Master Thesis

Eternal Inflation

Search for a Global Description

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Abstract

We present a review of the inflation scenario and explain how quantum fluctuations generated during inflation are responsible for the observed structure in the cosmic microwave background radiation. We show how in the slow-roll approximation quantum fluctuations can lead to a globally never-ending phase of inflation. The primary area of investigation is related to the back-reaction of quantum fluctuations on the classical background geometry of the (eternally) inflating universe. To this extent we give results from the literature for free and interacting fields in a fixed De Sitter background and we give a gauge invariant measure of the effect of back-reaction in the slow-roll approximation by means of an effective energy momentum tensor. Next we introduce the stochastic formalism to inflation in which quantum fluctuations are treated effectively as classical noise. We investigate how this approach is related to a global description of eternal inflation. Finally we consider a thermodynamic approach based on geometric quantities which have a thermodynamic interpretation in (quasi-)De Sitter space. We demonstrate that the first law of thermodynamics is equivalent to a combination of the Einstein equations. Perturbations are included to account for back-reaction and we investigate the behaviour in the eternal inflation regime.

Contents

Popular Scientific Summary

The further objects such as stars and galaxies are away from us, the faster they move away from us. This relation is known as Hubble's law after Edwin Hubble who was the first to provide empirical evidence for this law. From this observation it becomes clear that the universe *expands*. In fact, the universe is expanding at an accelerating rate. This was recently discovered through measurements of distant supernovae by Perlmutter, Schmidt and Riess, for which they earned the 2011 Nobel prize in physics. One of the things you could learn from these observations is that the universe is not some static background in which the stars, galaxies and Earth move, but it is evolving itself. This means that at different times, the universe could have expanded at different rates.

It is interesting to study this evolution of expansion of the universe. What will be the fate of the universe? And how did the universe begin? These are driving questions for the field of cosmology. The underlying theory is Einstein's theory for general relativity, which relates the geometry of spacetime to the energy, or mass, that is present inside the universe. Using this theory we deduce physical laws which govern this dynamic behaviour of the universe. These laws allow us to probe the early universe and make predictions for late times.

How did the universe begin? If the universe expands, then it makes sense that it has been smaller at early times. This was already suggested before Hubble came up with his law by the Belgian priest and physicist Lemaître. He found a particular solution to the Einstein equations in which he could show that the universe expands and also must have had a beginning: the 'Big Bang'. The solution actually breaks down really at the beginning, which has to do with a singularity (basically a division by zero), but there is no obstacle to describe the universe right after the beginning.

One can image that if you would put all the matter that exist in the universe into a small box, then early conditions must have been quite extreme. The energy density and temperature were enormous. Substantial supporting evidence for this picture was the discovery of the cosmic microwave background (CMB), detected by Penzias and Wilson in 1965. It was already predicted that a big bang with such extreme initial conditions must have left a trace in the present universe. Indeed, the CMB can be interpreted as the afterglow of the big bang, cooled down by the expansion of the universe. Right now, this radiation corresponds to a temperature of about 2.7 Kelvin, see figure 1.

Later, it was realized that there are some problems with this hot big bang picture. For example, this picture from the afterglow of the big bang looks almost exactly the same in all directions. This afterglow is basically a snapshot of the universe when it was quite young, approximately 380 000 years after the big bang. In this time light could have travelled only

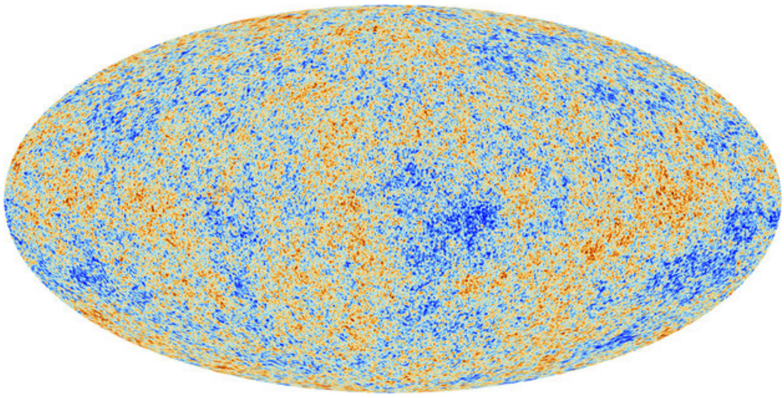

Figure 1 – The Cosmic Microwave Background; picture retrieved from PLANCK 2013 data. The average temperature of this thermal radiation is 2.7 K with relative fluctuations of $\mathcal{O}(10^{-5})$

over a distance of 380 000 light years. Therefore, regions of approximately this size could have interacted with each other, and reach a certain equilibrium temperature. Today, the universe is approximately 14 billion years old. Therefore we observe this afterglow ~ 13.5 billion years after it has been emitted. To observe the afterglow today means that we observe the radiation that was approximately ~ 13.5 billion light-years away from us at that time. This distance is much larger than the 380 000 light years that light has travelled since the big bang to the time of the afterglow. Therefore the present picture of the afterglow corresponds to a lot of regions which have not had the possibility to interact which each other and reach an equilibrium temperature. Still, we observe that the CMB is really uniform. Unless the Big Bang was very gentle and smooth we cannot explain why the CMB is so uniform.

This and other problems can be overcome by introducing an inflationary period. In this period the universe expands extremely fast, much faster then today's accelerated expansion. In this way a small region that has a uniform temperature becomes very large in a short amount of time. It can even become larger than the distance light travels between the time of the afterglow and today, such that it explains why the CMB is uniform.

The matter which we understand from the standard model cannot cause inflation. In fact we need a form of energy that behaves like a positive cosmological constant. A cosmological constant was earlier introduced by Einstein himself to his equations to allow for a static universe; something he has called his greatest mistake but nevertheless turned out to be quite important. Not only should this form of energy behave like a cosmological constant, there should also be a mechanism to end inflation. A simple method to obtain the right type of energy is to introduce a field: the *inflaton*. The physical nature of this field is unclear, and it is not likely that this field is detectable by particle accelerators such as the LHC. However this field provides us with just the right mechanism to have inflation at the early universe and end it at a later stage.

A period of inflation has interesting consequences. Especially quantum fluctuations during inflation are very important. For one thing, quantum fluctuations during inflation are responsible for the small variations present in the CMB. First think of a universe which is not inflating such as today. Classically, the vacuum is a not-so-interesting region of space with zero energy; However according to quantum mechanics (the uncertainty principle) this cannot be true. Constantly, there are pairs of particles and antiparticles created which annihilate immediately; they only exist for a really short period of time. During inflation the vacuum is also not empty, but now space-time expands really fast. Particles and their anti-particles are ripped apart, such that they do never annihilate and become real particles. This is of course not the whole story, but this is basically what causes the minor variations detected in the CMB, see figure 1.

Quantum fluctuations can even be more important; they can prevent the inflaton field from globally ending inflation. In some regions just as the universe we observe, inflation has ended; we call these regions *pocket universes* which have *thermalized*. However, the larger part of the universe is still inflating and inflation becomes *eternal*. This dramatically changes the global picture of the universe. This picture, in which there are pocket universes surrounded by an inflating space, is known as the *multiverse*.

One of the most interesting opportunities of this scenario has to do with string theory. According to string theory, the set of values for the physical constants that we are used to, such as particle masses and the strength of fundamental forces, is not unique. There are in fact a lot of possible sets for values, approximately 10^{1000} of them. This is a problem in string theory, because there is no reason why nature has picked out this specific set of physical constants that we observe for our universe. It is likely that only a small fraction of this large number of sets offers the possibility for the formation of stars, galaxies, planets and intelligent life. We have to praise ourselves lucky that we exist in this picture.

Eternal inflation, and the multiverse that generates, provides a more natural explanation than pure luck for our existence. Each of the pocket universes that are created can have a different set of physical constants. If there infinitely many pocket universes each of the possible set of constants will exist somewhere, such that a set which is capable of sustaining life will also exist. We live in one of these universes which have constants of nature which favour life, otherwise we wouldn't have lived and ask ourselves this question. This is known as the *anthropic principle*.

Although the idea of eternal inflation might sound appealing, the theory suffers from important shortcomings. One of the biggest problems is we do not have a consistent global description of eternal inflation. In this thesis we investigate this problem.

Eternal inflation would let itself naturally describe by means of a distribution function of the inflaton field. If we would have such a distribution we could calculate things such as the fraction of the universe which exist of thermalized regions like ours. We could even try to obtain a distribution function for the physical constants. We will look into this formalism, known as the *stochastic formalism*.

Another formalism which we will investigate is based on thermodynamics. Expanding space-times can be associated with interesting thermodynamic properties which have to do with the appearance of an *event horizon*. Similar relations exist for black holes. In particular we will consider one inflating pocket universe in an eternally inflating universe and see if we can apply a thermodynamic equation to this system which relates to Einstein equations.

Acknowledgements

Both the universe and my understanding of it have grown quasi-exponentially since I started working on this thesis. The former perhaps somewhat less quasi than the latter. In any case it has been fun to think about the subject, and I would like to thank my supervisor Jan Pieter van der Schaar for his guidance.

Notation and Conventions

Metrics are of the form $(-,+,+,+)$
Natural units are used in which:

$$\hbar = c = 1 \tag{1}$$

In this report the reduced Planck mass is used:

$$m_{\mathrm{p}} = (8\pi G)^{-1/2} = \frac{M_P}{\sqrt{8\pi}} \tag{2}$$

To obtain actual values one can reintroduce the constants using the following table

Table 1 – Constants

$8\pi G$	1.68×10^{-9}	$\mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$
c	3.00×10^{8}	$\mathrm{m\ s}^{-1}$
$\hbar$	1.06×10^{-34}	$\mathrm{m}^2\ \mathrm{kg\ s}^{-1}$

The following Fourier convention is used:

$$f(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \tilde{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \tilde{f}_{\mathbf{k}} = \int \mathrm{d}^3\mathbf{x} f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{3}$$

CHAPTER 1

Introduction

1.1 Brief History of the Universe

Although much insight has been gained in the physics of the early universe, the question regarding the beginning of the universe remains unresolved. Suppose the universe started out from a singularity at virtually infinite energy density, i.e. the big bang, and the energy density has decreased monotonically due to the expansion of the universe. Then the energy density at early times must have been tremendously high. At these energy densities, which are above the Planck scale, gravitational interactions become important and require a proper theory of quantum gravity. This remains one of the biggest problems in physics although theories like string theory have made a promising effort. Also at somewhat lower scales such as grand unification scale interesting physics is assumed which only exists in theory. In fact, the events that are hypothesized to have occurred during the first 10^{-10} seconds are based on physics that has not been experimentally verified.

The event which is the topic of this research also requires new physics. For reasons which will be explained later, the universe is believed the have gone through a period of *inflation* at its earliest stages. In this period the universe expanded exponentially fast. A primary candidate to have caused the acceleration is the *inflaton* field. According to this scenario, the energy content of the universe was dominated by the inflaton field, which in effect is similar to dark-energy or a cosmological constant. The universe expanded exponentially fast, thereby smearing out any pre-inflationary inhomogeneities. At the same time, quantum fluctuations of the inflaton field create new inhomogeneities which are responsible for the large scale structure of the present universe, and can be seen in the cosmic microwave background (CMB) radiation. In this period of inflation the *scale factor* of the universe increased approximately by a factor e^{60}. Since there are no real particles that oscillate the temperature is close to zero.

After this period of inflation, the inflaton field decays and transfers its energy into the particles we know today from the standard model. This period is called *reheating*. All the particles move relativistically, therefore this period is radiation dominated. This period is similar to what formerly was known as the big bang.

Observationally, an important event was the decoupling of photons and charged particles at around 380 000 years after the big bang. The photons that have not scattered since this event form the CMB, which was detected in 1964 by *Penzias* and *Wilson*. A summary of the events is given in table 1.1.

Table 1.1 – History of the inflationary universe. The events and numbers are taken from Baumann [2009]

	Time	
Planck epoch?	$< 10^{-43}$	s
String scale?	$\gtrsim 10^{-43}$	s
Grand Unification?	$\sim 10^{-36}$	s
Inflation?	$\gtrsim 10^{-34}$	s
Matter-Radiation equality	10^{4}	yrs
Last scattering	10^{5}	yrs
Dark-Energy domination	$\sim 10^{9}$	yrs
Present	14×10^{9}	yrs

Eternal Inflation

It has been shown that for a wide range of inflationary models, an interesting phenomenon emerges. When taking into account quantum effects during inflation, the inflaton field never decays completely; there will always exist regions which are still inflating, leading to the concept of *eternal inflation*. The majority of the universe exists of regions which are expanding at ever faster rates, at densities which can exceed the Planck density. This is sometimes referred to as *spacetime foam*. Furthermore, regions such as our own observable universe are created in which inflation has ended. The set of these regions is known as the *multiverse*.

There are interesting consequences of this picture. For example, the so-called pocket universes in which inflation has ended may have different vacua and therefore different properties of particles and even different fundamental constants of nature. This fits well with the landscape of vacua that iss predicted from string theory; eternal inflation provides a way to populate these vacua. Furthermore this theory allows for a anthropic explanation of the properties of our universe; we happen to live in a universe which is suitable for intelligent life.

On the other hand there are serious conceptual issues with eternal inflation. First of all, there is no consistent global description of eternal inflation. Quantum effects, which are responsible for eternal inflation, spoil the homogeneity of the universe which is one of the basic assumptions in ordinary cosmology. For this reason another framework to describe the universe (or multiverse) is required. Secondly, the quantum fluctuations carry energy and for this reason interact with spacetime through the Einstein equations thereby altering the background solution. This is known as *back-reaction* and problems associated with this will be discussed in chapter 6.

Two approaches which deal with these problems are investigated in this research. *Stochastic inflation* treats the quantum fluctuations as classical but random fluctuations. This is discussed in chapter 7. The second approach is based on the first law of thermodynamics, chapter 8. There seems to be a deep link between gravity and thermodynamics and this might shed new light on the aforementioned problems.

Before eternal inflation is investigated we review inflationary cosmology. In chapter 2 the basics of cosmology are considered and we elaborate on the theory and observations that have lead to the paradigm of inflation. Then in chapter 3 we see how inflation is realized by the introduction of the inflaton field. Chapter 4 considers quantum fluctuations during inflation and the relation of these quantum fluctuations with the CMB. Then eternal inflation is introduced and the problems discussed in this research are stated.

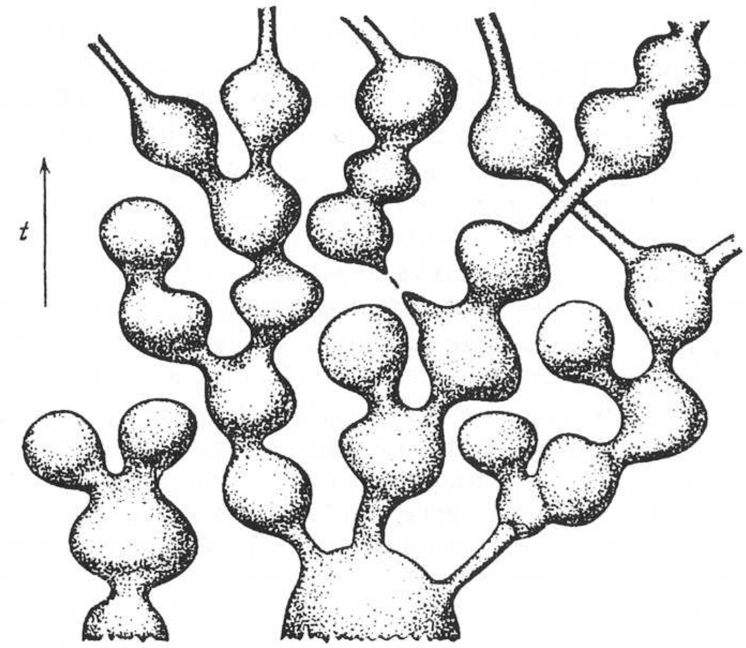

Figure 1.1 – Impression of eternal inflation [Linde 2005]

CHAPTER 2

Inflationary Cosmology

In this chapter the theory and observations which are responsible for the widespread acceptance of an inflationary period in the early universe will be briefly reviewed. This chapter mainly follows the introduction on cosmology given in Carroll [2004] and the TASI lectures on inflation by Baumann [2009].

2.1 Einstein Equations

Cosmology is concerned with describing the universe at large scales. For example an important topic is the expansion of the universe. For a large part the language that is used to discuss such topics is that of general relativity, which describes the relation between spacetime and energy. Let us start by writing down the action from which the Einstein equations can be derived.

$$S = \int \mathrm{d}^4x \, \sqrt{-g} \frac{R}{16\pi G} + S_M \tag{2.1}$$

The first term of this action is the Hilbert action. By extremizing this action one obtains the Einstein equations in vacuum. The second term is added to take matter into account. The Einstein equations read:

$$G^\mu_\nu = R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = 8\pi G T^\mu_\nu \tag{2.2}$$

On the left hand side is the Einstein tensor G^μ_ν, which is composed out of the the Ricci tensor R^μ_ν and the Ricci scalar R. All the terms on the left hand side are composed out of the metric and its first and second derivatives, thus describing the geometry of spacetime.

The energy-momentum tensor T^μ_ν appears on the right hand side. For convenience one index is raised. The energy-momentum tensor is defined:

$$T^\mu_\nu = g^{\mu\rho} T_{\rho\nu} \equiv g^{\mu\rho} \left(-\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\rho\nu}} \right) \tag{2.3}$$

The cosmological constant is included in the definition of the energy-momentum tensor. It can be made explicit by writing

$$T^\mu_\nu = T^\mu_{\nu,\phi} - \frac{\Lambda}{8\pi G}\delta^\mu_\nu \tag{2.4}$$

where $T^{\mu}_{\nu,\phi}$ denotes the EMT for the field content. In case there are no fields the EMT is proportional to the cosmological constant which can be interpreted as the vacuum energy.

2.2 FRW Spacetime

The Einstein equations are a set of differential equations governing the evolution of the metric. One can impose restrictions on the general form of the metric, for example by requiring to metric to contain certain isometries. The general form of the metric describing our universe is restricted by spatial homogeneity and isotropy on large scales. These restrictions are well motivated by observational evidence. From observations of the Cosmic Microwave Background (CMB) we find that typical temperature fluctuations $\frac{\delta T}{T}$ are of the order $\mathcal{O}(10^{-5})$ [Smoot et al. 1992], which establishes the isotropy. It is also an indication of the homogeneity at the time of last scattering. Density fluctuations would result in temperature fluctuations, therefore the universe at that time was homogeneous. Moreover, large sky surveys measure the position of galaxies and clusters as a function of red-shift. The result is that on scales $> 100\,\mathrm{Mpc}$, the universe is homogeneous. It is assumed that this homogeneity holds for the entire visible universe.

The Friedman Robertson Walker (FRW) metric takes these spatial symmetries into account:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\Omega^2 \right) \tag{2.5}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$ is the metric on the unit 2-sphere. The spatial part of the metric depends on an overall cosmic scale factor $a(t)$. We choose the convention in which the scale factor has the dimension of distance. The radial coordinate r is a *comoving* coordinate; to compute physical distances, one needs to multiply this with the scale factor at the time of interest. In the convention we have chosen this coordinate is unit-less. The dynamics of the scale factor is given by the Einstein equations (2.2). k is the curvature parameter of spatial surfaces. It is normalized by a redefinition of the scale factor such that the values are $+1$, 0 or -1 for closed, flat and open universes.

Using this metric we can calculate the Christoffel symbols, the Riemann tensor and finally the Ricci-tensor and Ricci-scalar which appear in the Einstein equations. The non-zero components of the Ricci-tensor are:

$$R^0_0 = 3\frac{\ddot{a}}{a} \tag{2.6}$$

$$R^i_i = \frac{\ddot{a}}{a} + 2H^2 + \frac{2k}{a^2} \tag{2.7}$$

(no summation over i in the second equation). $H = \frac{\dot{a}}{a}$ is the *Hubble constant.* The Ricci-scalar is:

$$R = 6\left[\frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2}\right] \tag{2.8}$$

The energy-momentum tensor of the universe is modelled as a perfect fluid. This means that the only parameters are the energy density ρ and the isotropic pressure p.

$$T^{\mu}_{\nu} = (\rho + p)u^{\mu}u_{\nu} + p\delta^{\mu}_{\nu} \tag{2.9}$$

In the rest frame of the fluid this becomes

$$T^{\mu}_{\nu} = \mathrm{diag}(-\rho, p, p, p) \tag{2.10}$$

The $(0,0)$ Einstein equation becomes:

$$-3H^2 - 3\frac{k}{a^2} = -8\pi G\rho \Longrightarrow H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{2.11}$$

This equation is known as the *Friedmann equation*. The (ii)-Einstein equation gives

$$-2\frac{\ddot{a}}{a} - H^2 - \frac{\kappa}{a^2} = 8\pi Gp \Longrightarrow \frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3p) \tag{2.12}$$

This last equation is known as the *Raychaudhuri* equation. These two equations (together sometimes denoted as the Friedmann equations) are essential in cosmology. They determine the evolution of the scale factor, which tells us if the universe is expanding or contracting, and at what rate. In the current model of the universe, ΛCDM, the universe is always expanding, although the rate varies. We will discuss this in the next section.

2.3 Filling the Universe with Matter

The expansion of the universe depends on the energy content of the universe. We will show that different types of energy sources depend differently on the scale factor, therefore resulting in a different rate of expansion. Furthermore, we will find that there have been periods which were dominated by a certain energy source, due to its different dependence on the scale factor.

The cosmologically relevant energy sources can be categorized into matter (M), radiation (R) and vacuum energy (Λ). The distinction is based on relation between a components energy density and the pressure it exerts in its rest frame. This relation is given by the *equation of state*.

$$p_i = \omega_i \rho_i \tag{2.13}$$

where i runs over the different components. The *equation of state parameter* (ω) for the different components is found as follows.

Matter

Firstly, matter in this context is defined as being collision-less and non-relativistic. Therefore it exerts no pressure and the equations of state parameter is zero. In the ΛCDM-model, the largest contribution to this term comes from cold dark matter. The baryons contribute a less significant part.

Radiation

For radiation we can determine the equation of state parameter by comparing the cosmological energy momentum tensor for a perfect fluid and the energy momentum tensor for electromagnetism. The latter is given by

$$T^{\mu\nu} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma} \tag{2.14}$$

The trace of this equation is zero. On the other hand, the trace of the energy momentum tensor for cosmological perfect fluid energy-momentum tensor equals $T^{\mu}_{\mu} = -\rho + 3p$. This results into the equation of state parameter for radiation: $\omega_R = \frac{1}{3}$. At present, CMB photons make up this energy density. The contribution from photons from the stars is negligible. When the universe was more dense, also neutrino's were considered radiation, and at the early universe most of the matter was ultra-relativistic and therefore equivalent to radiation.

Dark Energy

Finally there is dark energy. This form of energy is characterized by a negative equation of state. The microscopic nature of such a form of energy is still unclear. A possible candidate is the cosmological constant or, equivalently, vacuum energy. A form of energy with a negative equation of state is assumed to play the principal role in two different stages in the evolution of the universe: during inflation and at present. The effect of a dark energy-dominated universe is that the universe will expand at an accelerating rate. During inflation, this was per definition the case. For the present epoch, the acceleration was quite recently observationally confirmed; The 2011 Nobel prize was awarded to Saul Perlmutter, Brian P. Schmidt and Adam G. Riess for this observation.

During inflation the negative equation of state parameter is could be generated by the *inflaton* field. The details concerning this field are given in the next chapter. Usually it assumed that during inflation there is period when the equation of state parameter is close to $\omega = -1$. In that case the inflaton behaves as a cosmological constant. Spacetime can be described by a De Sitter universe. In this case the scale factor grows exponentially with time.

Time Evolution

Using the Friedmann equations in combination with the equation of state we find how the energy density for each component evolves as a function of the scale factor. Inserting the time-derivative of the first Friedmann equation

$$\dot{H} = \frac{4\pi G}{3H}\dot{\rho} + \frac{k}{a^2} \tag{2.15}$$

into the second Friedman equation, we obtain the *continuity equation.*

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{2.16}$$

This expression can also be found by setting the divergence of the time component of the energy-momentum tensor to zero.

$$0 = \nabla_\mu T^\mu{}_0 = \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu\lambda} T^\lambda{}_0 - \Gamma^\lambda_{\mu 0} T^\mu{}_\lambda = -\dot{\rho} - 3H(\rho + p) \tag{2.17}$$

Dividing the continuity equation by ρ, and using the equation of state we obtain:

$$\begin{aligned} \frac{\dot{\rho}}{\rho} &= -3(1+\omega)\frac{\dot{a}}{a} \\ \partial_t \log\rho &= -3(1+\omega)\partial_t \log a \\ \rho &\propto a^{-3(1+\omega)} \end{aligned} \tag{2.18}$$

Table 2.1 – Properties of single component universes

	$\rho(a)$	$a(t)$
De Sitter inflation / Vacuum energy	a^0	e^{Ht}
Radiation	a^{-4}	$t^{1/2}$
Matter	a^{-3}	$t^{2/3}$

Using this relation we construct table (2.1). The time dependence of the scale factor is found by solving the Friedmann in the single component cases.

We see that for De Sitter inflation, the universe is expanding exponentially. It is common to express the evolution of the universe in *e-folds* in this case. In one *Hubble time*, defined as $\Delta t = H^{-1}$, the scale factor increases by a factor e. The e-folds simply count the number of Hubble times, or $N = \log a$.

The ΛCDM model

It is common to write the Friedmann equation in terms of density parameters. First we write the curvature term which appears in the Friedmann equation as a density term $\rho_k = -3m_\mathrm{p}^2 \frac{k}{a^2}$, such that Friedmann reads:

$$H^2 = \frac{1}{3m_\mathrm{p}^2}\left(\rho + \rho_k\right) = \frac{1}{3m_\mathrm{p}^2}\rho_\mathrm{crit} \tag{2.19}$$

When the universe is flat: $\rho_k = 0$, such that $\rho = \rho_\mathrm{crit}$. We now define the density parameter by

$$\Omega_i = \frac{\rho_i}{\rho_\mathrm{crit}} \tag{2.20}$$

The Friedmann equation expressed in density parameters reads:

$$1 = \Omega + \Omega_k \tag{2.21}$$

$$\Omega \equiv \Omega_r + \Omega_m + \Omega_\Lambda \tag{2.22}$$

where Ω_r, Ω_m and Ω_Λ correspond to the density parameters of radiation matter and vacuum energy. To find the dependence on the scale factor we need equation (2.18) and notice that:

$$\rho_\mathrm{crit} = \rho_{\mathrm{crit},0}\frac{H^2}{H_0^2} \tag{2.23}$$

The scale factor dependence is given as follows:

$$\Omega_i = \Omega_{i,0}\left(\frac{H}{H_0}\right)^{-2}\left(\frac{a}{a_0}\right)^{-3(1+\omega_i)} \tag{2.24}$$

Since $\rho_k \propto a^{-2}$ it has effectively an equation of state parameter $\omega_k = -1/3$; sometimes referred to as dark fluid. The dependence is given by:

$$\Omega_k = \Omega_{k,0}\left(\frac{aH}{a_0 H_0}\right)^{-2} \tag{2.25}$$

This can also easily found using equation (2.20):

$$\Omega_k = -\frac{k}{(aH)^2} \tag{2.26}$$

The parameters of the ΛCDM model are approximately

$$\Omega_{r,0} = 8 \times 10^{-5} \qquad \Omega_{m,0} = 0.27 \qquad \Omega_{\Lambda,0} \approx 0.72 \tag{2.27}$$

A subscript 0 means in this case the value at present epoch. Using a combination of astronomical observations we find the curvature parameter today [Hinshaw et al. 2012]:

$$\Omega_{k,0} = -0.0027^{+0.0039}_{-0.0038} \tag{2.28}$$

Hot Big Bang Picture

With these parameters at hand and the Friedmann equation, one can obtain an overall history of the universe in standard (non-inflationary) cosmology. In this picture the universe starts out

Table 2.2 – Important epochs in standard cosmology. The scale factor is normalized to 1 at present.

a_i	$=$	0	Initial singularity
a_{rm}	$=$	3.0×10^{-4}	Radiation-Matter equality
$a_{\mathrm{m}\Lambda}$	$=$	0.72	Matter-Vacuum energy equality
a_0	$=$	1	Present

radiation dominated and becomes matter dominated after some time. At present, there is a large contribution coming from dark energy. Note that the current density parameters do not imply an early stage of inflation.

In section 2.5 we will discuss some problems with this picture. Will argue that a separate mechanism for inflation has to be added.

We would like remark that we can set the flat space approximation throughout in this picture. At present, the curvature parameter does not contribute significantly to the total energy density. The fraction of the curvature density parameter over the density parameter as a function of scale factor is:

$$\frac{\Omega_k}{\Omega} = \frac{\Omega_{k,0}}{\Omega_0}\left(\frac{a}{a_0}\right)^{1+3\omega} \Rightarrow \Omega \approx 1 \tag{2.29}$$

This means the contribution of the curvature term has only been smaller at earlier times so that we can safely use the approximation.

2.4 Causal Structure

In this paragraph we discuss some basics regarding the causal structure of the universe. This will particularly useful for the next section, in which we will see that the causal structure of standard cosmology leads to unnatural initial conditions, which can be solved by a period of inflation.

Conformal time

The causal structure of the universe is determined by the propagation of light. For clarity, we consider only radial propagation.

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\mathrm{d}r^2 = 0 \tag{2.30}$$

The trajectory of light in comoving coordinates is then given by:

$$\mathrm{d}r = \pm\frac{1}{a}\mathrm{d}t \tag{2.31}$$

It can be convenient to eliminate the factor a^{-1} in this last equation. For example, the scale factor will behave differently for the different epochs of the universe. We do not have to take this into account if the scale factor is embedded into the definition of time. We can do this by introducing *conformal time*:

$$\mathrm{d}\tau \equiv \frac{1}{a}\mathrm{d}t \tag{2.32}$$

The metric becomes $\mathrm{d}s^2 = a(\tau)^2\left(-\mathrm{d}\tau^2 + \mathrm{d}r^2\right)$ which is conformally related to Minkowski spacetime. To find the amount of conformal time that has past since some initial time we integrate:

$$\Delta\tau(t) = \Delta r = \int_{t_i}^{t} \frac{\mathrm{d}t'}{a(t')} \tag{2.33}$$

The conformal time is equal to the comoving distance light has travelled.

Horizons

We will distinguish between three types of horizons: the event horizon, the apparent horizon or Hubble radius, and the particle horizon.

First let us take a closer look at the Hubble parameter. The value today is [Hinshaw et al. 2012]

$$H_0 = (69.33 \pm 0.88)\,\mathrm{km\ s^{-1}\ Mpc^{-1}} \tag{2.34}$$

This says that an object at a fixed comoving distance, but at a physical distance 1 Mpc, will move away from us with $\sim 70\,\mathrm{km\ s^{-1}}$. We can calculate the distance at which an object would move away from us with the speed of light using Hubble's law. This gives the present *Hubble radius*:

$$a_0 r = H_0^{-1} \approx 14\,\mathrm{Gyr} \tag{2.35}$$

If the Hubble parameter does not change, any object that is further away than the Hubble radius cannot send signals which would arrive at the observer, since light that is send in the observers direction is moving away from him/her. In this case of a constant Hubble parameter, the Hubble radius would correspond to an *event horizon*.

An *event horizon* defines the set of points from which signals send at a given time will never be observed by an observer in the future.

However, in physical cosmology, the Hubble constant is changing as a function of time. In order to evaluate the event horizon, one needs to know the entire evolution of the Hubble parameter. For the radiation and matter dominated universe the Hubble radius increases, so all points can reach an observer eventually which means there is no event horizon. On the other hand, if the universe becomes vacuum energy dominated in the future there will be a finite event horizon. For an inflationary De Sitter phase which lasts forever there is similarly a finite event horizon.

A quantity that will be used frequently in the context of inflation is the *comoving Hubble radius*. It can be found by generalizing equation (2.35):

$$r_H = \frac{1}{aH} \tag{2.36}$$

This horizon is defined locally. When an event occurs outside this horizon, it appears to be out of causal contact with the observer. However, it might have been in causal contact in the past or will be in the future. Therefore, this horizon is also known as the comoving *apparent horizon*. In standard cosmology, at least for the radiation and matter epochs, the comoving Hubble radius increases as a function of time, while during inflation it decreases.

The last horizon we discuss is the *particle* horizon. This horizon is defined as the maximum distance light can have travelled since the big bang. This is given by equation (2.33) with the initial singularity as lower limit.

$$r_{\text{part. hor}} = \int_{t_i}^{t} \frac{\mathrm{d}t'}{a} \tag{2.37}$$

The particle horizon is equal to conformal time. $r_{\text{part. hor}} = \tau - \tau_i$.

It is possible to express the particle horizon as an integral over the comoving Hubble radius

$$\tau = \int_{t_i}^{t} \frac{\mathrm{d}t'}{a} = \int_{a_i}^{a} \frac{\mathrm{d}a'}{\dot{a}'a'} = \int_{a_i}^{a} \frac{\mathrm{d}a'}{H(a')^2} = \int_{a_i}^{a} \mathrm{d}\log a' \frac{1}{a'H} \tag{2.38}$$

The behaviour of this quantity is at the heart of both problems.

Table 2.3 – Comoving Hubble radius and particle horizon for single component universes

	r_H	τ
De Sitter inflation	a^{-1}	$\tau_i - a^{-1}$
Radiation	a	a
Matter	$a^{1/2}$	$2a^{1/2}$

2.5 Big Bang Puzzles

In the standard big bang picture summarized by table (2.2), the universe starts out radiation dominated. However, there are many puzzles associated with this picture, see for example Linde [Linde 2005], for a list of issues. Two major issues with this picture are the flatness and horizon puzzles. These problems are the reason to amend the conventional big bang picture with a preliminary period of inflation.

These two problems can both be explained by noticing that in the hot big bang picture, the comoving Hubble radius monotonically increases as a function of the scale factor. We already argued we can use the flat space approximation. The scale factor dependence of the comoving Hubble radius is:

$$(aH)^{-1} = (a_0H_0)^{-1}\left(\frac{a}{a_0}\right)^{1/2(1+3\omega)} \tag{2.39}$$

We see that for a radiation and matter dominated universe, this quantity evolves as a and $a^{1/2}$ respectively. We will show how this translates into the flatness and horizon problem. In the following we will disregard the contribution from vacuum energy which became important only recently.

Flatness problem

The flatness problem is of a fine tuning nature. It is a priori not to be expected that the initial density parameter (dominated by radiation) is approximately equal to the critical density. That is, there is no reason to expect the universe started out being very close to spatially flat.

The curvature parameter is given by equation (2.26). We can rewrite this equation as:

$$|\Omega_k| = \frac{|k|}{(aH)^2} = |\Omega_{k,0}|\left(\frac{a_0H_0}{aH}\right)^2 \tag{2.40}$$

We see that the curvature evolves as the square of the comoving Hubble radius. Since the comoving horizon increases as as a function of the scale factor, so does any deviation from flatness. This implicates that initially, the universe must have been extremely flat in order to comply with the current value of the curvature density parameter.

For example at the Planck-time, the degree of flatness has to be fine-tuned to an extremely small value. Any deviation from this small value would imply a dramatic change of the universe today. As somewhat larger initial curvature parameter would rapidly lead to a big crunch or big rip. To quantify to what degree the initial curvature parameter has to be fine-tuned let us calculate the curvature parameter at the Planck time, t_{pl}.

The curvature parameter increases $\propto a^2$, and a during the radiation and matter dominated epoch. The curvature today is related to the curvature at the Planck time according to:

$$\frac{\Omega_{k,0}}{\Omega_{k,\mathrm{pl}}} = \left(\frac{a_0}{a_{\mathrm{rm}}}\right)\left(\frac{a_{\mathrm{rm}}}{a_{\mathrm{pl}}}\right)^2 = (1+z_{\mathrm{rm}})\left(\frac{T_{\mathrm{pl}}g_{*,\mathrm{pl}}^{1/3}}{T_{\mathrm{rm}}g_{*,\mathrm{rm}}^{1/3}}\right)^2 \tag{2.41}$$

$$\approx 10^{59} \tag{2.42}$$

Where we have used $1 + z_{rm} = 3200$, $T_{\mathrm{rm}} = 0.75$ eV, $T_{\mathrm{pl}} = 10^{18}$ GeV. g_* denotes the effective number of degrees of freedom contributing to entropy which is $g_{*,\mathrm{rm}} = 3.9$ for radiation matter equality. When we assume a minimal super-symmetric model $g_{*,\mathrm{pl}} \sim 200$. [Komatsu et al. 2009].

This requires the flatness of the universe at t_{pl} to be $|\Omega_{k,\mathrm{pl}}| \sim 10^{-60}$ or smaller. Any deviation of this value would have had huge consequences for our present universe.

Horizon problem

The horizon problem is concerned with the observation that the CMB is quite homogeneous. The average temperature fluctuations observed in the CMB are [Smoot et al. 1992]:

$$\sqrt{\langle\left(\frac{\delta T}{T}\right)^2\rangle} \sim 10^{-5} \tag{2.43}$$

We will show that in the hot big bang picture, the last scattering surface exist of many causally disconnected regions. Therefore these regions cannot have established a thermal equilibrium, and make the homogeneity seem very peculiar.

The comoving distance to an event is given by equation (2.33). In table (2.3) we see that for radiation and matter, the comoving distance is proportional to the comoving Hubble radius. Since last scattering occurred when the universe was relatively young and the scale factor has increased quite a lot since that time, the comoving distance to this surface is relatively large; almost the size of the observable universe.

On the other hand, the comoving particle horizon at last scattering, which is also given by (2.33), but with $t_i = 0$, is much smaller. This means that when we look at the CMB, this surface encompasses many regions of the particle horizon at last scattering, leaving us with the question why we observe such an isotropic CMB.

To quantify the problem let us calculate how many independent particle sized regions fit within the current last scattering surface. This will also be useful to give a lower bound on the period of inflation.

We assume the universe has been radiation dominated in the period from last scattering until now. The present comoving distance to the last scattering surface becomes:

$$\Delta r = 2H_0^{-1}a_0^{-1}\left[1 - \left(\frac{a_{ls}}{a_0}\right)^{1/2}\right] \approx 2H_0^{-1}a_0^{-1} \tag{2.44}$$

Compare this to the particle horizon at last scattering. We assume the universe was completely radiation dominated before radiation matter equality, and completely matter dominated until last scattering:

$$\begin{aligned}\tau_{ls} &= H_{rm}^{-1}a_{rm}^{-1} + 2H_0^{-1}a_0^{-1}\left[\left(\frac{a_{ls}}{a_0}\right)^{1/2} - \left(\frac{a_{rm}}{a_0}\right)^{1/2}\right] \\ &= a_0^{-1}H_0^{-1}\left(\frac{a_{rm}}{a_0}\right)^{1/2}\left[2\left(\frac{a_{ls}}{a_{rm}}\right)^{1/2} - 1\right] = a_0^{-1}H_0^{-1}\frac{1}{\sqrt{1+z_{rm}}}\left[2\left(\frac{T_{rm}}{T_{ls}}\right)^{1/2} - 1\right] \\ &\approx 0.06H_0^{-1}a_0^{-1}\end{aligned} \tag{2.45}$$

We can estimate how many particle horizon sized volumes fit within the volume enclosed by the last scattering surface. From the previous results, we find $\left(\frac{2}{0.06}\right)^3 \sim 10^5$.

2.6 Inflation as a Solution

Comoving Hubble Radius

Before we move on to the inflation solution, let us make a remark about the flat space approximation. As was explained in the section on the flatness problem, it would be more natural if the initial conditions for the universe would allow for any curvature instead of an extremely fine-tuned flat universe. When we allow for any initial curvature we cannot at first sight use the approximation as made in equation (2.29) for the total energy density. However, it has been shown that for any curvature an inflationary phase will commence eventually [Damianos 2011]. Once exponential expansion is established, the curvature parameter becomes small within a few e-folds. So onwards from this point we can use the flat space approximation.

In the problems of the big bang we used that the comoving Hubble radius was always increasing. However, we will find that a decreasing comoving Hubble radius could solve both of the problems. From equation (2.39) we see that this is the case for $\omega < -\frac{1}{3}$. As was discussed in section 2.3 we can think of inflation as a period in a De Sitter universe such that $\omega = -1$. In this way, the expansion is exponential, and the Hubble horizon decreases exponentially:

$$(aH)^{-1} = \sqrt{\frac{3}{V_0}} a^{-1} \propto e^{-H_I t} \tag{2.46}$$

Flatness Problem

Since the comoving Hubble radius is decreasing during inflation, we can see from equation (2.40) that the curvature parameter is driven to flatness. The flatness problem is solved if the period of inflation lasts long enough. This gives a lower bound on the number of e-folds which we will now calculate. Will will assume the curvature parameter is of order one at the beginning of inflation $|\Omega_k(t_{\text{begin}})| \sim \mathcal{O}(1)$.

During inflation, the comoving Hubble radius is decreasing. The curvature evolves as the square of the Hubble radius and decreases $\propto a^{-2}$.

$$\frac{\Omega_{k,0}}{\Omega_{k,\text{begin}}} = \left(\frac{a_0}{a_{\text{rm}}}\right)\left(\frac{a_{\text{rm}}}{a_{\text{end}}}\right)^2\left(\frac{a_{\text{end}}}{a_{\text{begin}}}\right)^{-2} \tag{2.47}$$

$$= (1+z_{\text{rm}})\left(\frac{T_{\text{end}} g_{*,\text{end}}^{1/3}}{T_{\text{rm}} g_{*,\text{rm}}^{1/3}}\right)^2 e^{-2N} \tag{2.48}$$

If we estimate the temperature at the end of inflation to be $T_{\text{end}} \approx 1$ TeV, $g_{*,\text{end}} \approx g_{*,\text{pl}}$, $|\Omega_{k,begin}| = 1$, $|\Omega_{k,0}| = 0.01$ and using the same values as for the case without inflation we obtain:

$$N_{\text{min}} = \log\left(\frac{T_{\text{end}} g_{*,\text{end}}^{1/3}}{T_{\text{rm}} g_{*,\text{rm}}^{1/3}}\right) + \frac{1}{2}\log(1+z_{\text{rm}}) + \frac{1}{2}\log\frac{\Omega_{k,\text{begin}}}{\Omega_{k,0}} \tag{2.49}$$

$$= 36 \tag{2.50}$$

Horizon Problem

A period of inflation does not change the comoving distance to the last scattering surface. This remains $\Delta r = 2(a_0 H_0)^{-1}$. However, a decreasing comoving Horizon changes the particle horizon.

Since comoving Hubble radius increases during inflation, it can be arbitrary large initially for a long enough period of inflation. The particle horizon is a logarithmic integral over the Hubble radius, and gets it main contribution from the initial Hubble radius. The horizon problem is solved if the entire last scattering surface has been in causal contact, or $\Delta r < \tau_{begin}$, where τ_{begin} corresponds to the particle horizon at the beginning of inflation. It is approximately equal to the maximum comoving Hubble radius.

We will now determine the lower bound on the number of e-folds that is imposed by the solution to the horizon problem.

$$\tau_{\text{begin}} = (H_I a_{\text{end}})^{-1} \frac{a_{\text{end}}}{a_{\text{begin}}} = (H_I a_0)^{-1} \frac{a_{\text{end}}}{a_0} e^N > 2(a_0 H_0)^{-1} \tag{2.51}$$

$$e^N > 2 \frac{H_I}{H_0} \frac{a_{\text{end}}}{a_0} = 2 \frac{H_I}{T_{\text{end}} g_{*,\text{end}}^{1/3}} \frac{T_0}{H_0} \tag{2.52}$$

$$\Rightarrow N > \frac{1}{2} \log \frac{T_0}{H_0} - \log \frac{T_{\text{end}} g_{*,\text{end}}^{1/3}}{H_I} + \log 2 \tag{2.53}$$

$$\gtrsim 70 \tag{2.54}$$

In this approximation, the H_I and T_{end} should not be different up to many orders. So if inflation lasted for approximately 70 e-folds, we can explain the current isotropy of the CMB. It ensures our local Hubble patch is approximately homogeneous. If inflation lasted longer, a larger region will be nearly homogeneous, although outside this region the universe could be inhomogeneous.

These estimates of the minimal number of required e-folds are quite sketchy, however they are introduced to convince the reader that the mechanism of inflation works really well in order to solve the big bang problems.

CHAPTER 3

Inflation

A solution to the problems of the big bang scenario mentioned in the last chapter is an early stage of accelerated expansion, called inflation. This idea was pioneered by Alexei Starobinsky, Alan Guth and others at the end of the seventies. In this chapter we show how to implement the mechanism of inflation by means of the introduction of the classical *inflaton* field. Quantum effects will be discussed in the next chapter. The main reference for this chapter is the set of lecture notes from Baumann [2009].

3.1 The Classical Inflaton Field

In general relativity the curvature of spacetime is related to the energy and momenta of the system under consideration. In cosmology the system under consideration is the universe and the energies and momenta in principle belong to a set of quantum fields which should describe all the particles that exist in nature. At present, the field content as we understand it from the standard model is unable to explain the evolution of the universe or more specifically a period of inflation. To illustrate the discrepancy, recall that the energy density of particles we understand from the standard model only make up $\sim 4\,\%$ of the present energy density of the universe. For this reason, either the laws of cosmology are incomplete or we need to find this missing energy content (or a combination of both). The inflaton field is one of such fields which is introduced to explain inflation without the necessity of changing Einstein gravity.

This approach has been very effective so far. It is possible to come up with many models that give rise to an early period of inflation and provide a dynamical way to stop this process and let the universe proceed with its evolution known from standard cosmology. Furthermore, a particular nice property of this approach is that it can explain the origin of CMB density fluctuations and moreover predicts the power spectrum of the fluctuations. This will be discussed in the next chapter.

In the simplest model a classical homogeneous scalar field $\phi = \phi(t)$ is introduced into the action, the *inflaton* field. The properties of the scalar potential determine how inflation evolves.

The classical action of a scalar field minimally coupled to gravity is:

$$S_\phi = \int \mathrm{d}^4x\,\sqrt{-g}\mathcal{L}_\phi \tag{3.1}$$

$$\mathcal{L}_\phi = -\frac{1}{2}(\nabla\phi)^2 - V(\phi) \tag{3.2}$$

The energy momentum tensor for the scalar field is (see e.g. Carroll [2004]).

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi + g_{\mu\nu}\mathcal{L}_\phi \tag{3.3}$$

Since we consider only the homogeneous case, $\phi = \phi(t)$ the inflaton can be considered as a perfect fluid with the following energy density and pressure:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{3.4}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{3.5}$$

The equation of state is

$$\omega = \frac{p}{\rho} = \frac{\dot{\phi}^2 - V(\phi)}{\dot{\phi}^2 + V(\phi)} \tag{3.6}$$

We can write down the Friedmann equation for this field. We can assume the universe is flat, $k = 0$; In the last chapter we have seen that inflation drives a generally curved universe to flatness. We will assume that the curvature becomes negligible within a few e-folds. The Friedmann equations become:

$$H^2 = \frac{1}{3m_\mathrm{p}^2}\left(V(\phi) + \frac{1}{2}\dot{\phi}^2\right) \tag{3.7}$$

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = \frac{1}{3m_\mathrm{p}^2}\left(V(\phi) - \dot{\phi}^2\right) \tag{3.8}$$

For later reference we write down the derivative of the Hubble parameter:

$$\dot{H} = -\frac{1}{2}(\rho + p) = -\frac{1}{2}\rho(1+\omega) = -\frac{1}{2m_\mathrm{p}^2}\dot{\phi}^2 \tag{3.9}$$

Equation of Motion

Besides the Einstein equations the fields satisfies the Klein-Gordon equation in a curved background. This equation is determined by the Euler-Lagrange equation[1]:

$$0 = \nabla_\mu \frac{\partial\mathcal{L}_\phi}{\partial(\nabla_\mu\phi)} - \frac{\partial\mathcal{L}_\phi}{\partial\phi} = -\Box\phi + V_{,\phi} = \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} \tag{3.10}$$

This equation of motion is that of an harmonic oscillator with damping factor $3H$. This factor does not appear in the flat spacetime Klein Gordon equation. For large values of H, the field will rapidly decay and stop oscillating. We will discuss important consequences of this behaviour in chapter 4.

[1] One does not have to impose the equation separately from the Einstein equations. The Bianchi identities of the Einstein tensor imply a covariantly conserved energy momentum tensor. Reparametrization invariance of the matter action then implies that the equations of motion hold. See [Mukhanov and Winitzki 2007] p62.

3.2 Slow-Roll Inflation

A period of an accelerating expansion ($\omega < -1/3$) can in principle solve the big bang problems. However in order to have a long enough period of acceleration it is most efficient to have a period of (quasi-)exponential expansion. This is realized when the field has the equation of state similar to the cosmological constant ($\omega = -1$). In the limiting case:

$$H_0^2 = \frac{1}{3m_\mathrm{p}^2} V_0 = \frac{\Lambda}{3} \tag{3.11}$$

Such a universe, where the energy content is effectively given by a positive cosmological constant is known as a *De Sitter* universe. This type of space-time has other interesting properties, some of which we will discuss in chapter 8. Since $H_0 = \frac{\dot{a}}{a}$ is constant this means $a(t) = a_0 e^{H_0 t}$ i.e. exponential expansion. Since the universe is quickly driven to flatness we use the flat-space ($k = 0$) FRW metric:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2\right) \tag{3.12}$$

To realize such an phase in a physical model we introduce a potential which includes a *slow-roll* regime. An example of such a potential is given in figure 3.1.

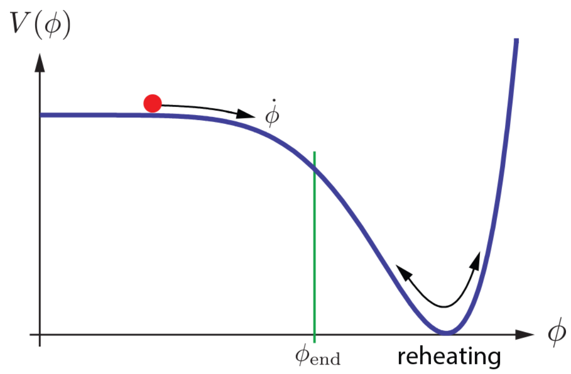

Figure 3.1 – Example of an inflaton potential with a slow-roll regime. As long as the position of the field is in the slow-rolling regime the expansion is (quasi-)exponential. Inflation ends when the universe is no longer in the slow-roll regime. It will roll down the potential and transfer its energy to other fields, reheating the universe in the process. Picture adapted from [Baumann 2009]

Two properties characterize the slow-roll approximation. Firstly, the equation-of-state parameter is close to the De Sitter limit $\omega = -1$. Secondly, it must retain that value for a long enough period to fulfil the requirements necessary to solve the big bang puzzles.

From equation (3.6) one can see that if the potential term dominates over the kinetic term the eos-parameter approximates the De Sitter limit. The second requirement can be realized when the kinetic term increases slow enough compared to the potential term. These states can be made more exact by introducing the *slow-roll parameters*:

$$\epsilon = \frac{3}{2}(1+\omega) = \frac{3}{2}\frac{\dot\phi^2}{V} = \frac{1}{2m_\mathrm{p}^2}\frac{\dot\phi^2}{H^2} = -\frac{\dot H}{H^2} \tag{3.13}$$

$$\eta = -\frac{\ddot\phi}{H\dot\phi} \tag{3.14}$$

Using equation (3.9) we have written ϵ in different forms which are common in the literature. Slow-roll inflation is defined by the approximation that these parameters are small[2] The smallness of ϵ is directly related to the equation of state parameter approximating the De Sitter limit $\omega \approx -1$, and the small η parameter guarantees that the kinetic contribution changes only slowly.

The slow-roll parameters can also be expressed in terms of the potential:

$$\epsilon_v = \frac{m_\mathrm{p}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2, \qquad \eta_v = m_\mathrm{p}^2\frac{V_{,\phi\phi}}{V} \tag{3.15}$$

During slow roll these definitions are related as follows:

$$\epsilon \approx \epsilon_v, \qquad \eta \approx \eta_v - \epsilon_v \tag{3.16}$$

The Friedmann equation becomes

$$H^2 = \frac{1}{3m_\mathrm{p}^2}\left(V + \frac{1}{2}\dot\phi^2\right) = \frac{V}{3m_\mathrm{p}^2}\left(1+\frac{\epsilon}{3}\right) \tag{3.17}$$

The equation of motion becomes:

$$3H\dot\phi(1-\frac{1}{3}\eta) = -V_{,\phi} \tag{3.18}$$

During slow-roll inflation terms all terms first order in slow-roll parameters are dropped:

$$H^2 = \frac{1}{3m_\mathrm{p}^2}V(\phi), \qquad 3H\dot\phi = -V_{,\phi} \tag{3.19}$$

The spacetime given by the above equations, or keeping leading order terms in the slow-roll parameters, is known as quasi De Sitter spacetime.

In quasi De Sitter we allow the Hubble constant to vary slowly. This can be realized by setting:

$$a(t) \propto e^{\int \mathrm{d}t\, H(t)} \tag{3.20}$$

It is often useful to work in conformal time. A quantity that will frequently be used is the conformal Hubble parameter.

$$\mathcal{H}(\tau) \equiv \frac{a'}{a} = aH \tag{3.21}$$

[2]In large field models such as chaotic inflation, a small parameter ϵ requires super-Planck values of the field ($\phi > m_P$). On the other hand, the potential energy can be kept below Planck scale when coupling constants are small, which agrees with experiment. See however the chapter on eternal inflation

The slow-roll parameter in terms of this quantity is

$$\epsilon = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} \tag{3.22}$$

If we assume the parameter ϵ is constant we can solve this differential equation: $\mathcal{H} = -\frac{1}{(1-\epsilon)\tau + c}$. We can set the constant to zero, by rescaling the conformal time.

$$\mathcal{H} = -\frac{1}{(1-\epsilon)\tau} \tag{3.23}$$

3.3 Models of Inflation

In the literature there exist many models of inflation. In this thesis we most of the considerations only assume there is a slow-roll regime regardless of the exact shape of the potential. However sometimes we will use a specific model. We will consider only single field models. Most of the models can be categorized according to whether the distance $\Delta\phi = \phi_{\text{cmb}} - \phi_{\text{end}}$ is large or small. ϕ_{cmb} is the value of the inflaton field at the time at which the quantum fluctuations were generated which later re-enter the horizon during last scattering, see next chapter. This distance corresponds to at approximately 60 e-folds of inflation.

In large field models, the field changes quite a lot $\Delta\phi > m_{\text{p}}$ during this time and the opposite for small field models.This division is useful because in contrast to small field models, gravitational waves are significantly produces in large field models, which would leave a detectable signature in the CMB.

An example of a small field model is given in figure 3.1. A more general form of the potential is

$$V(\phi) = V_0 + \frac{m^2}{2}\phi^2 + \frac{\nu}{3}\phi^3 + \frac{\lambda}{4}\phi^4 + \dots \tag{3.24}$$

As an example of a large field model we consider chaotic inflation, which has a potential of the form

$$V \sim c\phi^n \tag{3.25}$$

where c is a small coupling constant. An example of such a potential is given in figure 5.1. In these models the initial value of the field is somewhere high up the hill. To motivate this we can invoke Heisenberg's uncertainty principle: after a Planck-time since the initial singularity of the universe, the uncertainty in the energy(-density) is of the order $V \sim m_P^4$, such that an initial potential energy of $\lesssim m_P^4$ is not unlikely. In that case the potential (3.25) implies that the field can have super-Planckian values, since the coupling constant is small. Note that the energy density is still below the Planck energy.

It could be confusing that the potential does not look 'slow-roll' for high values of the field such that H^2 does not appear to be constant. However ϵ indeed goes to zero for large values of the field. This is most easily seen from the potential slow-roll parameter ϵ_v, eqn. (3.15) which for a potential of the form (3.25) goes as $\epsilon_v \propto \phi^{-2}$. Therefore the field indeed is slowly rolling for high values of the field.

New observational results by the PLANCK satellite have put restrictions on the models of inflation. The results are summarized in figure 3.2. For example it has been found that chaotic models where $V \sim \phi^n$, for $n > 3$ are not good fits to the observations.

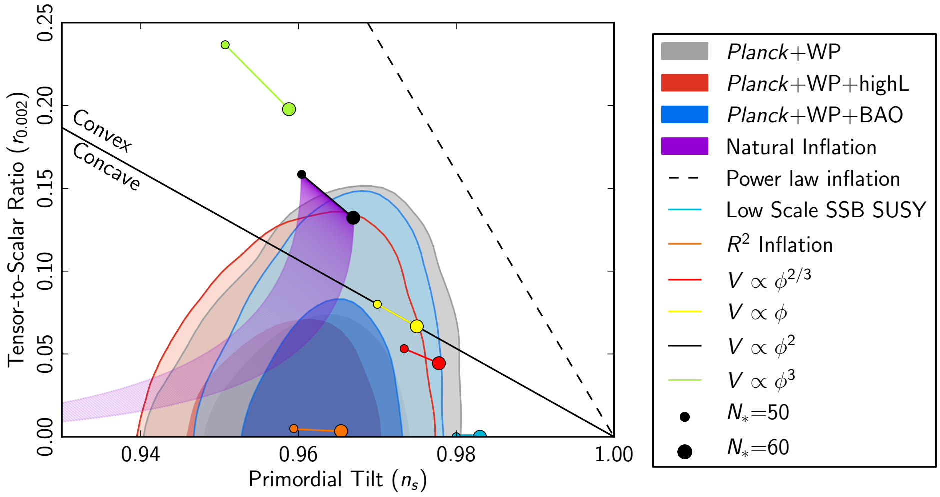

Figure 3.2 – Constraints on inflationary models according to the most recent PLANCK data [Ade et al. 2013].

CHAPTER 4

Including perturbations

One of the major achievements of inflation is that it provides an explanation for the observed density fluctuations in the CMB radiation. In this chapter we investigate this relation. Conceptually the CMB anisotropies are related to inflationary quantum fluctuations as follows.

1. We express quantum fluctuations which are created during inflation in a gauge invariant way through the curvature perturbation $\mathcal{R}$.
2. The amplitudes for modes that exceed the Hubble scale become classical and do not evolve in time.
3. When inflation ends[1] the curvature perturbations re-enter the horizon and appear as density fluctuations which are converted into the anisotropies of the CMB.

The goal of this chapter is to analyse these steps. The main articles that were used for this chapter are [Baumann 2009] and [Riotto 2002].

4.1 CMB Statistics

To relate the inflationary quantum fluctuations to the CMB observations we use statistics on the CMB data. An important statistical measure is the power spectrum, i.e. the amount of power per frequency interval. In the end we will be able to compare the observed power spectrum to the calculated power spectrum of inflationary fluctuations. If the CMB fluctuations are *Gaussian random* all statistical information is encoded the power spectrum. If not, then non-Gaussianities have to be taken into account. Current WMAP results indicate the non-Gaussianity is small such that we will disregard them for the present discussion.

In brief the CMB data is related to curvature perturbations as follows, see e.g. [Baumann 2009] for details. First the CMB is expanded in spherical harmonics. The multipole moments can be combined into an angular invariant power spectrum (as a function of frequency k). After

[1]Here one can make the following remark. In the transition period from inflation to radiation domination the universe goes through a phase of reheating which is not well understood. However, since the perturbations are super-horizon, local physics cannot have effected the perturbations. This is convenient because we can ignore the unclear process of reheating for our analysis.

applying a transfer function to account for sub-horizon evolution we can relate this spectrum to the power spectrum of curvature perturbations $\mathcal{R}$. The power spectrum $P_{\mathcal{R}}(k)$ is defined from:

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 P_{\mathcal{R}}(k) \delta(\mathbf{k} + \mathbf{k}') \tag{4.1}$$

The $\langle\rangle$ means ensemble average. We will later identify this ensemble average with the vacuum expectation value inflationary quantum fluctuations. A useful related quantity is the dimensionless power spectrum:

$$\Delta^2_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k) \tag{4.2}$$

From the Gaussian distribution function we can find for the two-point correlation function:

$$\langle \mathcal{R}(\mathbf{x}) \mathcal{R}(\mathbf{y}) \rangle = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} P_{\mathcal{R}}(k) = \int \mathrm{d}\ln k \, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \Delta^2_{\mathcal{R}} \tag{4.3}$$

Scale Invariance

The power spectrum for fluctuations is found to nearly *scale invariant*. Scale invariance is defined such that

$$\langle f(\lambda\mathbf{x}) f(\lambda\mathbf{y}) \rangle = \langle f(\mathbf{x}) f(\mathbf{y}) \rangle \tag{4.4}$$

From equation (4.3) we can deduce what it means for a power spectrum to be scale invariant.

$$\langle f(\lambda\mathbf{x}) f(\lambda\mathbf{y}) \rangle = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{i\lambda\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} P_f(k) \tag{4.5}$$

$$= \int \frac{\mathrm{d}^3\tilde{\mathbf{k}}}{(2\pi)^3} \frac{1}{\lambda^3} e^{i\tilde{\mathbf{k}}\cdot(\mathbf{x}-\mathbf{y})} P_f\left(\frac{\tilde{k}}{\lambda}\right) \tag{4.6}$$

Where we passed to coordinates $\tilde{k} = \lambda k$. For this to be equal to $\langle f(\mathbf{x}) f(\mathbf{y}) \rangle$ we need:

$$\frac{1}{\lambda^3} P_f\left(\frac{k}{\lambda}\right) = P_f(k) \qquad \Longrightarrow \qquad P_f \propto k^{-3}, \quad \text{or equivalently,} \quad \Delta^2_f \propto \text{constant} \tag{4.7}$$

WMAP data shows that at large scales (low multipoles) the dimensionless power spectrum is indeed close to being scale invariant. The departure of this scale invariance is given by the scalar spectrum index:

$$n_s - 1 = \frac{\mathrm{d}\ln \Delta^2_{\mathcal{R}}}{\mathrm{d}\ln k} \tag{4.8}$$

where $n_s = 1$ corresponds to scale invariance. The latest analysis of the WMAP data yields the value for the scalar spectrum index $n_s = 0.971 \pm 0.010$ [Hinshaw et al. 2012]. This nearly scale invariant spectrum was already predicted by Zel'Dovitch before the invention of inflation in order to explain the structure formation. However there was no mechanism to explain the scale invariance until the idea of inflation came about.

We will show later that inflation predicts this nearly scale invariant spectrum.

4.2 Linear Perturbation Theory

We know from observation that the typical amplitude of the CMB fluctuations is small ($\mathcal{O}(\sim 10^{-5})$). If the CMB fluctuations are to originate from inflationary fluctuations then these fluctuations are necessarily small as they leave the horizon. It will be shown later that fluctuations are small compared to the average value of the field even in early stages of inflation[2] Because the fluctuations are small linear perturbation theory approximates the full theory to high precision.

During inflation the only degrees of freedom are the metric and the inflaton field. We can decompose these degrees of freedom into a homogeneous part plus a small perturbation.

$$g_{\mu\nu}(t, \mathbf{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \mathbf{x}), \tag{4.9}$$

$$\phi(t, \mathbf{x}) = \bar{\phi}(t) + \delta\phi(t, \mathbf{x}) \tag{4.10}$$

The field and the metric are related through the Einstein equations. Since in general relativity we are free to choose a coordinate system we need to make sure the perturbations are physical and not a consequence of our gauge choice. First we will decompose the metric perturbations in a convenient way in order to single out the relevant degrees of freedom during inflation. Then we will show how to treat this in a gauge invariant way.

ADM formalism

The metric is a 4×4 symmetric object and hence consists of 10 components. Not all of these correspond to dynamic degrees of freedom. Reparametrization invariance of general relativity eliminates four of the components which are not degrees of freedom and in turn gives four constraint equations. An example of how reparametrization invariance leads to a constraint equation is given in appendix A. This separation into dynamic and constraint equations is most clearly seen in the Arnowitt-Deser-Misner (ADM) formalism[3].

We will make use of this formalism in section 4.3 to find the *Mukhanov* equation which will be useful when we quantize the perturbations. In this formalism the 4-geometry is determined using a variational principle which gives the path between 3-geometries at the endpoints. The four-dimensional metric will be decomposed in a spatial metric and *lapse* and *shift* functions which relate two infinitesimally separated 3-geometries.

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_k N^k & N_i \\ N_i & h_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^i/N^2 \\ N^i/N^2 & h^{ij} - N^i N^j/N^2 \end{pmatrix} \tag{4.11}$$

Here N and N_i represent the lapse and shift functions. In section 4.3 we write the Einstein-Hilbert action in a form in which N and N_i appear as *Lagrangian multipliers.* This allows us to split the Einstein equations into dynamical degrees of freedom and constraint equations as explained in appendix A.

[2] although the these fluctuations can be of major importance for the global dynamics of the universe, see the chapter on *eternal inflation.* Furthermore we will discuss the effect of the fluctuations on the background geometry in chapter 6.

[3] Originally the ADM formalism was developed in order to obtain a Hamiltonian version of general relativity. This formalism has interesting applications, for example in order to canonically quantize gravity. It is also important for quantum cosmology e.g. the Wheeler-De Witt equation (Schrödinger equation for the universe) can be obtained.

SVT decomposition

We will now decompose this metric into a background metric plus small perturbations. First we write the metric perturbations as spatial scalars, vectors and tensors:

$$g_{\mu\nu}^{ADM} \Rightarrow g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}, \qquad g_{\mu\nu}^{(0)} = \begin{pmatrix} -1 & 0 \\ 0 & a^2\delta_{ij} \end{pmatrix}, \quad \delta g_{\mu\nu} = \begin{pmatrix} -2\Phi & aB_i \\ aB_i & a^2 C_{ij} \end{pmatrix} \tag{4.12}$$

Note that Φ and B_i are related to the lapse and shift functions N and N^i. The spatial metric is given by δ_{ij}.

It will be convenient to further decompose this into *helicity* scalar, vector and tensor perturbations, known as the SVT decomposition. This will be a simplification because the different types of helicity spinors can be treated separately. The helicity depends on the transformation property of a rotation around the Fourier vector k in the helicity basis. We define helicity scalars, vectors and tensors by having helicity 0, ± 1 and ± 2. To avoid confusion we denote the spatial vector and tensors with a prefix '3-'. Because of translational and rotational symmetry of the homogeneous background it can be shown that the helicity scalars, vectors and tensors evolve independently, see Baumann [2009] for a proof.

The decomposition is as follows.

- A 3-scalar is also a helicity scalar such that Φ is a helicity scalar.
- According to Helmholtz' theorem a 3-vector can be decomposed into divergence of a scalar and a divergence-less vector:

$$B_i = B_{|i} + S_i, \qquad S_{i|i} = 0 \tag{4.13}$$

  The covariant derivatives denoted with a bar are with respect to the spatial metric δ_{ij}, and equal Roman indices are summed over. It can be shown that S_i is a helicity vector.

- The symmetric 3-tensor can be decomposed as follows.

$$C_{ij} = -2\Psi\delta_{ij} + 2E_{|ij} + F_{(i|j)} + H_{ij}, \qquad F_{i|i} = 0, \quad H_{ij|i} = H_{ii} = 0 \tag{4.14}$$

  Ψ and E are scalars, F_i is a vector and H_{ij} a tensor.

The metric perturbations are given by 10 parameters. There are four scalars: Φ, Ψ, B and E, four vector components: F_i and S_i and two tensor components: H_{ij}. We will not discuss the vector perturbations, which will not be created during inflation. The tensor degrees of freedom correspond to gravitational waves, which we will also not discuss. The lapse function corresponds to one helicity scalar component and the shift 3-vector decomposes into 2 helicity-vector components and a scalar. These components result in constraint equations and do not contribute to dynamical degrees of freedom. In total we end up with 2 scalar, 2 vector and 2 tensor degrees of freedom.

The most general form of the metric which includes scalar perturbations is:

$$\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + 2aB_{|i}\mathrm{d}x^i\mathrm{d}t + a^2\left[(1-2\Psi)\delta_{ij} + 2E_{|ij}\right]\mathrm{d}x^i\mathrm{d}x^j \tag{4.15}$$

Gauge Invariance

The scalar perturbations are not invariant under gauge transformations. Consider the following infinitesimal gauge transformations

$$t \to t + \alpha, \qquad x^i = x^i + \delta^{ij}\beta_{|j} \tag{4.16}$$

δ^{ij} is the inverse spatial metric. Note that the spatial metric δ_{ij} does not include the scale factor. For later usage we define $\partial^i f \equiv \delta^{ij}\partial_j$.

We find how the perturbations transform under these transformations by setting $\widetilde{\mathrm{d}s}^2 = \mathrm{d}s^2$ where we only keep terms up to first order in the infinitesimal parameters α and β, and the metric perturbations. This leads to

$$\Phi \to \Phi - \dot{\alpha} \tag{4.17}$$

$$\Psi \to \Psi + H\alpha \tag{4.18}$$

$$B \to B + a^{-1}\alpha - a\dot{\beta} \tag{4.19}$$

$$E \to E - \beta \tag{4.20}$$

Using that the field $\phi(t, \mathbf{x})$ transforms as a scalar we find the transformation rule for the inflaton perturbation:

$$\tilde{\phi}\,(\tilde{x}^\mu) = \phi(\tilde{x}^\mu) = \phi(x^\mu) \to \phi_0(\tilde{t}) + \tilde{\delta\phi} = \phi_0(t) + \dot{\phi}_0\alpha + \tilde{\delta\phi} = \phi_0(t) + \delta\phi \tag{4.21}$$

$$\delta\phi \to \tilde{\delta\phi} = \delta\phi - \dot{\phi}\alpha \tag{4.22}$$

By simple substitution one can check that the following quantity is invariant under transformation (4.16). This quantity is known as the curvature perturbation:

$$\mathcal{R} = \Psi + \frac{H}{\dot{\phi}}\delta\phi \tag{4.23}$$

Perturbed Einstein Equations

The general metric for scalar perturbations (4.15) consists of four scalar parameters of which there are only two degrees of freedom. Using a gauge transformation we can set 2 parameters to zero. A convenient gauge we will prove to be *Newtonian* gauge: $B = E = 0$.

$$g_{\mu\nu} = \begin{pmatrix} -(1+2\Phi) & 0 \\ 0 & a^2(1-2\Psi)\delta_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -(1+2\Phi) & 0 \\ 0 & a^{-2}(1-2\Psi)\delta^{ij} \end{pmatrix} \tag{4.24}$$

The Einstein tensor becomes

$$\delta G^0_0 = 6H\left(H\Phi + \dot{\Psi}\right) - 2a^{-2}\delta^{kl}\partial_k\partial_l\Psi \tag{4.25}$$

$$\delta G^0_i = -2\partial_i\left(H\Phi + \dot{\Psi}\right) \tag{4.26}$$

$$\delta G^i_0 = 2a^{-2}\partial^i\left(H\Phi + \dot{\Psi}\right) \tag{4.27}$$

$$\delta G^i_j = \delta^i_j\left(6H(H\Phi + \dot{\Psi}) + 4\dot{H}\Phi + 2H\dot{\Phi} + 2\ddot{\Psi} + a^{-2}\delta^{kl}\partial_k\partial_l(\Phi - \Psi)\right) \tag{4.28}$$

$$- a^{-2}\partial^i\partial_j(\Phi - \Psi) \tag{4.29}$$

From its definition we find the perturbed energy-momentum tensor.

$$\delta T_0^0 = \Phi\dot{\phi}_0^2 - \dot{\phi}_0\dot{\delta\phi} - V_{,\phi}\delta\phi \tag{4.30}$$

$$\delta T_i^0 = -\dot{\phi}_0\partial_i\delta\phi \tag{4.31}$$

$$\delta T_0^i = a^{-2}\partial^i\delta\phi \tag{4.32}$$

$$\delta T_j^i = \delta_j^i\left(\dot{\phi}_0\dot{\delta\phi} - \Phi\dot{\phi}_0^2 - V_{,\phi}\delta\phi\right) \tag{4.33}$$

Since the off-diagonal terms of T_j^i are zero, the same must hold for the off-diagonal terms of G_j^i. Therefore we find $\Phi = \Psi$, in this gauge. Later on, we will need the Einstein equations in this gauge so we give them here:

$$3H(H\Phi + \dot{\Phi}) - a^{-2}\delta^2\Phi = \frac{4\pi}{m_P^2}\left(\Phi\dot{\phi}_0^2 - \dot{\phi}_0\dot{\delta\phi} - V_{,\phi}\delta\phi\right) \tag{4.34}$$

$$\partial_i\left(H\Phi + \dot{\Phi}\right) = \frac{4\pi}{m_P^2}\partial_i\left(\dot{\phi}_0\delta\phi\right) \tag{4.35}$$

$$\ddot{\Phi} + 3H(H\Phi + \dot{\Phi}) + H\dot{\Phi} + 2\dot{H}\Phi = \frac{4\pi}{m_P^2}\left(\dot{\phi}_0\dot{\delta\phi} - \Phi\dot{\phi}_0^2 - V_{,\phi}\delta\phi\right) \tag{4.36}$$

4.3 Mukhanov-Sazaki Action

Now that we have found a gauge invariant parameter, $\mathcal{R}$, which we can use to describe the fluctuations we want to quantize this field. For this we need to find the action up to second order in perturbations. Note that the first order action leads to the equation of motion by the Euler Lagrange equation. The resulting action will be of the form of a simple harmonic oscillator, i.e. a free field with an effective mass term, which is well studied and therefore insightful for the study of cosmological perturbations. The literature used for this derivation can be found in [Maldacena 2003], [Baumann 2009], [Lim 2012] and [Misner et al. 1973]. We start from the Hilbert action plus the action for a single field inflaton field:

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{g}\left({}^{(4)}R - (\nabla\phi)^2 - 2V\right) \tag{4.37}$$

We set $m_\mathrm{p} = 1$ throughout this section. The second order action can be obtained most easily with the use of the ADM formalism. In this formalism physical degrees of freedom are singled out and subjective to constraint equations, as explained in section 4.2. This approach yields a computational simplification. The reason is that the lapse and shift functions of the ADM metric (4.11) appear as Lagrangian multipliers in the action. The terms which include the multipliers only have to be calculated up to first order.

One of the basic features of the ADM formalism is that the time and spatial components of the geometry are decomposed. The Ricci scalar corresponding to the four-geometry will be split into a three-geometry part and other terms. We can write the ${}^{(4)}R$ terms of ${}^{(3)}R$ and extrinsic curvature by the *Gauss-Codazzi* equations. First we expand

$${}^{(4)}R = {}^{(4)}R^{nn}{}_{nn} + 2\,{}^{(4)}R^{in}{}_{in} + {}^{(4)}R^{ij}{}_{ij} = 2\,{}^{(4)}R^{in}{}_{in} + {}^{(4)}R^{ij}{}_{ij} \tag{4.38}$$

The Gauss Codazzi equations read:

$$ {}^{(4)}R^{m}{}_{ijk} = {}^{(3)}R^{m}{}_{ijk} + K_{ik}K_{j}{}^{m} - K_{ij}K_{k}{}^{m} \tag{4.39} $$

$$ {}^{(4)}R^{n}{}_{ijk} = K_{ij|k} - K_{ik|j} \tag{4.40} $$

The *extrinsic curvature* K_{ij} is given by:

$$ K_{ij} = \frac{1}{2N}\left[\dot{h}_{ij} - 2N_{(i|j)}\right] \tag{4.41} $$

Raising the indices with metric (4.11):

$$ {}^{(4)}R^{ij}{}_{ij} = K^2 - K^{i}{}_{j}K^{j}{}_{i} + {}^{(3)}R \tag{4.42} $$

Here, K is the trace K^i_i. The ${}^{(4)}R^{in}{}_{in}$ is not so easily determined by the Codazzi Gauss equations. However it can be shown that this term can be written as a total derivative:

$$ {}^{(4)}R^{in}{}_{in} = -2\nabla_\mu(Kn^\mu - a^\mu), \quad a^\mu = n^{\mu}{}_{;\nu}n^\nu \tag{4.43} $$

We are only interested in deriving the equation of motion for the field. This allows us to disregard boundary terms in the action.

$$ S_{\text{geom}} = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}^3x\,\sqrt{h}N^{-1}\left(E^2 - E^{i}{}_{j}E^{j}{}_{i}\right) \tag{4.44} $$

Where $E_{ij} = NK_{ij}$. Using the metric (4.11) we find for the matter term.

$$ \mathcal{L}_\phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \tag{4.45} $$

$$ = \frac{1}{2}\left(N^{-2}(\dot{\phi} - N^i\partial_i\phi)^2 - h^{ij}\partial_i\phi\partial_j\phi\right) - V(\phi) \tag{4.46} $$

In total we obtain

$$ S = \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{h}\left[N\,{}^{(3)}R + N^{-1}(E_{ij}E^{ij} - E^2) + N^{-1}(\dot{\phi} - N^i\partial_i\phi)^2 - h^{ij}\partial_i\phi\partial_j\phi - 2NV\right] \tag{4.47} $$

This is the inflationary action in the ADM formalism. Note that the shift and lapse functions do not have any derivatives acting on them, therefore they act as Lagrangian multipliers.

The multipliers generate constraint equations. To find the second order action, we solve these constraint equations up to first order. Next we expand the other terms keeping only second order terms.

The constraints are obtained from varying this action with respect to the Lagrange multipliers. Varying with respect to N leads to the *Hamiltonian constraint*:

$$ \frac{\delta S}{\delta N} = {}^{(3)}R - N^{-2}(E_{ij}E^{ij} - E^2) - N^{-2}\dot{\phi}^2 - 2V = 0 \tag{4.48} $$

Varying with respect to N^j. First notice:

$$ \delta S = \delta[N^{-1}(E^{mn}E_{mn} - E^2)] = 2N^{-1}(E^{mn} - Eh^{mn})\delta E_{mn} \tag{4.49} $$

$$\delta E_{mn} = \frac{\partial E_{mn}}{\partial(\nabla_i N^j)}\delta(\nabla_i N^j) = -\frac{1}{2}(h_{nj}\delta^i_m + h_{mj}\delta^i_n)\delta(\nabla_i N^j) \tag{4.50}$$

Partially integrating yields the *momentum constraints*:

$$\frac{\delta S}{\delta N^j} = \nabla_i\left[N^{-1}(E^i{}_j - \delta^i_j E)\right] = 0 \tag{4.51}$$

We can choose a gauge, it will be comoving gauge, considering only the scalar components.

$$\delta\phi = 0, \quad h_{ij} = a^2(1-2\mathcal{R})\gamma_{ij} \tag{4.52}$$

Up to first order, the scalar components of the lapse and shift functions become:

$$N = 1+\alpha, \quad N_i = 0 + \psi_{,i} \tag{4.53}$$

Let us first solve the constraints of zeroth order. For a flat spacetime ${}^{(3)}R = 0$, so the last term can be omitted. The momentum constraints vanish identically whereas the Hamiltonian constraint becomes:

$$H^2 = \frac{1}{3}(V + \frac{1}{2}\dot{\phi}^2) \tag{4.54}$$

i.e. the Friedmann equation. This means the Friedmann equation is not an equation of motion for the scale factor, but it is a constraint equation. Solving the first order equation leads to:

$$\alpha = \tfrac{\dot{\mathcal{R}}}{H} \quad \partial^2\tilde{N}_i^{(1)} = 0 \tag{4.55}$$

$$\psi = -\frac{\mathcal{R}}{H} + \frac{a^2}{H}\epsilon_v\partial^{-2}\dot{\mathcal{R}} \tag{4.56}$$

Where the slow roll parameter is used as found in equation (3.13) ($\epsilon = \frac{1}{2}\frac{\dot{\phi_0}^2}{H^2}$). Putting these first order terms into the action and expanding the remaining terms up to second order leads to second order action:

$$S_{(2)} = \int \mathrm{d}^4x\, a^3\,\epsilon\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i\mathcal{R})^2\right] \tag{4.57}$$

This action is proportional to the slow roll parameter. This means that in pure De Sitter, the perturbation is a pure gauge mode. Passing to conformal time and switching to the following variables:

$$z = a\frac{\dot{\phi}_0}{H} = a\sqrt{2\epsilon}, \qquad \chi = z\mathcal{R} \tag{4.58}$$

The action becomes:

$$\begin{aligned} S_{(2)} &= \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}^3\mathbf{x}\, z^2\left[(\mathcal{R}')^2 - (\partial_i\mathcal{R})^2\right] \\ &= \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}^3\mathbf{x}\left[(\chi')^2 - 2\frac{z'}{z}\chi'\chi + \left(\frac{z'}{z}\right)^2\chi^2 - (\partial_i\chi)^2\right] \\ &= \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}^3\mathbf{x}\left[(\chi')^2 - \left(\frac{z'}{z}\chi^2\right)' + \frac{z''}{z}\chi^2 - (\partial_i\chi)^2\right] \end{aligned}$$

The total derivative on the right hand side can be written as a surface integral over the boundary by means of Stoke's theorem which is irrelevant for the determination of the equation of motion. The action becomes

$$S_{(2)} = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3\mathbf{x} \left[(\chi')^2 - (\partial_i\chi)^2 - m_{\text{eff}}^2\chi^2\right] \tag{4.59}$$

where the effective mass is given by:

$$m_{\text{eff}}^2(\tau) = -\frac{z''}{z} \tag{4.60}$$

The action (4.59) is known as the Mukhanov-Sazaki action [Mukhanov et al. 1992]. Note that this action explicitly depends on time. This means there is no energy conservation. In fact the mass term is negative, which will lead to particle production. The energy for the particles is supplied by the classical gravitational field.

Equation of Motion

To obtain the equation of motion we vary the second order action with respect to the Mukhanov variable χ. This is most easily done by first doing a partial integration:

$$S_{(2)} = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3\mathbf{x} \left[-\chi\chi'' + \chi\partial^2\chi - m_{\text{eff}}^2(\tau)\chi^2\right] \tag{4.61}$$

We see varying w.r.t. χ leads to the following equation of motion:

$$\frac{\delta S_{(2)}}{\delta\chi} = \chi'' - \Delta\chi + m_{\text{eff}}^2\chi = 0 \tag{4.62}$$

To tackle this equation we will pass to Fourier space such that the equation of motion decouples, i.e. the Fourier components can be treated independently.

$$\chi(\mathbf{x},\tau) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\chi_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} \tag{4.63}$$

The EOM becomes:

$$\chi_{\mathbf{k}}'' + \omega_k^2(\tau)\chi_{\mathbf{k}} = 0 \tag{4.64}$$

where the time dependent frequency is given by

$$\omega_k^2(\tau) \equiv k^2 + m_{\text{eff}}^2 \tag{4.65}$$

The general solution of (4.64) can be written as a superposition of two linearly independent solutions v_k and its complex conjugate v_k^*.

$$\chi_{\mathbf{k}}(\tau) = a_{\mathbf{k}}^- v_k(\tau) + a_{-\mathbf{k}}^+ v_k^*(\tau) \tag{4.66}$$

These mode functions should independently satisfy the equation of motion (4.64). The normalization of the mode function is not yet fixed, we will come back to this after the next section.

4.4 Quantisation

The first step to quantization is to promote the field to an operator. As an ansatz we expand in time-independent creation/annihilation operators with associated time-dependent mode functions. Combining equations (4.63) and (4.66) and using that we integrate over all $\mathbf{k}$ we obtain:

$$\hat{\chi}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left(e^{i\mathbf{k}\cdot\mathbf{x}} v_k(\tau) \hat{a}^-_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{x}} v^*_k(\tau) \hat{a}^+_{\mathbf{k}} \right) \tag{4.67}$$

From the second order action (4.59) we find the conjugate momentum:

$$\pi(\tau, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \chi'} = \chi' \tag{4.68}$$

Which we also promote to an operator and expand:

$$\hat{\pi} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left(e^{i\mathbf{k}\cdot\mathbf{x}} v_k(\tau) \hat{a}^-_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{x}} v^*_k(\tau) \hat{a}^+_{\mathbf{k}} \right) \tag{4.69}$$

Quantization is equivalent to imposing the canonical equal time commutation relation:

$$[\hat{v}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}) \tag{4.70}$$

We impose standard commutation relations for the creation annihilation annihilator operators:

$$[\hat{a}^-_{\mathbf{k}}, \hat{a}^+_{\mathbf{k}'}] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}'), \qquad [\hat{a}^-_{\mathbf{k}}, \hat{a}^-_{\mathbf{k}'}] = [\hat{a}^+_{\mathbf{k}}, \hat{a}^+_{\mathbf{k}'}] = 0 \tag{4.71}$$

Substituting these relations in the commutation relation (4.70) will lead to a normalisation condition for the mode functions. Since

$$[\hat{v}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left[e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} v_k \partial_\tau v^*_k - e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} v^*_k \partial_\tau v_k \right] \tag{4.72}$$

$$= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left[v_k \partial_\tau v^*_k - v^*_k \partial_\tau v_k \right] e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \tag{4.73}$$

To go from the first to the second line we used the fact that we integrate over all $\mathbf{k}$, such that we can switch from $\mathbf{k} \to -\mathbf{k}$ in the second term. The commutator should evaluate to a delta function as in equation (4.70):

$$i\delta(\mathbf{x} - \mathbf{y}) = i \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \tag{4.74}$$

We find the normalization condition

$$\langle v_k, v_k \rangle \equiv v_k \partial_\tau v^*_k - v^*_k \partial_\tau v_k = i \tag{4.75}$$

4.4.1 Results for De Sitter

We will now simplify the calculation by considering a pure De Sitter background. At the end of this section the results for quasi-de Sitter are given as well.

The curvature perturbation is ill-defined in the De Sitter space since it includes a factor ϵ which vanishes in this limit. The Mukhanov variable $\chi = z\mathcal{R}$ on the other hand behaves normal in the limit $\epsilon \to 0$. To see this consider flat gauge, $\Phi = 0$, such that $\mathcal{R} = \frac{H}{\dot{\phi}}\delta\phi$. Since $z = a\frac{\dot{\phi}}{H}$ we find

$$\chi = a\delta\phi \tag{4.76}$$

We need to determine the mode-functions. The mode functions satisfy the equation of motion. It depends only on the magnitude of $\mathbf{k}$.

$$v_k'' + \omega_k^2 v_k = 0, \qquad \omega_k^2 = k^2 + m_{\text{eff}}^2 \tag{4.77}$$

For De Sitter space the effective mass is given by

$$m_{\text{eff}}^2(\tau) = -\frac{z''}{z} = -\frac{a''}{a} = -\frac{2}{\tau^2} \tag{4.78}$$

In this case the general solution to eqn. (4.77) is a Bessel equation:

$$v_k = \alpha\frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + \beta\frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right) \tag{4.79}$$

Because of the free parameters α and β which appear in this equation the mode functions are not unique. We fix the mode functions by selecting an appropriate vacuum and furthermore use the normalization condition (4.75). Vacuum selection means that we demand that the vacuum state is the minimal energy state.

For the time-independent harmonic oscillator, or the Minkowski limit, a minimal energy state exists. The equation of motion in the Minkowski limit is as follows:

$$v_k'' + k^2 v_k = 0 \tag{4.80}$$

This fixes the Minkowski mode functions as follows:

$$v_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \tag{4.81}$$

For a time-dependent harmonic oscillator such a minimal energy state does not exist. However we can take the limit to early times when all fluctuations were far inside the horizon. In that case equation (4.77) reduces to the harmonic oscillator with time-independent frequency. The condition to fix the mode function becomes:

$$\lim_{\tau\to-\infty} v_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \tag{4.82}$$

The coefficients that satisfy this limit select the *Bunch-Davies* vacuum.

$$\alpha = 1, \beta = 0 \to v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) \tag{4.83}$$

4.5 Connection with Observations

In the introduction to this chapter we mentioned that super Hubble modes become classical and do not evolve in time. First we show the amplitude for the Mukhanov field approaches a constant for super-Hubble scales in the De Sitter limit.

$$\begin{aligned}\langle 0| \hat{\chi}_{\mathbf{k}}\hat{\chi}_{\mathbf{k}'} |0\rangle &= (2\pi)^3|v_k(\tau)|^2\delta(\mathbf{k}'-\mathbf{k}) = \frac{(2\pi)^3}{2k}\left[1+\frac{1}{(k\tau)^2}\right]\delta(\mathbf{k}'-\mathbf{k}) \\ &= \frac{(2\pi)^3}{2k}\left[1+\left(\frac{aH}{k}\right)^2\right]\delta(\mathbf{k}'-\mathbf{k}) = \frac{(2\pi)^3(aH)^2}{2k^3}\left[\left(\frac{k}{aH}\right)^2+1\right]\delta(\mathbf{k}'-\mathbf{k}) \end{aligned} \tag{4.84}$$

Now consider the De Sitter limit where $\chi = a\delta\phi$.

$$\langle 0| \delta\phi_{\mathbf{k}}\delta\phi_{\mathbf{k}'} |0\rangle = \frac{(2\pi)^3H^2}{2k^3}\left[\left(\frac{k}{aH}\right)^2+1\right]\delta(\mathbf{k}'-\mathbf{k}) \tag{4.85}$$

At super Hubble scales, $k \ll aH$ this approaches a constant:

$$\langle 0| \delta\phi_{\mathbf{k}}\delta\phi_{\mathbf{k}'} |0\rangle = \frac{(2\pi)^3H^2}{2k^3}\delta(\mathbf{k}'-\mathbf{k}) \tag{4.86}$$

This allows us to compare the fluctuations when they leave the horizon and again when they re-enter and show up in the CMB.[4]. From the last equation we can read off the power spectrum as defined by equations (4.1) and (4.2) but with the vacuum expectation value instead of classical ensemble averages of the perturbations:

$$P_{\delta\phi} = \frac{H^2}{2k^3}, \qquad \Delta^2_{\delta\phi} = \frac{H^2}{(2\pi)^2} \tag{4.87}$$

We can now extend this result to quasi-De Sitter space by calculating the power spectrum of the comoving curvature perturbation which includes the slow-roll parameter ϵ. The background values of H and $\dot{\phi}$ vary slowly with time but since the amplitude of perturbations remains constant at super horizon scales we only need to consider the power spectrum at horizon crossing. In the current gauge (flat gauge) the curvature perturbation $\mathcal{R}$ is related to the fluctuations $\delta\phi$ by $\mathcal{R} = \frac{H}{\dot{\phi}}\delta\phi$.

$$\Delta^2_{\mathcal{R}} = \frac{H_*^2}{\dot{\phi}_*^2}\frac{H_*^2}{(2\pi)^2} \tag{4.88}$$

where the $*$ denotes the value at horizon crossing, $k = aH$.

Quantum to Classical Transition

The quantum to classical transition is a complicated subject, even in ordinary quantum mechanics. To see how the quantum fluctuations during inflation can be considered classical we only show the occupation numbers of super-Hubble modes are very high; an important requirement for classical behaviour.

[4] One can show in a gauge invariant way that super-horizon adiabatic density fluctuations remain constant up to all orders in perturbation theory

The real space variance in De Sitter is given by

$$\langle \delta\phi(x)\delta\phi(x)\rangle = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}\langle \delta\phi_{\mathbf{k}}\delta\phi_{\mathbf{k}'}\rangle \tag{4.89}$$

$$= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{H^2}{2k^3}\left[\left(\frac{k}{aH}\right)^2 + 1\right] \tag{4.90}$$

$$= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{k}\left(\frac{1}{2a^2} + \frac{H^2}{2k^2}\right) \tag{4.91}$$

Passing to physical momenta $k_{\mathrm{ph}} = a^{-1}k$ we obtain:

$$\langle \delta\phi^2\rangle = \int \frac{\mathrm{d}^3\mathbf{k}_{\mathrm{ph}}}{(2\pi)^3}\frac{1}{k_{\mathrm{ph}}}\left(\frac{1}{2} + \frac{H^2}{2k_{\mathrm{ph}}^2}\right) \tag{4.92}$$

The first term is related the the vacuum energy of a scalar in Minkowski space and can be renormalized away. The second term however is relevant during inflation and can be seen as particle creation with occupation number

$$n_{\mathbf{k}_{\mathrm{ph}}} = \frac{H^2}{2k_{\mathrm{ph}}^2} \tag{4.93}$$

The occupation number takes on high values for super horizon modes and diverges in the limit $k_{\mathrm{ph}} \to 0$. Because of the high occupation numbers the field can be considered classical.

CHAPTER 5

Eternal Inflation

5.1 Introduction

In the last chapters we have discussed inflation: How it solves the big bang problems and how quantum fluctuations during inflation lead to the CMB anisotropies we observe today. While the inflationary stage has ended in our observable universe, in other parts of the universe it may persist. In the eternal inflation picture inflation never stops globally.

This leads to a different view on the universe. For example, the universe is not homogeneous on scales much larger than our observable universe. There can be many regions which (classically) evolve independently from each other, together forming the *multiverse*. The majority of the universe could be dominated by the maximum energy density possible, in the form of space-time foam.

The phenomenon of eternal inflation has interesting opportunities. For example it has an interesting application to string theory. In string theory there exist many different vacua due to the many possible ways to compactify extra dimensions. Different vacua may have different values for e.g. fundamental coupling constants, the cosmological constant and the fine-structure constant. There seems to be no preferable vacuum state, so why did nature pick out this particular vacuum we live in today? Eternal inflation could provide a way to realize all the different string vacua. The question stated before can be answered by the anthropic principle: our observable universe is just one that has the right kind of vacuum to allow for the existence of intelligent life.

On the other hand, when we adopt the anthropic principle we give up that there is a theory which predicts the value of observables. In that sense predictive power is lost. The study of eternal inflation therefore consists for a large part of dealing with probability functions instead of predicting.

There are two main approaches to the idea of eternal inflation, *Slow Roll Eternal Inflation* (SREI) and *False Vacuum Eternal Inflation* (FVEI). First we discuss how the concept of eternal inflation emerges from these models. Then we provide some problems.

5.2 Slow Roll Eternal Inflation

Slow-roll eternal inflation occurs for inflationary models which inflate due to presence of a slow-roll regime, when quantum fluctuations are more important than the change in the potential due to the slow rolling of the field.

In the last chapter we found that during inflation inhomogeneities are generated, which are small 60 e-folds before reheating. However, in the distant past the fluctuations may have been much larger and could be determining for the evolution of the universe.

In a generally slow roll type of inflation, the change of the field is a combination of classical rolling of the field plus quantum jumps in either direction:

$$\Delta\phi = \delta\phi_{\mathrm{qu}} + \Delta\phi_{\mathrm{cl}} \tag{5.1}$$

Here, $\Delta\phi_{\mathrm{cl}}$ is the classical movement of the field down the slope of the potential, and $\delta\phi_{\mathrm{qu}}$ is the typical amplitude of quantum fluctuations. When the quantum jumps become as large as the classical trajectory, the position of the field is as likely to move backwards as forwards.

During one e-fold the physical volume will expand a factor $e^3 \sim 20$. Since each of these regions will have the size given by the De Sitter horizon, they evolve independently. If on average the field does not decrease in one of these regions the process is eternal. Assuming a normal distribution for the quantum fluctuations, we can write the condition for eternal inflation as follows:

$$P(\delta\phi_{\mathrm{qu}} \geq |\Delta\phi_{\mathrm{cl}}|) = \int_{|\Delta\phi_{\mathrm{cl}}|}^{\infty} \frac{\mathrm{d}\delta\phi}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{\delta\phi^2}{2\sigma^2}} \geq e^{-3} \tag{5.2}$$

which leads to the following condition:

$$1.65\sigma \geq |\Delta\phi_{\mathrm{cl}}| \tag{5.3}$$

In figure 5.1 the different regimes for the inflaton field are indicated for a chaotic model. In the choatic scenario the field starts out at a high energy state, i.e. somewhere 'up the hill' of the potential. If this value is high enough then quantum fluctuations are more important than the rolling of the field towards its minimum. In this regime the probability for the field to move upwards for one of the ~ 20 newly created domains during one e-fold is large enough to establish eternal inflation.

Dispersion

Let us first compute the dispersion of the quantum fluctuations. We will see that the dispersion in De Sitter grows with time. In section 4.5 we already showed, eq. (4.92), that for super-Hubble fluctuations the equal-time correlator is:

$$\langle \phi^2 \rangle = \int \frac{\mathrm{d}^3\mathbf{k}_{\mathrm{ph}}}{(2\pi)^3} \frac{1}{k_{\mathrm{ph}}} \left(\frac{1}{2} + \frac{H^2}{2k_{\mathrm{ph}}^2} \right) \tag{5.4}$$

The first terms is the usual diverging term known from Minkowski space and can be renormalized. The second term becomes:

$$\langle \phi(t)^2 \rangle = \left(\frac{H_0}{2\pi} \right)^2 \int \mathrm{d}\ln k_{\mathrm{ph}} \tag{5.5}$$

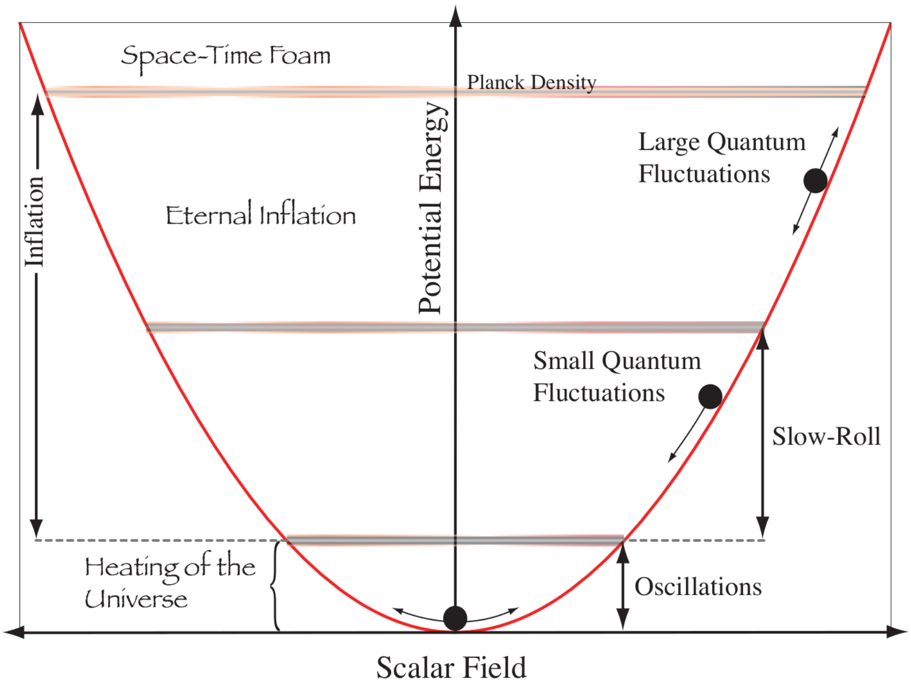

Figure 5.1 – Different stages of inflation in a chaotic model [Kuhnel 2009]

We denote this term with the same symbol ϕ since we will only be interested in this quantity. Note that the physical momentum depends on time. The main contribution to this expectation value will come from super-Hubble fluctuations, $k_{\rm ph} < H_0$, The characteristic length of a mode is given by $L_{\rm ph} = k_{\rm ph}^{-1}$ such that for super-Hubble we need $L_{\rm ph} > H_0^{-1}$. During one time-step Δt the physical wavelength increases by a factor $e^{H_0 \Delta t}$. In the integral we only need to consider wavelengths up to $L_{ph}(t + \Delta t) = e^{H_0 \Delta t} L_{ph}(t)$ or impose a cutoff $k_{ph} > k_{ph} e^{-H_0 \Delta t}$.

We find:

$$\sigma^2 = \langle \hat{\phi}(t + \Delta t)^2 \rangle - \langle \hat{\phi}(t)^2 \rangle = \left(\frac{H}{2\pi} \right)^2 \int_{He^{-H\Delta t}}^{H} \mathrm{d} \ln k_{ph} = \frac{H^3}{(2\pi)^2} \Delta t \tag{5.6}$$

In one Hubble time we have:

$$\sigma^2 = \frac{H^2}{(2\pi)^2} \tag{5.7}$$

Now we can estimate the size of quantum fluctuations which are created during one e-fold.

$$\delta\phi_{\rm qu} = \sqrt{\sigma^2} = \frac{H}{2\pi} \tag{5.8}$$

At the same time the classical trajectory of the field increases on average

$$\Delta\phi_{\rm cl} \sim \dot{\phi} H^{-1} \tag{5.9}$$

Using a normal distribution for the quantum fluctuations we can calculate when condition (5.2) is fulfilled:

$$\epsilon < \frac{1.65^2}{8\pi^2}\frac{H^2}{m_{\mathrm{p}}^2} \tag{5.10}$$

The relative slope of the potential should be small enough to ensure the onset of eternal inflation.

5.3 False Vacuum Eternal Inflation

In eternal inflation of the false vacuum type, bubble nucleation occurs by moving from one false vacuum to another. In these models, the potential is initially in some false vacuum, during which the equation is state is very much De Sitter-like. The field stays in the false vacuum, except in the rare event of tunneling to another vacuum. Several mechanisms exist for such tunneling events to occur. When the field tunnels to the true vacuum, or to another false vacuum, a new *bubble* universe is created. During the creation of such a bubble, the field rolls to its minimum. For an observer inside the bubble, the universe seems infinite. The next step in the evolution can be one of these three: Again De Sitter expansion, Minkowski space, or an Anti-De Sitter ('crunch) phase for tunneling to a vacuum with positive, zero (true vacuum) or negative vacuum energy density respectively.

Bubbles are nucleated inside the 'mother' De Sitter space, such that a region with lower effective cosmological constant is generated. The rate of nucleation is small compared to the physical volume expansion during a Hubble time, therefore bubbles will not catch up with the global expansion. Inside these bubbles, the false vacuum can again decay creating more bubbles. In comoving coordinates, eventually all of the mother space will decay to a lower vacuum. However, in physical coordinates this volume is infinite, such that the process is eternal. A conformal diagram of this scenario is given in figure (5.2).

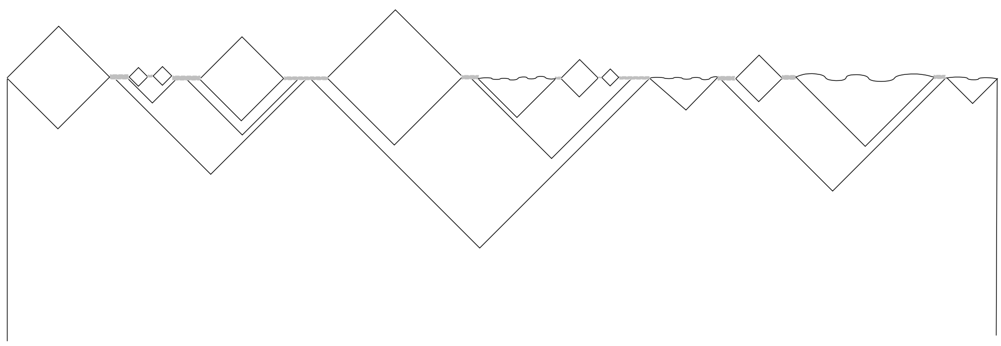

Figure 5.2 – Conformal diagram of false vacuum eternal inflation. The triangles on top correspond to bubbles with $\Lambda = 0$ and do not tunnel further. [Bousso et al. 2006]

5.4 Problems with Eternal Inflation

In this research the focus lies on the following problems.

Global description

One of the most important problems of eternal inflation is that there exist no consistent effective global description of this phenomenon. First of all, in what framework can we understand eternal inflation? The Friedmann equation is based on large scale homogeneity, which is violated by eternal inflation. Which cosmological laws govern eternal inflation?

A natural way to understand the global structure of the eternally inflating multiverse would be to know the distribution function for the inflaton field. However such a distribution function suffers from ambiguities due to the choice of the time parameter. We will see how such a distribution function emerges in the stochastic formalism in chapter 7.

Back-reaction

Back-reaction can be described as the effect of the quantum fluctuations on the background solution to the Einstein equations. It will be more clearly introduced in the next chapter.

Two major problems related to back-reaction will be addressed in this report. In the previous chapter we introduced linear perturbation theory. However, since the Einstein equations are non-linear this is not always a valid approximation. How do we tackle these effects? A second issue with back-reaction is that it involves a quantum process acting on the classical Einstein equations. For this reason semi-classical gravity will be introduced. The validity of this framework and other problems will be discussed in chapter 6.

To address these questions two approaches will be investigated. In the first place there is the *stochastic inflation* formalism. In this framework the quantum effects are accounted for by classical noise. We will treat this formalism in chapter 7. The other approach is based on *thermodynamics*. The relation between general relativity and thermodynamics will be investigated and we will see how this can be used with respect to the previously stated problems.

5.4.1 Comments on Eternal Inflation

Spacetime-foam

In some models of eternal inflation, for example chaotic models, the field is shifted upwards the potential such that eventually the Planck scale is reached. We do not have a field theory which holds in this regime because quantum gravitational effects should be accounted for; these regions are sometimes mentioned as *spacetime foam*. In chaotic eternal inflation seems that most of the spacetime volume is filled with quantum-foam. In order to overcome this one could fine-tune the potential in the chaotic scenario, as in figure 5.3.

Validity of Perturbation Theory

In the previous section we found that when the slope of the inflaton potential becomes nearly flat, quantum fluctuations start to dominate. In the last chapter we have calculated the spectrum in terms of the comoving curvature perturbation or equivalently the curvature perturbation on uniform-density hypersurfaces. We found for super Hubble scale perturbations

$$\langle \mathcal{R}^2 \rangle \sim \frac{H^4}{\dot{\phi}^2} \tag{5.11}$$

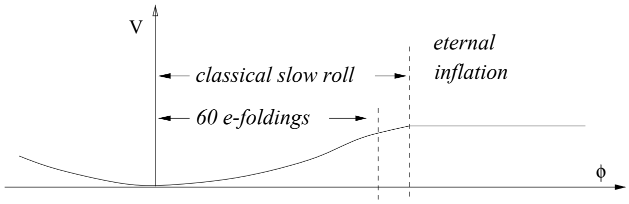

Figure 5.3 – Fine tuning of chaotic scenario in which eternal inflation corresponds to sub-Planckian De Sitter limit [Bousso et al. 2006]

In the eternally inflating regime $\dot{\phi} \sim H\delta\phi \sim H^2$ such that

$$\langle \mathcal{R}^2 \rangle \sim 1 \tag{5.12}$$

When this is the case one needs to take care that perturbations are still small when picking a gauge. In constant density gauge for example ($\delta\phi = 0$) the curvature perturbation is equal to the scalar metric perturbation: $\mathcal{R} = \Psi$. This implies $\Psi \sim 1$ while perturbation theory is based on this quantity to be small. However other gauges, such as longitudinal gauge and comoving gauge the linear perturbation theory still holds [Creminelli et al. 2008]. We will verify this for longitudinal gauge in section 8.4

Null Energy Condition

We want to remark that the quantum fluctuations which drive the field locally upwards the potential violate the null energy condition (NEC). This is classically not allowed but quantum processes are known to break this condition for brief periods of time. The NEC in this case can be put into the form[1]:

$$\rho + p = \dot{\phi}^2 \geq 0 \Longrightarrow \dot{H} \leq 0 \tag{5.13}$$

This can be seen from equation (3.9). However H *increases* when quantum fluctuation move upwards on the potential such that $\dot{H} > 0$. This means that scalar driven eternal inflation at least locally has to violate the NEC. However since eventually inflation will end for any Hubble patch, the NEC violation does not last permanently.

[1]The NEC reads: $N^\mu N^\nu T_{\mu\nu} \geq 0$ for all timelike vectors N^μ. It can be shown that $\dot{H} > 0$ corresponds to $N^\mu N^\nu T_{\mu\nu} < 0$ i.e. breaking of the NEC

CHAPTER 6

Back-reaction

Back-reaction is related to the effect of quantum fluctuations on the geometry of spacetime. There are many issues and possibilities related to the back-reaction effect. How do we include back-reaction in the classical Einstein equation? Are the effects of back-reaction observable? What are the implication for the global picture of the universe, e.g what are the consequences for eternal inflation? The possibilities include that suggestion that quantum gravitational back-reaction can to end inflation driven by a cosmological constant without the necessity of an inflaton field. It was also suggested that back-reaction could explain the current expansion of the universe, i.e. it could account for dark energy. This has been rejected however by most of the cosmologists. There have been a lot of research in this direction and different, sometimes contradicting, conclusions have been drawn. In this chapter a literature study to this subject is presented.

The chapter is build up as follows. First we explore classical back-reaction; the effective energy-momentum is introduced and we give results for De Sitter and slow-roll. Then we introduce semi-classical gravity and give results for the renormalized EMT in De Sitter.

6.1 Classical Back-reaction

Quantum fluctuations distort the homogeneity of the metric and/or the inflaton field, depending on the coordinate system. To describe this in a gauge invariant way we used the comoving curvature perturbation. In De Sitter space it suffices to consider only fluctuations of the scalar field ϕ. On super-Hubble scales, the fluctuations can become classical, because of the high occupation numbers. However, because they originate from quantum fluctuations we expect that the average is zero, such that the average of the classical field and metric remains the same as without the fluctuations. However, the variance $\langle \delta\phi^2 \rangle$ is non-zero. The question is how these fluctuations effect on the background solution of the Einstein equation.

To this end we are interested in an averaged version of the Einstein equations. The Einstein equations are non-linear. Terms that appear in the EMT like $\dot{\phi}^2$ and $V(\phi)$ are generally non-linear such that the variance of the classical fluctuations enters in the Einstein equations.

To see how perturbations can effect the background solution, consider the following (see [Mukhanov et al. 1997]). The universe is assumed to be homogeneous and isotropic with small perturbations. We can write the homogeneous metric and inflaton field and their inhomogeneous small perturbations as follows:

$$q^a = q_0^a + \delta q^a \tag{6.1}$$

In this notation q_0^a denote both the metric and the field homogeneous background values and δq^a are the inhomogeneous perturbations. In this notation the superscript a includes the spacetime indices.

Expanding the Einstein equations

$$\Pi_{\mu\nu} \equiv G_{\mu\nu} - \frac{8\pi}{m_P^2} T_{\mu\nu} = 0 \tag{6.2}$$

up to second order in perturbations and using the condensed notation we find:

$$\Pi\left(q_0^a\right) + \Pi_{,a}\delta q^a + \frac{1}{2}\Pi_{,ab}\delta q^a \delta q^b = 0 \tag{6.3}$$

If we take the spatial average of this equation $\langle \Pi_{,a}\delta q^a \rangle$ vanishes since isotropy ensures $\langle \delta q^a \rangle = 0$. We obtain

$$\Pi\left(q_0^a\right) = -\langle \frac{1}{2}\Pi_{,ab}\delta q^a \delta q^b \rangle \tag{6.4}$$

Therefore, second order perturbations effect the homogeneous background solution. However, the second order perturbations are not in general gauge invariant. In the article by Mukhanov et al. [1997] perturbations which are gauge invariant up to second order were introduced to make this equation gauge invariant. For scalar metric perturbations and using longitudinal gauge the equation (6.4) remains valid.

It is convenient to write the term on the right hand side of equation (6.4) as an effective energy momentum tensor (EEMT) which enters as a source to the background Einstein equations:

$$G_{\mu\nu} = \frac{8\pi}{m_P^2}\left(T_{\mu\nu} + \langle \tau_{\mu\nu} \rangle\right) \tag{6.5}$$

Where $\tau_{\mu\nu}$ is given by

$$\tau_{\mu\nu} = -\frac{m_P^2}{16\pi}\langle \Pi_{\mu\nu,ab}\delta q^a \delta q^b \rangle \tag{6.6}$$

Averaging

Recently it was found that the results given above actually depend on the averaging procedure in the calculation of the expectation value. The averaged Einstein equations are not the same as the Einstein equations with averaged geometry. In order to take this into account a gauge invariant way to average over spacetime has been introduced [Gasperini et al. 2009]. For this a clock field is introduced associated with an observer. So far there have been only results for long-wavelength perturbations although the formalism is capable of treating quantum averages as well [Finelli et al. 2011]

The back-reaction due to averaging effects the local expansion rate as follows:

$$H_{\text{eff}}^2 = \left(\frac{1}{a_{\text{eff}}}\frac{\partial a_{\text{eff}}}{\partial A_0}\right)^2 = H^2\left(1 + \frac{2}{H}\langle \phi\dot{\phi} \rangle - \frac{2}{H}\langle \dot{\phi^2} \rangle\right) \tag{6.7}$$

here the observer is specified by some auxiliary clock field A and ϕ is the linear order spatial metric perturbation, from the uniform curvature gauge. The long wave field perturbations can be related to these metric perturbations through constraint equations. For chaotic inflation it is found, for freely falling observers [Finelli et al. 2011]:

$$H_{\text{eff}}^2 = H^2 \left(1 + \mathcal{O}\left(\epsilon^2\right) \frac{H_i^4}{H^2 m_P^2} \ln a\right) \tag{6.8}$$

For these observers there is no leading order in slow-roll parameter backreaction effect. However, other observers find different results. E.g. for isotropic observers measure a decrease of the effective expansion, and have an equation of state which decreases from the De Sitter limit as a function of time [Marozzi and Vacca 2012]. Therefore different observers can probe different physical properties of the universe. Furthermore one should notice that short wavelength modes may become important but these calculations have yet not been done.

Observability of Back-reaction

For an observational point of view it interesting whether the backreaction effect can be measured. For single field models this is not possible, since a simple time-shift can mask such a backreaction effect [Unruh 1998]. Therefore , it is favourable to have another subdominant field present, which acts as a clock. The result can be observed by non-Gaussianity and a larger breaking of the scale invariance. Another model in which a decrease in effective Hubble parameter was found is in a two-field model, with a light test field [Marozzi et al. 2012].

Chaotic Inflation in the LW-limit

Although the EEMT of eq. (6.6) is not the full story we will nevertheless see what results we can obtain in the case of chaotic inflation. First we will consider the classical long-wavelength limit and in the next section we give some results from the literature where small wavelengths are taken into account. In the long wavelength regime the effect of *scalar* perturbations on the background is given by [Mukhanov et al. 1997]:

$$\rho_s = -\tau_0^0 = \left(2\frac{V''V^2}{V'^2} - 4V\right)\langle\Phi^2\rangle \tag{6.9}$$

$$\tag{6.10}$$

where Φ is the scalar metric perturbation[1]. Furthermore:

$$p_s = \frac{1}{3}\tau_i^i = -\rho_s \tag{6.11}$$

The equation of state parameter is that of De Sitter space. From the Einstein equations, the long-wavelength part of scalar perturbations is given by [Mukhanov 1985]:

$$\Phi = -\frac{V'}{2V}\delta\phi \tag{6.12}$$

[1] In the gauge that is used, i.e. longitudinal gauge, this corresponds to the gauge invariant Bardeen variable Φ

The result for $V = \frac{1}{2}m^2\phi^2$ is:

$$\frac{\rho_s(t)}{\rho_0} \simeq -\frac{3}{4\pi}\frac{m^2\phi_0^2(t_i)}{m_P^4}\left(\frac{\phi_0(t_i)}{\phi_0(t)}\right)^4 \tag{6.13}$$

The energy density is negative and therefore corresponds to a *negative* contribution to the effective cosmological constant:

$$\Lambda_{\text{eff}} = \Lambda + 8\pi G\rho_s \tag{6.14}$$

Furthermore, the absolute value of the effective density term due to back-reaction increases with time. If the field was initially above the self-reproduction scale, $\phi_0(t_i) \sim \phi_{\text{sr}} = m^{-1/2}m_P^{3/2}$), back-reaction becomes important before the end of inflation. This effect is slowing down inflation.

6.2 Semi-classical Back-reaction

6.2.1 De Sitter Space

First we will see that quantum zero-point fluctuations of a free scalar field with small mass induces constant back-reaction term. This is however *not* physically relevant, since it can be removed by a rescaling of the cosmological constant. Back-reaction would be relevant if it is time dependent, as we have seen in the classical case in the previous section. As examples we will evaluate the back-reaction of a free scalar field with zero and small mass and a self-interacting scalar field.

In the previous section we considered slowly evolving backgrounds. In this case it is hard to calculate vacuum expectation values. We circumvented this problem by considering only long-wavelengths, which are should give the dominant contribution to expectation values. Since long-wavelengths can be treated classically, we considered spatial averages instead of vacuum expectation values. In case of the Sitter space, there exist exact expressions for the mode-functions of quantum fields. Therefore we can do better than the long-wavelength classical approximation, and calculate renormalized vacuum expectation values. The formalism which can evaluate the back-reaction of quantum fluctuations on the background geometry is semi-classical gravity. The Einstein equations read

$$G^\mu_\nu = -\delta^\mu_\nu\Lambda + m_{\text{p}}^{-2}\langle T^\mu_\nu\rangle \tag{6.15}$$

In principle one should take the renormalized vacuum expectation value of the energy momentum tensor operator $\langle T^\mu_\nu\rangle_{\text{REN}}$. We will give some results from the literature and consider a simplified discussion in the sections below.

Small Mass

We consider the following Lagrangian for scalar field with small ($m_\phi^2 \ll H_0^2$) mass.

$$\mathcal{L}_\phi = -\frac{1}{2}\left[(\nabla\phi)^2 + m^2\phi^2\right] \tag{6.16}$$

If the vacuum is invariant under the De Sitter group vacuum expectation values have to be constant, for example $\langle\phi^2\rangle$ = constant and $\langle T_{\mu\nu}\rangle$ = constant $\times\, g_{\mu\nu}$. The vacuum expectation value of energy momentum tensor in the Bunch Davies vacuum is finite and given by [Bunch and Davies 1978]:

$$\langle T^{\mu}_{\nu} \rangle_{\text{REN}} = \frac{1}{4} \delta^{\mu}_{\nu} \langle T^{\sigma}_{\sigma} \rangle_{\text{REN}} = -\frac{1}{4} \delta^{\mu}_{\nu} m^2 \langle \phi^2 \rangle_{\text{REN}} \tag{6.17}$$

Here, $\langle \phi^2 \rangle_{\text{REN}}$ represents the renormalized two-point Green's function $\langle \phi(x)\phi(x') \rangle$. The renormalized result for $\langle \phi^2 \rangle$ with a small mass is [Birrell and Davies 1984][2]

$$\langle \phi^2 \rangle_{\text{REN}} = \frac{61 H_0^4}{240 \pi^2 m^2} \tag{6.18}$$

such that

$$\langle T^{\mu}_{\nu} \rangle_{\text{REN}} = -\frac{61 H_0^4}{960 \pi^2} \delta^{\mu}_{\nu} \tag{6.19}$$

This implies that the back-reaction of the light scalar field vacuum fluctuations contributes positively to the cosmological constant in eqn. (6.15). Interestingly this equation holds for $m \to 0$. Note that with the metric signature used in this report $\rho = -T^0_0$. In the literature there are different ways to denote the backreaction: as an effective Hubble parameter, effective cosmological constant or as an energy density.

No Mass

The previous result uses the renormalized value for zero-point fluctuations of a scalar field with small mass. In the end the mass does not enter in the expression for the energy momentum tensor. However the VEV of the light scalar field, eqn. (6.18) diverges for $m \to 0$. This can be explained by noticing that there exist no De Sitter invariant vacuum state for massless scalars. The resolution is to define the vacuum state as the minimal energy state in the infinite past or when inflation sets in, i.e. the Bunch Davies vacuum. Quantum fluctuations are generated and various correlators depend on time.

When considering the back-reaction of massless fluctuations the term $\langle \phi^2 \rangle$ does not appear in the energy-momentum tensor. For this reason there is no IR-enhanced contribution from this term. Only derivative terms appear in the EMT, and these are sub-dominant on large scales. We will suggest however that the back-reaction from massless scalars could be relevant in the next section.

Suggestion for Back-reaction from Massless Fluctuations

We consider a massless field in De Sitter space denoted by ϕ_{qu}. The subscript is there to remind us we are not dealing with the classical inflaton field but a spectator quantum field.

We assume that temporal derivatives dominate spatial derivatives such that the Lagrangian becomes $\mathcal{L} = \frac{1}{2}\dot{\phi}^2_{\text{qu}}$. This leads to density and pressure terms of $\rho = -p = \frac{1}{2}\dot{\phi}^2_{\text{qu}}$. To evaluate these terms we consider the time-dependent variance, eqn. (5.6):

$$\langle \phi_{\text{qu}} \rangle^2 = \frac{H^3}{(2\pi)^2} \Delta t = \left(\frac{H}{2\pi} \right)^2 N \tag{6.20}$$

[2]Based on point-splitting

where $N = \log a = H\Delta t$. The typical amplitude of quantum fluctuations is given by $|\phi_{\rm qu}| = \sqrt{\langle\phi_{\rm qu}^2\rangle}$. Absolute signs will be dropped in the following. The quantum fluctuations generated per e-fold are

$$\frac{\mathrm{d}\phi_{\rm qu}}{\mathrm{d}N} = \dot{\phi}_{\rm qu}\left(\frac{\mathrm{d}N}{\mathrm{d}t}\right)^{-1} = \dot{\phi}_{\rm qu}\left(\frac{\mathrm{d}\log a}{\mathrm{d}t}\right)^{-1} = \dot{\phi}_{\rm qu}H^{-1} \tag{6.21}$$

So for one e-fold we obtain $\phi_{\rm qu} = \dot{\phi}_{\rm qu}H^{-1}$. Since in De Sitter $\phi_{\rm qu} = \frac{H}{2\pi}$ we find

$$\rho = -p = \frac{H^4}{8\pi^2}. \tag{6.22}$$

Note that the sign and order of magnitude are the same as eqn. (6.18) which was found for a field with small mass and using renormalized VEV's. Also, we do not find a time-dependence of the back-reaction term. Therefore it can be removed by a rescaling of the cosmological constant.

Self-Interacting Field

For several common interactions it has been shown that back-reaction leads to negative contribution energy density, which effect becomes larger as a function of time [Ford 1985]. Self interaction requires renormalization in curved backgrounds which are quite complicated computations. We will present only the simplified conclusion of two articles which obtained results in this direction.

For self interacting $\frac{\lambda}{4}\phi^4$ theory in pure De Sitter background one and two-loop results can be found in [Onemli and Woodard 2002]. It was found that the back-reaction effect is not equal to a De Sitter-like equation of state. The energy density corresponding to the back-reaction is positive and increases with time. At the same time the equation of state becomes more like de Sitter. At some point both $p = -\rho$ and the back-reaction becomes constant. This value is simply an addition to the effective cosmological constant. This result has been verified by a stochastic procedure as well [Starobinsky and Yokoyama 1994].

The previous result however does not take tensor perturbations into account. In most of this report we have discarded tensor perturbations because observations show that at last scattering these contributions are small. However in the early stages of inflation they are important. Two-loop results have been calculated in [Tsamis and Woodard 1997]; these are perturbed quantum gravity results. It was found that the back-reaction effect negative and grows with time. This result offers the possibility to have inflation by a cosmological constant which ends due to the back-reaction, without the use of an inflaton field [Buchert and Obadia 2011]. Eternal inflation is not possible in this case.

6.2.2 Chaotic Inflation

In section (6.1) we found the classical long-wave limit results for the effect of back-reaction on a chaotic model with a quadratic potential. Similar results with the use of renormalized VEV's instead of the classical long wavelength approximation. It was found that for the combination of scalar and tensor perturbations, up to 2-loop effects the back-reaction is the following [Finelli et al. 2004]:

$$\rho_{\rm REN} \sim -p_{\rm REN} \sim -\frac{9}{8\pi^2}\epsilon H_0^4 \ln a \tag{6.23}$$

where H_0 is the value of the Hubble parameter at the onset of inflation. Furthermore, the slow-roll parameters ϵ and η were assumed to be of the same size. The time dependence is only valid when the quantum fluctuations are dominating the classical rolling of the potential. When the potential starts to dominate, the backreaction saturates to:

$$\rho_{\mathrm{REN}}^{\max} \sim -\frac{3}{16\pi^2}\frac{H_0^6}{H^2} \tag{6.24}$$

Just as for the long-wavelength regime, the back-reaction tends to cancel the effective cosmological constant. This is conflicting with the result from Starobinsky and Yokoyama [1994], who used the stochastic formalism to calculate VEV's. Their calculation however did not take gravitons into account. Note that in the De Sitter limit, the back-reaction of scalar perturbations vanishes, because of the dependency of ϵ in equation (6.23).

CHAPTER 7

Stochastic Inflation

In the previous chapters we have seen that quantum fluctuations are quite important during inflation; they might even prevent inflation from ending globally. Interestingly, super-Hubble quantum fluctuations are continuously generated and become effectively classical. But since the origin of the classical field is quantum-mechanical, the change of the classical field is not deterministic and should be governed by a stochastic process. In this chapter we discuss such a stochastic mechanism applicable during inflation. We will show that the background field, and therefore the geometry, behaves like Brownian motion. In this sense, the stochastic formalism deals with the back-reaction of the quantum fluctuations on the background geometry.

A typical example of a process which is subject to Brownian motion is the evolution of a large particle moving due to collisions with small randomly moving particles. This behaviour is typically described by two related equations: the *Langevin* and *Fokker-Planck* equations. In this chapter we will derive and interpret these equations.

7.1 Langevin Equation

A useful equation to describe Brownian motion is the Langevin equation, after Paul Langevin who published his findings in 1908. In the introduction we mentioned the example of a large particle moving in an environment of randomly moving smaller particles. The force acting on a larger particle is given by a friction term, proportional to the velocity of the large particle, and a noise term which is due to random motion of the small particles.

We will now derive how inflation corresponds to this picture. We follow the article by Starobinsky [1986], in which the Langevin equation for inflation was first derived. For this derivation we assume a small-field model of inflation (see section 3.3) which reduces to De Sitter (with H_0) for $\phi \to 0$. At the end we give the result for a general model as well, where only the slow-roll approximation is assumed.

To make the analogue with the particle example picture the averaged classical field in a certain Hubble patch as the large particle and the quantum fluctuations correspond to the small particles.

We now write the field as a quantum operator which we will split in a long and small wavelength part. As discussed before the super-horizon modes can be treated as classical because of high occupation numbers. We denote this field by $\phi_{\rm CG}(t, \mathbf{x})$. Because the long wavelengths are super-Hubble, regions with size of the order of the Hubble-scale can be assumed to be homogeneous; this is known as *coarse graining* the field. The field still depends on $\mathbf{x}$ but is homogeneous on roughly Hubble scales. The inhomogeneous small wavelength modes contribute to $\hat{\phi}_{\rm SW}(t, \mathbf{x})$. We consider now one such a homogeneous patch, $\phi_{\rm CG}(t, \mathbf{x}) = \phi_{\rm CG}(t)$ and define the full field:

$$\hat{\phi}(t, \mathbf{x}) \equiv \phi_{\rm CG}(t) + \hat{\phi}_{\rm SW}(t, \mathbf{x}) \tag{7.1}$$

where $\hat{\phi}_{\rm SW}$ is given by

$$\hat{\phi}_{\rm SW}(t, \mathbf{x}) = \int \frac{{\rm d}^3\mathbf{k}}{(2\pi)^3} \Theta(k - \epsilon a H_0) \left[\phi_k(t) \hat{a}^-_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + \phi^*_k(t) \hat{a}^+_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{7.2}$$

The step-function $\Theta(k - \epsilon a H_0)$ with $\epsilon \ll 1$ selects the small wavelengths but the range exceeds the Hubble scale.

$$\Theta(k - \epsilon a H_0) = \begin{cases} 1, & k > \epsilon a H_0 \\ 0, & k < \epsilon a H_0 \end{cases} \tag{7.3}$$

The parameter ϵ cannot be arbitrarily small. The lower bound depends on the specific model of inflation, but is chosen such that the small wavelength field obeys the massless Klein Gordon equation:

$$0 = -\Box \phi_{\rm SW} = \ddot{\phi}_{\rm SW} + 3H_0 \dot{\phi}_{\rm SW} - \frac{\nabla}{a^2} \phi_{\rm SW} \tag{7.4}$$

The homogeneous large scale field equation of motion in the slow-roll approximation is:

$$\dot{\phi}_{\rm CG} = -\frac{V_{,\phi_{\rm CG}}}{3H} \tag{7.5}$$

The full field, eqn. (7.1), satisfies the slow-roll equation of motion plus a term which is due to the inhomogeneous small wavelength field:

$$\dot{\phi}_{\rm CG} = -\frac{1}{3H_0} V_{,\phi_{\rm CG}} + N(t, \mathbf{x}) \tag{7.6}$$

$$N(t, \mathbf{x}) = -\dot{\phi}_{\rm SW} + \frac{\nabla}{3H_0 a^2} \phi_{\rm SW} \tag{7.7}$$

$N(t, \mathbf{x})$ is known as the *noise* operator. We are interested in the contribution from the noise to the homogeneous Hubble patch under consideration.

We will show that equation (7.6) corresponds to the *Langevin* equation in the context of slow-roll inflation:

$$\dot{\phi}_{\rm CG}(t, \mathbf{x}) = v(\phi_{\rm CG}) + N(t, \mathbf{x}) \tag{7.8}$$

The evolution of the coarse grained field is determined by a friction term $v(\phi_{\rm CG}) = -\frac{1}{3H_0} V_{,\phi_{\rm CG}}$ and a stochastic noise term $N(t, \mathbf{x})$. We will now prove that the noise, which originates from small scale quantum fluctuations, corresponds to a classical, stochastic noise term.

Equation (7.7) is:

$$N(t,\mathbf{x}) = \left(-\partial_t + \frac{\nabla}{3H_0a^2}\right)\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,\Theta(k-\epsilon aH_0)\left(\phi_k e^{i\mathbf{k}\cdot\mathbf{x}}a_{\mathbf{k}} + h.c.\right) \tag{7.9}$$

$$\begin{aligned} &= -\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Bigg\{\left[\partial_t\Theta(k-\epsilon aH_0)\right]\left(\phi_k e^{i\mathbf{k}\cdot\mathbf{x}}a_{\mathbf{k}} + h.c.\right) \\ &\qquad - \Theta(k-\epsilon aH_0)\left(-\partial_t + \frac{\nabla}{3H_0a^2}\right)\left(\phi_k e^{i\mathbf{k}\cdot\mathbf{x}}a_{\mathbf{k}} + h.c.\right)\Bigg\} \end{aligned} \tag{7.10}$$

$$\begin{aligned} &= -\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Bigg\{\left[\partial_t\Theta(k-\epsilon aH_0)\right]\left(\phi_k e^{i\mathbf{k}\cdot\mathbf{x}}a_{\mathbf{k}} + h.c.\right) \\ &\qquad - \Theta(k-\epsilon aH_0)\frac{\partial_t^2}{3H_0}\left(\phi_k e^{i\mathbf{k}\cdot\mathbf{x}}a_{\mathbf{k}} + h.c.\right)\Bigg\} \end{aligned} \tag{7.11}$$

We will now show that the second line, with the integral over a step-function, is much smaller than the first line, which is in fact an integral over a delta function. The integral runs over all $\mathbf{k}$, and the contribution to the second line of eqn. (7.11) from large k is not obviously small. However, we are interested in the effect of the noise on the homogeneous patch which is of Hubble-size. Measured on such a scale, it is easy to show that the contribution from large values of k, $(k \gg aH_0)$, is small, see e.g. [Mukhanov and Winitzki 2007] p8. The contribution coming from the integral over the step-function will be finite. However, we still need to show that this contribution is smaller than the first line.

The see this, consider the following. The mode functions are given by $\phi_k = a^{-1}v_k$, where v_k is given by equation (4.83) which we copy here:

$$v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) \tag{7.12}$$

We switch to cosmic time by setting $\tau = -(a(t)H_0)^{-1}$.

$$\phi_k(t) = \frac{H_0 e^{ik/aH_0}}{\sqrt{2}k^{3/2}}\left(\frac{k}{aH_0} + i\right) \tag{7.13}$$

The first and second derivatives of the mode function are

$$\dot{\phi}_k = \frac{-i\sqrt{k/2}}{a^2}e^{ik/aH_0}, \qquad \ddot{\phi}_k = \frac{i\sqrt{k/2}}{a^3}\left(ik+1\right)e^{ik/aH_0} \tag{7.14}$$

Such that

$$\ddot{\phi}_k = -\frac{1}{a}\left(ik+1\right)\dot{\phi}_k \tag{7.15}$$

We can write equation (7.11) as

$$\begin{aligned}
N(t,\mathbf{x}) = -\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \Bigg\{ &\partial_t \left[\Theta(k-\epsilon a H_0)\left(\phi_k e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}} + h.c.\right)\right] \\
&- \Theta(k-\epsilon a H_0)\left(\dot{\phi}_k e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}} + h.c.\right) \\
&- \Theta(k-\epsilon a H_0)\left(\frac{\ddot{\phi}_k}{3H_0} e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}} + h.c.\right) \Bigg\} && (7.16) \\
= -\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \Bigg\{ &\partial_t \left[\Theta(k-\epsilon a H_0)\left(\phi_k e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}} + h.c.\right)\right] \\
&- \Theta(k-\epsilon a H_0)\left(\dot{\phi}_k e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}} + h.c.\right) \\
&+ \Theta(k-\epsilon a H_0)\left[\dot{\phi}_k \left(\frac{ik}{aH_0} + \frac{1}{aH_0}\right) e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}} + h.c.\right] \Bigg\} && (7.17)
\end{aligned}$$

The integrals over the second and third line are finite when we consider the spatial average over a Hubble sized region. However, the last line is proportional to the comoving apparent horizon which decreases exponentially during inflation. The fact that the third line depends on an extra factor of k becomes irrelevant when considering the late time asymptotic where $(aH_0)^{-1} \to 0$. Therefore we can drop the second derivative term in equation (7.11). Finally, we arrive at

$$N(t,\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \delta(k-\epsilon a H_0) \frac{i\epsilon a H_0^3}{\sqrt{2}k^{3/2}} \left(e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}} - e^{-i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}}^{+}\right) \tag{7.18}$$

which agrees with equation (8) of the article [Starobinsky 1986].

To see that the noise is effectively classical, but random, consider the following. The conjugate momentum will be of the form $\dot{N}$ which does not change the structure of the creation/annihilation operators. Therefore the only non-zero terms which appear in the commutator $\left[N_{\mathbf{k}}, \dot{N}_{\mathbf{k}'}\right]$ will be of the form

$$-\left[a_{\mathbf{k}}, a_{\mathbf{k}'}^{+}\right] - \left[a_{\mathbf{k}}^{+}, a_{\mathbf{k}'}\right] = (2\pi)^3 \left(\delta(\mathbf{k}-\mathbf{k}') - \delta(\mathbf{k}'-\mathbf{k})\right) = 0 \tag{7.19}$$

Since the commutator vanishes the noise can be considered classical. However, we cannot ascribe a numerical value to the operators which appear in equation (7.18). Therefore the noise is random.

A more rigorous prove of randomness of the noise, is to calculate the correlation functions. This will give us the stochastic properties of the noise. If the correlation vanishes for times longer than the correlation time a variable can be considered random [Ma 1985].

$$C(t) \begin{cases} = 0, & t > \tau \\ \neq 0, & t < \tau \end{cases} \tag{7.20}$$

If there is a τ such that this holds then it defines the correlation time. Let us calculate the correlation function for the noise:

$$\begin{aligned}
\langle N(t_1) N(t_2) \rangle &= \frac{\epsilon^2 a(t_1) a(t_2) H_0^6}{2} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3} \delta[k-\epsilon a(t_1) H_0] \delta[k'-\epsilon a(t_2) H_0] \\
&\times k^{-3/2} (k')^{-3/2} (2\pi)^3 \delta(\mathbf{k}-\mathbf{k}') && (7.21) \\
&= \frac{\epsilon^2 a(t_1) a(t_2) H_0^6}{(2\pi)^2} \int \frac{\mathrm{d}k}{k} \delta[k-\epsilon a(t_1) H_0] \delta[k-\epsilon a(t_2) H_0] && (7.22)
\end{aligned}$$

We apply the second delta function to the integral, and use the scaling identity for the remaining delta function $\delta(cx) = \delta(x)/|c|$:

$$\begin{aligned}\langle N(t_1)N(t_2)\rangle &= \frac{\epsilon a(t_1)H_0^5}{(2\pi)^2}\delta[\epsilon H_0(a(t_2)-a(t_1))] = \frac{a(t_1)H_0^4}{(2\pi)^2}\delta[a(t_2)-a(t_1)] \\ &= \frac{H_0^4}{(2\pi)^2}\delta(e^{H_0\Delta t}-1) = \frac{H_0^3}{(2\pi)^2}\delta(\Delta t) = \frac{H_0^3}{4\pi^2}\delta\left(t_2-t_1\right) \end{aligned} \tag{7.23}$$

We find the noise correlator is a delta function for which the property (7.20) definitely holds. However, this derivation depends on the average procedure; where we used a sharp cut-off between long and short wavelengths one should have introduced a smooth window function. This leads to the result that the correlation time is $\tau \sim H_0^{-1}$ [Winitzki and Vilenkin 2000]. This results sets the time scale for which his formalism is valid.

Often, the noise term is written as normalized white noise with a model dependent coefficient. In this case:

$$N(t) = \frac{H_0^{3/2}}{2\pi}\xi(t) \tag{7.24}$$

$$\langle\xi(t)\rangle = 0, \qquad \langle\xi(t),\xi(t')\rangle = \delta(t-t') \tag{7.25}$$

For slow-roll models where the potential is flat when the field is small ($V(\phi) \to V_0$ for $\phi \to 0$), the Langevin equation becomes

$$\dot{\phi} = -\frac{1}{3H_0}V_{,\phi} + \frac{H_0^{3/2}}{2\pi}\xi(t) \tag{7.26}$$

With these properties satisfied, equation (7.8) corresponds to the stochastic differential equation which describes Brownian motion known as the Langevin equation. We have assumed a small-field model in this derivation, in which $H \approx H_0$, but it can similarly be done for models in which $H = H(\phi)$ is allowed to vary. In this case it is useful to write $\ln a = \int \mathrm{d}t\, H(t)$ as the independent variable in stead of time. The Langevin equation (7.8) becomes [Starobinsky 1986]

$$\frac{\partial\phi_{\mathrm{CG}}}{\partial\ln a} = H^{-1}\left(v(\phi_{\mathrm{CG}}) + N\right) \tag{7.27}$$

7.2 Fokker-Planck Equation

An important tool in the stochastic approach to inflation is the probability distribution $P_c(\phi, t)$. It can be interpreted in two ways. In the first place it describes for a field on a comoving worldline (fixed $\mathbf{x}$) the probability of finding the field ϕ at time t. Secondly this probability is a measure of the amount of comoving volume occupied by field ϕ.

The Fokker-Planck equation can be derived from the Langevin equation; we will follow the derivation found in [Ma 1985]. In the following we are only interested in the behaviour of the course grained field, such that we write $\phi_{\mathrm{CG}} \equiv \phi$.

Since we have shown the noise is uncorrelated for times longer than H^{-1} it automatically means the distribution after a time step longer then H^{-1} is Gaussian. Our first objective is to find the variance and the average.

Starting from the Langevin equation (7.8) we are interested in the change of the field at $t = \Delta t$, much larger then correlation length $\tau = H^{-1}$.

$$\Delta\phi = \int_0^{\Delta t} \mathrm{d}t\,[v(t) + N(t)] \tag{7.28}$$

Repeating this experiment an infinite times we find for the average

$$\langle \Delta\phi \rangle = v\Delta t \tag{7.29}$$

The variance is given by

$$\sigma^2 \equiv \langle \Delta\phi^2 \rangle - \langle \Delta\phi \rangle^2 = \left\langle \left(\int_0^{\Delta t} \mathrm{d}t\,[v(t) + N(t)] \right)^2 \right\rangle - \left\langle \int_0^{\Delta t} \mathrm{d}t\,[v(t) + N(t)] \right\rangle^2 \tag{7.30}$$

$$= \iint \mathrm{d}t\,\mathrm{d}t'\ \langle N(t)N(t') \rangle = \int_0^{\Delta t} \mathrm{d}t\,\frac{H_0^3}{(2\pi)^2} = \frac{H_0^3}{(2\pi)^2}\Delta t \tag{7.31}$$

In these steps we made use of equation (7.23). Note that this result agrees with equation (5.6), which was calculated using a quantum field theory approach.

The distribution of the field deviation is given by the following normal distribution:

$$P_{\mathrm{dev}}(\Delta\phi) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\Delta\phi - v\Delta t)^2}{2\sigma^2} \right] \tag{7.32}$$

We are interested in the distribution of the field, rather then the distribution of the deviation. We denote the field distribution by P_c, for comoving. Initially, the field distribution is given by $P_c(\phi', t')$. After a time step this distribution is convolved with the deviation distribution. The new values of the field and time are ϕ and t.

$$P_c(\phi, t) = \int \mathrm{d}\phi' P_{\mathrm{dev}}(\phi - \phi') P_c(\phi', t') \tag{7.33}$$

The deviation distribution depends on the time-step. We can differentiate equation (7.32) w.r.t. Δt. Use $\frac{\partial\sigma}{\partial\Delta t} = \frac{\sigma}{2\Delta t}$.

$$\frac{\partial P_{\mathrm{dev}}}{\partial \Delta t} = \frac{P_{\mathrm{dev}}}{2\sigma^2\Delta t}\left(\Delta\phi^2 - v^2\Delta t^2 - \sigma^2 \right) \tag{7.34}$$

This equals the following combination[1]

$$-v\frac{\partial P_{\mathrm{dev}}}{\partial\Delta\phi} + \frac{1}{2\sigma^2\Delta t}\frac{\partial^2 P_{\mathrm{dev}}}{\partial\Delta\phi^2} = \frac{P_{\mathrm{dev}}}{2\sigma^2\Delta t}\left[2v\Delta t\Delta\phi - 2v^2\Delta t^2 + (\Delta\phi - v\Delta t)^2 - \sigma^2 \right] \tag{7.35}$$

$$= \frac{P_{\mathrm{dev}}}{2\sigma^2\Delta t}\left(\Delta\phi^2 - v^2\Delta t^2 - \sigma^2 \right) = \frac{\partial P_{\mathrm{dev}}}{\partial \Delta t} \tag{7.36}$$

So

$$\frac{\partial P_{\mathrm{dev}}}{\partial \Delta t} = -v\frac{\partial P_{\mathrm{dev}}}{\partial\Delta\phi} + D\frac{\partial^2 P_{\mathrm{dev}}}{(\partial\Delta\phi)^2} \tag{7.37}$$

[1] There is a minus sign missing in [Ma 1985]

Here we introduced the diffusion coefficient

$$D \equiv \frac{\sigma^2}{2\Delta t} \tag{7.38}$$

We can now plug this in equation (7.33):

$$\frac{\partial P_c}{\partial t} = \int \mathrm{d}\phi' \left[v \frac{\partial P_{\mathrm{dev}}}{\partial \Delta\phi} + D \frac{\partial^2 P_{\mathrm{dev}}}{(\partial \Delta\phi)^2} \right] P_c(\phi', t') \tag{7.39}$$

$$= \partial_\phi \int \mathrm{d}\phi' v P_{\mathrm{dev}}(\phi - \phi') P_c(\phi', t') + \partial_\phi^2 \int \mathrm{d}\phi' D P_{\mathrm{dev}}(\phi - \phi') P_c(\phi', t') \tag{7.40}$$

It is only valid on time scales $\Delta t > H^{-1}$, but it is convenient to write it in continuous form. Since in a time step $\sim H_0^{-1}$ during slow roll the field does not change to much: $\phi \sim \phi'$. For slow-roll we can write $v = v(\phi)$ and $D = D(\phi)$. The Fokker Planck equation reads:

$$\frac{\partial P_c}{\partial t} = \partial_\phi \left(-v(\phi) P_c + \partial_\phi (D(\phi) P_c) \right) \tag{7.41}$$

In the next section we will investigate the relevance of this formula for the global description of the universe.

7.3 Global Description of the Universe?

From the Fokker-Planck equation one can determine the distribution function $P_c(\phi, t)$ and its evolution, once suitable initial conditions are specified[2]. Typical examples for initial conditions are $P_c(\phi, 0) = \delta(\phi)$ for models which the inflaton is initially in a false vacuum, and $P_c(\phi, 0) = \delta(\phi - \phi_0)$ for chaotic models, in which the inflaton field starts out ‘up the hill’ (see figure 5.1). From the obtained distribution function we can read off the fraction of comoving volume that is in a state ϕ at time t. Apparently a decent global description of the universe!

However, the comoving distribution function does not give us much information regarding the presence of eternal inflation. It can be seen that at late times, the comoving distribution will converge to a delta-peak located at the value ϕ where inflation has ended, even for eternal inflation. However, the larger part of the *physical* volume will still be inflating in the case of eternal inflation. To take the physical volume into account one should consider the volume weighted distribution function[3]:

$$P_V = \int \mathrm{d}a\, a^3 P(\phi, a, t) \tag{7.42}$$

where $P(\phi, a, t)$ denotes the joint comoving distribution function of the field ϕ and scale factor a. This distribution function can be interpreted as the fraction of the comoving volume where the field has value ϕ and the scale factor has value a at an equal-t hypersurface [Winitzki 2009]. In this case the Fokker-Planck equation will be modified:

$$\partial_t P_V = \partial_\phi [-v(\phi) P_V + \partial_\phi (D(\phi) P_V)] + 3H(\phi) P_V \tag{7.43}$$

[2]There exist also time-independent solutions, such that no initial conditions are necessary. However, physical interpretation meets all kinds of difficulties. For example, the distribution function is not normalizable. On the other hand, there might be an interesting link with quantum cosmology, since the stationary solution resembles the square of the Hawking-Hartle wavefunction of the universe. See [Linde 2005], p208 for a discussion.

[3]This distribution function can be normalized, but this is not necessary for our present considerations

Both the comoving and the volume weighted distribution function and its evolution have been solved for various inflationary models in e.g. [Martin and Musso 2006]. In fact, in this reference, the back-reaction from the gauge invariant second-order corrections, see section 6.1 have been taken into account[4]. We show a plot for a inflationary model with a potential $V \propto V_0 \left[1 - \left(\frac{\phi}{\mu}\right)^2\right]$ in figure 7.1.

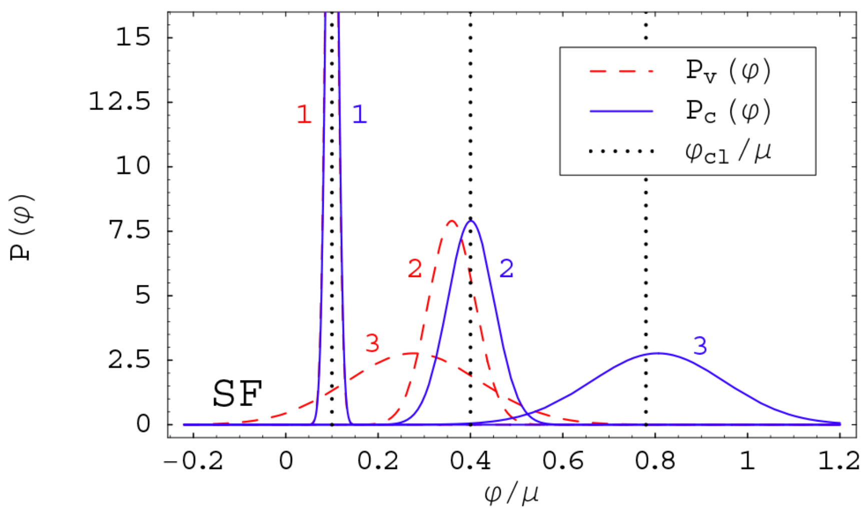

Figure 7.1 – Evolution of the field distribution for a small field model $V \propto 1 - \left(\frac{\phi}{\mu}\right)^2$. The initial value of the field is at $\phi_{\rm in}/\mu = 10^{-5}$. The numbers 1, 2 and 3 indicate three consequetive snapshots. Taken from Martin and Musso [2006]

Basically the evolution of the comoving distribution function (blue line) is as follows. The distributions starts out as a delta function at $\phi_{\rm in}/\mu = 10^{-5}$. Then the average position of the field rolls slowly towards the true vacuum while the dispersion grows linearly with time. Eventually, the probability of finding a field with a value of ϕ near the false vacuum becomes vanishingly small.

The volume weighted distribution (red line) on the other hand has a very different evolution. When the conditions for eternal inflation are met, see eqn. (5.2), the average of the volume weighted distribution will move towards the region of the initial false vacuum. At late times an increasingly large fraction the physical volume will be occupied by a field which is in the false vacuum.

An important issue regarding the volume weighted distribution function and the Fokker-Planck equation is that the physical interpretation depends on the choice of the time-slicing. In this section we have considered the distribution function of the inflaton field on equal-t hypersurfaces, where t is cosmic time. However, we could also have chosen a different time

[4] In the article [Martin and Musso 2006] the back-reaction was taken into account by an expansion in the noise up to second order.

coordinate, by a gauge transformation. Consider the following gauge choice:

$$\mathrm{d}T \equiv H(\phi)\mathrm{d}t, \qquad T = \int H\,\mathrm{d}t = \ln a \tag{7.44}$$

This is known as *e-folding time* since it measures the number of e-folds. In these coordinates, the comoving distribution function and volume weighted distribution function are the same [Winitzki 2009]. In this way, the presence of eternal inflation is not clear from the distribution function. This does not imply however, that the presence of eternal inflation, see [Winitzki 2009] p62. The question: How much of the physical universe is occupied by the field with value ϕ *does* depend on the gauge choice. In this sense, a global description of the universe by means of the volume-weighted distribution function is ambiguous.

CHAPTER 8

Thermodynamic approach

There seems to be an interesting correspondence between general relativity and thermodynamics. For example, thermodynamic laws have been generalized to systems which possess horizons [Gibbons and Hawking 1977] and Jacobson [1995] showed the Einstein equations can be written as a thermodynamic equation of state. Thermodynamics is also relevant for inflation; for example one can define an entropy and temperature for De Sitter space. In this chapter we will focus on quasi-De Sitter space and we show how in this case the Einstein equations are related to the first law of thermodynamics.

This approach will be useful in our investigation related to back-reaction. After we have established the thermodynamic law which is applicable in the quasi-De Sitter system we will include perturbations and consider their impact on the background geometry. Previously this has been investigated in Frolov and Kofman [2003] and recently in Galvez Ghersi et al. [2011]. We will include a discussion on the behaviour in the eternal inflation regime.

8.1 De Sitter Thermodynamics

As mentioned before, the thermodynamic laws have been generalized to systems possessing horizons. De Sitter space is such a system in which an event horizon is present. We will focus on thermodynamic properties of De Sitter space in this section. The main reference that has been used for this section is [Spradlin et al. 2001].

Geometrical Entropy

Entropy can loosely be defined as a measure of uncertainty in the actual internal configuration of a system[1]. This definition can also be applied to systems with horizons such as black holes and De Sitter space since they hide information from certain observers; there is an uncertainty in

[1]Suppose a system can be specified by some macroscopic parameters. The entropy is proportional to the number of internal configurations corresponding to this macroscopic system, and therefore inversely proportional to the probability of finding the actual internal configuration; In this sense, the entropy is proportional to the uncertainty in finding the actual internal state.

the actual internal configuration behind the horizon. In this sense one can associate an entropy to an event horizon.

Before we discuss De Sitter space we will consider a black hole, for which a thermodynamic interpretation is better understood. A black hole emits thermal radiation proportional to the surface gravity.

$$T = \frac{\kappa}{2\pi} \tag{8.1}$$

For a black hole the surface gravity is given by $\kappa = \frac{1}{4GM}$, where M is the black hole mass. The temperature at which a black hole emits radiation is then

$$T = \frac{1}{8\pi GM} \tag{8.2}$$

Now we can use the first law of thermodynamics:

$$T\mathrm{d}S = \mathrm{d}M \tag{8.3}$$

to find the entropy:

$$\int \mathrm{d}S = \int \frac{\mathrm{d}M}{T} = 8\pi G \int \mathrm{d}M\, M \tag{8.4}$$

Choosing the boundary conditions such that a black hole of zero mass has zero entropy we find:

$$S = 4\pi GM^2 = \frac{\pi R^2}{G} = \frac{A}{4G} \tag{8.5}$$

where we have used that the black hole event horizon is given by $R = 2GM$. Equation (8.5) is the famous Bekenstein-Hawking entropy law [Bekenstein 1973], [Hawking 1975]. Since the entropy was defined based on a geometrical property, i.e. the event horizon, we will refer to this entropy as the *geometrical* entropy. This entropy is not defined by counting the internal states although it would make sense if these two ways of determining the entropy would coincide. This is however not a well establish equivalence; only for some special cases of black holes has this equivalence been found.

It is assumed that the Bekenstein-Hawking entropy law holds for systems possessing horizons in general, including De Sitter space. However in De Sitter space the interpretation of the entropy is not clear. A black hole has a well defined position and and a finite size, which all observers can agree on. Therefore one can image that this object can have a well defined set of internal states to which we can associate an entropy. However, the De Sitter horizon depends on the observer and moreover, the information that is hidden from an observer corresponds to the region *outside* the horizon, which is possibly infinite in size. What microstates can we attribute to the De Sitter horizon? Let us continue to better understand the problems and possibilities of De Sitter thermodynamics.

Coordinate Systems

In the limit that the slow-roll parameters are zero the universe has the properties of De Sitter space. In this case the energy content is effectively given solely by a positive cosmological constant.

This space is actually larger then what our flat FRW-coordinates (3.12) can describe[2] We will write the coordinates here again for convenience:

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2 d\Omega^2\right) \tag{8.7}$$

The part of De Sitter space that is described by these coordinates can be seen from the Penrose diagram in figure 8.1 (left). However for our purposes we do not need all of De Sitter space, we only need the part of De Sitter space in which the universe is expanding.

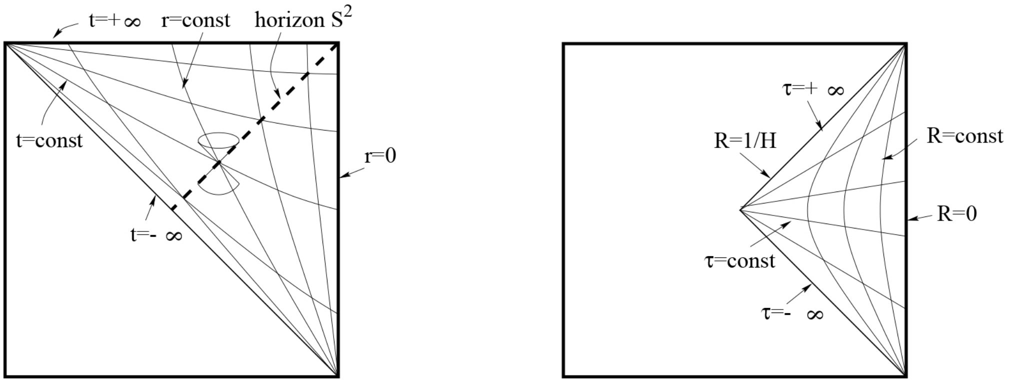

Figure 8.1 – Penrose diagram of de Sitter spacetime in time-dependent FRW coordinates (left) and static coordinates (right). A Penrose diagram is a convenient way to visualize the global causal structure of a spacetime, see e.g. Carroll [2004]. Light rays move at 45°. The full square corresponds to the full De Sitter space. The blank area in both figures corresponds to the part of De Sitter space which is not described by the coordinates used [Frolov and Kofman 2003]

Another interesting coordinate system on De Sitter space is the *static* coordinate system:

$$ds^2 = -(1 - H^2R^2)d\tau^2 + \frac{dR^2}{1 - H^2R^2} + R^2 d\Omega^2 \tag{8.8}$$

This coordinate system is also not complete, see the right image of figure 8.1. Note that this metric does not depend on time. However, it is not possible to define a global time-like Killing vector, such that we cannot define a conserved energy of De Sitter spacetime. The coordinates (8.8) show the appearance of a coordinate singularity which defines the De Sitter horizon: $R = H^{-1}$. It is similar to the black hole horizon, but now it is as if the observer is surrounded by a black hole.

De Sitter Entropy

We found that the De Sitter horizon is located at $R = H^{-1}$ in static coordinates. Just as in the case for a black hole we can define a surface gravity of the event horizon: $\kappa = H$ [Gibbons and

[2]The full De Sitter space can be represented as a 4-dimensional hypersurface in a 5-dimensional Minkowski space, with coordinates $(z_0, \dots, z_4)$. This hypersurface is defined through

$$-z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = H^{-2} \tag{8.6}$$

.

Hawking 1977]. Freely falling observers in De Sitter space will feel as if they are in a thermal bath of particles at temperature $T = \frac{H}{2\pi}$.

Because De Sitter space is associated with a temperature, we would like to follow the story of the beginning of this chapter and see if we can derive the entropy-law (8.5) in a similar fashion for De Sitter space. However, unlike in the case for a black hole, there is no infinitesimal mass-energy $\mathrm{d}M$ over which we can integrate to obtain the entropy. In De Sitter there is only a cosmological constant, which corresponds to a constant energy density throughout space. There is however a way to arrive at the entropy law by considering a generalization of the Sitter space, in which a black hole is present. In this way, at least for 3 dimensions, the Bekenstein-Hawking entropy law can be derived, see [Spradlin et al. 2001].

However we will take an another approach, based on the Misner-Sharp mass; a local mass function based on spherical symmetric spacetimes [Misner and Sharp 1964]. We will show that the entropy indeed is proportional to area of the De Sitter horizon. The approach differs from method described for the black hole where we varied the mass and therefore the horizon and temperature of the black hole. Instead we consider a constant horizon and temperature, and determine the entropy of a sphere around an observer as a function of the radius. At radii smaller than the horizon, the entropy scales with volume. However, exactly at the horizon, the entropy scales with area.

A general spherical metric can be written as:

$$\mathrm{d}s^2 = g_{ab}(t,r)\mathrm{d}x^a \mathrm{d}x^b + \rho^2(t,r)\mathrm{d}\Omega^2 \tag{8.9}$$

The Misner-Sharp mass is defined through:

$$1 - \frac{2GM(t,r)}{\rho(t,r)} = (\nabla\rho)^2 \tag{8.10}$$

Where $M(t,r)$ is the Misner-Sharp mass, which is a measure of the energy within a sphere of radius r at time t. ∇ corresponds to the covariant derivative with respect to the metric g_{ab}. The previous equation can be rewritten as:

$$M(t,r) = \frac{\rho}{2G}\left[1 - (\nabla\rho)^2\right] \tag{8.11}$$

For the static metric (8.8) this becomes[3]:

$$M(r) = \frac{R^3 H^2}{2G} \tag{8.14}$$

[3] We can also arrive at relation (8.14) by integrating over a volume with the energy density given by the vacuum energy:

$$M(r) = \int \mathrm{d}^3\mathbf{x}\,\sqrt{g}\,\rho_{\text{vac}} = \frac{4\pi}{3}R^3\rho_{\text{vac}} \tag{8.12}$$

where g is the determinant of the 3-dimensional spatial metric: $g_{ij} = \delta_{ij}$ in cartesian coordinates. The constant vacuum energy density is given by

$$H^2 = \frac{8\pi G}{3}\rho_{\text{vac}} \Rightarrow \rho_{\text{vac}} = \frac{3H^2}{8\pi G} \tag{8.13}$$

We now apply the thermodynamic first law:

$$\int \mathrm{d}S = \int \frac{\mathrm{d}M}{T} = \frac{2\pi}{H} \int \frac{\mathrm{d}M}{\mathrm{d}R} \mathrm{d}R = \frac{3\pi H}{G} \int \mathrm{d}R\, R^2 \tag{8.15}$$

We define the entropy to be zero when the radius and thus the Misner-Sharp mass is zero.

$$S = \frac{\pi H}{G} R^3 \tag{8.16}$$

At the horizon, $R = H^{-1}$ the entropy becomes:

$$S = \frac{\pi}{GH^2} = \frac{A}{4G} \tag{8.17}$$

The entropy in general scales with volume, according to equation (8.16), but exactly on the horizon the entropy is given by the area in stead of the volume.

8.2 Quasi-De Sitter Thermodynamics

The fact that entropy has an application to systems possessing an horizons led to the idea that there is more to the thermodynamic picture. In [Gibbons and Hawking 1977] it was shown that thermodynamic relations apply to such systems. Jacobson [1995] showed that the Einstein equations can be written as a thermodynamic equation of state. In this section we show how to implement the first law in the context of inflation which was done by Frolov and Kofman [2003]

The first law of thermodynamics states that the change of internal energy of the system equals heat that flows to the system plus work that is done on the system, $\mathrm{d}E = \mathrm{d}Q + \mathrm{d}W$. For reversible changes, we have $\mathrm{d}Q = T\mathrm{d}S$, $\mathrm{d}W = -P\mathrm{d}V$, and we can write down the *fundamental thermodynamic relation*:

$$T\mathrm{d}S = \mathrm{d}E + P\mathrm{d}V \tag{8.18}$$

We will discuss various thermodynamic quantities which have an interpretation in quasi-De Sitter space in the next sections.

8.2.1 The System

The thermodynamic system in the De Sitter context corresponds to the spacetime behind the horizon from the perspective of the observer at the origin, since this is the region which is associated with the uncertainty or lack of information due to the horizon.

We will work with a strictly spherical symmetric space, also for perturbations. Later we show this is equivalent to choosing longitudinal gauge. The reason for this choice is that in spherical symmetric systems it is possible to have a consistent definition of energy and energy flow, even in non-static spacetime. This is not possible in general, since it is not possible to define energy in terms of a conserved Noether charge associated with time-translation invariance. The spherically symmetric metric is of the form

$$\mathrm{d}s^2 = g_{ab}(\tau, r)\mathrm{d}x^a \mathrm{d}x^b + \rho^2(\tau, r)\mathrm{d}\Omega^2 \tag{8.19}$$

Here, g_{ab} is the two-dimensional metric associated with the coordinates $x^a = (\tau, r)$, $\rho(\tau, r)$ determines the physical radius and $\mathrm{d}\Omega^2$ is the line element on a unit 2-sphere. The unperturbed metric is defined as:

$$\mathrm{d}s^2 = a^2(-\mathrm{d}\tau^2 + \mathrm{d}r^2) + \rho^2 \mathrm{d}\Omega^2 \tag{8.20}$$

The field fluctuations can be expanded in spherical harmonics. Since expansion rapidly increases the size of the fluctuations it can be shown that in the spherical harmonics the $l = 0$ mode will dominate. This mode preserves the spherical symmetry and thereby simplifies the metric perturbation. It is not possible to define gauge-invariant variables in this case. However, it is possible to make a gauge choice which corresponds to the Newtonian gauge. In this case the metric becomes:

$$\mathrm{d}s^2 = a^2 \left[-(1+2\Phi)\mathrm{d}\tau^2 + (1-2\Phi)(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2)\right] \tag{8.21}$$

Apparent Horizon

Unlike De Sitter space there is not in general an event horizon associated to quasi-De Sitter space, in particular if the universe ends inflation at some point. To see if an event horizon is present one needs to know the entire future for some observer. In stead we will work with the apparent horizon, which can be defined at any instant.

In case of an expanding universe the apparent horizon is the same as the marginally trapped surface which can loosely be defined as follows: Consider a sphere at comoving radius r from the observer. Radial light-rays leaving the sphere form new spheres in both directions. Since the universe expands there is a radius r_{AH} where the physical area of both spheres start to increase; this defines the marginally trapped surface.

The in- and outgoing light-rays have solutions: $r_\pm = r_0 \pm \tau$ and physical area $A = 4\pi a^2 r_\pm^2$. The area changes according to

$$\partial_\tau A_\pm = 8\pi a^2 r_\pm^2 \left(\mathcal{H} \pm \frac{1}{r_\pm}\right) \tag{8.22}$$

The surface is trapped when $\partial_\tau A_\pm > 0$ for both directions of the light-rays. This requires $\mathcal{H} > r_\pm^{-1}$. We take the limit of the distance travelled by the light-rays to zero and change the inequality sign to an equal sign to find the distance at which the surface becomes marginally trapped. This is at $r = \mathcal{H}^{-1}$, which is already familiar to us.

Now we generalize this argument for a spherical symmetric metric, where $A = 4\pi\rho^2(\tau, r)$. The variation of the area along the paths of the in- and outward light-rays, parametrized by the time is given by

$$\frac{DA_\pm}{\mathrm{d}\tau} = k_\pm^a \nabla_a A = 8\pi\rho k_\pm^a \nabla_a \rho \tag{8.23}$$

where $k^a = \partial_\tau x^a(\tau)$ is the tangent vector along the paths of the light rays, and therefore is null ($k^a k_a = 0$) for either direction.

The condition for the marginally trapped surface is $\frac{DA_\pm}{\mathrm{d}\tau} = 0$. Since k^a is null this implies $\nabla_a \rho$ should be null. Therefore the condition for the apparent horizon is:

$$f \equiv (\nabla\rho)^2 = 0 \tag{8.24}$$

Where ∇ is the covariant derivative for the two-dimensional metric.

We find for the unperturbed metric (see section 3.2):

$$r^{(0)}_{\text{hor}} = \mathcal{H}^{-1} = -\tau(1-\epsilon) \tag{8.25}$$

Adding the perturbations yields:

$$r^{(1)}_{\text{hor}} = -\tau\left[1-\epsilon+2\Phi-\tau(\Phi_{,\tau}-\Phi_{,r})\right] \tag{8.26}$$

8.2.2 Entropy

From equation (8.5) we see that the variation of the entropy is proportional to the change in the area corresponding to the horizon radius:

$$\mathrm{d}A = k^a\nabla_a A \mathrm{d}\lambda \tag{8.27}$$

where k^a is the tangent vector at the horizon. For the unperturbed metric, if we take τ as the parameter, $k^a = \begin{pmatrix} 1 \\ -(1-\epsilon) \end{pmatrix}$ such that the variation is

$$\mathrm{d}A = \frac{8\pi\mathcal{H}\epsilon}{H^2}\mathrm{d}\tau \tag{8.28}$$

So in one e-fold, $\Delta\tau = \mathcal{H}^{-1}$, such that the area increases by $\Delta A = \frac{8\pi\epsilon}{H^2}$. However, to have a similar relation, but taking spatial perturbations into account, we need a different definition for the tangent vector at the horizon. We define a vector which will be tangent along the surface of the apparent horizon as follows[4]:

$$\zeta^a = -\epsilon^{ab}\nabla_b f \tag{8.29}$$

This tangent vector depends on the parameter used to parametrize the path of the horizon. We find $\zeta^a = B(\tau)k^a$, where $B(\tau) = 2H^2\mathcal{H}^{-1}(\tau)$ in this case. This parameter will cancel in the first law, but we keep it here in order to better understand the thermodynamic quantities.

$$\mathrm{d}S = \frac{2\pi}{G}\zeta^a\rho\nabla_a\rho(B^{-1}\mathrm{d}\lambda) \tag{8.30}$$

We find:

$$\mathrm{d}S^{(0)} = \frac{2\pi\epsilon\mathcal{H}}{GH^2}\,\mathrm{d}\tau, \qquad \mathrm{d}S^{(1)} = \frac{2\pi}{GH^2}\mathcal{H}^{-1}(\Phi_{,\tau\tau}-2\Phi_{,\tau r}+\Phi_{,rr})\mathrm{d}\tau \tag{8.31}$$

Since the physical position of the apparent horizon increases with time the entropy will increase accordingly.

8.2.3 Temperature

Uniformly accelerated observers in Minkowski space measure a temperature associated with Hawking-radiation proportional to their acceleration [Unruh 1976]. This is basically due to the fact that the observer does not have access to all of the spacetime, because of the existence of a

[4] for clarity we added a minus sign to this definition w.r.t. Galvez Ghersi et al. [2011]

(Rindler) horizon. Black holes and De Sitter space also have horizons associated which prevent an observer from measuring the complete state of the system. In this case the acceleration is replaced by the *surface gravity*, κ. For static black holes, the surface gravity is the acceleration of a point which is near the horizon, measured by an observer at spatial infinity, and for static De Sitter, it is similar, but measured by the observer at the origin. The temperature associated with horizon is

$$T = \frac{\kappa}{2\pi} \tag{8.32}$$

where $\kappa = H, (4M)^{-1}$, for respectively De Sitter space [Figari et al. 1975] [Gibbons and Hawking 1977] and a Schwartzschild black hole.

The surface gravity for dynamic, but spherically symmetric, spacetimes can be defined as [Hayward 1998]:

$$\kappa = \frac{1}{2}|\Box\rho| \tag{8.33}$$

Unperturbed

$$\mathcal{T}^{(0)} = \frac{H}{2\pi}\left(1 - \frac{\epsilon}{2}\right), \qquad \mathcal{T}^{(1)} = -\frac{H}{2\pi}\left[\Phi + \frac{\Phi_{,\tau} + \Phi_{,r}}{aH} + \frac{\Phi_{,\tau\tau} - \Phi_{,rr}}{2(aH)^2}\right] \tag{8.34}$$

8.2.4 Energy

We are interested in the change of the energy of the system. Energy in an expanding universe is ill-defined since there is no time translational symmetry. However, for spherically symmetric spacetimes it is possible to have a local notion of energy: the *Misner-Sharp* mass, see equation (8.11)

We can write the change of energy as the variation of this mass along the path of the horizon. We cannot say much about the total energy of the system outside the horizon but we assume we have $\delta E_{\text{out}} = -\delta E_{\text{in}}$, where δE_{out} is the change of energy of the system.

$$\delta E_{out} = -k^a \nabla_a m_{\text{MS}} \mathrm{d}\tau \tag{8.35}$$

To get something that resembles the Einstein equations we want to relate this to the energy momentum tensor. We could use the 2-dimensional Einstein equations to relate the Misner-Sharp mass to the energy momentum tensor, but this would defy the purpose of deriving the Einstein equations. In stead we define a conserved current related to the energy-momentum tensor that represents energy-flow in such a way that we can define the energy flux through the horizon as:

$$\delta E = j^\alpha \mathrm{d}\Sigma_\alpha \tag{8.36}$$

where j^α is the energy-flow which we will define shortly and $\mathrm{d}\Sigma_\alpha$ is the 3-volume element related to the infinitesimal change of the horizon. Note that we switched to 4 dimensional vectors.

The conserved current which represents the energy-flow has to be based on a divergence free vector. We cannot use time-like Killing vector since the patch of De Sitter space we are interested in has no time-like killing vector. Nevertheless, for spherical symmetric spaces Kodama [1980] proved the existence of a time-like vector which is conserved.

$$\xi^\alpha \equiv (\xi^a, 0, 0), \qquad \xi^a \equiv -\epsilon^{ab}\nabla_b\rho, \qquad \nabla_\alpha\xi^\alpha = 0 \tag{8.37}$$

This vector is null on the horizon. Furthermore, Kodama showed this vector can be related to the following conserved current:

$$K^\alpha = G^{\alpha\beta}\xi_\beta, \qquad \nabla_\alpha K^\alpha = \nabla_\alpha \left(G^{\alpha\beta}\xi_\beta\right) = 0 \tag{8.38}$$

Note that this result is *not* related to the Bianchi identities. When we write $T^{\alpha\beta}$ in stead of $G^{\alpha\beta}$ we know this current will be conserved as well. We will *define* the energy flow

$$j^\alpha \equiv -T^{\alpha\beta}\xi_\beta \tag{8.39}$$

The infinitesimal 3-volume element is given by

$$\mathrm{d}\Sigma_\alpha = (4\pi\rho^2 k^b \epsilon_{ab}, 0, 0)\mathrm{d}\tau \tag{8.40}$$

Using these definitions we find:

$$\delta E^{(\epsilon)}_{\text{out}} = -\frac{4\pi\mathcal{H}}{H^3}\left(\epsilon V - \frac{(\phi_{,\tau})^2}{a^2}\right)\mathrm{d}\tau \tag{8.41}$$

$$\delta E^{(1)}_{\text{out}} = -\frac{4\pi\mathcal{H}}{H^3}\left[\frac{V}{(aH)^2}\left(\Phi_{,\tau\tau} - 2\Phi_{,\tau r} + \Phi_{,rr}\right) - \frac{2\phi_{,\tau}}{a^2}\left(\delta\phi_{,\tau} - \delta\phi_{,r}\right)\right]\mathrm{d}\tau \tag{8.42}$$

8.2.5 Pressure and Volume

We identify the pressure of the system with half of the trace of the energy-momentum tensor:

$$\mathcal{P} = \frac{T^a_a}{2} \tag{8.43}$$

This statement will be justified in the next section. The change in volume is:

$$\mathrm{d}\mathcal{V}_{out} = -k^a\nabla_a\left(\frac{4\pi\rho^3}{3}\right) = -4\pi\rho^2 k^a \nabla_a \rho \tag{8.44}$$

For the system under consideration:

$$\mathcal{P}^{(0,\epsilon)} = -V, \qquad \mathcal{P}^{(1)} = -V_{,\phi}\delta\phi \tag{8.45}$$

$$\mathrm{d}\mathcal{V}^{(0,\epsilon)} = \frac{4\pi\epsilon\mathcal{H}}{H^3}\mathrm{d}\tau, \qquad \mathrm{d}\mathcal{V}^{(1)} = \frac{4\pi}{H^3}\frac{\Phi_{,\tau\tau} - 2\Phi_{,\tau r} + \Phi_{,rr}}{aH}\mathrm{d}\tau \tag{8.46}$$

8.3 Fundamental Thermodynamic Relation

Putting all the thermodynamic quantities together we can evaluate the thermodynamic relation. First we will apply this to the general spherical metric and then to the specific case of quasi-De Sitter.

8.3.1 General Spherical Metric

The thermodynamic quantities are listed below.

$$\mathcal{T} = \frac{|\Box\rho|}{4\pi} \tag{8.47}$$
$$\mathrm{d}S = \frac{2\pi}{G}B^{-1}\mathrm{d}\lambda\zeta^a\rho\nabla_a\rho \tag{8.48}$$
$$\delta E = -B^{-1}\mathrm{d}\lambda\zeta^a\nabla_a m \tag{8.49}$$
$$P = \frac{T}{2} \tag{8.50}$$
$$\mathrm{d}\mathcal{V} = -4\pi\rho^2(B^{-1}\mathrm{d}\lambda)\zeta^a\nabla_a\rho \tag{8.51}$$

The fundamental thermodynamic relation becomes

$$\mathcal{T}\mathrm{d}S = \delta E + P\mathrm{d}\mathcal{V}$$
$$\frac{1}{2G}\rho\zeta^a|\Box\rho|\nabla_a\rho = -\zeta^a\nabla_a M - 2\pi T\rho^2\zeta^a\nabla_a\rho \tag{8.52}$$

8.3.2 Derivation from GR

We can derive (8.52) as well by rewriting the Einstein equations. The 2d-Einstein equations of the general spherical metric are as follows:

$$G_{ab} = 2\rho^{-1}\left[g_{ab}\Box\rho - \nabla_a\nabla_b\rho\right] + \rho^{-2}g_{ab}\left[(\nabla\rho)^2 - 1\right] = 8\pi G T_{ab} \tag{8.53}$$

This can be written in terms of the Misner-Sharp mass, $m = \frac{\rho}{2G}\left[1 - (\nabla\rho)^2\right]$

$$g_{ab}\Box\rho - \nabla_a\nabla_b\rho - g_{ab}\frac{Gm}{\rho^2} = 4\pi GRT_{ab} \tag{8.54}$$

Using that $g_{ab}g^{ab} = 2$, we find the trace of this equation is:

$$\Box\rho - 2\frac{Gm}{\rho^2} = 4\pi G\rho T \tag{8.55}$$

Subtracting half of the trace times g_{ab} from equation (8.54) leads to:

$$-\nabla_a\nabla_b\rho + \frac{1}{2}g_{ab}\Box\rho = 4\pi G\rho(T_{ab} - Tg_{ab}) + 2\pi G\rho T g_{ab} \tag{8.56}$$

Next we contract this equation with $-\frac{\rho}{G}\zeta^a\nabla^b\rho$:

$$\frac{\rho}{2}\zeta^a\left[\nabla_a(\nabla\rho)^2 - \Box\rho\nabla_a\rho\right] = -\zeta^a 4\pi\rho^2(T_{ab} - Tg_{ab})\nabla^b\rho - 2\pi T\rho^2\zeta^a\nabla_a\rho \tag{8.57}$$

We can simplify this, by writing the first term on the right hand side in terms of (∇M). To see this, first we subtract the trace multiplied by g_{ab} from equation (8.54):

$$-\left[\nabla_a\nabla_b\rho - \frac{Gm}{\rho^2}g_{ab}\right] = 4\pi G\rho(T_{ab} - Tg_{ab}) \tag{8.58}$$

The derivative of the geometrical Misner-Sharp mass can be written:

$$\nabla_a m = \frac{1}{2G}\nabla_a \left[\rho - \rho(\nabla\rho)^2\right] \tag{8.59}$$
$$= \frac{1}{2G}\left[\nabla_a\rho - (\nabla\rho)^2\nabla_a\rho - \rho\nabla_a(\nabla\rho)^2\right] \tag{8.60}$$
$$= -\frac{\rho}{2G}\left[\nabla_a(\nabla\rho)^2 - \frac{1}{\rho}\left(1-(\nabla\rho)^2\right)\nabla_a\rho\right] \tag{8.61}$$
$$= -\frac{\rho}{2G}\left[2\nabla_a\nabla_b\rho\nabla^b\rho - 2\frac{Gm}{\rho^2}\nabla_a\rho\right] \tag{8.62}$$
$$= -\frac{\rho}{G}\left[\nabla_a\nabla_b\rho - \frac{Gm}{\rho^2}g_{ab}\right]\nabla^b\rho \tag{8.63}$$

The term between the braces is precisely equal to the left hand side of equation (8.58).

$$\nabla_a m = 4\pi\rho^2\left(T_{ab} - Tg_{ab}\right)\nabla^b\rho \tag{8.64}$$

Plugging in this result into equation (8.57)

$$\frac{\rho}{2}\zeta^a\left[\nabla_a(\nabla\rho)^2 - \Box\rho\nabla_a\rho\right] = -\zeta^a\nabla_a m - 2\pi T\rho^2\zeta^a\nabla_a\rho \tag{8.65}$$

For quasi-De Sitter, $\zeta^a\nabla_a(\nabla\rho)^2$ vanishes at the horizon, and $\Box\rho < 0$. Therefore we obtain:

$$\frac{1}{2G}\zeta^a|\Box\rho|\nabla_a\rho = -\zeta^a\nabla_a m - 2\pi T\rho^2\zeta^a\nabla_a\rho \tag{8.66}$$

8.3.3 Results for Quasi-De Sitter

Now we have shown the equivalence between the thermodynamic relation and general relativity in the general spherically symmetric case we can plug-in the results for quasi-De Sitter.

Without Perturbations

The thermodynamic quantities for quasi-De Sitter in the unperturbed case evaluate to:

$$\mathcal{T} = \frac{H}{2\pi}\left(1 - \frac{\epsilon}{2}\right) \tag{8.67}$$
$$\mathrm{d}S = \frac{4\pi}{G}B^{-1}\mathrm{d}\lambda\epsilon \tag{8.68}$$
$$\delta E = -\frac{8\pi}{H}B^{-1}\mathrm{d}\lambda\left(\epsilon V - \dot\phi^2\right) \tag{8.69}$$
$$\mathcal{P} = -V \tag{8.70}$$
$$\mathrm{d}\mathcal{V} = -\frac{8\pi}{H}B^{-1}\mathrm{d}\lambda\epsilon \tag{8.71}$$

The thermodynamic relation becomes:

$$\mathcal{T}\mathrm{d}S = \delta E + \mathcal{P}\mathrm{d}\mathcal{V} \tag{8.72}$$
$$2m_{\mathrm{p}}^2H^2\epsilon = \dot\phi^2 \tag{8.73}$$
$$-2m_{\mathrm{p}}^2\dot H = \dot\phi^2 \tag{8.74}$$

The $B^{-1}d\lambda$ terms cancel. Furthermore we have multiplied both sides with $\frac{H}{8\pi}$ to obtain a familiar result: the derivative of the Friedmann equation. This is (3.9) we calculated in section 3.1. The Friedmann equation can be recovered by integrating this equation up to a constant which could be the cosmological constant. Interestingly in the thermodynamic treatment this constant is irrelevant.

Including Perturbations

Evaluating all the thermodynamic quantities for the perturbed metric (8.21) at the horizon leads to the following result.

$$\mathcal{T} = \frac{H}{2\pi}\left[1 - \Phi - \frac{\epsilon}{2} - \frac{\ddot{\Phi}}{2H^2} + \frac{\Phi''}{2(aH)^2} - \frac{3\dot{\Phi}}{2H} - \frac{\Phi'}{aH}\right] \tag{8.75}$$

$$dS = \frac{4\pi}{G}B^{-1}d\lambda\,(\epsilon + \Delta) \tag{8.76}$$

$$\delta E = -\frac{8\pi}{H}B^{-1}d\lambda\left[V(\epsilon+\Delta) - \dot{\phi}^2 - 2\dot{\phi}\left(\dot{\delta\phi} - \frac{\delta\phi'}{a}\right)\right] \tag{8.77}$$

$$\mathcal{P} = -V \tag{8.78}$$

$$d\mathcal{V} = -\frac{8\pi}{H}B^{-1}d\lambda\,(\epsilon + \Delta) \tag{8.79}$$

where

$$\Delta \equiv \frac{\ddot{\Phi}}{H^2} + \frac{\Phi''}{(aH)^2} - \frac{2\Phi_{,tr}}{aH^2} + \frac{\dot{\Phi}}{H} \tag{8.80}$$

Writing the thermodynamic relation:

$$TdS = \delta E + \mathcal{P}d\mathcal{V} \tag{8.81}$$

$$m_{\mathrm{p}}^2 H^2(\epsilon + \Delta) = \frac{1}{2}\dot{\phi}^2 + \dot{\phi}\left(\dot{\delta\phi} - \frac{\delta\phi'}{a}\right) \tag{8.82}$$

Again, the $B^{-1}d\lambda$ terms cancels. Furthermore we have multiplied both sides by $\frac{H}{16\pi}$. Upon using relation (8.73) and explicitly writing out Δ we have:

$$m_{\mathrm{p}}^2 H^2\left(\frac{\ddot{\Phi}}{H^2} + \frac{\Phi''}{(aH)^2} - \frac{2\Phi_{,tr}}{aH^2} + \frac{\dot{\Phi}}{H}\right) = \dot{\phi}\left(\dot{\delta\phi} - \frac{\delta\phi'}{a}\right) \tag{8.83}$$

which is the final result.

The thermodynamic equation relates Φ to $\delta\phi$. A way to picture the effect of the fluctuations on the geometry is the following. Consider equation (8.26). We can drop the derivative terms for super-Hubble fluctuations. This yields:

$$A = \frac{4\pi}{H^2}(1 + 2\Phi) \tag{8.84}$$

The area of the horizon will fluctuate due to both quantum fluctuations and due to slow-roll, as pictured in figure (8.2).

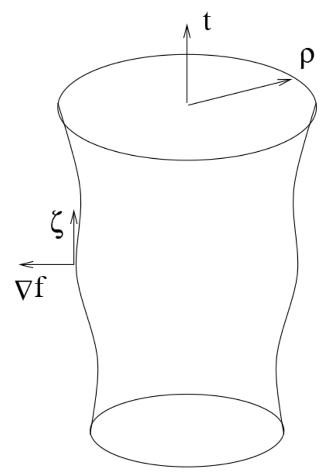

Figure 8.2 – Apparent horizon of quasi De Sitter space. It is affected by both slow-roll and quantum fluctuations. Figure taken from [Frolov and Kofman 2003]

Equivalence to GR

The thermodynamic relation which was found for the perturbations, eqn. (8.83) can also be found from the Einstein equations. This is to be expected since we have shown the two approaches are equivalent. From the $(0, 1)$ Einstein equation (4.35):

$$\Phi_{,tr} + H\Phi' = 4\pi G\dot{\phi}\delta\phi' \rightarrow \left(\dot{\Phi} + H\Phi\right)' = 4\pi G\left(\dot{\phi}\delta\phi\right)' \tag{8.85}$$

Such that we find up to a constant

$$\dot{\Phi} + H\Phi = 4\pi G\dot{\phi}\delta\phi \tag{8.86}$$

The perturbed $(0, 0)$ Einstein equation (4.34) reads

$$\frac{2}{a^2}\Delta\Phi - 6H\dot{\Phi} = 8\pi G\left(2V\Phi + \dot{\phi}\dot{\delta\phi} + V_{,\phi}\delta\phi\right) \tag{8.87}$$

where the Laplacian is given by $\Delta = \partial_r^2 + \frac{2}{r}\partial_r$. Writing $V = \frac{3H^2}{8\pi G} - \frac{\dot{\phi}^2}{2}$ and using the equation of motion $V_{,\phi} = -\ddot{\phi} - 3H\dot{\phi}$, we can write this equation as follows.

$$\frac{2}{a^2}\Delta\Phi - 6H(\dot{\Phi} + H\Phi) + 8\pi G\dot{\phi}^2\Phi = 8\pi G\left(\dot{\phi}\dot{\delta\phi} - \ddot{\phi}\delta\phi - 3H\dot{\phi}\delta\phi\right) \tag{8.88}$$

$$\frac{2}{a^2}\Delta\Phi + 8\pi G\dot{\phi}^2\Phi = 8\pi G\left(\dot{\phi}\dot{\delta\phi} - \ddot{\phi}\delta\phi\right) \tag{8.89}$$

$$\left(\frac{\Delta}{4\pi Ga^2} + \dot{\phi}_0^2\right)\Phi = \dot{\phi}_0^2\partial_t\left(\frac{\delta\phi}{\dot{\phi}_0}\right) \tag{8.90}$$

where to go from (8.88) to (8.89) eqn. (8.86) was used. Substituting the expressions (8.86) and (8.90) into the following identity leads to an expression which can be compared to the thermodynamic relation.

$$\partial_t(\dot{\phi}\delta\phi) + \dot{\phi}_0^2\partial_t\left(\frac{\delta\phi}{\dot{\phi}_0}\right) = 2\dot{\phi}_0\dot{\delta\phi} \tag{8.91}$$

Using the $(0,1)$ Einstein equation on the first term on the left hand side of the equation above we obtain

$$\partial_t(\dot{\phi}\delta\phi) = \frac{1}{4\pi G}\left(\ddot{\Phi} + H\dot{\Phi}\right) \tag{8.92}$$

Furthermore, subtracting $\frac{2}{a}$ times equation (8.85) leads to:

$$\ddot{\Phi} + \frac{\Phi''}{a^2} - 2\frac{\dot{\Phi}_{,r}}{a} + H\dot{\Phi} + \frac{2}{a^2}\left(\frac{1}{r} - \dot{a}\right)\Phi' = 8\pi G\dot{\phi}_0\left(\dot{\delta\phi} - \frac{1}{a}\delta\phi'\right) \tag{8.93}$$

So up to a term which vanishes at the horizon, formulas (8.83) and (8.93) are equal.

8.4 Eternal Inflation

Let us first verify that relation (8.83) holds in the eternally inflating regime. The condition for eternal inflation is that during one Hubble time, quantum fluctuations are of the same order as the classical rolling of the field:

$$|\delta\phi| \sim \Delta t|\dot{\phi}| = H^{-1}|\dot{\phi}| \tag{8.94}$$

This means we have the following relation

$$\sqrt{2\epsilon} = \frac{|\dot{\phi}|}{m_{\mathrm{p}}H} = \frac{|\delta\phi|}{m_{\mathrm{p}}} \tag{8.95}$$

We work in flat gauge such that $R = \frac{H}{\dot{\phi}}\delta\phi$. The real space variance of the curvature perturbation is, during eternal inflation

$$\sqrt{\langle R^2\rangle} = \frac{1}{\sqrt{2\epsilon}m_{\mathrm{p}}}\sqrt{\langle\delta\phi^2\rangle} = 1 \tag{8.96}$$

Since this is a gauge invariant parameter we can switch to another gauge. In the constant density $\delta\phi = 0$ gauge, where all the perturbations are characterized by metric fluctuations this leads to $\Phi \sim 1$ which would be problematic. There is however no issue with the longitudinal gauge in which we have calculated the thermodynamic relations. To see this we start by using the $(0,1)$ Einstein equation (8.86). Because the super-Hubble modes are frozen we can ignore the time-derivative term.

$$|\Phi| = \frac{1}{2m_{\mathrm{p}}^2}\frac{|\dot{\phi}|}{H}|\delta\phi| = \frac{H|\delta\phi|}{|\dot{\phi}|}\frac{\dot{\phi}^2}{2m_{\mathrm{p}}H^2} = \frac{H|\delta\phi|}{|\dot{\phi}|}\epsilon \tag{8.97}$$

This shows that although the first term is of order unity, Φ is still small and the system is perturbative.

A difficulty arises when we look at the thermodynamic equations for both the background and the perturbations, equation (8.82). First consider the following scaling for the eternal inflation regime

$$\dot{\delta\phi} = \frac{\mathrm{d}\delta\phi}{\mathrm{d}t} \sim \frac{\delta\phi}{H^{-1}} = H\delta\phi \sim \dot{\phi} \tag{8.98}$$

where we made use of eqn. (8.94). Substituting this scaling into the right hand side of eqn. (8.82) reveals that the equation cannot be decomposed into the average and its perturbations. This is problematic since we cannot split the system into a background geometry and superimpose quantum perturbations.

CHAPTER 9

Discussion

We have reviewed problems associated with the conventional big bang model and how a period of inflation solves these problems. The lower bound of the required duration of this period was found to be approximately 70 e-folds using CMB data. Quantum fluctuations during inflation are able to provide an explanation for the anisotropies measured in the CMB. We calculated the power spectrum of quantum fluctuations generated during inflation and discussed its relation to the angular power spectrum which is determined from CMB data.

To calculate the power spectrum in a gauge invariant way we reviewed linear perturbation theory. The quantization of the fluctuations requires the action up to second order in perturbations. To this end we used the ADM formalism which short-cuts this calculation and is also insightful on other grounds. For example it picks out the actual physical degrees of freedom while other equations appear as constraint equations such as the Friedmann equation. It was verified that the power spectrum is (very close to being) scale invariant which fits with observations.

Furthermore, we saw that for different inflationary models the inflationary quantum fluctuations lead to a globally never ending phase of inflation. The mechanisms of both slow-roll and false vacuum types of eternal inflation were discussed. We determined an upper bound of the slow-roll parameter ϵ for which inflation becomes eternal: $\epsilon < \frac{1.65^2}{8\pi^2}\frac{H^2}{m_{\mathrm{p}}^2}$.

Although inflation is widely accepted by the scientific community, the idea of eternal inflation is accompanied by scepticism. However, in the words of Guth [2007] who is considered one of the pioneers of inflation: *"I think the inevitability of eternal inflation in the context of new inflation is really unassailable. (...) The argument in the case of chaotic inflation is less rigorous, but I still feel confident that it is essentially correct."* In any case eternal inflation is intriguing. For example it has a natural solution to the problem of the extremely large landscape of string vacua and it provides an anthropic explanation for the existence of intelligent life.

Nevertheless the concept of eternal inflation has many difficulties associated with it. We discussed to problem of a global description of eternal inflation, and furthermore we have investigated the back-reaction of the quantum fluctuations on the classical background.

A literature study was performed on the subject of back-reaction. Due to the non-linearity of the Einstein equations, quantum fluctuations induce a back-reaction on the background geometry. We have seen how the back-reaction can be formulated in a gauge invariant way by introducing the effective energy momentum tensor. For chaotic inflation the effect of the back-reaction is that it slows down inflation although this is disputed by more recent research. In this research a different averaging procedure was adopted which depends on the observer. No significant back-reaction has been found when measured by freely-falling observers.

We have seen that back-reaction for free scalar fields with small mass in De Sitter space is irrelevant, since it can be removed by a rescaling of the cosmological constant. We have investigated the back-reaction effect of a massless scalar field. In this case the only contribution can come from derivative terms appearing in the Lagrangian. Using simple arguments we found that this contribution is similar to the back-reaction for a scalar field with small mass. When interactions are included, e.g. quartic self-interaction, the effect of back-reaction in De Sitter space is to lower the effective cosmological constant. This effect increases with time which means that De Sitter space is not stable to such fluctuations. This might be useful to explain why we observe such a small cosmological constant today. On the other hand, such a decrease in effective cosmological constant is problematic for the eternally inflating picture.

We derived the Langevin and Fokker-Planck equations which are central to the stochastic formalism to inflation. One of the benefits of the stochastic formalism is that it treats quantum fluctuations in an effectively classical manner by introducing a stochastic noise term. We have shown how the distribution function of the coarse grained field is effectively a description of the global geometry of the universe according to $H = H(\phi)$. However, when taking into account volume factors, which is certainly important during eternal inflation, the distribution function cannot unambiguously be defined. Different time reparametrizations yield different results. Still, the presence of eternal inflation is a gauge independent result.

We have briefly introduced thermodynamics of De Sitter space. We have shown that with the use of the Misner-Sharp mass it is possible to arrive at the Bekenstein-Hawking entropy law. Next we investigated the possibility to apply a thermodynamic first law to perturbed quasi-De Sitter spacetime, where we used definitions for thermodynamic quantities based on spherical spacetimes, following the article by Galvez Ghersi et al. [2011]. Indeed, the unperturbed thermodynamic law evaluated to an interesting result, which is the derivative of the Friedmann equation. We have shown that the approach remains valid in the eternally inflating regime.

APPENDIX A

Gauge Invariance and Constraint Equations

This section consists of a classical mechanics example to illustrate how the dynamical d.o.f. can be separated from the gauge modes. We will see the appearance of a constrained Hamiltonian.

Suppose we are dealing with a system of M particles. The EOM of the system can be found by extremizing the action which yield the Euler-Lagrange equations.

$$S = \int \mathrm{d}t \, L(q, \dot{q}, t) \tag{A.1}$$

Note that the general velocity and position are dependent variables. By a Legendre transformation, the action can be represented in its canonical form.

$$S = \int \mathrm{d}t \, \left(\sum_{i=1}^{M} p_i \dot{q}_i - H(p, q) \right) \tag{A.2}$$

Varying this action with respect to the independent variables p_i and q_i leads to the Hamiltonian equations of motion.

$$\dot{q}_i = \frac{\delta H}{\delta p_i}, \qquad \dot{p}_i = -\frac{\delta H}{\delta q_i} \tag{A.3}$$

Note that these equations are linear in time derivatives. This says that the p_i and q_i are dynamical d.o.f. We will find that constraint equations for the gauge modes do not have this property. (Also note that per particle the Euler-Lagrange equation gives a 2nd order differential equation while the Hamilton equations give 2 linear equations.)

Now we consider temporal reparametrization invariance. We consider t as an independent variable, lets call it the coordinate q_{M+1}, and we parametrize it by τ. Invariance means that physics remains the same under the gauge transformation $\tau \to \tilde{\tau}$. We write the action in so-called parametrized form plus a constraint equation.

$$S = \int \mathrm{d}\tau \, \tilde{L} = \int \mathrm{d}\tau \, \left(\sum_{i=1}^{M+1} p_i q_i' \right) \tag{A.4}$$

$$p_{M+1} = -H(p, q) \tag{A.5}$$

In the Hamiltonian, the p and q only run to M.

We can make the constraint equation explicit by inserting it into the action by means of a Lagrange multiplier.

$$S = \int \mathrm{d}\tau \left(\sum_{i=1}^{M+1} p_i q_i' - N(p_{M+1} + H(p,q) \right) \tag{A.6}$$

$$\frac{\delta S}{\delta N(\tau)} = 0 \tag{A.7}$$

This shows that if we start the other way around, i.e. with an action of the form (A.6) which has some reparameterization invariance, we can reduce it to an action of the form (A.2) plus a constraint.

APPENDIX B

Suggestion for a Stochastic Action Principle

We have shown that the back-reaction can be described by an effective energy momentum tensor, eqn. (6.6). It would be interesting to see if we could obtain a *stochastic* EEMT. In this way, the back-reaction from the statistical fluctuations on the geometry can be determined.

We will try to find the Lagrangian which reproduces the Langevin equation by varying w.r.t. ϕ. The noise-function $N(t)$ will be rewritten such that it contains model-specific terms and a normalized noise term ξ, as in equation (7.26). For the noise after a time Δt we have:

$$N(t) = \sqrt{\sigma^2}\xi(t) = \sqrt{2D(\phi)}\xi(t) \tag{B.1}$$

Where $D(\phi)$ is the diffusion coefficient (7.38) which is $D(\phi) = \frac{H(\phi)^3}{8\pi^2}$. The Langevin equation follows from the equation of motion in the slow-roll approximation:

$$\begin{aligned} 0 &= 3H\dot{\phi} + V_{,\phi} - 3H\sqrt{2D}\xi(t) && \text{(B.2)} \\ &= 3H\dot{\phi} + V_{,\phi} - 3\frac{H^{5/2}}{2\pi}\xi(t) && \text{(B.3)} \\ &= 3H\dot{\phi} + V_{,\phi} - \frac{3^{-1/4}}{2\pi}\frac{V^{5/4}}{m_{\mathrm{p}}^2}\frac{\xi(t)}{\sqrt{m_{\mathrm{p}}}} && \text{(B.4)} \end{aligned}$$

where we have identified $H(\phi)^2 = \frac{V(\phi)}{3m_{\mathrm{p}}^2}$ in the last step. This equation can be found from varying the following Lagrangian with respect to ϕ:

$$\mathcal{L} = -\frac{1}{2}(\nabla\phi)^2 - V(\phi) + \frac{3^{-1/4}}{2\pi m_{\mathrm{p}}^2}\frac{\xi(t)}{\sqrt{m_{\mathrm{p}}}}\int^{\phi} \mathrm{d}\phi'\, V^{5/4}(\phi') \tag{B.5}$$

This leads to an effective energy density:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{3^{-1/4}}{2\pi m_{\mathrm{p}}^2}\frac{\xi(t)}{\sqrt{m_{\mathrm{p}}}}\int^{\phi} \mathrm{d}\phi'\, V^{5/4}(\phi') \tag{B.6}$$

and the pressure is $p = -\rho$. In the De Sitter limit, $V(\phi) = V_0 = 3m_\mathrm{p}^2 H_0^2$ this becomes:

$$H_\mathrm{eff}^2 = H_0^2 \left(1 - \frac{1}{2\pi} \frac{\sqrt{H_0}\phi}{m_\mathrm{p}^{3/2}} \frac{\xi(t)}{\sqrt{m_\mathrm{p}}}\right) \tag{B.7}$$

For $V(\phi) \propto \phi^n$ we obtain:

$$V_\mathrm{eff} = V(\phi) \left(1 - \frac{\xi(t)}{\sqrt{m_\mathrm{p}}} \frac{m^{1-n/4}}{m_\mathrm{p}^2} \phi^{n/4+1}\right) \tag{B.8}$$

For $n = 2$:

$$V_\mathrm{eff} = V(\phi) \left(1 - \frac{\xi(t)}{\sqrt{m_\mathrm{p}}} \frac{m^{1/2}\phi^{3/2}}{m_\mathrm{p}^2}\right) \tag{B.9}$$

$n = 4$:

$$V_\mathrm{eff} = V(\phi) \left(1 - \frac{\xi(t)}{\sqrt{m_\mathrm{p}}} \frac{\phi^2}{m_\mathrm{p}^2}\right) \tag{B.10}$$

Stochastic Gravity

Within the line of thought of the previous section quite some effort has already been made in the framework of stochastic gravity. Not only the slow-roll equation of motion with a stochastic noise term (i.e. Langevin equation) is derived from an action principle, but a more general set of Einstein-Langevin equations are derived. For a review see Hu and Verdaguer [2008]. A treatment is however beyond the scope of this research.

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