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Discrete Painlevé system associated with Unitary matrix model

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Abstract. We discuss the emergence of discrete Painlevé system in the context of irregular conformal block which is also a $SU(2)$ $\mathcal{N}=2$ $N_f=2$ supersymmetric gauge theory. We develop a framework of orthogonal polynomials in unitary matrix model.

1. Introduction

In this paper, we consider the generic following type of the β-deformed "matrix model" depending on two integration contours C_L and C_R:

$$Z(N_L,N_R)=\mathcal{C}\left(\prod_{i=1}^{N_L}\int_{C_L}\mathrm{d}w_i\right)\left(\prod_{j=1}^{N_R}\int_{C_R}\mathrm{d}w_{N_L+j}\right)\Delta^{2\beta}(w)\exp\left(\sqrt{\beta}\sum_{I=1}^{N_L+N_R}W(w_I)\right), \quad (1.1)$$

where $\mathcal{C}$ is a normalization constant and $\Delta(w)$ is the Vandermonde determinant:

$$\Delta(w)=\prod_{1\le I<J\le N_L+N_R}(w_I-w_J). \quad (1.2)$$

Following [1], we introduce their generating function

$$\begin{aligned}\underline{Z}(N;\mu_L,\mu_R)&=\sum_{N_L+N_R=N}\frac{\mu_L^{N_L}}{N_L!}\frac{\mu_R^{N_R}}{N_R!}Z(N_L,N_R)\\&=\frac{\mathcal{C}}{N!}\int_C\mathrm{d}^Nw\,\Delta^{2\beta}(w)\,\exp\left(\sqrt{\beta}\sum_{I=1}^{N}W(w_I)\right),\end{aligned} \quad (1.3)$$

where $C=\mu_LC_L+\mu_RC_R$, i.e.,

$$\int_C=\mu_L\int_{C_L}+\mu_R\int_{C_R}. \quad (1.4)$$

As we recall from [2] (see, [3, 4, 5, 6, 7, 8] for earlier references), in the case of $\beta=1$ which we consider in the body of this paper, the $N_f=2$ matrix model of the above form with $\alpha_{1+2}\in\mathbb{Z}$ in fact reduces to the unitary matrix model with cosine + log potential in section 3.

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2. Unitary matrix model

In this section, we briefly review the unitary matrix model, the method of orthogonal polynomials and the string equations to explain our notation.

The partition function of the unitary matrix model is defined by

$$Z_{U(N)} := \frac{1}{\mathrm{vol}(U(N))} \int [\mathrm{d}U] \exp\Big(\mathrm{Tr}\, W_U(U)\Big), \tag{2.1}$$

where U is an $N \times N$ unitary matrix and $W_U(U)$ is a potential. We define a unitary Haar measure $[\mathrm{d}U]$ from the metric

$$\mathrm{d}s^2 = \mathrm{Tr}\big(\mathrm{d}U^\dagger\, \mathrm{d}U\big) = -\mathrm{Tr}\big(U^{-1}\mathrm{d}U\big)^2. \tag{2.2}$$

With this normalization of the measure, the volume of the unitary group $U(N)$ is given by

$$\mathrm{vol}(U(N)) = \int [\mathrm{d}U] = \frac{(2\pi)^{(1/2)N(N+1)}}{G_2(N+1)}, \tag{2.3}$$

where $G_2(z)$ is the Barnes function. Explicitly, $G_2(N+1)$ is given by

$$G_2(N+1) = \prod_{j=1}^{N-1} j! = \prod_{k=1}^{N-1} k^{N-k}. \tag{2.4}$$

If we diagonalize the unitary matrix U as

$$U = V^{-1}U_D V, \qquad U_D = \mathrm{diag}(z_1, z_2, \cdots, z_N), \qquad |z_i| = 1, \tag{2.5}$$

we have

$$Z_{U(N)} = \frac{1}{N!}\left(\prod_{i=1}^{N} \oint \frac{\mathrm{d}z_i}{2\pi\mathrm{i}\, z_i}\right) \Delta(z)\Delta(z^{-1}) \exp\left(\sum_{i=1}^{N} W_U(z_i)\right), \tag{2.6}$$

where

$$\Delta(z) = \prod_{1\le i<j\le N} (z_i - z_j). \tag{2.7}$$

Let

$$\mathrm{d}\mu(z) := \frac{\mathrm{d}z}{2\pi\mathrm{i}\, z} \exp\Big(W_U(z)\Big). \tag{2.8}$$

Then

$$Z_{U(N)} = \frac{1}{N!} \int \prod_{i=1}^{N} \mathrm{d}\mu(z_i)\, \Delta(z)\Delta(z^{-1}). \tag{2.9}$$

The partition function (2.6) expressed in eigenvalue integrals may be generalized to the form of the two contour model (1.1). A natural choice of the two contours C_L and C_R is take them as circles of radius r_L and r_R respectively. Suppose, $r_L < r_R$ and there is no singularity in the region $r_L \le |w| \le r_R$. Then the contours can be smoothly defomed to circles of the same radius, i.e., to the same contour: $C_L = C_R$. Then, for the two contour unitary matrix model, $Z_{U(N)}(N_L, N_R)$ depends only on $N = N_L + N_R$, and the generating function $\underline{Z}_{U(N)}$ is essentially $Z_{U(N)}(N, 0)$. Because $\underline{Z}_{U(N)} = (\mu_L + \mu_R)^N Z_{U(N)}$, we can set $\mu_L + \mu_R = 1$ without loss of generality. Hence $\underline{Z}_{U(N)} = Z_{U(N)}$.

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2.1. Orthogonal polynomials

The unitary matrix model can be solved [9, 10, 11] by the method of orthogonal polynomials [12, 13]. Let us use the monic orthogonal polynomials [9, 10] (In [11], orthogonal polynomials of different type have been introduced to solve the unitary matrix model).

Let p_n and $\tilde{p}_n$ $(n \geq 0)$ be monic polynomials satisfying orthogonality conditions with respect to the measure (2.8)

$$\int \mathrm{d}\mu(z) p_n(z) \tilde{p}_m(1/z) = h_n \delta_{n,m}, \tag{2.10}$$

where

$$p_n(z) = z^n + \sum_{k=0}^{n-1} A_k^{(n)} z^k, \qquad \tilde{p}_n(1/z) = z^{-n} + \sum_{k=0}^{n-1} B_k^{(n)} z^{-k}. \tag{2.11}$$

Let us introduce the moments μ_n for the measure (2.8) by

$$\mu_n := \int \mathrm{d}\mu(z) z^n, \qquad (n \in \mathbb{Z}). \tag{2.12}$$

For later convenience, we define $\mathcal{K}_k^{(n)}$ by

$$\mathcal{K}_k^{(n)} := \det\big(\mu_{j-i+k}\big)_{1\leq i,j\leq n}, \qquad (n \geq 0, k \in \mathbb{Z}). \tag{2.13}$$

From the definition, the orthogonal polynomials have the following properties:

$$\int \mathrm{d}\mu(z)\, p_n(z) z^{-k} = 0, \qquad \int \mathrm{d}\mu(z)\, z^k \tilde{p}_n(1/z) = 0, \qquad (k = 0, 1, \cdots, n-1), \tag{2.14}$$

Using these and the monic properties, the orthogonal polynomials are determined as

$$p_n(z) = \frac{1}{\tau_n} \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_{-1} & \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{-n+1} & \mu_{-n+2} & \mu_{-n+3} & \cdots & \mu_1 \\ 1 & z & z^2 & \cdots & z^n \end{vmatrix}, \tag{2.15}$$

$$\tilde{p}_n(1/z) = \frac{1}{\tau_n} \begin{vmatrix} \mu_0 & \mu_{-1} & \mu_{-2} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \mu_{-1} & \cdots & \mu_{-n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_{n-2} & \mu_{n-3} & \cdots & \mu_{-1} \\ 1 & z^{-1} & z^{-2} & \cdots & z^{-n} \end{vmatrix}, \tag{2.16}$$

where

$$\tau_n := \mathcal{K}_0^{(n)} = \det\big(\mu_{j-i}\big)_{1\leq i,j\leq n}. \tag{2.17}$$

(We set $\tau_0 = 1$). We can easily see that these polynomials obey (2.14).

The normalization constants h_n defined by (2.10) are given by

$$h_n = \frac{\tau_{n+1}}{\tau_n} = \frac{\mathcal{K}_0^{(n+1)}}{\mathcal{K}_0^{(n)}}. \tag{2.18}$$

The constant terms of these polynomials will play important roles.

$$A_n := p_n(0) = A_0^{(n)} = (-1)^n \frac{\mathcal{K}_1^{(n)}}{\mathcal{K}_0^{(n)}}, \qquad B_n := \tilde{p}_n(0) = B_0^{(n)} = (-1)^n \frac{\mathcal{K}_{-1}^{(n)}}{\mathcal{K}_0^{(n)}}. \tag{2.19}$$

Note that

$$\frac{h_n}{h_{n-1}} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \qquad 1 - \frac{h_n}{h_{n-1}} = \frac{\tau_n^2 - \tau_{n+1}\tau_{n-1}}{\tau_n^2}. \tag{2.20}$$

Using an identity

$$\tau_n^2 - \tau_{n+1}\tau_{n-1} = \big(\mathcal{K}_0^{(n)}\big)^2 - \mathcal{K}_0^{(n+1)}\mathcal{K}_0^{(n-1)} = \mathcal{K}_1^{(n)}\mathcal{K}_{-1}^{(n)}, \tag{2.21}$$

we can show that

$$1 - \frac{h_n}{h_{n-1}} = \frac{\mathcal{K}_1^{(n)}\mathcal{K}_{-1}^{(n)}}{(\mathcal{K}_0^{(n)})^2} = A_n B_n. \tag{2.22}$$

Thus we have the following relations:

$$\frac{h_n}{h_{n-1}} = 1 - A_n B_n. \tag{2.23}$$

Note that

$$\Delta(z) = \det\big(p_{j-1}(z_i)\big)_{1\le i,j\le N} = \sum_{\sigma\in S_N} (-1)^{\varepsilon(\sigma)} \prod_{k=1}^{N} p_{\sigma(k)-1}(z_k), \tag{2.24}$$

$$\Delta(z^{-1}) = \det\big(\tilde{p}_{j-1}(1/z_i)\big)_{1\le i,j\le N} = \sum_{\sigma\in S_N} (-1)^{\varepsilon(\sigma)} \prod_{k=1}^{N} \tilde{p}_{\sigma(k)-1}(1/z_k), \tag{2.25}$$

Using these relations, the partition function (2.6) is evaluated as

$$\underline{Z}_{U(N)} = \frac{1}{N!}\int \prod_{i=1}^{N} \mathrm{d}\mu(z_i)\, \Delta(z)\Delta(z^{-1}) = \prod_{k=0}^{N-1} h_k = \prod_{k=0}^{N-1} \frac{\tau_{k+1}}{\tau_k} = \tau_N. \tag{2.26}$$

Also, it can be written as

$$\underline{Z}_{U(N)} = h_0^N \prod_{j=1}^{N-1} \big(1 - A_j B_j\big)^{N-j}. \tag{2.27}$$

The orthogonal polynomials $p_n(z)$ obey the following relations:

$$z\, p_n(z) = p_{n+1}(z) + \sum_{k=0}^{n} C_k^{(n)}\, p_k(z), \tag{2.28}$$

where

$$C_k^{(n)} = (-1)^{n-k} \frac{\mathcal{K}_1^{(n+1)}\mathcal{K}_{-1}^{(k)}}{\mathcal{K}_0^{(n)}\mathcal{K}_0^{(k+1)}} = -\frac{h_n}{h_k} A_{n+1} B_k, \qquad (0 \le k \le n). \tag{2.29}$$

If we use p_k only, all lower degree polynomials appear in the expansion of $z\, p_n(z)$. Similarly, $\tilde{p}_n$ behave as follows:

$$z^{-1}\tilde{p}_n(1/z) = \tilde{p}_{n+1}(1/z) + \sum_{k=0}^{n} \widetilde{C}_k^{(n)} \tilde{p}_k(1/z), \tag{2.30}$$

where

$$\widetilde{C}_k^{(n)} = (-1)^{n-k} \frac{\mathcal{K}_1^{(k)} \mathcal{K}_{-1}^{(n+1)}}{\mathcal{K}_0^{(k+1)} \mathcal{K}_0^{(n)}} = -\frac{h_n}{h_k} A_k B_{n+1}, \qquad (0 \le k \le n). \tag{2.31}$$

The above recursion relations (2.28) and (2.30) can be rewritten as three-term relations:

$$p_{n+1}(z) = z\, p_n(z) + A_{n+1}\, z^n\, \tilde{p}_n(1/z), \tag{2.32}$$

$$\tilde{p}_{n+1}(1/z) = z^{-1}\, \tilde{p}_n(1/z) + B_{n+1}\, z^{-n}\, p_n(z). \tag{2.33}$$

2.2. String equations
Recall that

$$\mathrm{d}\mu(z) = \frac{\mathrm{d}z}{2\pi \mathrm{i}\, z} \exp\Big(W_U(z)\Big). \tag{2.34}$$

Using the following constraints for $k \in \mathbb{Z}$ and $\ell, m \ge 0$,

$$\begin{aligned} 0 &= \int \mathrm{d}z \frac{\partial}{\partial z} \left[\frac{z^k}{2\pi \mathrm{i}} \exp\Big(W_U(z)\Big) p_\ell(z) \tilde{p}_m(1/z) \right] \\ &= \int \mathrm{d}\mu(z)\, z^{k+1}\, W_U'(z) p_\ell(z) \tilde{p}_m(1/z) + \int \mathrm{d}\mu(z)\, z \frac{\partial}{\partial z} \big(p_\ell(z) z^k \tilde{p}_m(1/z)\big), \end{aligned} \tag{2.35}$$

we can obtain various polynomial equations for A_n and B_n.

In particular, let us consider the following three cases of (2.35): (i) $(k, \ell, m) = (-1, n, n-1)$, (ii) $(k, \ell, m) = (0, n, n)$ and (iii) $(k, \ell, m) = (1, n-1, n)$. They lead to the "string equations"

$$\int \mathrm{d}\mu(z) W_U'(z) p_n(z) \tilde{p}_{n-1}(1/z) = n(h_n - h_{n-1}), \tag{2.36}$$

$$\int \mathrm{d}\mu(z)\, z\, W_U'(z) p_n(z) \tilde{p}_n(1/z) = 0, \tag{2.37}$$

$$\int \mathrm{d}\mu(z)\, z^2\, W_U'(z) p_{n-1}(z) \tilde{p}_n(1/z) = -n(h_n - h_{n-1}). \tag{2.38}$$

3. Unitary matrix model with logarithmic potential

Let us consider the unitary matrix model with the following potential

$$W_U(z) = -\frac{1}{2\, \underline{g}_s} \left(z + \frac{1}{z} \right) + M \log z. \tag{3.1}$$

In the gauge theory parameters, $\underline{g}_s = g_s/\Lambda_2$ and $M = \alpha_{1+2} + N = (m_2 - m_1)/g_s$. We assume that M is an integer. Note that $1/(2\underline{g}_s) = \Lambda_2/(2g_s) = q_{02}$.

3.1. Moments and related quantities
The moments for this potential are given by

$$\begin{aligned} \mu_n &= \oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \exp\left(-\frac{1}{2\, \underline{g}_s} \left(z + \frac{1}{z} \right) \right) z^{M+n} \\ &= \left(-\frac{1}{2\, \underline{g}_s} \right)^{|M+n|} \sum_{k=0}^{\infty} \frac{1}{k!\, (k + |M+n|)!} \left(\frac{1}{2\, \underline{g}_s} \right)^{2k} = (-1)^{M+n} I_{|M+n|}(1/\underline{g}_s), \end{aligned} \tag{3.2}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind:

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k}. \tag{3.3}$$

Note that

$$\begin{aligned}\mathcal{K}_k^{(n)} &= \det(\mu_{j-i+k})_{1\le i,j\le n} \\ &= \det\Big((-1)^{M+j-i+k} I_{|M+j-i+k|}(1/\underline{g}_s)\Big)_{1\le i,j\le n} \\ &= (-1)^{n(M+k)} K_{M+k}^{(n)},\end{aligned} \tag{3.4}$$

where

$$K_\nu^{(n)} := \det\Big(I_{j-i+\nu}(1/\underline{g}_s)\Big)_{1\le i,j\le n}, \qquad (\nu\in\mathbb{C}; n=0,1,2,\cdots). \tag{3.5}$$

For an integer k, it holds that $I_{-k}(z) = I_k(z)$. Therefore, for $j-i+M+k\in\mathbb{Z}$, we have $I_{j-i+M+k}(1/\underline{g}_s) = I_{|j-i+M+k|}(1/\underline{g}_s)$. Also, we have $K_{-k}^{(n)} = K_k^{(n)}$ $(k\in\mathbb{Z})$. For later convenience, we have defined $K_\nu^{(n)}$ (3.5) as a determinant of $I_{j-i+\nu}(1/\underline{g}_s)$ such that the index M in $K_{M+k}^{(n)}$ can be analytically continued from an integer to any complex number.

Note that

$$\tau_n = \mathcal{K}_0^{(n)} = (-1)^{nM} K_M^{(n)}. \tag{3.6}$$

The normalization constants of the orthogonal polynomials are given by

$$h_n = \frac{\mathcal{K}_0^{(n+1)}}{\mathcal{K}_0^{(n)}} = (-1)^M \frac{K_M^{(n+1)}}{K_M^{(n)}}. \tag{3.7}$$

In particular, $h_0 = (-1)^M I_M(1/\underline{g}_s)$.

The constant term of the orthogonal polynomials are written in terms of $K_k^{(n)}$ as follows:

$$A_n = p_n(0) = (-1)^n \frac{\mathcal{K}_1^{(n)}}{\mathcal{K}_0^{(n)}} = \frac{K_{M+1}^{(n)}}{K_M^{(n)}}, \tag{3.8}$$

$$B_n = \tilde{p}_n(0) = (-1)^n \frac{\mathcal{K}_{-1}^{(n)}}{\mathcal{K}_0^{(n)}} = \frac{K_{M-1}^{(n)}}{K_M^{(n)}}. \tag{3.9}$$

The partition function (2.27) can be written in terms of these objects:

$$\underline{Z}_{U(N)} = (-1)^{MN} K_M^{(N)} = \prod_{k=0}^{N-1} h_k = h_0^N \prod_{j=1}^{N-1} \big(1 - A_j B_j\big)^{N-j}. \tag{3.10}$$

This partition function is essentially the tau function of the Painlevé III equation.

3.2. String equations

Let us write the string equations (2.36), (2.37) and (2.38) explicitly for the case of the potential (3.1). Since

$$W_U'(z) = -\frac{1}{2\,\underline{g}_s}\left(1 - \frac{1}{z^2}\right) + \frac{M}{z}, \tag{3.11}$$

we have

$$\int \mathrm{d}\mu(z)\, W_U'(z) p_n(z) \tilde{p}_{n-1}(1/z) = \frac{1}{2\,\underline{g}_s}\big(\widetilde{C}_n^{(n)} + \widetilde{C}_{n-1}^{(n-1)}\big) h_n + M h_n, \tag{3.12}$$

$$\int \mathrm{d}\mu(z)\, z\, W_U'(z) p_n(z) \tilde{p}_n(1/z) = -\frac{1}{2\,\underline{g}_s}\big(C_n^{(n)} - \widetilde{C}_n^{(n)}\big) h_n + M h_n, \tag{3.13}$$

$$\int \mathrm{d}\mu(z)\, z^2\, W_U'(z) p_{n-1}(z) \tilde{p}_n(1/z) = -\frac{1}{2\,\underline{g}_s}\big(C_n^{(n)} + C_{n-1}^{(n-1)}\big) h_n + M h_n. \tag{3.14}$$

Here we have used (2.28) and (2.30).

Then the string equations (2.36), (2.37), (2.38) for this potential become

$$\begin{aligned} \frac{1}{2\,\underline{g}_s}\big(\widetilde{C}_n^{(n)} + \widetilde{C}_{n-1}^{(n-1)}\big) + M &= n\left(1 - \frac{h_{n-1}}{h_n}\right), \\ -\frac{1}{2\,\underline{g}_s}\big(C_n^{(n)} - \widetilde{C}_n^{(n)}\big) + M &= 0, \\ -\frac{1}{2\,\underline{g}_s}\big(C_n^{(n)} + C_{n-1}^{(n-1)}\big) + M &= -n\left(1 - \frac{h_{n-1}}{h_n}\right), \end{aligned} \tag{3.15}$$

Using

$$\frac{h_n}{h_{n-1}} = 1 - A_n B_n, \qquad C_n^{(n)} = -A_{n+1} B_n, \qquad \widetilde{C}_n^{(n)} = -A_n B_{n+1}, \tag{3.16}$$

the string equations lead to the following recursion relations for A_n and B_n:

$$A_{n+1} = -A_{n-1} + \frac{2\, n \underline{g}_s\, A_n}{1 - A_n\, B_n}, \qquad B_{n+1} = -B_{n-1} + \frac{2\, n \underline{g}_s\, B_n}{1 - A_n\, B_n}, \tag{3.17}$$

$$A_n B_{n+1} - A_{n+1} B_n = 2\, M\, \underline{g}_s. \tag{3.18}$$

With the initial conditions $A_0 = B_0 = 1$, and

$$A_1 = \frac{I_{M+1}(1/\underline{g}_s)}{I_M(1/\underline{g}_s)}, \qquad B_1 = \frac{I_{M-1}(1/\underline{g}_s)}{I_M(1/\underline{g}_s)}, \tag{3.19}$$

the remaining constants A_n and B_n are characterized by the recursion relations (3.17), (3.18). We remark that one of recursion relations (3.17) can be obtained by combining the other of (3.17) with (3.18).

Recall that the modified Bessel function satisfies the following recursion relation:

$$I_{\nu-1}(z) - I_{\nu+1}(z) = (2\nu/z) I_\nu(z). \tag{3.20}$$

By examining (3.18) for $n = 0$, we can see that the range of the parameter M in the initial conditions (3.19) can be extended from the integers to any complex numbers. Furthermore,

$$A_n(M) = \frac{K_{M+1}^{(n)}}{K_M^{(n)}}, \qquad B_n(M) = \frac{K_{M-1}^{(n)}}{K_M^{(n)}}, \qquad (M \in \mathbb{C}) \tag{3.21}$$

indeed solve the string equations (3.17) and (3.18). Here $K_\nu^{(n)}$ is defined by (3.5).

Note that the partition function (3.10) depends on A_j and B_j only through their product A_jB_j. Let $A_n = R_nD_n$ and $B_n = R_n/D_n$. Then the partition function (3.10) becomes

$$\underline{Z}_{U(N)} = h_0^N \prod_{j=1}^{N-1} (1 - R_j^2)^{N-j}. \tag{3.22}$$

The equation (3.18) turns into

$$R_n R_{n+1} \left(\frac{D_n}{D_{n+1}} - \frac{D_{n+1}}{D_n} \right) = 2\, M\, \underline{g}_s. \tag{3.23}$$

This leads to

$$\frac{D_n}{D_{n+1}} = \frac{M\, \underline{g}_s + \sqrt{R_n^2\, R_{n+1}^2 + M^2\, \underline{g}_s^2}}{R_n\, R_{n+1}}, \tag{3.24}$$

$$\frac{D_{n+1}}{D_n} = \frac{-M\, \underline{g}_s + \sqrt{R_n^2\, R_{n+1}^2 + M^2\, \underline{g}_s^2}}{R_n\, R_{n+1}}. \tag{3.25}$$

By substituting these relations into the remaining relations (3.17), we find

$$(1 - R_n^2)\Big(\sqrt{R_n^2\, R_{n+1}^2 + M^2\, \underline{g}_s^2} + \sqrt{R_n^2\, R_{n-1}^2 + M^2\, \underline{g}_s^2}\Big) = 2\, n\, \underline{g}_s\, R_n^2. \tag{3.26}$$

This is equivalent to

$$\begin{aligned} 0 =&\eta_n^2 \Big[\xi_n^2 (1-\xi_n)^2 - \eta_n^2\, \xi_n^2 + \zeta^2 (1-\xi_n)^2\Big] \\ &+ \frac{1}{2}\, \eta_n^2\, \xi_n\, (1-\xi_n)^2 (\xi_{n+1} - 2\,\xi_n + \xi_{n-1}) - \frac{1}{16}(1-\xi_n)^4 (\xi_{n+1} - \xi_{n-1})^2, \end{aligned} \tag{3.27}$$

where $\xi_n \equiv R_n^2$, $\eta_n \equiv n\, \underline{g}_s$, $\zeta \equiv M\, \underline{g}_s$.

When $M = 0$ (i.e., with no logarithmic potential), (3.26) reduces to the string equation considered in [9]

$$(1 - R_n^2)\, R_n (R_{n+1} + R_{n-1}) = 2\, n\, \underline{g}_s\, R_n^2. \tag{3.28}$$

Let us introduce variables x_n and y_n by

$$x_n := \frac{A_{n+1}}{A_n}, \qquad y_n := \frac{B_{n+1}}{B_n}, \qquad (n = 0, 1, 2, \cdots). \tag{3.29}$$

They respectively obey the alternate discrete Painlevé II equation [14, 15] with different values of the parameter $\tilde{\mu}$. With the initial conditions $A_0 = 1$ and $B_0 = 1$, A_n and B_n can be expressed by these variables:

$$A_n = \prod_{k=0}^{n-1} x_k, \qquad B_n = \prod_{k=0}^{n-1} y_k. \tag{3.30}$$

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