Magnetic Dipoles and Quantum Topological Phases

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1. Introduction

Just as a magnetic dipole moving through in an electric field which has the appropriate geometry may acquire a topological quantum phase, the Aharonov-Casher (AC) phase [1], a electric dipole in a magnetic field which has the appropriate geometry may also acquire a topological quantum phase. This electromagnetic dual phenomenon was pointed out by He and McKellar [2], and independently by Wilkens [3] and is now know as the He-McKellar-Wilkens (HMW) phase [4]. This duality concept is illustrated in Fig. 1.

The proofs of both the AC and HMW phases require a two dimensional geometry — they are intrinsically planar effects. This raises the question *do the deviations from the idealised two dimensional situation in the experimental demonstration of these phases have any influence on the interpretation of the experiment?* In particular it is important to ask if the phase demonstrated experimentally is a *topological* phase. To be able to answer this question one needs to define the concept of *topological phase* or *geometric phase*[1]. My definition is in three parts:

(1) the phase is constructed from a position dependent phase $\phi(\mathbf{x})$ modifying the wavefunction $\psi(\mathbf{x})$ such that $\psi'(\mathbf{x}) = \exp\{i\phi(\mathbf{x})\}\,\psi(\mathbf{x})$ satisfies the free wave equation.

(2) the phase for a closed path is independent of the path, except for the number of times it circles some interior excluded region, and

(3) the phase of the closed path is determined by properties of the fields in the excluded region

The experimental realisation of the HMW effect requires an electric field to induce the electric dipole. As shown by Wei, Han and Wei [5] this electric field changes the topology of the configuration and allows a topological phase where, had the dipole had the same orientation without the electric field, the phase would not have been topological[2]. There is thus an important distinction between intrinsic and induced dipoles.

In the first two sections of this review I review the proofs of the HMW effect for intrinsic and then induced dipoles. I then review the experimental observations of the effect.

2. An Intrinsic Electric Dipole

The original derivation of the quantum topological phase acquired by an electric dipole [2] was repeated in a way which unified the AC and the HMW effects [7, 8], and shows that the effects occur for arbitrary spin. He and McKellar, in their later papers, used the Dirac equation (or the Bargmann-Wigner equation for spins greater than 1/2) to obtain an elegant derivation of the effect. For simplicity in this paper I concentrate on the spin-1/2 case. For a neutral spin half particle with an electric dipole moment μ_e the Dirac equation is

$$\left(i\gamma^\mu\partial_\mu + \frac{1}{2}\mu_e\sigma^{\mu\nu}\gamma_5 F^{\mu\nu} - m\right)\psi = 0. \tag{1}$$

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[1] I regard these terms as having the same meaning and use them interchangeably

[2] See the discussion in [6]

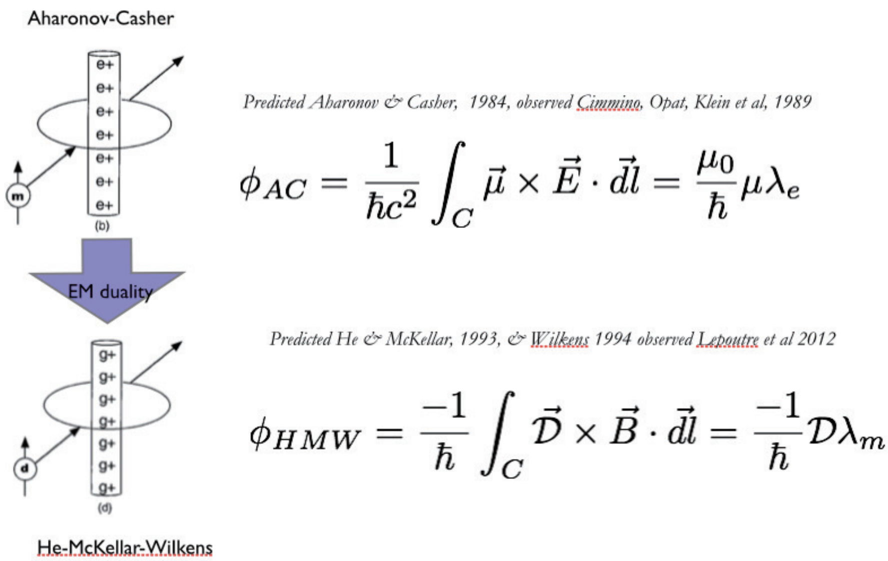

Fig. 1. The electromagnetic dual of the Aharonov Casher phase is the He McKellar Wilkens phase. Modified from Ref.[4].

Using the relationship that

$$-iF_{\mu\nu}\sigma^{\mu\nu}\gamma_5 = \tilde{F}_{\mu\nu}\sigma^{\mu\nu}, \quad \text{where} \quad \tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{2}$$

is the $3+1$ dimensional dual of the electromagnetic field tensor, in which the electric and magnetic fields are interchanged, the Dirac equation (1) may be written as

$$\left(i\gamma^\mu\partial_\mu + i\frac{1}{2}\mu_e\sigma^{\mu\nu}\tilde{F}^{\mu\nu} - m\right)\psi = 0. \tag{3}$$

In a situation where we have translational symmetry in the z direction the problem is reduced to motion in a plane, and the Dirac equation is reduced to $2+1$ dimensions.

I use the following conventions for the 2+1 dimensional metric $g_{\mu\nu}$ and the anti-symmetric tensor $\epsilon_{\mu\nu\alpha}$:

$$g_{\mu\nu} = \text{diag}(1,-1,-1) \quad \text{and} \quad \epsilon_{012} = +1. \tag{4}$$

In $2+1$ dimensions we need only 3 Dirac matrices, which are a suitably chosen set of Pauli matrices, and we will work with 2-spinors instead of 4-spinors.

There are two inequivalent representations of the Dirac matrices in $2+1$ dimensions which generate different Clifford algebras. The Clifford Algebra in $2+1$ dimensions has just 4 basis operators, the unit operator and the 3 two dimensional Dirac matrices, with the defining equation

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} + is\epsilon^{\mu\nu\lambda}\gamma_\lambda \quad \text{where} \quad s = \pm 1. \tag{5}$$

The two representations are distinguished by the value of the parameter s. The s values ±1 correspond to spin up and spin down in the "hidden" third spatial dimension. Possible representations of the two inequivalent sets of basis operators are

$$\gamma^0 = \sigma_3, \quad \gamma^1 = si\sigma_2, \quad \text{and} \quad \gamma^2 = i\sigma_1. \tag{6}$$

Note that the final results are independent of the representation[3].

The interaction term in the Dirac equation is proportional to

$$\tilde{F}_{\mu\nu}\sigma^{\mu\nu}\psi = -\tilde{F}^{\mu\nu}s\epsilon_{\mu\nu\lambda}\gamma^{\lambda}\psi,$$
$$\text{with} \quad \tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 \\ B^1 & 0 & E^3 \\ B^2 & -E^3 & 0 \end{pmatrix}. \quad (7)$$

Here E^i and B^i are the electric and magnetic fields, respectively. The indices "1" and "2" indicate the coordinates on the $x-y$ plane along the x and y directions. The index "3" indicates that the electric field in this configuration is normal to the $x-y$ plane, in the notional z direction, so that the electric dipole is parallel to the electric field.

The Dirac equation can now be rewritten as

$$\left(i\gamma^{\mu}\left[\partial_{\mu} + i\mu_e T_{\mu}\right] - m\right)\psi = 0, \quad (8)$$

with the "effective vector potential" T_μ as the $2+1$ dimensional dual of the $3+1$ dimensional dual $\tilde{F}^{\alpha\beta}$ of the electromagnetic field strength tensor $F^{\mu\nu}$

$$T_{\mu} = (1/2)\epsilon_{\mu\alpha\beta}\tilde{F}^{\alpha\beta}. \quad (9)$$

In the HMW configuration, the electric field vanishes and B_1, B_2 are constant in time. Then $T_\mu = (0, \mathbf{T}) = (0, T_1, T_2) = (0, B_2, -B_1) = (0, \boldsymbol{B} \times \boldsymbol{k})$, where $\boldsymbol{k}$ is a unit vector in the z direction, i.e. the direction of the electric moment.

Making a transformation

$$\psi' = \exp\left[-is\mu_e \int^{\boldsymbol{r}} \boldsymbol{T}(\boldsymbol{r'})\cdot \boldsymbol{ds'}\right]\psi \quad (10)$$

in Eq. (8), one finds that ψ' satisfies the free Dirac equation

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi' = 0. \quad (11)$$

[3] If one feels uncomfortable with handling the two inequivalent representations one can use 4-spinors and 4×4 Dirac matrices as was done by He and McKellar [7]. The 4×4 representation equivalent to Eq. (6) has the $s=+1$ representation in the $(1,1)$ place and the $s=-1$ representation in the $(2,2)$ place in the 2×2 block form of the Dirac matrices.

You see that the reduction of the problem to $2+1$ dimensions was a critical step towrads of this result, because the electric moment – electromagnetic field term in the Dirac equation can be converted into a γ^μ interaction with the vector dual of the tensor electromagnetic field. In $3+1$ dimensions the dual of the tensor electromagnetic field is a tensor and the dual of the $\sigma_{\mu\nu}$ tensor is also a tensor. We don't have the ability to transform the Dirac equation for an electric moment interacting with the the dual electromagnetic field into the vector current interaction with an a effective vector potential field, but we need to do that to be able to make the phase transformation to convert the wavefunction to one satisfying the free field equation.

However in $2+1$ dimensions we can make the phase transformation of Eq. (10) to recover the free Dirac equation and we now have a topological phase as long as

(1) curl $\boldsymbol{T} = 0$ in the interference region

(2) curl $\boldsymbol{T} \neq 0$ in the excluded region

As $\mathrm{curl}\,\boldsymbol{T} = \mathrm{curl}\,\boldsymbol{B}\times\boldsymbol{k} = \boldsymbol{k}(\mathrm{div}\,\boldsymbol{B}) - (\boldsymbol{k}\cdot\boldsymbol{\nabla})\boldsymbol{B}$, the simplest way for the excluded region to generate a non-vanishing contribution to the phase is for it to contain some magnetic charges, giving rise to $\mathrm{div}\,\mathbf{B} \neq 0$. To preserve the $2+1$ dimensional geometry, the magnetic charges should be extended uniformly and infinitely in the z direction. The simplest such charge configuration, that chosen by He and McKellar, is a line of magnetic monopoles on the z axis, with a linear magnetic monopole charge density λ_m. In this configuration, $\partial_z \boldsymbol{B} = 0$, so $(\boldsymbol{k}\cdot\boldsymbol{\nabla})\boldsymbol{B} = 0$.

Then the phase developed in the wave function when the particle travels along a closed path $\mathcal{P}$ which encircles the line of magnetic charge with a linear monopole density λ_m once is

$$\begin{aligned} \chi_{\mathrm{HMW}} &= s\mu_e \oint_{\mathcal{P}} \boldsymbol{T}\cdot \boldsymbol{dr} \\ &= -s\mu_e \int_{\mathcal{S}} (\boldsymbol{\nabla}\cdot\boldsymbol{B})\boldsymbol{k}\cdot d\mathcal{S} \\ &= -s\mu_e\lambda_m, \end{aligned} \quad (12)$$

as found by He and McKellar.

It is clear that, since no magnetic monopoles have yet been found[4] this manifestation of the HMW phase is not capable of experimental observation. Nevertheless it is important to emphasise that the model is mathematically consistent. The HMW system with magnetic monopoles as the source of the magnetic field, no electric field, electric dipoles and no electric charges is the precise electromagnetic dual of the Aharonov-Casher system, which has electric charges as the source of the electric field, no magnetic field, magnetic dipoles and no magnetic monopoles.

The concept of electromagnetic duality is described by Jackson [9]. Consider Maxwell's equations and the Lorentz force equation, extended to include magnetic monopoles:

$$\begin{aligned} \nabla\cdot\mathbf{D} &= \rho_e & \nabla\times\mathbf{H} &= \frac{\partial\mathbf{D}}{dt}+\mathbf{J}_e \\ \nabla\cdot\mathbf{B} &= \rho_m & \nabla\times\mathbf{E} &= \frac{\partial\mathbf{B}}{dt}+\mathbf{J}_m \\ \mathbf{F} &= q_e(\mathbf{E}+\mathbf{v}\times\mathbf{B}) + q_m(\mathbf{H}-\mathbf{v}\times\mathbf{D}) \end{aligned} \tag{13}$$

The source particles moving with velocity $\mathbf{v}$ carry both an electric and a magnetic charge. Normal Maxwellian electrrodynamics is the case that $q_m = 0, \rho_m = 0$ and $\mathbf{J}_m = 0$. The duality transformation

$$\begin{aligned} \mathbf{E} &= \mathbf{E}'\cos\xi + Z_0\mathbf{H}'\sin\xi \\ Z_0\mathbf{D} &= Z_0\mathbf{D}'\cos\xi + \mathbf{B}'\sin\xi \\ Z_0\mathbf{H} &= -\mathbf{E}'\sin\xi + \mathbf{B}'\cos\xi \\ \mathbf{B} &= -Z_0\mathbf{D}'\sin\xi + \mathbf{B}'\cos\xi \\ Z_0 q_e &= Z_0 q_e'\cos\xi + q_m'\sin\xi \\ q_m &= -Z_0 q_e'\sin\xi + q_m'\cos\xi \end{aligned} \tag{14}$$

retains the form of the equations, transforming them to

$$\begin{aligned} \nabla\cdot\mathbf{D}' &= \rho_e' & \nabla\times\mathbf{H}' &= \frac{\partial\mathbf{D}'}{dt}+\mathbf{J}_e' \\ \nabla\cdot\mathbf{B}' &= \rho_m' & \nabla\times\mathbf{E}' &= \frac{\partial\mathbf{B}'}{dt}+\mathbf{J}'_m \\ \mathbf{F} &= q_e'(\mathbf{E}'+\mathbf{v}\times\mathbf{B}') + q_m'(\mathbf{H}'-\mathbf{v}\times\mathbf{D}') \end{aligned} \tag{15}$$

The choice $\xi = \pi/2$ transforms electric charges into magnetic monopoles, magnetic dipole moments into electric dipole moments, magnetic fields into electric fields and electric fields into magnetic fields. This is just the transformation we need to transform the AC effect into the HMW effect. The equations of normal Maxwellian electrrodynamics are transformed to

$$\begin{aligned} \nabla\cdot\mathbf{D}' &= 0 & \nabla\times\mathbf{H}' &= \frac{\partial\mathbf{D}'}{dt} \\ \nabla\cdot\mathbf{B}' &= \rho_m' & \nabla\times\mathbf{E}' &= \frac{\partial\mathbf{B}'}{dt}+\mathbf{J}'_m \\ \mathbf{F} &= q_m'(\mathbf{H}'-\mathbf{v}\times\mathbf{D}'). \end{aligned} \tag{16}$$

The equations (16) lead to the HWM phase and show that the calculations are mathematically consistent.

As an amusing aside note that, if all particles have the same ratio of electric charge to magnetic charge, then the general equations (13) can be converted by duality transformations to either the usual Maxwell Equations or the magnetic monopole form of (16). In this sense our decision to describe the world in terms of electric charges and currents is purely an historical happenstance.

The alternative, independent, derivation by Wilkens [3] relied on the effective electric field[5] $\boldsymbol{E}_{\mathrm{R}} = \boldsymbol{v}\times\boldsymbol{B}$ felt by the moving dipole in the magnetic field, but Wilkens also suggested a line of magnetic dipoles as the source of the magnetic field, and proposed possible approximate realisations of this concept.

[4]The recent observation of a synthetic magnetic monopole in a Bose condensate [10] raises the question "is it possible to observe a synthetic HMW effect," That question has not yet been explored in detail.

[5]This effective electric field felt by a charge moving in a magnetic field was introduced by Röntgen, and is called the Röntgen field. I therefore use the subscript $_{\mathrm{R}}$ for it.

3. An Induced Electric Dipole

Wei, Han and Wei [5] pointed out that a practical realisation of the HWM effect would require an electric field to induce an electric dipole in a neutral atom. That is indeed how the HMW phase was measured. One may think it would be possible to avoid the electric field by using a molecule with an intrinsic dipole moment. However a beam of polarised molecules would rapidly depolarise in the absence of an electric field to maintain the alignment of the dipole. In either case a strong electric field is necessary for the realisation of the HMW phase. It is an important result of Wei, Han and Wei that the electric field changes the geometry, and allows a realisation of the HMW effect without a region in which div$\mathbf{B} \neq 0$.

Now the electric field felt by the moving atom is the sum of the applied field $\mathbf{E}$ and the Röntgen field $\mathbf{v} \times \mathbf{B}$, and so the induced dipole is $\mathbf{d} = \alpha(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, where α is the electric polarisability of the atom.

The Lagrangian is

$$\mathcal{L} = \frac{1}{2} m\mathbf{v}^2 + \frac{1}{2}\alpha(\mathbf{E} + \mathbf{v} \times \mathbf{B})^2. \tag{17}$$

Working non-relativistically the Schrödinger equation becomes

$$\frac{1}{2m}\left(-i\nabla - \alpha(\mathbf{B} \times \mathbf{E})\right)^2 \psi = 0, \tag{18}$$

after neglecting terms $\alpha\mathbf{E}^2, \alpha\mathbf{B}^2$, and for $\mathbf{v} \cdot \mathbf{B} = 0$.

Now it is clear that a phase factor

$$\exp\left(-i\alpha \int_P^{\mathbf{r}} \mathbf{B} \times \mathbf{E} \cdot \mathbf{ds}\right) \tag{19}$$

will convert the solution of the Schrödinger equation with in the presence of the fields to the free Schrödinger equation, in the same approximation.

The phase

$$\chi_{\text{HMW}} = \alpha \int_{\mathcal{C}} \mathbf{B} \times \mathbf{E} \cdot \mathbf{ds} \tag{20}$$

is topological if

$$\begin{aligned} &\text{curl}\ (\mathbf{B} \times \mathbf{E}) \\ &= \mathbf{B}\,\text{div}\,\mathbf{E} - \mathbf{E}\,\text{div}\,\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{E} - (\mathbf{E} \cdot \nabla)\mathbf{B} \end{aligned} \tag{21}$$

vanishes in the interference region and is non-zero in the excluded region. Now electric charges can generate the topological phase. Working with an induced dipole not only changes the topology, it also removes the need to have magnetic monopoles as sources of $\mathbf{B} \times \mathbf{E}$.

It is not immediately clear from this derivation whether or not it is necessary to have $\mathbf{v} \perp \mathbf{E}$ as well as $\mathbf{v} \perp \mathbf{B}$. However if one begins with the relativistic equivalent of the Lagrangian of an electrically polarisable particle moving in electric and magnetic fields, one finds that a term $-\alpha(\mathbf{v} \cdot \mathbf{E})^2$ is missing from the Lagrangian of Eq. (17). With this term in place the demonstration of the phase factor of Eq. (19) reduces the wavefunction in the presence of fields to that without fields requires

$$\mathbf{v} \cdot \mathbf{E} = 0 \tag{22}$$

The relativistic discussion of polarisable materials has it origin in the famous 1908 paper of Minkowski [11]. There are accessible accounts in Pauli [12] and Møller [13] and Becker and Sauter [14]. This subject is nowadays not often treated in courses on electromagnetism. For example, there is no relativistic discussion of polarisable materials in Jackson [9], and so I go into a little of the detail here.

Minkowski's proposal is that the relativistic version of $\mathbf{D}$ and $\mathbf{H}$ is the tensor $G_{\mu\nu}$ obtained by replacing $\mathbf{E}$ and $\mathbf{B}$ in $F_{\mu\nu}$ by $\mathbf{D}$ and $\mathbf{H}$. The Lagrangian density is then proportional to $-G_{\mu\nu}F^{\mu\nu}$. As

$$\mathbf{D} = \mathbf{E} + \mathbf{P}$$

and

$$\mathbf{H} = \mathbf{B} - \mathbf{M}$$

the relativistic description of the electric and magnetic moments is a tensor (which Becker and Sauter [14] call the moments tensor) $K_{\mu\nu}$ constructed from $F_{\mu\nu}$ by replacing $\mathbf{E}$ with $\mathbf{P}$ and $\mathbf{B}$ with -$\mathbf{M}$. Then $G_{\mu\nu} = F_{\mu\nu} + K_{\mu\nu}$ and the interaction Lagrangian involving the moments is then

$$\mathcal{L}_{\text{int}} = -\frac{1}{4} K_{\mu\nu}F^{\mu\nu}. \tag{23}$$

For now I will ignore intrinsic moments which are proportional to the spin of the particle, and only consider induced moments, which are proportional to the applied fields. We need the generalisation of

$$\mathbf{P} = \alpha\mathbf{E}, \quad \text{and} \quad \mathbf{M} = \chi\mathbf{B}, \tag{24}$$

which hold in the rest frame of the material. α is the electric polarisability and χ is the magnetic susceptibility. Following Minkowski I write

$$u^\mu K_{\mu\nu} = \alpha u^\mu F_{\mu\nu} \quad \text{and} \quad u^\mu \tilde{K}_{\mu\nu} = \chi u^\mu \tilde{F}_{\mu\nu}. \tag{25}$$

which is identical to Eq. (24) in the rest frame, and is a tensor equation, so it is the correct generalisation.

Equation (VI 58) in Møller:

$$\begin{aligned} & K_{\mu\nu} \\ = \; & u_\mu K_{\nu\lambda}u^\lambda - u_\nu K_{\mu\lambda}u^\lambda + \epsilon_{\mu\nu\kappa\lambda}\tilde{K}^{\kappa\sigma}u_\sigma u_\lambda \end{aligned} \tag{26}$$

shows how to construct $K_{\mu\mu}$ from $u^\mu K_{\mu\nu}$ and $u^\mu \tilde{K}_{\mu\nu}$. The result is

$$\begin{aligned} & K_{\mu\nu} \\ = \; & \alpha\left\{u_\mu F_{\nu\lambda}u^\lambda - u_\nu F_{\mu\lambda}u^\lambda\right\} + \chi\epsilon_{\mu\nu\kappa\lambda}\tilde{F}^{\kappa\sigma}u_\sigma u^\lambda \end{aligned} \tag{27}$$

Consider only the induced electric dipole moment, and set $\chi = 0$. Then with the auxiliary field 4-vector

$$F^\mu = F^{\mu\nu}u_\nu = \gamma\left(\mathbf{E}\cdot\mathbf{v}, \mathbf{E} + \mathbf{v}\times\mathbf{B}\right) \tag{28}$$

$$K_{\mu\nu} = \alpha\left\{u_\mu F_\nu - u_\nu F_\mu\right\}, \tag{29}$$

the interaction Lagrangian is

$$\mathcal{L}_{\text{int}} = -\frac{1}{4}K^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}\alpha F_\mu F^\mu \tag{30}$$

$$= \frac{1}{2}\alpha\gamma^2\left\{(\mathbf{E}+\mathbf{v}\times\mathbf{B})^2 - (\mathbf{E}\cdot\mathbf{v})^2\right\} \tag{31}$$

To $O(v^2)$ the first term of this is the interaction Lagrangian of Wei, Han and Wei, but the $(\mathbf{E}\cdot\mathbf{v})^2$ term is missing from their Lagrangian. For their argument to lead to the HMW phase not only must $\mathbf{B}\cdot\mathbf{v} = 0$, as they require, but also $\mathbf{E}\cdot\mathbf{v} = 0$. This second geometrical constraint is missing in their analysis, but it is satisfied in in their example and in the Toulouse experiment. We may have escaped the restriction to 2+1 dimensions in the non-relativistic limit, but there are still geometric constraints.

My attempts to give the relativistic derivation of the HMW effect with an induced electric dipole have not been successful, so I have to be satisfied with this improved non-relativistic derivation.

In the Aharonov Bohm effect there is no field and *a fortiori* no force on the charge in the interference region. In the HWM effect, as realised with an induced electric dipole, there are clearly both electric field, and magnetic fields but is there a force? A possible force comes from the interaction of the induced electric dipole $\mathbf{d} = \alpha(\mathbf{E}+\mathbf{v}\times\mathbf{B})$ with the sum of the applied and the Róntgen electric fields, $\mathbf{E}+\mathbf{v}\times\mathbf{B}$, so the force on the dipole is

$$\mathbf{F} = -\nabla\left\{\mathbf{d}\cdot(\mathbf{E}+\mathbf{v}\times\mathbf{B})\right\} = \alpha\nabla(\mathbf{E}+\mathbf{v}\times\mathbf{B})^{\mathbf{2}}. \tag{32}$$

When $|\mathbf{E}+\mathbf{v}\times\mathbf{B}|$ is constant there is no force on the electric dipole. That is a severe constraint on the experimental realisation of the HMW effect, requiring uniformity of both fields. However there are no additional restrictions on the geometry of the fields.

One should also consider whether or not the dipole will experience a torque. The torque will be

$$\begin{aligned} & \boldsymbol{\mathcal{T}} \\ = \; & \boldsymbol{d}\times(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) \\ = \; & \alpha(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\times(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) = 0 \end{aligned} \tag{33}$$

which vanishes for the induced dipole. The induced dipole is always parallel to the effective electric field, and thus experiences no torque.

To summarise: the HMW phase for an induced electric dipole, as given in Eq. (20) is a topological phase when

- $\mathbf{v}\perp\mathbf{B}$ and $\mathbf{v}\perp\mathbf{E}$.
- curl $(\mathbf{B}\times\mathbf{E})$ vanishes in the interference region.

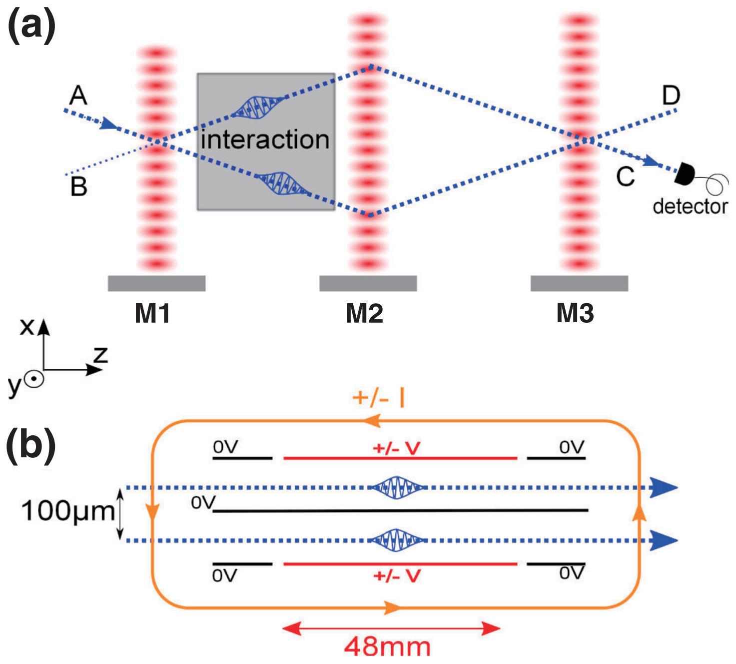

Fig. 2. The Toulouse experiment: (a) The atom interferometer, with two entrances A and B and two exits C and D (C is detected). An atomic beam (dotted lines) entering by A is diffracted by three quasiresonant laser standing waves produced by the mirrors Mi. The interaction region is placed where the distance between interferometer arms is largest, close to 100 μm. (b) The interaction region producing the electric and magnetic fields (not to scale — note the 100μm vertical scale and the 48mm horizontal scale). The interferometer arms (dotted lines) are separated by a septum, which is the common electrode of two plane capacitors producing opposite electric fields (high voltage electrodes labeled $\pm$V; grounded electrodes labeled 0V). Two rectangular coils (represented by a rectangle labeled $\pm I$) produce the magnetic field. After [15].

- curl $(\mathbf{B}\times\mathbf{E})$ is non zero in the excluded region.

Moreover although the the induced electric dipole is in electric and magnetic fields it experiences no torque, and, if the effective electric field $(\mathbf{E}+\mathbf{v}\times\mathbf{B})$ is constant in magnitude, it experiences no force.

4. Experimental Observations of the HMW Phase

In 2012, the Toulouse group led by Jacques Vigué succeeded in measuring the HMW phase [15–17] using an induced electric dipole moment in ^{7}Li ions, in a geometry which is a development of that proposed by Wei, Han and Wei. The experimental apparatus is summarised in Fig. 2.

The Toulouse group have taken great care to create uniform electric and magnetic fields, thus ensuring that there is no force on the atom. They analysed very carefully the uniformity of their fields and the forces that may be felt by the induced dipole to confirm that they do not contribute to the observed HWM effect [18]. They have also verified that the measured phase is independent of the velocity of the atoms.

The electric dipole moment is induced by applying an electric field to ^{7}Li ions, and is in the

Table I. Measured values of the HMW phase for different ion velocities, after Ref. [16]. VI is given in VA.

velocity in units ms^{-1}	phase in units 10^{-6} rad V I
744 ± 18	1.41 ± 0.24
1062 ± 20	1.315 ± 0.071
1520 ± 38	1.270 ± 0.072

plane of the path, not normal to it, and the magnetic field is normal to the plane of the path. However the dipole moment changes sign on the two sections of the path, and is not constant in direction. Were the central plate of the capacitor to shrink to a wire, the geometry would be just that of Wei, Han and Wei. The use of the double capacitor does not change the topology of the system but it both increases the magnitude of the possible the electric field, and increases the path over which the phase integral is performed. Both of these effects improve the observability of the phase.

The final results of this impressive experiment [16] is the observed phase for different ion velocities given in Table I.

The phase is clearly independent of the velocity as it should be, and the weighted mean value is $\phi_{\text{HMW, obs}} = (1.29 \pm 0.10) \times 10^{-6}$rad V I , to be compared to the calculated value $\phi_{\text{HMW, cal}} = (1.28 \pm 0.03) \times 10^{-6}$rad V I . The agreement of the measured and calculated values is well within the errors, and there is no doubt the the HMW phase has been successfully observed.

It is amusing to note that, if the ion has a magnetic moment, the same apparatus allows the measurement of the Aharonov Casher phase. The experiment has been performed by the Toulouse group, giving the first measurement of the topological Aharonov Casher effect with an atomic beam [19]. In this AC experiment it is necessary to disentangle the AC and HMW phases, which they successfully do.

5. Conclusion

The original suggestion, by He and McKellar, that the motion of an electric dipole around a line of magnetic monopoles would produce a topological phase which was the dual of the Aharonov Casher phase was made without any suggestion of how this esoteric phase could be measured. The realisation by Wei, Han and Wei that the need to induce the electric dipole with an electric field also eliminated the need for a monopole like magnetic field was an important step forward. This led to the measurement of the HMW phase, 20 years after the original theoretical model, by the Toulouse group.

To those of us who have been involved in this work, it is a satisfying story.

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