SYMMETRY RESTORATION AT FINITE TEMPERATURE

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Some field theories, possessing a symmetry, admit only solutions which do not respect the symmetry. This is the familiar Goldstone-Nambu phenomenon. However, when the theory is examined in an external environment, the hidden symmetry may be restored. We show how spontaneously broken global and local symmetries are restored in a temperature environment. The methods for studying this question are explained, and the critical temperature for symmetry restoration is computed in several models.

The formalism which we shall use is that of the generalized effective potential $V(\phi,G)$. That object, an extension of the familiar effective potential $V(\phi)$, is defined as follows, for a theory described by a Lagrangian $\mathcal{L}(\Phi)$[1]. Consider the vacuum persistence amplitude $Z(J,K)$ in the presence of space-time varying sources $J(x)$ and $K(x,y)$; $J(x)$ couples to $\Phi(x)$ and $K(x,y)$ to $\frac{1}{2}\Phi(x)\Phi(y)$. The generalized effective action $\Gamma(\phi,G)$ is defined by a double Legendre transform of $-i\ln Z(J,K) = W(J,K)$.

$$\frac{\delta W(J,K)}{\delta J(x)} = \phi(x), \quad \frac{\delta W(J,K)}{\delta K(x,y)} = \frac{1}{2}[\phi(x)\phi(y) + G(x,y)] \tag{1}$$

$$\begin{aligned}\Gamma(\phi,G) = W(J,K) - &\int dx\,\phi(x)J(x) \\ &-\frac{1}{2}\int dx\,dy\,[\phi(x)\phi(y) + G(x,y)]K(x,y)\end{aligned} \tag{2}$$

In the physical theory the sources are absent, hence $\Gamma(\phi,G)$ satisfies

$$\begin{aligned}\frac{\delta\Gamma(\phi,G)}{\delta\phi(x)} &= -J(x) - \int dy\,K(x,y)\phi(y) = 0 \\ \frac{\delta\Gamma(\phi,G)}{\delta G(x,y)} &= -\frac{1}{2}K(x,y) = 0\end{aligned} \tag{3}$$

It is clear from (1) that $\phi(x)$ is the expectation of the quantum field and $G(x,y)$ is the propagator of the theory. However translation invariance, which we do not expect to be spontaneously broken, implies that the field expectation is constant: $\phi(x) = \phi$, and that the propagator depends only on the coordinate difference: $G(x,y) = G(x-y)$. The generalized effective potential $V(\phi,G)$ is defined from $\Gamma(\phi,G)$ by using these translation invariant forms.

$$\Gamma(\phi,G)\Big|_{\substack{\text{translation}\\ \text{invariant}}} = -V(\phi,G)\int d^4x \tag{4}$$

It is a function of ϕ and a functional of $G(x-y)$; for later convenience G is expressed in the momentum representation: $G(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} G(k)$.

The stability equations (3) now become

$$\frac{\partial V(\phi,G)}{\partial\phi} = 0$$

$$\frac{\delta V(\phi,G)}{\delta G(k)} = 0 \tag{5}$$

For a definite example we chose a theory described by

$$\mathcal{L}(\Phi) = \tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - U(\Phi)$$

$$U(\Phi) = \frac{\lambda_o}{4!}\left(\frac{6\mu_o^2}{\lambda_o} - \Phi^2\right)^2 \tag{6}$$

and compute $V(\phi,G)$ in the Hartree-Fock approximation.[1]

$$V(\phi,G) = U(\phi) + \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\ell n G(k)$$
$$-\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}G(k)\left[k^2 + \mu_o^2 - \frac{\lambda_o\phi^2}{2}\right] + \frac{\lambda_o}{8}\left[\int\frac{d^4k}{(2\pi)^4}G(k)\right]^2 \tag{7}$$

The first term on the right hand side of (7) is the no-loop tree approximation, the next two terms represent the one-loop quantum correction, the last term arises from a two-loop graph. From (5), we find

$$\left(-\mu_o^2 + \frac{\lambda_o}{2}\int\frac{d^4k}{(2\pi)^4}G(k) + \frac{\lambda_o\phi^2}{6}\right)\phi = 0 \tag{8a}$$

$$iG^{-1}(k) = k^2 + \mu_o^2 - \frac{\lambda_o}{2} \int \frac{d^4k}{(2\pi)^4} G(k) - \frac{\lambda_o \phi^2}{2} \tag{8b}$$

Upon defining the renormalized mass parameter

$$-m^2 = -\mu_o^2 + \frac{\lambda_o}{2} \int \frac{d^4k}{(2\pi)^4} G(k)$$

the above equation becomes

$$(-m^2 + \frac{\lambda_o \phi^2}{6}) \phi = 0$$

$$iG^{-1}(k) = k^2 + m^2 - \frac{\lambda_o \phi^2}{2} \tag{9}$$

and the only consistent solution is the one which breakes spontaneously the $\Phi \leftrightarrow -\Phi$ symmetry of (6).

$$\phi^2 = \frac{6m^2}{\lambda_o}$$

$$iG^{-1}(k) = k^2 - 2m^2 \tag{10}$$

[The solution $\phi = 0$ is unacceptable, since it leads to a propagator with imaginary mass.] Similar calculations can be performed in an O(N) invariant Φ^4 theory, [for which the Hartree-Fock approximation dominates when N is large] in different dimensions.[2] The behavior of spontaneous symmetry breaking with varying dimensionality is thereby exposed: a continuous symmetry can be broken only when the dimension of spacetime is greater than 2.

Another interesting question is whether a spontaneously broken symmetry can be restored at finite temperature. Qualitative arguments indicating that this should happen were given by Kirzhnits and Linde,[3] and subsequent detailed computations have established the phenomenon.[4] This topic can be readily analyzed by a straight forward extension of the formalism.

A field theory at a finite temperature, proportional to $1/\beta$, is conveniently described by its finite-temperature Green's functions, which are defined by a statistical average: tr $e^{-\beta H} T\Phi(x_1) \ldots \Phi(x_n)/\text{tr } e^{-\beta H}$. These Green's functions satisfy the same differential equations as the corresponding ones at zero temperature. However, the boundary conditions are different: in the complex time interval $[0,-i\beta]$ there are periodicity requirements. It follows that in a theory with interactions, the formulas for temperature Green's functions in terms of the elementary free-field propagator and vertices are the same as at zero temperature, except that the free-field propagators have a "momentum" representation consistent with the periodicity conditions.

$$D_\beta(x) = \int_k e^{-ikx} \frac{i}{k^2-m^2} \tag{11}$$

where $\int_k$ stands for $\frac{1}{-i\beta} \sum_n \int \frac{d^3k}{(2\pi)^3}$, $n = 0, \pm 1 \ldots; k^2 = k_o^2 - \underset{\sim}{k}^2$; and k_o equals $\frac{2\pi n}{-i\beta}$. Consequently symmetry behavior at finite temperature can be studied in terms of solution to eqs. (8) which are also valid at finite temperature.[5]

We are thus led to the system of equations

$$[-\mu_o^2 + \frac{\lambda_o}{2} \int_k G_\beta(k) + \frac{\lambda_o \phi^2{}_\beta}{6}]\ \phi_\beta = 0 \tag{12a}$$

$$iG_\beta^{-1}(k) = k^2 + \mu_o^2 - \frac{\lambda_o}{2} \int_k G_\beta(k) - \frac{\lambda_o \phi_\beta^2}{2} \tag{12b}$$

$$\phi_\beta = \mathrm{tr}\ e^{-\beta H}\ \Phi(x)/\mathrm{tr}\ e^{-\beta H}$$

$$\phi_\beta^2 + \int_k e^{-ik(x-y)} G_\beta(k) = \mathrm{tr}\ e^{-\beta H} T\Phi(x)\Phi(y)/\mathrm{tr}\ e^{-\beta H}$$

The symmetric solution $\phi_\beta = 0$ will be acceptable if the mass parameter in the propagator is positive.

Eq. (12b) is solved by $G_\beta^{-1}(k) = i[-k^2 + m_\beta^2]$ where the temperature dependent mass m_β satisfies [at $\phi_\beta = 0$] the gap equation.

$$m_\beta^2 = -\mu_o^2 + \frac{\lambda_o}{2} \int_k \frac{i}{k^2 - m_\beta^2}$$

$$= -\mu_o^2 + \frac{\lambda_o}{2\beta} \sum_n \int \frac{d\underset{\sim}{k}}{(2\pi)^3} \frac{1}{\frac{(2\pi n)^2}{\beta^2} + \underset{\sim}{k}^2 + m_\beta^2}$$

$$= -\mu_o^2 + \frac{\lambda_o}{2}\int\frac{d\underset{\sim}{k}}{(2\pi)^3} \; \frac{1}{2\sqrt{\underset{\sim}{k}^2 + m_\beta^2}} + \frac{\lambda_o}{\beta^2} f(m_\beta^2\beta^2) \tag{13a}$$

$$f(a^2) = \frac{1}{4\pi^2} \int_o^\infty \frac{x^2 dx}{\sqrt{x^2 + a^2}} [e^{\sqrt{x^2 + a^2}} - 1]^{-1} \tag{13b}$$

The integral occurring in (13a) is divergent; the gap equation must be renormalized. It is important that renormalization does not involve any counterterms beyond those of the zero temperature theory. To carry out the renormalization we evaluate the integral in (13a) with a cutoff, and rewrite that expression.

$$m_\beta^2 = -m^2 + \frac{\lambda}{\beta^2} f(m_\beta^2\beta^2) + \frac{\lambda}{32\pi^2} m_\beta^2 \ell n \frac{m_\beta^2}{m^2} \tag{13c}$$

Here m^2 is the renormalized mass parameter, and λ is the renormalized coupling constant; they are given in terms of the bare parameters and cutoff Λ.

$$-m^2 = -\mu_o^2 + \frac{\lambda_o}{32\pi^2} [\frac{\Lambda^2}{2} + m^2 \ell n \frac{\Lambda^2}{m^2} - m^2]$$

$$\frac{1}{\lambda} = \frac{1}{\lambda_o} + \frac{1}{32\pi^2}[\ell n \frac{\Lambda^2}{m^2} - 1]$$

Eq. (13c) is the renormalized gap equation for m_β^2 and we seek solutions for positive m_β^2. At low temperature $[\beta\to\infty]$ there is symmetry breaking and no positive solution exists. At high temperature $[\beta\to 0]$ one can satisfy (13c) with $m_\beta^2>0$. The critical temperature β_c^{-1} is that value of the temperature for which m_β^2 vanishes. Therefore from (13b) and (13c) we have

$$0 = -m^2 + \frac{\lambda}{\beta_c^2} f(0)$$

$$\frac{1}{\beta_c^2} = \frac{24m^2}{\lambda} \tag{14}$$

For $\beta^{-1}>\beta_c^{-1}$ symmetry is manifest, for $\beta^{-1}<\beta_c^{-1}$ the symmetry is broken.

In this calculation we have ignored higher loop corrections, which are proportional to higher powers of the coupling constant. Hence the entire approach is correct only if the coupling is small, in which case β_c^{-1}, given by (14), is indeed large. One can estimate that for small λ, the corrections are $0(\lambda^{1/2})$ i.e. $\beta_c^{-2} = \frac{24m^2}{\lambda}[1+0(\lambda^{1/2})]$. Eq. (13c) can be solved for m_β when $\beta \gtrsim \beta_c$.

$$m_\beta = \frac{2\pi}{3}(\beta^{-1}-\beta_c^{-1}) \tag{15}$$

The computation may be extended to an O(N) invariant Φ^4. [The Hartree-Fock approximation then dominates for large N.] Once again (14) and (15) are found, except that λ is replaced by $\lambda\,\frac{(N+2)}{3}$. Also if gauge fields are included, so that the theory is locally O(N) invariant, one gets

$$\frac{1}{\beta_c^2} = \frac{m^2}{\lambda\frac{(N+2)}{72} + e^2\,\frac{(N-1)}{4}} \tag{16}$$

where e is the gauge coupling strength.[6]

What is the significance of phase transitions in field theory which restore a spontaneously broken symmetry? One answer addresses itself to questions of principle. It may be thought that a theory with a hidden, spontaneously broken symmetry is equivalent to a theory without any symmetry at all, and that the various relationships that exist between masses, coupling constants etc. are merely consequences of perturbative unitary or renormalizability. However, if the physical environment can be arranged so that the symmetry becomes manifest, one can not doubt the existence of the symmetry. Practical application of this phenomenon must be confined to speculations about the early universe, since only in that environment were there temperatures sufficiently high to effect a phase transition. [The order of magnitude of β_c^{-1} may be estimated as follows. For global symmetries, we identify the vacuum expectation value of the field with f_π, the pion decay constant. Hence $f_\pi^2 \approx \frac{6m^2}{\lambda}$ and, $\beta_c^{-1} = 0(f_\pi) \gtrsim 100$ MeV. For local symmetries, recall that the spontaneously generated vector meson mass is $\sqrt{2}ef_\pi$ and the mass of the Higgs particle is $\sqrt{2m^2}$. Hence (16) is also given by $\beta_c^{-2} = \left[\frac{e^2}{m_W^2}\,\frac{N+2}{6} + \frac{e^2}{m_H^2}\,\frac{N-1}{2}\right]^{-1}$. With $e^2 \approx 10^{-2}$ and $m_H < m_W$ we get $\beta_c^{-1} \approx 0(10m_H)$.] Thus a temperature environment for field theoretic phase transitions is not readily available. However, One may study other environments which effect phase transitions described by critical parameters that have more immediate experimental consequences.[7]

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5. This brief synopsis of field theory at finite temperature is of course an inadequate summary of the beautiful work of Martin,Schwinger and others. For a fuller account see for example L. P. Kadanoff and G. Baym Quantum Statistical Mechanics, (W. A. Benjamin,Menlo Park, 1962). Discussions of this topic which focus on the present application are also given in Refs.4.
6. The approximation and results of this section are familiar in statistical mechanics, where they are known as "spherical model","mean-field theory" etc. For a review see E. Stanley, Introduction to Phase Transitions and Critical Phenomena (Oxford Univ.Press, New York,1971).
7. That phase transitions can be induced by high matter density has been shown by T.D.Lee and G.C.Wick, Phys.Rev. D 9, 2291 (1974). A.Salam and J. Strathdee, Nature 252, 569 (1974) and ICTP preprint, have studied field theoretic phase transitions in electromagnetic environments.

DISCUSSION

K. Symanzik: Could you comment on the gauge dependence or independence of your results for a gauge theory?
The critical temperature was calculated in a variety of renormalizable gauges parametrized by gauge constants (covariant gauges, R_ξ gauges, etc.) and the answer comes out independent of these constants, hence gauge invariant within that class. For non-renormalizable gauges (the unitary gauge and others) a well defined answer does not emerge. We attribute this to a failure of the perturbation theory.

J. Zittartz: Could you not get your answers just by dimensional arguments?
We give a numerical value for the critical temperature. Dimensional arguments, which were used initially by Kirzhnits and Linde, will yield the dependence of β_c^2 on m^2 and λ, but will not provide the numerical coefficient.

K.R. Ito: You consider an equation for the temperature dependent propagator. On the other hand, we can define phase transitions by the order parameter. Are these equivalent?
Yes; indeed Dolan has rederived our results by calculating the order parameter.

T. Kugo: What happens in massless scalar electrodynamics, where Coleman and Weinberg find symmetry breaking at zero temperature?
We have not done calculations for this massless theory, since our perturbation theory requires a mass parameter. However, I would speculate that finite temperature again restores the symmetry, and that a mass is generated for the charged particle, but not for the photon.