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MASS FORMULAE AND THE ALGEBRAIC STRUCTURE OF SUPERCONVERGENCE RELATIONS*

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ABSTRACT

Sets of $t = 0$ current algebra and superconvergence sum rules are treated as equations in the coupling constants and masses of states which are assumed to dominate the sum rules. The solutions of these sets lead to new mass relations among particles of different spins and parities. The algebraic structure of the sum rules and their solutions is discussed.

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In a previous paper[1] (hereafter denoted by I) we have suggested that the complete set of current algebra and superconvergence sum rules for forward scattering of pions on a hadronic target x leads to a determination of masses and coupling constants of various states which are assumed to dominate the sum rules. We have shown[1] that the complete set of π-ρ $t = 0$ sum rules is approximately saturated by the π, ω and A_1 intermediate states and that the obtained predictions for m_ω, m_{A_1}, $g_{\omega\rho\pi}$ and $\Gamma(A_1 \to \rho\pi)$ are in good agreement with experiment. In this paper we analyse the algebraic structure of the $t = 0$ sum rules for π-x scattering and apply our technique to a few additional cases. We use the same assumptions as in I and find:

(a) If all $t = 0$ superconvergence and current algebra sum rules[2] for π-x scattering are saturated by states forming an irreducible representation (IR) of the $SU(2) \times SU(2)$ chiral algebra of charges, the complete set of equations in the masses and coupling constants has a unique, non-trivial solution in the limit of zero pion mass.

(b) If additional states contribute to the sum rules we always find a consistent solution. However, the uniqueness is lost and we can express all masses and coupling constants in terms of a few free parameters, corresponding to the mixing coefficients of the additional IR's which contribute to the sum rules.

(c) All states in a given IR of $SU(2) \times SU(2)$ have the same m^2 value.[3] If the $SU(2) \times SU(2)$ states are mixtures of single particle states, their m^2 values are given by the appropriate weighted averages of the m^2 values of the mixed physical states.

(d) The application of these considerations to various simple cases leads to many new mass relations among particles of different spins and parities.

As in I, we assume (1) chiral $SU(2) \times SU(2)$ algebra of charges, (2) $[D^i, Q_5^j] = \delta^{ij} S$ where $D^i = \frac{d}{dt} Q_5^i$, (3) PCAC, (4) $s^{\alpha_I(0) - |\Delta h|}$ high energy behavior for a t-channel amplitude with helicity change Δh and isospin I, where $\alpha_I(0)$ is the $t = 0$ intercept of the leading Regge trajectory, and (5) $\alpha_2(0) < 0$.

The only non-vanishing s-channel helicity amplitudes for π-x scattering at $t = 0$ are the amplitudes $A_{0\lambda,0\lambda}$. The helicity crossing matrices indicate that the only t-channel helicity amplitudes which may contribute to $A_{0\lambda,0\lambda}$ at $t=0$ are $A_{00,\mu\nu}$ where $(\mu-\nu)$ is even. It is therefore convenient to divide all $t = 0$ superconvergence relations into two classes: Sum rules of class I involve amplitudes (with even Δh in the t-channel) which contribute to the non-vanishing helicity amplitudes at $t = 0$. These are "pure" $t = 0$ sum rules and the corresponding amplitudes can, in principle, be measured directly. Other sum rules ("class II") involve amplitudes (corresponding to odd Δh in the t-channel) which do not vanish at $t = 0$ but do not contribute to any of the non-vanishing $t = 0$ helicity amplitudes.[4] In principle, such an amplitude $B(s,0)$ can be determined only by extrapolating $B(s,t)$ to $t = 0$. The algebraic analysis presented in this paper refers mainly to the "pure" (class I) sum rules which are the ones related to the physical forward scattering amplitude. Class II sum rules may, however, give additional information and enable us in a few cases to determine parameters (mixing angles) which are left free by the set of class I sum rules.

The current algebra $t = 0$ sum rules can be derived only by using PCAC. We will therefore study the self-consistency of the complete set of equations only in the limit $m_\pi = 0$. We realize that the superconvergence relations can be derived without taking this limit. We find, however, that the overall consistency of the saturation assumption requires $m_\pi^{ext} = 0$ even if we consider only the superconvergence relations. This may mean that to the extent that these relations give symmetry results, they do so only because of their connection to the algebra of currents. If this is really the case, we clearly have to consider all our sum rules in the limit implied by PCAC or by vector meson dominance which are the crucial links between the algebra of weak and electromagnetic currents and the strong interaction sum rules. Notice, however, that whenever the pion appears as an intermediate state, its mass is not necessarily zero, and we consider it as an additional physical quantity.

We now proceed to discuss a few specific sets of sum rules which enable us both to demonstrate our general conclusions and to present those predictions which can be immediately tested by experiment.

(a) We first discuss the case of a pure IR of $SU(2) \times SU(2)$. We consider the set of $t = 0$ sum rules for π-ρ scattering[1] and assume that the π and ω intermediate states saturate the sum rules. We solve the set of equations in masses and coupling constants and find[5]:

$$m_\pi = m_\omega = m_\rho \tag{1}$$

$$g^2_{\omega\rho\pi} = \frac{4g^2_{\rho\pi\pi}}{m^2_\rho} = \frac{8}{f^2_\pi} \tag{2}$$

While it is clear that Eqs. (1),(2) do not agree very well with experiment, it is interesting to understand algebrically why we have obtained such a solution. In order to do so we notice that our saturation assumption is equivalent to assuming that, at infinite momentum, the $h = 1$ components of ρ and ω are in the $(\frac{1}{2},\frac{1}{2})$ IR of $SU(2) \times SU(2)$ while the $h = 0$ ρ and π are in $(1,0) \pm (0,1)$. In this case, the axial charge Q_5, which is a generator of the algebra, connects ρ only to ω and π. The matrix elements of the operator

D^i between particle states at infinite momentum satisfy[6]:

$$\lim_{p_z \to \infty} p_z (\alpha|D^i|\beta) = -\frac{i}{2}(m_\beta^2 - m_\alpha^2)(\alpha|Q_5^i|\beta) \qquad (3)$$

If $(\alpha|D^i|\beta) = 0$ and $(\alpha|Q_5^i|\beta) \neq 0$, Eq. (3) leads to $m_\beta = m_\alpha$. The commutation relation $[D^i, Q_5^j] = \delta^{ij}S$ implies that the operators D^i and S transform according to the $(\frac{1}{2},\frac{1}{2})$ representation $SU(2) \times SU(2)$. Consequently, for any IR (k,ℓ):

$$((k,\ell)|D^i|(k,\ell)) = 0 \qquad (4)$$

We conclude that if ρ and ω (or ρ and π, for $h = 0$) are in the same $SU(2) \times SU(2)$ representation, $(\rho_1|D|\omega_1) = 0$ and $(\rho_0|D|\pi) = 0$ where the subscripts denote the helicities. Eq. (3) then leads to the prediction of equal masses for ρ,ω and π (Eq. 1). This is actually a much more general result: If we saturate all $t = 0$ sum rules[2] for π-x scattering by states forming an IR of $SU(2) \times SU(2)$, we find that all matrix elements of D vanish. The masses of all intermediate states are then predicted to be the same as the mass of x and the sum rules for $I = 2$ t-channel amplitudes become trivial identities while the $I = 1$ sum rules lead to the ordinary predictions of the charge algebra.

(b) In order to study the case of a reducible, finite, representation we now allow the φ meson to contribute to the same set of π-ρ sum rules. The solution is not unique and it depends on a free parameter θ which we define by:

$$\frac{g_{\varphi\rho\pi}}{g_{\omega\rho\pi}} = \tan\theta \qquad (5)$$

The general solution is:

$$m_\pi^2 = m_\rho^2 = m_\omega^2 \cos^2\theta + m_\varphi^2 \sin^2\theta \tag{6}$$

$$\frac{4g^2_{\rho\pi\pi}}{m_\rho^2} = \frac{g^2_{\omega\rho\pi}}{\cos^2\theta} = \frac{g^2_{\varphi\rho\pi}}{\sin^2\theta} = \frac{8}{f_\pi^2} \tag{7}$$

We immediately see that as $g_{\varphi\rho\pi} \to 0$, $\theta \to 0$, $m_\rho \to m_\omega$. The predictions for m_π and $g_{\rho\pi\pi}$ are not affected by φ since φ contributes only to the transverse sum rules while π contributes only to longitudinal sum rules.

From the algebraic point of view the solution (6)-(7) can be understood in the following way: The addition of φ is equivalent to assigning the $h = 1$ ω and φ to orthogonal mixtures of the $(\frac{1}{2},\frac{1}{2})$ and $(0,0)$ IR's while ρ_1, ρ_0 and π are classified as before. We define:

$$|\varphi_1 > = \cos\theta\,|(0,0) > + \sin\theta\,|(\tfrac{1}{2},\tfrac{1}{2}) > \tag{8}$$

$$|\omega_1 > = -\sin\theta\,|(0,0) > + \cos\theta\,|(\tfrac{1}{2},\tfrac{1}{2}) > \tag{9}$$

Q_5 connects ρ_1 only to states in the $(\frac{1}{2},\frac{1}{2})$ representation while D connects ρ_1 only to $(0,0)$. We therefore find:

$$\frac{(\rho_1^+|Q_5^+|\varphi_1)}{(\rho_1^+|Q_5^+|\omega_1)} = \tan\theta \tag{10}$$

$$\frac{(\rho_1^+|D^+|\varphi_1)}{(\rho_1^+|D^+|\omega_1)} = -\cot\theta \tag{11}$$

Eq. (10) is identical to (5) and leads to (7). Eq. (11) together with (3) leads to Eq. (6). The angle θ that was arbitrarily introduced in Eq. (5)

is now interpreted as the mixing angle between the $(\frac{1}{2},\frac{1}{2})$ and (0,0) representations. Its experimental value is close to zero, and the absence of $\varphi \rightarrow \rho\pi$ decay therefore leads to the approximate equality between m_ρ and m_ω.

The degeneracy of m_π and m_ρ was removed in I by introducing the A_1 as an additional state in the sum rules. The experimental value for $\Gamma(\rho \rightarrow \pi\pi)$ determined the π-A_1 mixing angle (denoted by ψ in I) to be approximately 45° and the components of the $h = 0$, $(1,0)$-$(0,1)$ representation of SU(2) $\times$ SU(2) to be $\frac{1}{\sqrt{2}} |\pi^i> + \frac{1}{\sqrt{2}} |A_1^i>$. The obtained mass formula is consequently:

$$\tfrac{1}{2}(m_\pi^2 + m_{A_1}^2) = m_\rho^2 \tag{12}$$

(c) Our third example is the set of all[2] $t = 0$ sum rules for $\pi N \rightarrow \pi N$, $\pi N \rightarrow \pi N^*$, $\pi N^* \rightarrow \pi N^*$ where N^* is the $\frac{3}{2}^+$ resonance at 1236 MeV. If we saturate these sum rules by N and N* only we find a unique solution in which $m_N = m_{N^*}$ and all coupling constants satisfy the usual chiral algebra (or SU(6)) relations such as $G_A = \frac{5}{3}$ etc.[7] We know, however, that the saturation by one IR does not agree with experiment and that many additional states have non-negligible contributions. The mixing coefficients for N and N* can be determined from the experimental weak, electromagnetic and pionic transitions. These indicate[8] that the $h = \frac{1}{2}$, $(1,\frac{1}{2})$ representation of SU(2) $\times$ SU(2) includes the "pure" N*(1236) and a mixed $I = \frac{1}{2}$ state $\{\cos\theta|N> + \sin\theta|X>\}$ where X includes components from the $P_{11}(1400)$, $D_{13}(1530)$, $S_{11}(1550)$, $F_{15}(1688)$, $D_{15}(1688)$, $S_{11}(1700)$ $I = \frac{1}{2}$ nucleon resonances. We therefore obtain the following mass formula:

$$\cos^2\theta\, m_N^2 + \sin^2\theta\, m_X^2 = m_{N^*}^2 \tag{13}$$

where m_X^2 is a weighted average of the m^2-values of the $I = \frac{1}{2}$ resonances. The actual contribution of any one of these states can be determined only from the so far unknown decay rates $N^*_{\frac{1}{2}} \to N^*(1236) + \pi$. Substituting the experimental values of m_N, m_{N^*} and[8] the $\cos\theta = 0.8$ we predict $m_X = 1.64$ BeV, clearly within the expected mass range.

(d) We next consider all $t = 0$ sum rules for π-δ scattering where δ is a $J^P = 0^+$, $I^{CG} = 1^{+-}$ state which may or may not be identified with the observed narrow peak at 960 MeV.[9] We have only two such sum rules, one for the $I = 1$ and one for the $I = 2$ t-channel amplitudes. The only known particles that could contribute[10] are η and $X^0(960)$. The saturated sum rules read:

$$g^2_{\pi\delta\eta} + g^2_{\pi\delta X^0} = \frac{8}{f_\pi^2} \tag{14}$$

$$(m_\eta^2 - m_\delta^2)\, g^2_{\pi\delta\eta} + (m^2_{X^0} - m_\delta^2)\, g^2_{\pi\delta X^0} = 0 \tag{15}$$

If $\Gamma(\delta \to \pi\eta) \leq 5$ MeV (as is the case if δ is the 960 MeV state) we find that η contributes less than 2% of the sum rule (14). Eq. (15) then leads to $m_\delta \cong m_{X^0}$ in strong support of the assignment of the 960 MeV peak. The $SU(2) \times SU(2)$ classification then assigns δ and X^0 to $(\frac{1}{2},\frac{1}{2})$ while η is mostly in $(0,0)$. This allows us to determine the sign of the η-X^0 octet-singlet mixing angle. The sign is the one which identifies the η as an almost pure $\lambda\bar{\lambda}$ quark structure while X^0 is mostly $p\bar{p} + n\bar{n}$.

(e) Our last example is the set of $t = 0$ sum rules for π-A_1 scattering. There are five sum rules (including one of class II) similar to the five π-ρ sum rules.[1] We assume that the sum rules are dominated by the following states[11]: $\sigma(J^P = 0^+, I^{CG} = 0^{++})$, ρ, $D(J^P = 1^+, I^{CG} = 0^{++})$, $B(J^P = 1^+, I^{CG} = 1^{-+})$. We use the A_1 and ρ couplings and masses abtained in I, and find a unique solution for the π-A_1 sum rules. The masses of σ,D,B are predicted to satisfy:

$$m_\rho = m_\sigma \tag{16}$$

$$m_B = m_D \tag{17}$$

The coupling constant relations are cumbersome and cannot be directly tested. We will present them elsewhere, together with a detailed discussion of the sum rules. At this point we only remark that there are some vague indications for a σ-type resonance around the ρ-mass[12] which, if verified, will agree with Eq. (16). The D-particle is the isosinglet of the A_1 octet (or nonet) and therefore will probably be found in the 1.1 - 1.2 BeV region, not very far from the B mass.

Additional applications and analysis of the t = 0 sets of sum rules may enable us to have a better understanding of the mass spectrum of the various resonances and of their chiral algebra classification. A particularly interesting (and open) question is the role played by the class II t = 0 superconvergence relations with respect to the determination of free mixing angles of the chiral algebra. We hope to return to this problem in a future publication.

FOOTNOTES AND REFERENCES

1. F. J. Gilman and H. Harari, Phys. Rev. Letters **18**, 1150 (1967).

2. By "All t = 0 sum rules" we refer to all sum rules involving amplitudes which actually contribute to the scattering at t = 0. These include the charge algebra sum rules and part of the complete set of superconvergence relations. We later refer to these as "class I superconvergence rules".

3. This does not imply $SU(2) \times SU(2)$ invariance. There are $SU(2) \times SU(2)$ symmetry breaking contributions to the masses of the states. These transform according to the $(\frac{1}{2},\frac{1}{2})$ representation and do not split the masses in an IR.

4. The first two superconvergence relations of V. de-Alfaro, S. Fubini, G. Furlan and C. Rossetti, Phys. Letters **21**, 576 (1966) belong to the two calsses defined here. The I = 1 sum rule for the amplitude A is a class I sum rule (and corresponds to the difference between two Adler-Weisberger sum rules for π-ρ scattering). The I = 2 sum rule for the amplitude B is a class II sum rule since B does not contribute to t = 0 π-ρ scattering. (See also Eqs. (3)-(13) and the related discussion in Reference 1.)

5. The explicit equations can be trivially obtained from Eqs. (7a)-(13a) of Reference 1 by setting $g_L = g_T = 0$.

6. The operator D was used by S. Fubini, G. Furlan and C. Rossetti, Nuovo Cimento **40**, 1171 (1965) in deriving SU(3) mass formulae. See also, V. de-Alfaro, S. Fubini, G. Furlan and C. Rossetti, to be published in Nuovo Cimento.

7. The class II superconvergence relations for $\pi N^* \to \pi N^*$ are inconsistent with this solution. The N-N* sum rules were discussed by P. H. Frampton,

Oxford preprint and H. F. Jones and M. D. Scadron, Imperial College preprint.

8. R. Gatto, L. Maiani and G. Preparata, Phys. Rev. Letters **16**, 377 (1966), H. Harari, Phys. Rev. Letters **16**, 964 (1966), **17**, 56 (1966), I. S. Gerstein and B. W. Lee Phys. Rev. Letters **16**, 1060 (1966).

9. The $J^P = 0^+, I^{CG} = 1^{+-}$ assignment is the most probable for this peak, if it is a genuine resonance. See also R. H. Dalitz, Proceedings of the 13th Conference on High Energy Physics, Berkeley, 1966.

10. If the B(1220) meson exists it could also contribute. We find, however, that for $\Gamma(B \to \delta\pi) = 10$ MeV our predicted δ mass changes only by 40 MeV. Needless to say, there is no evidence, so far, for the decay $B \to \delta\pi$.

11. This should really be regarded as a speculative excercise since the only state that is really known here is the ρ-meson. We present the results here mainly with the idea of predicting the approximate masses of other expected states.

12. See e.g. A. H. Rosenfeld, et. al., Rev. Mod. Phys. **39**, 1 (1967)(p. 21).