcal{N}}{2}\frac{\sqrt{\epsilon}}{\pi T R}\int d^4k\sqrt{-h}\left[\eta_-^\dagger(-k)\eta_+(k) + \eta_+^\dagger(-k)\eta_-(k)\right] \tag{2.100}$$

The correlator that we want to calculate is not Hermitian and therefore we should keep only the first term in the action to extract the correlator. This procedure is equivalent to the recipe proposed in [Iqbal and Liu, 2009a]. Substituting the expansion of the field on the boundary the action becomes:

$$S_{\mathrm{bdy}} = \frac{\mathcal{N}}{2}(\pi T R)^3\int d^4k\left(\epsilon^{-1/2}\varphi^\dagger\tilde{\chi} + \epsilon^{1/2}\varphi^\dagger\chi\right) . \tag{2.101}$$

To substract the divergencies in the limit $\epsilon \to 0$ we introduce the counterterm:

$$S_{\mathrm{ct}} = -S_{\mathrm{bdy}}^{\mathrm{div}} = -\frac{\mathcal{N}}{2}(\pi T R)^3\int d^4k\epsilon^{-1/2}\varphi^\dagger\tilde{\chi} \tag{2.102}$$

The counterterm is written in covariant form as:

$$S_{\rm ct} = -\frac{\mathcal{N}R}{4}\int d^4x\sqrt{-h}\bar{\eta}_-\partial\!\!\!/_h\eta_-\,, \qquad (2.103)$$

where $\bar{\eta}_- = \eta^\dagger\Gamma^1$ and $\partial\!\!\!/_h = \Gamma^\mu\partial_\mu$. This counterterm does not introduce any finite correction to the correlator since there is no $u^{7/4}$ term in the boundary expansion of η_-. The correlator is calculated from the renormalized action.

$$S_{\rm ren} = S_{\rm bdy} + S_{\rm ct} =$$
$$\frac{\mathcal{N}}{2}(\pi TR)^3\int\varphi^\dagger\,{\rm diag}(-i\frac{\sqrt{2}}{4} - 4q + 4(1-\sqrt{2}L\!\!\!/)\omega, -i\frac{\sqrt{2}}{4} + 4q + 4(1-\sqrt{2}L\!\!\!/)\omega)\varphi\,.$$

The retarded correlator of the operator dual to η is therefore:

$$G^{\mathcal{R}} = \frac{\mathcal{N}}{2}(\pi TR)^3\,{\rm diag}(-i\frac{\sqrt{2}}{4} - 4q + 4(1-\sqrt{2}L\!\!\!/)\omega, -i\frac{\sqrt{2}}{4} + 4q + 4(1-\sqrt{2}L\!\!\!/)\omega)\,, \qquad (2.104)$$

where we have absorbed the extra ϵ power in a field redefinition. The normalization of the action is calculated in the appendix and is found to be:

$$\mathcal{N} = \frac{N_c^2}{\pi^2}\,. \qquad (2.105)$$

From the analysis of the projections of the correlators we can deduce that with i indices summed $G^{ii}_{\alpha\dot{\alpha}} = 4G^{\mathcal{R}}_{\alpha\dot{\alpha}}$ where $G^{ii}_{\alpha\dot{\alpha}}$ is the correlator that comes in the Kubo formula (2.42) for the diffusion constant. The energy density is (cf. equation (1.68)):

$$\epsilon = \frac{3\pi^2}{8}N_c^2T^4\,.$$

We finally compute the diffusion constant from the Kubo formula and find:

$$D_s = \frac{2\sqrt{2}}{9\pi T}\,. \qquad (2.106)$$

This result is in agreement with the diffusion constant calculated in [Policastro, 2009].

2.5.1 Running of the correlator

In this section we give an alternative derivation in the spirit of the membrane paradigm approach to the question of universality of the diffusion constant [Iqbal and Liu, 2009b]. The idea is that the correlator can be formally computed on any radial slice of the geometry. When evaluated on the boundary it corresponds to the real field-theory correlator, and one can derive a flow equation that expresses the running from the boundary to the horizon. In favorable cases, at low frequency the flow is trivial and the UV correlator is equal to the value at the horizon.

Let us consider again the Rarita-Schwinger equation (2.90); at zero momentum and frequency, it becomes

$$\eta'_+ = (-A+B)\eta_+\,, \quad \eta'_- = (-A-B)\eta_- \qquad (2.107)$$

where $B = m\sqrt{g_{uu}}$, and $mR = 3/2$ for the gravitino. The ratio $w = \eta_+/\eta_-$ satisfies $\partial_u w = 2Bw$, and we have the flow equation

$$\mathcal{F}' \equiv \partial_u \left(e^{-2\int B(u)du} \frac{\eta_+}{\eta_-} \right) = 0\,. \tag{2.108}$$

The integral converges at the horizon, so we have $\mathcal{F}(u=1) = \eta_+/\eta_-(u=1) = -i$. At the boundary we have $\mathcal{F} \sim u^{-m}\eta_+/\eta_- = \chi/\phi$, but the coefficient of proportionality requires the computation of $\int_1^u \sqrt{g_{uu}}$ so we cannot express it purely in terms of horizon data. We find

$$D_s = \frac{8}{9\pi T} \exp\left(-m\log\epsilon - 2m \int_\epsilon^1 \sqrt{g_{uu}} du \right) \tag{2.109}$$

For the the near-extremal background we have $g_{uu} = \frac{R^2}{4(1-u^2)u^2}$ and performing the integral we get:

$$D_s = \frac{8}{9\pi T} \exp(-mR\log 2)\,, \tag{2.110}$$

which reproduces the result (2.106), and allows to compute the diffusion constant in any asymptotically AdS black hole background. However, as we will see, in presence of a finite charge density for a $U(1)$ under which the gravitino is charged (typically an R-symmetry), the equations of motion are more complicated and do not become diagonal even at zero momentum.

2.6 Supercharge diffusion at finite density

2.6.1 The background

We want to consider the change in the diffusion constant when there is a finite background charge density. In $\mathcal{N}=4$ SYM there is an $SU(4)$ R-symmetry group under which the supercurrents transform in the fundamental representation. It is possible to give a vev simultaneously to three charges in the Cartan subalgebra of $SU(4)$. The corresponding solution is an asymptotically AdS black hole carrying electric charges, known as the STU black hole.

The STU solution was first constructed as a solution of $d=5, \mathcal{N}=2$ gauged supergravity with 2 matter multiplets where the gauging is performed by introducing a linear combination of the Abelian fields already present in the ungauged theory $A_\mu = V_I A^I_\mu$. The construction of the theory is explained in [Gunaydin et al., 1985] and the solutions of the bosonic part are presented in [Behrndt et al., 1999]. The solution can be embedded in type IIB supergravity, the uplift to 10 dimensions is given in [Cvetic et al., 1999] but for our purposes it is sufficient to work with the 5-dimensional solution.

The bosonic part of the gauged $\mathcal{N}=2$ supergravity Lagrangian reads:

$$\mathcal{L} = e\left[R + \frac{2}{L^2}V - \frac{1}{2}G_{IJ}F^I_{\mu\nu}F^{\mu\nu\,I} - G_{IJ}\partial_\mu X^I \partial_\nu X^J + \frac{1}{24e}\epsilon^{\mu\nu\rho\sigma\lambda}C_{IJK}F^I_{\mu\nu}F^{\rho\sigma\,J}A^K_\lambda \right], \tag{2.111}$$

where $F^I_{\mu\nu}$, $I = 1, 2, 3$ are the field strengths corresponding to the three Abelian gauge fields A^I_μ, $e = \sqrt{-g}$ and the potential V is given by

$$V(X) = 9V_I V_J \left(X^I X^J - \frac{1}{2} G^{IJ} \right) . \tag{2.112}$$

The X^I are scalar fields which have to satisfy the constraint

$$\frac{1}{6} C_{IJK} X^I X^J X^K = 1 \, , \tag{2.113}$$

where C_{IJK} are constants and G_{IJ} is the metric on the scalar manifold.
We are interested in finding flat black hole solutions and therefore we adopt an ansatz for the metric of the form:

$$ds^2 = -e^{-4U(r)} f(r) dt^2 + e^{2U(r)} \left(\frac{dr^2}{f(r)} + \frac{r^2}{L^2}(dx^2 + dy^2 + dz^2) \right) \tag{2.114}$$

with

$$f(r) = -\frac{m}{r^2} + \frac{r^2}{L^2} e^{6U(r)} \, . \tag{2.115}$$

For the scalar fields we adopt the ansatz

$$X_I = \frac{1}{3} e^{-2U(r)} H_I \qquad H_I = h_I + \frac{q_I}{r^2} \, , \tag{2.116}$$

The functions H_I are harmonic functions and there is no guarantee that the equations of motion will have a solution in terms of these functions.
The gauge field equations read:

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} G_{IJ} F^{J\,\mu\nu} \right) \propto \epsilon^{\nu\alpha\beta\gamma\lambda} F^J_{\alpha\beta} F^K_{\gamma\lambda} C_{IJK} \, . \tag{2.117}$$

Since we are interested only in electrically charged solutions the right hand side of the equation vanishes and the equation is solved by:

$$F^J_{rt} = -a^J \, e^{-4U(r)} G^{JI} \partial_r \tilde{H}_I \, , \tag{2.118}$$

where

$$\tilde{H}_I = 1 + \frac{\tilde{q}_I}{r^2} \tag{2.119}$$

are harmonic functions and $\tilde{q}_I$ represent the physical charges and a^J are constants to be determined later. The Einstein equations read:

$$R_{\mu\nu} = F^2_{\mu\nu} - \frac{1}{6} g_{\mu\nu} F^2 + \partial_\mu X^I \partial_\nu X^J G_{IJ} - \frac{2}{3L^2} V(X) g_{\mu\nu} \tag{2.120}$$

with $F^2_{\mu\nu} = G_{IJ} F^I_{\mu\rho} F^J_{\nu\lambda} g^{\rho\lambda}$. Analyzing these equations the requirement $h_I = 3V_I$ appears as an additional constraint.
For a general choice of the constants C_{IJK} the equations cannot be solved using the harmonic function ansatze and one needs to regard H_I and $\tilde{H}_I$ as general functions. For

a special choice of the constants, $C_{123} = 1$ and the rest of the constants zero the metric on the scalar manifold is given by:

$$G^{IJ} = 2\,\mathrm{diag}[(X^1)^2, (X^2)^2, (X^3)^2] \tag{2.121}$$

For this configuration there is a solution in terms of harmonic functions, the STU black hole [Behrndt et al., 1999]. In this case the scalar potential is $V = 2\sum_I 1/X^I$ and the scalar field satisfy the constraint $X^1X^2X^3 = 1$. The equations are solved by

$$e^{6U(r)} = H_1H_2H_3 \tag{2.122}$$

identifying the charges $\tilde{q}_I = q_I$ and setting $a^I = \sqrt{m/q_I}/2$. Furthermore we choose $h_I = 1$ so that the harmonic functions are canonically normalized. The metric can now be written in the form:

$$ds^2 = -\mathcal{H}^{-2/3}f(r)dt^2 + \mathcal{H}^{1/3}\left(\frac{dr^2}{f(r)} + r^2(dx^2 + dy^2 + dz^2)\right) \tag{2.123}$$

$$f(r) = -\frac{m}{r^2} + \frac{r^2}{L^2}\mathcal{H} \qquad H_I = 1 + \frac{q_I}{r^2} \qquad \mathcal{H} = H_1H_2H_3 \tag{2.124}$$

$$X^I = \frac{\mathcal{H}^{1/3}}{H_I(r)} \qquad A_t^I = \sqrt{\frac{m}{q_I}}(1 - H_I^{-1})\,. \tag{2.125}$$

Introducing a new coordinate $u = r_+^2/r^2$ where r_+ is the largest root of $f(r) = 0$ the metric of the STU black hole becomes:

$$ds^2 = -\mathcal{H}^{-2/3}\frac{(\pi TL)^2}{u}f(u)dt^2 + \mathcal{H}^{1/3}\frac{(\pi TL)^2}{u}(dx^2 + dy^2 + dz^2) + \mathcal{H}^{1/3}\frac{L^2}{4f(u)u^2}du^2$$

$$f(u) = \mathcal{H}(u) - u^2\prod_{I=1}^{3}(1 + \kappa_I) \qquad H_I(u) = 1 + \kappa_I u$$

$$\kappa_I = \frac{q_I}{(\pi TL^2)^2} \qquad \mathcal{H}(u) = \prod_{I=1}^{3} H_I(u)\,.$$

In the zero charge limit this metric is exactly (2.43). The scalar and gauge fields are given by:

$$X^I = \frac{\mathcal{H}^{1/3}}{H_I(u)} \qquad A_t^I = \pi TLu\frac{\kappa_I}{H_I(u)}\prod_{J=1}^{3}(1 + \kappa_J) \tag{2.126}$$

The Hawking temperature of the background is:

$$T_H = \frac{2 + \kappa_1 + \kappa_2 + \kappa_3 - \kappa_1\kappa_2\kappa_3}{2\sqrt{(1 + \kappa_1)(1 + \kappa_2)(1 + \kappa_3)}}T\,. \tag{2.127}$$

The energy density, entropy density, charge density and chemical potentials are respectively:

$$\begin{aligned}\epsilon &= \frac{3\pi^2 N^2 T^4}{8}\prod_{I=1}^{3}(1+\kappa_I)\\ s &= \frac{\pi^2 N^2 T^3}{2}\prod_{I=1}^{3}(1+\kappa_I)^{1/2}\\ \rho_I &= \frac{\pi N^2 T^3}{8}\sqrt{\kappa_I}\prod_{J=1}^{3}(1+\kappa_J)^{1/2}\\ \mu_I = A_t^I(u)|_{u=1} &= \frac{\pi T L\sqrt{\kappa_I}}{1+\kappa_I}\prod_{J=1}^{3}(1+\kappa_J)^{1/2}\end{aligned} \tag{2.128}$$

Imposing the condition of thermodynamic stability on the background implies a restriction on the charges of the background [Son and Starinets, 2006]

$$2-\kappa_1-\kappa_2-\kappa_3+\kappa_1\kappa_2\kappa_3 > 0\,. \tag{2.129}$$

The relevant part of the Lagrangian for the gravitino is [Gunaydin et al., 1985]:

$$\begin{aligned}\mathcal{L} =& e\,[\ \bar{\Psi}_\mu\Gamma^{\mu\nu\rho}\mathcal{D}_\nu\Psi_\rho + i\bar{\lambda}^a\Gamma^\mu\Gamma^\nu\Psi_\mu f_i^a\partial_\nu\phi^i - \frac{1}{2}h_I^a\bar{\lambda}^a\Gamma^\mu\Gamma^{\lambda\rho}\Psi_\mu F_{\lambda\rho}^I\\ &+\frac{1}{8X^I}i\left(\bar{\Psi}_\mu\Gamma^{\mu\nu\rho\sigma}\Psi_\nu F_{\rho\sigma}^I + 2\bar{\Psi}^\mu\Psi^\nu F_{\mu\nu}^I\right) + \frac{3}{2L}\bar{\Psi}_\mu\Gamma^{\mu\nu}\Psi_\nu\mathcal{V}_o - \frac{3}{L}\bar{\lambda}^a\Gamma^\mu\Psi_\mu\mathcal{V}_a\ ]\,,\end{aligned} \tag{2.130}$$

where $I=1,2,3$, $i,a=1,2$. h_I^a, $\mathcal{V}_a$ and $\mathcal{V}_o$ are functions of the scalar fields , ϕ^i are the scalars fields on the constrained scalar manifold and f_i^a is the vielbein on this manifold. The covariant derivative acts on a spinor as

$$\mathcal{D}_\mu\psi = D_\mu\psi - i\frac{3}{2L}V_I A_\mu^I\psi\,,$$

where V_I are constants and should be equal to 1/3 for the H_I to be properly normalized [Behrndt et al., 1999]. $\mathcal{V}_o$ is given by:

$$\mathcal{V}_o = \frac{1}{2}G^{IJ}\frac{V_I}{X^J} \tag{2.131}$$

and $F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I$ are the gauge field strengths.The equations of motion for the gravitino are:

$$\begin{aligned}&\Gamma^{\mu\nu\rho}\mathcal{D}_\nu\Psi_\rho + i\Gamma^\mu\Gamma^\nu\lambda_a f_i^a\partial_\nu\phi^i - \frac{1}{2}h_I^a\Gamma^\mu\Gamma^{\lambda\rho}\lambda^a F_{\lambda\rho}^I + \frac{i}{8X^I}\left(\Gamma^{\mu\nu\rho\sigma}\Psi_\nu F_{\rho\sigma}^I + 2\Psi^\nu F_{\mu\nu}^I\right)\\ &\qquad + \frac{3}{2L}\Gamma^{\mu\nu}\Psi_\nu\mathcal{V}_o - \frac{3}{L}\Gamma^\mu\lambda^a\mathcal{V}_a = 0\end{aligned} \tag{2.132}$$

The supersymmetry transformations act on the gravitino as:

$$\delta\Psi_\mu = \mathcal{D}_\mu\epsilon + \frac{i}{24X^I}(\Gamma_{mu}^{\nu\rho} - 4\delta_\mu^\nu\Gamma^\rho)\hat{F}_{\nu\rho}^I\epsilon + \frac{1}{6L}\sum_I X^I\Gamma_\mu\epsilon\,, \tag{2.133}$$

where

$$\hat{F}^I_{\mu\nu} = F^I_{\mu\nu} + \frac{i}{4X^I}\bar{\Psi}_{[\mu}\Psi_{\nu]}$$

and we neglected the terms containing the spinors λ in the two previous formulas.
We are again interested in the transverse component of the gravitino, which can be defined as $\eta = \Gamma^x\Psi_x - \Gamma^y\Psi_y$. Using the same strategy as in the zero-charge case, one can show that the transverse part of the gravitino is decoupled, both from the longitudinal part and from the spinors λ. The equation for the transverse component is therefore:

$$\eta' + \left(\frac{\gamma^5 \not{K}}{\sqrt{fu}} + F(u) - \frac{3i}{2L}\sqrt{g_{uu}g^{tt}}\gamma_5\gamma^1 V_I A^I_t + \frac{i}{4X^I}\sqrt{g^{tt}}\gamma^1 F^I_{ut} + \frac{3}{2L}\sqrt{g_{uu}}\gamma^5\mathcal{V}_o\right)\eta = 0 \tag{2.134}$$

with

$$\begin{aligned} \not{K} &= -i\sqrt{\frac{\mathcal{H}}{f}}\gamma^1\omega + iq\gamma^4 \\ F(u) &= \frac{1}{4}\frac{g'_{tt}}{g_{tt}} + \frac{3}{4}\frac{g'_{xx}}{g_{xx}} . \end{aligned} \tag{2.135}$$

Note that the transverse component of the gravitino is gauge invariant and that we can again split our equation in two different systems by projecting on the eigenspaces of γ_{23}.

2.6.2 Analytic solution for small charge

The system of differential equations obtained above cannot be solved analytically. Since calculating the diffusion constant requires knowing the retarded correlator for zero momenta we set the momentum to zero and attempt to find a solution as an expansion in the charges. We consider in turn the cases where only one charge is turned on, and when there are three equal charges.

One charge

In this case we take $\kappa_1 = \kappa,\ \kappa_2 = \kappa_3 = 0$ and attempt to find a solution of the form:

$$\eta(u,\kappa) = \eta^{(0)}(u) + \sqrt{\kappa}\,\eta^{(1)}(u) + \kappa\,\eta^{(2)}(u) + \dots \tag{2.136}$$

Solving the system of equations near the horizon we identify the exponents:

$$U_\pm = -\frac{1}{4} \pm \frac{i}{4}\frac{\sqrt{\kappa} + 4\sqrt{1+\kappa}\,\omega}{2+\kappa} . \tag{2.137}$$

The choice of U_- yields incoming-wave boundary conditions. Near the boundary we find the following behavior:

$$\begin{aligned} \eta_1 &= \omega\phi(\omega,\kappa)u^{3/4} + \chi(\omega,\kappa)u^{7/4} + \phi(\omega,\kappa)\frac{\sqrt{\kappa(1+\kappa)} + 2\omega^3}{2}u^{7/4}\log u + \dots , \\ \eta_3 &= \phi(\omega,\kappa)u^{1/4} + \tilde{\phi}(\omega,\kappa)u^{5/4} + \dots , \end{aligned} \tag{2.138}$$

This expansion implies that there will be a logarithmic divergence of the correlator that would need to be regularized with an appropriate counterterm. However this will not affect the diffusion constant that is computed from the imaginary part of the correlator, so we will ignore this divergence.

The integration constants that appear at every order in the $\sqrt{\kappa}$ expansion (2.136) can be fixed at zeroth order by imposing incoming boundary conditions at the horizon and at the i-th order by imposing $\eta^{(i)} = 0$ on the horizon. The near boundary expansion up to order $\sqrt{\kappa}$ for the two components is [2]:

$$\begin{aligned} \eta_1 &= u^{7/4}[2^{-3/4} i + \sqrt{\kappa}(\alpha + \beta \log u)] \\ \eta_3 &= u^{1/4}(2^{3/4} - i\sqrt{\kappa}\gamma) \end{aligned} \tag{2.139}$$

where α, β, γ are *real* constants. Notice that there is no term going like $u^{3/4}$ so we don't need to regularize the boundary action. In terms of the source ϕ and the normalizable mode χ the correlator is then simply given by:

$$G^{\mathcal{R}} = \frac{\pi N_c^2 T^3}{2} \frac{\chi}{\phi} = \frac{-\sqrt{2}}{4} i + \left(\frac{\alpha}{2^{3/4}} - \frac{\gamma}{4\, 2^{3/4}} + \frac{\beta}{2^{3/4}} \log u \right) \sqrt{\kappa} + ... \tag{2.140}$$

The coefficient of $\sqrt{\kappa}$ is real and therefore the diffusion constant is not corrected to the order $\sqrt{\kappa}$.

Three charges

We consider here the case of three equal charges, namely $\kappa_1 = \kappa_2 = \kappa_3 = \kappa$. For this special choice of the charges the thermodynamic stability condition (2.129) is saturated for $\kappa = 2$ but it is not violated even for $\kappa > 2$. The Hawking temperature (2.127) on the other hand becomes negative for $\kappa > 2$ and so in order to have a physical theory we must restrict the value of the charge to $\kappa < 2$. For $\kappa = 2$ we have an extremal black hole as it can be seen from equation 2.127.
We attempt to find a solution as an expansion in $\sqrt{\kappa}$ as in the case of one charge. The indices near the horizon are

$$V_{\pm} = -\frac{1}{4} \pm i \frac{3\sqrt{\kappa(1+\kappa)} + 4\omega}{4(2-\kappa)\sqrt{1+\kappa}} \tag{2.141}$$

and the incoming wave boundary condition corresponds to V_-. The near boundary expansion is written in this case:

$$\begin{aligned} \eta_1 &= \omega\phi(\omega,\kappa)u^{3/4} + \chi(\omega,\kappa)u^{7/4} + \phi(\omega,\kappa)\frac{3\sqrt{\kappa(1+\kappa)} - 8\omega^3}{8} u^{7/4} \log u + \dots \\ \eta_3 &= \phi(\omega,\kappa)u^{1/4} + \tilde{\phi}(\omega,\kappa)u^{5/4} + \dots \end{aligned} \tag{2.142}$$

The calculation done for the one charged case goes through for the three charges case and the end result is similar to (2.139) with different numerical values for the constants

2. For q=0 the system for (η_1, η_3) is identical to the system for (η_2, η_4)

α, β, γ. In this case too, then, the diffusion constant does not receive corrections to order $\sqrt{\kappa}$.
The explicit computation of higher orders in this expansion becomes too complicated, therefore we revert to a numerical analysis.

2.6.3 Numerical Solution

In order to solve numerically the system of equations we need to provide initial conditions in form of a near horizon expansion where we have imposed the incoming boundary conditions. We will work with the components (η_1, η_3) since the equations for the other two are exactly identical for $q = 0$. We solve the system near the horizon and find a expansion of the form:

$$\eta_{hor}(\omega, u, \kappa) = \eta_{hor}^{(0)}(\omega, \kappa) + \sqrt{1-u}\ \eta_{hor}^{(1)}(\omega, \kappa) + \dots \tag{2.143}$$

Using the expansion above as an initial condition we integrate the system of equations from the horizon to the boundary. Then we can extract the correlator from the near boundary region of the solution.

The result for the one-charge case is presented in figure 2.1. At $\kappa = 0$ it reproduces correctly the value calculated analytically for zero charge.

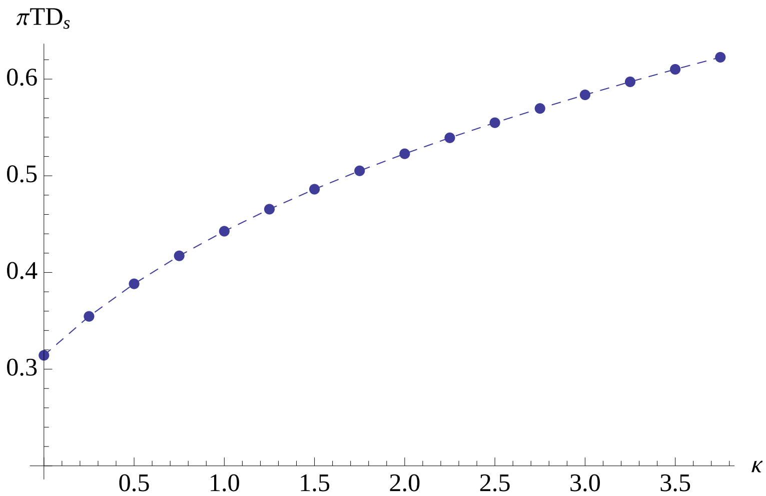

Figure 2.1: The diffusion constant as a function of the charge for the one charge case $\kappa_1 = \kappa,\ \kappa_2 = \kappa_3 = 0$.

In figure 2.2 we present the diffusion constant as a function of T/μ and find a behavior similar to the one observed in [Gauntlett et al., 2011] for an AdS-RN solution of N=2, D=4 supergravity. For large values of T/μ the diffusion constant approaches the zero density value which is represented by the horizontal line.

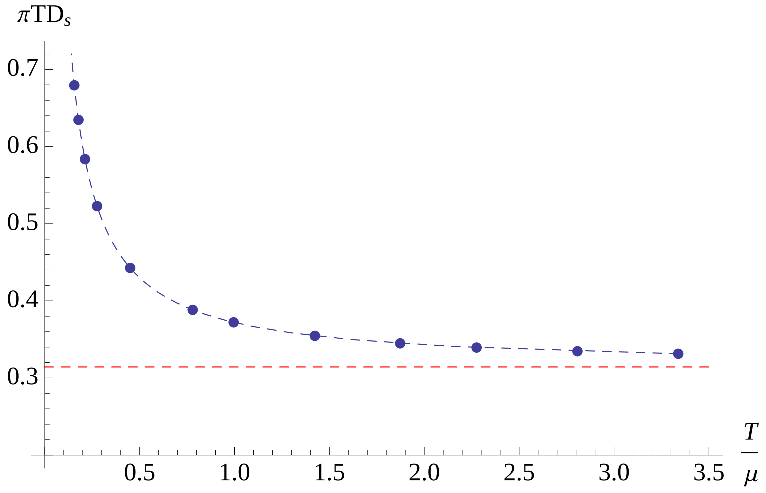

Figure 2.2: The diffusion constant as a function of T/μ for the one charge case. The horizontal line represents the zero density analytical result.

In figure 2.3 we give the result for three charges as a function of κ, and in figure 2.4 as function of T/μ. We observe that the behavior is qualitatively similar to that of the one charge case for high temperature, but the diffusion constant (or rather the combination TD_s) approaches a finite limit as the temperature goes to zero for the extremal black hole which appears for $\kappa = 2$.

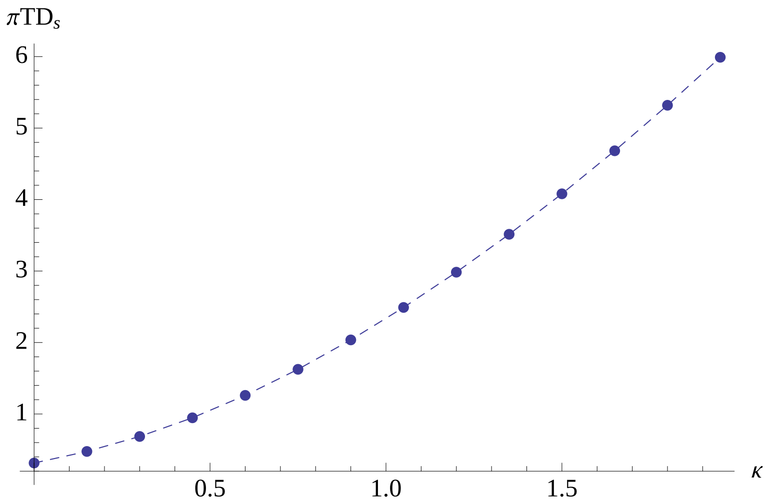

Figure 2.3: The diffusion constant in the three equal charges case as a function of the charge.

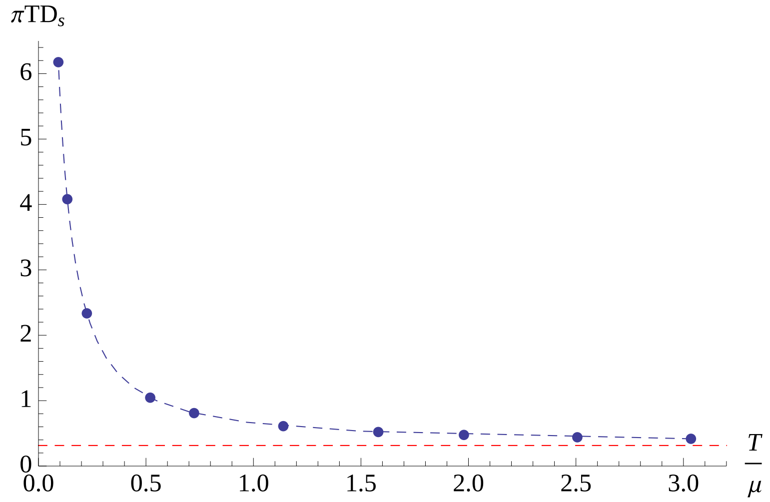

Figure 2.4: The diffusion constant as a function of T/μ for the three charges case. The horizontal line represents the zero density analytical result.

To summarize, we have considered the hydrodynamic limit of the correlators of supercharges in $\mathcal{N} = 4$ SYM at zero and finite R-charge density. From the transverse part of the correlator the supercharge diffusion constant was extracted using a Kubo formula. In the zero charge case we confirmed the result found previously from the calculation of the supersymmetric sound attenuation, using the longitudinal channel of the correlator. The main motivation for the present work has been to answer the question of the presence of universality in the hydrodynamic supersymmetric sector of theories with a holographic dual. The conclusion appears to be negative: the diffusion constant depends on the value of the $U(1)$ charge density.

In a recent paper [Erdmenger and Steinfurt, 2013] it has been proposed that the transport coefficient which multiplies the spin 3/2 part of the constitutive relation (2.19) could be universal. In fact this transport coefficient is the combination $D_s + D_\sigma/(d-2)$ in $d+1$ spacetime dimensions and was proven to be proportional to the absorption cross section for fermions in analogy with the proof of the viscosity universality. However the absorption cross section was related to the horizon data of a metric conformal to the black brane metric and not the original one, thus its computation still requires the knowledge of the full metric, in order to determine the conformal factor, and not only the near-horizon part. The analysis of [Erdmenger and Steinfurt, 2013] was limited to a neutral gravitino. It would be interesting to extend it to gauged supergravity models where the gravitino is charged under the R-symmetry, so that its coupling to the gauge field has to be taken into account.

Chapter 3

Holographic entanglement entropy

In this chapter after reviewing quantum field theory techniques and results for the entanglement entropy (EE) we introduce the holographic recipe and analyze the behavior of EE in theories with massive flavor fields.

3.1 Entanglement entropy in field theory

Entanglement is one of the most distinctive properties of quantum systems. Informally speaking, it corresponds to the fact that a measurement performed on a part of the system will affect another part, or alternatively it quantifies the amount of information on a subsystem that is accessible by performing measurements on another subsystem. There exist several measures of entanglement; the most commonly used is the *entanglement entropy*. It can be naturally introduced in a quantum system divided into two subsystems A and B. Consider an observer that has only access to the subsystem A; the results of all the possible measurements he can make are encoded in the reduced density matrix ρ_{red} obtained by integrating out the degrees of freedom in B. The entanglement entropy (EE) of the subsystem A with B is defined as the von Neumann entropy associated to the reduced density matrix:

$$S_A = -\mathrm{tr}(\rho_{\mathrm{red}} \log \rho_{\mathrm{red}})\,. \tag{3.1}$$

Very often one considers the case in which the subsystems are the degrees of freedom living in different regions of space. The definition is completely general and can be in principle applied to any system, provided that the degrees of freedom are local, so that one can associate a Hilbert space to a given region of spacetime. On the other hand, EE is a very non-local observable, therefore it provides different information compared to local quantities such as correlators; for instance, it has been used as a probe of long-range topological order in two-dimensional systems with a mass gap [Kitaev and Preskill, 2006]. It is also useful in many other contexts ranging from condensed matter physics to quantum information.

EE has been the subject of intensive study in the last few years; its computation is generally a very challenging problem and few exact results are known. In a quantum field theory, EE is a UV divergent quantity and its computation requires the introduction of an ultraviolet regulator a. In terms of this cutoff the structure of the divergence, for a theory in $d+1$ spacetime dimensions, can be summarized as follows (see [Calabrese and Cardy, 2004] for a more extended review of known properties):

$$S_A = \frac{c_{d-1}}{a^{d-1}} + ... + \frac{c_1}{a} + c_0 \log a + S_f \,, \tag{3.2}$$

where S_f is finite for $a \to 0$; the coefficients c_i depend in general on the geometric properties of the boundary surface Σ separating the regions A and B, and have been computed in a limited number of cases (a review of the computational tools used to compute EE in free quantum field theories can be found in [Casini and Huerta, 2009]). The leading divergent term is proportional to the area of Σ, a fact known as the "area law". Most of the terms in the expansion are actually ambiguous, as they are not invariant under a rescaling of the cutoff. One exception is the coefficient of $\log a$; in a conformal field theory, it has been shown to be related to the central charges appearing in the trace anomaly. In particular in a 1+1 dimensional CFT entanglement entropy is given by

$$S = \frac{c}{3} \log \frac{\ell}{a} \tag{3.3}$$

for an interval of length ℓ and c denoting the central charge of the conformal field theory.

We describe now a general technique for computing the entanglement entropy in a quantum field theory [Calabrese and Cardy, 2004]; this technique goes under the name of replica trick. We start by computing the $\mathrm{tr}_A \rho_A^n$ with $\rho_A = \mathrm{tr}_B \rho$ being the reduced density matrix in a system divided in A and B. Then we differentiate it with respect to n and take the limit $n \to 1$, so that EE is given by

$$S = -\frac{\partial}{\partial n} \mathrm{tr}_A \rho_A^n \Big|_{n=1} = -\frac{\partial}{\partial n} \log \mathrm{tr}_A \rho_A^n \Big|_{n=1} . \tag{3.4}$$

The second equality holds because the reduced density matrix is normalized so that $\mathrm{tr}_A \rho_A = 1$.
We will see now how we can compute $\mathrm{tr}_A \rho_A^n$ using the path integral formalism in an 1+1 dimensional Euclidean quantum field theory. Suppose that A is the interval $x \in [u, v]$ at $t_E = 0$. The ground state wave functional Ψ is given by the path integral

$$\Psi(\phi_0(x)) = \int_{t_E=-\infty}^{\phi(t_E=0,x)=\phi_0(x)} \mathcal{D}\phi \, e^{-S(\phi)} \,. \tag{3.5}$$

The density matrix is then given by the product $[\rho]_{\phi_0 \phi_0'} = \Psi(\phi_0)\bar{\Psi}(\phi_0')$. The complex conjugate of the wave functional can be obtained by the integrating from $t_E = \infty$ to $t_E = 0$. Then to obtain the reduced density matrix we have to integrate ϕ_0 with $\phi_0(x) = \phi_0'$ on B; the resulting matrix element will depend only on the fields $\phi_+(x)$ and $\phi_-(x)$ living in A. Following this logic the reduced density matrix can be computed as

$$[\rho_A]_{\phi_+ \phi_-} = \frac{1}{Z_1} \int_{t_E=-\infty}^{t_E=\infty} \mathcal{D}\phi \; e^{-S(\phi)} \prod_{x \in A} \delta(\phi(+0, x) - \phi_+(x)) \delta(\phi(-0, x) - \phi_-(x)) \,, \tag{3.6}$$

where we divided by the vacuum partition function to normalize the density matrix. This path integral is represented in fig. 3.1(a).
Now that we have the reduced density matrix we can compute $\mathrm{tr}_A\rho_A^n$. To take the trace the fields should be successively identified in each copy of the reduced density matrix

$$[\rho_A]_{\phi_{1+}\phi_{1-}}[\rho_A]_{\phi_{2+}\phi_{2-}}\cdots[\rho_A]_{\phi_{n+}\phi_{n-}}\,. \tag{3.7}$$

In the path integral formalism this trace corresponds to gluing together the fields in A as $\phi_{i-}(x)=\phi_{(i+1)+}(x)$ and integrating over ϕ_{i+}. This procedure can be visualized as follows: we take n copies of the (t_E,x) plane with a cut along A and then we glue them together along this cut (cf. fig. 3.1(b)). Computing the trace of ρ_A^n will then be equivalent to computing the path integral on the n-sheeted Riemann surface that will be created by this gluing procedure. The trace will be computed then as

$$\mathrm{tr}_A\rho_A^n=\frac{1}{(Z_1)^n}\int_{(t_E,x)\in\mathcal{R}_n}\mathcal{D}\phi\;e^{-S(\phi)}\equiv\frac{Z_n}{(Z_1)^n}\,. \tag{3.8}$$

Here we illustrated the path integral computation for a field theory in 2 dimensions but the technique can be generalized to higher dimensional theories; an example of such a generalization will be given in what follows.

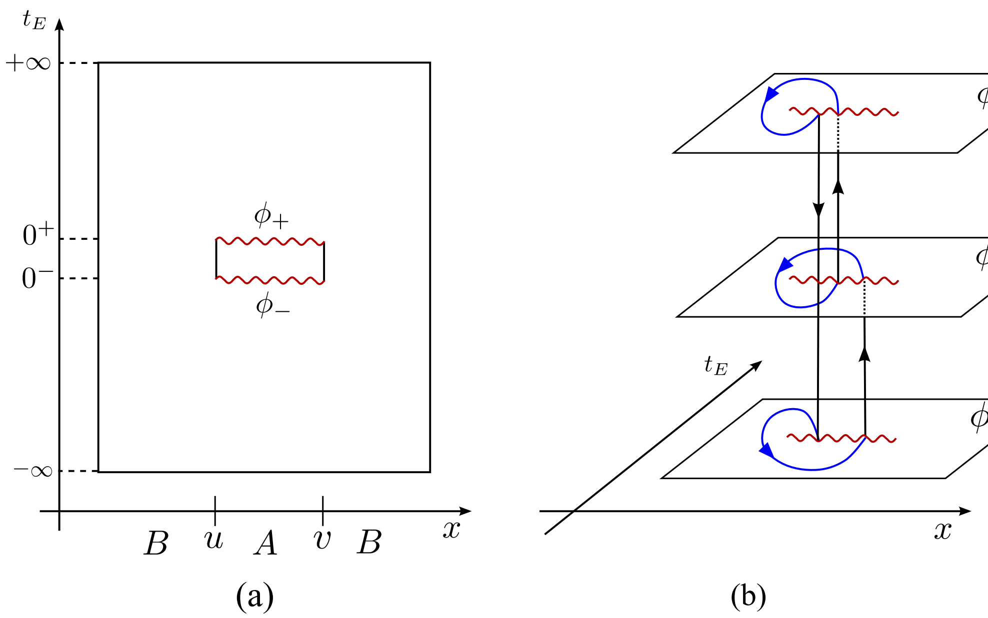

Figure 3.1: (a) The path integral representation of $[\rho_A]_{\phi_+\phi_-}$. (b) The n-sheeted Riemann surface $\mathcal{R}_n$ for $n=3$.

3.1.1 Mass corrections in the waveguide geometry

The structure of the entanglement entropy presented in (3.2) is valid for conformal theories. When we move away from conformality the result can depend also on the intrinsic scales of the theory, such as masses. We will concentrate on the corrections that appear in a massive deformation of a CFT. Such corrections have been studied in [Hertzberg and Wilczek, 2011] for free scalar field theory with finite correlation length $\xi = 1/m$ in a waveguide geometry and it has been found that there is a finite contibution to the entropy of the form, in $d = 3$,

$$S_f = \frac{A_\Sigma}{24\pi} m^2 \log m + f_0 \log m + f_1 m \,, \tag{3.9}$$

where the coefficients f_i depend on the geometrical characteristics of the waveguide, and A_Σ is the area of the entangling surface. The terms appearing in (3.9) are finite and independent of the ultraviolet regulator. They can be isolated from the UV-divergent part by taking derivatives with respect to the correlation length (see [Liu and Mezei, 2013a, Liu and Mezei, 2013b] for an alternative proposal for defining finite universal parts).
In [Hertzberg, 2013] the first term of (3.9) has been computed perturbatively in a scalar field theory with ϕ^3 and ϕ^4 interactions, with the result that the structure remains the same but the bare mass is replaced by the renormalized mass.

We review here the computation of the mass corrections of equation (3.9) following the presentation in [Hertzberg, 2013].

A straightforward generalization of the replica trick in higher dimensional theories is to consider fields in a waveguide geometry which is the product space $\mathbb{R} \times \mathcal{M}_{d-1}$. In our case we will consider a scalar field and we will divide the system in two complementary subsystems A an $\bar{A}$ as shown in fig. 3.2(a). To identify the Riemann surface on which we have to compute the path integral we can proceed as follows. We can think of n copies of a disc in (t_E, x) space (cf. fig 3.2(b)) which we have to glue along the cut; the resulting space will be a cone C_δ of deficit angle $\delta = 2\pi(1 - n)$. Then we can send the radius of the cone to infinity to recover the infinite length of the $\mathbb{R}$ direction. So, to summarize we need to compute the partition function on $C_\delta \times \mathcal{M}_{d-1}$ which we will denote Z_n.

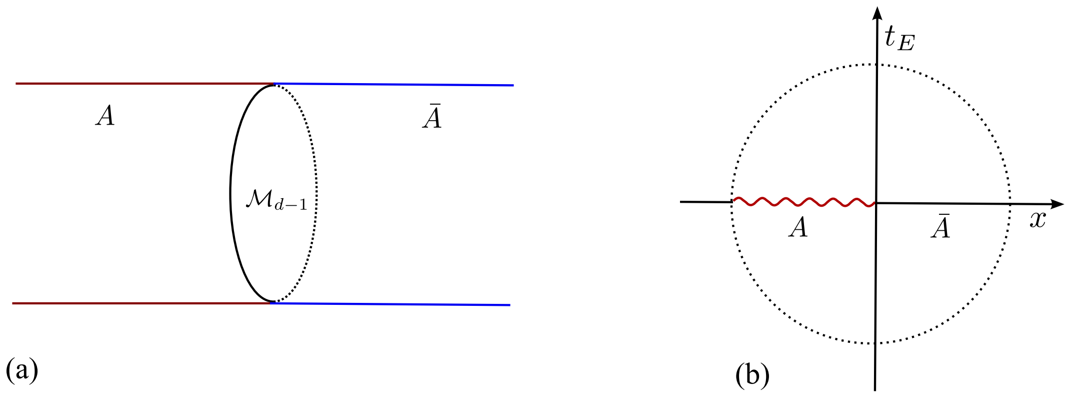

Figure 3.2: (a) The waveguide geometry. (b) For the replica trick we need to glue together n copies of the disk along the cut, the resulting Riemann surface will be a cone.

From equation (3.8) we have that

$$\log \mathrm{tr}_A \rho_A^n = \log Z_n - n \log Z_1 \tag{3.10}$$

and so according to (3.4) we have for the entanglement entropy

$$S = -\frac{\partial}{\partial n} \left(\log Z_n - n \log Z_1\right)\Big|_{n=1} . \tag{3.11}$$

In free field theory the partition function can be expressed symbolically as

$$\log Z_n = -\frac{1}{2} \log \det(-\Box + m^2) . \tag{3.12}$$

Then, if we consider the derivative of this quantity with respect to m^2, we can express it in terms of the Green's function as

$$\frac{\partial}{\partial m^2} \log Z_n = -\frac{1}{2} \int\limits_n d^{d+1}x \, G_n(x,x) , \tag{3.13}$$

where the subscript n on the integral denotes that we have to integrate over the cone.

We start our analysis by computing the Green's function on the cone; it satisfies the equation

$$(-\Box + m^2) G_n(x,x') = \delta^{(d+1)}(x,x') . \tag{3.14}$$

We can write the coordinates as $x = (r, \theta, \tilde{x})$ where (r, θ) are the coordinates on the cone and $\tilde{x}$ are the coordinates on the cross-section of the waveguide. The Green's function was calculated on the cone in [Calabrese and Cardy, 2004] and is given by

$$G_n(\tilde{x}, \tilde{x}') = \frac{1}{2\pi n} \int \frac{d^{d-1}\tilde{p}}{(2\pi)^{d-1}} \sum_{k=0}^{\infty} d_k \int\limits_0^{\infty} dq \, q \frac{J_{k/n}(qr) K_{k/n}(qr)}{q^2 + m^2 + p^2} \cos((\theta - \theta')/n) e^{i\tilde{p}\cdot(\tilde{x} - \tilde{x}')} , \tag{3.15}$$

where J is the Bessel function of the first kind and the constants d_k are $d_{k=0} = 1$ and $d_{k\geq 1} = 2$.

For the free theory partition function we need the Green's function in the limit $x' \to x$; after integrating over q we get:

$$G_n(x,x) = \frac{1}{2\pi n} \int \frac{d^{d-1}\tilde{p}}{(2\pi)^{d-1}} \sum_{k=0}^{\infty} d_k \, I_{k/n}\left(\sqrt{m^2 + \tilde{p}^2}\, r\right) K_{k/n}\left(\sqrt{m^2 + \tilde{p}^2}\, r\right) \tag{3.16}$$

where I and K are the modified Bessel functions of the first and second kind respectively. Using the Euler-Maclaurin formula

$$\sum_{k=0}^{\infty} d_k F(k) = 2 \int\limits_0^k dk F(k) - \frac{1}{6} F'(0) - 2 \sum_{j>1}^{\infty} \frac{B_{2j}}{(2j)!} F^{(2j-1)}(0) \tag{3.17}$$

we can express the sum over k in terms of an integral over k. We have then

$$G_n(x,x) = \frac{1}{2\pi n}\int \frac{d^{d-1}\tilde{p}}{(2\pi)^{d-1}}\left[2\int_0^\infty dk\, I_{k/n}\left(\sqrt{m^2+\tilde{p}^2}\, r\right) K_{k/n}\left(\sqrt{m^2+\tilde{p}^2}\, r\right) + \frac{1}{6n}K_0^2\left(\sqrt{m^2+\tilde{p}^2}\, r\right)\right] + (j>1) \tag{3.18}$$

where $(j>1)$ stands for the contribution of the last term in (3.17). Given that for flat space corresponding to $n=1$ we have translational invariance $G_1(x,x) = G_1(0)$ and defining $\bar{k} = k/n$ we can extract the n dependence of the Green's function writing in the following way:

$$G_n(x,x) = G_1(0) + f_n(r)\,. \tag{3.19}$$

The function f_n that we introduced os given by

$$f_n(r) = \frac{1}{2\pi n}\frac{1-n^2}{6n}\int \frac{d^{d-1}\tilde{x}}{(2\pi)^{d-1}}K_0^2\left(\sqrt{m^2+\tilde{p}^2}\, r\right) + (j>1)\,. \tag{3.20}$$

We can now use equation (3.10) and the results for the Green's function to compute the entropy. The derivative of the reduced density matrix is given by

$$\begin{aligned}\frac{\partial}{\partial m^2}\log \mathrm{tr}[\rho_A^n] &= \frac{\partial}{\partial m^2}\log Z_n - n\frac{\partial}{\partial m^2}\log Z_1 \\ &= -\frac{1}{2}\left[\int_n d^{d+1}x\, G_n(x,x) - n\int d^{d+1}x\, G_1(0)\right]\,.\end{aligned} \tag{3.21}$$

The angular contribution for the first integral is $2\pi n$ and for the second 2π so that the contribution from $G_1(0)$ cancels between the two integrals. The derivative of the reduced density matrix becomes now

$$\begin{aligned}\frac{\partial}{\partial m^2}\log \mathrm{tr}[\rho_A^n] &= -\frac{1-n^2}{12n}\int d^{d-1}\tilde{x}\int_0^\infty dr\, r\int \frac{d^{d-1}\tilde{p}}{(2\pi)^{d-1}}K_0^2\left(\sqrt{m^2+\tilde{p}^2}\, r\right) \\ &= -\frac{1-n^2}{24n}A_\Sigma\int \frac{d^{d-1}\tilde{p}}{(2\pi)^{d-1}}\frac{1}{m^2+\tilde{p}^2}\,,\end{aligned} \tag{3.22}$$

where A_Σ is area of the waveguide's cross-section. Integrating this expression with respect to m^2 and deriving with respect to n we find for the entanglement entropy

$$S = -\frac{1}{12}A_\Sigma\int \frac{d^{d-1}\tilde{p}}{(2\pi)^{d-1}}\log(m^2+\tilde{p}^2) + (\text{m independent terms})\,. \tag{3.23}$$

The integration over the momenta diverges and therefore requires the introduction of a UV cutoff $1/a$. performing the integration and identifying the UV cutoff independent terms for $d=3$ we find the log term appearing in (3.9).

3.2 The holographic prescription

In a seminal paper [Ryu and Takayanagi, 2006b] Ryu and Takayanagi proposed a remarkably simple recipe for the computation of EE in theories with a holographic dual gravity description. The quantum field theory lives on the boundary of AdS; consider a region on a constant time slice of the boundary A enclosed by the entangling surface $\partial A = \Sigma$. According to the proposal, the EE of the region is proportional to the area $\mathcal{A}$ of a minimal surface that extends in the bulk of AdS and whose restriction to the boundary of AdS is ∂A:

$$S = \frac{\mathcal{A}}{4G_N^{(d+2)}} \,. \tag{3.24}$$

Among the various applications of this formula (see [Nishioka et al., 2009]) it is worth mentioning the identification of the exact contribution of the central charges to the $\log a$ term [Solodukhin, 2008]. This proposal has been proved in the case of a spherical entangling surface by mapping the problem of computing entanglement entropy to that of computing thermal entropy using a conformal transformation [Casini et al., 2011]. A more general proof, that should be applicable to any geometry of the entangling surface, has been recently proposed [Lewkowycz and Maldacena, 2013] based on arguments about the solutions of gravitational theories with a boundary and their relation to the entropy of the density matrix.
The log term appearing in equation (3.9) has been identified in a holographic computation of the entanglement entropy in [Hung et al., 2011] by introducing a massive scalar in AdS that sourced a relevant deformation of the CFT.

We will not give a proof of this prescription here but we will rather hint to its correctness by verifying that it gives the right result in a 1+1 dimensional CFT. The dual geometry to a 1+1 dimensional field theory is AdS_3 with metric:

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + dx^2 + dz^2) \,. \tag{3.25}$$

We compute the EE for an interval of length ℓ on the boundary. The minimal surface in this case will be the minimal length given by the length of a geodesic that extends between the point $(x, z) = (-\ell/2, a)$ and $(x, z) = (\ell/2, a)$ with a being the UV cutoff. The geodesic with these boundary conditions is given by

$$(x, z) = \frac{\ell}{2}(\cos\sigma, \sin\sigma), \qquad \frac{2a}{\ell} \leq \sigma \leq \pi - \frac{2a}{\ell} \,. \tag{3.26}$$

Then the minimal length can be calculated as

$$L = 2R \int_{\frac{2a}{\ell}}^{\pi/2} \frac{d\sigma}{\sin\sigma} = 2R \log\frac{\ell}{a} \,. \tag{3.27}$$

Using now the holographic prescription (3.24) we arrive at the expression for the entanglement entropy:

$$S = \frac{A}{4G_N^{(3)}} = \frac{c}{3}\log \ell a, \tag{3.28}$$

where we used the fact that the central charge is related to the AdS_3 radius by [Brown and Henneaux, 1986]:

$$c = \frac{3R}{2G_N^{(3)}} \, . \tag{3.29}$$

We have therefore confirmed the result stated in (3.3).

3.2.1 Review of the pure AdS case

For future reference we recall here the computation of the entanglement entropy for a slab and a ball geometry in pure AdS_{d+2} (see e.g. [Ryu and Takayanagi, 2006a]). The metric is given by:

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + \sum_{i=1}^{d} dx_i^2 + dz\right) \, . \tag{3.30}$$

The slab is defined on a constant time slice on the boundary as:

$$x_1 \in [-\ell/2, \ell/2] \; ; \quad x_{2,3,\ldots,d} \in (-\infty, \infty)$$

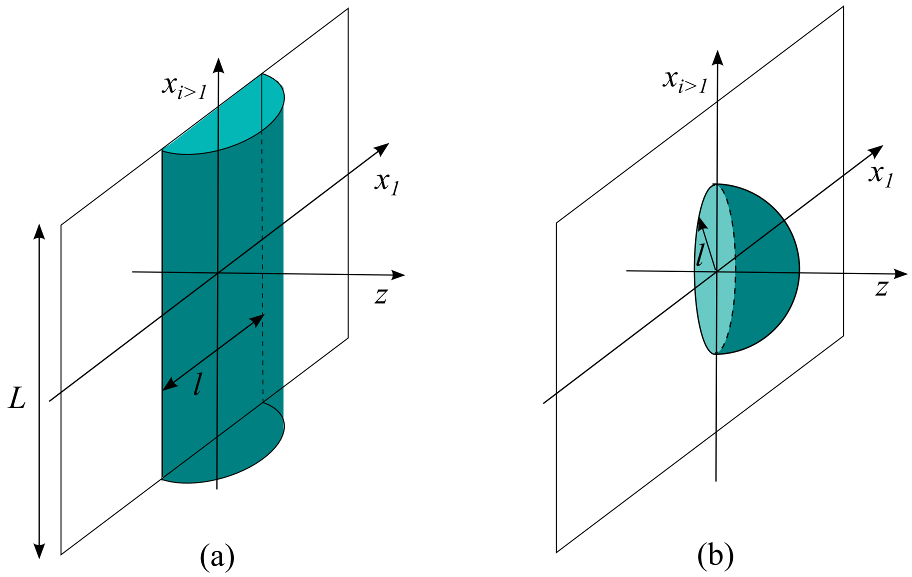

Figure 3.3: The slab geometry (a) and the ball geometry (b) and the corresponding minimal surfaces in AdS space.

We will use the regularized length L for the infinite directions as shown in the geometrical construction in fig. 3.3 (a). The holographic entanglement entropy can be computed as the area $\mathcal{A}$ of the minimal surface extending in the AdS bulk and whose boundary lies on the entangling surface separating the slab and the rest of the boundary. We start by minimizing the area functional for the surface extending in the bulk. Choosing an embedding of the form $z = z(x_1) = z(x)$ for the surface we have:

$$S_{\text{area}} = R^d L^{d-1} \int_{-\ell/2}^{\ell/2} dx \, \frac{\sqrt{1+z'^2}}{z^d} \,. \tag{3.31}$$

Given that the integrand does not depend explicitly on x we can compute the constant of motion and get

$$\frac{dz}{dx} = \frac{\sqrt{\tilde{z}^{2d} - z^{2d}}}{z^d} \,, \tag{3.32}$$

where $\tilde{z}$ is the turning point of the surface. The minimal area is therefore given by:

$$\mathcal{A} = 2R^d L^{d-1} \int dz \, \frac{\tilde{z}^d}{z^d \sqrt{\tilde{z}^{2d} - z^{2d}}} \,. \tag{3.33}$$

To compute the integral we need to introduce a UV cutoff a and also satisfy the constraint:

$$\frac{\ell}{2} = \int_{-\ell/2}^{0} dx = R^2 \frac{\sqrt{\pi}\, \Gamma\left(\frac{d+1}{2d}\right)}{\Gamma\left(\frac{1}{2d}\right)} \tilde{z} \,. \tag{3.34}$$

The area of the minimal surface after regularization is given by:

$$\mathcal{A}^{\text{sl}}_{\text{AdS}} = \frac{2R^d}{d-1} \left(\frac{L}{a}\right)^{d-1} - \frac{2^d \pi^{d/2} R^d}{d-1} \left(\frac{\Gamma\left(\frac{1+d}{2d}\right)}{\Gamma\left(\frac{1}{2d}\right)}\right)^d \left(\frac{L}{\ell}\right)^{d-1} \,. \tag{3.35}$$

We move on now to the computation for the ball geometry where the entangling surface is a sphere of radius ℓ (fig. 3.3 (b)). It is convenient to write the metric in spherical coordinates; introducing the coordinate $r^2 = \sum_{i=1}^{d} x_i$ the metric becomes

$$ds^2 = \frac{R^2}{z^2} \left(-dt^2 + dr^2 + r^2 d\Omega^2_{d-1} + dz^2\right) \,. \tag{3.36}$$

Choosing an embedding of the form $r = r(z)$ the area functional is given by

$$S_{\text{area}} = R^d \text{vol}(S^{d-1}) \int dz \, \frac{r^{d-1}}{z^d} \sqrt{1 + r'^2} \,. \tag{3.37}$$

The equations of motion of this area functional admit the solution

$$r^2 + z^2 = \ell^2 \,. \tag{3.38}$$

The minimal area is therefore given by:

$$\mathcal{A} = R^d \text{vol}(S^{d-1}) \int_{a/\ell}^{1} du \, \frac{(1-u^2)^{\frac{d-2}{2}}}{u^d} \,, \tag{3.39}$$

where a is the UV cutoff. For small values of the cutoff and for d odd the minimal area can be expressed as a series of the following form:

$$\mathcal{A}^{\text{sp}}_{\text{AdS}} = \frac{2\pi^{d/2} R^d}{\Gamma(d/2)} \left[p_1 \left(\frac{\ell}{a}\right)^{d-1} + p_3 \left(\frac{\ell}{a}\right)^{d-3} + \ldots + p_{d-2} \left(\frac{\ell}{a}\right)^2 + p_o \log \frac{\ell}{a} \right] \,. \tag{3.40}$$

The values of the coefficients for $d = 3$, which will be of interest to us are $p_1 = 1/2$ and $p_o = -1/2$.

3.3 Corrections from massive flavor degrees of freedom

3.3.1 The slab case

We move now to the computation of the entanglement in the backreacted D3/D7 geometry given in (1.89). For the case of the slab geometry, we choose an embedding of the form $\rho = \rho(x)$; the area functional of the surface is then given by [1]:

$$S_{\text{area}} = \frac{L^2}{R^5} \int_{-\ell/2}^{\ell/2} dx\; h^{1/2} F S^4 \sqrt{1 + h F^2 \rho'^2}\,. \tag{3.41}$$

The embedding function satisfies the equation

$$\frac{d\rho}{dx} = -\frac{\sqrt{hF^2S^8 - \tilde{h}\tilde{F}^2\tilde{S}^8}}{\sqrt{h\tilde{h}F\tilde{F}\tilde{S}^4}}\,, \tag{3.42}$$

where we denote $\tilde{\rho}$ the turning point of the surface and the tilded functions are the values of the functions at the turning point. Using this relation the minimal area can be computed as follows:

$$\mathcal{A} = 2\frac{L^2}{R^5} \int_{\tilde{\rho}}^{\infty} d\rho\; \frac{h^{3/2} F^3 S^8}{\sqrt{hF^2S^8 - \tilde{h}\tilde{F}^2\tilde{S}^8}}\,, \tag{3.43}$$

$$\ell = 2 \int_{\tilde{\rho}}^{\infty} d\rho\; \frac{\sqrt{h\tilde{h}F\tilde{F}\tilde{S}^4}}{\sqrt{hF^2S^8 - \tilde{h}\tilde{F}^2\tilde{S}^8}}\,. \tag{3.44}$$

For convenience we switch to the z coordinates given in terms of ρ by eq. (1.97) and (1.98) in the regions $\rho > \rho_q$ and $\rho < \rho_q$ respectively. To regularize the area integral we introduce a UV cutoff at $z = a$. We compute the width of the slab and the minimal area to first order in the perturbation parameter ϵ_q:

$$\ell = \ell_0 + \epsilon_q \ell_1 \tag{3.45}$$

$$\mathcal{A} = \mathcal{A}_0 + \epsilon_q \mathcal{A}_1\,. \tag{3.46}$$

I. Turning point located at $\tilde{\rho} > \rho_q$ ($\tilde{z} < z_q$)

We start by computing the entropy for the case $\tilde{\rho} > \rho_q$ i.e. the turning point is located in the region where the D7 branes extend. We can express both the length ℓ and the area $\mathcal{A}$ in terms of the parameter $b = \tilde{z}/z_q$.

$$\frac{\ell}{z_q} = \gamma_1 b + \epsilon_q \left[\frac{1}{720} b^5 \left(48\gamma_3 - 15\gamma_2\right) + \frac{1}{720} b^3 \left(40\gamma_2 - 160\right) + \frac{b\gamma_2}{8}\right]\,, \tag{3.47}$$

1. We divide the area by $R^5 \text{vol}(S^5)$ to make the results comparable with the AdS case where there is no internal five sphere.

$$\frac{z_q^2}{L^2R^3}\mathcal{A}^I = -\frac{\gamma_1}{2b^2} + \epsilon_q\left\{\frac{1}{144}b^2\left(24\gamma_3 - 3\gamma_2\right) - \frac{\gamma_2}{8b^2} + \frac{1}{144}\left[8\gamma_2 - 16(6\log(bz_q) + 1 + \log 4)\right]\right\} \tag{3.48}$$

where

$$\gamma_1 = \frac{2\sqrt{\pi}\Gamma(2/3)}{\Gamma(1/6)}; \qquad \gamma_2 = \frac{\Gamma(2/3)\Gamma(5/6)}{\sqrt{\pi}}; \qquad \gamma_3 = \frac{\Gamma(1/3)\Gamma(7/6)}{\sqrt{\pi}}.$$

The divergent piece of the area is given by:

$$\mathcal{A}_{\text{div}} = \frac{L^2R^3}{a^2} - \epsilon_q L^2R^3\left[\frac{1}{4a^2} - \frac{2}{3z_q^2}\log a\right]. \tag{3.49}$$

The zeroth order term of the area matches the result for the AdS case eq. (3.35) for $d = 3$ as expected. To express the area in terms of ℓ we can pertubatively invert the relation (3.47) which leads to:

$$\mathcal{A}^I(\ell) = -\frac{\gamma_1^3L^2R^3}{2\ell^2} + \epsilon_q L^2R^3\left[\frac{6\log\gamma_2 + 1 - 2\log 2}{9z_q^2} + \frac{\gamma_3\ell^2}{10\gamma_2^2 z_q^4} - \frac{\gamma_2^3}{4\ell^2}\right]. \tag{3.50}$$

II. Turning point located at $\tilde{\rho} < \rho_q$ ($\tilde{z} > z_q$)

To compute the length and area integrals in this case we must split them in two parts: one from the boundary to ρ_q and another one from ρ_q to the turning point. The results that we find for the length and area are:

$$\begin{aligned}\frac{\ell}{z_q} = \gamma_1 b + \frac{\epsilon_q}{2160}&\left[-\frac{30}{b^7} - \frac{15\gamma_1}{b^3} - 480b^3 + \frac{30}{b^3} + 48b^5 B\left(\frac{1}{b^6}; \frac{1}{3}, \frac{1}{2}\right)\right.\\ &\left.+\frac{6\sqrt{b^2-1}\left(56b^4 + 71b^2 + 31\right)}{\sqrt{b^4+b^2+1}} + \left(-15b^5 + 40b^3 + \frac{5}{b^3} + 90b\right)B\left(\frac{1}{b^6}; \frac{2}{3}, \frac{1}{2}\right)\right],\end{aligned} \tag{3.51}$$

$$\begin{aligned}\frac{z_q^2}{L^2R^3}\mathcal{A}^{II} = -\frac{\gamma_1}{2b^2} + \frac{\epsilon_q}{432b^7}&\left[24b^9 B\left(\frac{1}{b^6}; \frac{1}{3}, \frac{1}{2}\right) + \left(-3b^8 + 8b^6 - 18b^4 + 1\right)bB\left(\frac{1}{b^6}; \frac{2}{3}, \frac{1}{2}\right)\right.\\ &-48b^7 - 48b^7\left(-2\cosh^{-1}\left(b^3\right) + 6\log(bz_q) + 2\log 2\right)\\ &\left.-\frac{6\sqrt{b^2-1}\left(12b^8 + 9b^6 + 17b^4 - b^2 - 1\right)}{\sqrt{b^4+b^2+1}} - 3b\gamma_1\right].\end{aligned} \tag{3.52}$$

where $B(z; a, b)$ is the incomplete Beta function defined as:

$$B(z; a, b) = \int_0^z t^{a-1}(1-t)^{b-1}dt\,. \tag{3.53}$$

The counterterms used for the regularization of the area are the same as for the surface extending only in the $\rho > \rho_q$ region since the fact that the surface extends further in the interior does not affect the ultraviolet behavior of the integrals. Now we can invert again the relation $\ell(b)$ to express the area in terms of ℓ:

$$\begin{aligned}\frac{z_q^2}{L^2R^3}\mathcal{A}^{II}(\ell) = -\frac{\gamma_1}{2b^2} + \frac{\epsilon_q}{720}\Bigg[-\frac{90\gamma_1^6 z_q^6}{\ell^6}\,{}_2F_1\left(\frac{1}{2},\frac{2}{3};\frac{5}{3};\frac{z_q^6\gamma_1^6}{\ell^6}\right) + 72\,{}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{4}{3};\frac{z_q^6\gamma_1^6}{\ell^6}\right) \\ + \frac{10\gamma_1^{10}z_q^{10}}{\ell^{10}} - \frac{10\gamma_1^6 z_q^6}{\ell^6} + 160\cosh^{-1}\left(\frac{\ell^3}{\gamma_1^3 z_q^3}\right) + \frac{10\gamma_1^4 z_q^4\sqrt{\ell^6-\gamma_1^6 z_q^6}}{\ell^7} + 80\log\left(\frac{\gamma_1^6}{4\ell^6}\right) \\ - \frac{232\gamma_1^2 z_q^2\sqrt{\ell^6-\gamma_1^6 z_q^6}}{\ell\left(\gamma_1^4 z_q^4+\gamma_1^2\ell^2 z_q^2+\ell^4\right)} - \frac{232\ell\sqrt{\ell^6-\gamma_1^6 z_q^6}}{\gamma_1^4 z_q^4+\gamma_1^2\ell^2 z_q^2+\ell^4} - \frac{242\gamma_1^4 z_q^4\sqrt{\ell^6-\gamma_1^6 z_q^6}}{\ell^3\left(\gamma_1^4 z_q^4+\gamma_1^2\ell^2 z_q^2+\ell^4\right)} + 80\Bigg]\,.\end{aligned} \tag{3.54}$$

We are interested in the behavior of the theory for large values of ℓ in order to probe the cutoff independent mass corrections to the entanglement entropy. Following Hertzberg and Wilczek [Hertzberg and Wilczek, 2011], we can extract these cutoff-independent contributions; identifying $\xi^{-1} = m = 1/z_q$, the cutoff independent part is

$$S_\xi = (-\xi^{-2})^2\frac{\partial S}{\partial(\xi^{-2})^2}\,. \tag{3.55}$$

We can check that indeed this quantity is UV-finite, and it is a function of $\Lambda^2 \equiv \ell^2/z_q^2$. The large Λ expansion, $\ell \gg \xi$, reveals the following term:

$$S_\xi \approx \epsilon_q\frac{L^2R^3}{G_N}\frac{1}{3\xi^2} = \frac{1}{2\pi^2}\lambda_q N_f N_c\frac{A_\Sigma}{48\pi\xi^2}\,. \tag{3.56}$$

Note that an entropy of the form

$$S = -\frac{A_\Sigma}{24\pi}\frac{1}{\xi^2}\log\xi - \frac{4b_1}{\xi} + 2b_0\log\xi \tag{3.57}$$

produces an S_ξ of the following form

$$S_\xi = \frac{\mathcal{A}_\Sigma}{48\pi\xi^2} + \frac{b_1}{\xi} + b_0\,. \tag{3.58}$$

Therefore the term that we found for the slab geometry corresponds to the $m^2\log m$ term in (3.9). The constant term and the $1/\xi$ terms are missing compared to (3.58) which was identified as the free field theory result in a waveguide geometry in [Hertzberg and Wilczek, 2011]. The coefficient b_1 is related to the perimeter of the waveguide and the b_0 is related to curvature; the fact that there are no analogs of these geometric quantities in the slab geometry is probably the reason of the absence of these terms.

3.3.2 The ball case

In this section we consider the case where the entangling surface is a sphere of radius ℓ. The embedding of the minimal surface is given in terms of a function $r(z)$ where r is the radial coordinate in the boundary, $r^2 = \sum x_i^2$. It is convenient to make a change of variable to $r^2 = y(z)^2 - z^2$. The AdS solution then reads simply $y = \text{const} = \ell$. The area functional is now

$$S_{\text{area}} = \frac{4\pi}{R^5} \int dz\; h^{1/2} S^4 F (y^2 - z^2)^{1/2} \sqrt{(yy' - z)^2 + hF^2 \rho'(z)^2 (y^2 - z^2)}\,. \tag{3.59}$$

The corresponding equations of motion

$$\begin{aligned}&4F^5h^3S\rho'^4\left(y^2-z^2\right)+2F^3h^2\rho'\left\{\rho' S\left[z^2\left(y'^2+2\right)-y^3y''+yz\left(zy''-6y'\right)+y^2\left(2y'^2+1\right)\right]\right.\\&\left.-(4S'\rho'-S\rho'')\left(z^2-y^2\right)\left(z-yy'\right)\right\}+2h\left(SF'+4FS'\right)\left(z-yy'\right)^3+FSh'\left(z-yy'\right)^3=0\end{aligned} \tag{3.60}$$

can be solved at first order in ϵ_q. We denote the perturbed solution by $y = y_0 + \epsilon_q y_1$. Again we have to distinguish the case where the surface extends in the bulk only in the region $\rho > \rho_q$ from the case where it goes further in the bulk.

I. Turning point located at $\tilde{\rho} > \rho_q$ ($\tilde{z} < z_q$)

In this case the perturbed solution is

$$y_1^I(z) = w(z) + \frac{C_1^I\left(z^2 - 2\ell^2\right)}{\sqrt{\ell^2 - z^2}} + C_2^I\,, \tag{3.61}$$

$$w(z) \equiv \frac{4\ell^3 \log z}{3z_q^2} + \frac{\left(4\ell^4 - 2\ell^2 z^2\right)\log\left(\frac{\sqrt{\ell^2 - z^2}+\ell}{z}\right)}{3z_q^2\sqrt{\ell^2 - z^2}} - \frac{z^6}{80\ell z_q^4} - \frac{\ell z^4}{30 z_q^4} + \frac{z^4}{18\ell z_q^2} + \frac{\ell z^2}{3z_q^2} - \frac{z^2}{8\ell}\,.$$

In order for the solution to be regular at $z = \ell$ we must set $C_1^I = 0$. The other constant is fixed by the boundary condition $y_1^I(z = 0) = 0$:

$$C_2^I = -\frac{4\ell^3}{3z_q^2}\log(2\ell)\,.$$

The integral for the area can be calculated analytically, and we have

$$\mathcal{A}^I = 4\pi R^3\left[\frac{1}{2}\frac{\ell^2}{a^2} - \frac{1}{2}\log(\frac{\ell}{a}) + \epsilon_q\left(\frac{\ell^2}{8a^2} + \frac{4\ell^2 + 3z_q^2}{12z_q^2}\log(\frac{a}{2\ell}) + \frac{\ell^4}{30z_q^4} + \frac{7\ell^2}{18z_q^2} - \frac{1}{16}\right)\right]\,. \tag{3.62}$$

II. Turning point located at $\tilde{\rho} < \rho_q$ ($\tilde{z} > z_q$)

In this case the embedding is described by two different functions:

$$y_1^{II} = \begin{cases} w(z) + C_1^{II} \dfrac{z^2 - 2\ell^2}{\sqrt{\ell^2 - z^2}} + C_2^{II} \,, & 0 < z < z_q \\ \dfrac{z_q^4}{144 \ell z^2} + D_1 \left(\sqrt{z^2 - \ell^2} - \dfrac{\ell^2}{\sqrt{z^2 - \ell^2}} \right) + D_2 \,, & z_q < z < \tilde{z} \end{cases} \tag{3.63}$$

As in case I, regularity of the solution at $z = \ell$ fixes $D_1 = 0$, and the boundary condition $y_1^{II}(z = 0) = 0$ fixes

$$C_1^{II} = \frac{C_2^{II}}{2\ell} + \frac{2\ell^2 \log(2\ell)}{3 z_q^2} \,.$$

The two remaining constants are fixed by requiring the continuity of the solution and of the first derivative at the matching point $z = z_q$. Notice that even though we are matching the solutions in two different regions, the point z_q is not really a boundary as the metric is smooth across this point, therefore there is no "refraction" and the geodesics are smooth curves. The matching condition gives:

$$\begin{aligned} C_2^{II} &= \frac{4}{45 z_q^4} \left(\sqrt{\ell^2 - z_q^2} \left(14 \ell^2 z_q^2 - 2 z_q^4 + 3 \ell^4 \right) + 15 \ell^3 z_q^2 \log \left(\frac{2 \ell z_q}{\sqrt{\ell^2 - z_q^2} + \ell} \right) \right) \,, \\ D_2 &= \frac{1}{90 z_q^2 \left(\ell \left(\sqrt{\ell^2 - z_q^2} + \ell \right) - z_q^2 \right)} \Bigg[- 83 \ell^2 z_q^2 \sqrt{\ell^2 - z_q^2} + 16 z_q^4 \sqrt{\ell^2 - z_q^2} \\ &\quad + 12 \ell^4 (10 \log 2 - 1) \sqrt{\ell^2 - z_q^2} - 120 \ell^3 \left(\ell \left(\sqrt{\ell^2 - z_q^2} + \ell \right) - z_q^2 \right) \log \left(\sqrt{1 - \frac{z_q^2}{\ell^2}} + 1 \right) \\ &\quad - 29 \ell z_q^4 + 12 \ell^5 (1 + 10 \log 2) + \ell^3 (17 - 120 \log 2) z_q^2 \Bigg] \,. \end{aligned} \tag{3.64}$$

Once again the integration can be performed analytically, with the following result

$$\begin{aligned} \frac{\mathcal{A}^{II}}{4\pi R^3} = & \frac{1}{2} \frac{\ell^2}{a^2} - \frac{1}{2} \log \frac{\ell}{a} + \epsilon_q \Bigg\{ \frac{\ell^2}{8a^2} - \frac{1}{720 \ell z_q^4} \Bigg[4 \ell^2 z_q^2 \Bigg(60 \ell \log \frac{a \left(\sqrt{\ell^2 - z_q^2} + \ell \right)}{2 \ell z_q} \\ & - 83 \sqrt{\ell^2 - z_q^2} + 70 \ell \Bigg) + z_q^4 \Bigg(180 \ell \log \frac{a \left(\sqrt{\ell^2 - z_q^2} + \ell \right)}{2 \ell z_q} - 45 \ell - 64 \sqrt{\ell^2 - z_q^2} \Bigg) \\ & + 24 \ell^4 \left(\ell - \sqrt{\ell^2 - z_q^2} \right) \Bigg] \Bigg\} \,. \end{aligned} \tag{3.65}$$

The turning point, both in case I and II, is modified from its zeroth order value and is determined by $\tilde{z} = \ell + \epsilon_q y_1(\ell)$. However this shift does not affect the area, to first order in ϵ_q, since the integrand of the action functional evaluated on the zeroth order solution vanishes at $\tilde{z}$.

The divergent terms in the last formula are the same as in (3.62), as it must be since the divergence comes only from the $z \sim 0$ region. We extract the mass-dependent universal part using (3.55). Again we find that it is a function of $\Lambda^2 \equiv \ell^2/z_q^2$. In the limit of large Λ, $\ell \gg \xi$, it has an expansion

$$S_\xi \approx 4\pi R^3 \epsilon_q \left(\frac{1}{6}\Lambda^2 - \frac{1}{8}\right) = \frac{\lambda_q}{2\pi^2} N_f N_c \left(\frac{\mathcal{A}_\Sigma}{48\pi\xi^2} - \frac{1}{16}\right) . \tag{3.66}$$

Comparing with (3.58), we see that we find the leading term and the constant term, while the term proportional to $1/\xi$ is once again missing.

We have computed the corrections to the entanglement entropy due to the massive flavor fields coupled to $\mathcal{N} = 4$ SYM in 3+1 dimensions; from these we could extract the UV-divergent terms and the universal mass-dependent finite terms. The main results of this chapter are contained in eqs. (3.54),(3.56),(3.65),(3.66), giving the exact result for the area and the finite mass-dependent terms for the slab and the ball, respectively.

It is instructive to compare what we found with the previously known results. As already mentioned, the mass-dependent terms have been computed for the first time in [Hertzberg and Wilczek, 2011] for a free field; the contribution is

$$S_{\text{free}} \sim \gamma\, \mathcal{A}_\Sigma\, m^2 \log m$$

with $\gamma = \dfrac{1}{24\pi}$ for a scalar, and $\gamma = \dfrac{1}{48\pi}$ for a Dirac fermion (in 3+1 dimensions).

Subsequently, in [Hertzberg, 2013] the coupling constant dependence of the coefficient γ was studied at one loop in perturbation theory for cubic and quartic scalar interactions; the result was that γ is unchanged if m is taken to be the renormalized mass.

In [Lewkowycz et al., 2013] the entanglement was computed in the $\mathcal{N} = 2^*$ SYM theory, which is a deformation of $\mathcal{N} = 4$ SYM by relevant operators $m_b^2 \mathcal{O}_2 + m_f \mathcal{O}_3$ that give mass to the scalars and to the fermions. The theory is supersymmetric only for $m_b = m_f$, otherwise SUSY is broken and for $m_b > m_f$ there is a tachyonic mode, however the computation of the entanglement is insensitive to these issues. The result they found is that adding the operator $m_f \mathcal{O}_3$,

$$\mathcal{O}_3 = -i\text{Tr}\,\psi_1\psi_2 + \frac{2}{3} m_f \sum_{i=1}^{3} \text{Tr}|\phi_i|^2 \,,$$

which gives mass m_f to fermions and $2/3 m_f^2$ to bosons, the entanglement computed holographically is

$$S_{\mathcal{N}_\rfloor = 2^*} \sim \frac{N^2}{12\pi} \mathcal{A}_\Sigma\, m^2 \log m \,.$$

At weak coupling the result for the same theory reads:

$$S_{\mathcal{N}_\rfloor = 2^*} \sim \frac{N^2}{4\pi} \mathcal{A}_\Sigma\, m^2 \log m \,.$$

There is then a disagreement between weak and strong coupling, the two results differ by a finite multiplicative factor.

In the theory we considered, the massive degrees of freedom are $\mathcal{N}=2$ hypermultiplets $Q_I, \tilde{Q}^I$ in the bifundamental representation of $U(N_f)\times U(N_c)$. Each hypermultiplet contains two complex scalars and two Weyl fermions. The weak-coupling computation would give then

$$S_{\mathcal{N}=2} \sim \frac{6N_f N_c}{24\pi}\mathcal{A}_\Sigma\, m^2 \log m\,.$$

The holographic computation we performed gave the following result for the $\log m$ term:

$$S_{\mathcal{N}=2} \sim \frac{\lambda_q}{2\pi^2}\frac{N_f N_c}{24\pi}\mathcal{A}_\Sigma\, m^2 \log m\,,$$

as can be seen from equation (3.56) or (3.66). We notice that like for the $\mathcal{N}=2^*$ case we have a disagreement: at strong coupling the factor 6 in the numerator is replaced by $\lambda_q/2\pi^2$. These results cast some doubt on the conjecture of [Hertzberg, 2013] even though both cases are not very conclusive: in [Lewkowycz et al., 2013] the operator $\mathcal{O}_3$ actually does not contain only mass terms but also Yukawa couplings (that we didn't write). In our case also one source of ambiguity comes from the difficulty in defining precisely the flavor mass, since the quarks are not gauge-invariant operators and one should more properly talk about meson masses.

Finally, let us note that the results presented in [Chang and Karch, 2013, Jensen and O'Bannon, 2013] for a system of D3/D7 branes also exhibit a linear dependence on the 't Hooft coupling at first order in the backreaction of the flavor branes. The exact coefficient do not much though, a disagreement possibly related to the fact that the solutions we used resulted from the smearing technique whereas their result was based on linearized backreaction.

Conclusions and perspectives

In this thesis we focussed on two applications of the AdS/CFT correspondence: the transport properties of fermionic excitations in strongly coupled field theories and the mass correction to entanglement entropy. Our results were presented in the second and third chapter while in the first chapter we gave an introduction to the subject of AdS/CFT correspondence.

In the second chapter we started by reviewing known results and techniques concerning the shear viscosity which constituted our motivation and inspiration for our subsequent work on the diffusion of supercharges. We have computed the transverse part of supercharge correlators in $\mathcal{N} = 4$ SYM at finite temperature at zero and finite density; then we extracted the phonino diffusion constant using a Kubo formula. In the zero charge case we could confirm the result found previously from the calculation of the supersymmetric sound attenuation, using the longitudinal channel of the correlator.

For the finite density computation we used the background of the so called STU black hole, a solution of $\mathcal{N} = 2$, $D = 5$ gauged supergravity. Our computation revealed that the diffusion constant depends on the chemical potential of the dual theory and therefore does not have a universality property analogous to the one of the shear viscosity. The absence of universality is perhaps not surprising, since we saw that the transverse gravitino is coupled to the scalars and gauge fields that are running in the solution. Furthermore we observed that the dimensionless diffusion constant TD_s tends to a finite value for the extremal black hole. It would be interesting to know if this is true in all cases of extremal black hole, and perhaps find if this value can be predicted by some sort of attractor mechanism. Moreover it would be interesting to examine the behavior of the diffusion constant in non-relativistic backgrounds in view of applications to condensed matter systems with emergent supersymmetry [Lee, 2010].

In the third chapter we treated the subject of entanglement entropy. After a general overview of field theory and holographic results we focussed on the corrections induced by mass terms in the field theory. We detailed the computation of mass correction in the case of a holographic model constructed out of D3/D7 branes where the backreaction of the D7 branes was given in terms of an expansion parameter proportional to N_f/N_c for two geometries: the slab and the ball.

Our result seam to invalidate the conjecture that the log mass term maintains the form $m^2 \log m$ at strong coupling with the mass replaced by the renormalized mass. A disagreement was also found in other holographic computations [Lewkowycz et al., 2013, Bea et al., 2013]. It would be interesting nevertheless to pursue the perturbative computation of [Hertzberg, 2013] to higher order, to see if the discrepancy persists.

It would also be interesting to consider other cases of massive theories obtained by top-down string constructions (for instance in the flavored Klebanov-Strassler model

[Benini et al., 2007a, Bigazzi et al., 2009b]), as well as considering the setup of $D3/D7$ branes at finite temperature and density; the background geometries are known also in this case [Bigazzi et al., 2011].

Finally, as we mentioned, there are other mass-dependent terms with coefficients that depend on the geometry of the entangling region. In the case we studied we found one coefficient related to the curvature of the entangling surface, namely the constant term in (3.58), that is non zero for the ball. It would be interesting to compute the entanglement for other cases, e.g. in the case of a waveguide geometry to identify all of the coefficients.

In conclusion, we can say that there are still a lot of open questions related to hydrodynamics and entanglement apart from the ones already mentioned. This thesis makes a small step towards answering these questions but there is still plenty of room for future work.

Appendix A

Gamma matrix conventions

We will use the following explicit representation of the five-dimensional flat gamma-matrices as 2x2-block matrices :

$$\begin{aligned} \gamma^1 &= i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^{k+1} = i\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, k = 1,2,3 \\ \gamma_5 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \tag{A.1}$$

where σ^k are the usual Pauli matrices. They satisfy $\{\gamma_i, \gamma_j\} = 2\eta_{ij}$ with $\eta_{ij} = \mathrm{diag}(-1,1,1,1,1)$. Notice that we call 1 the time component.
In the text we often write a gamma matrix acting on a chiral spinor, it is then understood that the matrix should be replaced by the corresponding block with the right chirality: $(\gamma^i\psi)_\alpha = (\sigma^i)_{\alpha\dot{\beta}}\psi^{\dot{\beta}}$. Chiral indices can be raised and lowered with $\epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}}$.

Appendix B

Normalization of the boundary action

In this appendix we fix the normalization of the boundary action using the zero temperature calculation of the stress-energy tensor superfield correlator. Exceptionally in this appendix we use the conventions of [Gates et al., 1983].
The leading term of the energy-momentum tensor superfield OPE has the following structure up to a constant $\mathcal{A}$ [Anselmi et al., 1997]:

$$J_{\alpha\dot{\alpha}}(z)J_{\beta\dot{\beta}}(0) = \mathcal{A}\frac{s_{\alpha\dot{\beta}}\bar{s}_{\beta\dot{\alpha}}}{(s^2\bar{s}^2)^2}\,, \tag{B.1}$$

where

$$s_{\alpha\dot{\alpha}} = (x-x')_{\alpha\dot{\alpha}} + \frac{i}{2}[\theta_\alpha(\bar{\theta}-\bar{\theta}')_{\dot{\alpha}} + \bar{\theta}'_\alpha(\theta-\theta')_\alpha]\,,$$
$$\bar{s}_{\alpha\dot{\alpha}} = (x-x')_{\alpha\dot{\alpha}} + \frac{i}{2}[\bar{\theta}_{\dot{\alpha}}(\theta-\theta')_\alpha + \theta'_\alpha(\bar{\theta}-\bar{\theta}')_\alpha]\,.$$

The super energy-momentum tensor is a superfield containing the axial current (at zeroth order in θ) , the supercurrent (at linear order in θ) and the energy-momentum tensor (at second order in θ). To fix the proportionality constant we can calculate the energy-momentum tensor correlator by applying twice the operator $\frac{1}{8}([\bar{D}_{\dot{\alpha}}, D_\alpha]J_{\beta\dot{\beta}} + [\bar{D}_{\dot{\beta}}, D_\beta]J_{\alpha\dot{\alpha}})$ on the super energy-momentum tensor OPE and compare with the energy momentum correlator [Arutyunov and Frolov, 1999]:

$$T_{\mu\nu}T_{\rho\sigma} = \frac{5N_c^2}{\pi^4}\frac{J^\kappa_\rho J^\lambda_\sigma}{|x|^4}\left(\delta_{\mu\kappa}\delta_{\nu\lambda} + \delta_{\mu\lambda}\delta_{\nu\kappa} - \frac{1}{2}\delta_{\mu\nu}\delta_{\kappa\lambda}\right)\,, \tag{B.2}$$

where $J^\nu_\mu = \delta^\nu_\mu - 2x_\mu x^\nu/|x|^2$. From the comparison we deduce that:

$$\mathcal{A} = -\frac{32}{\pi^4}N_c^2\,. \tag{B.3}$$

Knowing the exact expression for the super energy-momentum tensor OPE we can compute the supercurrent correlator by applying the operator $D_\gamma\bar{D}'_{\dot{\gamma}}$ and doing the proper symmetrizations. Then we can compare with the zero temperature correlator of the supercurrent calculated in [Corley, 1999]:

$$S^\alpha_\mu\bar{S}^{\dot{\alpha}}_\nu = \frac{16\mathcal{N}}{\pi^2}\Pi^\rho_\mu\frac{\sigma^{\alpha\dot{\alpha}}_\lambda x^\lambda}{|x|^8}\left(\eta_{\rho\nu} - 2\frac{x_\rho x_\nu}{|x|^2}\right)\,, \tag{B.4}$$

where $\Pi^\rho_\mu = \left(\delta^\rho_\mu - \frac{1}{4}\gamma_\mu\gamma^\rho\right)$ and find the value of the normalization of the action:

$$\mathcal{N} = \frac{N_c^2}{\pi^2}\,. \tag{B.5}$$

Appendix C

Résumé détaillé (en français)

Nous présentons dans cet appendice un résumé de la thèse en français.

C.1 Introduction

La correspondance AdS/CFT [Maldacena, 1998] a été utilise pour répondre à des questions concernant les théories de jauge fortement couplées [Casalderrey-Solana et al., 2011] et les systèmes de matière condensée [Hartnoll, 2009]. Elle a été proposée par Maldacena et elle stipule que la théorie de supergravité classique de type IIB est duale à la théorie de jauge $\mathcal{N} = 4$ SYM fortement couplée. Dans cette thèse nous nous intéresserons à deux problèmes que nous analyserons en utilisant la correspondance AdS/CFT : les propriétés des modes fermioniques dans des théories fortement couplées et le comportement de l'entropie d'intrication dans un système contenant des degrés de liberté massifs. Les travaux présentés dans cette thèse ont été publiées dans [Kontoudi and Policastro, 2012] et [Kontoudi and Policastro, 2013].

Durant la dernière décennie les coefficients de transport des théories fortement couplées ont été calculés en utilisant la correspondance AdS/CFT (voir par exemple [Policastro et al., 2001, Policastro et al., 2002a, Policastro et al., 2002b] ou [Casalderrey-Solana et al., 2011] pour une description plus récente). La théorie duale utilisée est la supergravité dans la limite hydrodynamique qui offre une description effective de la théorie à des échelles de temps et de longueur grandes par rapport aux échelles microscopiques du système. Un résultat très marquant de ces investigations a été la découverte de l'universalité du rapport de la viscosité sur la densité d'entropie qui prend la valeur $1/4\pi$ dans toutes les théories qui peuvent être décrites par une théorie duale de gravité d'Einstein couplée à des champs de matière.

La plupart du travail dans ce domaine a porté sur les degrés de liberté bosoniques. Il est tout de même naturel de se demander si la supersymétrie impose suffisamment de contraintes pour que les fluctuations fermioniques aient aussi des propriétés d'universalité. En ayant pour but de répondre à cette question nous nous sommes intéressés [Kontoudi and Policastro, 2012] à la constante de diffusion du phonino, une excitation reliée aux fluctuations des supercharges, dans une théorie à densité finie. D'abord nous avons développé une méthode pour calculer la constante de diffusion en utilisant une formule de Kubo. Cette formule relie la constante de diffusion à la fonction de corrélation retardée de la composante transverse du gravitino. Avec cette méthode nous avons confirmé le résultat auparavant obtenue [Policastro, 2009] en calculant la fonction de corrélation retardée dans un fond de brane noire.

La géométrie de fond que nous avons utilisés pour le calcul à densité finie est le trou noir STU. Il s'agit d'un trou noir asymptotiquement AdS qui a été identifié comme une solution de la supergravité $\mathcal{N} = 2$, $D = 5$. Cette solution peut être incorporée dans la supergravité de type IIB et est duale à $\mathcal{N} = 4$ SYM avec trois charges appartenants à un sous-groupe du groupe de R-symétrie $SU(4)$. Nous avons étudiés des configurations différentes de trois charges et nous avons obtenue des résultats qualitativement similaires. La constante de diffusion dépend des trois charges et sa valeur approche le résultat de densité nulle pour $T/\mu \ll 1$ comme prévu. Le fait que la constante de diffusion dépend des charges nous indique qu'elle n'est pas universelle.

L'intrication est un des phénomènes qui caractérisent les systèmes quantiques. On peut définir une entropie associée à ce phénomène, l'entropie d'intrication, qui mesure l'intrication entre deux sous-systèmes. D'habitude on divise le système quantique en deux sous-systèmes qui correspondent à deux régions de l'espace. Une des propriétés les plus intéressantes de l'entropie d'intrication est qu'elle fournit de l'information non locale sur le système. Son comportement peut être utilisé pour identifier de phénomènes tels que l'ordre topologique ou la présence d'une surface de Fermi. Le comportement de l'entropie d'intrication a été analysé dans les théories conformes [Calabrese and Cardy, 2004] où la longueur de corrélation est infinie. Nous pouvons alors nous demander quel sera l'effet d'une longueur de corrélation finie sur l'entropie d'intrication. En pratique nous pouvons introduire une longueur de corrélation finie en ajoutant des excitations massives dans la théorie ; des corrections émanants des champs massifs ont été présentés dans [Hertzberg and Wilczek, 2011] et [Hertzberg, 2013].

Il existe une prescription holographique pour calculer l'entropie d'intrication dans le contexte de la correspondance AdS/CFT [Nishioka et al., 2009] et puisque la correspondance nous permet de sonder le comportement de la théorie des champs duale fortement couplée ça serait intéressant de voir comment l'entropie d'intrication dépends du couplage. Nous nous sommes intéressés au comportement des termes induites par les degrés de liberté massifs. Nous avons utilisé la prescription holographique pour évaluer dans [Kontoudi and Policastro, 2013] l'entropie d'intrication pour la théorie $\mathcal{N} = 4$ SYM couplée à N_f hypemultiplets massifs : c'est la théorie qui vit à l'intersection de N_c D3 et de N_f D7 branes [Karch and Katz, 2002]. Dans la limite $N_f \ll N_c$ cette théorie peut être décrite par de D7 branes sondes dans AdS_5. Afin d'observer l'effet de degrés de liberté massifs il faut prendre en compte la “backreation" de D7 branes sur la géométrie de fond. La solution qui prend en compte cet effet est connu sous forme d'un développement perturbait en $\epsilon = N_f/N_c$ [Bigazzi et al., 2009a]. Notre analyse a montrée que que le résultats à couplage fort diffèrent de résultats à couplage faible en accord avec d'autres résultats holographiques [Lewkowycz et al., 2013, Bea et al., 2013].

C.2 Elements de la correspondance AdS/CFT

Dans cette section nous présentons quelques éléments de base de la correspondance AdS/CFT.

Théories superconformes

Les théories superconformes et en particulier $\mathcal{N} = 4$ Super-Yang-Mills jouent un rôle très important dans la correspondance AdS/CFT. Les théories superconformes sont des théories des champs qui possèdent une symétrie superconforme : une symétrie qui comporte la symétrie conforme et la supersymétrie.

L' algèbre du groupe des symétries conformes est donnée par :

$$\begin{aligned}
[L_{\mu\nu}, P_\rho] &= -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} - \eta_{\mu\rho}L_{\nu\sigma} + \eta_{\nu\sigma}L_{\rho\mu} - \eta_{\mu\rho}L_{\rho\nu}) \\
[D, P_\mu] &= -iP_\mu \\
[L_{\mu\nu}, K_\rho] &= -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \\
[D, K_\mu] &= iK_\mu \\
[P_\mu, K_\nu] &= 2iL_{\mu\nu} - 2i\eta_{\mu\nu}D
\end{aligned} \tag{C.1}$$

où $\mu = 0, ..., 4$, $\eta_{\mu\nu}$ est la métrique de Minkowski, $L_{\mu\nu}$ est le générateur de transformations de Lorentz, P_μ est le générateur de translations, D est le générateur de transformations d'échelle et K_μ est le générateur des transformations conformes spéciales. Cette algèbre est isomorphe à l'algèbre du groupe $SO(4, 2)$. Le caractéristique commun des toutes les transformations conformes est que la métrique est invariante sous cettes transformations à un facteur multiplicatif près. L'invariance conforme impose des contraintes sur la forme des fonctions de corrélation à deux points. Pour un opérateur scalaire ϕ, par exemple, le corrélateur prend la forme

$$\langle\phi(x)\phi(0)\rangle \propto \frac{1}{x^{2\Delta}} \tag{C.2}$$

où Δ est la dimension conforme de l'opérateur.

La supersymétrie est une symétrie entre les fermions et les bosons (des références standards sur le sujet sont [Wess and Bagger, 1992] et [Gates et al., 1983]), ces générateurs Q^a_α et $\bar{Q}_{\dot{\alpha}a}$ sont respectivement des spineurs de Weyl gauche et droit . Les lettres grecques sont des indices spinoriels et $a = 1, ..., \mathcal{N}$ où $\mathcal{N}$ est le nombre de supersymétries. L'algèbre de supersymétrie est donnée par

$$\{Q^a_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}b}\} = 2\sigma^\mu_{\alpha\dot{\beta}}P_\mu\delta^a_b \qquad \{Q^a_\alpha, Q^b_\beta\} = 2\epsilon_{\alpha\beta}Z^{ab} \tag{C.3}$$

où $\sigma^\mu_{\alpha\dot{\beta}}$ sont les matrices de Pauli. Les Z^{ab} sont les charges centrales de l'algèbre. Pour $\mathcal{N} > 1$ on peut transformer les générateurs sous le groupe $SU(\mathcal{N})$ sans affecter l'algèbre ; cette symétrie de l'algèbre est connue sous le nom de R-symétrie.

L'algèbre superconforme est réalisé en ajoutant aux deux algèbres présentés précédemment les générateurs superconformes $S_{\alpha a}$ et $\bar{S}^a_{\dot{\alpha}}$. Les relations de commutation qui combinent les générateurs superconformes avec les algèbres de supersymétrie et de symétrie conforme sont :

$$\begin{aligned}
&\{S_{\alpha a}, \bar{S}^b_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}}K_\mu\delta^b_a \\
&\{Q^a_\alpha, S_{\beta b}\} = \epsilon_{\alpha\beta}(\delta^a_b D + T^a_b) + \frac{1}{2}\delta^a_b L_{\mu\nu}\sigma^{\mu\nu}_{\alpha\beta}
\end{aligned} \tag{C.4}$$

où T_b^a est le générateur de R - symétrie.

Une théorie qui est invariante sous les transformations superconformes en 4 dimensions est la $\mathcal{N} = 4$ Super-Yang-Mills (SYM). Son Lagrangien est donné par

$$\mathcal{L} = \mathrm{tr}\left\{ -\frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - \sum_i D_\mu X^i D^\mu X^i \right. \\ \left. + \sum_{a,b,i} g C_i^{ab} \lambda_a [X^i, \lambda_b] + \sum_{a,b,i} g \bar{C}_{iab} \bar{\lambda}^a [X^i, \bar{\lambda}^b] + \frac{g^2}{2} \sum_{i,j} [X^i, X^j]^2 \right\} \tag{C.5}$$

où $F_{\mu\nu}$ est l'intensité de champ associée à A_μ, les λ_α^a, $a = 1, ..., 4$ sont des spineurs de Weyl gauches et les X^i, $i = 1, ..., 6$ sont des scalaires réels. Sous la symétrie $SU(4)_R$, A_μ est un singulier, λ_α^a est dans la représentation **4** et les X^i sont dans la **6**. Tous les champs sont dans la représentation adjointe du groupe de jauge et forment le multiplet de jauge de l'algèbre de supersymétrie. La théorie st invariante sous les transformations du groupe superconforme qui, dans ce cas est $SU(2,2|4)$ et contient le groupe conforme en 4 dimensions $SO(2,4) \sim SU(2,2)$, le groupe de R-symétrie $SO(6) \sim SU(4)$ et leurs extensions supersymétriques.

Théorie des cordes et supergravité

Le point de départ de la théorie des cordes est une corde classique qui se propage dans l'espace temps. Il existe des cordes fermées et des cordes ouvertes. La théorie des cordes fermées peut être décrite en termes des champs suivants : la métrique $G_{\mu\nu}$, un tenseur antisymétrique $B_{\mu\nu}$, le dilaton Φ, qui est un champ scalaire, la métrique induite sur le worldsheet γ_{ab}, la courbure associée $R_\gamma^{(2)}$ et α', la racine carrée de la longueur de Planck.

L'action de la théorie sur le worldsheet est un modèle sigma non - linéaire. Afin d'avoir une théorie de cordes cohérente on doit imposer l'invariance conforme sur le worldsheet, ce qui implique une invariance d'échelle de la théorie quantique. Cela veut dire que le fonctions β (eq. 1.15 dans le texte) du groupe de renormalisation doivent s'annuler pour le modèle sigma. Les équations $\beta_{\mu\nu}^G = \beta_{\mu\nu}^B = 0$ calculés au premier ordre en α' sont en effet les équations de mouvement de la théorie de supergravité de type IIB. Le fait que les fonctions β ont été calculés au premier ordre en α' fait de la théorie de supergravité une théorie effective de la théorie des cordes à bases énergies.

La partie bosonique de l'action de supergravité de type IIB est

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}|H_3|^2 \right) \right. \\ \left. - \frac{1}{2}|F_1|^2 - \frac{1}{2}|\tilde{H}_3|^2 - \frac{1}{4}|\tilde{F}_5|^2 \right] - \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3 \,. \tag{C.6}$$

Les champs qui y apparaissent sont la métrique $g_{\mu\nu}$, le dilaton Φ et les n-formes C_0, C_2, C_4. On définit $F_{n+1} = dC_n$, $H_3 = dB_2$, $\tilde{F}_3 = F_3 - C_0 H_3$ et $\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3$. Les équations du mouvement doivent être complétées par la condition $F_5 = *F_5$.

Une classe importante de solutions sont les p-branes, des solutions qui contiennent une hypersurface $(p+1)$-dimensionnelle plate. La solution peut être mise sous la forme suivante :

$$ds^2 = H(\vec{y})^{-1/2} dx^\mu dx_\mu + H(\vec{y})^{1/2} d\vec{y}^2 \tag{C.7}$$

$$e^\Phi = H(\vec{y})^{(3-p)/4} \tag{C.8}$$

$$F_{p+2} = -d(H^{-1}) \wedge \epsilon_{1,p} \tag{C.9}$$

où x^μ, $\mu = 0, 1, ..., p$ sont les coordonnées parallèles à la p-brane, $y^a = x^{p+a}$, $a = 1, ..., 9-p$ sont les cordonnées perpendiculaire à la p-brane, $\epsilon_{1,p}$ est la forme de volume parallèle à la brane et H est une fonction harmonique donnée par :

$$H(y) = 1 + \frac{R^{7-p}}{y^{7-p}} \, . \tag{C.10}$$

Ces solutions sont supersymétriques et sont appelées solutions extrêmes. Il existe aussi des solutions qui brisent la supersymétrie et sont appelées non-extrêmes. Nous présentons ici une solution de ce type, donnée par :

$$ds_{10}^2 = \frac{-f(r)dt^2 + dx_p^2}{\sqrt{H(r)}} + \sqrt{H(r)} \left(\frac{dr^2}{f(r)} + r^2 d\Omega_{8-p}^2 \right) \tag{C.11}$$

$$H(r) = 1 + \frac{R^{7-p}}{r^{7-p}} \qquad f(r) = 1 - \frac{r_o^{7-p}}{r^{7-p}} \tag{C.12}$$

où r_o est la position de l'horizon. Cette solution est une généralisation de type brane noire de la solution extreme.

Dans la théorie des cordes ouvertes il faut imposer des conditions aux bords des cordes. Les Dp-branes sont des objets de dimension $p+1$ sur lesquels sont fixées les extrémités des cordes ouvertes. L'action effective à base énergie qui nous permet de décrire ces objets, en analogie avec (C.6) pour les cordes fermées, est l'action de Dirac-Born-Infeld (DBI) :

$$S_{\text{DBI}} = -T_{Dp} \int d^{p+1}\xi \; e^{-\Phi} \sqrt{-\det(\gamma_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta} + \hat{B}_{\alpha\beta})} \tag{C.13}$$

où ξ sont les coordonnées sur la Dp-brane, $\gamma_{\alpha\beta}$ set la métrique induite sur la Dp-brane, $\hat{B}_{\alpha\beta}$ est la pullback de la 2-forme $B_{\mu\nu}$, $F_{\alpha\beta}$ est l'intensité du champ de jauge qui vit sur la Dp-brane et T_{Dp} est la tension de la Dp-brane.

Enoncé de la correspondance

La correspondance AdS/CFT a été motivée par l'identification des descriptions effectives à base énergie de deux secteurs de la théorie des cordes : le secteur de cordes ouvertes et de cordes fermées et leur descriptions en termes de N_c Dp-branes coïncidentes et de p-branes respectivement. Il s'agit d'une conjecture proposée par Maldacena [Maldacena, 1998] qui stipule que les deux théories suivantes sont duales :

– Théorie des supercordes de type IIB dans $AdS_5 \times S^5$, où AdS_5 et S^5 ont le meme rayon R, $N_c = \int_{S^5} F_5^+$ et le couplage des cordes est g_s.

– La théorie $\mathcal{N} = 4$ SYM en 4 dimensions, avec groupe de jauge $SU(N_c)$ et couplage de Yang-Mills g_{YM}.

Les paramètres de deux théories sont identifiés comme suit :

$$g_s = g_{YM}^2 \qquad R^4 = 4\pi g_s N_c \alpha'^2. \tag{C.14}$$

Une deuxième version, plus faible, de la correspondance peut être établit dans la limite de 't Hooft :

$$\lambda \equiv g_{YM}^2 N_c = \text{fixed}; \qquad N_c \to \infty. \tag{C.15}$$

Nous pouvons aussi prendre la limite de λ grand dans la théorie de $\mathcal{N} = 4$ SYM. Cette limite correspond à la limite classique de la supergravité $\alpha' = 0$. Cette forme de la correspondance est celle que nous allons utiliser par la suite.

Une indication de la correspondance est que les symétries des deux théories duales correspondent. Nous pouvons identifier les représentations des groupes de symétrie de deux théories et établir ainsi une correspondance entre les opérateur de la théorie $\mathcal{N} = 4$ SYM et les champs de la théorie de supergravité. La théorie de supergravité vit dans l'espace $AdS_5 \times S^5$ dont la métrique est donnée par :

$$ds^2 = R^2 \left(\frac{1}{z^2} dx_\mu dx^\mu + \frac{dz^2}{z^2} \right) + R^2 d\Omega_5^2 \,. \tag{C.16}$$

Le groupe d'isométries de l'espace AdS est $SO(2,4)$ et celui de la 5-sphère est $SO(6)$. Ces groupes sont en parfaite adéquation avec la partie bosonique du groupe des symétries de la théorie $\mathcal{N} = 4$ SYM. Les opérateurs de la théorie SYM vivent dans un espace 4-dimensionnel. Les champs de la théorie de supergravité vivent initialement dans un espace 10-dimensionnel mais nous pouvons les décomposer en une partie angulaire sur la 5-sphère et une partie sur l'espace AdS_5. Les opérateurs de la théorie des champs vivent sur le bord conforme de l'espace AdS_5 tandis que les champs de supergravité sont étendues aussi dans le "bulk" de l'espace AdS_5. Nous nous sommes concentrés ici sur le cas standard de la correspondance mais en général on peut dire qu'une théorie des champs conforme en d dimensions est duale à une théorie de gravité asymptotiquement AdS en $d+1$ dimensions.

Une implication très importante de la correspondance est qu'elle nous permet de calculer des fonctions de corrélations de la théorie de champs en utilisant les solutions de la théorie de supergravité. Afin de calculer des fonctions de corrélation on introduit un terme de source dans la théorie des champs

$$\int d^4x \, J(x) \mathcal{O}(x) \,, \tag{C.17}$$

où $J(x)$ est la source de l'opérateur $\mathcal{O}(x)$. La fonction génératrice est alors donnée par :

$$Z[J] = \left\langle \exp \left(- \int d^4x J \mathcal{O} \right) \right\rangle \,. \tag{C.18}$$

La correspondance stipule que dans la théorie de supergravité duale la fonction génératrice est donnée par :

$$Z[J] = \exp\left(-S^{\text{on-shell}}[\phi]\right) \tag{C.19}$$

où $S^{\text{on-shell}}$ est l'action de supergravité on-shell et la source $J(x)$ est identifiée avec le mode non-renormalisable du champ ϕ près du bord de l'espace AdS. Nous pouvons alors évaluer la fonction à 2 points comme suit :

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = -\frac{\delta S^{\text{on-shell}}}{\delta J(x)\delta J(y)} \,. \tag{C.20}$$

Généralisations de la correspondance

La correspondance peut être généralisée afin de décrire des théories à température non-nulle, densité non-nulle et contentant des degrés de liberté de saveur.

Une température non-nulle peut être introduite dans la théorie des champs en introduisant un trou-noir où une brane-noire dans la théorie de gravité correspondante. La solution (C.11) pour $p = 3$ dans la limite $r_o \ll R$ prend la forme suivante :

$$ds^2_{10} = \frac{(\pi T R)^2}{u}(-f(u)dt^2 + dx^i dx_i) + \frac{R^2}{4u^2 f(u)}du^2 + R^2 d\Omega_5^2 \tag{C.21}$$

avec $f(u) = 1 - u^2$ et $T = r_o/(\pi R)$ est la température qui caractérise la brane noire. Cette brane noire a aussi une densité d' entropie de Bekenstein-Hawking qui est proportionnelle à l'aire de l'horizon et est donnée par :

$$s = \frac{\pi^2}{2} N_c^2 T^3 \,. \tag{C.22}$$

L'énergie libre de la théorie peut être calculée comme $F = T S_E^{\text{on-shell}}$ en fonction de l'action on-shell dans la signature Euclidienne. Son calcul requiert une procédure de renormalisation de l'action qui est détaillée dans le texte principal. La densité d'énergie libre de la solution est

$$f = \frac{F}{V_3} = -\frac{\pi^2}{8} N_c^2 T^4 \,. \tag{C.23}$$

Enfin, la densité d'énergie de la solution peut être calculée par le lois de la thermodynamique comme :

$$\epsilon = f + Ts = \frac{3\pi^2}{8} N_c^2 T^4 \,. \tag{C.24}$$

Une fois qu'on a rajouté une température non-nulle dans la théorie on peut aussi avoir une densité non-nulle. Cela correspond à des solutions des branes noires chargées, c.-à-d. des solutions qui contiennent des champs de jauge. Ceci peut être justifié en comparant la fonction de partition dans l'ensemble canonique $Z = e^{-\beta \int (\mathcal{H} - \mu \rho)}$ avec celle d'une théorie contenant des champs de jauge $Z[A_\mu] = e^{-\beta \int (\mathcal{H} - A_\mu J^\mu)}$; nous pouvons alors identifier le potentiel chimique avec la composante temporelle du champ de jauge $A_0 = \mu$.

La version standard de la correspondance contient que des champs dans la représentation adjointe. Afin d'introduire des champs dans la représentation fondamentale il faut rajouter une deuxième pile des branes dans la construction. Nous pouvons rajouter plusieurs types de branes mais une construction typique, proposé dans [Karch and Katz, 2002] est de rajouter de D7 branes qui sont étendues dans les directions suivantes :

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	×	×	×	×						
D7	×	×	×	×	×	×	×	×		

Les D7 branes se trouvent dans la géométrie de fond $AdS_5 \times S^5$ générée par les D3 branes. Dans la limite des branes tests (probe branes) nous considérons que les branes n'altèrent pas la géométrie de fond. Les D7 branes sont décrite par l'action (1.24) mais sans champ de jauge et champ B :

$$S_{D7} = -N_f T_{D7} \int d^8x \, e^{-\Phi} \sqrt{-g_8} \, . \tag{C.25}$$

Cette action nous fournit une équation qui détermine la façon dont les D7 branes sont immergées dans la géométrie de fond.

Théories avec des degrés de liberté de saveur "unquenched"

La construction que nous venons de présenter peut être étudier davantage en prenant en compte l'effet des branes sur la géométrie de font, la "backreaction". En effet, dans la limite de 't Hooft la "backreaction" est négligeable et il faut plutôt étudier la théorie dans la limite de Veneziano [Veneziano, 1976] :

$$N_f \to \infty \qquad \frac{N_f}{N_c} = \text{fixed} \, . \tag{C.26}$$

Dans cette limite, appelée "unquenched", les champs de saveur peuvent circuler dans les boucles des diagrammes de Feynman tandis que dans la limite des branes test les champs de saveur ne sont pas dynamiques. Pour la théorie de gravité correspondante cela implique que nous devons résoudre le système couplé qui comporte l'action pour le fond et pour les D7 branes :

$$S = S_b + S_{fl} \tag{C.27}$$

$$S_b = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_{10}} \left[R - \frac{1}{2} \partial_M \Phi \partial^M \Phi - \frac{1}{2} e^{2\Phi} F_{(1)}^2 - \frac{1}{2} \frac{1}{5!} F_{(5)}^2 \right] \tag{C.28}$$

$$S_{fl} = -T_7 \sum_{N_f} \left(\int d^8 \, x e^{\Phi} \sqrt{-g_8} - \int C_8 \right) \, . \tag{C.29}$$

Nous nous intéresserons au cas des degrés de liberté massifs, ce qui correspond au fait que le D7 branes soient séparées de D3 branes. De plus nous faisons usage des solutions trouvées en utilisant la technique d'étalement ("smearing technique") [Nunez et al., 2010] qui consiste à remplacer les branes localisées par une densité homogène dans l'espace transverse aux branes. L'ansatz pour la métrique est donnée par [Bigazzi et al., 2009a] :

$$ds_{10}^2 = h^{-1/2}(-dt^2 + d\vec{x}_3^2) + h^{1/2} \left[F^2 d\rho^2 + S^2 ds_{CP^2}^2 + F^2 (d\tau + A_{CP^2})^2 \right] \, , \tag{C.30}$$

$$ds^2_{CP^2} = \frac{1}{4}d\chi^2 + \frac{1}{4}\cos^2\frac{\chi}{2}(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{1}{4}\cos^2\frac{\chi}{2}\sin^2\frac{\chi}{2}(d\psi + \cos\theta d\varphi)^2\,, \quad \text{(C.31)}$$

$$A_{CP^2} = \frac{1}{2}\cos^2\frac{\chi}{2}(d\psi + \cos\theta d\varphi)\,. \quad \text{(C.32)}$$

Les D7 branes sont étendues dans la direction radiale ρ jusqu'à ρ_q et par conséquent les solutions pour les fonctions qui apparaissent dans la métrique sont différentes avant et après ρ_q. Les solutions sont trouvées sous forme d'une série perturbative en ϵ_q donnée par :

$$\epsilon_q = \frac{1}{8\pi^2}\lambda_q\frac{N_f}{N_c} \quad \text{(C.33)}$$

où λ_q est le couplage de 't Hooft à l'échelle de ρ_q.

Nous pouvons fixer l'invariance sous reparametrisations de la métrique en définissant une nouvelle coordonnée z et en imposant que la fonction h soit definit comme suit :

$$h(z) = \frac{z^4}{R^4}; \qquad R^4 \equiv \frac{1}{4}Q_c\,. \quad \text{(C.34)}$$

Pour $\rho > \rho_q$ la coordonnée z est donnée en fonction de ρ par :

$$\begin{aligned} z_>(\rho) =& \frac{e^{-\rho}R^2}{\sqrt{\alpha'}}\left[1 + \frac{\epsilon_q}{720}\left(\frac{8e^{-6\rho}R^{12}}{\alpha'^{\,3}z_q^6} - \frac{45e^{-4\rho}R^8}{\alpha'^2 z_q^4} + \frac{30e^{-4\rho_*}R^8}{\alpha'^2 z_q^4}\right.\right. \\ &\left.\left. + \frac{120e^{-2\rho}R^4}{\alpha' z_q^2} - \frac{120e^{-2\rho_*}R^4}{\alpha' z_q^2} + 120\rho - 120\rho_* + 10\right)\right] \end{aligned} \quad \text{(C.35)}$$

où $z_q = z(\rho_q)$. Nous pouvons par la suite inverser cette relation pour obtenir $F_>(z)$ et $S_>(z)$ sous forme des séries perturbatives en ϵ_q :

$$F_>(z) = \frac{R^2}{z} + \frac{R^2\epsilon_q}{240zz_q^6}\left(-45z^4z_q^2 + 40z^2z_q^4 - 10z_q^6 + 16z^6\right)$$

$$S_>(z) = \frac{R^2}{z} + \frac{R^2\epsilon_q}{240zz_q^6}\left(15z^4z_q^2 - 20z^2z_q^4 + 10z_q^6 - 4z^6\right)\,.$$

En imposant la condition de continuité de la fonction h à $\rho = \rho_q$ nous obtenons les expressions suivantes pour la coordonnée z et les fonctions $F_<(z)$ et $S_<(z)$ pour $\rho < \rho_q$:

$$z_<(\rho) = \frac{e^{-\rho}R^2}{\sqrt{\alpha'}}\left[1 + \epsilon_q\left(\frac{e^{-4\rho_*}R^8}{24\alpha^2 z_q^4} + \frac{\alpha^2 e^{4\rho}z_q^4}{240R^8} - \frac{e^{-2\rho_*}R^4}{6\alpha z_q^2} - \frac{1}{6}\log\left(\frac{\sqrt{\alpha'}z_q}{R^2}\right) - \frac{\rho_*}{6} + \frac{1}{8}\right)\right]\,, \quad \text{(C.36)}$$

$$F_<(z) = S_<(z) = \frac{R^2}{z} + \epsilon_q\frac{R^2z_q^4}{720z^5}\,.$$

Ces formules décrivent la géométrie après la "backreaction" et nous les utiliserons plus tard pour évaluer l'entropie d'intrication dans cette géométrie.

C.3 Hydrodynamique holographique

Dans cette section nous présentons l'hydrodynamique dans la correspondance AdS/CFT et principalement le travail de l'auteur sur la constante de diffusion du gravitino [Kontoudi and Policastro, 2012].

Hydrodynamique dans la correspondance AdS/CFT

L'hydrodynamique est une théorie effective qui décrit un système à des échelles de longueur et de temps grandes par rapport à l'échelle microscopique du système. Les courants conservés jouent un rôle primordial dans cette description effective et la description de la théorie est faite par les équations de conservation de courants et les relations constitutives. Les relations constitutives sont des expressions des courants conservés en fonction des dérivés des variables thermodynamiques.

Dans le cas du tenseur énergie impulsion $T_{\mu\nu}$ l'équation de conservation est donnée par

$$\partial_\mu T^{\mu\nu} = 0 \tag{C.37}$$

et la relation constitutive par

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} - \sigma^{\mu\nu} \tag{C.38}$$

avec

$$\sigma^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} \left[\eta \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta} \partial_\lambda u^\lambda \right) + \zeta g_{\alpha\beta} \partial_\lambda u^\lambda \right] . \tag{C.39}$$

La vitesse du fluide est notée u_μ et $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ est le projecteur aux directions perpendiculaires à la vitesse. Les coefficients η et ζ sont des paramètres libres de la théorie effective et sont respectivement la viscosité de cisaillement et la viscosité de volume. Ces paramètres constituent des exemples de coefficients de transport.

Dans les théories de champs supersymétriques on peut avoir des courants conservés fermioniques, les supercourants S^μ_α et $\bar{S}^\mu_{\dot\alpha}$ qui sont des vecteurs - spineurs de Weyl gauche et droit respectivement. Dans ce cas l'équation de conservation s'écrit :

$$\partial_\mu S^\mu_\alpha = \partial_\mu \bar{S}^\mu_{\dot\alpha} = 0 \tag{C.40}$$

et les relations constitutives sont données par :

$$\begin{aligned} S^i_\alpha &= -D_s \partial^i \rho_\alpha + D_\sigma (\gamma^{ij} \partial_j \rho)_\alpha - \frac{P}{\epsilon} (\gamma^i \gamma^1 \rho)_\alpha \\ \bar{S}^i_{\dot\alpha} &= -D_s \partial^i \bar\rho_{\dot\alpha} + D_\sigma (\partial_j \bar\rho \gamma^{ji})_{\dot\alpha} - \frac{P}{\epsilon} (\bar\rho \gamma^1 \gamma^i)_{\dot\alpha} \end{aligned} \tag{C.41}$$

où ρ_α et $\bar\rho_{\dot\alpha}$ sont les densités des supercharges et D_s et D_σ sont des constantes de diffusion. Ces constantes de diffusion caractérisent l'atténuation d'une excitation collective, le phonino. Dans les théories conformes les deux constantes de diffusion sont égales.

Une manière efficace de calculer les paramètres libres de l'hydrodynamique est de les relier aux fonctions de corrélation de la théorie via des formules de Kubo. En effet, en utilisant la théorie des réponses linéaires on peut considérer la fonction de corrélation des perturbations de la métrique comme la réponse du système à l'introduction d'un tenseur énergie impulsion. En utilisant ce principe on peut prouver la formule de Kubo suivante pour la viscosité :

$$\eta = -\lim_{\omega\to 0}\lim_{k\to 0}\frac{1}{\omega}\mathrm{Im}G^R_{xy,xy}(\omega,\vec{k})\,. \tag{C.42}$$

Pour le supercourants nous pouvons trouver une formule de Kubo en partant de la définition de la fonction de corrélation retardée avec le résultat suivant :

$$\epsilon D_s = \frac{1}{12}\lim_{\omega\to 0}\left[\lim_{k\to 0}(i\gamma^1)^{\alpha\dot{\alpha}}\,\mathrm{Im}\,G^{ii}_{\alpha\dot{\alpha}}(\omega,\vec{k})\right]\,. \tag{C.43}$$

Nous allons maintenant étudier quelques applications de la correspondance au calcul des coefficients de transport. Plusieurs phénomènes de transport ont été étudiés et leur propriétés in été découvertes. En particulier il a été prouvé que le rapport de la viscosité de cisaillement sur la densité d'entropie est universel, c.-à-d. il a la meme valeur pour tout système qui peut être décrit par un dual gravitationnel. L'étude des propriétés hydrodynamiques à travers AdS/CFT a été initiée dans [Policastro et al., 2001, Policastro et al., 2002a, Policastro et al., 2002b]. Ici nous présentons comme exemple le cas de la viscosité.

Nous allons maintenant évaluer le corrélateur de la composante transverse du tenseur énergie impulsion dans la géométrie de fond (C.21). Nous utiliserons l'action de supergravité pour le champ dual : la métrique. Nous pouvons choisir des coordonnées de façon que les perturbations de la métrique se propagent selon la troisième coordonne spatiale, c.-à-d. l'impulsion est $\vec{k}=(0,0,q)$. Nous pouvons par la suite classifier les perturbations de la métrique par rapport au représentations du groupe de rotations $O(2)$ sur le plan (x,y). La perturbation transverse h_{xy} découple des perturbations longitudinales h_{tx} and h_{ty}. En définissant

$$\phi = h^x_y \qquad a_t = \phi^x_t \qquad a_z = h^t_z\,, \tag{C.44}$$

et en supposant que toutes les autres composantes sone nulles nous pouvons écrire l'action comme suit :

$$S = \frac{N_c^2}{8\pi^2 R^3}\int d^4x du\,\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4g^2_{\text{eff}}}g^{\mu\rho}g^{\nu\sigma}f_{\mu\nu}f_{\rho\sigma}\right) \tag{C.45}$$

Nous avons definit $f_{\mu\nu}=\partial_\mu a_\nu - \partial_\nu a_\mu$ et $g^2_{\text{eff}} = g_{xx}$. La composante transverse de la métrique vérifie l'equation suivante dans l'espace de Fourier :

$$\phi_k'' - \frac{1+u^2}{uf}\phi_k' + \frac{\omega^2 - k^2 f}{(2\pi T)^2 u f^2}\phi_k = 0\,, \tag{C.46}$$

où les primes symbolisent les dérivées par rapport à u. Suivant 1.4.1 afin de résoudre l'équation dans un fond à température non-nulle il faut imposer des conditions entrantes à l'horizon. Les équations ne peuvent pas être résolues analytiquement mais elle peuvent être résolues perturbetivement pour des petites fréquences. La solution qui a le comportement voulu à l'horizon est :

$$\phi_k(u) = (1-u)^{-i\omega/(4\pi T)}F_k(u) \tag{C.47}$$

où la fonction $F_k(u)$ peut être écrite comme un développement sur la fréquence :

$$F_k(u) = 1 - \frac{i\omega}{4\pi T}\ln\frac{1+u}{2} + \frac{\omega^2}{32\pi^2T^2}\left[\left(\ln\frac{1+u}{2} + 8(1-q^2/\omega^2)\right)\ln\frac{1+u}{2} - 4\mathrm{Li}_2\frac{1+u}{2}\right] \tag{C.48}$$

La partie pertinente de l'action (C.45) devient :

$$S = -\frac{\pi^2N^2T^4}{8}\int dud^4x\ \frac{f}{u}\phi'^2 + ... \tag{C.49}$$

Donc, la fonction de corrélation du tenseur énergie impulsion est donnée par :

$$G^R_{xy,xy}(\omega,k) = \frac{N_c^2T^2}{16}(i2\pi T\omega + q^2)\,. \tag{C.50}$$

En utilisant la formule de Kubo (C.42) en combinaison avec le résultat précèdent nous avons pour la viscosité :

$$\eta = \frac{\pi}{8}N_c^2T^3\,. \tag{C.51}$$

En utilisant la valeur de la densité d'entropie (C.22) nous obtenons le fameux résultat :

$$\frac{\eta}{s} = \frac{1}{4\pi}\,. \tag{C.52}$$

Ce résultat a été calculé ici pour un cas particulier de théorie à température non-nulle. Nous présentons dans la section 2.3.1 du texte principal une explication du fait que cette valeur reste inchangée dans toutes les théories qui peuvent être décrites par un dual gravitationnel.

Le corrélateur du supercourant

La fonction de corrélation retardée des supercourants S_i^α, est défini comme :

$$G_{ij}^{\alpha\dot\beta}(k) = \int d^4xe^{-ik\cdot x}i\theta(x^0)\langle\{S_i^\alpha(x),\bar S_j^{\dot\beta}(0)\}\rangle\,. \tag{C.53}$$

La conservation de supercourants et l'invariance superconforme impliquent :

$$k^iG_{ij} = 0 \qquad \gamma^iG_{ij} = G_{ij}\gamma^j = 0\,. \tag{C.54}$$

Ces contraintes nous permettent d'établir les relations suivantes entre les parties transverses et longitudinales du corrélateur dans la limite des petites impulsions et fréquences :

$$\begin{aligned} G^T_{22} = G^T_{33} = -a'/2 \qquad & G^T_{44} = 0 \\ G^L_{22} = G^L_{33} = -a'/6 \qquad & G^L_{44} = -2a'/3\,. \end{aligned} \tag{C.55}$$

Avant de détailler le calcul du corrélateur à température finie, notons ici que dans le cas d'une théorie à température nulle avec signature Euclidienne le corrélateur des supercourants a été calculé [Corley, 1999, Rashkov, 1999, Volovich, 1998] avec le résultat suivant :

$$G_{ij}(k) \propto \Pi_i^l\frac{\not k}{k}\left(\delta_{ln} - \frac{2(2m+1)}{2m+5}\frac{k_lk_n}{k^2}\right)\Pi_j^n \tag{C.56}$$

où $\Pi_i^j = \delta_i^j - \gamma_i\gamma^j/4$ est le projecteur sur la direction transverse.

La diffusion de supercharge à temperature finie

Pour le calcul de la fonction de corrélation de supercourants nous utiliserons la géométrie de fond de brane noire (voir eq. C.21). Sa métrique est donnée par :

$$ds^2 = \frac{\pi^2 T^2 R^2}{u}(-f(u)dt^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{4f(u)u^2}du^2 \tag{C.57}$$

où $f(u) = 1 - u^2$. L'operateur dual au supercourant est le gravitino et nous avons besoin de l'action du gravitino en supergravité ; elle donnée par :

$$S = \int d^4x\sqrt{-g}(\bar{\Psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \Psi_\rho - m\bar{\Psi}_\mu \Gamma^{\mu\nu}\Psi_\nu)\,, \tag{C.58}$$

où la dérivée covariance agit sur le spineurs comme $D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$. L'équation du mouvement est l'équation de Rarita-Schwinger :

$$\Gamma^{\mu\nu\rho} D_\nu \Psi_\rho - m\Gamma^{\mu\nu}\Psi_\nu = 0\,. \tag{C.59}$$

Dans la jauge $\Gamma^\mu \Psi_\mu = 0$ les équations pour les composantes spatiales du champ prennent la forme [Policastro, 2009] :

$$\gamma^5 \psi_k' + \frac{1}{2\pi T\sqrt{uf}}\left(\frac{1}{\sqrt{f}}\gamma^1\partial_t + \gamma^j\partial_j\right)\psi_k + \frac{u^2-2}{2uf}\gamma^5\psi_k - \frac{1}{2u}\gamma_k\psi_5 + \frac{mR}{2u\sqrt{f}}\psi_k = 0 \tag{C.60}$$

où $\psi_a = e_a^\mu \Psi_\mu$, $\gamma_a = e_a^\mu \Gamma_\mu$ et $\psi' = \partial\psi/\partial u$.
Nous pouvons definir la composantes transverse du gravitino comme $\eta = \gamma^2\psi_2 - \gamma^3\psi_3$. L'équation obéis par ce champ dans l'espace de Fourier est découplée des autres champs et devient :

$$\eta' = -\gamma^5\left(\frac{\not{P}}{\sqrt{uf}} + \frac{u^2-2}{2uf}\gamma^5 - \frac{3}{4u\sqrt{f}}\right)\eta \tag{C.61}$$

avec $\not{P} = -i\omega/\sqrt{f}\gamma^1 + iq\gamma^4$.
Nous ne pouvons pas trouver une solution analytique de cette équation mais nous pouvons la résoudre comme une série perturbative pour des petites impulsions. L'equation (C.61) commute avec $\gamma_{23} = \mathrm{diag}(1, -1, 1, -1)$ et par conséquent on peut obtenir une équation différentielle matricielle pour (η_1, η_3) (ou pour (η_2, η_4)). Nous trouvons les solutions suivantes près de l'horizon :

$$\begin{aligned} &(1-u)^{-\frac{1}{4}-i\frac{\omega}{2}}(-i, 1) \\ &(1-u)^{-\frac{1}{4}+i\frac{\omega}{2}}(i, 1) \end{aligned} \tag{C.62}$$

Nous devons imposer des conditions entrantes à l'horizon afin de calculer le corrélateur retardé [Son and Starinets, 2002]. Cela correspond à imposer la relation $\eta^+ = -i\eta^-$. Le comportement de la solution près du bord est donnée par :

$$\eta = \begin{pmatrix} u^{3/4}(-i3^{3/4})(q-\omega)\alpha + u^{7/4}\frac{3^{3/4}}{2\sqrt{2}}\beta(\omega, q) \\ u^{3/4} i 3^{3/4}(\omega+q)\gamma + u^{7/4}\frac{3^{3/4}}{2\sqrt{2}}\delta(\omega, q) \\ u^{1/4}3^{3/4}[i\beta(\omega, q) + \sqrt{2}(q - (1-\sqrt{2}\not{L})\omega)\alpha] \\ u^{1/4}3^{3/4}[i\delta(\omega, q) - \sqrt{2}(q + (1-\sqrt{2}\not{L})\omega)\gamma] \end{pmatrix} \tag{C.63}$$

où $\not{L} = \log(1+\sqrt{2})$. Nous pouvons identifier la source φ avec la composante de chiralité négative $\eta^- = u^{1/4}\varphi$ et identifier χ et $\tilde{\chi}$ comme les coefficients du développement limité de la partie de chiralité positive $\eta^+ = u^{3/4}\tilde{\chi} + u^{7/4}\chi$. Cela nous permet d'établir les relations suivantes qui sont valables au premier ordre aux impulsions :

$$\tilde{\chi} = \frac{i}{2\pi T}\not{k}\varphi\,, \tag{C.64}$$

$$\chi = \mathrm{diag}(-i\frac{\sqrt{2}}{4} - 4q + 4(1-\sqrt{2}\not{L})\omega, -i\frac{\sqrt{2}}{4} + 4q + 4(1-\sqrt{2}\not{L})\omega)\varphi\,. \tag{C.65}$$

Sachant que, pour le gravitino, l'action on-shell est nulle, la seule contribution pour le corrélateur proviendra de l'action sur le bord :

$$S_{bdy} = \mathcal{N}\int d^4x\sqrt{-h}h^{ij}\bar{\Psi}_i\Psi^j\,, \tag{C.66}$$

où h est la métrique induite sur le bord et $\bar{\Psi} = i\Psi^\dagger\Gamma^1$. Définissant les champs comme $\phi = \gamma^2\psi_2 + \gamma^3\psi_3$ et $\eta = \gamma^2\psi_2 - \gamma^3\psi_3$ et posant $\phi = 0$, $\psi_1 = 0$, $\psi_4 = 0$ nous avons :

$$S_{bdy} = \frac{\mathcal{N}}{2}\int d^4x\sqrt{-h}\frac{\sqrt{u}}{\sqrt{f}\pi TR}\bar{\eta}\eta \tag{C.67}$$

L'action dans l'espace de Fourier à la limite $u = \epsilon \to 0$ prend la forme :

$$S_{bdy} = \frac{\mathcal{N}}{2}(\pi TR)^3\int d^4k\left(\epsilon^{-1/2}\varphi^\dagger\tilde{\chi} + \epsilon^{1/2}\varphi^\dagger\chi\right) \tag{C.68}$$

Suivant la recette proposée dans [Iqbal and Liu, 2009a] pour le calcul de la fonction de corrélation et après la renormalisation holographique l'action devient :

$$S_{ren} = \frac{\mathcal{N}}{2}(\pi TR)^3\int \varphi^\dagger\,\mathrm{diag}(-i\frac{\sqrt{2}}{4} - 4q + 4(1-\sqrt{2}\not{L})\omega, -i\frac{\sqrt{2}}{4} + 4q + 4(1-\sqrt{2}\not{L})\omega)\varphi$$

Par conséquent la fonction de corrélation de l'opérateur dual à η est donnée par :

$$G^{\mathcal{R}} = \frac{\mathcal{N}}{2}(\pi TR)^3\,\mathrm{diag}(-i\frac{\sqrt{2}}{4} - 4q + 4(1-\sqrt{2}\not{L})\omega, -i\frac{\sqrt{2}}{4} + 4q + 4(1-\sqrt{2}\not{L})\omega)\,. \tag{C.69}$$

La normalisation de l'action est calculée dans l'appendice B et est egale à $\mathcal{N} = N_c^2/\pi^2$. D'apres l'analyse sur les parties transverses et longitudinales des corrélateurs on peut en déduire que (avec sommation des indices i) $G^{ii}_{\alpha\dot{\alpha}} = 4G^{\mathcal{R}}_{\alpha\dot{\alpha}}$ où $G^{ii}_{\alpha\dot{\alpha}}$ est le corrélateur qui apparaît dans la formule de Kubo (C.43) pour la constante de diffusion. En utilisant la valeur de la densité d'énergie (C.24) on obtient pour la constante de diffusion :

$$D_s = \frac{2\sqrt{2}}{9\pi T}\,. \tag{C.70}$$

Ce résultat est en accord avec la constante de diffusion calculée dans [Policastro, 2009].

La diffusion de supercharge à densité finie

Nous aimerions étudier l'influence d'un fond de densité finie sur la constante de diffusion. Dans la théorie $\mathcal{N} = 4$ SYM il existe un groupe de R-symétrie sous lequel les supercourants se transforment dans la représentation fondamentale. Il est possible de donner simultanément une valeur attendue non-nulle aux trois charges de la sous-algebre de Cartan de $SU(4)$. La solution correspondante est un trou noir asymptotiquement AdS avec des charges électriques, connu sous le nom de trou noir STU. Elle a été construite pour la première fois comme une solution de la supergravité $d = 5, \mathcal{N} = 2$ avec deux multiplets de matière.

La théorie contient trois champs de jauge abéliens A_I et trois champs scalaires qui obéissent la contrainte $X^1 X^2 X^3 = 1$, avec une métrique sur la variété scalaire donnée par :

$$G^{IJ} = 2\,\mathrm{diag}[(X^1)^2, (X^2)^2, (X^3)^2]\,, \tag{C.71}$$

et potentiel scalaire $V = 2\sum_I 1/X^I$. La métrique du trou noir avec un horizon invariant par translations est :

$$ds^2 = -\mathcal{H}^{-2/3}\frac{(\pi TL)^2}{u}f(u)dt^2 + \mathcal{H}^{1/3}\frac{(\pi TL)^2}{u}(dx^2+dy^2+dz^2) + \mathcal{H}^{1/3}\frac{L^2}{4f(u)u^2}du^2$$

$$f(u) = \mathcal{H}(u) - u^2\prod_{I=1}^{3}(1+\kappa_I) \qquad H_I(u) = 1+\kappa_I u$$

$$\kappa_I = \frac{q_I}{(\pi TL^2)^2} \qquad \mathcal{H}(u) = \prod_{I=1}^{3} H_I(u)$$

où q_I sont les trois charges. Dans la limite de charge nulle la métrique est exactement (2.43). Le champ scalaire et le champ de jauge sont donnés par :

$$X^I = \frac{\mathcal{H}^{1/3}}{H_I(u)} \qquad A_t^I = \pi TLu\frac{\kappa_I}{H_I(u)}\prod_{J=1}^{3}(1+\kappa_J) \tag{C.72}$$

La température de Hawking du fond est :

$$T_H = \frac{2+\kappa_1+\kappa_2+\kappa_3-\kappa_1\kappa_2\kappa_3}{2\sqrt{(1+\kappa_1)(1+\kappa_2)(1+\kappa_3)}}T \tag{C.73}$$

La densité d'énergie, la densité d'entropie, la densité de charge et les potentiels chimiques sont respectivement :

$$\begin{aligned}
\epsilon &= \frac{3\pi^2 N^2 T^4}{8} \prod_{I=1}^{3} (1+\kappa_I) \\
s &= \frac{\pi^2 N^2 T^3}{2} \prod_{I=1}^{3} (1+\kappa_I)^{1/2} \\
\rho_I &= \frac{\pi N^2 T^3}{8} \sqrt{\kappa_I} \prod_{J=1}^{3} (1+\kappa_J)^{1/2} \\
\mu_I = A_t^I(u)|_{u=1} &= \frac{\pi T L \sqrt{\kappa_I}}{1+\kappa_I} \prod_{J=1}^{3} (1+\kappa_J)^{1/2} .
\end{aligned} \tag{C.74}$$

En imposant la condition de stabilité thermodynamique sur le fond on obtient une condition sur les charges [Son and Starinets, 2006] :

$$2 - \kappa_1 - \kappa_2 - \kappa_3 + \kappa_1\kappa_2\kappa_3 > 0 . \tag{C.75}$$

La partie de l'action qui est pertinente pour le gravitino est donnée par [Gunaydin et al., 1985] :

$$\begin{aligned}
\mathcal{L} = e\, [\, & \bar{\Psi}_\mu \Gamma^{\mu\nu\rho} \mathcal{D}_\nu \Psi_\rho + i\bar{\lambda}^a \Gamma^\mu \Gamma^\nu \Psi_\mu f_i^a \partial_\nu \phi^i - \frac{1}{2} h_I^a \bar{\lambda}^a \Gamma^\mu \Gamma^{\lambda\rho} \Psi_\mu F_{\lambda\rho}^I \\
& + \frac{1}{8X^I} i \left(\bar{\Psi}_\mu \Gamma^{\mu\nu\rho\sigma} \Psi_\nu F_{\rho\sigma}^I + 2\bar{\Psi}^\mu \Psi^\nu F_{\mu\nu}^I \right) + \frac{3}{2L} \bar{\Psi}_\mu \Gamma^{\mu\nu} \Psi_\nu \mathcal{V}_o - \frac{3}{L} \bar{\lambda}^a \Gamma^\mu \Psi_\mu \mathcal{V}_a \,]
\end{aligned} \tag{C.76}$$

où $I = 1,2,3,\ i,a = 1,2.\ h_I^a$, $\mathcal{V}_a$ et $\mathcal{V}_o$ sont des fonctions des champs scalaires, ϕ^i sont les champs scalaires sur la variété scalaire contrainte et f_i^a est la vielbein sur cette variété. La dérivée covariante agit sur les spineurs comme

$$\mathcal{D}_\mu \psi = D_\mu \psi - i \frac{3}{2L} V_I A_\mu^I \psi ,$$

où les V_I sont des constantes et doivent être égales à 1/3 pour faire end sorte que les H_I soient proprement normalisés [Behrndt et al., 1999]. $\mathcal{V}_o$ est donné par :

$$\mathcal{V}_o = \frac{1}{2} G^{IJ} \frac{V_I}{X^J} , \tag{C.77}$$

et $F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I$ sont les intensités de champ associées au champs de jauge. les équations de mouvement pour le gravitino sont :

$$\begin{aligned}
\Gamma^{\mu\nu\rho} \mathcal{D}_\nu \Psi_\rho + i\Gamma^\mu \Gamma^\nu \lambda_a f_i^a \partial_\nu \phi^i - \frac{1}{2} h_I^a \Gamma^\mu \Gamma^{\lambda\rho} \lambda^a F_{\lambda\rho}^I + \frac{i}{8X^I} \left(\Gamma^{\mu\nu\rho\sigma} \Psi_\nu F_{\rho\sigma}^I + 2\Psi^\nu F_{\mu\nu}^I \right) \\
+ \frac{3}{2L} \Gamma^{\mu\nu} \Psi_\nu \mathcal{V}_o - \frac{3}{L} \Gamma^\mu \lambda^a \mathcal{V}_a = 0
\end{aligned} \tag{C.78}$$

Les transformations de supersymétrie agissent sur le gravitino comme

$$\delta\Psi_\mu = \mathcal{D}_\mu \epsilon + \frac{i}{24X^I} (\Gamma_{mu}^{\nu\rho} - 4\delta_\mu^\nu \Gamma^\rho) \hat{F}_{\nu\rho}^I \epsilon + \frac{1}{6L} \sum_I X^I \Gamma_\mu \epsilon \tag{C.79}$$

où

$$\hat{F}^I_{\mu\nu} = F^I_{\mu\nu} + \frac{i}{4X^I}\bar{\Psi}_{[\mu}\Psi_{\nu]}$$

et nous avons omis les termes contenant les spineurs λ dans les deux formules précédentes.

Nous sommes intéressés à la combosante transverse du gravitino, qui peut être défini comme $\eta = \Gamma^x\Psi_x - \Gamma^y\Psi_y$. Comme on l'a fait dans la section précédente nous pouvons démontrer que la partie transverse du gravitino decouple de la partie longitudinale et des spineurs λ. L'equation pour la composante transverse est donc :

$$\eta' + \left(\frac{\gamma^5 \not{K}}{\sqrt{fu}} + F(u) - \frac{3i}{2L}\sqrt{g_{uu}g^{tt}}\gamma_5\gamma^1 V_I A^I_t + \frac{i}{4X^I}\sqrt{g^{tt}}\gamma^1 F^I_{ut} + \frac{3}{2L}\sqrt{g_{uu}}\gamma^5\mathcal{V}_o\right)\eta = 0 \tag{C.80}$$

avec

$$\begin{aligned} \not{K} &= -i\sqrt{\frac{\mathcal{H}}{f}}\gamma^1\omega + iq\gamma^4 \\ F(u) &= \frac{1}{4}\frac{g'_{tt}}{g_{tt}} + \frac{3}{4}\frac{g'_{xx}}{g_{xx}} \end{aligned} \tag{C.81}$$

Notons ici que la composante transverse du gravitino est invariant de jauge et nous pouvons séparer notre équation en deux systèmes en projetant sur les espaces propres de γ_{23}.

Cette équation peut être résolu comme en faisant un développement limité pour des petites impulsions comme on l'a fait dans le cas de charge nulle. Cependant il faut aller à des ordres plus élevés pour pouvoir tirer des conclusions sur l'universalité de la constante de diffusion. Une solution au premier ordre est présentée dans le texte principal. Nous nous concentrons ici sur la solution numérique de l'équation qui nous permet d'avoir une solution valable pour toutes les valeurs de la fréquence. Afin de résoudre numériquement le système d'équations nous devons spécifier des conditions initiales sous la forme d'un développement limité près de l'horizon. Nous travaillerons avec les composantes (η_1, η_3) puisque les équations pour le reste de composantes sont identiques pour $q = 0$ (limite à laquelle nous devons connaitre la solution pour utiliser la formule de Kubo). Tout d'abord nous résolvons le système asymptotiquement près de l'horizon et nous trouvons un développement de la forme générale :

$$\eta_{hor}(\omega, u, \kappa) = \eta^{(0)}_{hor}(\omega, \kappa) + \sqrt{1-u}\ \eta^{(1)}_{hor}(\omega, \kappa) + \dots \tag{C.82}$$

En utilisant ce développement comme une condition initiale nous intégrons le système d'équations de l'horizon vers le bord. Ensuite nous pouvons extraire le corrélateur à partir de la forme de la solution près du bord.

Le résultat pour le cas d'une charge $\kappa_1 = \kappa$, $\kappa_2 = \kappa_3 = 0$ est présenté sur la figure 2.1. A $\kappa = 0$ la valeur trouvé dans le cas de charge nulle est correctement reproduite. Sur la figure 2.2 nous présentons la constante de diffusion en fonction de T/μ et nous trouvons un comportement similaire à celui observé dans [Gauntlett et al., 2011] pour une solution AdS-RN de la supergravité $\mathcal{N} = 2$, $D = 4$. Pour des grandes valeurs de T/μ la constante de diffusion approche la valeur de densité nulle représentée par la ligne horizontale.

Sur la figure 2.3 nous présentons le résultat pour le cas de trois charges $\kappa_1 = \kappa_2 = \kappa_3 = \kappa$ en fonction de κ, et sur la figure 2.4 en fonction de T/μ. Nous remarquons que le comportement est qualitativement similaire à celui d'une charge pour des températures grandes mais TD_s tend vers une limite finie quand la température tends vers zéro.

C.4 Entropie d'intrication holographique

Introduction à l'entropie d'intrication

L'intrication est l'une des propriétés distinctives des systèmes quantiques. Elle reflète le fait qu'une mesure effectuée sur une partie d'un système va affecter le reste du système. Il existe plusieurs mesures de l'intrication, la plus couramment utilisée est *l'entropie d'intrication*. Elle peut être naturellement introduite dans un système quantique divisé en deux sous-systèmes A et B. Considérons un observateur qui a accès seulement au sous-système A ; les résultats de toutes les mesures qu' il peut faire sont codés dans la matrice de densité réduite ρ_{red} obtenue en intégrant les degrés de liberté dans B. L'entropie d'intrication du sous-système A est définie comme l'entropie de Von Neumann associé à la matrice de densité réduite :

$$S_A = -\,\text{tr}(\rho_{red} \log \rho_{red})\,. \tag{C.83}$$

Très souvent, on considère le cas dans lequel les sous-systèmes sont les degrés de liberté qui vivent dans des régions différentes de l'espace. La définition est tout à fait générale et peut être, en principe, s'appliquer à tout système, à condition que les degrés de liberté soient locaux, de sorte que l'on peut associer un espace de Hilbert à une région donnée de l'espace-temps. D'autre part, l'entropie d'intrication est une observable non-locale, et par conséquent, elle fournit des informations différentes par rapport aux quantités locales comme par exemple les corrélateurs . Elle est également utile dans de nombreux autres contextes, allant de la matière condensée à l'information quantique.

L'entropie d'intrication a fait l'objet d'études intensives au cours des dernières années, son calcul est généralement un problème très difficile et peu de résultats exacts sont connus. Dans une théorie quantique des champs, l'entropie d'intrication est une quantité divergente dans le UV et son calcul nécessite la mise en place d'un régulateur ultraviolet a. En termes de ce régulateur la structure de la partie divergente, dans une théorie en $d+1$ dimensions de l'espace-temps , peut être résumée comme suit (voir [Calabrese and Cardy, 2004] pour une étude plus approfondi des propriétés connues) :

$$S_A = \frac{c_{d-1}}{a^{d-1}} + ... + \frac{c_1}{a} + c_0 \log a + S_f\,, \tag{C.84}$$

où S_f est fini pour un $a \to 0$; les coefficients c_i dépendent en général les propriétés géométriques de la surface Σ séparant les régions A et B, et ont été calculés dans un nombre limité de cas (un examen des outils informatiques utilisés pour calculer l'entropie d'intrication dans les théories quantiques des champs libres se trouve dans [Casini and Huerta, 2009]) . Le terme divergent dominant est proportionnel à la surface de Σ, un fait connu comme la lois de l'aire. La plupart des termes du développement sont en fait ambigus, car ils ne sont pas invariants par changement d'échelle du régulateur. Une exception est le coefficient de $\log a$; dans une théorie conforme des champs , il a été démontré qu'il est relié aux charges centrales qui apparaissent dans l'anomalie conforme.

Dans un article très influent [Ryu and Takayanagi, 2006b] Ryu et Takayanagi ont proposé une recette remarquablement simple pour le calcul de l'entropie d'intrication dans les théories avec un dual holographique de gravité. La théorie quantique des champs vit au bord de l'espace AdS ; considérons une région du bord A délimitée par la surface d'intrication $\partial A = \Sigma$. Selon la proposition , l' entropie d'intrication de la région est proportionnelle à l'aire $\mathcal{A}$ de la surface minimale qui s'étend dans le "bulk" de AdS et dont la restriction sur le bord de AdS est ∂A :

$$S = \frac{\mathcal{A}}{4G_N^{(d+2)}} . \tag{C.85}$$

La structure de l'entropie d'intrication présentée dans (C.84) est valable pour les théories conformes. Lorsque nous nous éloignons de la conformalité le résultat peut dépendre aussi des échelles intrinsèques de la théorie, comme la masse. Nous allons nous concentrer sur les corrections qui apparaissent dans une déformation massive d'une théorie conforme des champs. Ces corrections ont été étudiés dans [Fursaev, 2006] pour la théorie de champ scalaire libre avec une longueur de corrélation finie $\xi = 1/m$ dans une région cubique, et par [Hertzberg and Wilczek, 2011] dans une géométrie de guide d'onde, c.-à-d. un cylindre dont la section transversale présente une forme arbitraire . Il a été trouvé qu'il existe une contribution fini à l' entropie de la forme, en $d = 3$,

$$S_f = \frac{A_\Sigma}{24\pi} m^2 \log m + f_0 \log m + f_1 m \tag{C.86}$$

où les coefficients f_i dépendent des caractéristiques géométriques du guide d'ondes, et A_Σ est l'aire de la surface d'intrication. Les termes figurant dans (C.86) sont finis et indépendants du régulateur ultraviolet . Ils peuvent être isolés de la partie divergente - UV en prenant des dérivées par rapport à la longueur de corrélation (voir [Liu and Mezei, 2013a, Liu and Mezei, 2013b] pour une proposition alternative pour définir pièces universelles finis) .

Dans [Hertzberg, 2013] le premier terme de (C.86) a été calculé perturbativement dans une théorie scalaire avec des interactions ϕ^3 et ϕ^4, est il a été trouvé que la structure reste la même , mais la masse est remplacé par la masse renormalisée.

Nous allons par la suite étudier un autre exemple d'entropie d'intrication dans une théorie des champs massive. Nous utiliserons la prescription holographique pour calculer l' entropie d'intrication pour $\mathcal{N} = 4U(N_c)$ SYM couplé à N_f hypermultiplets massives. Cette théorie a été présentée dans la partie introductive. Nous calculons l'entropie d'intrication dans deux cas : une région infinie délimitée par deux hyperplans (une dalle) et une boule, délimité par une sphère. Pour référence on note ici l'aire minimale dans un espace AdS dans le cas d'une géométrie de dalle de largeur ℓ :

$$\mathcal{A}^{\rm sl}_{\rm AdS} = \frac{2R^d}{d-1}\left(\frac{L}{a}\right)^{d-1} - \frac{2^d\pi^{d/2}R^d}{d-1}\left(\frac{\Gamma\left(\frac{1+d}{2d}\right)}{\Gamma\left(\frac{1}{2d}\right)}\right)^d\left(\frac{L}{\ell}\right)^{d-1} , \tag{C.87}$$

et d'une boule de rayon ℓ :

$$\mathcal{A}^{\rm sp}_{\rm AdS} = \frac{2\pi^{d/2}R^d}{\Gamma(d/2)}\left[p_1\left(\frac{\ell}{a}\right)^{d-1} + p_3\left(\frac{\ell}{a}\right)^{d-3} + ... + p_{d-2}\left(\frac{\ell}{a}\right)^2 + p_o \log\frac{\ell}{a}\right]. \qquad (C.88)$$

Les valeurs des coefficients pour $d = 3$, sont $p_1 = 1/2$ et $p_o = -1/2$.

Corrections dues aux degrés de liberté de saveur massifs

Nous présentons ici le calcul de l'entropie d'intrication dans le cas de la géométrie de dalle et de la sphère dans les fond produit par des champs de saveur massifs. La géométrie de fond est donnée par (C.30).

Geometrie de dalle

Dans le cas de la géométrie de dalle nous choisissions une paramètrisation de la surface de la forme $\rho = \rho(x)$; la fonctionnelle de l'aire de la surface est donnée par :

$$S_{\rm area} = \frac{L^2}{R^5}\int_{-\ell/2}^{\ell/2} dx\; h^{1/2}FS^4\sqrt{1+hF^2\rho'^2}\,. \qquad (C.89)$$

La fonction $\rho(x)$ obéit l'équation :

$$\frac{d\rho}{dx} = -\frac{\sqrt{hF^2S^8 - \tilde{h}\tilde{F}^2\tilde{S}^8}}{\sqrt{h\tilde{h}F\tilde{F}\tilde{S}^4}} \qquad (C.90)$$

où nous notons $\tilde{\rho}$ l'extrémité de la surface et le fonctions marquées avec un tilde sont les valeurs des fonction à $\tilde{\rho}$. En utilisant cette relation l'aire minimal peut être évaluer comme suit :

$$\mathcal{A} = 2\frac{L^2}{R^5}\int_{\tilde{\rho}}^{\infty} d\rho\;\frac{h^{3/2}F^3S^8}{\sqrt{hF^2S^8 - \tilde{h}\tilde{F}^2\tilde{S}^8}}\,, \qquad (C.91)$$

$$\ell = 2\int_{\tilde{\rho}}^{\infty} d\rho\;\frac{\sqrt{h\tilde{h}F\tilde{F}\tilde{S}^4}}{\sqrt{hF^2S^8 - \tilde{h}\tilde{F}^2\tilde{S}^8}}\,. \qquad (C.92)$$

A notre convenance nous utiliserons la coordonnée z donnée en fonction de ρ dans les équations (C.35) et (C.36) dans les régions $\rho > \rho_q$ et $\rho < \rho_q$ respectivement. Afin de régulariser l'intégral de l'aire nous introduisons un régulateur UV à $z = a$. Nous calculons la largeur de la dalle et l'aire minimal au premier ordre en ϵ_q :

$$\ell = \ell_0 + \epsilon_q\ell_1 \qquad (C.93)$$

$$\mathcal{A} = \mathcal{A}_0 + \epsilon_q\mathcal{A}_1\,. \qquad (C.94)$$

I. Extrémité de la surface à $\tilde{\rho} > \rho_q$ $(\tilde{z} < z_q)$: Nous commençons par le cas ou l'extrémité de la surface est située dans la région où sont étendues les D7 branes. Nous pouvons exprimer la largeur ℓ et l'aire $\mathcal{A}$ en fonction du paramètre $b = \tilde{z}/z_q$:

$$\frac{\ell}{z_q} = \gamma_1 b + \epsilon_q \left[\frac{1}{720} b^5 (48\gamma_3 - 15\gamma_2) + \frac{1}{720} b^3 (40\gamma_2 - 160) + \frac{b\gamma_2}{8} \right], \tag{C.95}$$

$$\frac{z_q^2}{L^2R^3}\mathcal{A}^I = -\frac{\gamma_1}{2b^2} + \epsilon_q \left\{ \frac{1}{144} b^2 (24\gamma_3 - 3\gamma_2) - \frac{\gamma_2}{8b^2} + \frac{1}{144}[8\gamma_2 - 16(6\log(bz_q) + 1 + \log 4)] \right\} \tag{C.96}$$

où

$$\gamma_1 = \frac{2\sqrt{\pi}\Gamma(2/3)}{\Gamma(1/6)}; \qquad \gamma_2 = \frac{\Gamma(2/3)\Gamma(5/6)}{\sqrt{\pi}}; \qquad \gamma_3 = \frac{\Gamma(1/3)\Gamma(7/6)}{\sqrt{\pi}}.$$

La partie divergente de l'aire est donnée par :

$$\mathcal{A}_{\text{div}} = \frac{L^2R^3}{a^2} - \epsilon_q L^2R^3 \left[\frac{1}{4a^2} - \frac{2}{3z_q^2}\log a \right]. \tag{C.97}$$

Le résultat d'ordre zéro est en accord avec le cas AdS dans l'eq. (C.87) pour $d = 3$ comme prévu. Afin d'exprimer l'aire en fonction de ℓ nous pouvons inverser perturbativement la relation (C.95) et nous avons :

$$\mathcal{A}^I(\ell) = -\frac{\gamma_1^3 L^2R^3}{2\ell^2} + \epsilon_q L^2R^3 \left[\frac{6\log\gamma_2 + 1 - 2\log 2}{9z_q^2} + \frac{\gamma_3\ell^2}{10\gamma_2^2 z_q^4} - \frac{\gamma_2^3}{4\ell^2} \right]. \tag{C.98}$$

II. Extrémité de la surface à $\tilde{\rho} < \rho_q$ $(\tilde{z} > z_q)$: Pour évaluer les intégrales de la largeur et l'aire dans ce cas nous devons les diviser en deux parties : une partie du bord jusqu'à ρ_q et une autre de ρ_q jusqu'à l'extrémité. Les résultats trouvés sont :

$$\begin{aligned}\frac{\ell}{z_q} = \gamma_1 b + \frac{\epsilon_q}{2160}\Bigg[&-\frac{30}{b^7} - \frac{15\gamma_1}{b^3} - 480b^3 + \frac{30}{b^3} + 48b^5 B\left(\frac{1}{b^6}; \frac{1}{3}, \frac{1}{2}\right) \\ &+ \frac{6\sqrt{b^2-1}\left(56b^4 + 71b^2 + 31\right)}{\sqrt{b^4+b^2+1}} + \left(-15b^5 + 40b^3 + \frac{5}{b^3} + 90b\right) B\left(\frac{1}{b^6}; \frac{2}{3}, \frac{1}{2}\right)\Bigg],\end{aligned} \tag{C.99}$$

$$\begin{aligned}\frac{z_q^2}{L^2R^3}\mathcal{A}^{II} = -\frac{\gamma_1}{2b^2} + \frac{\epsilon_q}{432b^7}\Bigg[&24b^9 B\left(\frac{1}{b^6}; \frac{1}{3}, \frac{1}{2}\right) + \left(-3b^8 + 8b^6 - 18b^4 + 1\right) bB\left(\frac{1}{b^6}; \frac{2}{3}, \frac{1}{2}\right) \\ &-48b^7 - 48b^7\left(-2\cosh^{-1}\left(b^3\right) + 6\log(bz_q) + 2\log 2\right) \\ &-\frac{6\sqrt{b^2-1}\left(12b^8 + 9b^6 + 17b^4 - b^2 - 1\right)}{\sqrt{b^4+b^2+1}} - 3b\gamma_1\Bigg].\end{aligned} \tag{C.100}$$

où $B(z;a,b)$ est la fonction Beta incomplète. Nous pouvons maintenant inverser la relation $\ell(b)$ pour exprimer l'aire en fonction de ℓ :

$$\frac{z_q^2}{L^2R^3}\mathcal{A}^{II}(\ell) = -\frac{\gamma_1}{2b^2} + \frac{\epsilon_q}{720}\Bigg[-\frac{90\gamma_1^6 z_q^6}{\ell^6}\,{}_2F_1\left(\frac{1}{2},\frac{2}{3};\frac{5}{3};\frac{z_q^6\gamma_1^6}{\ell^6}\right) + 72\,{}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{4}{3};\frac{z_q^6\gamma_1^6}{\ell^6}\right)$$
$$+\frac{10\gamma_1^{10}z_q^{10}}{\ell^{10}} - \frac{10\gamma_1^6 z_q^6}{\ell^6} + 160\cosh^{-1}\left(\frac{\ell^3}{\gamma_1^3 z_q^3}\right) + \frac{10\gamma_1^4 z_q^4\sqrt{\ell^6-\gamma_1^6 z_q^6}}{\ell^7} + 80\log\left(\frac{\gamma_1^6}{4\ell^6}\right)$$
$$-\frac{232\gamma_1^2 z_q^2\sqrt{\ell^6-\gamma_1^6 z_q^6}}{\ell\left(\gamma_1^4 z_q^4+\gamma_1^2\ell^2 z_q^2+\ell^4\right)} - \frac{232\ell\sqrt{\ell^6-\gamma_1^6 z_q^6}}{\gamma_1^4 z_q^4+\gamma_1^2\ell^2 z_q^2+\ell^4} - \frac{242\gamma_1^4 z_q^4\sqrt{\ell^6-\gamma_1^6 z_q^6}}{\ell^3\left(\gamma_1^4 z_q^4+\gamma_1^2\ell^2 z_q^2+\ell^4\right)} + 80\Bigg]\,. \tag{C.101}$$

Nous sommes intéressés au comportement de la théorie pour des grandes valeur de ℓ afin d'en déduire les corrections massives sur l'entropie d'intrication. En utilisant la méthode de [Hertzberg and Wilczek, 2011] nous pouvons extraire les contributions qui sont indépendantes du régulateur ultraviolet. En faisant l'identification $\xi^{-1} = m = 1/z_q$ la partie indépendante du régulateur peut être identifier en calculant :

$$S_\xi = (-\xi^{-2})^2\frac{\partial S}{\partial(\xi^{-2})^2}\,. \tag{C.102}$$

Nous pouvons vérifier que cette quantité est en effet fini dans l'ultraviolet et est une fonction de $\Lambda^2 \equiv \ell^2/z_q^2$. Le développement limité pour Λ grand , $\ell \gg \xi$, révèle les termes suivants :

$$S_\xi \approx \epsilon_q\frac{L^2R^3}{G_N}\frac{1}{3\xi^2} = \frac{1}{2\pi^2}\lambda_q N_f N_c\frac{A_\Sigma}{48\pi\xi^2}\,. \tag{C.103}$$

A titre de comparaisons notons qu'un entropie de la forme

$$S = -\frac{A_\Sigma}{24\pi}\frac{1}{\xi^2}\log\xi - \frac{4b_1}{\xi} + 2b_0\log\xi \tag{C.104}$$

donne lieu à un S_ξ de la forme suivante :

$$S_\xi = \frac{\mathcal{A}_\Sigma}{48\pi\xi^2} + \frac{b_1}{\xi} + b_0\,. \tag{C.105}$$

Par conséquent le terme que nous avons trouvé corresponds au terme $m^2\log m$ dans (C.86). Le terme constant et le terme $1/\xi$ n'apparaissent pas en comparaison avec (C.105).

Geometrie de boule

Nous étudions ici le cas dans lequel la surface d'intrication est une sphère de rayon ℓ. La parametrisation de la surface est donne en termes de fonction $r(z)$ où r est la coordonnée radiale sur le bord, $r^2 = \sum x_i^2$. Nous faisons un changement de variable en $r^2 = y(z)^2 - z^2$. Alors la solution pour un espace AdS est simplement donnée par $y = \text{const} = \ell$. La fonctionnelle de l'aire est dans ce cas :

$$S_{\text{area}} = \frac{4\pi}{R^5}\int dz\; h^{1/2}S^4F(y^2-z^2)^{1/2}\sqrt{(yy'-z)^2 + hF^2\rho'(z)^2(y^2-z^2)}\,. \tag{C.106}$$

Les équations de mouvement correspondantes peuvent être résolues au premier ordre en ϵ_q. Nous notons la solution $y = y_0 + \epsilon_q y_1$. Nous devons aussi dans ce cas faire la distinction entre les surfaces dont l'extrémité se situe à $\rho > \rho_q$ et celles qui sont étendues plus dans le "bulk".

I. Extrémité de la surface à $\tilde{\rho} > \rho_q$ $(\tilde{z} < z_q)$: Dans ce cas la solution est :

$$\begin{aligned} y_1^I(z) &= w(z) + \frac{C_1^I\left(z^2 - 2\ell^2\right)}{\sqrt{\ell^2 - z^2}} + C_2^I\,, \qquad \text{(C.107)} \\ w(z) &\equiv \frac{4\ell^3 \log z}{3z_q^2} + \frac{\left(4\ell^4 - 2\ell^2 z^2\right)\log\left(\frac{\sqrt{\ell^2-z^2}+\ell}{z}\right)}{3z_q^2\sqrt{\ell^2 - z^2}} - \frac{z^6}{80\ell z_q^4} - \frac{\ell z^4}{30 z_q^4} + \frac{z^4}{18\ell z_q^2} + \frac{\ell z^2}{3z_q^2} - \frac{z^2}{8\ell}\,. \end{aligned}$$

Pour que la solution soit régulière en $z = \ell$ nous devons imposer $C_1^I = 0$. Les autres constantes sont fixées par la condition au bord $y_1^I(z = 0) = 0$:

$$C_2^I = -\frac{4\ell^3}{3z_q^2}\log(2\ell)\,.$$

L'intégrale de l'aire peut être calculée analytiquement et nous avons :

$$\mathcal{A}^I = 4\pi R^3\left[\frac{1}{2}\frac{\ell^2}{a^2} - \frac{1}{2}\log(\frac{\ell}{a}) + \epsilon_q\left(\frac{\ell^2}{8a^2} + \frac{4\ell^2 + 3z_q^2}{12z_q^2}\log(\frac{a}{2\ell}) + \frac{\ell^4}{30z_q^4} + \frac{7\ell^2}{18z_q^2} - \frac{1}{16}\right)\right]\,. \qquad \text{(C.108)}$$

II. Extrémité de la surface à $\tilde{\rho} < \rho_q$ $(\tilde{z} > z_q)$: Dans ce cas la parametrisation est exprimée par deux fonctions :

$$y_1^{II} = \begin{cases} w(z) + C_1^{II}\dfrac{z^2 - 2\ell^2}{\sqrt{\ell^2 - z^2}} + C_2^{II}\,, & 0 < z < z_q \\[2ex] \dfrac{z_q^4}{144\ell z^2} + D_1\left(\sqrt{z^2 - \ell^2} - \dfrac{\ell^2}{\sqrt{z^2 - \ell^2}}\right) + D_2\,, & z_q < z < \tilde{z} \end{cases} \qquad \text{(C.109)}$$

Comme dans le cas I, la condition de régularité de la surface en $z = \ell$ fixe $D_1 = 0$ et la condition au bord $y_1^{II}(z = 0) = 0$ fixe

$$C_1^{II} = \frac{C_2^{II}}{2\ell} + \frac{2\ell^2\log(2\ell)}{3z_q^2}\,.$$

Les constantes supplémentaires sont fixées en imposant la condition de continuité de la solution et de sa dérivée première au point $z = z_q$. Cette condition implique :

$$C_2^{II} = \frac{4}{45z_q^4}\left(\sqrt{\ell^2 - z_q^2}\,(14\ell^2 z_q^2 - 2z_q^4 + 3\ell^4) + 15\ell^3 z_q^2 \log\left(\frac{2\ell z_q}{\sqrt{\ell^2 - z_q^2} + \ell}\right)\right),$$

$$D_2 = \frac{1}{90z_q^2\left(\ell\left(\sqrt{\ell^2 - z_q^2} + \ell\right) - z_q^2\right)}\Bigg[-83\ell^2 z_q^2\sqrt{\ell^2 - z_q^2} + 16z_q^4\sqrt{\ell^2 - z_q^2}$$
$$+ 12\ell^4(10\log 2 - 1)\sqrt{\ell^2 - z_q^2} - 120\ell^3\left(\ell\left(\sqrt{\ell^2 - z_q^2} + \ell\right) - z_q^2\right)\log\left(\sqrt{1 - \frac{z_q^2}{\ell^2}} + 1\right)$$
$$- 29\ell z_q^4 + 12\ell^5(1 + 10\log 2) + \ell^3(17 - 120\log 2)z_q^2\Bigg]. \tag{C.110}$$

L'intégration est encore une fois faite analytiquement avec le résultat suivant :

$$\frac{\mathcal{A}^{II}}{4\pi R^3} = \frac{1}{2}\frac{\ell^2}{a^2} - \frac{1}{2}\log\frac{\ell}{a} + \epsilon_q\Bigg\{\frac{\ell^2}{8a^2} - \frac{1}{720\ell z_q^4}\Bigg[4\ell^2 z_q^2\Bigg(60\ell\log\frac{a\left(\sqrt{\ell^2 - z_q^2} + \ell\right)}{2\ell z_q}$$
$$- 83\sqrt{\ell^2 - z_q^2} + 70\ell\Bigg) + z_q^4\Bigg(180\ell\log\frac{a\left(\sqrt{\ell^2 - z_q^2} + \ell\right)}{2\ell z_q} - 45\ell - 64\sqrt{\ell^2 - z_q^2}\Bigg)$$
$$+ 24\ell^4\left(\ell - \sqrt{\ell^2 - z_q^2}\right)\Bigg]\Bigg\}. \tag{C.111}$$

Les termes divergents dans la dernière équation sont le mêmes que dans (C.108), ce qui est attendu puisque les divergences proviennent uniquement de la région $z \sim 0$. Nous déduisons les termes universels dépendants de la masse en utilisant (C.102). A la limite de grand Λ, $\ell \gg \xi$, S_ξ prend la forme suivante :

$$S_\xi \approx 4\pi R^3 \epsilon_q \left(\frac{1}{6}\Lambda^2 - \frac{1}{8}\right) = \frac{\lambda_q}{2\pi^2} N_f N_c \left(\frac{\mathcal{A}_\Sigma}{48\pi\xi^2} - \frac{1}{16}\right). \tag{C.112}$$

En comparant avec (C.105), nous remarquons que nous avons trouvé le terme dominant et le terme constant, tandis que le terme proportionnel à $1/\xi$ est absent. Dans les deux géométries nous remarquons que le coefficient du terme $m^2 \log m$ dépend du couplage de 't Hooft, ce qui est en contradiction avec la conjecture de [Hertzberg, 2013].

C.5 Conclusions et perspectives

Dans cette thèse nous nous sommes concentrés sur deux applications de la correspondance AdS/CFT : les propriétés de transport des excitations fermioniques dans de théories des champs fortement couplées et les corrections de masse à l'entropie d'intrication.

Dans la première partie nous avons présenté quelques éléments de base da le correspondance.

Dans la deuxième partie nous avons commencé en présentant les notions de base sur l'hydrodynamique holographique. Ensuite nous avons présenté l'évaluation de la composante transverse de la fonction de corrélation des supercharges dans $\mathcal{N} = 4$ SYM à température non-nulle et densité nulle et non-nulle. Par la suite nous en avons déduit la constante de diffusion du phonino en utilisant une formule de Kubo. Dans le cas de charge nulle nous avons confirmé le résultat auparavant obtenu en étudiant l'atténuation du son supersymétrique. Pour le calcul à densité non-nulle nous avons utilisé la solution de trou noir STU, une solution de la supergravité à champ de jauge $\mathcal{N} = 2$, $D = 5$. Notre calcul a révélé que la constante de diffusion dépend du potentiel chimique et par conséquent n'est pas universelle. L'absence d'universalité n'est peut être pas une surprise puisque la composante transverse du gravitino est couplée à des scalaires et des champs de jauge qui dépendent de la direction radiale. De plus nous avons remarqué que la combinaison TD_s tend vers une valeur finie dans le cas du trou noir extreme. Ca serait intéressant de voir si cela est vrai pour tous les trous noirs extrêmes et vérifier si sa valeur pourrait être prédite par une variante du mécanisme d'attracteur. En outre, calculer la constante de diffusion dans de géométries non-relativistes pourrait s'avérer utile en vue d'applications dans des systèmes de matière condensée avec supersymétrie émergente.

Dans la troisième partie nous avons traité le sujet de l'entropie d'intrication. Apres une introduction sur le sujet nous avons présenté la calcul de l'entropie d'intrication dans un modèle holographique contenant des D3 et des D7 branes avec "backreaction". Nos résultats ne confirment pas la conjecture que le terme logarithmique de masse $m^2 \log m$ ne change pas de forme à couplage fort mis à part le fait que la masse est remplacée par la masse renormalisée puisque le coefficient dépend dans notre cas du couplage de 't Hooft. Un désaccord a également été trouvé dans d'autres calculs holographiques. Ca serait tout de même intéressant de poursuivre le calcul perturbatif de [Hertzberg, 2013] afin de voir si le désaccord persiste à l'ordre plus élevé. Nous pourrions aussi envisager de calculer l'entropie d'intrication dans d'autres théories massives émanant de la théorie des cordes (comme par exemple le modèle Klebanov-Strassler) ou encore dans le modèle de D3/D7 branes à température et densité non-nulles.

Enfin, il existe d'autres termes dépendants de la masse avec des coefficients qui dépendent de la géométrie de la surface d'intrication. Dans le cas que nous avons étudié nous avons trouvé un coefficient en rapport avec la courbure de la surface d'intrication, la terme constant de l'équation (C.105) qui est non-null pour la sphère. Ca serait intéressant de calculer l'entropie d'intrication pour des géométries différentes, comme par exemple pour la géométrie du guide à ondes, afin d'identifier tous les coefficients.

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