SLAC–PUB–4515
PUPT–1084
HUTP–87/A085
December 1987
(N)

$\hat{c} = 1$ Superconformal Field Theory*

L. Dixon[1], P. Ginsparg[2], and J. Harvey[3]

[1,3]Physics Dept.
Princeton University
Princeton, N.J. 08544

[2]Stanford Linear Accelerator Center
Stanford, CA 94305

We consider superconformal field theories with central charge $\hat{c} = \frac{2}{3}c = 1$. We find five continuous one-parameter families of theories all interconnected via a set of multicritical points that are reached by modding out theories with enlarged symmetries. We find as well 6 theories that have no integrable marginal operators and thus constitute isolated points of superconformal invariance in the $\hat{c} = 1$ moduli space. We briefly discuss $c = 3/2$ conformal theories that contain a twisted superconformal algebra, including 3 isolated theories with a twisted N=3 superconformal algebra, and theories constructed as the tensor product of the $c = 4/5$ and $c = 7/10$ minimal theories.

~1/88 *(submitted to Nucl. Phys. B)*

⋆ Work supported in part by the Department of Energy, contract DE–AC03-76SF00515.

[1] (ljd@pupthy.princeton.edu or ljd@pucc.bitnet)
[2] Current address: Lyman Lab. of Physics, Harvard Univ., Cambridge, MA 02138
(ginsparg@huhepl.hepnet, ginsparg@huhepl.bitnet, or ginsparg@huhepl.harvard.edu)
[3] (jah@pupthy.princeton.edu or jah@pucc.bitnet)

1. Introduction

Conformal field theories in 2 dimensions are partially characterized by the value of the central charge c appearing in the operator product of the stress-energy tensor with itself [1]

$$T(z_1)\,T(z_2) \sim \frac{c/2}{(z_1-z_2)^4} + \frac{2}{(z_1-z_2)^2}\,T(z_2) + \frac{1}{(z_1-z_2)}\partial_2 T(z_2). \tag{1.1}$$

Equivalent operator products are satisfied by the anti-holomorphic pieces ($z_1, z_2, T \to \bar{z}_1, \bar{z}_2, \overline{T}$) and the two sets of operator products are equivalent to two commuting copies of the Virasoro algebra. In what follows, we shall only need to consider left-right symmetric systems and so shall frequently omit mention of the anti-holomorphic pieces. We shall also frequently borrow the terminology of string theory and refer to z and $\bar{z}$ as "worldsheet" coordinates. Systems with $N=1$ superconformal symmetry have a super stress-energy tensor $\mathbf{T}(z) = T_F(z) + \theta T_B(z)$ whose components satisfy the operator products

$$\begin{aligned}
T_B(z_1)\,T_B(z_2) &\sim \frac{3\hat{c}/4}{(z_1-z_2)^4} + \frac{2}{(z_1-z_2)^2}\,T_B(z_2) + \frac{1}{(z_1-z_2)}\partial_2 T_B(z_2)\\
T_B(z_1)\,T_F(z_2) &\sim \frac{3/2}{(z_1-z_2)^2}\,T_F(z_2) + \frac{1}{z_1-z_2}\partial_2 T_F(z_2)\\
T_F(z_1)\,T_F(z_2) &\sim \frac{\hat{c}/4}{(z_1-z_2)^3} + \frac{1/2}{z_1-z_2}\,T_B(z_2)\ ,
\end{aligned} \tag{1.2}$$

where $\hat{c} = \frac{2}{3}c$. The conventional normalization is such that the energy-momentum tensor for a single bosonic field has central charge $c=1$ in (1.1), and that for a single free superfield has central charge $\hat{c}=1$ in (1.2).

For $c<1$, there exists a complete classification[2][3] of the unitary representations of the Virasoro algebra, and as well a classification[4] of their modular invariant combinations. Analogously, for $N=1$ superconformal systems with $\hat{c}<1$ there is a classification[5] of the unitary representations of the super Virasoro algebra and of their modular invariant combinations[6]. No such classifications exist respectively for $c \geq 1$ and $\hat{c} \geq 1$, and the boundary cases $c=1$ and $\hat{c}=1$ thus provide the simplest opportunities for probing uncharted properties of more general conformal systems alleged to be of interest both for

their own sake and for their role in characterizing the classical solution space of string theory.

The simplest $c=1$ conformal field theory is constructed in terms of a single scalar field x parametrizing a circle of radius r, i.e. with the identification $x \equiv x + 2\pi r$. In general, operators V_i of dimension $(1,1)$ in a conformal field theory are called marginal operators, and can be used to generate deformations of a theory that preserve conformal invariance at the classical level. (To leading order, the change in the action is given by $\delta S = \delta g_i \int d^2z\, V_i(z,\bar{z})$.) Deformations by marginal operators that also preserve conformal invariance at the quantum level (i.e. that remain marginal in the deformed theory) may be "integrated" to generate a family of conformal theories continuously connected to the original theory. Such integrable marginal operators are referred to as exactly marginal, truly marginal, or critical. At generic values of the compactification radius r, the $c=1$ circle theory contains one exactly marginal operator, $V = \partial x \bar{\partial} x$, and perturbing the theory by this operator simply corresponds to changing the value of r.

At $r=\sqrt{2}$, another exactly marginal operator appears, giving by definition a multicritical point, from which emerges a further line of $c=1$ conformal field theories. These latter can be constructed in terms of a single scalar field x with the identifications $x \equiv x + 2\pi r$ and $x \equiv -x$, i.e. parametrizing an $S^1/\mathbf{Z}_2$ orbifold. The orbifold line contains no exactly marginal operators other than the one that corresponds to changing the radius.

It was recently observed [7] that there exist in addition to the above two lines three isolated points of conformal symmetry with $c=1$, disconnected from the remaining known $c=1$ conformal systems. The new theories were constructed from the circle theory at the self-dual radius $r=1/\sqrt{2}$, where it has an affine $SU(2)\times SU(2)/\mathbf{Z}_2$ symmetry, by modding out by the tetrahedral, octahedral, and icosahedral groups $\mathbf{T}, \mathbf{O}, \mathbf{I} \subset SO(3)$.

These results suggest the following strategy for mapping out the moduli space of conformal field theories for any given central charge c. First one chooses a known theory and identifies the exactly marginal operators. These generate a multi-parameter moduli space of conformal field theories with one parameter associated to each exactly marginal operator. Then one identifies any multicritical points in this space, and constructs all continuously connected families of conformal field theories generated by integrating the new exactly marginal operators that appear at these points. One next identifies any discrete symmetries that exist both generically and at non-generic points in the space of theories thus far constructed, and mods out by these symmetries to construct all further theories allowed by modular invariance. The procedure may then be pursued to exhaustion. This procedure can also be applied to superconformal theories, although in general the perturbation will break supersymmetry, leaving only a conformal field theory. A sufficient condition for the perturbation to preserve superconformal invariance is that the exactly marginal operator be the highest component of a dimension $(\frac{1}{2},\frac{1}{2})$ superfield. The perturbation then takes the manifestly supersymmetric form $\delta S = \delta g_i \int d^2z\, d\theta\, d\overline{\theta}\, \Phi_i(z,\overline{z},\theta,\overline{\theta})$, where the Φ_i are a set of marginal superfields. In practice one must be careful in checking that this condition is satisfied, because there may be several possible definitions of the worldsheet supersymmetry generator, and hence several possible ways of organizing the fields into supermultiplets. Examples of this phenomenon will appear in the course of our analysis.

In this paper we will apply the above procedure to begin charting the space of $\hat{c}=1$ superconformal field theories. The picture we find is qualitatively similar to that described above for the case $c=1$, but in sec. 2 we will find instead 5 continuous lines of theories, and in sec. 3 show that they are interconnected at 4 multicritical points. Features qualitatively different from the $c=1$ case occur as well. "Discrete torsion" will emerge to act non-trivially in the $\hat{c}=1$ case, and some of the $\hat{c}=1$ isolated points constructed in sec. 4, for example, appear with a two-fold multiplicity. In addition, one of the multicritical points in the $\hat{c}=1$ case has two inequivalent marginal operators and no $\mathbf{Z}_2$ symmetries relating either operator to minus itself, so that multicritical point admits four inequivalent directions of deformation. Our treatment here points to many open questions, some of which are mentioned in our concluding sec. 5.

2. Lines of theories

A. The circle line

A natural realization of superconformal systems at $\hat{c}=1$ is given by a single free superfield, consisting of a free boson and a free Majorana fermion. When the complex worldsheet coordinates $z,\overline{z}$ live on a torus, we must specify the sum over the spin structures of the fermion. To fix our notation, we first describe the partition function $Z_{\rm circ}(r)$ for the superconformal circle line. This is constructed by tensoring the theory of a single free boson $x(z,\overline{z})=\frac{1}{2}(x(z)+x(\overline{z}))$ compactified on a circle of radius r, together with the theory of a single Majorana fermion with left and right components $\psi(z)$ and $\overline{\psi}(\overline{z})$.[1] The partition function for the single bosonic field is

$$\Gamma(r)=\frac{1}{\eta\overline{\eta}}\sum_{m,n\in\mathbf{Z}} q^{\frac{1}{2}(\frac{m}{2r}+nr)^2}\,\overline{q}^{\frac{1}{2}(\frac{m}{2r}-nr)^2}, \tag{2.1}$$

where $q=e^{2\pi i\tau}$ (τ is the modular parameter for the torus) and $\eta = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$. The partition function for the fermion, summed over equal spin structures for left and right components (i.e. GSO projected by $(-1)^F$), is[2]

$$Z_{\rm Ising}=\frac{1}{2}\left({\rm A}\underset{\rm A}{\square}+{\rm P}\underset{\rm A}{\square}+{\rm A}\underset{\rm P}{\square}+{\rm P}\underset{\rm P}{\square}\right)=\frac{1}{2}\left(\left|\frac{\vartheta_3}{\eta}\right|+\left|\frac{\vartheta_4}{\eta}\right|+\left|\frac{\vartheta_2}{\eta}\right|\right) \tag{2.2}$$

(equal to the critical point partition function for the Ising model on a torus with modular parameter τ). The product of the two partition functions

$$Z_{\rm circ}(r)=\Gamma(r)\,Z_{\rm Ising}, \tag{2.3}$$

[1] Our normalization conventions are such that the operator products satisfy $\partial x(z)\partial x(w)\sim-\frac{1}{(z-w)^2}$, $\psi(z)\psi(w)\sim-\frac{1}{(z-w)}$ (and similarly for antiholomorphic components).

[2] Recall that the Jacobi theta functions satisfy $\sqrt{\vartheta_3/\eta}=q^{-1/48}\prod_{n=1}^{\infty}(1+q^{n-1/2})$, $\sqrt{\vartheta_4/\eta}=q^{-1/48}\prod_{n=1}^{\infty}(1-q^{n-1/2})$, and $\sqrt{\vartheta_2/\eta}=q^{1/24}\sqrt{2}\prod_{n=1}^{\infty}(1+q^n)$.

is then the partition function for a GSO projected system with $N = 1$ superconformal symmetry generated by left and right worldsheet supersymmetry generators $T_F = -\frac{1}{2}\psi\partial x(z)$, $\overline{T}_F = -\frac{1}{2}\overline{\psi}\,\overline{\partial} x(\overline{z})$.

We note from (2.1) that $Z_{\rm circ}(r)$ shares the same duality[3] as the bosonic partition function $\Gamma(r)$ of (2.1), namely $Z_{\rm circ}(r) = Z_{\rm circ}(1/2r)$. At the self-dual point $r = 1/\sqrt{2}$, the affine $SU(2)$ symmetry, generated by $J^3 = i\partial x$, $J^{\pm} = e^{\pm i\sqrt{2}x(z)}$, combines with the $N = 1$ superconformal symmetry to form not a super-affine symmetry but rather an $N = 3$ superconformal algebra [9] in which the three supersymmetry generators $T_F^i = \frac{i}{2}\psi J^i$ transform as a triplet under the affine $SU(2)$ subalgebra (ditto for anti-holomorphic).

To construct new theories, we recall that given a modular invariant theory that possesses some symmetry group G, we can frequently construct another modular invariant theory by a procedure known as "modding out" by the symmetry G [10]. In the Hamiltonian picture, this procedure involves projecting onto group invariant states in the original (untwisted) Hilbert space sector, and then adding on twisted sectors, again with a group invariant projection, corresponding to strings which close only up to an element of G. (An orbifold modification, changing the boundary conditions of fields in the theory while leaving the the energy-momentum tensor invariant, does not alter the value of the central charge as determined by (1.1).) In general, the symmetry group G need not act in a left-right symmetric fashion. For the small values of c considered here, however, the constraint of modular invariance (equivalent to left-right level matching in the case of G abelian[10][11]) only allows left-right symmetric twists.

In the Lagrangian picture, we use $g\underset{h}{\square}$ to denote the path integral for a conformal field theory on the torus twisted by elements $g, h \in G$ in the "time" and "space" directions. For the case when G is an abelian group with N elements, the modding out procedure above corresponds to defining the partition function

$$Z = \frac{1}{N}\sum_{g,h\in G} g\underset{h}{\square}\ \epsilon(g,h)\ , \tag{2.4}$$

[3] For recent discussions and further references, see [7],[8] (the latter reference including a treatment of interactions).

where $\epsilon(g,h)$ is a phase factor we shall discuss in a moment. The sum over h is seen to implement the sum over twisted sectors in the Hilbert space, and the sum over g (normalized by $1/N$) the group invariant projection in each sector. When G is non-abelian, boundary conditions twisted by non-commuting elements $g, h \in G$ are not consistent, and the correct prescription is to take the summation in (2.4) only over mutually commuting group elements: $g, h \in G$ with $gh = hg$ [10][12]. In the case that G has a normal subgroup $N \subset G$, the modding out may be performed in two steps, first constructing the theory modded out by N, and then modding out that theory by the quotient group G/N.[4]

The phase factor $\epsilon(g,h)$ in (2.4), termed "discrete torsion" in [11], is constrained to satisfy one-loop modular invariance, $\epsilon(g,h) = \epsilon(g^a h^b, g^c h^d)$ for $ad - bc = 1$, and as well additional conditions stemming in the Hamiltonian picture from the requirement of a consistent operator interpretation, and in the Lagrangian picture from the requirement of two-loop modular invariance and factorization [13]. These conditions may be concisely expressed in the relations[11] $\epsilon(g,g) = 1$, $\epsilon(g,h) = \epsilon(h,g)^{-1}$, and $\epsilon(g_1 g_2, h) = \epsilon(g_1,h)\,\epsilon(g_2,h)$. The phase factors $\epsilon(g,h)$ can be non-trivial if and only if $H^2(G,U(1))$, the second cohomology group of G with $U(1)$ coefficients, is non-trivial. This cohomology group is perhaps more familiar in its role of classifying allowed projective representations of the group G (where it is also known as the Schur multiplier $M(G)$). In Hamiltonian language, the inclusion of discrete torsion corresponds to changing the projections in twisted sectors.

If the symmetry group G exists at all values of some set of moduli for a given family of conformal field theories, then we can clearly mod out by G at all these values to produce another family of theories with the same set of moduli. From the standpoint of correlation functions of operators (on the genus zero worldsheet), this is easily understood by first noting that exactly marginal

[4] An example of this is given by modding out by a space group S acting on $\mathbf{R}^n$, in which case[10] one can first mod out $\mathbf{R}^n$ by the normal subgroup Λ of translations, and then mod out the resulting torus by the group $\overline{P} = S/\Lambda$, which gives the action of the point group P on the torus $\mathbf{R}^n/\Lambda$.

operators invariant under G survive the projection in the untwisted sector of the twisted theory. Since correlation functions for fields in the untwisted sector of the twisted theory are the same as in the original theory, the G-invariant exactly marginal operators will be exactly marginal as well in the twisted theory. We can therefore construct new lines of superconformal theories starting from the circle theory by identifying those symmetries that exist for all values of the radius and leave the worldsheet supersymmetry generator invariant (up to a sign).

We shall now introduce the generic symmetry generators for the theories of interest. First we consider two symmetries, each of order two, that act at generic radii. The first is the $\mathbf{Z}_2$ symmetry generated by

$$R: \quad x \to -x, \quad \psi \to -\psi. \tag{2.5}$$

This is a symmetry both of the action for a free boson and fermion and also of the worldsheet supersymmetry generator $T_F = -\frac{1}{2}\psi\partial x$. The second symmetry is a $\mathbf{Z}_2$ symmetry of any superconformal theory, defined to act as $+1$ on states in the antiperiodic (NS,NS) sector of the worldsheet supersymmetry generator, and as -1 on states in the periodic (R,R) sector. We will refer to this symmetry as $(-1)^{F_s}$ since it resembles the action of an operator $(-1)^{F_s}$ in string theory where F_s would be the spacetime fermion number.[5] In appendix A we give a brief description of the result of modding out a general superconformal theory by $(-1)^{F_s}$.

We now recall that the summation over winding numbers and momenta in the circle theory partition function (2.1) is conveniently viewed as a summation over an even Lorentzian self-dual lattice of signature (1,1), with lattice vectors $p = (p_L, p_R) = m(\frac{1}{2r}, \frac{1}{2r}) + n(r, -r)$. Modding out the circle theory at any radius r by the group generated by $\exp(2\pi i p \cdot (r, -r)/\ell)$, a $\mathbf{Z}_\ell$ subgroup of the diagonal $U(1)$ symmetry of the circle theory, gives the same theory at radius r/ℓ. To give something new, we must accompany these translations by the symmetries R and $(-1)^{F_s}$ described above. Since these other twists are each of order two, it is sufficient to couple only to the translation of the circle coordinate x by half its period, generated by the operator $e^{2\pi i p\cdot\delta}$ where δ is the shift vector $\frac{1}{2}(r, -r)$. The reason is that if we couple one of the order two elements to a shift with ℓ odd, then the group generated is $\mathbf{Z}_2 \times \mathbf{Z}_\ell$, and the effect of modding out is the same as modding out the theory at radius r/ℓ by the $\mathbf{Z}_2$ symmetry. For ℓ even, on the other hand, the group generated is isomorphic to $\mathbf{Z}_\ell$ and has a $\mathbf{Z}_{\ell/2}$ normal subgroup. The modding out can then be performed in two stages, first by $\mathbf{Z}_{\ell/2}$ and then by $\mathbf{Z}_\ell/\mathbf{Z}_{\ell/2} = \mathbf{Z}_2$, and is thus equivalent to modding out the theory at radius $r/(\ell/2)$ by the $\mathbf{Z}_2$ symmetry. Finally the symmetry $Re^{2\pi i p\cdot\delta}$ is equivalent to the action of R alone, as can be seen by working with a shifted coordinate $x' = x + 2\pi r/4$.

To summarize, the generic symmetries that we shall employ to try to construct new theories are the $\mathbf{Z}_2$ symmetries R, $(-1)^{F_s}$, and their non-trivial combinations,

$$S_\delta \equiv (-1)^{F_s} e^{2\pi i p\cdot\delta} \qquad \text{and} \qquad S_R \equiv (-1)^{F_s} R\,, \tag{2.6}$$

with $e^{2\pi i p\cdot\delta}$ and with each other. These symmetries are all consistent with worldsheet superconformal invariance. We are now ready to construct new theories by modding out the circle theory by groups generated by the $\mathbf{Z}_2$ elements we have defined. We shall use the notation $(g_1, \ldots, g_n)\, Z$ to denote symbolically the theory resulting from modding out the theory Z by the group generated by the elements $(g_1, \ldots, g_n)$.

B. The orbifold line

A second line of theories is obtained by twisting the circle theory (2.3) by the reflection R of (2.5):

$$Z_{\rm orb}(r) = R Z_{\rm circ}(r). \tag{2.7}$$

[5] More precisely spacetime fermion number coincides with the chiral analog of this operator for conventional NSR fermions in string theory, where spin structures of left- and right-movers can be decoupled. In the theories considered here, on the other hand, modular invariance requires that left- and right-movers have the same spin structure, and will tell us to project onto states with $(-1)^{F_L+F_R}$ odd rather than even in the sector twisted by $(-1)^{F_s}$ (analogous to the case of spacetime theories considered in [14]).

Since the sum over the spin structures for the fermion is invariant under $\psi \to -\psi$, we see that the partition function $Z_{\mathrm{orb}}(r)$ is equal to that for an ordinary $S^1/\mathbf{Z}_2$ bosonic orbifold[6] times the partition function (2.2) for the fermion:

$$Z_{\mathrm{orb}}(r) = \frac{1}{2}\left(\Gamma(r) + \frac{|\vartheta_3\vartheta_4|}{\eta\bar\eta} + \frac{|\vartheta_2\vartheta_3|}{\eta\bar\eta} + \frac{|\vartheta_2\vartheta_4|}{\eta\bar\eta}\right) Z_{\mathrm{Ising}}. \tag{2.8}$$

$Z_{\mathrm{orb}}(r)$ shares the same duality as $Z_{\mathrm{circ}}(r)$, namely $Z_{\mathrm{orb}}(r) = Z_{\mathrm{orb}}(1/2r)$. At its self-dual point $r = 1/\sqrt{2}$, $Z_{\mathrm{orb}}(r)$ satisfies

$$Z_{\mathrm{orb}}\left(\frac{1}{\sqrt{2}}\right) = Z_{\mathrm{circ}}(\sqrt{2}) \tag{2.9}$$

due to the equivalence[10] (see also [8][7][16]) between the twist by reflection and shift by half period at the affine $SU(2)^2$ point. In the superconformal case the theories at the point (2.9) incorporate a (twisted) $N = 3$ algebra.

C. The super-affine line

Another line of theories with $N = 1$ superconformal invariance is given by twisting the circle theory by S_δ:

$$Z_{\mathrm{s\text{-}a}}(r) = S_\delta Z_{\mathrm{circ}}(r) \tag{2.10}$$

(the subscript s-a stands for "super-affine" for reasons that will be clarified shortly). These are our first superconformal theories at $\hat{c} = 1$ that are not simply products of $c = 1$ theories with the $c = 1/2$ theory Z_{Ising}. The theories defined by (2.10) have partition function

$$\begin{aligned} Z_{\mathrm{s\text{-}a}}(r) = &\frac{1}{2}\left(\left|\frac{\vartheta_3}{\eta}\right| + \left|\frac{\vartheta_4}{\eta}\right|\right)\Gamma^+ + \frac{1}{2}\left|\frac{\vartheta_2}{\eta}\right|\Gamma^- \\ &+ \frac{1}{2}\left(\left|\frac{\vartheta_3}{\eta}\right| - \left|\frac{\vartheta_4}{\eta}\right|\right)\Gamma^-_\delta + \frac{1}{2}\left|\frac{\vartheta_2}{\eta}\right|\Gamma^+_\delta , \end{aligned} \tag{2.11}$$

where Γ^+ and Γ^- are defined as is Γ in (2.1) but with the lattice summation restricted to $m \in 2\mathbf{Z}$, $m \in 2\mathbf{Z}+1$ (i.e. m even and odd) respectively; the shifted lattices $\Gamma^\pm_\delta$ are defined similarly but now also with the lattice shifted by δ so that $n \in \mathbf{Z} + \frac{1}{2}$. Modular invariance of $Z_{\mathrm{s\text{-}a}}(r)$ is easily verified, as is the property

$$S_\delta Z_{\mathrm{s\text{-}a}}(2r) = Z_{\mathrm{circ}}(r),$$

leaving neither theory the more "fundamental". Under $r \leftrightarrow 1/r$, we see that Γ^+ and Γ^-_δ are invariant, and $\Gamma^- \leftrightarrow \Gamma^+_\delta$, so $Z_{\mathrm{s\text{-}a}}(r)$ satisfies the duality $Z_{\mathrm{s\text{-}a}}(r) = Z_{\mathrm{s\text{-}a}}(1/r)$. The theory at the self-dual point $r = 1$ has partition function

$$Z_{\mathrm{s\text{-}a}}(1) = \frac{1}{2}\left(\left|\frac{\vartheta_3}{\eta}\right|^3 + \left|\frac{\vartheta_4}{\eta}\right|^3 + \left|\frac{\vartheta_2}{\eta}\right|^3\right), \tag{2.12}$$

and turns out to have a super-affine $SO(3)^2$ symmetry, as discussed in the next section. In appendix B, we provide a bit of extra intuition for the modding out prescription (2.10) by considering its effect at $r = 1$.

D. The super-orbifold line

The super-affine line of theories inherits all of the $\mathbf{Z}_2$ symmetries possessed by the circle line. We can therefore construct a fourth line of theories by the twist

$$Z_{\mathrm{s\text{-}orb}}(r) = R\, Z_{\mathrm{s\text{-}a}}(r). \tag{2.13}$$

These theories can also be constructed by twisting the circle theory by the $\mathbf{Z}_2 \times \mathbf{Z}_2$ group generated by R and S_δ. For a $\mathbf{Z}_2 \times \mathbf{Z}_2$ twist with generators α, β, discrete torsion corresponds to choosing the sign of the modular orbit

$$\alpha \underset{\beta}{\square} + \beta \underset{\alpha}{\square} + \alpha\beta \underset{\alpha}{\square} + \alpha \underset{\alpha\beta}{\square} + \alpha\beta \underset{\beta}{\square} + \beta \underset{\alpha\beta}{\square} . \tag{2.14}$$

It is simple to see however that for $\alpha = R$ and $\beta = S_\delta$, each term in this orbit actually vanishes since the first term, for example, corresponds in Hamiltonian language to taking the trace of an off-diagonal operator, and the remaining terms are all related by modular transformations. The torsion therefore has no effect on the one-loop partition function in this case, and

$$Z_{\mathrm{s\text{-}orb}}(r) = (S_\delta, R)\, Z_{\mathrm{circ}}(r)$$

[6] For a derivation of the partition function for the $S^1/\mathbf{Z}_2$ orbifold at $c = 1$, see for example [15].

independent of the sign of the discrete torsion. It is not hard to show that twisting the circle theory by the group generated by (S_δ, S_R) also yields $Z_{\text{s-orb}}$ independent of the sign of the discrete torsion.

The partition function for this theory,

$$Z_{\text{s-orb}}(r) = \frac{1}{2}\Big(Z_{\text{s-a}}(r) + \frac{1}{2|\eta|^3}\big(|\vartheta_2\vartheta_3^2| + |\vartheta_2\vartheta_4^2| + |\vartheta_3\vartheta_2^2| + |\vartheta_3\vartheta_4^2| + |\vartheta_4\vartheta_2^2| + |\vartheta_4\vartheta_3^2|\big)\Big), \tag{2.15}$$

is easily determined by calculating the terms $R\underset{1}{\square}$ and $RS_\delta\underset{1}{\square}$ in the circle theory and adding on their modular orbits. We see that the super-orbifold theory shares the duality $Z_{\text{s-orb}}(r) = Z_{\text{s-orb}}(1/r)$ of the super-affine theory.

E. The orbifold-prime line

The final line of theories is constructed by twisting the circle theory by the group generated by S_R

$$Z_{\text{orb}'}(r) = S_R Z_{\text{circ}}(r), \tag{2.16}$$

and has partition function

$$Z_{\text{orb}'}(r) = Z_{\text{orb}}(r) - 3, \tag{2.17}$$

where $Z_{\text{orb}}(r)$ is as given in (2.8). We note that this theory shares the duality $Z_{\text{orb}'}(r) = Z_{\text{orb}'}(1/2r)$ of the circle and orbifold lines.

The close relation (2.17) between the partition functions of the orbifold and orbifold-prime theories suggests other relations between them. Indeed the result of modding out the super-affine line by its remaining $\mathbf{Z}_2$ symmetries yields

$$Z_{\text{orb}}(r) = (S_\delta, R)_+ Z_{\text{s-a}}(2r) = (S_\delta, S_R)_+ Z_{\text{s-a}}(2r),$$
$$Z_{\text{orb}'}(r) = (S_\delta, R)_- Z_{\text{s-a}}(2r) = (S_\delta, S_R)_- Z_{\text{s-a}}(2r),$$

where the subscript on the group generators indicates the relative sign of the discrete torsion specified by $\epsilon(S_\delta, R) = \pm 1$ and $\epsilon(S_\delta, S_R) = \pm 1$ in the two cases. We see that the Z_{orb} and $Z_{\text{orb}'}$ theories are simply related by a $\mathbf{Z}_2$ torsion.

There is yet another relation between the orbifold and orbifold-prime theories, given by modding out by $(-1)^{F_s}$. According to the results of appendix A, this action has the effect of reversing the sign of the contribution of the ${}_{\mathrm{P}}\underset{\mathrm{P}}{\square}$ term in the partition function (where the P's indicate the Ramond sector for the worldsheet supersymmetry generator). Now the orbifold theory has precisely three $(1/16, 1/16)$ operators in the Ramond sector: one spin field for the fermion and two twist fields for the boson (recall that the P (Ramond) sector includes the sector with x and ψ both periodic and also the sector with both antiperiodic). Keeping track as well of the GSO-unprojected states in the construction of the orbifold, we find that the ${}_{\mathrm{P}}\underset{\mathrm{P}}{\square}$ term in the orbifold partition function is equal to three. It follows that

$$Z_{\text{orb}'}(r) = ((-1)^{F_s})Z_{\text{orb}}(r) \quad \text{and} \quad Z_{\text{orb}}(r) = ((-1)^{F_s})Z_{\text{orb}'}(r). \tag{2.18}$$

This completes our introduction of the theories we have been able to generate by modding out by symmetries that exist for generic values of the moduli. There are some remaining twists of the three orbifold type lines $Z_{\text{orb}}(r)$, $Z_{\text{s-orb}}(r)$, and $Z_{\text{orb}'}(r)$ not yet considered, but these result in no new theories. The orbifold lines are distinguished from the circle and super-affine lines by the absence of dimension $(1,0)$ and $(0,1)$ currents and by the fact that R acts trivially on them. Thus in twisting these theories we only need to consider $e^{2\pi i p\cdot\delta}$ (which reduces the radius by half), S_δ, and $(-1)^{F_s}$. But twisting by S_δ gives

$$Z_{\text{s-orb}}(r) = S_\delta\, Z_{\text{orb}}(r) = S_\delta\, Z_{\text{orb}'}(r) \qquad \text{and} \qquad Z_{\text{orb}}(r) = S_\delta\, Z_{\text{s-orb}}(2r).$$

Twisting by $(-1)^{F_s}$, according to the results of appendix A, has a nontrivial action on the one-loop partition function only for theories with a nonvanishing ${}_{\mathrm{P}}\underset{\mathrm{P}}{\square}$ term in the partition function. Of the five lines considered here, this is the case only for the orbifold and orbifold-prime theories, and these two lines, as discussed above, are simply interchanged by the $(-1)^{F_s}$ twist.

For the other three lines, we wish to show that the $(-1)^{F_s}$ twisted theory is equivalent to the untwisted theory; i.e. that it has the same operator algebra as well as the same partition function. To do this we need to establish an isomorphism between operators in the Ramond sector with even and odd worldsheet fermion numbers. For the circle theory this isomorphism is given by multiplying operators in the Ramond sector by the fermion zero mode ψ_0. This leaves invariant the relevant three-point couplings between two Ramond states and a Neveu-Schwarz state, as can be seen by calculating the three-point function with the Ramond states as in and out states. Since the three-point couplings determine the operator algebra, this establishes the isomorphism. Moreover since the super-orbifold line intersects the circle line and has the same supersymmetry generator at the crossing point, the same argument implies as well that the super-orbifold line is isomorphic to its $(-1)^{F_S}$ twisted version at that point. But then integrating the effect of the marginal operator that deforms the theory in the super-orbifold direction shows that the super-orbifold line and its $(-1)^{F_S}$ twisted version are isomorphic throughout. A similar argument works for the super-affine line at $r = 1$, where it can be written in terms of three free fermions (see below). The operator giving the isomorphism between the ± 1 Ramond states in this case is $\psi_0^1 \psi_0^2 \psi_0^3$.

3. Super-affine symmetry and multicriticality

To understand the additional symmetry at the self-dual point $Z_{\text{s-a}}(r)$ of the super-affine line, we first recall that in general dim G free Majorana fermions can be used to realize a super-affine G algebra with an enveloping super Virasoro algebra[17]. At $c = 3/2$, three free fermions ψ^i taken to transform as the vector of $SO(3)$ can be used to represent an $N = 1$ superconformal algebra with a super-affine $SO(3)$ symmetry. The superconformal current is given by $T_F = -\frac{1}{2}\psi^1\psi^2\psi^3 = -\frac{1}{12}\epsilon_{ijk}\psi^i\psi^j\psi^k$, and the components of the superfield affine generators $\mathbf{J}^i = \mathcal{J}^i + \theta J^i = i\psi^i + \theta\frac{i}{2}\epsilon_{ijk}\psi^j\psi^k$ satisfy the super-affine operator products

$$J^i(z_1)J^j(z_2) \sim \frac{\frac{1}{2}k\delta^{ij}}{(z_1 - z_2)^2} + \frac{i\epsilon_{ijk}}{(z_1 - z_2)}J^k(z_2)$$

$$J^i(z_1)\mathcal{J}^j(z_2) \sim \frac{i\epsilon_{ijk}}{(z_1 - z_2)}\mathcal{J}^k(z_2)$$

$$\mathcal{J}^i(z_1)\mathcal{J}^j(z_2) \sim \frac{\frac{1}{2}k\delta^{ij}}{(z_1 - z_2)}$$

(at affine level $k = 2$). A modular invariant theory on the torus is given by taking left and right fermions ψ^i and $\overline{\psi}^i$ and summing over the same spin structure for all six fermions (GSO projecting on $(-1)^{F_L+F_R}$). This symmetric sum over spin structures gives a theory that manifests the full super-affine $SO(3)^2$ symmetry and has partition function

$$\frac{1}{2}\left(\mathrm{AAA}\,\square_{\mathrm{AAA}} + \mathrm{PPP}\,\square_{\mathrm{AAA}} + \mathrm{AAA}\,\square_{\mathrm{PPP}} + \mathrm{PPP}\,\square_{\mathrm{PPP}}\right), \tag{3.1}$$

equal to (2.12). We have thus identified the extra symmetry possessed by the $Z_{\text{s-a}}(r)$ theory at its self-dual point as a super-affine $SO(3)^2$ symmetry. (We shall frequently refer to the theory $Z_{\text{s-a}}(1)$ with super-affine symmetry as the "$SO(3)^2$ theory" to distinguish it from the "$SU(2)^2$ theory" $Z_{\text{circ}}(1/\sqrt{2})$ with ordinary affine $SU(2)^2$ symmetry (embedded in an $N = 3$ superconformal algebra).)

We now consider the multicritical points that occur on the lines introduced in the previous section. It is straightforward from examination of the partition functions to determine the points at which additional (1,1) operators appear. The first order integrability conditions (as reviewed in [8]) can be used to show that the points at which these operators can be truly marginal are at radii $r = \frac{1}{\sqrt{2}}$, 1, and $\sqrt{2}$ on any of these lines. We shall see that there are indeed exactly marginal operators at each of these points. The reader may find it useful to refer to fig. 1 to follow better the discussion of these points.

The point $r = \sqrt{2}$ on the circle line, for example, corresponds to the point $r = 1/\sqrt{2}$ on the orbifold line for the same reason that this equivalence results for the $c = 1$ circle and orbifold lines. The correspondence between the theories at the crossing point, as mentioned in the previous section, is most easily established by recognizing that each of the theories can be constructed by $\mathbf{Z}_2$ twists, either by a reflection $x \to -x$ or by a shift $x \to x + 2\pi/2\sqrt{2}$ of the theory at the self-dual point $r = 1/\sqrt{2}$ of the circle line, and that the two $\mathbf{Z}_2$ modifications are equivalent due to the enhanced affine $SU(2)^2$ symmetry at that point. From this we find

$$Z_{\text{circ}}(\sqrt{2}) = Z_{\text{orb}}\left(\frac{1}{\sqrt{2}}\right), \tag{3.2}$$

not only at the level of the partition function but also as an isomorphism between the operator algebras of the theories. From the standpoint of either theory, for example, it might not be immediately obvious that the additional (1,1) operator that appears at the multicritical point is integrable to all orders, but from the standpoint of the other theory, we have the manifestly exactly marginal operator $\partial x \bar{\partial} x$. It might appear surprising at first that both marginal deformations are consistent with superconformal invariance, given that there is only a single $(\frac{1}{2},\frac{1}{2})$ state in the spectrum at the crossing point with which the (1,1) operators could be paired by supersymmetry. The paradox is resolved by recognizing that there are three $(\frac{3}{2},0)$ operators (and three $(0,\frac{3}{2})$'s) in the non-local $\mathbf{Z}_2$ covering theory at that point, one given by $\psi\partial x$ and the other two by $\psi\exp(\pm i\sqrt{2}x)$. The first acts as the supersymmetry generator for the circle line. The circle line (and also the super-affine line) at $r=\sqrt{2}$ actually possesses an untwisted $N=2$ superconformal algebra with the other two dimension $(\frac{3}{2},0)$ fields acting as $T_F^{\pm}$. The $U(1)$ part of this algebra rotates T_F^{+} and its complex conjugate T_F^{-} by opposite phases, and any hermitian combination of the two can be used as the $N=1$ supersymmetry generator for the orbifold line.

We employ a similar logic to establish equivalences between theories at the other potential multicritical points mentioned above. With the super-affine theory at $r=1$ represented in terms of three fermions ψ^i, we can regard $\psi^1\psi^2$ say as the fermionized current ∂x. The reflection R of (2.5) can be realized in these terms as $\psi^2\to-\psi^2$, $\psi^3\to-\psi^3$. But this fermionic twist also has the interpretation of decoupling the sum over spin structures to independent sums over the (left-right) spin structures for ψ^1 and for ψ^2,ψ^3. Rebosonizing the last two fermions, we find an Ising model tensored with a circle at radius $r=1$ (for more details, see appendix B). It thus follows that

$$Z_{\text{s-orb}}(1)\equiv RZ_{\text{s-a}}(1)=Z_{\text{circ}}(1), \tag{3.3}$$

and the circle and super-orbifold lines cross at a multicritical point, as depicted in fig. 1. We can regard the equivalence (3.3) as resulting from the permutation symmetry among the fermions at the $SO(3)^2$ point. The enhanced symmetry

at $Z_{\text{s-a}}(1)$ will be further exploited in the next section when we mod out by other subgroups of the diagonal $SO(3)$.

Other multicritical point equivalences follow from the symmetries of the circle line at $r=1/\sqrt{2}$, which imply $RZ_{\text{circ}}(1/\sqrt{2})=e^{2\pi i p\cdot\delta}Z_{\text{circ}}(1/\sqrt{2})$. From this and the $r\leftrightarrow 1/r$ duality of the super-affine line we find

$$Z_{\text{orb}'}\left(\frac{1}{\sqrt{2}}\right)\equiv S_R Z_{\text{circ}}\left(\frac{1}{\sqrt{2}}\right)=S_\delta Z_{\text{circ}}\left(\frac{1}{\sqrt{2}}\right)=Z_{\text{s-a}}\left(\frac{1}{\sqrt{2}}\right)=Z_{\text{s-a}}(\sqrt{2}), \tag{3.4}$$

so that $Z_{\text{orb}'}(1/\sqrt{2})$ and $Z_{\text{s-a}}(\sqrt{2})$ specify identical theories at a crossing point between the super-affine and orbifold-prime lines. Similarly, we find that the super-orbifold and orbifold-prime lines cross at $Z_{\text{s-orb}}(\sqrt{2})=Z_{\text{orb}'}(\sqrt{2})$,

$$\begin{aligned}Z_{\text{s-orb}}(\sqrt{2})&\equiv RZ_{\text{s-a}}(\sqrt{2})=RZ_{\text{s-a}}\left(\frac{1}{\sqrt{2}}\right)=R\left(S_\delta Z_{\text{circ}}\left(\frac{1}{\sqrt{2}}\right)\right)\\&=R\left(S_R Z_{\text{circ}}\left(\frac{1}{\sqrt{2}}\right)\right)=S_R\left(RZ_{\text{circ}}\left(\frac{1}{\sqrt{2}}\right)\right)=S_R Z_{\text{circ}}(\sqrt{2})\\&=Z_{\text{orb}'}(\sqrt{2}).\end{aligned} \tag{3.5}$$

This last crossing point is distinguished as the only multicritical point at $\hat{c}=1$ for which neither of the theories at the crossing point is self-dual. The junction between the circle and super-orbifold theories, on the other hand, is the only one at which the supersymmetry generator need not be redefined. At the other three multicritical points, there is a phenomenon similar to that discussed at the circle-orbifold crossing. Finally we point out that the equivalences (3.2)–(3.5) are also easily confirmed at the level of the partition functions from their explicit forms given in the previous section.

The four multicritical points above, all reflecting the enhanced symmetry at either the $SU(2)^2$ or $SO(3)^2$ theory, turn out to exhaust the multicritical behavior of the $\hat{c}=1$ theories considered here. The remaining (1,1) operators that appear at $r=1$ and $\sqrt{2}$ of the orbifold line, and at $r=1$ on the orbifold-prime line, are integrable but generate no new theories because of discrete symmetries at these points that relate the different marginal directions. The theory $Z_{\text{orb}}(1)$ for example is comprised of three copies of the Ising model and has an S_3 permutation symmetry that relates the three marginal operators, leaving all deformations equivalent to the generic deformation of radius along the orbifold line. The same discrete symmetry relates the three potentially different marginal directions at $r = 1$ on the orbifold-prime line. The partition function on this line differs from that of the orbifold line by the removal of the three (1/16,1/16) operators, but the operator algebra of the marginal operators, coming from the untwisted sector, turns out to be unaffected (see appendix B).

Finally, the point $Z_{\rm orb}(\sqrt{2})$ on the orbifold line is equivalent to a 4-state Potts model tensored with the Ising model, and has an S_4 permutation symmetry (under which the three marginal operators transform as a three dimensional representation), again rendering all of the marginal deformations equivalent.[7] On the other hand, there is a multicritical point at $r = \sqrt{2}$ on the orbifold-prime line. This point may be obtained from the S_4 symmetric point at $r = \sqrt{2}$ on the orbifold line by modding out by S_δ, which breaks the symmetry down to S_2, leaving two of the three surviving marginal operators inequivalent. This gives a good example of an inequivalence of the operator algebras for two theories, despite the close relation between their partition functions.

4. Modding out by enhanced symmetries

Having modded out by the symmetries that exist at generic radii, we now turn to a more systematic consideration of modding out by the enlarged symmetries that occur at specific points on the lines in fig. 1. Even in the $c = 1$ case, enlarged discrete symmetries that exist at special values of the radii $r_{\rm circ}$ and $r_{\rm orb}$ have been only partially identified. In all known cases, twisting by discrete symmetries gives either a point on the circle or orbifold lines or one of the three isolated polyhedral theories mentioned in the introduction. In what follows we shall restrict attention to modding out by discrete symmetries only at particular highest symmetry points. At $c = 1$ the point of highest symmetry is the affine $SU(2)^2$ point at which it is possible to mod out by all the finite subgroups of a diagonal $SO(3)$ symmetry. The finite subgroups of $SO(3)$ are the cyclic groups $\mathbf{C}_n$ of order n, the dihedral groups $\mathbf{D}_n$ of order $2n$, and the polyhedral groups $\mathbf{T}$, $\mathbf{O}$, and $\mathbf{I}$, of orders 12, 24, and 60 respectively. At $\hat{c} = 1$, the point of highest symmetry is the super-affine $SO(3)^2$ point, at which dividing out by the finite subgroups of the diagonal $SO(3)$ preserves the world-sheet superconformal symmetry, generated by $T_F = -\frac{1}{12}\epsilon_{ijk}\psi^i\psi^j\psi^k$ and its anti-holomorphic partner.

Twisting by each of these groups G allows in principle additional new theories related by insertion of discrete torsion for both $c = 1$ and $\hat{c} = 1$. To characterize the non-trivial possibilities we need to determine $H^2(G, U(1))$, as mentioned in the discussion following (2.4). In the language of projective representations of groups it is easy to see that $H^2(G, U(1))$ is at least $\mathbf{Z}_2$ for many of these groups. A projective representation is a representation $U(g)$ of group elements $g \in G$ that satisfies the group multiplication law only up to a phase, $U(g_1)U(g_2) = \omega(g_1, g_2)\, U(g_1 g_2)$.[8] In terms of these ω's, the phase ϵ of (2.4) is given by $\epsilon(g_1, g_2) = \omega(g_1, g_2)\omega^{-1}(g_2, g_1)$ [11][18]. Now the projective spinor representation of $SO(3)$ automatically determines a set of $\mathbf{Z}_2$ phases $\omega(g_1, g_2) = \pm 1$ that restrict to a consistent (i.e. associative) set of phases $\omega(g_1, g_2)$ for any finite subgroup of $SO(3)$. For the subgroups $\mathbf{C}_n$ and $\mathbf{D}_{2n+1}$, these phases are easily shown to be removable by a phase redefinition, whereas for the remaining groups, $\mathbf{D}_{2n}$, $\mathbf{T}$, $\mathbf{O}$, and $\mathbf{I}$, this possibility is easily excluded: they all have abelian $\mathbf{D}_2 \approx \mathbf{Z}_2 \times \mathbf{Z}_2$ subgroups for which $\omega(g_1, g_2) = -\omega(g_2, g_1)$ when $g_1, g_2 \neq 1$, and consequently no phase redefinition can exist. To confirm that the $\mathbf{Z}_2$ central extensions of these groups to their double covers in $SU(2)$ indeed saturate all possibilities for non-trivial second cohomology with $U(1)$

[7] It turns out (see the end of sec. 4) that there are also three supersymmetry generators at this point, which likewise transform as a three dimensional representation of the S_4 symmetry. The three marginal operators are each the highest component of a dimension $(\frac{1}{2}, \frac{1}{2})$ superfield with lowest component $\psi\overline{\psi}$ with respect to one of these three supersymmetry generators.

[8] The condition of associativity provides a notion of "closure" that the phases ω must satisfy, $\omega(g_1, g_2)\,\omega(g_1 g_2, g_3) = \omega(g_1, g_2 g_3)\,\omega(g_2, g_3)$. An "exact" phase $\omega(g_1, g_2) = \rho^{-1}(g_1)\rho^{-1}(g_2)\rho(g_1 g_2)$ is one removable by the redefinition $U(g) \to \rho(g)U(g)$ for all $g \in G$. There follows a natural notion of cohomology H^2.

coefficients requires a bit more effort. Conveniently, Schur[9] (1911) has established that $H^2(G,U(1)) = \mathbf{Z}_2$ for $G = \mathbf{D}_{2n}$, $\mathbf{T}$, $\mathbf{O}$, $\mathbf{I}$, and $H^2(G,U(1)) = 1$ for $G = \mathbf{C}_n$, $\mathbf{D}_{2n+1}$.

At $c = 1$ however the modular orbits affected by the discrete torsion all vanish (for the same reason as did (2.14)), so the one-loop partition function is unaffected. This does not necessarily mean that the two torsion-related theories are identical.[10] One can argue that they are identical for the $\mathbf{D}_{2n}$-twisted $SU(2)^2$ theories at $c = 1$, because these theories are points on the $c = 1$ orbifold line. This line is more auspiciously viewed as modding out $\mathbf{R}^1$ by $\mathbf{D}_\infty$ (the translation-reflection group of the line), for which $H^2(\mathbf{D}_\infty, U(1)) = 1$ [21], and so non-trivial torsion is excluded. From the $\mathbf{D}_{2n}$ point of view, it is easy to verify that the different operators selected by the two choices of projections in twisted sectors nonetheless satisfy isomorphic algebras. For the polyhedral theories $\mathbf{T}$, $\mathbf{O}$, and $\mathbf{I}$, we have been unable to formulate a definitive argument for interactions between different twisted sectors.[11]

At $\hat{c} = 1$, on the other hand, we shall find the inclusion of discrete torsion to have a non-trivial effect. We denote by $Z[G]$ the theory given by modding out the super-affine $SO(3)^2$ theory $Z_{\text{s-a}}(1)$ by the finite subgroup $G \subset SO(3)$. (Each such theory will have its states classified by a twisted super-affine $SO(3)^2$

[9] Quoted in [19] in terms of permutation groups – recall that $\mathbf{T} \approx A_4$, $\mathbf{O} \approx S_4$, and $\mathbf{I} \approx A_5$.

[10] To see that there is an issue here, we recall that affine Spin(32)/$\mathbf{Z}_2$ and affine $E_8 \times E_8$ have identical one-loop partition functions but are different conformal field theories. In the other direction, however, there are as well ways of constructing theories that differ by discrete torsion but are nonetheless the same theory. For example, if one mods out the tachyonic $SO(32)$ heterotic string theory by a $(\mathbf{Z}_2)^4$ acting on the internal fermions as four groups of eight, then there are $2^6 = 64$ discrete phase choices. 3 of these give identically the $E_8 \times E_8$ theory, 5 the Spin(32)/$\mathbf{Z}_2$ theory, and 33 the $SO(16) \times SO(16)$ theory[20].

[11] An analysis of this question from the standpoint of the associated statistical mechanical models may be found in [22].

symmetry, isomorphic to the untwisted algebra.) For the cyclic groups $G = \mathbf{C}_n$, we shall employ the special notation

$$Z_n \equiv Z[\mathbf{C}_n] \tag{4.1}$$

for the partition functions. The generator of $\mathbf{C}_n$ may be taken to act on the super-affine generators as $\mathbf{J}^3 \to \mathbf{J}^3$, $\mathbf{J}^\pm \to e^{\pm 2\pi i/n}\mathbf{J}^\pm$ (and simultaneously $\overline{\mathbf{J}}^3 \to \overline{\mathbf{J}}^3$, $\overline{\mathbf{J}}^\pm \to e^{\pm 2\pi i/n}\overline{\mathbf{J}}^\pm$). For n odd, we find the action of modding out to be equivalent to decreasing the radius to discrete values along the line (2.11),

$$Z_{2m+1} = Z_{\text{s-a}}\left(\frac{1}{2m+1}\right) = Z_{\text{s-a}}(2m+1). \tag{4.2}$$

For n even, $n = 2m$, we find instead that we are taken to discrete points along the line (2.3),

$$Z_{2m} = Z_{\text{circ}}\left(\frac{1}{m}\right) = Z_{\text{circ}}\left(\frac{m}{2}\right). \tag{4.3}$$

(This latter result is easily understood by recognizing that $\mathbf{C}_{2m}$ has as normal subgroup $\mathbf{C}_2$, and the modding out of $Z_{\text{s-a}}(1)$ may proceed in two steps: first mod out by $\mathbf{C}_2$ and then by $\mathbf{C}_{2m}/\mathbf{C}_2 = \mathbf{C}_m$. The first step gives the theory $Z_{\text{circ}}(1)$, and the second step reduces the radius by a factor of m.) We have indicated the points corresponding to these theories in fig. 1. Note that $Z[\mathbf{C}_2]$ and $Z[\mathbf{C}_4]$ according to (4.3) are equivalent, due to the $r \leftrightarrow \frac{1}{2r}$ duality of the circle line. The partition function $Z_2 = Z_4 = Z_{\text{s-orb}}(1) = Z_{\text{circ}}(1)$ corresponds to the multicritical point at which the circle and super-orbifold lines cross.

With the convention $Z_1 = Z_{\text{s-a}}(1)$, the partition functions for the orbifold, orbifold-prime, and super-orbifold theories of (2.8), (2.17), and (2.15) may be usefully written

$$\begin{aligned} Z_{\text{orb}}(r) &= \frac{1}{2}\big(Z_{\text{circ}}(r) + 2Z_2 - Z_1 + 3\big), \\ Z_{\text{orb}'}(r) &= \frac{1}{2}\big(Z_{\text{circ}}(r) + 2Z_2 - Z_1 - 3\big), \\ Z_{\text{s-orb}}(r) &= \frac{1}{2}\big(Z_{\text{s-a}}(r) + 2Z_2 - Z_1\big), \end{aligned} \tag{4.4}$$

similar in forms to the expression for the partition function at $c = 1$ for the bosonic orbifold $S^1/\mathbf{Z}_2$ given in [7][8] (although the Z_n's have different definitions here). Counting the number of (1,1) and (1/16,1/16) states in the theories to follow is simplified by noting that Z_1 has nine (1,1) and zero (1/16,1/16) states; Z_2 has five (1,1) and one (1/16,1/16) states; and otherwise Z_n has three (1,1) and zero (one) (1/16,1/16) states for n odd (even).

The partition functions for the remaining non-abelian groups may frequently be calculated in two ways. For the $\mathbf{D}_n$, we can take the reflection added to $\mathbf{C}_n$ to act as $J^3 \to -J^3$, $J^\pm \to J^\mp$. The method employed in [7] evaluates the sum (2.4) by identifying all mutually commuting group elements to reduce the calculation of the partition function to a sum over abelian subsectors, each in turn expressible in terms of the Z_n of (4.1). For the $\mathbf{D}_{2n+1}$, for example, the mutually commuting elements lie in a cyclic group, $\mathbf{C}_{2n+1}$, and $(2n+1)$ disjoint $\mathbf{C}_2$'s. From (2.4) we find for the partition functions,

$$\begin{aligned} Z[\mathbf{D}_{2n+1}] &= \frac{1}{4n+2}\Big((2n+1)Z_{2n+1} + (2n+1)(2Z_2 - Z_1)\Big) \\ &= \frac{1}{2}(Z_{2n+1} + 2Z_2 - Z_1) \\ &= Z_{\text{s-orb}}\left(\frac{1}{2n+1}\right) = Z_{\text{s-orb}}(2n+1), \end{aligned} \tag{4.5}$$

where $(2n+1)Z_{2n+1}$ represents the contribution of $\sum_{g,h\in\mathbf{C}_{2n+1}} g\underset{h}{\square}$ and we have subtracted $Z_1 = 1\underset{1}{\square}$ from the second term to to avoid overcounting. By the comments following (4.4), we see that the $Z[\mathbf{D}_{2n+1}]$ have two marginal operators, one of which is integrable, and a single (1/16,1/16) operator. The result (4.5) may also be derived via the quotient construction $\mathbf{D}_{2n+1}/\mathbf{C}_{2n+1} = \mathbf{Z}_2$, passing first through the theories (4.2). We see also that the equivalence (3.3) in the language of subgroups of $SO(3)$ is due to the coincidence between $\mathbf{D}_1$ and $\mathbf{C}_2$.

The partition functions for the $\mathbf{D}_{2n}$'s may be obtained by the above method (see [7]) or via the normal subgroup embedding $\mathbf{D}_2 \subset \mathbf{D}_{2n}$. In the latter procedure we first calculate the result for the abelian twist by $\mathbf{D}_2$, giving

$Z[\mathbf{D}_2] = Z_{\text{orb}}(1) = \frac{1}{2}(3Z_2 - Z_1 + 3)$ (equivalent to the (Ising)^3 model). Modding out by the remaining $\mathbf{D}_{2n}/\mathbf{D}_2 = \mathbf{C}_n$ then gives an orbifold at radius $r = 1/n$. The result of either procedure is

$$Z[\mathbf{D}_{2n}] = \frac{1}{2}(Z_{2n} + 2Z_2 - Z_1 + 3) = Z_{\text{orb}}\left(\frac{1}{n}\right) = Z_{\text{orb}}\left(\frac{n}{2}\right). \tag{4.6}$$

We see that the partition functions (4.5) and (4.6) for the odd and even dihedral groups lie on the super-orbifold and orbifold lines respectively (the points labelled by $\mathbf{D}_{2n+1}$ and $\mathbf{D}_{2n}$ in fig. 1). The $Z[\mathbf{D}_{2n}]$ have two marginal operators, one of which is integrable, and three (1/16,1/16) operators.

We are now in a position to consider the effect of adding $\mathbf{Z}_2$ torsion to the $\mathbf{D}_{2n}$ theories. This requires adding $\mathbf{Z}_2$ torsion to each of the n $\mathbf{D}_2 \approx \mathbf{Z}_2 \times \mathbf{Z}_2$ subgroups of $\mathbf{D}_{2n}$. For a single $\mathbf{D}_2$, the modular orbit whose sign is changed is identically a constant, equal to 3/2 (see appendix B). The partition function for the torsion-related theory is therefore $Z'[\mathbf{D}_2] = Z_{\text{orb}'}(1) = \frac{1}{2}(3Z_2 - Z_1 - 3)$. The partition functions for the remaining $Z'[\mathbf{D}_{2n}]$'s, related to those of (4.6) by $\mathbf{Z}_2$ torsion, can again be calculated either by the method of [7] or via the normal embedding $\mathbf{D}_2 \subset \mathbf{D}_{2n}$ passing this time through the theory $Z_{\text{orb}'}(1)$. The result is

$$Z'[\mathbf{D}_{2n}] = \frac{1}{2}(Z_{2n} + 2Z_2 - Z_1 - 3) = Z_{\text{orb}'}\left(\frac{1}{n}\right) = Z_{\text{orb}'}\left(\frac{n}{2}\right), \tag{4.7}$$

not surprisingly giving image points of (4.6) on the $Z_{\text{orb}'}(r)$ line, as indicated by the $\mathbf{D}'_{2n}$ in fig. 1. The spectrum of the theories (4.7) differs from those of (4.6) by the removal of the three (1/16,1/16) states.

Next we discuss the superconformal analogs of the isolated polyhedral theories at $c = 1$. Theories based on the tetrahedral and octahedral groups $\mathbf{T}$ and $\mathbf{O}$ can be constructed either directly or via one of the normal subgroup embeddings $\mathbf{T}/\mathbf{D}_2 = \mathbf{Z}_3$, $\mathbf{O}/\mathbf{D}_2 = S_3$, or $\mathbf{O}/\mathbf{T} = \mathbf{Z}_2$. In the latter procedure we first construct the (Ising)^3 theory $Z[\mathbf{D}_2]$, then mod out by the $\mathbf{Z}_3$ normal subgroup of the S_3 permutation symmetry to obtain the $Z[\mathbf{T}]$ theory, and finally by a residual $\mathbf{Z}_2$ symmetry to give the $Z[\mathbf{O}]$ theory.[12] The partition functions, constructed by either method, are

$$Z[\mathbf{T}] = \frac{1}{2}(2Z_3 + Z_2 - Z_1 + 1),$$
$$Z[\mathbf{O}] = \frac{1}{2}(Z_4 + Z_3 + Z_2 - Z_1 + 2). \tag{4.8a}$$

The icosahedral group $\mathbf{I}$, on the other hand, has no non-trivial normal subgroups, and we have available only the method used in [7] to construct the partition function $Z[\mathbf{I}]$. The mutually commuting elements of the icosahedral group lie in 6 $\mathbf{C}_5$'s acting about axes through opposite faces of a dodecahedron, 10 $\mathbf{C}_3$'s acting about axes through antipodal vertices, and 5 non-overlapping $\mathbf{D}_2$'s comprised of the rotations of order 2 about axes through the centers of (the 15 pairs of) opposite edges. Substitution in (2.4) gives the partition function

$$Z[\mathbf{I}] = \frac{1}{60}\Big(6(5Z_5 - Z_1) + 10(3Z_3 - Z_1) + 5\big(4Z[\mathbf{D}_2] - Z_1\big) + Z_1\Big)$$
$$= \frac{1}{2}(Z_5 + Z_3 + Z_2 - Z_1 + 1). \tag{4.8b}$$

We read off from (4.8a, b) that $Z[\mathbf{T}]$ and $Z[\mathbf{I}]$ have one (1/16,1/16) operator while $Z[\mathbf{O}]$ has two, and each of the three theories has a single (1,1) operator.

Just as was the case for the polyhedral theories at $c = 1$, the marginal operator does not satisfy the first order integrability condition and the theories (4.8a, b) constitute isolated points of superconformal symmetry in the $\hat{c} = 1$ moduli space. This follows from the observation that of the nine marginal operators $J^i\overline{J}^j$ of the $Z_{\text{s-a}}(1)$ theory, only the diagonal combination $V = \sum_{i=1}^{3} J^i\overline{J}^i$ survives the orbifold projection for $G = \mathbf{T}, \mathbf{O}, \mathbf{I}$. But the operator product of V with itself contains a term proportional to $V/|z_1 - z_2|^2$, and consequently V cannot remain marginal in the infinitesimally deformed theory. Our intuition for the isolated nature of these theories remains the same as in the $c = 1$ case:

[12] A similar procedure gives the polyhedral theories based on $\mathbf{T}$ and $\mathbf{O}$ at $c = 1$, except in that case the starting point is the $SU(2)^2$ theory modded out by $\mathbf{D}_2$, i.e. the point $r = \sqrt{2}$ on the orbifold line. That theory is the 4-state Potts model, and has a full S_4 permutation symmetry.

these are theories constructed by modding out by symmetries that exist only at a given fixed radius, $r = 1$, of the super-affine line, so modding out effectively freezes the radius.

Our discussion of non-trivial H^2's suggests that we next consider additional theories related to those of (4.8a, b) by $\mathbf{Z}_2$ torsion. As for the $\mathbf{D}_{2n}$'s, we find that the modular orbit whose sign is changed contributes a constant to the partition function, and the torsion-related theories have the partition functions

$$Z'[\mathbf{T}] = \frac{1}{2}(2Z_3 + Z_2 - Z_1 - 1),$$
$$Z'[\mathbf{O}] = \frac{1}{2}(Z_4 + Z_3 + Z_2 - Z_1 - 2), \tag{4.9}$$
$$Z'[\mathbf{I}] = \frac{1}{2}(Z_5 + Z_3 + Z_2 - Z_1 - 1).$$

($Z'[\mathbf{T}]$ and $Z'[\mathbf{O}]$ may also be constructed by modding out the $Z'[\mathbf{D}_2]$ theory at $Z_{\text{orb}'}(1)$ by a $\mathbf{Z}_3$, and then a $\mathbf{Z}_2$ symmetry.) The marginal operators in these theories are identical to those in their torsion-related counterparts; hence these theories constitute three additional isolated points of superconformal invariance, indicated as $\bullet'$'s in fig. 1. The spectra of the theories (4.9) are related to those of (4.8a, b) by the removal of the one or two (1/16,1/16) states of each theory. Just as the action of $(-1)^{F_s}$ toggled between $Z_{\text{orb}}(r)$ and $Z_{\text{orb}'}(r)$ in (2.18), modding out by $(-1)^{F_s}$ in the polyhedral context takes the theories of (4.8a, b) to the corresponding theories of (4.9), and vice-versa.

To close this section, we catalog the theories at $\hat{c} = 1$ whose states are classified by a (twisted) $N = 3$ superconformal invariance. These are the theories reached by modding out the $N = 3$ theory $Z_{\text{circ}}(1/\sqrt{2})$ by the finite subgroups G of $SO(3)$. Their partition functions $\tilde{Z}[G]$ are simply the products of all the $c = 1$ partition functions given in [7] with the $c = 1/2$ partition function Z_{Ising}. The cyclic groups $\mathbf{C}_n$ give the points[13]

$$\tilde{Z}[\mathbf{C}_n] \equiv \tilde{Z}_n = Z_{\text{circ}}\left(\frac{1}{n\sqrt{2}}\right) = Z_{\text{circ}}\left(\frac{n}{\sqrt{2}}\right) \tag{4.10}$$

[13] Note that the Z_n's utilized in [7] are given here by $\tilde{Z}_n/Z_{\text{Ising}}$.

on the circle line. The dihedral groups $\mathbf{D}_n$ give the points

$$\tilde{Z}[\mathbf{D}_n] = Z_{\rm orb}\left(\frac{n}{\sqrt{2}}\right) = \frac{1}{2}(\tilde{Z}_n + 2Z_2 - Z_1 + 3) \\ = \frac{1}{2}(\tilde{Z}_n + 2\tilde{Z}_2 - \tilde{Z}_1) \tag{4.11}$$

on the orbifold line. (Except for $\tilde{Z}[\mathbf{D}_2]$, we have not marked the theories (4.10) and (4.11) in fig. 1.) Finally the polyhedral groups give the three isolated theories

$$\begin{aligned} \tilde{Z}[\mathbf{T}] &= \frac{1}{2}(2\tilde{Z}_3 + \tilde{Z}_2 - \tilde{Z}_1), \\ \tilde{Z}[\mathbf{O}] &= \frac{1}{2}(\tilde{Z}_4 + \tilde{Z}_3 + \tilde{Z}_2 - \tilde{Z}_1), \\ \tilde{Z}[\mathbf{I}] &= \frac{1}{2}(\tilde{Z}_5 + \tilde{Z}_3 + \tilde{Z}_2 - \tilde{Z}_1), \end{aligned} \tag{4.12}$$

indicated by $\tilde{\bullet}$'s in fig. 1. The theories (4.12) are probably less interesting from the string theory point of view since despite having an untwisted $N = 1$ superconformal symmetry in each twisted sector, the role of supersymmetry generator is exchanged between different $(\frac{3}{2}, 0)$ fields in different twisted sectors and it is not clear how to couple such a structure to a single worldsheet gravitino.

5. Comments

In sect. 1, we outlined a method for generating conformal field theories — identifying exactly marginal operators and modding out by symmetries consistent with modular invariance — that has been used to generate a set of (super) conformal theories at $c = 1$ ($\hat{c} = 1$). It leads to all known conformal (superconformal) theories at the two respective values of c. We lack a classification theorem that would ensure the completeness of these sets of theories. In its absence, we have recourse only to exploring other known ways of constructing conformal field theories to see if anything new arises. Two such methods are the GKO coset algebra construction[3] and tensoring together theories with smaller values of c.

All possibilities using the GKO construction[3] that we have checked lie among the theories already considered. Modular invariant combinations of the

characters from the $c = 1$ theories $SO(n)_1 \times SO(n)_1/SO(n)_2$, for example, give points on the circle and orbifold lines at radii $r = \sqrt{n}/2$ (generalizing the $SU(2)_2 \times SU(2)_2/SU(2)_4$ construction of the $N = 2$ supersymmetric point at $r = \sqrt{3}$). A systematic exploration of these constructions at $c = 1$ ($\hat{c} = 1$) would be welcome.

Tensoring together theories with smaller c, on the other hand, does lead to something new. At $c = 1$ the only such possibility is to tensor together two copies of the $c = \frac{1}{2}$ Ising model, known to lead to the points $r = 1$ on the circle and orbifold lines. But at $c = 3/2$, we find that there exist conformal, but not superconformal, theories constructed as the tensor product of minimal theories with $c = 4/5$ and $c = 7/10$. To describe these, we label the six characters of the $c = 7/10$ system as χ_i, where $i = 0$, 7/16, 3/80, 3/2, 3/5, 1/10, and the ten characters of the $c = 4/5$ system as λ_a, $a = 0$, 2/5, 1/40, 7/5, 21/40, 1/15, 3, 13/8, 2/3, 1/8. The $c = 4/5$ system by itself has two modular invariant combinations, one diagonal $(\sum_a \lambda_a \bar{\lambda}_a)$ and the other corresponding to the 3-state Potts model. Taking the product of either of these with the single (diagonal) modular invariant of the $c = 7/10$ system gives two modular invariant partition functions, neither of which is that of a superconformal system. To show this we first identify the unique dimension $(\frac{3}{2}, 0)$ field that could serve as the supersymmetry generator T_F for the $\hat{c} = 1$ system (it is a linear combination of the 3/2 operator from the $c = 7/10$ system and the product of the 1/10 and 7/5 operators from the two systems). We find that either tensor product theory contains primary fields whose singularities with the candidate T_F differ from the local or square root behavior allowed in the Z_2 non-local cover of a superconformal theory. We also find a single marginal operator in each theory, constructed as the product of the dimension (3/5, 3/5) operator in the $c = 7/10$ system and the dimension (2/5, 2/5) operator in the $c = 4/5$ system, which turns out to be integrable to first order. Thus there may exist lines of theories connected to these two (although we see no obvious reason why the marginal operator should be integrable in higher orders of perturbation theory).

Other less trivial combinations of these two minimal models constitute partition functions of superconformal systems. We have identified for example the modular invariant combination

$$\begin{aligned}&\left(\chi_0\overline{\chi}_0+\chi_{7/16}\overline{\chi}_{7/16}+\chi_{3/2}\overline{\chi}_{3/2}\right)\left((\lambda_0+\lambda_3)(\overline{\lambda}_0+\overline{\lambda}_3)+2\lambda_{2/3}\overline{\lambda}_{2/3}\right)\\&+\left(\chi_{3/5}\overline{\chi}_0+\chi_{1/10}\overline{\chi}_{3/2}+\chi_{3/80}\overline{\chi}_{7/16}\right)\left((\lambda_{2/5}+\lambda_{7/5})(\overline{\lambda}_0+\overline{\lambda}_3)+2\lambda_{1/15}\overline{\lambda}_{2/3}\right)\\&+\left(\chi_0\overline{\chi}_{3/5}+\chi_{3/2}\overline{\chi}_{1/10}+\chi_{7/16}\overline{\chi}_{3/80}\right)\left((\lambda_0+\lambda_3)(\overline{\lambda}_{2/5}+\overline{\lambda}_{7/5})+2\lambda_{2/3}\overline{\lambda}_{1/15}\right)\\&+\left(\chi_{1/10}\overline{\chi}_{1/10}+\chi_{3/5}\overline{\chi}_{3/5}+\chi_{3/80}\overline{\chi}_{3/80}\right)\left((\lambda_{2/5}+\lambda_{7/5})(\overline{\lambda}_{2/5}+\overline{\lambda}_{7/5})+2\lambda_{1/15}\overline{\lambda}_{1/15}\right)\\&\qquad=Z_{\text{s-a}}(\sqrt{3})\end{aligned}$$

as the partition function for the theory at $r=\sqrt{3}$ on the super-affine line. A related combination gives the theory at $r=\sqrt{3}$ on the super-orbifold line. The problem of finding all modular invariant combinations of the characters of tensor products of even just two members of the discrete unitary series remains open, so we cannot say for certain whether new superconformal theories may also arise in this way. As a non-trivial check on the conjecture that all conformal field theories at a given value of c can be constructed by the procedure outlined earlier, we are investigating the possibility that there exist discrete quantum (non-superconformal invariant) symmetries that would allow the two non-superconformal $7/10+4/5$ combinations to be viewed as twisted versions of the super-affine theory at $r=\sqrt{3}$.

It would also be interesting to determine what happens when we perturb the systems considered here with relevant operators. Note that there are (1/16,1/16) relevant perturbations in many of these theories that preserve supersymmetry, so we might hope to find systems that exhibit supersymmetry away from the critical point (as emphasized for $c=1$ systems in [23]). This could prove useful in the pressing search for experimentally realizable superconformal orbifolds in nature. Of potential further interest would be to follow the renormalization group flows to other conformal and superconformal theories generated by these and other relevant operators (as in [24]).

Another issue of interest involves the existence of statistical mechanics models that realize the conformal field theories considered here at their critical

points. For the $c=1$ systems, such models are known[7][25]. For $\hat{c}=1$, the candidates include 19 vertex models[26].

Finally, $c=1$ and $\hat{c}=1$ are the boundary points for the unitary discrete representations of the Virasoro and $N=1$ super-Virasoro algebras respectively. Are there generic features of conformal field theories that occur as well at the boundary points for other discrete series (e.g. [27]) of conformal field theories? An analysis at the level presented here is likely to be quite complicated for larger values of c. For example, the boundary point for $N=2$ superconformal theories occurs at $c=3$. The natural realization of such theories is in terms of a complex superfield, and there are four continuously variable parameters (corresponding to the metric and torsion which define the toroidal compactification of the bosonic components of the superfield). There is also a point of higher symmetry in the moduli space at which it is possible to write the two supersymmetry generators in terms of complex fermions ψ^a as $T_F^+=-\frac{1}{12}\epsilon_{abc}\psi^a\psi^b\psi^c$ and T_F^- as its complex conjugate. These are left invariant by $SU(3)$ rotations of the fermions and modding out by large enough subgroups of $SU(3)$ can give $N=2$ superconformal theories that are disconnected from the generic four parameter moduli space and/or have lower dimensional moduli spaces (including completely isolated theories).

A more general understanding of the nature of isolated conformal field theories and of the connectivity of the moduli space of conformal field theories may also be of importance in phenomenological applications of string theory, plagued as they are by the large multiplicity of possible string vacuum states.

Acknowledgements

L.D. is supported by NSF contract PHY80-19754; P.G. by DOE contract DE-AC03-76SF00515, DOE OJI grant FG-84ER40171, the A. P. Sloan foundation, and by NSF contract PHY-82-15249; J.H. by NSF contract PHY80-19754, PYI award PHY86-58033, and by the A. P. Sloan foundation. We thank C. Vafa and E. Witten for discussions of group cohomology, T. Banks for discussions of supersymmetry, and M. Strassler for prompting us to think about the 7/10+4/5 theories. One of us also thanks N. Seiberg for continued admonishment to start reading his papers (and sincerely regrets not having done so), and we thank G. Moore for suggesting that we compactify moduli space (and apologize for not doing so).

Appendix A.

The partition function for a general left-right symmetric superconformal field theory can be written as

$$Z_{\rm sc} = \frac{1}{2}\left(\mathrm{A}\underset{\mathrm{A}}{\square} + \mathrm{P}\underset{\mathrm{A}}{\square} + \mathrm{A}\underset{\mathrm{P}}{\square} + \mathrm{P}\underset{\mathrm{P}}{\square} \right) , \tag{A.1}$$

where A and P refer to antiperiodic and periodic boundary conditions on the worldsheet supersymmetry generator. The partition function for the theory twisted by $(-1)^{F_s}$ is most simply determined by first calculating $(-1)^{F_s}\underset{1}{\square}$ and then summing over the modular orbits. This gives

$$\left((-1)^{F_s}\right) Z_{\rm sc} = \frac{1}{2}\left(\mathrm{A}\underset{\mathrm{A}}{\square} + \mathrm{P}\underset{\mathrm{A}}{\square} + \mathrm{A}\underset{\mathrm{P}}{\square} - \mathrm{P}\underset{\mathrm{P}}{\square} \right) = Z_{\rm sc} - \mathrm{P}\underset{\mathrm{P}}{\square} . \tag{A.2}$$

In the Ramond sector, the zero mode G_0 of the worldsheet supersymmetry generator, satisfying $G_0^2 = L_0 - \hat{c}/16$, annihilates the Ramond ground states with conformal weight $\hat{c}/16$, and assigns all remaining states into pairs with opposite worldsheet fermion number $(-1)^F$. The Ramond ground states give the only contribution to $\mathrm{P}\underset{\mathrm{P}}{\square}$, which is therefore a constant equal to the sum of their $(-1)^F$ eigenvalues. We see that the modification (A.2) in general changes the partition function only by a constant. That the partition function could be consistently changed by this constant was also observed from a slightly different viewpoint in the second of ref. [6].

The modding out (A.2) also relates certain of the superconformal theories in the $c = 1$ moduli space. The two $N = 2$ superconformal theories at $r = \sqrt{3}$ and $r = \sqrt{3}/2$ on the circle line for example are related by this modification, as are the twisted $N = 2$ theories at $r = \sqrt{3}$ and $r = \sqrt{3}/2$ on the orbifold line.

In these two cases the partition functions differ by an additive constant equal to the number of Ramond ground states, respectively 2 and 1.

We emphasize again that although the partition functions of theories related by (A.2) in general differ only by a constant, the operator product coefficients and correlation functions for fields in the Ramond sector are completely different.

Appendix B.

In this appendix we try to provide some intuition for the relations between some of the lines of fig. 1 by considering their properties at the free fermion points $r = 1$.

First we consider the effect of the modding out prescription (2.10) at $r = 1$ on the circle line. The theory (2.3) at $r = 1$ can be understood as a Dirac $\times$ Majorana theory with two independent sums over spin structures

$$\begin{aligned} Z_{\rm circ}(1) &= \frac{1}{2}\left(\mathrm{AA}\underset{\mathrm{AA}}{\square} + \mathrm{PP}\underset{\mathrm{AA}}{\square} + \mathrm{AA}\underset{\mathrm{PP}}{\square} + \mathrm{PP}\underset{\mathrm{PP}}{\square} \right) \\ &\qquad \cdot \frac{1}{2}\left(\mathrm{A}\underset{\mathrm{A}}{\square} + \mathrm{P}\underset{\mathrm{A}}{\square} + \mathrm{A}\underset{\mathrm{P}}{\square} + \mathrm{P}\underset{\mathrm{P}}{\square} \right) \\ &= \frac{1}{4}\left(\left|\frac{\vartheta_3}{\eta}\right|^2 + \left|\frac{\vartheta_4}{\eta}\right|^2 + \left|\frac{\vartheta_2}{\eta}\right|^2 \right)\left(\left|\frac{\vartheta_3}{\eta}\right| + \left|\frac{\vartheta_4}{\eta}\right| + \left|\frac{\vartheta_2}{\eta}\right| \right) . \end{aligned} \tag{B.1}$$

The action (2.10) has the effect of coupling together the two independent spin structures to give (3.1). By contrast, the action (2.7) has the opposite effect of producing a fully decoupled sum over three independent spin structures, thereby resulting in the $(\text{Majorana})^3 = (\text{Ising})^3$ theory

$$\begin{aligned} Z_{\rm orb}(1) = Z^3_{\rm Ising} &= \frac{1}{2^3}\left(\mathrm{A}\underset{\mathrm{A}}{\square} + \mathrm{P}\underset{\mathrm{A}}{\square} + \mathrm{A}\underset{\mathrm{P}}{\square} + \mathrm{P}\underset{\mathrm{P}}{\square} \right)^3 \\ &= \frac{1}{8}\left(\left|\frac{\vartheta_3}{\eta}\right| + \left|\frac{\vartheta_4}{\eta}\right| + \left|\frac{\vartheta_2}{\eta}\right| \right)^3 . \end{aligned} \tag{B.2}$$

at $r = 1$ on the orbifold line (2.8).

The $\mathbf{Z}_2$ torsion that relates the theory $Z_{\rm orb}(1)$ to $Z_{\rm orb'}(1)$ has a particularly straightforward description in this fermionic language. We can consider the (Ising)^3 theory at $Z_{\rm orb}(1)$ as arising from modding out the theory $Z_{\rm s\text{-}a}(1)$ by the $\mathbf{Z}_2 \times \mathbf{Z}_2$ that decouples the sum over three identified spin structures to a sum over three independent spin structures (this is the action of modding out by $\mathbf{D}_2 \subset SO(3)$). Then the discrete torsion changes only the sign of the modular orbit for which all three spin structures are different. (This is precisely the phase change that distinguishes the $SO(16) \times SO(16)$ heterotic theory from the $E_8 \times E_8$ theory in their construction from a $\mathbf{Z}_2 \times \mathbf{Z}_2$ modification of the tachyonic $SO(32)$ theory[28].) The corresponding term in the partition function with changed sign is thus $(6/8)|\vartheta_2\vartheta_3\vartheta_4|/|\eta|^3 = 3/2$, and we recover the relation $Z_{\rm orb'}(1) = Z_{\rm orb}(1) - 3$.

The interpretation in Hamiltonian language is as follows. The (Ising)^3 theory has an independent GSO projection for each of the three (left-right) fermions onto say $(-1)^{F_i} = +1$ in each Hamiltonian sector. We denote this projection by $(+++)$. Then the effect of the torsion is to change the projections to $(--+)$ in the (AAP) sector and to $(+--)$ in the (APP) sector (and similarly for permutations), while the projections in the untwisted sectors (AAA) and (PPP) are left unchanged. Note that this theory, $Z_{\rm orb'}(1) = Z'[\mathbf{D}_2]$, has the same S_3 permutation symmetry as $Z_{\rm orb}(1) = Z[\mathbf{D}_2]$. Modding it out by $Z_3 \subset S_3$ or by S_3 gives respectively $Z'[\mathbf{T}]$ or $Z'[\mathbf{O}]$.

References

[1] For a review with references to the original literature, see P. Goddard and D. Olive, Int. Jnl. of Mod. Phys. A1 (1986) 303.

[2] D. Friedan, Z. Qiu, and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575.

[3] P. Goddard, A. Kent, and D. Olive, Comm. Math. Phys. 103 (1986) 105.

[4] J. Cardy, Nucl. Phys. B270 [FS16] (1986) 186;
D. Gepner, Nucl. Phys. B287 (1987) 11;
A. Cappelli, C. Itzykson, and J.-B. Zuber, Comm. Math. Phys. 113 (1987) 1, and further references therein.

[5] D. Friedan, Z. Qiu, and S. Shenker, Phys. Lett. 151B (1985) 37;
M. Bershadsky, V. Knizhnik and M. Teitelman, Phys. Lett. 151B (1985) 31;
Z. Qiu, Nucl. Phys. B270 (1986) 205.

[6] A. Cappelli, Phys. Lett. 185B (1987) 82;
D. Kastor, Nucl. Phys. B280[FS18] (1987) 304.

[7] P. Ginsparg, Nucl. Phys. B295[FS21] (1988) 153.

[8] R. Dijkgraaf, E. Verlinde, and H. Verlinde, "c=1 Conformal Field Theories on Riemann Surfaces," Utrecht preprint THU-87/17 (1987).

[9] A. Schwimmer and N. Seiberg, Phys. Lett. B184 (1987) 191.

[10] L. Dixon, J. Harvey, C. Vafa, and E. Witten, Nucl. Phys. B261 (1985) 678; Nucl. Phys. B274 (1986) 285.

[11] C. Vafa, Nucl. Phys. B273 (1986) 592.

[12] D. S. Freed and C. Vafa, Comm. Math. Phys. 110 (1987) 349.

[13] N. Seiberg and E. Witten, Nucl. Phys. B276 (1986) 272.

[14] R. Rohm, Nucl. Phys. B237 (1984) 553, and Princeton Univ. Ph.D. thesis (1985), unpublished;
L. Dixon and J. Harvey, Nucl. Phys. B274 (1986) 93;
P. Ginsparg and C. Vafa, Nucl. Phys. B289 (1987) 414.

[15] S. Elitzur, E. Gross, E. Rabinovici, and N. Seiberg, Nucl. Phys. B283 (1987) 431.

[16] K. Kobayashi, Phys. Rev. Lett. 58 (1987) 2507;
P. Di Francesco, H. Saleur, and J.-B. Zuber, "Critical Ising Correlation Functions in the Plane and on the Torus," Saclay preprint SPhT/87-097, submitted to Nucl. Phys. B.

[17] P. Goddard, W. Nahm, and D. Olive, Phys. Lett. 160B (1985) 111;
P. Goddard, A. Kent, and D. Olive, Comm. Math. Phys. 103 (1986) 105;
I. Antoniadis, C. Bachas, C. Kounnas, and P. Windey, Phys. Lett. 171B (1986) 51;
P. Windey, Comm. Math. Phys. 105 (1986) 511.

[18] W. Goldman, Adv. Math. 54 (1984) 200.

[19] G. Karpilovsky, *Projective Representations of Finite Groups* (Marcel Dekker Inc., N.Y, 1985), p. 171, 179, 183.

[20] P. Ginsparg, "Informal string lectures," HUTP-87/A077, to appear in proceedings of the U.K. Summer Institute (Aug. 1987), edited by P. Landshoff and B. Webber.

[21] F. R. Beyl and J. Tappe, *Group Extensions, Representations, and the Schur Multiplicator*, Lecture Notes in Mathematics #958 (Springer-Verlag, New York, 1982), p. 184.

[22] P. Ginsparg, to appear.

[23] D. Friedan and S. Shenker, "Supersymmetric critical phenomena and the two dimensional gaussian model," EFI 86-17, (March, 1986) unpublished.

[24] A. B. Zamolodchikov, Sov. J. Nucl. Phys. 44 (1986) 529;
A. Ludwig and J. Cardy, Nucl. Phys. B285[FS19] (1987) 687;
D. Kastor, E. Martinec, and S. Shenker, to appear.

[25] V. Pasquier, J. Phys. A20 (1987) L1229.

[26] P. di Francesco, H. Saleur, and J.-B. Zuber, to appear.

[27] D. Kastor, E. Martinec, and Z. Qiu, "Current algebra and conformal discrete series", Chicago preprint EFI-87-58;
J. Bagger, D. Nemeschansky, and S. Yankielowicz, "Virasoro algebras with central charge $c > 1$", Harvard preprint HUTP-87/A073;
M. Douglas, "G/H conformal field theory", Caltech preprint CALT-68-1453 (1987).

[28] L. Alvarez-Gaumé, P. Ginsparg, G. Moore, and C. Vafa, Phys. Lett. 171B (1986) 155.

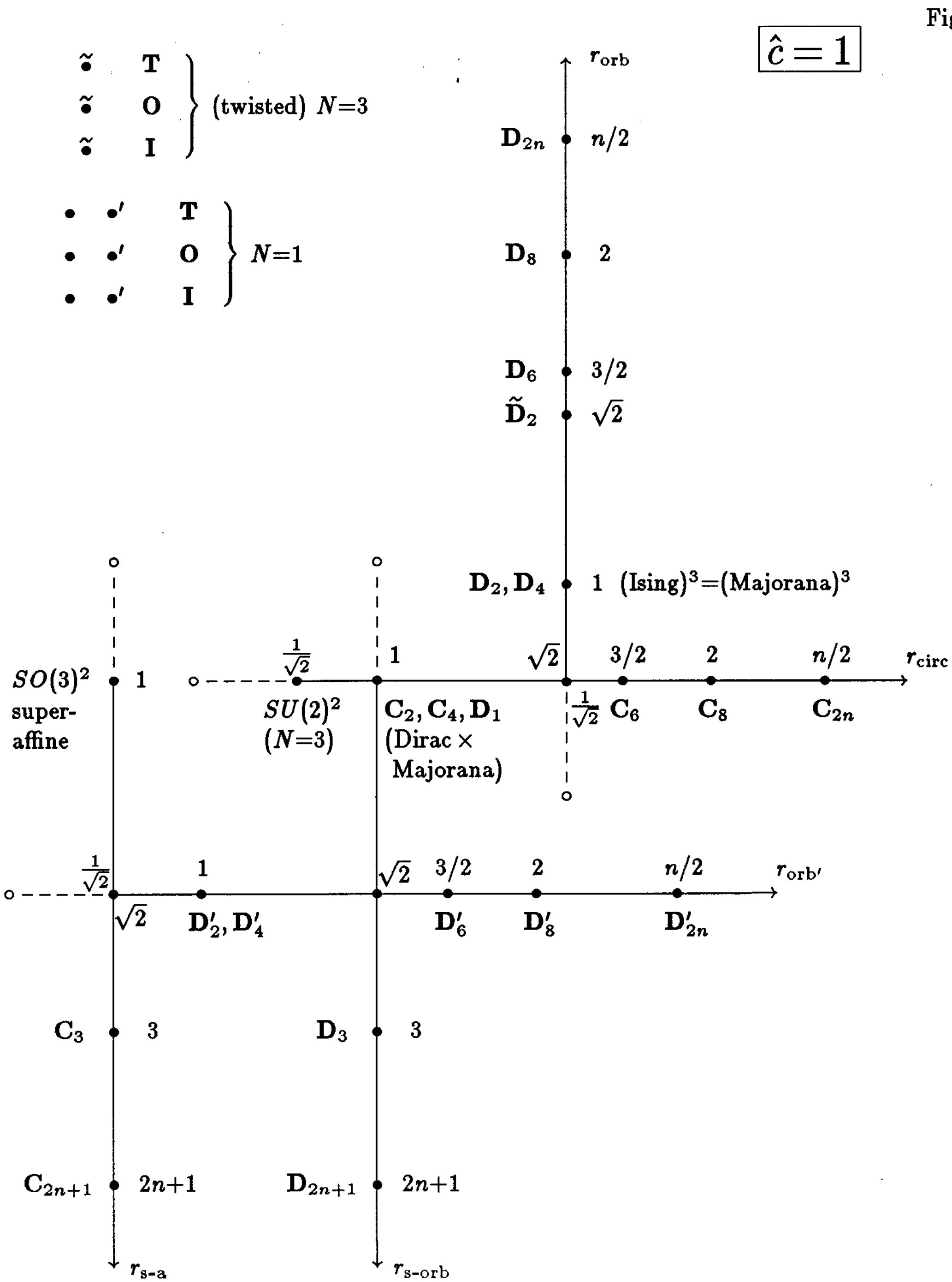

Fig. 1: Survey of $\hat{c}=1$ theories. The axes labelled $r_{\rm circ}$ and $r_{\rm orb}$ represent the lines of ordinary $c=1$ compactifications on the circle S^1 and orbifold $S^1/\mathbf{Z}_2$ tensored with the $c=1/2$ Ising model. The regions represented by dotted lines are determined by the dualities described in the text. The points labelled by groups G are theories that result from modding out the super-affine $SO(3)^2$ theory $Z_{\rm s\text{-}a}(1)$ by the finite subgroups of the diagonal $SO(3)$. The points labelled by G''s are related to the preceding theories by a $\mathbf{Z}_2$ torsion. The points labelled by $\tilde{\ }$'s are given by modding out the $SU(2)^2$ theory $Z_{\rm circ}(1/\sqrt{2})$ by the indicated subgroup of $SO(3)$, and possess twisted $N=3$ supersymmetry. For clarity we have not indicated most of these latter since the remaining simply appear on the circle and orbifold lines at the same positions as on the $c=1$ diagram given in [7].