



Excitation of the Synchro-Betatron Resonances by the Beam-Beam Interaction in the Tevatron Run II Lattice

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Abstract

Effect of different mechanisms of the synchro-betatron coupling (dispersion function at the interaction points, chromatic tune modulation, finite bunch length) on the \bar{p} -bar stability in the Tevatron Run II configuration is considered. It is found that the long-range interactions in the presence of large dispersion produce large contribution (~ 10) to the chromaticity of the betatron tunes. This chromaticity, in its turn, can give rise to multiple synchrotron satellites of the betatron resonances increasing their effective width.

Novel formulae are presented which permit to significantly speed up analytical calculations of the beam-beam tuneshifts and resonance strength.

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1 Introduction

There are several reasons owing to which the beam-beam driven resonances (the synchro-betatron ones in particular) can present in the Tevatron Run II configuration [1] a greater danger than in the previous runs. Besides some increase in the beam-beam tuneshift due to a larger number of protons per bunch and a smaller proton emittance (see Table 1), there will be a factor of seven larger number of the parasitic long-range interactions, some of them at only $\sim 7\sigma_{\perp}$ separation (see Fig.1) with the Run II lattice version available on the Web [2]. These numerous parasitic encounters will strongly enhance excitation of the odd-order resonances.

	Run Ib	Run II	Run II, SC RF
$N_p/\text{bunch}, 10^{11}$	2.32	2.7	2.7
ϵ_p (95% normal.), $\pi\cdot\text{mm}\cdot\text{mrad}$	23	20	20
$\xi/\text{nominal IP}$	0.0074	0.01	0.01
N parasitic IPs	10	70	70
ξ , total	~ 0.015	~ 0.025	~ 0.025
σ_s , cm	63	37	14
σ_E , 10^{-4}	1.5	0.9	3.1
v_s , 10^{-3}	0.7	0.7	6.4

Table 1. Comparison of the proton beam parameters for different Tevatron runs.

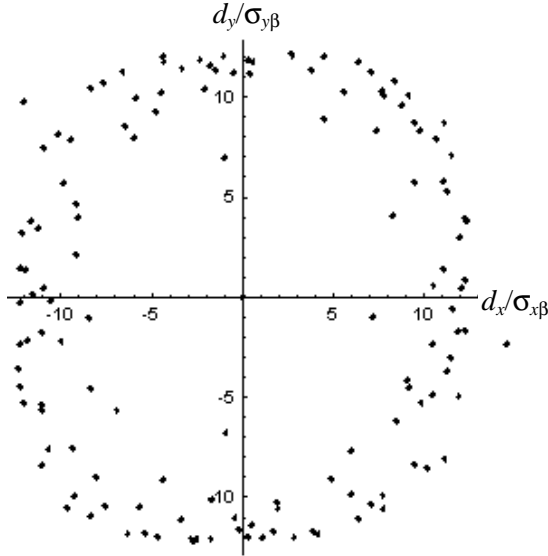


Figure 1. Normalized beam separation at all possible interaction points with 36×36 bunches.

Another important effect which the long-range interactions give rise to is the beam-beam induced chromaticity (observed experimentally at LEP [3]). It depends on the particle transverse amplitudes and the bunch collision schedule and therefore can not be eliminated with the help of sextupoles. The large chromatic tune modulation of the off-momentum particles will produce synchrotron satellites increasing the effective width of the betatron resonances. There is also an additional contribution to the satellite excitation owing to the finite dispersion at the location of parasitic encounters.

The effect of the head-on interactions can be more pronounced in the Run II configuration as well due to a smaller bunch length (see Table 1), especially with introduction of the superconducting RF which will provide $V_{\text{RF}}=20\text{MV}$ at 212MHz as compared to $V_{\text{RF}}=1\text{MV}$ at 53MHz with the existing RF system. The reduction in the bunch length is big enough to weaken the beneficial phase averaging effect [4, 5, 6] but may be insufficient to avoid excitation of the synchrotron satellites by the betatron phase modulation of the beam-beam kicks at the low-beta interaction points (IPs).

The present report is devoted to the analytical study of the beam-beam driven synchro-betatron resonances in the Run II configuration and to some development of the theoretical tools which occurred necessary to perform this study.

2 Beam-beam potential

Let us choose the generalized azimuth $\theta = s/R$, R being the mean radius of the orbit, as the independent variable (so that the longitudinal momentum plays the role of the Hamiltonian) and introduce canonically conjugated coordinates and momenta $\underline{q} = (x, y, \sigma)$, $\underline{p} = (p_x, p_y, \delta_p)$, where in the ultra-relativistic case $\sigma = s - ct$ for one beam (let it be p-bar) and $\sigma = -s - ct$ for the other.

Due to the synchrotron motion the actual collision point (CP) of two particles belonging to the counter-rotating bunches is shifted w.r.t. the interaction point (IP) of the reference particles by the amount

$$s_{CP} - s_{IP} = \frac{\sigma_1 - \sigma_2}{2}. \quad (2.1)$$

If the r.m.s. bunch length σ_s is comparable with the beta-function this shift leads to: i) modulation of the betatron phase by synchrotron motion thus provoking SBRs; ii) phase averaging of high order betatron resonances.

To analyze these effects (well-known in principle) we will apply here a technique permitting to lump the interaction with a long bunch to a single point.

It was shown in Ref.[7] how to ascribe the interaction which takes place at a shifted point to the position of the reference IP with the help a similarity transformation and construct a symplectic “synchro-beam” map. The generating Hamiltonian for this map can be found [8] and presented in the form

$$H_{bb} = -\frac{r_p N}{\gamma} \delta_p(\theta - \theta_{IP}) \int G(\underline{q}, \underline{p} | \underline{q}', \underline{p}') F_0(\underline{q}', \underline{p}', \theta) d^3 q' d^3 p', \quad (2.2)$$

where $\delta_p(\theta)$ is the periodic δ -function, N is the number of particles in the bunch, F_0 is the normalized to unity distribution function, r_p is the classical radius of the particles, γ is the relativistic mass factor, G is Green's function.

In the case of a finite horizontal crossing angle, 2α , Green's function can be written in the form

$$G = -\ln \left\{ \left[x - x' + \left(\alpha + \frac{p_x + p'_x}{2} \right) (\sigma - \sigma') \right]^2 + \left[y - y' + \frac{p_y + p'_y}{2} (\sigma - \sigma') \right]^2 \right\}, \quad (2.3)$$

its dependence on the momenta being the price paid for lumping all the interaction at the point $\theta = \theta_{IP}$.

Let us assume that the particle motion in the strong (proton) beam is regular so that invariants of motion $\underline{I} = (I_x, I_y, I_s)$ exist. They can be used in the construction of the equilibrium distribution function which we presume to be Gaussian:

$$F_0 = \frac{1}{(2\pi)^3 V} \exp \left(-\underline{\epsilon}^{-1} \cdot \underline{I} \right), \quad \underline{\epsilon} = \langle \underline{I} \rangle, \quad V = \epsilon_x \epsilon_y \epsilon_s, \quad \underline{\epsilon}^{-1} = (\epsilon_x^{-1}, \epsilon_y^{-1}, \epsilon_s^{-1}). \quad (2.4)$$

where ϵ_i , $i = x, y, s$, are the r.m.s. emittances (mean values of the action variables).

To evaluate the beam-beam potential (2.2) let us express the coordinates and momenta via the action-angle variables. First we extract the closed orbit and dispersion

$$\begin{aligned}
x &= x_{co} + D_x \delta_p + \tilde{x}, \quad p_x = x'_{co} + (D'_x + x'_{co}) \delta_p + \tilde{p}_x, \\
\sigma &= \tilde{p}_x D_x - \tilde{x} (D'_x + x'_{co}) + \tilde{p}_y D_y - \tilde{y} (D'_y + y'_{co}) + \tilde{\sigma},
\end{aligned} \tag{2.5}$$

where the tilde marks variables describing betatron and synchrotron oscillations (uncoupled in the linear approximation) and prime denotes differentiation by s . The most important nonlinear term for the subsequent analysis arises from the betatron tune modulation by the synchrotron motion (see Ref.[8] for the rigorous introduction via a Lie-transform)

$$\begin{aligned}
\tilde{x} &= \sqrt{2\beta_x I_x} \sin(\psi_x + \phi_x - \eta_x \tilde{\sigma}), \\
\tilde{p}_x &= \sqrt{2I_x / \beta_x} [\cos(\psi_x + \phi_x - \eta_x \tilde{\sigma}) - \alpha_x \sin(\psi_x + \phi_x - \eta_x \tilde{\sigma})], \\
\tilde{\sigma} &= \sqrt{2\beta_s I_s} \sin \psi_s, \quad \delta_p = \sqrt{2I_s / \beta_s} \cos \psi_s,
\end{aligned} \tag{2.6}$$

where $\alpha_x = -\beta'_x/2$ is the Twiss parameter, $\beta_s = R\alpha_M/v_s$ is the longitudinal β -function with α_M being the momentum compaction factor,

$$\phi_x(\theta) = \int_0^{R\theta} \frac{ds'}{\beta_x} - v_{x0} \theta \tag{2.7}$$

is the periodic betatron phase function and

$$\eta_x = \frac{1}{\alpha_M R} \frac{dv_x}{d\delta_p}. \tag{2.8}$$

The formulas for the vertical plane are analogous to (2.5-8).

Let us note that parameters α_x, x_{co}', D_x' entering Green's function (2.3) via momenta can be important only for low-beta interaction points where they should be zero (in an absence of misalignments). Therefore we may put $\alpha_x = x_{co}' = D_x' = 0$ everywhere (and so for the vertical plane). Also, we make a number of assumptions to simplify the formulae:

1) the betatron functions at the low-beta IPs to be equal, $\beta_x^* = \beta_y^* = \beta^*$, so that the finite bunch length effect in both planes is described by the same variable

$$b = \frac{\sigma - \sigma'}{2\beta^*}, \tag{2.9}$$

2) the dispersion functions for both beams are equal (which is not really so in Tevatron);

3) the vertical dispersion function and crossing angle are negligible.

Now let us introduce the r.m.s. beam sizes of the strong bunch and their ratio r :

$$\begin{aligned}
\sigma_i &= \sqrt{\epsilon_i \beta_i}, \quad i = x, y, s, \\
\sigma_{xt} &= \sigma_x \sqrt{1 + \kappa^2}, \\
\kappa &= \sqrt{\sigma_x^2 + \alpha^2 \sigma_s^2 + D_x^2 \sigma_E^2} / \sigma_x, \quad \sigma_E = \sigma_s / \beta_s, \\
r &= \sigma_y / \sigma_{xt}.
\end{aligned} \tag{2.10}$$

Representing Green's function as a Fourier transform

$$G = -\ln(\Delta x^2 + \Delta y^2) = \frac{r}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(ik_1 \frac{\Delta x}{\sigma_{xt}} + ik_2 \frac{\Delta y}{\sigma_y}) \frac{dk_1 dk_2}{r^2 k_1^2 + k_2^2} \quad (2.11)$$

and performing in (2.2) integration by the transverse variables we obtain

$$H_{bb} = -\frac{r_p N}{\pi \gamma} \delta_p(\theta - \theta_{IP}) \iint \frac{r du_1 du_2}{r^2 u_1^2 + u_2^2} \exp\left[-\frac{u_1^2 \sigma_x^2}{2 \sigma_{xt}^2} - \frac{u_2^2}{2}\right] \iint d\sigma' d\sigma'_p F_L(\sigma', \sigma'_p) \times \quad (2.12)$$

$$\exp\left\{i \frac{u_1}{\sigma_{xt}} \left[\frac{d_x + D_x(\delta_p - \delta'_p) + \alpha(\sigma - \sigma')}{\sqrt{1+b^2}} + \tilde{x}\right] + i \frac{u_2}{\sigma_y} \left[\frac{d_y}{\sqrt{1+b^2}} + \tilde{y}\right]\right\}$$

where F_L is the longitudinal distribution function (normalized to unity), $u_i = k_i (1+b^2)^{1/2}$, double tilde denotes additional phase shift due to displacement (2.1) of the actual collision point:

$$\tilde{x} = \exp\left[-i\varphi \frac{\partial}{\partial \psi_x}\right] \tilde{x}, \quad \varphi = -\arctan b. \quad (2.13)$$

In the following we will neglect the variable b appearing anywhere else in eq.(2.12) besides the phase φ . This is justified in the absence of large crossing angle α since at the nominal IPs, where b can reach appreciable values, parameters $d_{x,y}, D_x$ should be zero (in the absence of imperfections) whereas at parasitic IPs b is small. Let us note, in passing, that the terms we are going to neglect provide extra suppression of the effect of imperfections in addition to the phase averaging described by φ .

When the finite bunch length effect can be ignored altogether, the beam-beam potential can be reduced to the form

$$H_{bb} = -\frac{2r_p N}{\gamma} \delta_p(\theta - \theta_{IP}) \int_0^1 \frac{dt}{t \sqrt{1 + (r^2 - 1)t^2}} \times \quad (2.14)$$

$$\exp\left\{-\frac{(d_x + \tilde{x} + D_x \delta_p + \alpha \tilde{\sigma})^2}{2 \sigma_{xt}^2} t^2 - \frac{(d_y + \tilde{y})^2}{2 \sigma_y^2 [1 + (r^2 - 1)t^2]} r^2 t^2\right\}$$

which was the starting point for analysis in Ref.[9]¹. In its derivation we made use of the formula

$$\frac{1}{r^2 u_1^2 + u_2^2} = \int_0^\infty e^{-(r^2 u_1^2 + u_2^2) \tau} d\tau \quad (2.15)$$

with subsequent integration by the longitudinal variables and $u_{1,2}$ and finally setting $\tau = (t^{-2} - 1)/2r^2$.

¹ In the Mathematica notebook inc_bb.nb cited in Ref.[9] the sign of the potential had been chosen opposite to that in eq.(2.14) to describe the p - p interaction. Also, there was a minor misprint in the formula not affecting the actual calculations. Let us note that a similar representation of the beam-beam potential was used by J.Jowett [10].

3 Beam-beam contribution to the tunes and chromaticity

The beam-beam tuneshifts are defined by the formula

$$\Delta v_i^{(bb)} = \frac{\partial}{\partial I_i} \langle H_{bb} \rangle, \quad i = x, y, s, . \quad (3.1)$$

where the brackets denote averaging over all phase variables, ψ_i , and the azimuth, θ . The averaging eliminates dependence on the phase ϕ , so there is no finite bunch length effect on the tunes under the assumptions made (equal β -functions and small offsets and dispersion at low-beta IPs, large β -functions at parasitic IPs).

In the simplest case $\alpha = d_{x,y} = D_x = 0$ averaging over ψ_i can be done analytically.

Normalizing the action variables of the *weak* beam particles by the r.m.s. emittances of the *strong* beam:

$$J_i = I_i / \epsilon_i, \quad i = x, y, s, \quad (3.2)$$

and introducing the beam-beam parameter

$$\xi_x = \frac{r_p N}{2\pi\gamma\epsilon_x(1+\kappa^2)(1+r)} \quad (3.3)$$

($\kappa = 0$ for now) we obtain the formula

$$\begin{aligned} \Delta v_x^{(bb)}(J_x, J_y) = \xi_x \int_0^{1/2} \frac{(1+r) du}{\sqrt{1+2(r^2-1)u}} \exp\left[-J_x u - J_y \frac{r^2 u}{1+2(r^2-1)u}\right] \times \\ [I_0(J_x u) - I_1(J_x u)] \cdot I_0\left[J_y \frac{r^2 u}{1+2(r^2-1)u}\right], \end{aligned} \quad (3.4)$$

which was given without derivation in Ref.[11] and, in the case of a round beam, $r = 1$, in Ref.[12].

In the general case the averaging over ψ_i can be performed quite efficiently by direct numerical integration as proposed in Ref.[9]. We will employ this method in the present study as well. Let us remind the expression for the averaged beam-beam Hamiltonian

$$\begin{aligned} \langle H_{bb} \rangle = -\frac{r_p N}{\gamma\pi^4} \int_0^1 X\left[0, 0, 0, t\delta_x, t\sqrt{\frac{J_x}{1+\kappa^2}}, t\sqrt{\frac{\kappa^2 J_s}{1+\kappa^2}}\right] \times \\ Y\left[0, 0, \frac{rt\delta_y}{\sqrt{1+(r^2-1)t^2}}, \frac{rt\sqrt{J_y}}{\sqrt{1+(r^2-1)t^2}}\right] \frac{dt}{t\sqrt{1+(r^2-1)t^2}}, \end{aligned} \quad (3.5)$$

where $\delta_x = d_x/\sigma_{xt}\sqrt{2}$, $\delta_y = d_y/\sigma_y\sqrt{2}$ and

$$\begin{aligned} X(l, m, n, d, x, z) = \\ \int_0^\pi \int_0^\pi \sin^{2m} \phi \sin^{2n} \psi e^{-(d+x\cos\phi+z\cos\psi)^2} H_{l+m+n}(d+x\cos\phi+z\cos\psi) d\phi d\psi, \quad (3.6) \\ Y(l, m, d, y) = \int_0^\pi \sin^{2m} \phi e^{-(d+y\cos\phi)^2} H_{l+m}(d+y\cos\phi) d\phi, \end{aligned}$$

$H_n(x)$ being the Hermite polynomials. Now we can calculate the tunes with the help of the relation

$$\frac{\partial}{\partial x} X(l, m, n, d, x, z) = \frac{x}{2m+1} X(l+1, m+1, n, d, x, z) \quad (3.7)$$

and an analogous formula for Y .

Fig.2 shows the footprint of the antiproton bunch #6 in the tune diagram obtained with the standard optics [2] at the working point (WP) $v_{x0}=20.585$, $v_{y0}=20.575$. The 5th, 7th and 12th order sum resonance lines² are also shown. The arc lines in the footprints correspond to equidistant step 2 values of the normalized transverse amplitude

$$a_{\perp} = (a_x^2 + a_y^2)^{1/2}, \quad a_{x,y} = \left(\frac{2}{\epsilon_{\bar{p}}} I_{x,y} \right)^{1/2} = \left(\frac{2\epsilon_p}{\epsilon_{\bar{p}}} J_{x,y} \right)^{1/2}; \quad (3.8)$$

the radial lines being equidistant in $\arctan(a_x/a_y)$ value.

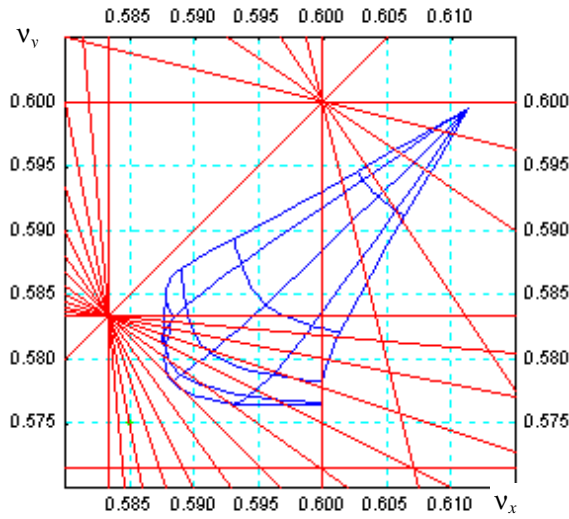


Figure 2. Footprint of the p-bar bunch #6 in the tune diagram obtained with the optics [2]

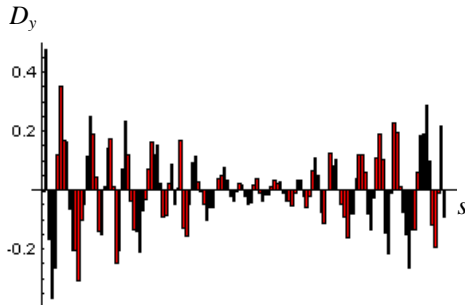


Figure 3. Vertical dispersion on the p-bar helix

dispersion as well which can be quite large³ (see Fig.3).

Due to the finite dispersion at parasitic IPs there should be some dependence of the beam-beam tunes on the synchrotron motion. Calculations for the standard Run II configuration show that for the average values of the tunes over the synchrotron period this dependence is negligible.

However, there can be significant variation of the instant values of the tunes which we will treat in terms of the beam-beam contribution to chromaticity. To get a feeling of the order of the effect let us consider a single long-range interaction point at horizontal separation d_x . The tunes for a particle with constant momentum deviation δ_p is

$$\Delta v_x^{(bb)} = -\frac{r_p N \beta_x}{4\pi\gamma(d_x + D_x \delta_p)^2} \quad (3.9)$$

Correspondingly, we have for the beam-beam contribution to chromaticity (which we will call just “the beam-beam chromaticity” for brevity)

$$\frac{\partial}{\partial \delta_p} \Delta v_x^{(bb)} = D_x \frac{\partial}{\partial d_x} \Delta v_x^{(bb)} = -\frac{2D_x}{d_x} \Delta v_x^{(bb)} \quad (3.10)$$

The ratio $|D_x/d_x|$ can be as large as 10^3 , so that for the tunes of the order of 10^{-3} the shift in chromaticity may be ~ 1 .

Calculating the beam-beam chromaticity on the Tevatron helix one should take into account the vertical

² analysis shows that the difference resonances are less important at the considered WPs.

³ Still its contribution to the beam sizes is not significant which justifies its exclusion from all other expressions.

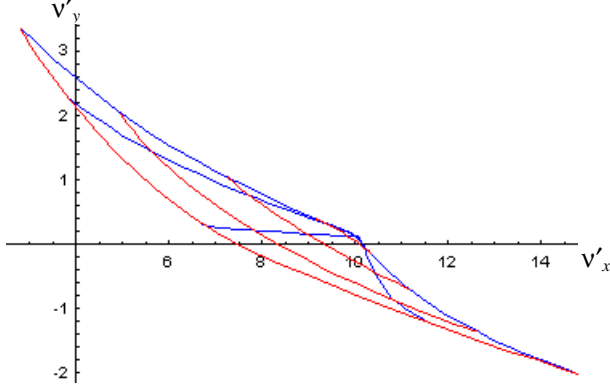


Figure 4. Footprint of the p-bar bunch #6 in the chromaticity diagram

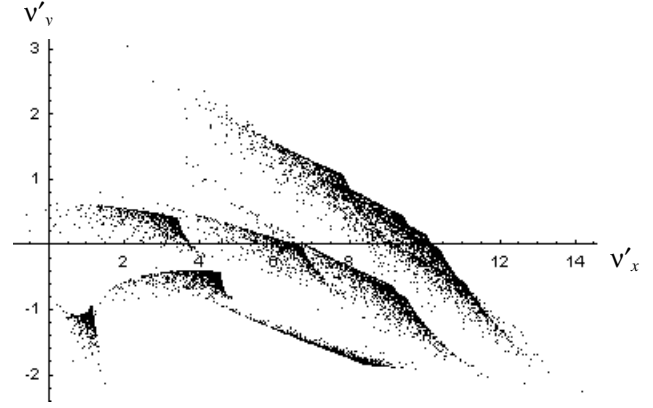


Figure 5. Total p-bar distribution in the chromaticity plane

The necessary formulas can be obtained from eq.(3.5) with account of the differentiation rule

$$\frac{\partial}{\partial d} X(l, m, n, d, x, z) = -X(l+1, m, n, d, x, z) \quad (3.11)$$

and its counterpart for Y .

Fig.4 shows the footprint of the antiproton bunch #6 in the plane of the beam-beam chromaticities

$$\chi_i(J_x, J_y) = \frac{\partial}{\partial \delta_p} \Delta v_i^{(bb)} = D_x \frac{\partial}{\partial d_x} \Delta v_i^{(bb)} + D_y \frac{\partial}{\partial d_y} \Delta v_i^{(bb)}, \quad i = x, y, \quad (3.12)$$

the arc lines now corresponding to the values $a_{\perp} = 1, 2, 3, 4$. The beam-beam contribution to chromaticity is really very large and strongly depends on the betatron amplitudes.

There is also significant bunch-to-bunch variation. Fig.5 shows the total distribution of antiprotons (represented by 1000 points/bunch) in the plane of chromaticities.

We see that the chromaticities can not be made small simultaneously for all particles in all bunches with the help of correction sextupoles.

4 Beam-beam resonances

In the present report we include the effects of the synchrotron motion into the analysis of the beam-beam driven resonances in the Tevatron Run II configuration started in Ref.[9]. Since the longitudinal emittance $\epsilon_s = \sigma_s \sigma_E \approx 3.5 \cdot 10^{-5} \text{m}$ is by four orders of magnitude larger than the transverse one we may ignore variation of the longitudinal action in the resonance exchange of energy and consider it as a constant of motion:

$$I_s = \text{const.} \quad (4.1)$$

Again we limit ourselves to a single resonance approximation considering one resonance at a time and then applying the Chirikov criterion to access the cooperative effect of multiple resonances.

In the vicinity of the resonance⁴

$$\Delta_m = \underline{m} \cdot \underline{v} - n \equiv m_x v_x + m_y v_y + m_s v_s - n \approx 0 \quad (4.2)$$

we can transform to the rotating frame and write the main terms in the Hamiltonian in the form

⁴ Here the synchrotron tune should be understood as an algebraic value negative above the transition energy.

$$H \approx \int_{I_1}^{I_2} \Delta_{\underline{m}}(I'_1, I_2, I_s) dI'_1 + C_{\underline{m}}(I_1, I_2, I_s) \cos(\Psi_1 + \Psi_0) \quad (4.3)$$

where the new transverse action-angle variables were introduced

$$I_1 = \frac{m_x I_x + m_y I_y}{m_x^2 + m_y^2}, \quad I_2 = \frac{m_y I_x - m_x I_y}{m_x^2 + m_y^2} = \text{const}, \quad (4.4)$$

$$\Psi_1 = m_x \Psi_x + m_y \Psi_y + m_s \Psi_s - n\theta, \quad \Psi_2 = m_y \Psi_x - m_x \Psi_y,$$

the resonance coefficient $C_{\underline{m}}$ and phase Ψ_0 are defined by the Fourier transform of the beam-beam potential (2.12) with account of the phase shift (2.13) as

$$C_{\underline{m}} = 2 |H_{bb}^{(\underline{m})}|, \quad \Psi_0 = \text{Arg } H_{bb}^{(\underline{m})}, \quad (4.5)$$

$$H_{bb}^{(\underline{m})} = \frac{1}{(2\pi)^4} \iint e^{in\theta - i\underline{m} \cdot \underline{\Psi}} H_{bb} d\theta d^3 \underline{\Psi}.$$

The synchrotron motion affects the resonance behavior in a number of ways. First, it somewhat weakens the principal resonance, $m_s = 0$, by the virtue of: i) dephasing the successive beam-beam kicks in the result of chromatic tune modulation; ii) phase averaging of every kick due to a finite bunch length. On the other hand, these very same mechanisms give rise to the synchrotron satellites, $m_s \neq 0$, which can effectively widen the resonance if the Chirikov overlap condition is met. In addition to these mechanisms, the crossing angle and finite dispersion can contribute to excitation of the synchrotron satellites.

Let us consider these factors in some detail.

4.1 Effect of finite crossing angle and dispersion

Let us first consider the effect of finite crossing angle and dispersion ignoring for a while the chromaticity and finite bunch length. Using representation (2.14) we obtain in this case for the contribution of one IP

$$H_{bb}^{(\underline{m})} = \exp(in\theta_{IP} + i\underline{m} \cdot \underline{\phi}_{IP} + im_s \mu_{IP}) \frac{r_p N}{\pi^4 \gamma} \frac{J_x^{m_x/2} (r^2 J_y)^{m_y/2} (\kappa^2 J_s)^{m_s/2}}{(2|m_x| - 1)!! (2|m_y| - 1)!! (2|m_s| - 1)!! (1 + \kappa^2)^{(m_x + m_s)/2}} \times \quad (4.6)$$

$$\int_0^1 \frac{t^{|m_x| + |m_y| + |m_s| - 1} dt}{[1 + (r^2 - 1)t^2]^{(|m_y| + 1)/2}} X(0, |m_x|, |m_s|, \delta_x t, t \sqrt{\frac{J_x}{1 + \kappa^2}}, t \sqrt{\frac{\kappa^2 J_s}{1 + \kappa^2}}) Y(0, |m_y|, \frac{\delta_y r t}{\sqrt{1 + (r^2 - 1)t^2}}, \frac{r t \sqrt{J_y}}{\sqrt{1 + (r^2 - 1)t^2}})$$

where the functions X, Y are defined by eqs.(3.6), the phase μ_s is defined by the relations

$$\sin \mu_s = \frac{D_x \sigma_E}{\sqrt{D_x^2 \sigma_E^2 + \alpha^2 \sigma_s^2}}, \quad \cos \mu_s = \frac{\alpha \sigma_s}{\sqrt{D_x^2 \sigma_E^2 + \alpha^2 \sigma_s^2}}, \quad (4.7)$$

also, we have dropped the constant phase common to all IPs.

Calculations for the Tevatron lattice show that effect of the horizontal dispersion at parasitic IPs is not significant even in the case of large synchrotron amplitudes characteristic to the superconducting RF parameters. Table 2 gives the strength $C_{\underline{m}}$ of the synchrotron satellites of the $m_x = 5$, $m_y = 0$ resonance at $J_x = 16$, $J_y = 0$, $J_s = 1$ for the two RF systems.

$ m_s $	0	1	2	3	4	5
Cu RF	$4.367 \cdot 10^{-12}$	$4.103 \cdot 10^{-13}$	$8.989 \cdot 10^{-14}$	$1.717 \cdot 10^{-14}$	$3.019 \cdot 10^{-15}$	$4.266 \cdot 10^{-16}$
SC RF	$3.678 \cdot 10^{-12}$	$8.453 \cdot 10^{-13}$	$2.115 \cdot 10^{-13}$	$2.859 \cdot 10^{-14}$	$3.561 \cdot 10^{-14}$	$1.919 \cdot 10^{-15}$

Table 2. The strength C_m of the synchrotron satellites of the $5Q_x$ resonance with two RF systems.

The effect of the dispersion on the strength of the principal resonance, $m_s = 0$, also being rather small, we will exclude the horizontal dispersion from the following analysis (as well as the crossing angle which is not foreseen for the first stage of Run II).

4.2 Finite bunch length effect on resonance excitation (phase averaging)

Let us now consider the effect of rapid phase advance variation in the vicinity of low-beta IP on the resonance excitation. In the case of $\beta_x^* = \beta_y^* = \beta^*$ and of no offsets, no crossing angle nor dispersion at the IP, we can factorize the resonance coefficient introducing the longitudinal factor⁵

$$L_m(J_s) = \frac{1}{(2\pi)^2} \iiint e^{-im_s\psi_s - i(m_x+m_y)\phi - i(m_x\eta_x+m_y\eta_y)\sigma} F_L(J'_s) dJ'_s d\psi_s d\psi'_s, \quad (4.8)$$

where F_L is the normalized to unity distribution function in the longitudinal action variable, which we again choose to be Gaussian, $F_L = \exp(-J_s)$. It is obvious that $|L_m(J_s)| \leq 1$ always.

In the case of a small bunch length, $\sigma_s \ll \beta^*$, we may put $\phi = -b$ and obtain (assuming $\eta_x = \eta_y = \eta = \text{const}$ for simplicity) the generalization of the Bessel satellite formula:

$$L_m(J_s) = J_{m_s}[(m_x + m_y)(1 - 2\beta^*\eta)\lambda\sqrt{J_s}] \cdot \exp\left[-\frac{1}{4}(m_x + m_y)^2\lambda^2\right], \quad (4.9)$$

$J_n(x)$ being the Bessel function and

$$\lambda = \frac{\sigma_s}{\beta^*\sqrt{2}}. \quad (4.10)$$

For infinitely short bunches, $\sigma_s \rightarrow 0$, the longitudinal factor is just the Kronecker delta: $L_m(J_s) = \delta_{m_s,0}$.

It is interesting to note that while the finite bunch length provides most of the phase averaging described by the exponential factor in eq.(4.9), the chromaticity could suppress excitation of the synchrotron satellites at $\eta\beta^* = 1/2$ or, recalling the definition (2.8), at

$$\frac{d}{d\delta_p} v_{x,y} = \frac{\alpha_M R}{2\beta^*} \approx 4 \quad (4.11)$$

for the Tevatron parameters $\alpha_M = 2.3 \cdot 10^{-3}$, $R = 1\text{km}$, $\beta^* = 35\text{cm}$. However, the large spread in the beam-beam chromaticity makes such suppression impossible even for short bunches.

Let us continue analysis of an arbitrary bunch length case remembering that in the absence of the offsets only even-order resonances (and their satellites) can be excited by the beam-beam interaction:

$$m_x + m_y = 2m. \quad (4.12)$$

Making notice that

⁵ In fact it may depend on the transverse amplitudes due to the beam-beam contribution to the chromaticity related parameters $\eta_{x,y}$.

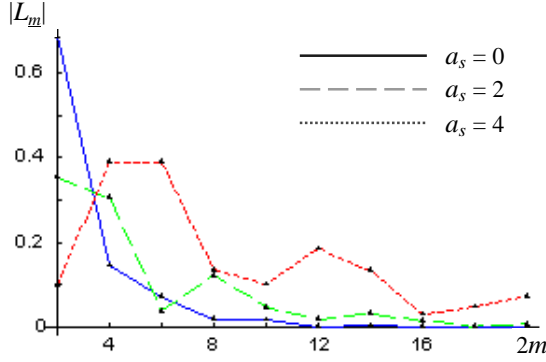


Figure 6. Longitudinal factor vs. resonance order

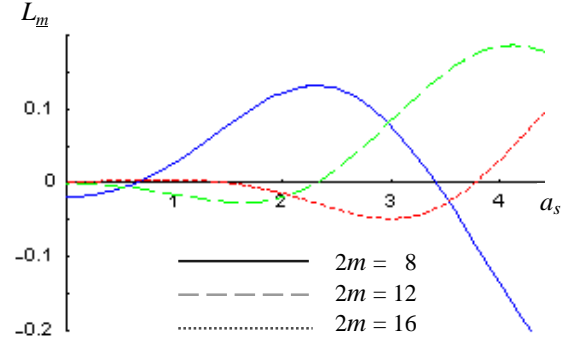


Figure 7. Longitudinal factor vs. synchrotron amplitude

$$e^{-2im\varphi} = \left(\frac{2}{1 + i \tan \varphi} - 1 \right)^m = (-1)^m \left[1 + \int_0^\infty dt e^{-(1+i \tan \varphi)t} \sum_{k=1}^m \frac{(-2)^k m! t^{k-1}}{(m-k)!(k-1)!k!} \right], \quad (4.13)$$

we can perform integration in (4.8) and obtain

$$L_m(J_s) = (-1)^m \{ (-1)^{m_s} J_{m_s}(4m\beta^* \eta \lambda \sqrt{J_s}) + \int_0^\infty dt e^{-t-\lambda^2 t^2/4} J_{m_s}[\lambda(t-4m\beta^* \eta) \sqrt{J_s}] \sum_{k=1}^m \frac{(-2)^k m! t^{k-1}}{(m-k)!(k-1)!k!} \}, \quad (4.14)$$

For the graphical representation it is more convenient to introduce the normalized synchrotron amplitude, a_s , related to the normalized action variable by the formula (we assume the longitudinal emittances in the two beams to be equal)

$$a_s = \sqrt{2J_s} \quad (4.15)$$

First we consider the finite bunch length effect on the principal resonance, $m_s = 0$, at zero chromaticity, $\eta = 0$. Fig.6 shows the dependence of the absolute value of longitudinal factor (4.14) on the resonance order, $2m$, for several values of the normalized synchrotron amplitude, a_s , at $\sigma_s = \beta^*$. For an on-momentum particle, $a_s = 0$, the longitudinal factor rapidly falls off with the resonance order due to the phase averaging, reaching as small value as 10^{-3} at $2m = 12$. However, with increasing a_s the phase averaging is less effective, as found in Ref.[6]. It may occur not sufficient to suppress the head-on contribution to excitation of high order resonances.

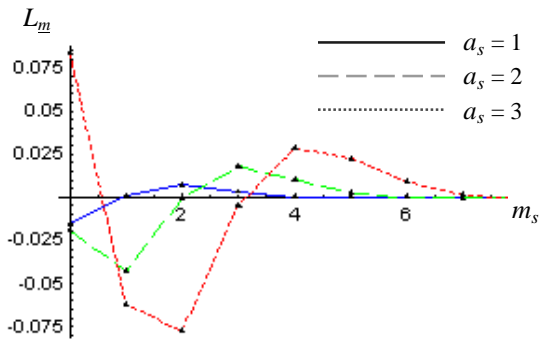


Figure 8. Longitudinal factor of 12th order resonances vs. synchrotron satellite order.

Fig.7 shows the longitudinal factor as a function of a_s for the resonances of the order $2m = 8, 12, 16$, the dependence qualitatively reminds that found analytically in Ref.[6], though quantitatively it is rather different.

Now let us turn to the synchrotron sidebands, $m_s \neq 0$. Fig.8 shows the dependence of the longitudinal factor of the $2m = 12$ resonances on the sideband number for several values of a_s at $\sigma_s/\beta^* = 1$, $\eta = 0$. We see that the strength of the synchrotron satellites can be comparable with that of the principal resonance effectively increasing its width. Fig.9 shows beatings in the transverse amplitudes due to the 12th order betatron

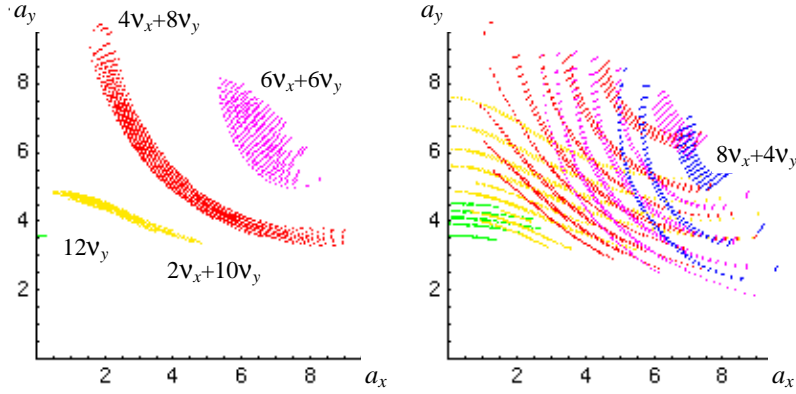


Figure 9. Swing of the transverse amplitudes due to the 12th order resonances and their synchrotron satellites excited by the head-on interactions in the case of the existing RF (left) and SC RF (right).

resonances and their synchrotron satellites excited by the beam-beam interaction at two Tevatron low-beta IPs at $J_s = 1$, $\eta = 0$, $\sigma_s/\beta^* = 37/35$, $v_s = 7 \cdot 10^{-4}$ (left) and $\sigma_s/\beta^* = 14/35$, $v_s = 6.4 \cdot 10^{-3}$ (right). In the first case the effect, although quite pronounced, should not pose significant problems in the Tevatron operation since the resonance groups are well separated. Let us note that the lines representing effect of a particular resonance are not shown if their length does not exceed a certain threshold value (\sim the line width) or if they lie outside the circle $a_\perp = 11$.

In the second case, which corresponds to the 20 MV, 212 MHz SC RF, i) the width of the resonances becomes larger due decrease in the phase averaging, ii) satellites are farther apart due to a larger synchrotron tune. In the result the 12th order resonances and their satellites form a maze through which particles, with some assistance from the external noise and higher order resonances, can escape starting from amplitudes possibly as low as $a_\perp \sim 4$.

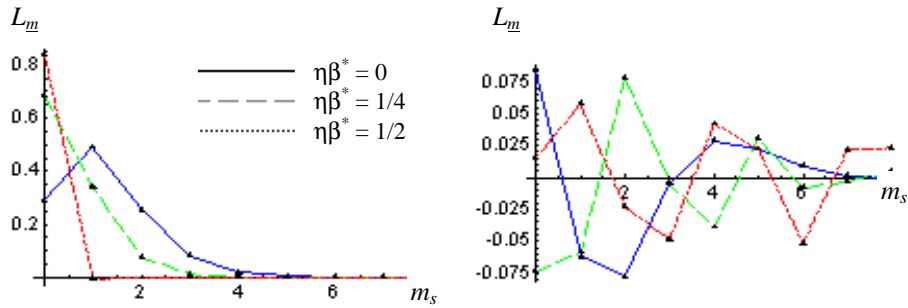


Figure 10. Longitudinal factor of 12th order resonances vs. synchrotron satellite order at the indicated values of the chromaticity parameter η and $\sigma_s/\beta^* = 0.1$ (left) and $\sigma_s/\beta^* = 1$ (right)

As noted earlier, in the case of a short bunch the chromaticity can (in principle) compensate the effect of the finite length on the synchrotron satellite excitation. It is no longer so in the case of a long bunch as illustrated by Fig.10 which shows the dependence of the longitudinal factor of the 12th order resonances on the sideband number for $a_s = 3$ and several values of η at $\sigma_s/\beta^* = 0.1$ (left) and $\sigma_s/\beta^* = 1$ (right). In any case, this compensation will not work in Tevatron due to a large beam-beam chromaticity.

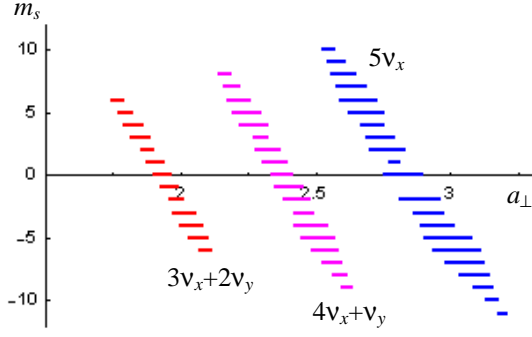


Figure 11. Swing of the transverse amplitude due to the 5th order resonances and their synchrotron satellites at $a_s = \sqrt{2}$.

In the result the synchrotron satellites up to a high order,

$$m_s \approx \left| \frac{m_x v'_x + m_y v'_y}{v_s} \right| \sigma_E a_s, \quad (4.17)$$

can be excited leading to an appreciable increase in the effective resonance width.

Fig.11 shows the swing of the transverse amplitude, a_{\perp} , defined by eqs.(3.8), due to the 5th order resonances and their satellites excited by the long-range interactions for particles in the 6th antiproton bunch with initially $a_x = a_y$ and $J_s = 1$ ($a_s = \sqrt{2}$) in the absence of the bare lattice chromaticity ($\eta_{x0} = \eta_{y0} = 0$). The sidebands of different resonances overlap which means the possibility of fast particle transport over the entire set of resonances almost doubling the particle amplitude (from $a_{\perp} \approx 1.75$ to $a_{\perp} \approx 3.25$). This can affect not only the particle lifetime but the luminosity as well.

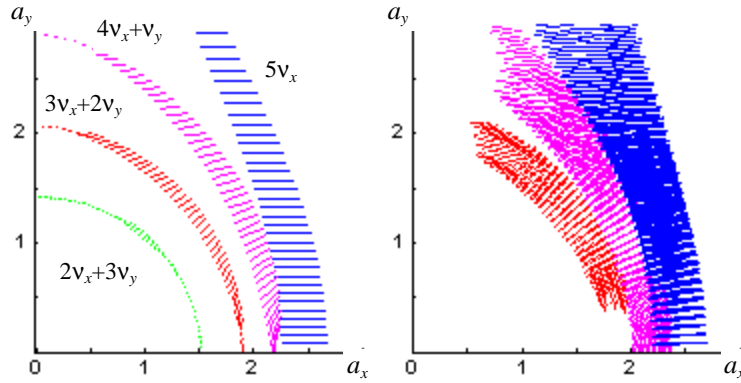


Figure 12. Swing of the transverse amplitudes due to the 5th order resonances and their synchrotron satellites at $a_s = 0$ (left) and $a_s = \sqrt{2}$ (right).

A 2D picture of the resonance satellites overlap is presented in Fig.12. The left plot shows the width of the betatron resonances for on-momentum particles ($J_s = 0$) when no satellite is excited. At large synchrotron amplitudes ($J_s = 1$) these resonances are, in accordance with eq.(4.16), noticeably suppressed (e.g. the $2v_x + 3v_y$ resonance width falls below the chosen threshold). However, due to excitation of numerous satellites the effective resonance width is substantially increased.

With the superconducting RF the distance between the satellites of each of the 5th order resonances becomes larger than the resonance width (Fig.13) but now the satellites of different

4.3 Betatron tune modulation

At parasitic interaction points the betatron functions are usually so large in comparison with the bunch length that one may put $\phi \rightarrow 0$ in eq.(4.8) and obtain the classical formula for the synchrotron satellites excitation by the betatron tune modulation

$$L_m(J_s) = J_{m_s} \left[\frac{(m_x v'_x + m_y v'_y) \sigma_E}{v_s} \sqrt{2J_s} \right] \quad (4.16)$$

where the prime means differentiation of the tunes by δ_p , the relation $\sigma_s = \beta_s \sigma_E$ was used.

Due to the large beam-beam contribution discussed in Section 3 the total chromaticity can not be made small by the conventional means of the sextupoles.

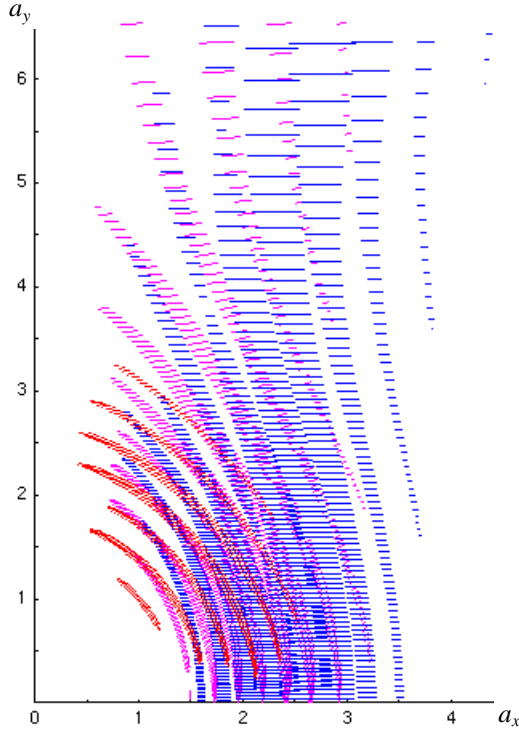


Figure 13. The same as in Fig.12 (right) but with the superconducting RF.

resonances overlap so that the amplitude range over which the fast diffusion can occur is further increasing.

5 Summary

The effects of the synchro-betatron coupling via the beam-beam interaction appear to be quite significant in the Tevatron Run II configuration with 36×36 bunches, the main conclusions being:

- ◆ There is a strong beam-beam contribution to chromaticity rendering the total (intrabunch + bunch-to-bunch) spread in chromaticity of ~ 14 units horizontally and ~ 5 units vertically.

- ◆ The chromatic tune modulation presents the major driving mechanism of the synchro-betatron resonances; the betatron phase modulation at low-beta IPs also provides a noticeable synchro-betatron coupling, whereas the effect of a finite dispersion at the parasitic IPs appears to be less important.

- ◆ With the standard bare lattice tunes, $\nu_{x0}=20.585$, $\nu_{y0}=20.575$, there is a strong excitation of the 5th order resonances and their synchrotron satellites by the long-range beam-beam interactions leading to fast particle

diffusion over the range of normalized betatron amplitudes from $a_{\perp} \approx 1.75$ to $a_{\perp} \approx 3.25$. This can affect not only the particle lifetime but the luminosity as well.

- ◆ Excitation of the satellites of the 12th order resonances by the betatron phase modulation, although quite pronounced, should not pose significant problems in the Tevatron operation with the existing RF. However, with the planned upgrade to the superconducting RF it will lead to particle diffusion from amplitudes possibly as low as $a_{\perp} \sim 4$.

- ◆ It is necessary to extend the present analysis on the TEV33 configuration (140×121 bunches, crossing angle, SC RF) where the effect of the odd-order resonances should be even stronger.

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