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1 Texture Zero Mass Matrices and Their Implications

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Abstract. We have made an attempt to briefly address the issue of texture zero fermion mass matrices from the 'bottom-up' perspective. Essentials pertaining to texture zero mass matrices have been summarized and using the facility of Weak Basis transformations, the implications of the texture zero mass matrices so obtained have been examined for the quark as well as the lepton sector.

Povzetek. Avtorica obravnava masne matrike za kvarke in leptone, ki imajo ničelne elemente razporejene po določenih vzorcih. Povzame bistvene značilnosti takih masnih matrik, ki jih transformira v šibko bazo ter določi proste parametre iz eksperimentalnih podatkov.

Keywords: Texture zero mass matrices, Weak Basis transformations, Quark mass matrices, Lepton mass matrices

1.1 Introduction

Understanding fermion masses and mixings is of paramount importance in the field of High Energy Physics. Regarding the quark case, at present one has a fairly good idea of the masses as well as the mixing angles [1]. In particular, one finds that both the quark masses as well as the mixing angles exhibit a clear cut hierarchy. For the case of neutrinos, although, recently refinements of the reactor mixing angle s_{13} [2,3], the solar mixing angle s_{12} and the atmospheric mixing angle s_{23} have been carried out, however, regarding the neutrino masses, in the absence of their absolute measurements, one has their interpretation only in terms of the neutrino mass-squared differences [4].

In order to understand the underlying pattern of fermion masses and flavor mixings, experimental efforts in the form of continuous refinements of the fermion mixing data are being carried out regularly. Along with these attempts, large amounts of efforts at the phenomenological end are also being made. In the present context, we have followed the "bottom-up" approach which involves phenomenological formulation of mass matrices which may eventually provide clues for the efforts carried out through the "top-down" approach. In this context, an interesting idea being investigated in the quark as well as leptonic sector is that of the texture zero mass matrices [5]-[8]. In the present paper, after presenting a brief outline of the essentials pertaining to the texture zero mass matrices in

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Section 2, the details of the analyses corresponding to the quark and leptonic sectors have been presented in Sections 3.1 and 3.2 respectively. Finally, Section 4, summarizes our conclusions.

1.2 Essentials pertaining to texture zero mass matrices

Fermion masses, along with fermion mixings, provide a good opportunity to hunt for physics beyond the SM. In view of the relationship of fermion mixing phenomenon with that of the fermion mass matrices, understanding flavor physics essentially implies formulating fermion mass matrices. The lack of a viable approach from the top-down perspective brings up the need for formulating fermion mass matrices from a bottom-up approach. In this context, initially, incorporating the texture zero approach, several ansatze were suggested for quark mass matrices.

1.2.1 Quark mass matrices

In the Standard Model (SM), the fermion mass matrices, having their origin in the Higgs fermion couplings, are completely arbitrary, therefore, the number of free parameters available with a general mass matrix is larger than the physical observables. For example, if no restrictions are imposed, there are 36 real free parameters in the two 3×3 general complex mass matrices, M_U and M_D, which in the quark sector need to describe 10 physical observables, i.e., 6 quark masses, 3 mixing angles and 1 CP violating phase. Similarly, in the leptonic sector, physical observables described by lepton mass matrices are 6 lepton masses, 3 mixing angles and 1 CP violating phase for Dirac neutrinos (2 additional phases in case neutrinos are Majorana particles). Therefore, to develop viable phenomenological fermion mass matrices, as a first step, one needs to constrain the number of free parameters associated with the mass matrices so as to obtain valuable clues for developing an understanding of fermion mixing phenomenology.

In the SM and its extensions in which righthanded quarks are singlets, the above mentioned task is accomplished by considering the fermion mass matrices to be Hermitian. This brings down the number of real free parameters from 36 to 18, which however, is still a large number compared to the number of observables. To this end, Weinberg implicitly and Fritzsch [9,10] explicitly proposed the idea of texture zero mass matrices which imparted considerable predictability to the fermion mass matrices. This approach involves assuming certain elements of the Hermitian quark mass matrices to be zero, e.g., the typical Fritzsch texture zero Hermitian quark mass matrices are given by

$$M_U = \begin{pmatrix} 0 & A_U & 0 \\ A_U^* & 0 & B_U \\ 0 & B_U^* & C_U \end{pmatrix}, \qquad M_D = \begin{pmatrix} 0 & A_D & 0 \\ A_D^* & 0 & B_D \\ 0 & B_D^* & C_D \end{pmatrix}, \tag{1.1}$$

where M_U and M_D refer to the mass matrices in the up and down sector respectively. Such matrices were named as texture zero mass matrices with a particular matrix defined as texture 'n' zero if the sum of the number of diagonal zeros and half the number of the symmetrically placed off diagonal zeros is 'n'. Each of the above matrix is texture three zero type, together these are known as texture six zero Fritzsch mass matrices. On lines of these ansatze, by considering lesser number of texture zeros, several possible Fritzsch like texture zero mass matrices can be formulated. Also, one can get non Fritzsch like mass matrices by shifting the position of $C_i (i = U, D)$ on the diagonal as well as by shifting the position of zeros among the non diagonal elements. One can thus obtain a very large number of possible texture zero mass matrices.

An analysis of these mass matrices involves firstly diagonalizing them using bi-unitary orthogonal transformations and then obtaining the fermion mixing matrix using the relationship between the mass matrices and the mixing matrices. The corresponding mixing matrix is compared with the experimentally available mixing matrix which then determines the viability of a given texture zero mass matrix. As an example, we present here essentials pertaining to the diagonalization of texture 4 zero mass matrices. A general Fritzsch-like texture 2 zero mass matrix can be expressed as

$$M_k = \begin{pmatrix} 0 & A_k & 0 \\ A_k^* & D_k & B_k \\ 0 & B_k^* & C_k \end{pmatrix}, \tag{1.2}$$

where $k = l, \nu D$, for neutrino case and $k = U, D$, for quark case. Considering both the matrices of either the up and the down sector for quarks or the charged lepton or neutrino sector for leptons to be the texture 2 zero type, one essentially obtains the case of texture 4 zero mass matrices. Texture 6 zero mass matrices can be obtained from the above mentioned matrices by taking both D_k to be zero in both sets of mass matrices. Texture 5 zero matrices can be obtained by taking $D_k = 0$ in one of the two mass matrices.

To fix the notations and conventions, we detail the formalism connecting the mass matrix to the mixing matrix. The mass matrices, for Hermitian as well as symmetric case, can be exactly diagonalized. To facilitate diagonalization, the mass matrix M_k can be expressed as

$$M_k = Q_k M_k^r P_k, \tag{1.3}$$

or

$$M_k^r = Q_k^\dagger M_k P_k^\dagger, \tag{1.4}$$

where M_k^r is a real symmetric matrix with real eigenvalues and Q_k and P_k are diagonal phase matrices. For the Hermitian case $Q_k^\dagger = P_k$, whereas for the symmetric case under certain conditions $Q_k = P_k$. In general, the real matrix M_k^r is diagonalized by the orthogonal transformation O_k, e.g.,

$$M_k^{diag} = O_k^T M_k^r O_k, \tag{1.5}$$

which on using equation (4) can be written as

$$M_k^{diag} = O_k^T Q_k^\dagger M_k P_k^\dagger O_k. \tag{1.6}$$

Using the method, mentioned above, we reproduce the general diagonalizing transformation O_k, e.g.,

$$\begin{pmatrix} \pm O_k(11) & \pm O_k(12) & \pm O_k(13) \\ \pm O_k(21) & \mp O_k(22) & \pm O_k(23) \\ \mp O_k(31) & \pm O_k(32) & \pm O_k(33) \end{pmatrix}, \tag{1.7}$$

where

$$O_k(11) = \sqrt{\frac{m_2 m_3 (m_3 - m_2 - D_k)}{(m_1 - m_2 + m_3 - D_k)(m_3 - m_1)(m_1 + m_2)}},$$

$$O_k(12) = \sqrt{\frac{m_1 m_3 (m_1 + m_3 - D_k)}{(m_1 - m_2 + m_3 - D_k)(m_3 + m_2)(m_1 + m_2)}},$$

$$O_k(13) = \sqrt{\frac{m_1 m_2 (m_2 - m_1 + D_k)}{(m_1 - m_2 + m_3 - D_k)(m_3 + m_2)(m_3 - m_1)}},$$

$$O_k(21) = \sqrt{\frac{m_1 (m_3 - m_2 - D_k)}{(m_3 - m_1)(m_1 + m_2)}},$$

$$O_k(22) = \sqrt{\frac{m_2 (m_3 + m_1 - D_k)}{(m_2 + m_3)(m_1 + m_2)}},$$

$$O_k(23) = \sqrt{\frac{m_3 (m_2 - m_1 + D_k)}{(m_3 + m_2)(m_1 + m_2)}},$$

$$O_k(31) = \sqrt{\frac{m_1 (m_2 - m_1 + D_k)(m_1 + m_3 - D_k)}{(m_1 - m_2 + m_3 - D_k)(m_3 - m_1)(m_1 + m_2)}},$$

$$O_k(32) = \sqrt{\frac{m_2 (m_2 - m_1 + D_k)(m_3 - m_2 - D_k)}{(m_1 - m_2 + m_3 - D_k)(m_3 + m_2)(m_1 + m_2)}},$$

$$O_k(33) = \sqrt{\frac{m_3 (m_3 - m_2 - D_k)(m_1 + m_3 - D_k)}{(m_1 - m_2 + m_3 - D_k)(m_3 - m_1)(m_3 + m_2)}}, \tag{1.8}$$

$m_1, -m_2, m_3$ being the eigenvalues of M_k.

While carrying out the analysis of texture zero mass matrices, the viability of the formulated mass matrices is ensured by checking the compatibility of the mixing matrices so obtained from these with the low energy data. In order to obtain the mixing matrix, we note that in the SM, the quark mass terms for three generations of quarks can be expressed as

$$\overline{q}_{U_L} M_U q_{U_R} + \overline{q}_{D_L} M_D q_{D_R}, \tag{1.9}$$

where $q_{U_{L(R)}}$ and $q_{D_{L(R)}}$ are the left (right) handed quark fields for the up sector (u, c, t) and down sector (d, s, b) respectively. M_U and M_D are the mass matrices for the up and the down sector of quarks. In order to re-express above equation in terms of the physical quark fields, one can diagonalize the mass matrices by the following bi-unitary transformations

$$V_{U_L}^{\dagger} M_U V_{U_R} = M_U^{diag} \equiv Diag(m_u, m_c, m_t), \tag{1.10}$$

$$V_{D_L}^{\dagger} M_D V_{D_R} = M_D^{diag} \equiv Diag(m_d, m_s, m_b), \tag{1.11}$$

where $M_{U,D}^{diag}$ are real and diagonal, while V_{U_L}, V_{U_R} etc. denote the eigenvalues of the mass matrices, i.e., the physical quark masses. Using the above equations, one can rewrite equation (9) as

$$\overline{q}_{U_L} V_{U_L} M_U^{diag} V_{U_R}^{\dagger} q_{U_R} + \overline{q}_{D_L} V_{D_L} M_D^{diag} V_{D_R}^{\dagger} q_{D_R} \tag{1.12}$$

which can be re-expressed in terms of physical quark fields as

$$\overline{q}_{U_L}^{phys} M_U^{diag} q_{U_R}^{phys} + \overline{q}_{D_L}^{phys} M_D^{diag} q_{D_R}^{phys}, \tag{1.13}$$

where $\overline{q}_{U_L}^{phys} = V_{U_L}^{\dagger} q_{U_L}$ and $\overline{q}_{D_L}^{phys} = V_{D_L}^{\dagger} q_{D_R}$ and so on. The mismatch of diagonalizations of up and down quark mass matrices leads to the quark mixing matrix V_{CKM}, referred to as the Cabibbo-Kobayashi-Maskawa (CKM) matrix [11] given as

$$V_{CKM} = V_{U_L}^{\dagger} V_{U_R}. \tag{1.14}$$

Over the past few years, both in the quark as well as lepton sector, a large number of analyses [5]-[8] have been carried out which establish the texture zero approach as a viable one for explaining the fermion mixing data. However, as mentioned earlier, since the number of possible texture zero mass matrices is very large, one has to carry out an exhaustive analysis of all possible texture zero mass matrices. To account for this limitation, therefore, Branco *et al*. [12,13] and Fritzsch and Xing [14,15] have proposed the concept of 'Weak Basis (WB) transformations'.

Within the SM and some of its extensions, one has the facility of making Weak Basis (WB) transformations W on the quark fields, e.g., $q_L \to Wq_L$, $q_R \to Wq_R$, $q'_L \to Wq'_L$, $q'_R \to Wq'_R$. These are unitary transformations which leave the gauge currents real and diagonal but transform the mass matrices as

$$M_U \to W^{\dagger} M_U W,\ M_D \to W^{\dagger} M_D W. \tag{1.15}$$

Without loss of generality, this approach introduces zeros in the quark mass matrices leading to a reduction in the number of parameters defining the mass matrices. Following this, one can arrive at two kinds of structures of the mass matrices, e.g., Branco *et al.* [12,13] give

$$M_q = \begin{pmatrix} 0 & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix},\ M_{q'} = \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix}, \quad q,\ q' = U, D, \tag{1.16}$$

whereas Fritzsch and Xing [14,15] give

$$M_q = \begin{pmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}, \qquad q = U, D. \tag{1.17}$$

The mass matrices so obtained can thereafter be considered texture zero mass matrices and same methodology can be used to analyze these. Interestingly, one now has an additional advantage that the large number of possible structures are not all independent. Several of these are related through WB transformations and therefore yield the same structure of the diagonalizing transformations leading to similar mixing matrices, making the number of matrices to be analyzed much less than before. However, there is a limitation too, i.e, this idea does not result in constraining the parameter space of the elements of the mass matrices. To overcome this, one can further impose a condition on the elements of the mass matrices by considering the following hierarchy for these [8]

$$(1,i) \lesssim (2,j) \lesssim (3,3); \qquad i = 1,\ 2,\ 3,\ \ j = 2,\ 3. \tag{1.18}$$

1.2.2 Lepton mass matrices

Keeping in mind the quark lepton universality [16], similar to the case of texture zero quark mass matrices discussed in the previous section, it becomes desirable to carry out a corresponding analysis in the lepton sector also. In the case of leptons, several attempts have been made to formulate the phenomenological mass matrices considering charged leptons to be diagonal, usually referred to as the flavor basis case [17]. However, in the present work, we have considered the non flavor basis [18], wherein, texture is imposed on both the charged lepton mass matrix as well as on the neutrino mass matrix. The 'smallness' of the neutrino masses is best described in terms of 'seesaw mechanism' [19] given by

$$M_\nu = -M_{\nu D}^T M_R^{-1} M_{\nu D}, \tag{1.19}$$

with M_ν, $M_{\nu D}$ and M_R corresponding to the light Majorana neutrino mass matrix, the Dirac neutrino mass matrix and the heavy right handed Majorana neutrino mass matrix respectively.

The methodology of analyzing the texture zero lepton mass matrices remains essentially the same as that for the case of quarks. One can impose texture on the charged lepton mass matrix M_l and on the Dirac neutrino mass matrix $M_{\nu D}$. Equation (1.19) can then be used to obtain the Majorana neutrino matrix M_ν which along with the matrix M_l allows the construction of the Pontecorvo Maki Nakagawa Sakata (PMNS) matrix [20] for examining the viability of the mass matrices. Using these ideas, in the following we have briefly summarized the results of the analyses in the case of quarks [21] as well as leptons [22].

1.3 Results and discussion

1.3.1 Texture zero quark mass matrices

We begin with the the most general Hermitian mass matrices, given by

$$M_q = \begin{pmatrix} E_q & A_q & F_q \\ A_q^* & D_q & B_q \\ F_q^* & B_q^* & C_q \end{pmatrix} \qquad (q = U, D). \tag{1.20}$$

Invoking WB transformations, zeros can be introduced in these matrices using a unitary matrix W, leading to

$$M_U = \begin{pmatrix} E_U & A_U & 0 \\ A_U^* & D_U & B_U \\ 0 & B_U^* & C_U \end{pmatrix}, \qquad M_D = \begin{pmatrix} E_D & A_D & 0 \\ A_D^* & D_D & B_D \\ 0 & B_D^* & C_D \end{pmatrix}. \tag{1.21}$$

One may note that these matrices are, in fact, texture one zero each, together these are referred as texture two zero mass matrices.

To check the viability of these mass matrices, one needs to examine the compatibility of the CKM matrix reproduced through these with the recent quark mixing data. Results of a detailed analysis of these matrices, carried out in Ref. [21], reveal that using the following quark masses and the mass ratios at the M_Z scale as inputs [23]

$$\begin{gathered} m_u = 1.38^{+0.42}_{-0.41}\,\text{MeV}, \quad m_d = 2.82 \pm 0.48\,\text{MeV}, \quad m_s = 57^{+18}_{-12}\,\text{MeV}, \\ m_c = 0.638^{+0.043}_{-0.084}\,\text{GeV}, \quad m_b = 2.86^{+0.16}_{-0.06}\,\text{GeV}, \quad m_t = 172.1 \pm 1.2\,\text{GeV}, \\ m_u/m_d = 0.553 \pm 0.043, m_s/m_d = 18.9 \pm 0.8 \end{gathered} \tag{1.22}$$

and imposing the latest values [1] of the three mixing angles as constraints for the construction of the CKM matrix, one arrives at

$$V_{CKM} = \begin{pmatrix} 0.9739 - 0.9745 & 0.2246 - 0.2259 & 0.00337 - 0.00365 \\ 0.2224 - 0.2259 & 0.9730 - 0.9990 & 0.0408 - 0.0422 \\ 0.0076 - 0.0101 & 0.0408 - 0.0422 & 0.9990 - 0.9999 \end{pmatrix}, \tag{1.23}$$

this being fully compatible with the one given by Particle Data Group (PDG) [1]. In order to examine whether these mass matrices can accommodate CP violation in the quark sector, in the present work we have made an attempt to reproduce the CP violating Jarlskog's rephasing invariant parameter J. One obtains a range of J = $(2.494 - 3.365) \times 10^{-5}$, this again being compatible with its latest experimental value $(3.04^{+0.21}_{-0.20}) \times 10^{-5}$ [1].

1.3.2 Texture zero lepton mass matrices

Similar to the quark case, using the facility of WB transformations, wherein it is possible to make a unitary transformation, one can reduce the general lepton mass matrices to

$$M_l = \begin{pmatrix} E_l & A_l & 0 \\ A_l^* & D_l & B_l \\ 0 & B_l^* & C_l \end{pmatrix}, \qquad M_{\nu D} = \begin{pmatrix} E_{\nu D} & A_{\nu D} & 0 \\ A_{\nu D}^* & D_{\nu D} & B_{\nu D} \\ 0 & B_{\nu D}^* & C_{\nu D} \end{pmatrix}. \tag{1.24}$$

A detailed analysis of these mass matrices has been carried out in Ref. [22]. In the present work, for the normal and inverted ordering of neutrino masses, we have first examined the viability of these mass matrices and then we have investigated their implications for CP violation in the leptonic sector.

The latest situation regarding neutrinos masses and mixing angles at 3σ C.L. is summarized as follows [24]

$$\Delta m_{21}^2 = (7.02 - 8.09) \times 10^{-5} eV^2; \quad \Delta m_{23}^2 = (2.325 - 2.599) \times 10^{-3} eV^2; \tag{1.25}$$

$$\sin^2\theta_{12} = 0.270 - 0.344; \quad \sin^2\theta_{23} = 0.385 - 0.644; \quad \sin^2\theta_{13} = 0.0188 - 0.0251. \tag{1.26}$$

The 3σ C.L. ranges of the PMNS matrix elements recently constructed by Garcia *et al.*[24] are as follows

$$U_{PMNS} = \begin{pmatrix} 0.801 - 0.845 & 0.514 - 0.580 & 0.137 - 0.158 \\ 0.225 - 0.517 & 0.441 - 0.699 & 0.164 - 0.793 \\ 0.246 - 0.529 & 0.464 - 0.713 & 0.590 - 0.776 \end{pmatrix}. \tag{1.27}$$

For the inverted and normal neutrino mass orderings, the mass matrices mentioned in equation (1.24) yield the following magnitudes of the corresponding PMNS matrix elements [22] respectively

$$U_{PMNS}^{IO} = \begin{pmatrix} 0.034 - 0.859 & 0.0867 - 0.593 & 0.135 - 0.996 \\ 0.250 - 0.971 & 0.068 - 0.812 & 0.043 - 0.808 \\ 0.103 - 0.621 & 0.395 - 0.822 & 0.088 - 0.810 \end{pmatrix}. \tag{1.28}$$

$$U_{PMNS}^{NO} = \begin{pmatrix} 0.444 - 0.993 & 0.123 - 0.837 & 0.004 - 0.288 \\ 0.061 - 0.816 & 0.410 - 0.941 & 0.047 - 0.872 \\ 0.012 - 0.848 & 0.049 - 0.779 & 0.460 - 0.992 \end{pmatrix}. \tag{1.29}$$

For both the mass orderings, one finds that the 3σ C.L. ranges of the PMNS matrix elements given by Garcia *et al.* are inclusive in the ranges of the PMNS matrix elements found here, thereby ensuring the viability of texture two zero mass matrices considered here. Further, analogous to the case of quarks, we have made an attempt to find constraints for the CP violating Jarlskog's rephasing invariant parameter in the leptonic sector also. For the inverted mass ordering, one obtains a range of J from $-0.05 - 0.05$, whereas, for the normal mass ordering the, parameter J is obtained in the range $-0.03 - 0.03$. These observations, therefore, lead one to conclude that the texture two zero leptonic mass matrices are not only compatible with the recent leptonic mixing data but also provide interesting bounds for the Jarlskog's rephasing invariant parameter.

1.4 Summary and Conclusions

To summarize, in the present work, we have made an attempt to provide an overview of texture zero fermion mass matrices. For the case of both quarks and leptons, incorporating the texture zero approach as well as using the WB transformations, analyses of the "general" fermion mass matrices have been discussed.

After examining the viability of these mass matrices, we have obtained interesting bounds on the Jarlskog's rephasing invariant parameter in the quark and leptonic sector.

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