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SIMPLIFICATION OF THE COMPUTATION OF REAL HADAMARD TRANSFORMS*

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ABSTRACT

The computation of a real Hadamard (or Walsh) transform can be simplified by first computing a pseudotransform, employing an easily derived auxiliary matrix each of whose elements is either 1 or 0 (instead of 1 or -1), and then applying some very simple linear corrections to obtain the desired transform. The auxiliary matrix is not orthogonal, so its square, unlike that of the Hadamard (or Walsh) matrix, is not a scalar matrix. However, subject to one easily satisifed restriction, it has an inverse, which, within a scalar multiplier, differs from that of the Hadamard (or Walsh) matrix of the same order in only a single element.

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Introduction

Computation of Hadamard transforms by hand is very tedious, but not every engineer has instant access to a computer and individual assistance of a programmer. A hand or desk calculator is of much help, but it is so easy to commit sign errors that even then the task is tedious. The methods developed herein alleviate these difficulties appreciably, yet the computations are exact, not merely approximate. Although their objective is to simplify hand computation, they may also be beneficially applicable to machine programming.

In matrix form the Hadamard transform F of a discrete-data function f is

$$F = Hf \tag{1}$$

where the real Hadamard matrix H is a square orthogonal matrix of order 4 m (where m is a positive integer), each of whose elements is either 1 or -1, and F and f are column matrices of 4m elements [1]. In the following development the Hadamard matrices include the Walsh matrices W of order 2^n (where n is a positive integer), and the formulas can be modified to apply to the Walsh transform specifically by making the simple substitution

$$4m = 2^n \qquad (n > 1) \tag{2}$$

and by writing W in place of H.

It is customary to arrange the rows and columns of W ($4m = 2^n$) in some standard order, such that all the elements in the 0-th row and 0-th column of W are 1, and such that W is symmetric and possesses certain other desirable structural properties [2].

The more general H ($4m \neq 2^n$) does not possess all of these special properties, and as yet no standard form has been suggested. It is convenient and always possible, of course, to make all the elements in the 0-th row and 0-th column 1. It is also convenient, if and when possible, to make H symmetric.

It can be shown that by modifications of Paley's methods [3] it is possible to construct all known Hadamard matrices of order 200 or less, except at most four (92, 156, 172, and 184), in symmetric form, with all elements in the 0-th row and 0-th column 1, and with the same number of 1's as -1's on the principal diagonal [4]. The four possible exceptions are of the Williamson type [5], which the author has not yet studied thoroughly. The only orders 200 or less for which Hadamard matrices are still unknown are 116 and 188.

However, these details are irrelevant herein, except one restriction that will be introduced in due course. The Hadamard (or Walsh) functions may be in any desired order, and the set (although not individual functions) may be shifted if desired. In other words, the rows and columns of H may be permuted arbitrarily, and H may be either symmetric or asymmetric. In the following development it is assumed that the rows and columns have been so permuted that H is asymmetric and that all the elements of some row r and some column c, not necessarily the 0-th, are all 1. The resulting formulas can be simplified slightly in obvious ways for either or both of the special (and more customary) cases.

Simplified Computation of Transform

The computation of (1) can be simplified by first computing a pseudotransform

$$X = Af \tag{3}$$

employing an auxiliary matrix

$$A = \frac{1}{2}(H + U) \tag{4}$$

where U is the universal matrix, a square matrix of order 4m <u>all</u> of whose elements are 1, and then applying some simple linear corrections to X to obtain F, using (1) rewritten as

$$
\begin{aligned}
F &= (H + U - U)f \\
&= \left\{2\left[\tfrac{1}{2}(H + U)\right] - U\right\}f \\
&= 2Af - Uf \\
&= 2X - D \qquad (5)
\end{aligned}
$$

where the correction term

$$D = Uf \qquad (6)$$

is a column matrix all of whose elements are simply the algebraic sum of all the elements of f. Thus the spectrum F is obtained by simply multiplying each element of X by 2 and subtracting an easily determined constant from it.

The computation of (3) is much simpler than the direct computation of (1), because there are no signs of products to account for, only signs of terms, and also because nearly half the elements of A are 0. (In fact, if negative Hadamard or Walsh functions are used — i.e., if every row of H is first multiplied by -1, which is permissible because the orthogonality of a matrix is invariant to elementary matrix operations — more than half the elements of A are 0. The result is simply a negative transform, which can be corrected if desired simply by changing the sign of every element in F.)

These features of A facilitate hand computation of transforms of somewhat higher order than can be done efficiently using H directly.

Simplified Computation of Inverse Transform

The inverse transform

$$f = H^{-1} F \qquad (7)$$

can be obtained in essentially the same way. Because H is orthogonal

$$H^{-1} = (4m)^{-1} H^t \tag{8}$$

where H^t denotes the transpose of H. The computation of (7) can be simplified by first computing a pseudotransform

$$x = A^t F \tag{9}$$

employing the auxiliary matrix

$$A^t = \frac{1}{2}(H^t + U) \tag{10}$$

where, because all of the elements of U are 1,

$$U = U^t \tag{11}$$

and then applying some simple linear corrections to x to obtain f, using (7) rewritten as

$$\begin{aligned} f &= (4m)^{-1}(H^t + U - U)F \\ &= (4m)^{-1}\left\{2\left[\frac{1}{2}(H^t + U)\right] - U\right\}F \\ &= (4m)^{-1}(2A^tF - UF) \\ &= (4m)^{-1}(2x - d) \end{aligned} \tag{12}$$

where the correction term

$$d = UF \tag{13}$$

is a column matrix all of whose elements are simply the algebraic sum of all the elements of F. Thus the function f is obtained by simply multiplying each element of x by 2 and subtracting an easily determined constant from it, and then multiplying each element of $(2x - d)$ by $(4m)^{-1}$.

Inverse of Auxiliary Matrix

For a largely qualitative study in which the interest is more in relative than in absolute magnitudes, the pseudospectrum X from (3) may be adequate without the correction (6), inasmuch as X is but a linear distortion of F. In such a case it is desirable to be able to obtain the inverse transform directly from

$$f = A^{-1}X \tag{14}$$

The only restriction (alluded to earlier) is that H not contain a row and/or column all of whose elements are -1, because A would then have a row and/or column of all whose elements are 0, and, thus being singular, could not have an inverse.

Because A is nonorthogonal (as can be verified readily by inspection of any simple numerical example), A^{-1}, unlike H^{-1}, does not satisfy such a simple relation as (8). It can be shown that if A is nonsingular,

$$A^{-1} = 2(4m)^{-1}\left[H^t - \frac{1}{2}(4m)^{-1}(HUH)^t\right] \tag{15}$$

Later on (15) will be rewritten in a form more suitable for the desired computational purpose.

To verify (15) assume that

$$A^{-1} = 2(4m)^{-1}\left[H^t - k(HUH)^t\right] \tag{16}$$

where k is a constant to be determined. From (4) and (16),

$$AA^{-1} = (4m)^{-1}(H + U)\left[H^t - k(HUH)^t\right] \tag{17}$$

which, upon expansion and substitution of (11), can be written

$$AA^{-1} = (4m)^{-1}\left\{HH^t + UH^t - k\left[(HH^t)(UH^t) + (UH^t)^2\right]\right\} \tag{18}$$

As is well known, or as can be obtained by premultiplication of both sides of (8) by 4m H,

$$HH^t = 4\,m\,I \qquad (19)$$

where I denotes the identity matrix. Upon substitution of (19), (18) becomes

$$AA^{-1} = I + (4\,m)^{-1}\left\{UH^t - k\left[4\,m\,UH^t + (UH^t)^2\right]\right\} \qquad (20)$$

Now, if u_{ij} denotes the elements of U, and h^t_{jk} those of H^t, the elements of UH^t are

$$p_{ik} = \sum_{j=0}^{4m-1} u_{ij}\, h^t_{jk} \qquad (i,\ k = 0,\ 1,\ \ldots\ 4\,m-1) \qquad (21)$$

Since all the elements of U are 1, and since all the elements of one column of H^t (say the c-th) are 1, while all other columns of H contain an equal number of 1's and -1's, (21) reduces to

$$p_{ik} = \sum_{j=0}^{4m-1} h_{jk}\,\delta_{kc} = 4m\,\delta_{kc} \qquad (22)$$

where δ_{kc} is the Kronecker delta. Thus UH^t is a square matrix of order 4m all of whose elements are 0, except in the c-th column, all of whose elements are 4m.

If q_{ik} are the elements of $(UH^t)^2$, then it follows from (22) that

$$q_{ik} = \sum_{j=0}^{4m-1} p_{ij}\, p_{jk} = \sum_{j=0}^{4m-1} p_{ij}\,\delta_{jc}\, p_{jk}\,\delta_{kc}$$

$$= p_{ic} \sum_{j=0}^{4m-1} p_{jk}\,\delta_{kc} = (4m)^2\,\delta_{kc} \qquad (23)$$

Thus $(UH^t)^2$ is a square matrix of order 4m all of whose elements are 0, except in the $\underline{c}$-th column, all of whose elements are $(4m)^2$. Consequently

$$(UH^t)^2 = 4m\, UH^t \tag{24}$$

and (20) can be written

$$AA^{-1} = I + (4m)^{-1} \left[1 - 2k(4m)\right] UH^t \tag{25}$$

If A and A^{-1} in (17) are interchanged, the resulting equation is

$$A^{-1}A = I + (4m)^{-1} \left[1 - 2k(4m)\right] H^tU \tag{26}$$

in which the roles of rows and columns are interchanged, and H^tU is a square matrix of order 4m all of whose elements are 0, except in one row (say the r-th), all of whose elements are 4m.

It can be seen from (25) and (26) that if

$$k = \frac{1}{2}(4m)^{-1} \tag{27}$$

then

$$AA^{-1} = A^{-1}A = I \tag{28}$$

and (15) is verified.

<u>Simplified Computation of Pseudotransform</u>

For computation (15) can be rewritten in a more suitable form by observing that HUH is a matrix all of whose elements are 0, except a single element. If h_{ij}, u_{jk}, and $h_{k\ell}$ denote the elements of the respective factor matrices, the elements of the product are

$$s_{i\ell} = \sum_{j=0}^{4m-1} \sum_{k=0}^{4m-1} h_{ij}\, u_{jk}\, h_{k\ell}$$

$$= \sum_{j=0}^{4m-1} h_{ij}\, u_{jk} \sum_{k=0}^{4m-1} h_{k\ell} \quad (i, \ell = 0, 1, \ldots 4m-1) \tag{29}$$

Since all the elements of U are 1, and since all the elements of one row of H (say the r-th) are 1, while all other rows of H contain an equal number of 1's and -1's,

$$s_{i\ell} = \sum_{j=0}^{4m-1} h_{rj} \sum_{k=0}^{4m-1} h_{k\ell} = 4m\,\delta_{rj} \sum_{k=0}^{4m-1} h_{k\ell} \tag{30}$$

and since all the elements of one column of H (say the c-th) are 1, while all other columns of H contain an equal number of 1's and -1's,

$$s_{i\ell} = 4m\,\delta_{rj} \sum_{k=0}^{4m-1} h_{kc}$$

$$= (4m\,\delta_{rj})\,(4m\,\delta_{kc})$$

$$= (4m)^2\,\delta_{rj}\,\delta_{kc} \tag{31}$$

Therefore,

$$HUH = (4m)^2\,\delta_{rj}\,\delta_{kc}\,U \tag{32}$$

is a square matrix of order 4m all of whose elements are 0, except the single element in the r-th row and c-th column, whose value is $(4m)^2$. Consequently (15) can be rewritten as

$$A^{-1} = 2(4m)^{-1} \left[H^t - \frac{1}{2}(4m)\,(\delta_{rj}\,\delta_{kc}\,U)^t\right] \tag{33}$$

Interestingly as well as conveniently, A^{-1} can be obtained from H^t (within the scalar multiplier) simply by subtracting 2m from the single element of H^t in the c-th row and r-th column (carefully observing the transposition of $\delta_{rj}\,\delta_{kc}\,U$).

Finally, in essentially the same way as before, the computation of (14) can be simplified by first computing a pseudotransform

$$y = A^t X \tag{34}$$

employing the auxiliary matrix A^t as defined by (10), and then applying some simple linear corrections to y to obtain f, using (14) rewritten as

$$\begin{aligned} f &= 2(4m)^{-1}\left[H^t + U - U - \frac{1}{2}(4m)\,(\delta_{rj}\,\delta_{kc}\,U)^t\right] X \\ &= 2(4m)^{-1}\left\{2\left[\frac{1}{2}\,(H^t + U)\right] - U - \frac{1}{2}\,(4m)\,(\delta_{rj}\,\delta_{ke}\,U)^t\right\} X \\ &= 2(4m)^{-1}\,(2A^tX - UX) - (\delta_{rj}\,\delta_{kc}\,U)^t\,X \\ &= 2(4m)^{-1}\,(2y - e_1) - e_2 \end{aligned} \tag{35}$$

where the correction term

$$e_1 = UX \tag{36}$$

is a column matrix all of whose elements are simply the algebraic sum of all the elements of X, and the correction term

$$e_2 = (\delta_{rj}\,\delta_{kc}\,U)^t\,X \tag{37}$$

is a column matrix all of whose elements are 0, except the single element in the c-th row, whose value is the value of X in the r-th row. Thus the function f is obtained by simply multiplying each element of y by 2 and subtracting an easily determined constant from it, then multiplying each element of $(2y - e_1)$ by $2(4m)^{-1}$, and then subtracting from the c-th element of $2(4m)^{-1}(2y - e_1)$ the r-th element of X.

Conclusion

The foregoing formulas are more general than is required in most applications. As mentioned in the introduction, the Hadamard or Walsh matrix is written most often in some form that is symmetric, with the 0-th row and 0-th column those all of whose elements are 1.

If H is symmetric, then the superscript t may be deleted wherever it appears. Also, as mentioned in the introduction, the formulas can be altered slightly to apply especially to the Walsh matrices of order 2^n if desired, simply by substituting (2) wherever applicable.

The reader can readily verify, by means of a numerical example of his own, the author's claim that these formulas and methods greatly facilitate the hand computation, with or without the help of a hand or desk calculator, of transforms of somewhat higher order than can be done efficiently using H directly.

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