Supersymmetric Configurations in The Rotating D1-D5 System and PP-Waves

A thesis presented

by

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to

The Department of Physics

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in the subject of

Physics

Harvard University

Cambridge, Massachusetts

May, 2003

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Abstract

Two families of supersymmetric configurations are considered. One is the 1/4 supersymmetric D1-D5 system with angular momentum, and the other is a family of pp-waves of type IIB string theory with some supersymmetry .

In the first part of the thesis some configurations of the D1-D5 system are examined which give conical singularities in AdS_3 as their near horizon limit. It is shown that they can be made non-singular by adding angular momentum to the brane system. The smooth asymptotically flat solutions constructed this way are used to obtain *global* AdS_3 as the near horizon geometry.

Using the relation of the D1-D5 system to the oscillating string, a large family of supergravity solutions is constructed which describe BPS excitations on $AdS_3 \times S^3$ with angular momentum on S^3. These solutions take into account the full back-reaction on the metric, and can be viewed as Kaluza-Klein monopole "supertubes", which are completely non-singular geometries. The different chiral primaries of the dual CFT are identified with these different supergravity solutions. This part is adapted from the papers [1], [2].

In its second part, a general class of supersymmetric pp-wave solutions of type IIB string theory is constructed, such that the superstring worldsheet action in light cone gauge is that of an interacting massive field theory. It is shown that when the light cone Lagrangian has (2,2) supersymmetry, one can find backgrounds that lead to arbitrary superpotentials on the worldsheet. Both flat and curved transverse spaces are considered. In particular, the background giving rise to the $N = 2$ sine Gordon theory on the worldsheet is analyzed. Massive mirror symmetry relates it to the deformed CP^1 model (or sausage model) which seems to elude a purely

supergravity target space interpretation. These are results which appeared in the paper [3].

Contents

Acknowledgments

I am deeply grateful to my advisor, Juan Maldacena, for his devoted guidance throughout my research, for sharing with me his excitement and passion for physics and string theory, and for constantly reminding me of their beauty. I greatly appreciate all the time and energy he has spent with me, and his constant inspiration. I thank him for his warm and encouraging attitude, for teaching me so many things in physics, and for showing me the importance of clarity, preciseness and simplicity. I am very privileged to have been his student.

I also want to thank Andrew Strominger who was the first to welcome me to Harvard, and has been a constant source of support and guidance ever since. I have enjoyed very interesting and insightful discussions with him.

I am very thankful to Princeton University for hosting me for the past five semesters as an exchange scholar, and to the Institute for Advanced Study at Princeton, which treated me as one of their own. I have had the rare opportunity of interacting and collaborating with many faculty and fellow graduate students both at Harvard and also at Princeton and at the Institute for Advanced Study. I am especially grateful to Nathan Seiberg, Oleg Lunin, Kostas Skenderis, Kentaro Hori, Igor Klebanov, Horatiu Nastase, Carlos Nunez, Shiraz Minwalla, Matthew Headrick and Sameer Murthy.

I have a special debt to Adam Schwimmer, my M.Sc. advisor, and David Kutasov, who both introduced me to this field in Israel. I would also like to thank the other string theorists in Israel, particularly Ofer Aharony, Micha Berkooz, Jacob Sonnenschein, Shmuel Elitzur, Barak Kol and Joan Simón, for interesting discussions and collaborations and for their hospitality.

Finally, I want to thank my family and friends, who helped me and supported me all the way. I dedicate this thesis to my late grandmother, Esther Farhy.

Part I

The Rotating D1-D5 System
Introduction

1. Introduction

The formulation of a consistent theory of quantum gravity is one of the greatest challenges modern physics faces today. Although such a theory is yet unknown, it has been argued by 'tHooft, motivated by black hole entropy considerations [4] that such a theory should be holographic in nature [5,6]. This remarkable suggestion qualitatively means that a quantum gravity theory, describing the physics within some volume of spacetime, should be describable as some theory on the boundary, with no more than one degree of freedom per Planck area. This general principle plays a prominent role in string theory in the context of the AdS/CFT correspondence, relating string theory in AdS spacetimes (times a compact space) with conformal field theories of one less dimension, which can be thought of as living on the boundary of AdS. The correspondence and its first concrete example were suggested by Maldacena relating type IIB string theory compactified on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ super Yang-Mills theory [7]. Further, it has been argued by Gubser, Klebanov and Polyakov [8]for this case and by Witten [9] studying the special properties of general AdS spaces, that observables of the quantum gravity theory in AdS correspond to correlation functions in a local CFT.

Since this outstanding conjecture has been made, it has been successfully tested in many different settings. The one which we choose to focus on is the AdS_3/CFT_2 correspondence, which is special and interesting for a few reasons. First, the supergravity theory on AdS_3 can be shown to be equivalent to a two-dimensional superconformal field theory. This was first shown by Brown and Henneaux for pure gravity on AdS_3 [10] , and later was shown for all extended supergravity theories [1] on AdS_3 [14]. The construction relies on the observation that the asymptotic dynamics and symmetries of the supergravity theory are generated by a left and right Virasoro and Kac-Moody algebras. Manipulating the 3d supergravity action, in its Chern-Simons form, an equivalent super-Liouville two dimensional action was derived [2], which is superconformal invariant [3]. Second, the string theory on AdS_3 can actually be studied using worldsheet methods beyond the supergravity approximation [21,22]. This is due to the fact that the string theory can be defined without turning on Ramond-Ramond fields, and that the symmetry group associated with two-dimensional conformal theories is infinite-dimensional. This way the correspondence has been established in perturbative string theory.

In string theory, the duality can be derived by thinking of a brane configuration whose near-horizon geometry is $AdS_3 \times S^3 \times M^4$ (where M^4 is either T^4 or $K3$). Working in type IIB string theory, one can take a system of Q_1 D1 branes along a circle y of radius R, and Q_5 D5 branes wrapping the compact M^4 and stretching along y as well. All the branes coincide in the transverse directions. This configuration has $SO(1,1) \times SO(4) \approx SO(1,1) \times SU(2)_L \times SU(2)_R$ Lorentz symmetries

[1] these theories were classified in [11,12,13.]

[2] It is interesting to note that the Liouville action is unable to reproduce the Bekenstein-Hawking entropy of the black holes. The reason being that its lowest dimension operators have a nonzero conformal dimension, and thus its effective central charge is not $c = 6k$, but $c_{eff} = 1$. There are many approaches to this problem. Some, like Martinec [15], claim that the Chern-Simons or Liouville actions are only an effective description, which does not capture the underlying microscopic degrees of freedom of the system. Others, like Carlip [16] try to count the degrees of freedom on the horizon with specific boundary conditions and claim to reproduce the right entropy. However such approaches are problematic. For a discussion of some of these issues, see [17].

[3] Actually, the superconformal symmetry algebra is not always an algebra in the strict sense. For some Lie superalgebras of the bulk gravity, the resulting superconformal anticommutators also include bilinears in the currents. Such non-linear superconformal algebras have been discussed and classified in [18,19,20]

and $\mathcal{N} = (4,4)$ supersymmetry. The supergravity solution involves a dilaton, a 3-form RR field strength, and a metric which is asymptotically flat. In the case that $M^4 = T^4$ it is given by [23,24,25,26]:

$$\begin{aligned} e^{-2\phi} &= \frac{1}{g^2 h^2(r)} \\ F_3 &= \frac{2r_5^2}{g}\epsilon_3 + 2r_1^2 g e^{-2\phi} *_6 \epsilon_3 \\ ds^2 &= \frac{1}{h(r)}[-dt^2 + dy^2 + (1+\frac{r_1^2}{r^2})dx_i dx^i] + h(r)[dr^2 + r^2 d\Omega_3^2] \\ h(r) &\equiv (1+\frac{r_1^2}{r^2})^{1/2}(1+\frac{r_5^2}{r^2})^{1/2} \end{aligned} \tag{1.1}$$

where $*_6$ is taken in the six dimensions t, y, r, Ω_3, where ϵ_3 is the volume form on the 3-sphere , r_1, r_5 are constants and x^i are the directions along the torus, each of period $2\pi v^{1/4}\alpha'^{1/2}$. In the near horizon scaling limit $\alpha' \to 0$ with $U = \frac{r}{\alpha'}$ and R kept fixed, the solution becomes

$$\begin{aligned} e^{-2\phi} &= \frac{Q_5}{g_6^2 Q_1} \\ ds^2/\alpha' &= \frac{U^2}{\ell^2}(-dt^2 + dy^2) + \frac{\ell^2}{U^2}dU^2 + \sqrt{\frac{Q_1}{vQ_5}}dx_i dx^i + \ell^2 d\Omega_3^2 \\ g_6 &= g^2/v \quad ; \quad \ell^2 \equiv g_6\sqrt{Q_1 Q_5} \\ Q_1 &= \frac{v}{4\pi g^2 \alpha'}\int e^{2\phi} *_6 F_3 \;\; ; \;\; Q_5 = \frac{1}{4\pi\alpha'}\int F_3 \end{aligned} \tag{1.2}$$

i.e. $AdS_3 \times S^3 \times M^4$, where the AdS radius and the sphere radius are given by $R^2 = g_6\sqrt{Q_1 Q_5}\ell_s^2$, and the M^4 volume is proportional to $\frac{Q_1}{Q_5}$.

The dual superconformal field theory is the IR fixed point of the field theory living on the D1-D5 brane system. It can be viewed as the 1+1 sigma model whose target space is the instanton moduli space, with the Q_1 D1-branes acting as instantons of the low energy super Yang-Mills theory on the D5 branes. It was argued [27,28,29,30] that the instanton moduli space is a deformation of the symmetric product of k copies of M^4: $Sym(M^4)^k$ ($k = Q_1 Q_5$ for $M^4 = T^4$ and

$k = Q_1 Q_5 + 1$ for $M^4 = K3)^4$.

Using this explicit form of the CFT, further evidence for the correspondence has been given, comparing the elliptic genus of the orbifold conformal field theory to a sum over geometries on the supergravity side [32,33].

One may wonder what happens when we add angular momentum to the D1-D5 system, i.e. let the branes rotate in the four dimensional transverse space. Solutions parameterized by two angular momentum parameters $J_{L,R}$ were given in [34,35] , and will be discussed in chapter 2. As we will show, the rotation corresponds in the CFT to changing the R-charges of the chiral primaries. For specific values of angular momentum we will see that we get a supersymmetric solution with an asymptotically flat metric, with non-singular *global* $AdS_3 \times S^3$ as its near-horizon limit.

More general exact supersymmetric gravity solutions for the rotating D1-D5 system can be constructed by relating it through dualities to the oscillating string. We will explain and employ this method in chapter 3, and use it to match the chiral primaries of the conformal field theory to exact six-dimensional supergravity solutions.

This de-singularization can also be viewed as a Myers type blow up effect [36](induced by angular momentum rather than by an external R-R field), where the D1-D5 branes blow up into a Kaluza-Klein monopole. Actually, as we will show on chapter 3, this is dual to the angular-momentum induced blow up of the D0-F1 system to a D2 supertube [37]. For the above mentioned special values of angular momenta, the Kaluza-Klein monopole radius is related to its charge in a way which makes the geometry non-singular.

Below we provide some more details about AdS_3 spaces and their properties,

[4] The deformation of the model consists of blowing up the fixed points of the orbifold and modifying the B-field living at the fixed points.

about the 3-dimensional supergravity action and about the 2-dimensional CFT and its chiral primaries.

1.1. The AdS_3 Spacetime - Global and Local Properties

AdS_3 is a three-dimensional spacetime of constant negative curvature. It is the group manifold of the universal covering of $SL(2,R)$ which can be clearly seen writing its metric as

$$ds^2 = \frac{\ell^2}{2} Tr(g^{-1}dgg^{-1}dg) \tag{1.3}$$

where $g \in SL(2,R)$. In the explicit parameterization

$$g = \begin{pmatrix} 1/y & w^-/y \\ w^+/y & y + w^+w^-/y \end{pmatrix}$$

the metric becomes

$$ds^2 = \frac{\ell^2}{y^2}(dy^2 + dw^+dw^-) \tag{1.4}$$

This is AdS_3 in **Poincaré coordinates**. In its form (1.3) it is easy to see the metric is invariant under right and left multiplication of g by constant $SL(2,R)$ matrices $h_{L,R}$: $g \to h_L g h_R$. This isometry group $SL(2,R) \times SL(2,R) \approx SO(2,2)$ is also manifest when one represents AdS_3 as a hyperboloid embedded in $R^{2,2}$:

$$\begin{aligned} X_0^2 + X_3^2 - X_1^2 - X_2^2 &= \ell^2 \\ ds^2 &= -dX_0^2 - dX_3^2 + dX_1^2 + dX_2^2 \end{aligned} \tag{1.5}$$

The relation between the Poincaré coordinates and the embedding coordinates is

$$g = \begin{pmatrix} 1/y & w^-/y \\ w^+/y & y + w^+w^-/y \end{pmatrix} = \frac{1}{\ell}\begin{pmatrix} X_0 - X_2 & X_1 - X_3 \\ X_1 + X_3 & X_0 + X_2 \end{pmatrix} \tag{1.6}$$

It is important to note that the Poincaré coordinates only cover half of the hyperboloid.

Another useful set of coordinates which covers all of the hyperboloid is the **global coordinates**, in which the metric takes the form:

$$ds^2 = -[1 + (\frac{r}{\ell})^2]dt^2 + \frac{dr^2}{1 + (\frac{r}{\ell})^2} + r^2 d\varphi^2 \tag{1.7}$$

These are related to the hyperboloid coordinates by

$$X_0 + iX_3 = \ell\sqrt{1+(\frac{r}{\ell})^2}e^{it/\ell} \;\; ; \;\; X_1 + iX_2 = re^{i\varphi}$$

A 3-dimensional space which is a solution of Einstein's equations with a negative cosmological constant, unlike spaces of higher dimensions, has to have a metric which is locally AdS_3. If we take the coordinates in (1.7) to have the range $t \in R$; $r \in [0, \infty)$; $\varphi \in S^1$ then the space is *Globally* AdS_3. It has the structure of a solid cylinder with a timelike two-dimensional boundary at $r = \infty$ (or $y = 0$ in the Poincaré coordinates).

Timelike geodesics moving in global AdS_3 never reach its boundary and oscillate between a maximal and minimal r (the minimal r can be the origin if the geodesic has no angular momentum). Lightlike geodesics, on the other hand, can reach the boundary in finite proper time.

However, keeping the local structure, one could quotient the space, changing its global structure. As AdS_3 is the group manifold of $SL(2, R)$, one can quotient the space by any discrete subgroup of $SL(2, R)$, as long as by doing this one gets a space with an admissable causal structure, i.e. that no closed timelike or null curves are created by the identifications. All such possible quotients have been classified by Bañados, Teitelboim, Zanelli and Henneaux [38] , and are characterized by two parameters M, J. The different *locally* AdS_3 spaces are:

* Global AdS_3 with no quotient : $M = -1$; $J = 0$.
* Conical Singularities: $-1 < M < 0$; $J = 0$: These spaces have a deficit angle of $\beta = 2\pi - M$.
* The zero mass BTZ black hole : $M = 0$; $J = 0$.
* General BTZ black holes: $M > 0$; $|J| \leq M$: These black holes have mass M and angular momentum J.

The BTZ black hole [39] has an inner cauchy horizon and an outer killing event horizon, given in global coordinates by $r_\pm^2$ such that $r_+^2 - r_-^2 = M\ell^2$; $-2r_+r_- = J\ell$.

In coordinates with natural identifications ($\varphi \sim \varphi + 2\pi$) the metrics take the form:

$$
\begin{aligned}
ds^2 &= -N^2 dt^2 + \frac{dr^2}{N^2} + r^2 (N^\varphi dt + d\varphi)^2 \\
N^2 &\equiv -M + (\frac{r}{\ell})^2 + (\frac{J}{2r})^2 \ ; \ N^\varphi = -\frac{J}{2r^2}
\end{aligned}
\tag{1.8}
$$

1.2. Three Dimensional Supergravity as a Chern-Simons Theory

Pure Einstein gravity in three dimensions with a cosmological constant is described by the action

$$
S = \frac{1}{16\pi G} \int_M d^3x \sqrt{-g} [R - 2\Lambda] \tag{1.9}
$$

It was shown by Witten [40] that for a negative cosmological constant $\Lambda = -\frac{1}{\ell^2}$, this theory can be re-written as a Chern-Simons action with an $SL(2,R) \times SL(2,R)$ gauge group.

To see this, one works with first order formalism, introducing the dreibeins e^a_μ and spin connections $\omega^a_\mu \equiv \frac{1}{2} \epsilon^a{}_{bc} \omega^{bc}_\mu$, where the action is re-written as

$$
S = \frac{2}{16\pi G} \int_M d^3x [e^a \wedge (d\omega^a + \frac{1}{2} \epsilon_{abc} \omega^b \wedge \omega^c) + \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c]
$$

and then changes variables from e^a, ω^a to $\mathcal{A}^a_{L,R} = \omega^a \pm \frac{1}{\ell} e^a$. In these new variables the action (1.9) becomes

$$
S = S_{CS}[\mathcal{A}_L] - S_{CS}[\mathcal{A}_R] \ ; \ S_{CS}[\mathcal{A}] \equiv \frac{k}{4\pi} \int_M Tr(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}) \tag{1.10}
$$

where we regard $\mathcal{A}$ as taking value in the Lie algebra $sl(2,R)$, and $k \equiv \ell/4G$.

The Chern-Simons action is proportional to a topological invariant - the secondary Chern class, and does not depend on the choice of metric. Thus all observables of the theory are topological invariants [41]. Under the gauge transformation $\mathcal{A} \to U \mathcal{A} U^{-1} - dU U^{-1}$ (U being a group element) the action changes by a Wess-Zumino term, which is zero for non-compact groups and an integer winding number of the transformation for compact groups. Thus the theory is Gauge invariant for any k in the first case, and for integer k's in the second case. As we are dealing with $SL(2,R)$ Chern-Simons theory, for which H^3 vanishes, the level k need is not quantized.

Using this equivalence, it was shown [40] that pure gravity in 1+2 dimensions is solvable, and that it has asymptotic $Vir \times Vir$ symmetries [10].

In addition the Chern-Simons formalism makes it clear that the theory has no bulk degrees of freedom. As the $\mathcal{A}_t$ component has no kinetic term, it effectively acts as a Lagrange multiplier, and when integrated over, one is left with a two-dimensional chiral WZW action on the boundary of M. Thus the theory has only degrees of freedom living on the boundary.

Now, we would like to go beyond pure gravity, and try to describe supergravity on AdS_3 in a similar fashion. This was first done by Achucarro and Townsend [42] for $Osp(N|2;R) \times Osp(N|2;R)$ supergravity, by introducing the concept of a super-connection Γ i.e. a connection which takes values in a superalgebra (a Z_2 graded algebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$) . In order to represent AdS_3 geometries G must contain $sl(2,R)$ as a bosonic subalgebra ($G_{\bar{0}} \cong sl(2;R) \oplus g$). There are only seven families of such superalgebras, which were classified in [11,12,13]. Then the 3-dimensional supergravity action can be written as [14]

$$S_{sugra} = S_{scs}[\Gamma_L] - S_{scs}[\Gamma_R] \;\; ; \;\; S_{scs}[\Gamma] \equiv \frac{k}{4\pi}\int_M STr(\Gamma \wedge d\Gamma + \frac{2}{3}\Gamma \wedge \Gamma \wedge \Gamma) \quad (1.11)$$

and the supertrace is a Killing nondegenerate bilinear form on the superalgebra. The field content can be seen by decomposing Γ to its superalgebra components:

$$\Gamma = (\mathcal{A}^a \sigma^a) + (A^a T^a) + (\psi_{+\alpha} R^{+\alpha} + \psi_{-\alpha} R^{-\alpha})$$

(Here σ^a are the $sl(2,R)$ generators, T^a the generators of g, the internal bosonic symmetry algebra , and $R^{\pm\alpha}$ the fermionic generators , transforming in the spinor representation of $sl(2,R)$). Thus we have the following field content: $\mathcal{A}^a_L, \mathcal{A}^a_R$ which make up the dreibeins and connections , A^a_L, A^a_R which are the gauge bosons, and $\psi^{\pm\alpha}_L, \psi^{\pm\alpha}_R$ which are the gravitini.

As in the pure gravity case, here also Γ_t in (1.11) functions as a Lagrange multiplier, and the action can be re-written as a chiral super-WZW living on the boundary of M. All the fields, in particular the A^L_μ, A^R_μ, have degrees of freedom only on the boundary.

The internal symmetry group g is determined in string theory by the manifold on which one compactifies the 10-dimensional theory to get AdS_3. In the case we consider, the D1-D5 system wrapped on the M^4, the near-horizon geometry is $AdS_3 \times S^3 \times M^4$, so the 3-dimensional gauge fields come from Kaluza-Klein modes on the S^3 [43,32] and the symmetry algebra of the space is $(sl(2,R) \times su(2)) \times (sl(2,R) \times su(2))$. The corresponding symmetry superalgebra is $psu(1,1|2)_L \times psu(1,1|2)_R$, and the internal symmetry gauge connections A^L_μ, A^R_μ take values in $su(2)$.

The infinitesimal form of gauge transformations in this formalism is $\delta\Gamma = d\Lambda + [\Gamma, \Lambda]$ where Λ is in the superalgebra. The subset of gauge transformations where Λ has only $G_{\bar{1}}$ components (Λ is fermionic) are the supersymmetry transformations of the model. Using this formalism Izquierdo and Townsend [44] classified all supersymmetric vacuum spacetimes of (2,0) AdS_3 supergravity i.e. AdS_3 with one extra $U(1)$ gauge field. In the next chapter we would discuss specific supersymmetric configurations with 3-dimensional conical singularities, and show how by changing the $su(2)$ gauge fields, one can obtain supersymmetric nonsingular configurations.

1.3. The Two Dimensional Conformal Field Theory and its Chiral Primaries

As mentioned before, the two dimensional conformal field theory is a 1+1 dimensional sigma model with a target space which is a deformation of the symmetric product $Sym(M^4)^k$. With this deformation the gravity description is valid. One can change the moduli of the CFT and reach the symmetric product point (with no deformation), where gravity is strongly coupled, and the supergravity approximation is not good. However, if we discuss chiral primary states of the CFT (states with $L_0 = J_0^3$ and $\bar{L}_0 = \bar{J}_0^3$), these have protected energies, and so allow treatment at the orbifold point. Thus, dealing with such states, we can employ the 'gas of strings' description of the theory. This is a description valid for orbifold conformal field theories, in which the strings are winding around the circle y, and moving in M^4, with total winding number k [45].

The CFT contains left and right bosonic Virasoro and SU(2) Kac-Moody generators and fermionic currents which are charged under the $SU(2)$s, and has central charge $c = 6k$. The supercharges obey the following anticommutation relations:

$$\begin{aligned} \{Q_r^{++}, Q_s^{--}\} &= 2L_{r+s} + 2(r-s)J_{r+s}^3 + \frac{c}{3}\delta_{r+s}(r^2 - \frac{1}{4}) \\ \{Q_r^{+-}, Q_s^{-+}\} &= 2L_{r+s} + 2(r-s)J_{r+s}^3 + \frac{c}{3}\delta_{r+s}(r^2 - \frac{1}{4}) \end{aligned} \tag{1.12}$$

The theory has two sectors, depending on the periodicity condition one assigns to the fermions around the circle y. Periodic boundary conditions are described by the R-R sector, and anti-periodic by the NS-NS sector. The modding of the supercharges r, s are integer in the first case, and half-integer in the second. Thinking of the CFT as the dual of a gravity theory, the periodicity conditions of the CFT should correspond to those in the spacetime around the y-circle. Thus, if we are discussing asymptotically flat supergravity configurations (like the D1-D5 system) we expect to be in the R-R sector, while if we are discussing other asymptotics (like the $AdS_3 \times S^3$ - the near horizon geometry of the brane configuration) we can be in the NS-NS sector.

For concreteness, let us work with the case $M = T^4$ (The $K3$ case follows similar lines). To describe the orbifold CFT, we choose to work with a specific realization of the superconformal algebra in terms of 4k free bosons describing the k coordinates on the T^4 : $X_I^{a\dot{a}}$ $I = 1..k$; $a, \dot{a} = 1, 2$ are spinorial indices on the T^4, and 4k free fermions $\psi_I^{\alpha\dot{\alpha}}$, $\alpha, \dot{\alpha} = \pm$ spinorial indices on $SO(4)$ [46]. The superconformal generators can be realized in terms of these free fields, and in particular the $SU(2)$ Kac Moody currents are bilinears in the fermion fields.

The periodicity conditions we wish to impose on the fields are that they cyclically permute when one goes around the origin, i.e.

$$X^I(ze^{2\pi i}, \bar{z}e^{-2\pi i}) = X^{I+1}(z,\bar{z}) \quad ,I = 1..n-1 \; ; \quad X^n(ze^{2\pi i}, \bar{z}e^{-2\pi i}) = X^1(z,\bar{z})$$

A simple way to impose this is to define new fields $Y^m(z,\bar{z})$ by $Y^m \equiv \frac{1}{\sqrt{n}} \sum_{I=1}^n e^{-2\pi i \frac{mI}{n}} X^I$. These fields have periodicity conditions $Y^m(ze^{2\pi i}, \bar{z}e^{-2\pi i}) = e^{2\pi i \frac{m}{n}} Y^m(z,\bar{z})$, which are easier to treat [47]. Their OPEs with a basic Z_n twist operator $\tilde{\sigma}_n$ are given by:

$$\partial Y^m(z)\tilde{\sigma}_{(1..n)}(0) = z^{\frac{m}{n}-1}\tau_{(1..n)}(0) + ... \tag{1.13}$$

(τ is an excited twist field).

The conformal weight of the twist field can be obtained by calculating the expectation value of the energy momentum tensor for the twisted vacuum and is $h_{\tilde{\sigma}} = \frac{1}{24}(n - \frac{1}{n})$. [5]

In order to create charged twist fields, it is convenient to bosonize the left-moving and right-moving fermions to the bosonic $\Phi^I = \phi^I(z) + \bar{\phi}^I(\bar{z})$:

$$\psi_I^{+a}(z) = e^{i\phi_a^I(z)} \; ; \; \psi_I^{-a}(z) = \epsilon_{ab} e^{-i\phi_I^b(z)}$$

$$\bar{\psi}_I^{+a}(\bar{z}) = e^{i\bar{\phi}_a^I(\bar{z})} \; ; \; \bar{\psi}_I^{-a}(\bar{z}) = \epsilon_{ab} e^{-i\bar{\phi}_b^I(\bar{z})}$$

Then to get $U(1)$ charged operators in twisted sectors, we add momenta for the bosonized fermions, and define the operators:

$$O_{(1..n)}^{(k_L,k_R)}(z,\bar{z}) \equiv e^{\frac{i}{n}[k_L^j \sum_{I=1}^n \phi_j^I(z) + k_R^j \sum_{I=1}^n \bar{\phi}_j^I(\bar{z})]} \tilde{\sigma}_{(1..n)}(\Phi, X)(z,\bar{z}) \tag{1.15}$$

---

[5] Getting back to the X^I fields, gives the OPE:

$$\partial X^I(z)\tilde{\sigma}_{(1..n)}(0) = z^{\frac{1}{n}-1} e^{-\frac{2\pi i}{n} I} \tau_{(1..n)}(0) + ... \tag{1.14}$$

where we only write the most singular term in the OPE, but there are other less singular terms as well.

Here $k^j_{L,R}$ $j=1,2$ are constant vectors, and $\tilde{\sigma}_{(1..n)}(\Phi, X)$ involve 6 twist fields as in (1.14)- four for the X^I's and two for the Φ^I's.

In (1.15), note that the exponential has no OPE with the twist fields $\tilde{\sigma}$, as we only have the center-of-mass modes $\sum_I \phi^I$, $\sum_I \bar{\phi}^I$.

The operator in (1.15) has conformal weights

$h = n\frac{(k^1_L)^2+(k^2_L)^2}{2n^2} + \frac{6}{24}(n-\frac{1}{n})$, and $\bar{h} = n\frac{(k^1_R)^2+(k^2_R)^2}{2n^2} + \frac{6}{24}(n-\frac{1}{n})$

and R-charges: $j_3 = \frac{1}{2}(k^1_L + k^2_L)$, $\bar{j}_3 = \frac{1}{2}(k^1_R + k^2_R)$

Now, the possible values of the momenta $k^j_{L,R}$ should be determined by the periodicity conditions on the fermions. When one goes once around the origin each fermion field ψ^I becomes ψ^{I+1}, so the product $\Pi^n_{I=1}\psi^I$ picks up a phase $(-1)^{n+1}$. This product involves only the center of mass boson $\sum_I \phi^I$. In order to reproduce this OPE we need that $2\pi i n \frac{k^{1,2}_L}{n} = \pi i(n+1+2p_{1,2})$ for $p_{1,2}$ some integers, i.e. $k^{1,2}_L = \frac{n+1}{2} + p_{1,2}$. Similarly for k_R we get $k^{1,2}_R = \frac{n+1}{2} + q_{1,2}$ for $q_{1,2}$ some integers.

We find that for the operators (1.15) , $h - j_3 = \frac{p_1(p_1+1)+p_2(p_2+1)}{2n}$ and $\bar{h} - \bar{j}_3 = \frac{q_1(q_1+1)+q_2(q_2+1)}{2n}$, meaning these are chiral primaries in any of the 8 combinations where $p_{1,2}, q_{1,2}$ equal 0 or -1. However only 4 of these combinations - the ones where $p_1 = p_2$ and $q_1 = q_2$ would exist also for the field theory describing the symmetric product of $K3$'s. We therefore focus on these and end up with four chiral primary twist operators for each n-permutation: (1.15) with $k^1_L = k^2_L = \frac{n\pm1}{2}$, $k^1_R = k^2_R = \frac{n\pm1}{2}$. By summing over all n-permutations, one can construct the following chiral primaries with conformal weights and charges:

$$\sigma_n^{--}(z,\bar z) = \frac{1}{\sqrt{k!(k-n)!n}} \sum_{f\in S_k} O_{f(1..n)f^{-1}}^{(\frac{n-1}{2},\frac{n-1}{2})}(z,\bar z) \ ; \ h = j_3 = \frac{n-1}{2} \ , \ \bar h = \bar j_3 = \frac{n-1}{2}$$
$$\sigma_n^{+-}(z,\bar z) = \frac{1}{\sqrt{k!(k-n)!n}} \sum_{f\in S_k} O_{f(1..n)f^{-1}}^{(\frac{n+1}{2},\frac{n-1}{2})}(z,\bar z) \ ; \ h = j_3 = \frac{n-1}{2} \ , \ \bar h = \bar j_3 = \frac{n+1}{2}$$
$$\sigma_n^{-+}(z,\bar z) = \frac{1}{\sqrt{k!(k-n)!n}} \sum_{f\in S_k} O_{f(1..n)f^{-1}}^{(\frac{n-1}{2},\frac{n+1}{2})}(z,\bar z) \ ; \ h = j_3 = \frac{n-1}{2} \ , \ \bar h = \bar j_3 = \frac{n+1}{2}$$
$$\sigma_n^{++}(z,\bar z) = \frac{1}{\sqrt{k!(k-n)!n}} \sum_{f\in S_k} O_{f(1..n)f^{-1}}^{(\frac{n+1}{2},\frac{n+1}{2})}(z,\bar z) \ ; \ h = j_3 = \frac{n+1}{2} \ , \ \bar h = \bar j_3 = \frac{n+1}{2} \tag{1.16}$$

The last 3 operators can be obtained from the first by successive actions with J^+_{-1} and $\bar J^+_{-1}$.

In the gas of strings picture, a twist operator $\sigma_n^{s,\bar s}$ describes a string with n winding around the y-circle (and spins $s, \bar s$). A general chiral primary in the theory (restricting to the cohomology classes of M^4 we have discussed) is described by the operator

$$\prod_{i=1}^{p} [\sigma_{n_i}^{s_i,\bar s_i}]^{m_i} \quad , \quad \sum_{i=1}^{p} n_i m_i = k$$

and corresponds to a partition of the string to p groups, each has m_i copies of winding n_i ($s_i, \bar s_i$ take the values +,-).

The NS-NS vacuum corresponds to the global AdS_3 vacuum (where constant time surfaces are smooth discs), and is described by the chiral primary $(\sigma_1^{--})^k$, where there are k singly-wound strings.

The two sectors - the RR and the NS-NS are related by spectral flow - an automorphism of the $\mathcal{N} = 4$ algebra [48] :

$$\begin{aligned} h^R &= h^{NS} - j_3^{NS} + \frac{c}{24} \\ j_3^R &= j_3^{NS} - \frac{c}{12} \end{aligned} \tag{1.17}$$

This maps the chiral primaries of the NS-NS sector (with $h = j_3, \bar h = \bar j_3$) to the ground states of the R-R sector (with $h = \bar h = \frac{c}{24}$).

In chapter 3, we will find for each chiral primary of the CFT, its dual supergravity configuration.

Chapter 2

De-singularization by Rotation

2. De-singularization by Rotation

It is well known that some BPS states in four and five dimensional supergravity theories can be realized as non-singular extremal black holes with non-zero horizon area. This is the situation for generic black hole charges. However, there are some cases where the area of the horizon becomes zero and the geometry becomes singular. For example, this happens for 1/4 BPS states of string theory on T^5. In this chapter we show that by considering 1/4 BPS states with maximal angular momentum we can produce a completely non-singular geometry once we suitably include one of the internal dimensions. We were led to this solution by thinking about supersymmetric conical singularities in AdS_3. So first we analyze various aspects of supersymmetric AdS_3 spaces and conical singularities [44]. When we are dealing with AdS_3 we can consider the theory with NS-NS or RR boundary conditions on the spatial circle. It is known that the $M = 0$ BTZ black hole is a RR ground state [50]. We show that by introducing Wilson lines for $U(1)$ gauge fields in AdS_3 we can also interpret other conical singularities as RR ground states. Even pure AdS_3 with a suitable Wilson line can be interpreted as a RR ground state. All these ground states are different in their $U(1)$ charges.

If we view global AdS_3 as the near horizon region of a six dimensional rotating black string of string theory on $R^{1,4} \times S^1 \times M^4$ coming from D1-D5 branes wrapped on S^1, [6] then we can match the smooth *global* AdS_3 solution to asymptotically flat space in such a way that it preserves supersymmetry. In other words, by adding angular momentum we can find a smooth supergravity solution that corresponds

[6] The D5 branes also wrap M^4.

to the D1-D5 system. These are solutions which have maximal angular momenta $J_L = \pm J_R = Q_1 Q_5/2 \equiv k/2$. [7]

These AdS_3 geometries with Wilson lines can also have the interpretation of "giant gravitons" in AdS_3.

The proper interpretation of these solutions will involve a precise statement and understanding of the possible boundary conditions for the gauge fields that live on AdS_3. So in section 2.1 we review some facts about gauge fields and Chern Simons theory. In section 2.2 we describe the interpretation of the solutions from the AdS/CFT point of view. In section 2.3 we match the AdS_3 solutions to the asymptotically flat region. In section 2.4 we briefly remark about the interpretation of these configurations as giant gravitons.

2.1. Some Facts about Wilson Lines and Chern Simons Theory

Let us start by describing some facts about $U(1)$ gauge fields. Suppose we have a plane described by coordinates ρ, φ, $ds^2 = d\rho^2 + \rho^2 d\varphi$. Then consider a gauge field with the connection $A_\varphi = a$ where a is any constant. We see that $F = 0$ everywhere in the plane except at the origin where it is a delta function. This is of course the familiar gauge field of a Bohm-Aharonov vortex. The interaction with the gauge field is normalized so that we get the phase $e^{i\int A}$ for the field with the minimal quantum of charge. We see that if a is an integer particles do not feel any field and indeed we can set A to zero by a gauge transformations $A \to A + d\epsilon$ where $\epsilon(\varphi + 2\pi) = \epsilon(\varphi) + 2\pi n$, with n integer. We need to specify the boundary conditions for the charged fields when we go around the origin. We will work with fixed boundary conditions for the fields and we will vary a. Suppose we have a fermionic field and we impose the boundary condition that ψ is periodic as it goes around the

[7] One can also view the system in the S-dual picture, involving $F1 - NS5$. Conformal models describing the $F1 - NS5$ system with couplings representing the angular momenta have been discussed in [51].

circle. Then if we set $a = 1/2$, the field will effectively become antiperiodic. This implies that the fermionic field will be totally continuous at the origin, since the minus sign is what we expect for a rotation by 2π.

Now let us suppose that we have Chern Simons theory on a solid cylinder $D_2 \times R$, where D_2 is a disk. Then we need to impose some boundary conditions on the gauge field. As shown in [41][52] we can impose the boundary condition only on one component of A along the boundary. One way to understand this is to view the direction orthogonal to the boundary as time so that one realizes that the two components of A along the boundary are canonically conjugate variables. We will be interested in setting boundary conditions of the form $2A_- = A_0 - A_\varphi = 0$. It is easy to see that these boundary conditions are consistent. We choose these boundary conditions because it was shown in [14] that they are appropriate for gauge fields in AdS supergravities. Once we give these boundary conditions we can have a variety of states in the theory with various values of A_+ on the boundary. These values are $2A_+ = q/2k$, with q integer [52], and k the level of the Chern Simons theory. These various states can arise by inserting various Wilson lines in the interior. States with $q \to q \pm 2k$ are related by a large gauge transformation which does not vanish at the boundary. These transformations map physical states to other physical states in the boundary theory. From the point of view of the topological theory in the bulk, states with q and $q \pm 2k$ are equivalent. The $U(1)$ charge of the state has the value of $\frac{1}{2\pi}\int A$ along the spatial circle. If we have a Wilson line of charge q in the interior, this value is $A_\varphi = q/(2k)$.

Similar remarks about CS theory apply when the gauge group is non-compact, such as $SL(2,R)$. In this case we consider again configurations with vanishing field strength and with the same asymptotic boundary conditions. This implies that the space is locally AdS but not globally. For example, we can consider the conical space

$$ds^2 = -(r^2 + \gamma^2)dt^2 + r^2 d\varphi^2 + \frac{dr^2}{r^2 + \gamma^2} \tag{2.1}$$

Locally this is an AdS_3 space, but at $r = 0$ we have a conical singularity if $\gamma \neq 1$.

2.2. Conical Singularities and AdS/CFT

In this section we will apply some of the above remarks to supergravity theories on AdS_3. What we will describe is mainly contained in [44][53]. We will consider supergravity theories with extra $U(1)$ gauge fields on AdS_3. One example we have in mind is the case of string theory on $AdS_3 \times S^3 \times K3$, but other examples could be treated in a similar way. We will consider gravity theories on AdS_3 with at least $(2, 2)$ supersymmetry. This implies that we will have $U(1)_L \times U(1)_R$ gauge fields. Pure three dimensional gravity on AdS_3 is given by an $SL(2, R)^2$ Chern Simons theory, which we will use to describe the conical spaces. In this situation we could consider solutions with arbitrary Wilson lines for the $U(1)_{L,R}$ gauge field as well as the $SL(2, R)_{L,R}$ gauge fields. In principle these solutions are singular in the interior and we should not consider them, unless we have a good reason to think that the singularity will be resolved in the full theory.

In this chapter we will consider singularities which preserve at least (2,2) supersymmetry. We will impose RR boundary conditions on the fields and we consider arbitrary Wilson lines. In order for the solution to be supersymmetric the Wilson line in the $SL(2, R)$ part and the $U(1)$ part should be essentially the same, we will later make this statement more precise. The boundary of AdS_3 is $R \times S^1$. We normalize charges so that a fermion carries integer charge under $U(1)_{R,L}$. As standard in AdS/CFT, the boundary conditions on all supergravity fields correspond to the microscopic definition of the "Lagrangian" of the CFT, including the periodicities of the fields as we go around the circle, etc. We can then consider all solutions to the supergravity equations with given boundary conditions. Different solutions correspond to different states in the boundary CFT. Now let us choose RR boundary conditions for the CFT and sugra fields on the spatial boundary circle. We will impose the boundary condition $A^L_- = A^R_+ = 0$ for $U(1)_{L,R}$.[8]

We will consider flat gauge fields with $U(1)_{R,L}$ connections given by constant values $A^L_+ = a_+$, $A^R_- = a_-$. Supersymmetry determines the three dimensional geometry. We consider spinors generating supersymmetry that are periodic when we go around the circle, since we said we are interested in the RR sector. The solution is then:[9]

$$
\begin{aligned}
ds_3^2 =& -[(r - \frac{a_+^2 - a_-^2}{r})^2 + 4a_+^2]dt^2 + \frac{dr^2}{(r - \frac{a_+^2-a_-^2}{r})^2 + 4a_+^2} + \\
&+ r^2(d\varphi + \frac{a_+^2 - a_-^2}{r^2}dt)^2 \\
&A^L_+ = a_+ \quad , \quad A^R_- = a_- \quad , \quad A^L_- = A^R_+ = 0
\end{aligned} \tag{2.2}
$$

In the particular case of $a_+ = a_- = \gamma/2$ the solution is

$$
\begin{aligned}
ds_3^2 =& -(r^2 + \gamma^2)dt^2 + r^2 d\varphi^2 + \frac{dr^2}{r^2 + \gamma^2} \\
&A^L_+ = A^R_- = \gamma/2 \quad , \quad A^L_- = A^R_+ = 0
\end{aligned} \tag{2.3}
$$

All these configurations have zero energy, as implied by the RR sector superalgebra. The AdS_3 space in (2.3) seems to have negative energy, but one should add to this the energy that comes from the Wilson line. This additional energy comes from the "singleton" that lives at the boundary of AdS which encodes this degree of freedom. This was explicitly shown in [14]. So we have $L_0 = \bar{L}_0 = 0$. The angular momenta are half the $U(1)$ charges, $J_L = J_R = k\gamma/2$. So we see that γ

---

8 Actually one could impose the boundary condition $A^L_- = \epsilon_L$, $A^R_+ = \epsilon_R$, where $\epsilon_{L,R}$ are some constants. These would correspond to left and right spectral flows with the parameters $\epsilon_{L,R}$.

9 We use conventions where $R_{AdS} = 1$.

should be quantized as $\gamma = n/k$. We get zero energy states with various amounts of angular momenta.

So what is the interpretation of these spaces? which ones are allowed and which ones are not? All these are supersymmetric solutions. Almost all of them are singular. Only if $\gamma = 1$ we see from (2.3) that we get a nonsingular solution. Let us discuss this solution first. The three dimensional geometry is that of AdS_3. The Wilson line around the origin of AdS_3 is such that it effectively changes the periodicity of fermionic fields from periodic to anti-periodic, so that they are smooth at the origin. This solution has angular momenta $J_L = J_R = k/2$. What is this state in the boundary CFT?. We know that the boundary CFT has a large number of RR vacua [54]. These vacua have angular momenta $|J_{L,R}| \leq k/2$. We see that the non-singular solution corresponds to a state with the maximal value of the angular momentum. From general arguments [49]we know that there is a single RR state with maximal value of the RR charge, it is the state that maps to the NS vacuum under spectral flow. Here we indeed see that the state we find is essentially the same as global AdS_3 which was identified as the NS vacuum. The only difference is that the Wilson lines imply that particle energies are shifted as they are shifted under spectral flow. Now we turn to the solutions with $\gamma \neq 1$. All those solutions contain a singularity at the origin. It is clear that starting from the solution with $\gamma = 1$ we can add supergravity particles that decrease the angular momentum and leave $L_0 = \bar{L}_0 = 0$, these particles, are of course, the chiral primaries discussed in [26], see also [32]. If we have particles with high values of the angular momentum, $l \gg 1$, l/k fixed, they will appear like very massive particles from the AdS_3 point of view and will give rise to the conical spaces with $\gamma < 1$. It is not possible to get the conical spaces with $\gamma > 1$ in this fashion, since all those supergravity particles would increase the energy and will remove us from the RR vacuum. In other words, by adding supergravity particles to the state with $J_L = J_R = k/2$ we can decrease the angular momentum while preserving the zero energy condition. If we try to increase J we would increase the energy.

We could imagine decreasing J by adding supergravity particles with low values of the spin, those gravity particles have wavefunctions which are quite extended in AdS. If we added them in a coherent state, we should be able to find classical solutions which are also smooth and do not have these conical singularities. Finding these solutions would require us to use the full six dimensional gravity equations. In other words, the fact that for $J < k/2$ we only found singular solutions does not mean that there are no non-singular solutions. A trivial example is the following. Consider the $AdS_3 \times S^3$ case. Now we have $SU(2)_L$ and $SU(2)_R$ symmetry groups. Let us pick the $U(1)$'s in the above discussion to be in the direction $\hat{3}$. Take the solution with maximal angular momentum and perform an $SU(2)_{L,R}$ rotation in the $\hat{1}$ axis so that now the angular momentum points in the $\hat{2}$ direction. We get exactly the same AdS_3 space but now with a Wilson line $A_+^{L,2} = A_-^{R,2} = 1/2$ and the rest zero. This is a solution with zero $U(1)$ charges but with no singularity, as opposed to the solution in (2.3) with $\gamma = 0$. Of course here we are treating these Wilson lines in a classical fashion. This is correct in the large k limit where we deal with macroscopic amounts of angular momentum.

It is easy to see that any solution which is AdS_3 and a Wilson line of the form $A_+^L = 1/2+n$, $A_-^R = 1/2+n'$ with integer n, n' will be non-singular. These solutions correspond to the spectral flow of the state with $n = n' = 0$. These solutions do not preserve the supersymmetries that the RR ground state preserve, but they do preserve other supersymmetries. These are the configurations that are related by spectral flow to the NS sector ground state.

2.3. Non-singular Solutions in Asymptotically Flat Space

In this section we point out that the conical spaces, including the non-singular

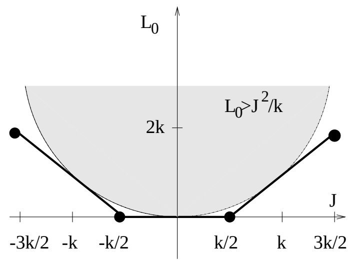

Figure 1: Spectrum of the theory in the RR sector. RR ground states have spins $|J| \leq k/2$. Quantum numbers that lie within the shaded region, with $L_0 > J^2/k$ can be carried by black holes. We have a similar figure for $\bar{L}_0$ and $\bar{J}$.

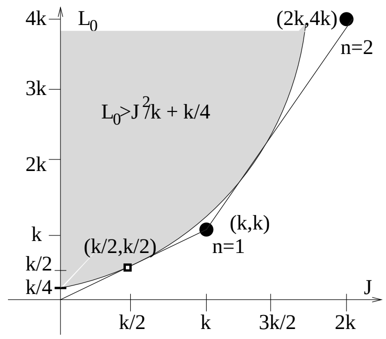

Figure 2: Spectrum of the theory in the NS sector sector. Quantum numbers that lie within the shaded region, with $L_0 > J^2/k + k/4$ can be carried by black holes. States with $J = nk$, $L_0 = n^2k$ are AdS_3 spaces with Wilson lines.

AdS_3 space with a Wilson line, can be extended to supersymmetric solutions of six dimensional supergravity that are asymptotic to $R^5 \times S^1$, with periodic boundary conditions on S^1. In other words, they represent BPS solutions in this six dimensional string theory.

We can find the solution by starting with the most general five dimensional black hole solution written in [55], lifting it up to six dimensions as in [35] and taking the extremal limit with zero momentum charge while keeping the angular momenta nonzero. The solution we obtain is parameterized by two angular momentum parameters which we take as $\gamma_{1,2}$: $J_{L,R} = \frac{k}{2}(\gamma_1 \mp \gamma_2)$ and can be written in the form[10]:

[10] To relate our parameters and coordinates to the ones in eq. (4) of Cvetic and Larsen

$$
\begin{aligned}
\frac{ds_6^2}{\sqrt{k}} =& \frac{1}{h}(-dt^2 + d\varphi^2) + hf(d\theta^2 + \frac{r^2 dr^2}{(r^2+\gamma_1^2)(r^2+\gamma_2^2)}) \\
&- \frac{2}{hf}[(\gamma_2 dt + \gamma_1 d\varphi)\cos^2\theta d\psi + (\gamma_1 dt + \gamma_2 d\varphi)\sin^2\theta d\phi] + \\
&+ h[(r^2+\gamma_2^2) + (\gamma_1^2 - \gamma_2^2)\frac{\cos^2\theta}{h^2 f^2}]\cos^2\theta d\psi^2 + \\
&+ h[(r^2+\gamma_1^2) - (\gamma_1^2 - \gamma_2^2)\frac{\sin^2\theta}{h^2 f^2}]\sin^2\theta d\phi^2
\end{aligned}
\tag{2.4}
$$

where:

$$
\begin{aligned}
f &= f(r,\theta) \equiv r^2 + \gamma_1^2\cos^2\theta + \gamma_2^2\sin^2\theta \\
h &= h(r,\theta) \equiv \frac{\sqrt{k}}{R_y^2}(1 + \frac{R_y^2 Q_1}{kf})^{1/2}(1 + \frac{R_y^2 Q_5}{kf})^{1/2}
\end{aligned}
\tag{2.5}
$$

and R_y is the radius of the S^1 parameterized by φ.

Setting the two angular momenta equal ($\gamma_2 = 0 \;\;,\;\; \gamma \equiv \gamma_1$): $J_L = J_R = k\gamma/2$, we get the solution:

$$
\begin{aligned}
\frac{ds_6^2}{\sqrt{k}} =& -\frac{1}{h}(dt + \frac{\gamma\sin^2\theta}{r^2+\gamma^2\cos^2\theta}d\phi)^2 + \frac{1}{h}(d\varphi - \frac{\gamma\cos^2\theta}{r^2+\gamma^2\cos^2\theta}d\psi)^2 + \\
&+ h\frac{r^2+\gamma^2\cos^2\theta}{r^2+\gamma^2}dr^2 + \\
&+ h[(r^2+\gamma^2\cos^2\theta)d\theta^2 + (r^2+\gamma^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\psi^2]
\end{aligned}
\tag{2.6}
$$

In the decoupling near-horizon limit the metric reduces to a locally $AdS_3 \times S^3$, where the S^3 angles are defined as $\tilde{\psi} = \psi - \gamma\varphi$, $\tilde{\phi} = \phi - \gamma t$ [35].

Since the original angles are identified as: $\varphi \sim \varphi + 2\pi$, $\theta \sim \theta + \pi/2$, $\psi \sim \psi + 2\pi$, $\phi \sim \phi + 2\pi$, these new coordinates have the global identifications:

$$
\begin{aligned}
(\varphi,\tilde{\psi}) &\sim (\varphi,\tilde{\psi}) + 2\pi(1,-\gamma) \sim (\varphi,\tilde{\psi}) + 2\pi(0,1) \\
\theta &\sim \theta + \pi/2 \\
\tilde{\phi} &\sim \tilde{\phi} + 2\pi
\end{aligned}
\tag{2.7}
$$

---

[35], we have $\gamma_{1,2} = \frac{R_y}{\sqrt{k}}(\cosh\delta_0\ell_{1,2} - \sinh\delta_0\ell_{2,1})$, $k = \lambda^4$ and $r = \frac{\sqrt{k}}{R_y}r^{C.L.}$, $t = \frac{1}{R_y}t^{C.L.}$, $\varphi = \frac{1}{R_y}y^{C.L.}$

For general (noninteger) values of the parameter γ the periodicities of the AdS_3 and the S^3 parts are still coupled, and the geometry obtained is singular.

The most interesting solution is the one with angular momenta $J_L = J_R = k/2$, when $\gamma = 1$, since it is non-singular. It is a non-singular, geodesically complete geometry. In its decoupling near-horizon limit, the space is globally a direct product $AdS_3 \times S^3$, as can be seen looking at the periodicities of the angles in (2.7).

It seems that the fact that this solution is non-singular is related to the fact that there are very few states in the CFT with similar values of the angular momenta.

It would be interesting to see if other BPS states in the AdS_3 region could be matched to the asymptotically flat region. Natural candidates are states with $L_0 = 0, J^R = k/2$ and $J_L = k/2 + nk$. In the near horizon limit these states have $A^R_- = 1/2$, $A^L_+ = 1/2 + n$. The elliptic genus formula tells us that there is a single BPS state with these values of the angular momenta[11] (it is just the left spectral flow of the state we found above). If we tried to take a limit of the solutions in [55][35], we would find that $J_R = 0$. It could be that we need to make a more general ansatz.

2.4. Super Giant Gravitons

In this section we consider NS-NS boundary conditions on the circle at the boundary. The ground state is AdS_3 with no Wilson lines for the $U(1)$ gauge fields. We can consider the spectrum of chiral primaries, i.e. states with $L_0 = J_L$, $\bar{L}_0 = J_R$ as in [26]. From the CFT point of view we can calculate how many of these states we expect. It turns out that there is a single state with $J_{L,R} = 0$ and the number of states increases as we increase the values of $J_{L,R}$, it reaches a maximum at $J_L = J_R = k/2$ and then it starts decreasing again so that for $J_L = J_R = k$ we

[11] In principle, we also need to add the center of mass motion of the string in the transverse four dimensions [56].

find just a single state again. In other AdS compactifications there is a maximum value for the single particle BPS states. In [57] it was shown that these states are realized as expanded branes, see also [58]. In $AdS_5 \times S^5$ the cutoff appears at $J = N$ [54], where J is the angular momentum on S^5. In AdS_3 the situation is different [26], there is an absolute cutoff on J at $J_L = J_R = k$, there are no chiral primary states beyond this value of J. By using the previous ideas about Wilson lines it is easy to see that this state is just AdS_3 with $U(1)_L \times U(1)_R$ Wilson lines equal to $A^L_+ = A^R_- = 1$. We could roughly think about it as an AdS space which is just rotating as a whole. Only the "singleton" field is excited. The singleton is the mode that appears at the boundary from the Chern Simons theory in the interior. We can say that gravitons became so big that they live at the boundary of AdS.

So in the $AdS_3 \times S^3$ case the graviton with maximum angular momentum is not an expanded brane but just a different classical solution. This is in agreement with the fact that the maximal spin, k, is of the order of the inverse six dimensional Newton's constant, while in the $AdS_5 \times S^5$ this maximal value, N, is proportional to the square root of Newton's constant. Notice that objects such as long strings [59][30] are not of concern here since we can work at a point in moduli space where there is no finite energy long string at infinity. This is possible if Q_1 and Q_5 are coprime [30].

In summary, as we pile up chiral primary particles on AdS_3 we get to a point at $J_L = J_R = k/2$ where we are on the verge of making a black hole [54]. If J_L, J_R approach their maximal values we have again a smooth geometry with a small number (if $J_{L,R}$ are sufficiently close to k) of chiral primary particles.

Chapter 3

Gravity Solutions for the D1-D5 System with Angular Momentum

3. Gravity Solutions for the D1-D5 System with Angular Momentum

Perhaps one of the most distinctive aspects of gravity is that time slows down near heavy objects due to gravitational redshift.

We now have many cases where we have dual descriptions of gravitational theories in terms of ordinary quantum field theories via the AdS/CFT correspondence. It is interesting then to find situations where this effect is under some degree of control, so that we can understand it from the field theory point of view. Black holes are extreme examples where this redshift factor goes to zero. In this chapter we consider configurations where this redshift factor is important but does not go to zero.

We focus on $AdS_3 \times S^3$ compactifications, and we consider states with angular momentum on S^3 that are BPS. These states are also called "chiral primary" states. When these states carry large amounts of angular momentum, their back-reaction on the metric cannot be ignored. In this chapter we construct exact gravity solutions which take this backreaction into account. We indeed find that there is an important redshift effect that implies, among other things, that the energy gap to the next non-BPS excitation decreases as we increase the angular momentum. This gap goes to zero for certain states that are on the verge of forming black holes.

These solutions can be found by noticing that the D1/D5 system with angular momentum blows up into a Kaluza-Klein monopole supertube, U-dual to the one described in [37]. Since the Kaluza-Klein monopole is non-singular, these geometries are non-singular. The configuration with maximal angular momentum, which corresponds to a supertube with circular shape, has a near horizon geometry equal to $AdS_3 \times S^3$ in global coordinates. Supertubes with non-circular shapes correspond to chiral primary excitations on the $AdS_3 \times S^3$ vacuum.

The solutions are also interesting since they provide non-singular gravity solutions for configurations that are 1/4 BPS in toroidally compactified string theory. Different gravity solutions are related to different microscopic states.

Previous work on the subject focused on gravity solutions with conical singularities. We show that these conical singularities are not a good description of the long distance properties of generic chiral primaries, i.e. the non-singular solutions are different, even at long distances. Some very special chiral primaries can give conical metrics with opening angles of the form $2\pi/N$. Conical metrics with non-integer angles are not a good approximation to any of the non-singular metrics. Singular geometries more closely related to chiral primaries can be found in [60]. We will show that our solutions look like the solutions in [60] at long distances.

In this chapter we also analyze some aspects of the geometry of supertubes in other dimensions and in various limits.

In section 3.1 we describe the construction of the gravity solutions. In section 3.3 we discuss the relation of these gravity solutions to the problem of chiral primaries in $AdS_3 \times S^3$. In section 3.4 we describe general non-singular solutions with plane wave asymptotic boundary conditions, which can be thought of as arising from particles propagating on plane wave backgrounds. In section five we discuss some aspects of the gravitational geometry of supertubes in various dimensions. This section is a bit disconnected from the previous parts of the chapter.

3.1. The Solutions

In this section we consider ten dimensional supergravity compactified on $S^1 \times T^4$, and consider a system of Q_1 D1 branes wrapped on S^1 and Q_5 D5 branes wrapped on all the compact directions[12]. We are interested in constructing solutions which carry angular momentum and are 1/4 BPS. In other words, they are as BPS as the D1 and D5 branes with no angular momentum. Since there are four non-compact transverse directions, the angular momenta live in $SO(4) \sim SU(2)_L \times SU(2)_R$. The angular momentum is bounded by $J_L, J_R \leq k \equiv Q_1 Q_5$ [61]. For large values of Q the angular momentum can be macroscopic and can have an important effect on the geometry of the configuration. This was initially explored in [1][62] who found that the geometry with maximal angular momentum was non-singular. In the meantime, studies of other 1/4 BPS configurations with angular momentum have given rather interesting results. The best known example is the so called "supertube" which is a configuration carrying D0 and fundamental string charges with angular momentum, which is described in terms of a tubular D2 brane with electric and magnetic fields on its worldvolume [37]. The configuration with maximal angular momentum consists of a tubular D2 brane with a radius square proportional to the product of the two charges. The configuration does not carry any net D2 brane charge. Tubes with arbitrary cross sections are also possible, but they carry less angular momentum [37].

The D0-F1 system is U-dual to D1-D5. Under this U-duality the above D2 brane goes over to a Kaluza Klein monopole which is wrapped on T^4 and a circle in the four non-compact dimensions. The special circle of the KK monopole is the S^1 common to the D1 and D5 branes. The gravity solution for a circular KK monopole was found in [1,62] (based on the general solutions in [34]) though it was not given this description (which is not that obvious by just looking at the metric). This solution is non-singular because the KK monopole has a non-singular geometry.

Now we construct similar solutions with arbitrary shapes which are also non-singular. The technique we use to find the solution is based on the observation

[12] In appendix B we explain how to obtain the solutions for the $K3$ case.

that this system is U-dual to fundamental strings with momentum along the string. Microscopic configurations of the system are given by strings carrying travelling waves along them, in other words, strings with only left (or only right) moving excitations. For this case there are gravity solutions that closely correspond to given microscopic states [63,64] Namely these solutions describe an oscillating string with an arbitrary profile $\mathbf{F}(v)$, where v is a lightcone coordinate along the string. By a chain of dualities these can be mapped to the D1-D5 system so that we find the solution [65] (see appendix B)

$$\begin{aligned}
ds^2 &= f_1^{-1/2} f_5^{-1/2} \left[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2 \right] + f_1^{1/2} f_5^{1/2} d\mathbf{x} \cdot d\mathbf{x} \\
&\quad + f_1^{1/2} f_5^{-1/2} d\mathbf{z} \cdot d\mathbf{z} \\
e^{2\Phi} &= f_1 f_5^{-1}, \\
C^{(2)}_{ti} &= \frac{B_i}{f_1}, \qquad C^{(2)}_{ty} = f_1^{-1} - 1, \\
C^{(2)}_{iy} &= -\frac{A_i}{f_1}, \qquad C^{(2)}_{ij} = \mathcal{C}_{ij} + f_1^{-1}(A_i B_j - A_j B_i)
\end{aligned} \tag{3.1}$$

The functions $f_{1,5}$ and A_i appearing in this solution are related to the profile $\mathbf{F}(v)$

$$f_5 = 1 + \frac{Q_5}{L} \int_0^L \frac{dv}{|\mathbf{x} - \mathbf{F}|^2}, \quad f_1 = 1 + \frac{Q_5}{L} \int_0^L \frac{|\dot{F}|^2 dv}{|\mathbf{x} - \mathbf{F}|^2}, \quad A_i = -\frac{Q_5}{L} \int_0^L \frac{\dot{F}_i dv}{|\mathbf{x} - \mathbf{F}|^2} \tag{3.2}$$

and the forms B_i and $\mathcal{C}_{ij}$ are defined by the duality relations

$$d\mathcal{C} = - *_4 df_5, \qquad dB = - *_4 dA. \tag{3.3}$$

where the $*_4$ is defined in the four non-compact spatial dimensions. The one brane charge is given by

$$Q_1 = Q_5 \langle |\dot{F}|^2 \rangle = Q_5 \frac{1}{L} \int_0^L |\dot{F}|^2 dv \tag{3.4}$$

The length L that appears in these formulas is

$$L = \frac{2\pi n_5}{R} = 2\pi n_5 R' \tag{3.5}$$

where n_5 is the number of fivebranes and R is the radius of the y circle, while R' is the radius in the original fundamental string description[13]. n_5 is the original number of strings which becomes the number of fivebranes. We see that we are taking a configuration where the string is multiply wound. This will be important for later considerations. Configurations where the string consists of independent pieces can be obtained by adding the corresponding contributions in the harmonic functions (3.2). The solutions are parameterized by the profile $\mathbf{F}(v)$ which describes a trajectory in the four non-compact dimensions. Note that the final solution (3.1) is time independent. We will see that the v dependence of F translates into a dependence of the solution on the non-compact dimensions. The angular momentum of the solution (3.1) is given by

$$J_{ij} = \frac{Q_5 R}{L} \int_0^L (F_i \dot{F}_j - F_j \dot{F}_i) dv \qquad (3.6)$$

It can be checked that the angular momentum is always smaller than $n_1 n_5$. We will later concentrate on the two $U(1)$ components $J_\phi = J_{12}$, $J_\psi = J_{34}$ and define $2J_{L,R} = J_\phi \pm J_\psi$.

Note that all these solutions correspond to different ground states of the D1/D5 system. This system has a large degeneracy, of order $e^{2\pi\sqrt{2Q_1Q_5}}$.[14].

3.2. An Argument Showing the Solution is Non-singular

Looking at the metric (3.1), one might think that it is singular if $\mathbf{x} = \mathbf{F}(v_0)$

[13] For simplicity we have set $g = \alpha' = V_4 = 1$ in the above formulas. In that case $Q_1 = n_1$ and $Q_5 = n_5$, otherwise Q_i have dimensions of length square and denote the contribution of the onebranes and fivebranes to the gravitational radius of the configuration, while $n_{1,5}$ are integers.

[14] More precisely, these solutions are particular combinations of states of the theory. Classically there is an infinite number of solutions since they are parameterized by continuous parameters. In the quantum theory we should quantize the moduli space of solutions and that will give us a finite number. This quantization is expected to give us the same as quantizing the left movers on a string, though we did not verify it explicitly.

for some value of v_0, since the harmonic functions (3.2) diverge there. However, it was shown in [1,62] that the maximally rotating solution is non-singular. The maximally rotating solution corresponds to a circular profile

$$F_1 + iF_2 = ae^{i\omega v} \qquad F_3 = F_4 = 0, \qquad \text{with} \quad \omega = \frac{2\pi}{L} = \frac{R}{n_5} \tag{3.7}$$

From the expression for the charges (3.4) we get that the radius is

$$a = \frac{\sqrt{Q_1 Q_5}}{R} \tag{3.8}$$

On the other hand if we have a circular profile with a frequency $\omega' = n\omega$ (and $a' = a/n$), we would get a geometry which has a conical singularity of opening angle $2\pi/n$.

Let us now look at the geometry corresponding to a more general profile $F(v)$. We will analyze the metric near the potential singularity $\mathbf{x} = \mathbf{F}(v_0)$ and show that for a generic profile $\mathbf{F}(v)$ the solution is completely regular. By generic we mean a profile satisfying two conditions:

(i) the profile does not have self–intersections (if $v_1 \neq v_2$, then $\mathbf{F}(v_1) \neq \mathbf{F}(v_2)$);

(ii) the derivative $\dot{\mathbf{F}}(v)$ never vanishes.

Looking at the vicinity of the singularity for such profile, we find

$$\begin{aligned} f_5 &\approx \frac{Q_5}{L}\int_{-L/2}^{L/2}\frac{dv}{|\mathbf{x}-\mathbf{F}|^2} \approx \frac{Q_5}{L}\int_{-L/2}^{L/2}\frac{dv}{x_\perp^2 + (\dot{F})^2 v^2} = \frac{Q_5}{L}\frac{\pi}{|\dot{F}|x_\perp}, \\ f_1 &\approx \frac{Q_5}{L}\frac{\pi|\dot{F}|}{x_\perp}, \qquad A_i \approx -\frac{Q_5}{L}\frac{\pi\dot{F}_i}{|\dot{F}|x_\perp} \end{aligned} \tag{3.9}$$

We have split the coordinates of the transverse space around the point $\mathbf{F}(v_0)$ into a longitudinal piece, x_l, along $\dot{F}(v_0)$ and a transverse piece $x_\perp$.

The asymptotics (3.9) can be used to show that there are no singularities in the longitudinal piece of the metric

$$ds_l^2 \equiv |\dot{F}|\left[f_5 dx_l^2 - f_1^{-1}|A_i|^2 dx_l^2\right], \tag{3.10}$$

but they are not good enough for finding the finite contribution to ds_l. We will refer to the appendix F of [65] where more careful analysis was done, and give the result

$$ds_l^2 = |\dot{F}| C dx_l^2 \tag{3.11}$$

where C is a positive numerical coefficient whose value depends on *global* properties of the profile

$$C(v_0) = \frac{1}{|\dot{\mathbf{F}}(v_0)|^2} \left\{ \frac{Q_5}{L} \int_0^L \frac{dv(\dot{\mathbf{F}}(v) - \dot{\mathbf{F}}(v_0))^2}{(\mathbf{F}(v) - \mathbf{F}(v_0))^2} + (1 + |\dot{\mathbf{F}}(v_0)|^2) \right\} \tag{3.12}$$

Let us now analyze the metric in the space transverse to the singularity

$$ds_\perp^2 \equiv |\dot{F}| \left[f_5 (dx_\perp^2 + x_\perp^2 d\Omega_2^2) + f_1^{-1} (B_i dx^i)^2 \right] \tag{3.13}$$

In order to compute the leading order terms in the metric it is important to compute B_i which is dual to A_i. We only need to compute this to leading order in $x_\perp$ so that we find

$$B_\psi \sim -(\cos\theta - 1) \frac{\pi Q_5}{L} \tag{3.14}$$

where the metric in the flat transverse space is parameterized as

$$ds_0^2 = dx_l^2 + dx_\perp^2 + x_\perp^2 (d\theta^2 + \sin^2\theta d\psi^2) \tag{3.15}$$

Note that the range of θ is $0 \le \theta < \pi$. Then the transverse metric (3.13) becomes

$$ds_\perp^2 = \frac{4Q_5\pi}{L} \left[(d\sqrt{x_\perp})^2 + x_\perp \left\{ \left(\frac{d\theta}{2} \right)^2 + \sin^2 \frac{\theta}{2} d\psi^2 \right\} \right] \tag{3.16}$$

Let us now look at the complete metric

$$ds^2 = |\dot{F}| C dx_l^2 + ds_\perp^2 + \frac{L x_\perp}{\pi Q_5} \left\{ dy^2 + 2B_i dx^i dy - dt^2 + 2A_i dx^i dt \right\}, \tag{3.17}$$

Near the singularity we get

$$\begin{aligned} ds^2 = \frac{4Q_5\pi}{L} \{ & (d\sqrt{x_\perp})^2 + \\ & + x_\perp \left[\left(\frac{d\theta}{2} \right)^2 + \sin^2 \frac{\theta}{2} \left(d\psi + \frac{dy}{R} \right)^2 + \cos^2 \frac{\theta}{2} \frac{dy^2}{R^2} \right] \} \\ & + |\dot{F}| C (dx_l - \frac{1}{C|\dot{F}|} dt)^2 - \frac{1}{C|\dot{F}|} dt^2 \end{aligned} \tag{3.18}$$

where we used (3.5). Let us introduce new coordinates

$$\chi = \frac{y}{R}, \qquad \tilde{\psi} = \psi + \chi, \qquad \tilde{\theta} = \frac{\theta}{2}, \qquad \rho = \sqrt{x_\perp} \tag{3.19}$$

We see that $\chi = \frac{y}{R}$ has periodicity 2π, and thus the change of coordinates from ψ to $\tilde{\psi}$ is well defined. We also note that $\tilde{\theta}$ has a range $0 \leq \tilde{\theta} < \pi/2$ and the metric (3.18) becomes:

$$\begin{aligned} ds^2 = & \frac{4Q_5\pi}{L}\left\{d\rho^2 + \rho^2\left[d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\psi}^2 + \cos^2\tilde{\theta}d\chi^2\right]\right\} \\ & + |\dot{F}|C(dx_l - \frac{1}{C|\dot{F}|}dt)^2 - \frac{1}{C|\dot{F}|}dt^2 \end{aligned} \tag{3.20}$$

The first line in the above expression gives the metric of a flat four dimensional space from which we conclude that the geometry is regular near the string profile. Note also that the Killing vector ∂_t becomes light like at $\rho = 0$.

To summarize, what is happening is that the circle y is shrinking to zero size as we approach the string source but the presence of the field B_i implies that it is non-trivially fibered over the S^2 that is transverse to the line described by $\mathbf{F}$ in R^4 and we see that the y circle combines with the two sphere and $x_\perp$ to give a non-singular space, precisely as it happens for the Kaluza Klein monopole. Note also that a non-trivial condition on the harmonic functions characterizing the solution arises from demanding that the B_i field leads to a well defined fibration of the y circle on the four dimensional space away from the sources. This quantization condition on B_i translates into a quantization condition for the field A_i, which is obeyed with a unit coefficient if we take and A_i field as in (3.9) with $2\pi Q_5/L = R_y$.

In fact we can view these metrics as "supertubes" analogous to the ones described in [37] where the D1-D5 system blows up to a KK monopole that wraps the four directions of T^4 and a curve of shape given by the profile $\mathbf{F}(v)$ in the non-compact directions. The circle of the KK monopole is the y circle where the initial D1 and D5 are wrapped. The final geometry is non-singular if the system blows up into a single KK monopole. This is ensured if the function $\mathbf{F}$ does not self intersect.

One would like to know what the topology of these solutions is. We can ignore the T^4 for this question. Before we put the D1-D5 system, the topology of the six dimensional space is $R \times R^4 \times S^1$. The topology of a fixed radius surface far away is $R \times S^3 \times S^1$. When we go in the radial direction the S^3 is filled in so that we have R^4. Let us start with the non-singular maximally rotating circular solution. The topology of a fixed radius surface far away is the same as above but when we go in we fill in the S^1 so that the final topology is $R \times R^2 \times S^3$. This is shown explicitly in appendix A . The topology for geometries that can be obtained as continuous deformations of the circle is still going to be the same. The actual metric and geometry of the solution will of course depend on many parameters. For example in the regime that $a^2 \gg Q_1, Q_5$ we find that the gravity solution has the shape of a ring whose gravitational thickness (of order $\sqrt{Q_1}$, or $\sqrt{Q_5}$) is much smaller than its radius.

An important lesson is that these configurations with D-brane charge can change the topology of the spacetime where they live. This situation is common in many examples of AdS/CFT. It is an example of a so called “geometric transition”.

3.3. Geometries with A_N Singularities

There are some singular geometries that are believed to be allowed in string theory. A particular example arises when we multiply each non-constant piece in the harmonic functions (3.2) by N. In that case we get a Z_N singularity on the ring. In other words, instead of a single KK monopole we have N coincident ones giving rise to an A_{N-1} singularity. The resulting geometry is a Z_N quotient of the geometries we discussed in the previous subsection. It is easy to see that we can find non-singular deformations of these geometries by separating the N copies of the harmonic functions in the transverse directions. When this separation is small we get that the A_{N-1} singularity becomes locally a smooth ALE space. This situation was explored in detail in [66] and we refer the reader to it for the details. Notice that if n_1 and n_5 are coprime and we are at a generic point in moduli space [30], then it is not possible to deform the A_{N-1} singularity by displacing the "center of mass" of the rings in the non-compact directions, but it is possible to deform it by combining the whole system into a single string. When we start with N coincident rings as in (3.7) we get a geometry that is a Z_N quotient of $AdS_3 \times S^3$. This has fixed points along an equator of S^3 and the origin of AdS_3. The metric can be written as the conical metric in Appendix C, with a coefficient $\gamma^{-1} = N$. Conical metrics with arbitrary non-integer γ^{-1} have been considered in the literature. These metrics are suspicious since they would correspond to KK monopoles which do not obey the proper quantization condition. Indeed, if one looks at those metrics one finds that there are extra conical singularities as compared to the A_N case. These are easy to understand in the space parameterized by the flat coordinates x^i. In this case we have "Dirac-strings" coming out of the KK monopole extended along the disk in the 12 plane with boundary on the ring. If the quantization condition is not obeyed, then the metric will be singular on this disk. Since we found a large family of non-singular metrics one might wonder if one could take a smooth metric which approximates these conical spaces with γ^{-1} not integer arbitrarily well. We argue in appendix C that this is not possible. In conclusion, conical metrics with γ^{-1} not integer are not a good approximation to the real solutions.

3.4. Geometries Corresponding to Chiral Primaries.

We first need to take the decoupling limit of the solutions that we considered above (3.1). This amounts to dropping the ones in the harmonic functions f_1 and f_5 in (3.2). We can then see that the asymptotic geometry for large $|x|$ is that of $AdS_3 \times S^3$. If we take the standard periodic conditions on the spinors along S^1 that preserve supersymmetry in the asymptotically flat context then we see that we are in the Ramond sector of the theory. Different solutions correspond to different Ramond vacua. These vacua can have various values of angular momenta ranging over $-k/2 \leq J_{L,R} \leq k/2$. The solution with a circular profile in the 12 plane corresponds to the Ramond vacuum with maximal angular momentum $J_L = J_R = k/2$. Under spectral flow this state goes over to the NS vacuum. Spectral flow in the CFT is an operation that maps states in the R sector to states in the NS sector. It is a rather trivial operation involving only the overall U(1) R-charge of the states so we do not expect a significant change in the properties of the state when we perform it. In fact spectral flow amounts to a simple coordinate redefinition

$$\tilde{\phi} = \phi - \frac{t}{R} \qquad \tilde{\psi} = \psi + \frac{y}{R} \tag{3.21}$$

with these new variables the time independent configurations that we had in the Ramond sector can become time dependent if they depended on ϕ. For example, the ϕ independent ring solution (3.7) becomes the time independent $AdS_3 \times S^3$ vacuum. Ramond sector solutions where the ring is deformed to other shapes, like an ellipse, for example, become time dependent when they are viewed as NS sector solutions. This is related to the fact that chiral primaries carry non-zero energy in the NS sector. Under spectral flow all RR vacua correspond to chiral primary states with

$$\begin{aligned} J^{NS}_{L,R} =& J^{R}_{L,R} - \frac{k}{2} \\ L^{NS}_0 =& |J^{NS}_R| \ , \qquad \bar{L}^{NS}_0 = |J^{NS}_L| \ . \end{aligned} \tag{3.22}$$

The physical properties of the solution in the interior of the space do not change when we do spectral flow, since it is just a redefinition of what we mean by energy and angular momentum[15]. Note, in particular, that the statement that the $M = 0$

[15] Note, in particular, that spectral flow acting on the NS vacuum *does not* produce conical singularities as was asserted in [67][53] . A more detailed discussion about the action of spectral flow can be found in [62][1].

BTZ black hole is the Ramond ground state is imprecise. There are many Ramond ground states and they look quite different in the supergravity description depending on their angular momenta.

As discussed in [26][32][43] chiral primary states close to the NS ground state can be obtained by adding perturbative gravity modes on the NS ground state. They are particular gravity modes that are BPS. First we restrict to deformations of the ring into a more general shape in the 12 plane. There are two classes of deformations we can consider. One is a change in the shape of the ring and the other is small changes in the velocity with which we go around the ring. These two correspond to two towers of chiral primary states.

The supergravity chiral primary states are given as follows [26][32][43]. It is convenient to separate the three form field strengths in six dimensions into self dual and anti-self dual parts. The background fields in $AdS_3 \times S^3$ are self dual. The chiral primary fields correspond to fluctuations in the anti-self dual part of the three form field strengths on the S^3 (which also mix with fluctuations of scalar fields). These fields produce gravity modes with $(J_L, J_R) = 1/2, 1, 3/2, \cdots$. There is also one special tower of supergravity fields which starts at $(J_L, J_R) = 1, 3/2, \cdots$. These come from certain fluctuations in the metric of the three sphere.

The two classes of deformations of the ring that we discussed above correspond to two of these towers of chiral primaries. More precisely, changes in the velocity correspond to the tower associated to the anti-self dual component of the field strength whose self dual component is turned on in the background. The changes in shape correspond to the tower associated with deformations of the sphere. This can be seen by noticing that the lowest angular momentum deformation of the shape we can do is to deform the circle to an ellipse. This has angular momentum $J_\phi = 2$ which corresponds to $J_L = J_R = 1$. On the other hand we can change the velocity by $v \to v + \epsilon \cos v$ which will introduce a mode with angular momentum $J_\phi = 1$

which corresponds to $(J_L, J_R) = (1/2, 1/2)$. In summary, different Fourier modes in the expansion of $F_{1,2}$ around the circular profile are in direct correspondence with chiral primary gravity modes with different values of angular momentum on S^3 (i.e. different values of J_ϕ).

There is another chiral primary tower with $(J_L, J_R) = (m+1, m), \quad m = 0, 1/2, \ldots$ and a similar one with $J_L \leftrightarrow J_R$. These corresponds to oscillations of the ring into the 34 plane.

Finally we should consider many other chiral primaries that come from the anti-self dual components of other field strengths. These are easily described in the T^4 theory as oscillations in the internal T^4 directions of the initial fundamental string which we used to construct the solutions. We give the general solutions for those in appendix B. One nice aspect of those solutions is that we can easily find some solutions for which the metric is ϕ independent. For example, choosing a simple profile where the string is also oscillating with frequency $m\omega$ in the internal torus, we obtain in the near-horizon-limit the six dimensional metric (see appendix B)

$$\begin{aligned} \frac{ds^2}{\sqrt{Q_1 Q_5}} &= (r^2 + \beta\cos^2\theta)\frac{1}{\sqrt{\alpha}}\left[-(dt - \frac{\beta\sin^2\theta d\phi}{r^2+\beta\cos^2\theta})^2 + (d\chi + \frac{\beta\cos^2\theta d\psi}{r^2+\beta\cos^2\theta})^2\right] \\ &+ \frac{\sqrt{\alpha}dr^2}{r^2+\beta} + \sqrt{\alpha}d\theta^2 + \frac{\sqrt{\alpha}}{r^2+\beta\cos^2\theta}(r^2\cos^2\theta d\psi^2 + (r^2+\beta)\sin^2\theta d\phi^2) \end{aligned} \tag{3.23}$$

where the function α is given by

$$\alpha = 1 - (1-\beta)\left(\frac{\beta\sin^2\theta}{r^2+\beta}\right)^m \tag{3.24}$$

and β and m are two parameters characterizing the solution. m is the angular momentum of the single particle chiral primary we are exciting with $(J_L, J_R) = (\frac{m}{2}, \frac{m}{2})$. In other words we are considering a coherent state associated to this single particle chiral primary. The parameter $0 \leq \beta \leq 1$ measures the total angular momentum of the solution which is

$$J_L^{NS} = J_R^{NS} = \frac{n_1 n_5}{2}(1-\beta) \tag{3.25}$$

So that for $\beta = 1$ we get global $AdS_3 \times S^3$ and for $\beta = 0$ we get the singular geometry corresponding to the $M = 0$ BTZ black hole which from the NS point of view could be described as the extremal limit of a black hole that is rotating in the internal S^3. For $\beta > 0$ the geometry is non-singular. We can ask if the metric (3.23) goes over to the metric of a conical defect. Since the angular momentum is given by (3.25) we would expect that the opening angle of the corresponding conical defect should be $2\pi\beta$, or $\gamma = \beta$ in the notation of appendix C. These conical metrics were considered in [53,1,62]. It turns out that the conical metric is not a good approximation to the long distance behavior of (3.23) since (3.23) contains terms that decay slowly at $r \to \infty$. In fact there is a massless field in AdS with conformal weight two ($\Delta = \bar{\Delta} = 1$) which has a vev, this implies that the metric (3.23) differs from the AdS metric by terms of order $1/r^2$. That is precisely the order of the difference between the conical metrics and the AdS metric. This is discussed in more detail in appendix B.

It is interesting to consider the limit of very large m with β fixed. In that limit we can set $\alpha = 1$ as long as we are at a distance of order $R_{AdS}/\sqrt{m}$ from the line at $r = 0,\ \theta = \pi/2$. The limiting metric is obtained from (3.23) by setting $\alpha = 1$. In this limit the solution has singularity along the circle $r = 0, \theta = \frac{\pi}{2}$ like the one present in the Aichelburg-Sexl metric [68]. The metric coincides with the solution in [69], which was expected to describe the metric of high momentum particles moving along a maximum circle of S^3.

Another interesting limit that we can take is a "plane wave" limit where we concentrate on distances which are small compared to the AdS radius. In this limit, and in the region where $\alpha = 1$, the metric is

$$
\begin{aligned}
ds^2 &= 2dx^+dx^- - (s^2+u^2)(dx^+)^2 + ds^2 + du^2 + u^2 d\tilde{\psi}^2 + s^2 d\chi^2 \\
&+ (\beta - 1)\left[2dx^+dx^- - (s^2+u^2)(dx^+)^2 - \frac{(dx^-)^2}{s^2+u^2}\right] \\
&+ \frac{\beta-1}{s^2+u^2}\left[u^4 d\tilde{\psi}^2 - 2s^2u^2 d\chi d\tilde{\psi} + s^4 d\chi^2\right]
\end{aligned}
\tag{3.26}
$$

where $x^+ = t$, $x^- = \phi R^2_{AdS}$ and $\tilde{\psi} = \psi + \chi$. This metric is singular at $s = u = 0$, but close to this point it is also necessary to take into account the full form for α which we give in appendix B. The final metric is non-singular and is explicitly written in appendix B. We see that the behavior near $r = s = 0$ is of the form we expect for a metric which is carrying momentum density $p_- \sim (\beta - 1)$ in the ϕ or x^- direction.

An interesting aspect of (3.26) is that the metric does not asymptote to the plane wave metric at large $u^2 + s^2$. This is due to the fact that from the plane wave point of view we have constant p_- *density* and therefore infinite total p_-. We show in section 4 that solutions with excitations localized in the x^- direction which carry finite total p_- are indeed asymptotic to the standard plane wave.

An interesting question we would like to understand is the behavior of the energy gap in these geometries. If we concentrate on excitations that are ϕ and ψ independent then the energy gap can be computed easily in the case of $m = 1$ where we obtain (see appendix B)

$$\omega_0 = 2\sqrt{\beta} \tag{3.27}$$

for the energy of the lowest energy excitation. The energy gap for large m is harder to estimate but we prove in appendix B that it is always lower than (3.27).

We see that as we increase the energy of the solution (by decreasing β, see (3.25)) the redshift factor at the origin decreases, so that a clock runs more slowly there and also the energy gap to the next excitation is very low.

3.5. Remarks on the CFT Description

A semi-quantitative explanation of this fact was given in [65][69]. The idea is that these chiral primaries will involve multiply wound strings. The energy gap for exciting such states becomes smaller as $1/w$ where w is the winding. This decrease of the energy gap can be seen even when the CFT is at its orbifold point, where the theory becomes a free CFT whose target space is $Sym(T^4)^{n_1 n_5}$ (for a full discussion of this system see [54]). The NS ground state is in the untwisted sector and can be interpreted as consisting of $n_1 n_5$ singly wound strings. The energy gap to the next BPS excitation is of order one (we normalized the circle of the CFT to have radius one). High angular momentum single particle chiral primaries involve strings that are multiply wound. Non-BPS excitations on a string of winding number w go as $2/w$. In order to obtain a more precise match with the particular energy gaps we obtained above it seems that we need to go away from the orbifold point otherwise we can easily run into contradictions. For example, let us consider the chiral primaries with $(J_L, J_R) = (1/2, 1/2)$ that come from the internal torus. These would naively correspond, in the free orbifold picture, to states in the untwisted sector (singly wound strings) where we have some excitations on some of the $n_1 n_5$ singly wound strings. More precisely, each single particle excitation corresponds to exciting one string by adding a left and right moving fermion in the lowest state. Since all strings are singly wound we get an energy gap of order one at the orbifold point independently of β. On the other hand, the gravity description of such states is given by (3.23) with $m = 1$, for which the energy gap is (3.27). Clearly we need to take into account that the supergravity picture is valid only when we get away from the orbifold point in the CFT, which blurs the distinction between singly wound and multiply wound strings. It would be highly desirable to understand better this effect from the CFT point of view. In [65] some agreement was found with this naive picture, since only very special geometries and chiral primaries were used.

From the general solutions in (3.1) it is also possible to obtain a general family of solutions with plane wave asymptotic boundary conditions. The final prescription is that in order to obtain such solutions we should drop the ones in (3.2) and consider a profile $\mathbf{F}(v)$ which is a straight line in R^4 with small wiggles in the various R^4 and T^4 coordinates. This is rather analogous to our previous discussion where we took a profile that was a circle with some wiggles.

First we discuss how to obtain this as a limit of the general metric (3.1) corresponding to a circular profile with some small oscillations. It is a limit where we zoom in into a small section of the circle where this section looks like a straight line with some oscillations. More precisely, we rescale the coordinates and the profile:

$$x_1 = a + \frac{x_1'}{R}, \quad x_2 = \frac{x_2'}{R}, \quad x_3 = \frac{x_3'}{R}, \quad x_4 = \frac{x_4'}{R}, \quad t = t'R, \quad y = \chi R, \quad R = \frac{\hat{R}}{\epsilon},$$
$$F_1 = a\cos\omega v + \frac{\hat{F}_1(vR)}{R}, \quad F_2 = a\sin\omega v + \frac{\hat{F}_2(vR)}{R}, \quad F_3 = \frac{\hat{F}_3(vR)}{R}, \quad F_4 = \frac{\hat{F}_4(vR)}{R} \tag{3.28}$$

and we define the new rescaled functions

$$\hat{f}_1 = \epsilon^2 f_1, \quad \hat{f}_5 = \epsilon^2 \hat{R}^4 f_5, \quad \hat{A}_i = \frac{1}{R^2} A_i, \quad \hat{B}_i = \frac{1}{R^2} B_i,$$

Then we take a limit $\epsilon \to 0$ while the new coordinates and $\hat{R}$ remain fixed.

Defining the new parameter $\sigma = a\omega\hat{R}^2$ and dropping the primes in the new coordinates we find

$$\begin{aligned}
\hat{f}_5 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dv'}{(x_2 - \hat{F}_2 - \sigma v')^2 + (x_\perp - \hat{F}_\perp)^2} \\
\hat{f}_1 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\left[(\sigma + \dot{\hat{F}}_2)^2 + |\dot{\hat{F}}_\perp|^2\right] dv'}{(x_2 - \hat{F}_2 - \sigma v')^2 + (x_\perp - \hat{F}_\perp)^2} \\
\hat{A}_2 &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(\sigma + \dot{\hat{F}}_2) dv'}{(x_2 - \hat{F}_2 - \sigma v')^2 + (x_\perp - \hat{F}_\perp)^2}, \\
\hat{A}_\perp &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\dot{\hat{F}}_\perp dv'}{(x_2 - \hat{F}_2 - \sigma v')^2 + (x_\perp - \hat{F}_\perp)^2}
\end{aligned} \tag{3.29}$$

Here we introduced a new integration parameter $v' = vR$. Note that in terms of this parameter the argument of $\hat{F}_i$ is ϵ–independent. In the derivation of (3.29) we also used the relation (3.5)

$$\frac{Q}{L} = \frac{R}{2\pi}$$

Note that the functions $\hat{F}^i(v')$ are not required to be periodic, they are arbitrary functions of v'. In terms of the functions (3.29) the metric becomes

$$ds^2 = \frac{1}{\sqrt{\hat{f}_1 \hat{f}_5}} \left[-(dt - \hat{A}_i dx^i)^2 + (dy + \hat{B}_i dx^i)^2\right] + \sqrt{\hat{f}_1 \hat{f}_5} dx^i dx^i \tag{3.30}$$

where $\hat{B}$ is defined by

$$d\hat{B} = -{}^* d\hat{A}$$

Let us examine the asymptotic behavior of this metric at large $x_\perp$. In this limit ($x_\perp \gg |\hat{F}|$) we get the following leading contributions to the harmonic functions[16]

$$\hat{f}_5 \approx \frac{1}{2\sigma x_\perp}, \quad \hat{f}_1 \approx \frac{1}{\beta}\frac{\sigma}{2x_\perp}, \quad \tilde{A}_2 \approx -\frac{1}{2x_\perp}, \quad \hat{B}_\psi \approx -\frac{(\cos\theta - 1)}{2}, \tag{3.31}$$

all other components of the gauge fields $\hat{A}$ and $\hat{B}$ are subleading. In (3.31) we introduced a parameter $\beta \leq 1$:

$$\beta \equiv \left(1 + \frac{1}{\sigma^2}\langle(\dot{\hat{F}}_2)^2 + |\dot{\hat{F}}_\perp|^2\rangle\right)^{-1} \tag{3.32}$$

Substituting the expressions (3.31) in (3.30) and introducing the new coordinates:

$$\begin{aligned} x^+ &= t\ , \quad x^- = \frac{x_2}{\sqrt{\beta}}, \quad \tilde{\psi} = \psi + \chi \\ u &= \beta^{-1/4}\sqrt{2x_\perp}\sin\frac{\theta}{2}, \quad s = \beta^{-1/4}\sqrt{2x_\perp}\cos\frac{\theta}{2}, \end{aligned} \tag{3.33}$$

we find

$$\begin{aligned} ds^2 = &-\beta\left[2dx^+dx^- + (u^2+s^2)(dx^+)^2\right] + (1-\beta)\frac{(dx^-)^2}{u^2+s^2} \\ &+ du^2 + ds^2 + \frac{u^2 s^2}{u^2+s^2}(d\tilde{\psi} - d\chi)^2 + \frac{\beta}{u^2+s^2}(s^2 d\chi + u^2 d\tilde{\psi})^2 \end{aligned} \tag{3.34}$$

[16] Here we assumed that $\langle\dot{\hat{F}}_i\rangle = 0$. If this condition is not true, then string moves along the direction $\langle\dot{\hat{F}}_i\rangle$ and we can account for this motion by redefining coordinate x_2.

which is indeed the same as (3.26) after some simple changes of signs. This is the general behavior of the metric for a configurations with uniform momentum density in the x^- direction. If the excitation is localized in the x^- direction, as we expect it to be for a finite p_- wavepacket, then the profile for the corresponding vibration will be such that $|\dot{\hat{F}}|$ differs significantly from zero only in a finite range of v'. Then the averages entering the expression (3.32) vanish and $\beta = 1$. So in this case the metric asymptotically goes to the usual plane wave.[17]

So far we have been looking at profiles which oscillate only in the noncompact directions. For the case of oscillations on the torus the six dimensional Einstein metric is still given by (3.30), but the function $\hat{f}_1$ should be replaced by $\bar{f}_1$

$$\begin{aligned} \bar{f}_1 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\left[(\sigma + \dot{\hat{F}}_2)^2 + |\dot{\hat{F}}_\perp|^2 + |\dot{\hat{\mathcal{F}}}|^2\right] dv'}{(x_2 - \hat{F}_2 - \sigma v')^2 + (x_\perp - \hat{F}_\perp)^2} - \hat{f}_5^{-1}\hat{\mathcal{A}}_a\hat{\mathcal{A}}_a, \\ \hat{\mathcal{A}}_a &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\dot{\hat{\mathcal{F}}}_a dv'}{(x_2 - \hat{F}_2 - \sigma v')^2 + (x_\perp - \hat{F}_\perp)^2} \end{aligned} \tag{3.35}$$

The large $x_\perp$ limit of the resulting solution still has the form (3.34), but β is now defined by

$$\beta \equiv \left(1 + \frac{1}{\sigma^2}\langle(\dot{\hat{F}}_2)^2 + |\dot{\hat{F}}_\perp|^2 + |\dot{\hat{\mathcal{F}}}|^2\rangle\right)^{-1}$$

3.7. The Solution for a Localized Excitation

In order to understand the asymptotic behavior of the solution when the excitation is localized in x^-, we write down the explicit form for one such solution with a very simple profile. We pick a profile with a perturbation only in the torus direction.

$$\hat{\mathcal{F}}_1 = \begin{cases} 0, & |v'| > v_0 > 0 \\ b(v_0 - |v'|), & |v'| < v_0 \end{cases}$$

[17] Note, in particular, that the coefficient of the term that goes as $(dx^-)^2/(s^2+u^2)$ goes to zero, while in flat space it goes over to some function of x^-. The difference is due to the rapid decay in the transverse coordinates of wavefunctions with fixed p_-.

with all other components of $\hat{\mathcal{F}}_a$ and $\hat{F}_i$ equal to zero. This profile gives following harmonic functions

$$\begin{aligned}
&\hat{f}_5 = \frac{1}{2\sigma x_\perp}, \quad \tilde{A}_2 = -\frac{1}{2x_\perp}, \quad \hat{B}_\psi = -\frac{(\cos\theta - 1)}{2}, \\
&\mathcal{A}_1 = \frac{b}{2\pi\sigma x_\perp} \arctan\left\{ \frac{2(\sigma v_0)^2 x_\perp x^-}{[x_\perp^2 + (x^-)^2]^2 + (\sigma v_0)^2 (x_\perp^2 - (x^-)^2)} \right\} \\
&\bar{f}_1 = \frac{\sigma}{2x_\perp} + \frac{b^2}{2\pi\sigma x_\perp} \arctan\left(\frac{2\sigma v_0 x_\perp}{x_\perp^2 + (x^-)^2 - (\sigma v_0)^2} \right) - \hat{f}_5^{-1} \hat{\mathcal{A}}_a \hat{\mathcal{A}}_a,
\end{aligned} \tag{3.36}$$

In particular for $\sigma v_0 \ll x_\perp$ and arbitrary value of x^- we get

$$\bar{f}_1 = \frac{\sigma}{2x_\perp} + \frac{b^2}{2\pi\sigma x_\perp} \frac{\sigma v_0}{x_\perp} \left\{ \frac{2x_\perp^2}{(x^-)^2 + x_\perp^2} + O\left(\frac{\sigma v_0}{x_\perp} \right) \right\}$$

Thus in the leading order at large $x_\perp$ we get a usual plane wave [70],[71], i.e. the metric (3.34) with $\beta = 1$.

3.8. The Supertubes in Different Dimensions

The previous analysis of the D1-D5 system is special because the configuration blows up to a Kaluza-Klein monopole and leads to a non-singular situation. This will be the case also for all configurations with two charges that result from doing U-duality on the T^4. Of course, if we did T-duality on the S^1 where the D1 and D5 are wrapped we would get a singular metric since the KK monopole would become an NS 5 brane. The fact that the metric is non-singular is related to the fact that the theory has a non-zero energy gap for generic non-BPS excitations around the state with maximal angular momentum. This energy gap is also non-zero for other two charge systems in different number of dimensions. So we considered similar supergravity solutions in different dimensions but we found that they were all singular. In this section we summarize this discussion. For simplicity we have concentrated on the solutions with maximal angular momentum in a given two plane of the non-compact transverse directions. Another question we consider is the following. We take the large radius limit of the ring and then look at the resulting geometry. It turns out that the following two limits do not commute, the near ring limit for fixed ring radius and the large radius first and then the small distance limit. The physical reason why they do not commute is that when one is approaching the ring one is exploring the IR region of the field theory living on the branes so that one is sensitive to the long distance geometry of the branes.

3.9. Solutions in Different Dimensions

In order to analyze such systems, we start with the F1-P1 system in $R^{1,d}\times S^1\times T^{8-d}$ (with the appropriate powers in the harmonic functions) , integrate the string sources along a ring (as explained in [72]), and then perform some dualities on it to get the desired system. We can write the solutions in different U-dual frames. We choose to work with the D0-F1 system blowing up to a D2 - the supertube of [37].

We start then with the 1/4 supersymmetric supergravity solution describing an oscillating string, wound around the S^1 and carrying right moving momentum [63], [64] in d non-compact transverse dimensions

$$\begin{aligned} ds^2 &= -e^{2\Phi}dudv - (e^{2\Phi}-1)\dot{F}^2dv^2 + 2(e^{2\Phi}-1)\dot{\mathbf{F}}\cdot d\mathbf{x}dv + d\mathbf{x}_d^2 + d\mathbf{z}_{8-d}^2 \\ &\equiv H(\mathbf{x},v)(-dudv + K(\mathbf{x},v)dv^2 + 2A_i(\mathbf{x},v)dx^idv) + d\mathbf{x}_d^2 + d\mathbf{z}_{8-d}^2 \\ B_{uv} &= \frac{1}{2}(e^{2\Phi}-1) \qquad ; \qquad B_{vi} = -\dot{F}_i(e^{2\Phi}-1) = HA_i \\ e^{-2\Phi} &= 1 + \frac{Q}{|\mathbf{x}-\mathbf{F}(v)|^{d-2}} \end{aligned} \tag{3.37}$$

where the light cone coordinates are $u, v = t \pm y$ with $y \sim y + R_y$ and $\mathbf{x}$ are d noncompact directions and $\mathbf{z}$ parameterize a T^{8-d}. $\mathbf{F}(v)$ is a d-dimensional vector describing the location of the string.

Taking the ring profile: $(F_1^\alpha + iF_2^\alpha)(v) = ae^{i(\omega v+\alpha)}$, $F_3^\alpha = F_4^\alpha = 0$, and integrating the harmonic functions along α, we get functions describing oscillating strings uniformly distributed along a ring:

$$\begin{aligned}\langle H^{-1}(\mathbf{x})\rangle &= 1+\frac{Q}{2\pi}\int_0^{2\pi}\frac{d\alpha}{|\mathbf{x}-\mathbf{F}^\alpha|^{d-2}} = 1+\frac{Q}{\sigma^{d-2}}I_1^{(d-2)}(-\frac{2as}{\sigma^2})\\ \langle K(\mathbf{x})\rangle &= \frac{Q}{2\pi}\int_0^{2\pi}d\alpha\frac{|\partial_v\mathbf{F}^\alpha|^2 d\alpha}{|\mathbf{x}-\mathbf{F}^\alpha|^{d-2}} = a^2\omega^2(\langle H^{-1}\rangle -1) = \frac{a^2\omega^2 Q}{\sigma^{d-2}}I_1^{(d-2)}(-\frac{2as}{\sigma^2})\\ \langle A_\phi(\mathbf{x})\rangle &= as\omega\frac{Q}{2\pi}\int_0^{2\pi}\frac{\partial_v F_\phi d\alpha}{|\mathbf{x}-\mathbf{F}^\alpha|^{d-2}} = \frac{a\omega Qs}{\sigma^{d-2}}I_2^{(d-2)}(-\frac{2as}{\sigma^2})\end{aligned} \tag{3.38}$$

where $s^2 \equiv x_1^2+x_2^2$ is the radial coordinate in the ring plane, $w^2 \equiv x_3^2+\cdots+x_d^2$ is the perpendicular distance from the ring plane, $\sigma^2 \equiv a^2+s^2+w^2$, and where we defined the integrals:

$$I_1^{(n)}(k) \equiv \frac{1}{2\pi}\int_0^{2\pi}\frac{d\alpha}{(1+k\cos\alpha)^{n/2}} \quad ; \quad I_2^{(n)}(k) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos\alpha d\alpha}{(1+k\cos\alpha)^{n/2}} \tag{3.39}$$

for integer n's. The integrals above can be easily evaluated, and appear in appendix D. For odd n they involve elliptic functions.

Now, if we perform S-duality and then T_y duality on (3.37), we get the $F1D0 \to D2$ supertube solution for any dimension d, with H, K, A_ϕ, I_1, I_2 as in (3.38), (3.39)

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{H^{-1}}(1+K)}\Big[-dt^2+2A_\phi dt d\phi + H^{-1}dy^2+\\ &\quad + s^2(\frac{a\omega Q}{\sigma^{d-2}})^2\{((I_1^{(d-2)})^2-(I_2^{(d-2)})^2)+\frac{1+a^2\omega^2}{a^2\omega^2 Q}\sigma^{d-2}I_1+(\frac{\sigma^{d-2}}{a\omega Q})^2\}d\phi^2\Big]+\\ &\quad +\sqrt{H^{-1}}\Big(ds^2+d\mathbf{w}_{d-2}^2+d\mathbf{z}_{8-d}^2\Big)\\ e^{-2\Phi} &= H^{3/2}(1+K)\\ B_2 &= -\frac{K}{1+K}dt\wedge dy - \frac{A_\phi}{1+K}d\phi\wedge dy\\ C_1 &= -(H-1)dt + HA_\phi d\phi \quad ; \quad C_3 = -\frac{A_\phi}{1+K}dt\wedge d\phi\wedge dy \end{aligned} \tag{3.40}$$

where in the second line we wrote $g_{\phi\phi} = \frac{1}{\sqrt{H^{-1}}(1+K)}(s^2H^{-1}(1+K)-A_\phi^2)$ in terms of the integrals I_1, I_2. [18]

[18] Note that $I_1^{(d-2)}, I_2^{(d-2)}$ in (3.40) are evaluated at $-\frac{2as}{\sigma^2}$.

Analyzing the behavior of the different functions in the metric above, one finds that these solutions are everywhere regular, except maybe on a ring of radius a in the x-space ($w = 0, s = a$). In appendices C and D we show that these solutions are indeed singular on the ring in all dimensions except the for the D1-D5 system described above. We did not find any U-dual frame where the solutions were regular, except for $d = 4$ which is U-dual to the D1-D5 system [19].

3.10. The Large Ring Limit

In this subsection we take the limit where the radius of the ring becomes very large. This can be achieved by taking very large values of the charges. From the formula of the ring radius (3.8) we see that in this limit we expect to have a finite energy density per unit length along the ring. In this limit the ring becomes a straight line. We want to find the metric near the ring in this case. This metric has the form of the metric of a brane with some fluxes on it. In the D1-D5 it will be a KK monopole with some fluxes on it. These fluxes have a special value such that the supersymmetry that is preserved is independent of the orientation of the brane. Below we explain this in detail.

We can try take the limit where $a \to \infty$ in (3.40), and see if the solutions we obtain really describe a flat $D2$ with fluxes.

From a worldvolume analysis [37] one finds that the radius of the supertube scales with the D0,F1 charges as $a \sim \sqrt{Q_0 Q_s}$ where the two charges in our notations are $Q, a^2\omega^2 Q$. Keeping the ratio of the charges fixed, the scaling is

$$a \sim Q \to \infty \quad ; \quad \delta \equiv a\omega \ fixed \tag{3.41}$$

Taking this limit for a fixed ρ in (3.40) and defining the coordinate $x_\parallel \equiv a\phi$

[19] Recently the $d = 3$ case was analyzed in [73].

gives the following metrics and fields for $d \geq 4$:

$$
\begin{aligned}
ds^2 &= \frac{[1+\frac{\tilde{q}}{\rho^{d-3}}]^{1/2}}{[1+\delta^2\frac{\tilde{q}}{\rho^{d-3}}]}\{dy^2 + \frac{1+\frac{\tilde{q}}{\rho^{d-3}}(1+\delta^2)}{1+\frac{\tilde{q}}{\rho^{d-3}}}dx_\parallel^2 + \frac{2\delta\frac{\tilde{q}}{\rho^{d-3}}}{1+\frac{\tilde{q}}{\rho^{d-3}}}dx_\parallel dt - \frac{dt^2}{1+\frac{\tilde{q}}{\rho^{d-3}}}\}+ \\
&\quad + [1+\frac{\tilde{q}}{\rho^{d-3}}]^{1/2}[d\rho^2 + \rho^2 d\Omega_{d-2}^2 + dz_{18-d}^2] \\
B_2 &= -\frac{\delta\frac{\tilde{q}}{\rho^{d-3}}}{1+\delta^2\frac{\tilde{q}}{\rho^{d-3}}}(\delta dt\wedge dy + dx_\parallel \wedge dy) \\
C_1 &= \frac{\frac{\tilde{q}}{\rho^{d-3}}}{1+\frac{\tilde{q}}{\rho^{d-3}}}(dt+\delta dx_\parallel) \quad ; \quad C_3 = -\frac{\delta\frac{\tilde{q}}{\rho^{d-3}}}{1+\delta^2\frac{\tilde{q}}{\rho^{d-3}}}dt\wedge dx_\parallel \wedge dy \\
e^{-2\phi} &= (1+\frac{\tilde{q}}{\rho^{d-3}})^{-3/2}(1+\delta^2\frac{\tilde{q}}{\rho^{d-3}})
\end{aligned}
\tag{3.42}
$$

where the effective charge $\tilde{q}$ is given by

$$
\tilde{q} = \frac{Q}{a}\cdot \lim_{\rho\to 0}[(\frac{\rho}{a})^{d-3} I_1^{(d-2)}(1-\frac{\frac{\rho^2}{2a^2}}{1+\frac{\rho}{a}\sin\Theta+\frac{\rho^2}{2a^2}})]
$$

which for the different dimensions is :

$$
\begin{array}{cccccc}
d = & 4 & 5 & 6 & 7 & 8 \\
\tilde{q} = & \frac{Q}{2a} & \frac{Q}{\pi a} & \frac{Q}{4a} & \frac{2Q}{3\pi a} & \frac{3Q}{16a}
\end{array}
\tag{3.43}
$$

For $d = 3$ we get a logarithmic singularity .

We would like to compare (3.42) with the metric and fields describing a D2-brane with F1 and D0 fluxes on a T^{8-d}. These can be generated by starting with the supergravity solution of a $D2$ in the $\tilde{t}, \tilde{y}, \tilde{x_p}$ directions [20]. Then T-dualizing in $\tilde{x_p}$ to obtain a D1 in the $\tilde{y}, \tilde{t}$ directions, smeared on the $\tilde{x}_p$ direction (with a harmonic function $f = 1 + \frac{q}{r^{d-3}}$) .Then making a boost and a rotation with parameters α, θ mixing $\tilde{t}, \tilde{y}, \tilde{x}_p$ to give t, y, x_p [21], and finally making a T-duality in the y-direction

[20] We choose a gauge where the Ramond-Ramond Gauge field vanishes at spatial infinity.

[21] so that

$$
\begin{aligned}
\tilde{t} &= \cosh\alpha t + \sinh\alpha(\cos\theta x_p + \sin\theta y) \\
\tilde{x_p} &= \cosh\alpha(\cos\theta x_p + \sin\theta y) + \sinh\alpha t \\
\tilde{y} &= (\cos\theta y - \sin\theta x_p)
\end{aligned}
$$

[22] .

This gives the following metric and gauge fields :

$$
\begin{aligned}
ds^2 &= f^{-1/2}[-(1-h^{-1}\frac{q\sinh^2\alpha}{r^{d-3}})dt^2 + (1+h^{-1}\frac{q\cosh^2\alpha\cos^2\theta}{r^{d-3}})dx_p^2 + \\
&\quad + 2h^{-1}\frac{q\sinh\alpha\cosh\alpha\cos\theta}{r^{d-3}}dtdx_p + fh^{-1}dy^2] + f^{1/2}[dr^2 + r^2 d\Omega_{d-2}^2 + dz_{8-d}^2] \\
B_2 &= -h^{-1}\frac{q\sin\theta\cosh\alpha}{r^{d-3}}[\sinh\alpha dy\wedge dt + \cosh\alpha\cos\theta dy\wedge dx_p] \\
C_1 &= (f^{-1}-1)[\cos\theta\cosh\alpha dt + \sinh\alpha dx_p] \quad ; \quad C_3 = h^{-1}\frac{q\sin\theta\cosh\alpha}{r^{d-3}}dt\wedge dx_p\wedge dy \\
e^{2\phi} &= g^2 f^{3/2}h^{-1} \\
f &\equiv 1+\frac{q}{r^{d-3}} \quad ; \quad h\equiv 1+\frac{q\cosh^2\alpha\sin^2\theta}{r^{d-3}}
\end{aligned}
\tag{3.44}
$$

Comparing (3.44) to (3.42) we find exact agreement if we choose $\sinh\alpha = \tan\theta = \delta$. All of the solutions (3.44) are 1/2 supersymmetric as they are dual to a D2. However only the subfamily of such solutions with $\sinh\alpha = \tan\theta$ would continue being supersymmetric (with 1/4 supersymmetry) if we start curving the brane, taking the direction x_p and putting it on some closed curve, e.g. the ring. (The exact supersymmetries that this curvely shaped D2 with fluxes preserves can be found doing a worldvolume analysis , as done in [37] , or as done for the $D2\overline{D2}$ system in [75][76]). Under a T-duality in the S^1 circle this system becomes a D1 brane that winds along the S^1, and moves along the S^1 as it stretches in the x_p direction. The velocity is such that a brane that is stretched in the opposite direction along x_p but with the same winding and velocity intersects the original brane at a point that moves with the speed of light [77]. These configurations preserve 1/4 of the supersymmetries. These configurations are intimately related to the oscillating strings we started with. In fact strings carrying oscillations only in one direction will intersect with each other at points that move at the speed of light.

In the $d = 4$ case we can make a U-duality to the D1-D5 system so that the

[22] this procedure was explained for example in [74]

large ring radius (3.42) becomes a straight KK monopole carrying D1 and D5 fluxes

$$\begin{aligned} ds^2 &= [1+\frac{q_1}{\rho}]^{-1/2}[1+\frac{q_5}{\rho}]^{-1/2}[-(dt-\frac{\sqrt{q_1q_5}}{\rho}dx_{\|})^2+(dy-\sqrt{q_1q_5}(1-\cos\Theta)d\psi)^2] \\ &\quad +[1+\frac{q_1}{\rho}]^{1/2}[1+\frac{q_5}{\rho}]^{1/2}[d\rho^2+\rho^2d\Theta^2+\rho^2\sin^2\Theta d\psi^2+dx_{\|}^2]+\sqrt{\frac{\rho+q_1}{\rho+q_5}}dz_{(4)}^2 \\ e^{2\phi} &= \frac{\rho+q_1}{\rho+q_5} \\ C_2 &= -\frac{q_1}{\rho+q_1}(dt+\sqrt{\frac{q_5}{q_1}}dx_{\|})\wedge(dy+\sqrt{\frac{q_5}{q_1}}\rho(1-\cos\Theta)d\psi) \end{aligned} \tag{3.45}$$

where we have defined the charge densities $q_i = Q_i/(2a)$ which are finite in the limit. This metric is non-singular if $R_y = 2\sqrt{q_1q_5}$. This is a condition on the fluxes for a given radius R_y. If we U-dualize (3.44) we can get solutions which represent KK monopole with arbitrary values of the fluxes that are 1/2 BPS. What is special about the fluxes in (3.45) is that we can reverse the KK monopole charge, keeping the same values for q_1, q_5 so that the configuration with KK and anti-KK charges still preserve 1/4 of the supersymmetries. As shown in [77] this configuration is U-dual to configurations with intersecting D-branes where the intersection point moves at the speed of light (see also [78]).

Note that in the limit that we drop the 1 in the harmonic functions that appear in (3.45) we obtain a plane wave in six dimensions. We can get this as a limit where we scale the charges to infinity and the rest of the coordinates appropriately. The geometry (3.45) thus provides us with a spacetime which is asymptotically flat and that looks like a plane wave in a suitable near horizon limit.

Part II

Strings on PP-Waves and Massive Two Dimensional Field Theories Introduction

4. Introduction

Ramond-Ramond backgrounds are a very important piece of string theory and they play a prominent role in the string theory/gauge theory correspondence. Backgrounds of the plane wave type are particularly interesting since they are exactly solvable backgrounds [79]. These backgrounds are very useful for studying the relation between large N gauge theory and string theory [71]. The existence of a covariantly constant null Killing vector greatly simplifies the quantization of a string in light cone gauge [80].

Naturally it would be extremely interesting if one could find more Ramond-Ramond backgrounds, for which the superstring worldsheet action would be simple. In this part of the thesis we study backgrounds of the pp-wave type which lead to *interacting* theories in light cone gauge. For this purpose we consider type IIB string theory with a five-form field strength which has the form $F_5 = dx^+ \wedge \varphi_4$. If φ_4 is a constant form in the transverse space it leads to masses for the Green-Schwarz light cone fermions. By taking non-constant four forms φ_4 we find that the light cone action becomes an interacting theory with a rather general potential. The mass scale in the light cone theory is set by p_-. Boosts in the x^+, x^- directions corresponds to an RG flow transformation on the worldsheet. Low values of $|p_-|$ correspond to the UV of the worldsheet theory while large values of $|p_-|$ explore the IR of the worldsheet theory. We study solutions that preserve some supersymmetries. We find that we can have an $N = (2, 2)$ theory on the worldsheet with an arbitrary superpotential. Similarly we can get $N = (1, 1)$ theories as long as the real superpotential is a harmonic function. We discuss solutions where the transverse space is curved or flat. One interesting result is that we can find backgrounds that lead to integrable models on the worldsheet in light cone gauge. Using results for integrable models we can compute some non-trivial features of the string spectrum. We can consider for example Toda theories. We discuss explicitly the case where we get the $N = 2$ sine Gordon model on the worldsheet. Soliton solutions of the massive theory correspond to strings that interpolate between different "potential wells" in the target space. Now that we have massive interacting theories on the worldsheet we see that various dualities of these theories are worldsheet dualities which lead to interesting dualities in the target space. The $N = 2$ sine Gordon theory is dual to the supersymmetric CP^1 theory [81,82,83,84,85], via a mirror symmetry transformation. The size of the CP^1 depends on the energy scale of the worldsheet theory. The size of the worldsheet circle is proportional to p_-. Thus, we find that strings with very small p_- feel they are on a big space while strings with large p_- feel they are on a smaller space.

Other backgrounds that lead to interacting theories in lightcone gauge were described in [86,87].

In chapter 5 we discuss the gravity backgrounds that lead to supersymmetric interacting theories on the worldsheet. In chapter 6 we describe the actions we get on the worldsheet from the backgrounds discussed in chapter 5. We then discuss in more detail some particular backgrounds. First we discuss the background leading to the $N = 2$ sine Gordon model on the worldsheet and the associated duality to the CP^1 model. We then discuss what happens if we have an A_N singularity transverse to a pp-wave and we resolve it.

Before we embark on this path, let us review a few concepts we would use later. In the following subsections we would first review plane-wave and more general pp-wave backgrounds, then we would survey the different formalisms devised for describing the superstring worldsheet action, focusing on the Green-Schwarz formalism in light-cone gauge. Finally we would review a few points regarding Landau-Ginzburg models and mirror symmetry, that would become relevant when we discuss the worldsheet actions in pp-wave backgrounds.

4.1. PP-Wave Backgrounds

Plane fronted parallel-ray waves , or *pp-waves*, are spacetimes which admit a covariantly constant null Killing vector. One can always choose coordinates where the Killing vector is $\xi_\mu = \partial_\mu(x^+)$, and write the metric in the Brinkmann form :

$$ds^2 = -2dx^+dx^- + g_{++}(x^+, x^k)(dx^+)^2 + A_{+i}(x^+, x^k)dx^+dx^i + g_{ij}(x^+, x^k)dx^i dx^j \tag{4.1}$$

In the following we shall only be working with spacetimes where $A_{+i} = 0$.

In the particular case where $g_{++}(x^+, x^k) = -\mu_{ij}(x^+)x^i x^j$ and $g_{ij} = \delta_{ij}$, these are also called *plane waves*.

We focus first on the case where $g_{ij} = \delta_{ij}$. One can see that for the metrics (4.1), all curvature invariants vanish [23], yet they differ from flat space in their global and causal structures, as we would discuss later.

For $\partial^i\partial_i g_{++} = 0$, these are naturally solutions of pure gravity [24]. However, as *all* curvature invariants vanish, it can be argued that these must be exact solutions of perturbative string theory, that are not corrected in higher orders of α' [80].

One can also add NS-NS and R-R fields to this background. As long as their energy momentum tensor satisfies $T_{+i} = T_{ij} = 0$, and $T_{++} = R_{++}$ (which now

[23] The only nonvanishing component of the Riemann curvature tensor when $g_{ij} = \delta_{ij}$ is $R_{+i+j} \sim \partial_i\partial_j g_{++}$, and therefore of the Ricci tensor $R_{++} \sim \partial^i\partial_i g_{++}$.

[24] and if one wants to consider the more general spaces where g_{ij} is not necessarily δ_{ij} then the condition becomes that that g_{ij} is Ricci flat and that $\nabla^2 g_{++} = 0$, the laplacian taken with respect to the transverse metric.

replaces $\partial^i\partial_i g_{++} = 0$), and that they have a zero Lie derivative by the same null Killing vector ξ_μ, these are still exact solutions of perturbative string theory, by the same considerations. [25]

The geodesic motion equations on theses spaces are :

$$\begin{aligned} \dot{x}^+ = p^+ \quad ; \quad \ddot{x}^- = p^+\dot{g}_{++} - \frac{1}{2}(p^+)^2\partial_+ g_{++} \\ \ddot{x}^i = (p^+)^2\partial_i g_{++} \\ -2p^+\dot{x}^- + (p^+)^2 g_{++} + \sum_i (\dot{x}^i)^2 = \epsilon \end{aligned} \tag{4.2}$$

where dot denotes derivative by the affine parameter, and ϵ is a negative, zero or positive constant for time-like, null and spacelike geodesics respectively. Thus light-cone momentum $p^+ = -p_-$ is conserved, and the transverse components feel a tidal force $p_-^2\partial_i g_{++}$. For plane waves this force is just linear in the displacement, so in that case the transverse coordinates perform harmonic oscillations or an exponential motion (depending on the sign of the eigenvalues of μ_{ij}).

Investigating the causal structure of pp-wave spacetimes, it was shown that they cannot admit event horizons, as every point in them is causally connected to infinity [89]. In [89]also the geodesically completeness of plane-wave and pp-wave backgrounds was studied, as well as their boundary structure.

Note that under rescalings of the form $x^+ \to \lambda x^+$, $x^- \to \lambda^{-1}x^-$, g_{++} and p_- both rescale, and the combination $p_-^2 g_{++}$ is invariant. This property will have implications later when we analyze the action of strings propagating on such backgrounds.

Plane-waves have some very interesting supersymmetry properties. It turns out that by choosing different constant matrices μ_{ij} one can produce solutions with

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[25] This situation changes when one introduces a curved metric in the transverse space ($g_{ij} \neq \delta_{ij}$), as now there are nonzero components with transverse index in the curvature tensor. α' corrections can exist even if the transverse space is a Ricci flat Kahler manifold [88].

a peculiar fractional amount of supersymmetry (a fraction $\frac{1}{2} \leq \nu \leq 1$ [90,91,92,93]). It also turns out that by taking

$$\begin{aligned} ds^2 &= -2dx^+dx^- - 4\mu^2(x^ix^i)(dx^+)^2 + dx^idx^i \\ F_5 &= \mu(1+*)(dx^+ \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4) \end{aligned} \tag{4.3}$$

one gets maximally supersymmetric solutions of type IIB string theory [94]. A similar construction in 11-dimensions :

$$\begin{aligned} ds^2 &= -2dx^+dx^- - \frac{\mu^2}{36}[4\sum_{i=1}^{3}(x^i)^2 + \sum_{i=4}^{9}(x^i)^2](dx^+)^2 + dx^idx^i \\ F_4 &= \mu dx^+ \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{aligned} \tag{4.4}$$

gives a maximally supersymmetric solution of M-theory [95].

This is quite remarkable, as there are only very few solutions of string theory which are maximally supersymmetric. In 11 dimensions these are flat Minkowski space, $AdS_4 \times S^7$, $AdS_7 \times S^4$ and the maximally supersymmetric plane wave (4.4), and in 10 dimensional type IIB string theory, these are flat Minkowski space, $AdS_5 \times S^5$ and the maximally supersymmetric plane wave (4.3). In [96] it was proved that for M-theory and type IIB these are all the maximally supersymmetric solutions, and that for type IIA, type I and heterotic string theories, the maximally supersymmetric solutions are all locally isometric to flat space with no fluxes and constant dilaton.

The IIB maximally supersymmetric plane wave solution (4.3) has the following 32 real Killing spinors (where we use our conventions as explained in appendix F, and we are working with real coordinates x^i , $i = 1, ..8$):

$$\epsilon = e^{i\mu\Gamma_-\Gamma^{1234}x^i\Gamma_i}\Gamma_+\Gamma_-\psi_0 + e^{-2i\mu\Gamma^{1234}x^+}\Gamma_-\Gamma_+\eta_0 \tag{4.5}$$

ψ_0,η_0 are constant spinors. Note the 16 supersymmetries parameterized by the ψ_0 piece are not annihilated by Γ^+, while the 16 parameterized by the η_0 piece are.

The 32 killing spinors (4.5) are associated with 32 supersymmetry charges, which anti-commute to the bosonic symmetry generators of the algebra. These are:

* Twelve $SO(4) \times SO(4)$ rotations J^{ij}, $i,j = 1,2,3,4$ or $i,j = 5,6,7,8$, [26] the killing vectors being $x^i \partial_j - x^j \partial_i$.
* Two translations in the lightcone directions $P_\pm$, with the killing vectors $\partial_\pm$.
* Sixteen more x^+-dependent translations and rotations :$P_i = -2\mu \sin(2\mu x^+) x^i \partial_- + \cos(2\mu x^+)\partial_i$, and $J^{+i} = -\cos(2\mu x^+) x^i \partial_- + \frac{1}{2\mu} \sin(2\mu x^+)\partial_i$.

Having the killing spinors and vectors, the spacetime superalgebra is easy to derive and has been written down in [94].

Actually, there is another way to derive the supersymmetry algebra of the maximally supersymmetric plane wave, thinking of it as a Penrose limit of $AdS_5 \times S^5$. In this limiting procedure, one blows up the metric and fields in the neighborhood of a null geodesic [97][98][99]. Taking the metric of $AdS_5 \times S^5$

$$ds^2 = R^2[-dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + d\psi^2 \cos^2 \theta + d\theta^2 + \sin^2 \theta d\Omega_3^{'2}]$$

and blowing it up around the null geodesic at $\rho = \theta = 0$ and $dt = d\psi$ by the rescaling:

$$2\mu x^+ = \frac{1}{2}(t+\psi) \ ; \ \frac{1}{2\mu} x^- = R^2(t-\psi) \ ; \ \rho = \frac{r}{R} \ ; \ \theta = \frac{y}{R} \ ; \ R \to \infty$$

, gives exactly the metric (4.3) with $r^2 = x_1^2 + ... + x_4^2$, $y^2 = x_5^2 + ... + x_8^2$ [27] [71]. As $AdS_5 \times S^5$ is also a maximally supersymmetric space, one can take its supersymmetry algebra $su(2,2|4)$ and perform the same Penrose limit on the generators. Doing that one obtains the maximally supersymmetric pp-wave superalgebra [100].

Taking plane waves where $\mu_{ij} \neq \mu \delta_{ij}$ results with solutions with less supersymmetry and less bosonic symmetries, but always with at least 16 killing spinors.

[26] Note that although the metric in (4.3) has $SO(8)$ rotation symmetry, the 5-form field strength breaks that to $SO(4) \times SO(4)$.

[27] And doing the same to the $AdS_5 \times S^5$ 5-form field strength, gives the F_5 in (4.3)

Looking at general pp-wave backgrounds (4.1), supersymmetry is no longer guaranteed. In the next chapter we will find all type IIB x^+-independent pp-wave backgrounds, with a 5-form field strength, which preserve some spacetime supersymmetry. We will distinguish ones which preserve only (1,1) supersymmetry from ones which preserve (2,2) supersymmetry or more. We will also analyze string propagation on such backgrounds. To do that we first review some ideas regarding the superstring worldsheet actions and their symmetries.

4.2. Light Cone Gauge and the World Sheet Supersymmetry Algebra

The description of superstrings in general backgrounds is a fundamental problem in string theory, and yet a highly nontrivial task. Up to date a few alternative formalisms have been developed, trying to address this problem, yet each one has its own drawbacks. The two standard ones are the Ramond-Neveu Schwarz (RNS) formalism [101] and the Green-Schwarz (GS) formalism [102].

The RNS description exhibits a manifest worldsheet symmetry. The worldsheet fields are the bosonic space-time coordinates $X^\mu(\sigma,\tau)$ and their fermionic partners - two component worldsheet spinors $\psi^\mu(\sigma,\tau)$. In a flat background the action can be explicitly written and quantized. The action in conformal gauge is

$$S = -\frac{1}{2\pi}\int d^2\sigma\{\partial_\alpha X^\mu\partial^\alpha X_\mu - i\bar{\psi}^\mu \not{\partial}\psi_\mu\} \tag{4.6}$$

However, in this formulation the existence of spacetime symmetry is not manifest. Also, although it is possible to extend the RNS formalism to more general backgrounds with NS-NS fields, it is not clear how to extend it to include R-R background fields.

The GS formalism on the other hand does exhibit manifest spacetime supersymmetry, but not manifest worldsheet supersymmetry . The worldsheet field content in this case is ten bosonic worldsheet coordinates $X^\mu(\sigma,\tau)$ and two anticommuting

Majorana-Weyl space-time spinor coordinates $\theta^{Aa}(\sigma,\tau)$, $A = 1,2$, $a = 1..16$ (a being a spinor index in 10-dimensions).

Other than these two, there were also a few more formalisms developed by Berkovits, and which later have been shown to be related to each other and to the RNS and GS formulations (a review of these can be found in [103]).

One of these is the covariant pure spinor formalism [104]. This formalism manifestly preserves all SO(9,1) Poincaré symmetries of the background, and in flat space reduces to a quadratic action, which is easy to quantize (as the RNS action). It introduces fermionic canonical momenta d_α conjugate to the worldsheet spinors θ^α into the action, it uses bosonic pure spinor variables λ^α, which function as ghost variables [28], and makes use of a nilpotent BRST operator $Q = \int \lambda^\alpha d_\alpha$ to define the physical states of the theory. However this formalism in curved backgrounds usually gives non-linear actions, which are hard to solve.

Another formalism introduced by Berkovits is the hybrid formalism [105,106], which like the GS action uses spacetime spinor variables on the worldsheet, and which reduces to a free action for flat backgrounds, where quantization is simple. The action contains $N = 2$ worldsheet supersymmetry, replacing the kappa symmetry of the GS action. However it does not have manifest ten dimensional Lorentz invariance, and can at most retain a manifest $U(5)$ subset of the Lorentz group. Actions were built for $SO(3,1)\times U(3)$, $SO(5,1)\times U(2)$, $SO(1,1)\times U(4)$ and $U(5)$ symmetry groups.

The $U(4)$ hybrid formalism, which is of special interest in the context of pp-waves, exhibits manifest (2,2) worldsheet supersymmetry. It is a critical N=2 superconformal field theory, related by field redefinition to the RNS superstring (as shown in [107]). In light-cone gauge , when the fermionic superfields are gauged

[28] These are the bosonic superpartners of the fermionic worldsheet spinors θ^α obeying $\lambda\gamma^m\lambda = \bar\lambda\gamma^m\bar\lambda = 0$ for all spacetime indices m.

away, it reduces to the light-cone gauged GS action [108]. It has also been related to the pure spinor formalism, both being different gauge fixings [109] of the same action - the "doubly supersymmetric action" [110]. Its field content is four chiral and four antichiral bosonic superfields ($X^i, X^{\bar{i}}$, i=1..4), two chiral and antichiral fermionic superfields $(\Theta^+, \Theta^-), (\bar{\Theta}^+, \bar{\Theta}^-)$ and two semi-chiral and semi-anti-chiral fermionic superfields $(W^+, W^-), (\bar{W}^+, \bar{W}^-)$. The action in a flat background exhibits manifestly only 25 of the 45 SO(9,1) Lorentz transformations, 9 of the 10 translations, and 20 of the 32 type IIB supersymmetries. In curved backgrounds, the action becomes more complicated and in general non quadratic, yet one expects the supergravity equations of motion to still come from requiring quantum $(2,2)$ superconformal invariance of the action. For some special curved backgrounds, the action was explicitly written [111,106].

For the rest of this section we focus on the GS formalism, and in the end make some more remarks about the Berkovits $U(4)$ hybrid formalism.

As we are primarily interested in type IIB string theories, we take $\theta^{1,2}$, the GS worldsheet Majorana-Weyl spinors, to have the same handedness. The GS action in a flat background is

$$\begin{aligned} S = &-\frac{1}{2\pi}\int \sqrt{h} h^{\alpha\beta}\Pi_\alpha \cdot \Pi_\beta \\ &+ \frac{1}{\pi}\int d^2\sigma\{-i\epsilon^{\alpha\beta}\partial_\alpha X^\mu(\bar{\theta}^1\Gamma_\mu\partial_\beta\theta^1 - \bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2) + \epsilon^{\alpha\beta}\bar{\theta}^1\Gamma^\mu\partial_\alpha\theta^1\bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2\} \\ \Pi^\mu_\alpha \equiv &\ \partial_\alpha X^\mu - i\bar{\theta}^A\Gamma^\mu\partial_\alpha\theta^A \end{aligned} \tag{4.7}$$

and its equations of motion are:

$$\begin{aligned} &\Pi_\alpha \cdot \Pi_\beta = \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta}\Pi_\gamma \cdot \Pi_\delta \\ &\Gamma \cdot \Pi_\alpha P_-^{\alpha\beta}\partial_\beta\theta^1 = 0 \\ &\Gamma \cdot \Pi_\alpha P_+^{\alpha\beta}\partial_\beta\theta^2 = 0 \\ &\partial_\alpha[\sqrt{h}(h^{\alpha\beta}\partial_\beta X^\mu - 2iP_-^{\alpha\beta}\bar{\theta}^1\Gamma^\mu\partial_\beta\theta^1 - 2iP_+^{\alpha\beta}\bar{\theta}^2\Gamma^\mu\partial_\beta\theta^2)] = 0 \end{aligned} \tag{4.8}$$

where $P_{\pm}^{\alpha\beta}$ are projection operators defined by $P_{\pm}^{\alpha\beta} \equiv \frac{1}{2}(h^{\alpha\beta} + \epsilon^{\alpha\beta}/\sqrt{h})$. The action has a local fermionic symmetry - kappa symmetry - which is parameterized by $\kappa^{A\alpha a}$ ($\alpha = 0, 1$ being a worldsheet vector index) , such that $\kappa^{1\alpha} = P_{-}^{\alpha\beta}\kappa_{\beta}^{1}$, $\kappa^{2\alpha} = P_{+}^{\alpha\beta}\kappa_{\beta}^{2}$. The symmetry transformation is given by

$$\begin{aligned} \delta\theta^A &= 2i\Gamma \cdot \Pi_\alpha \kappa^{A\alpha} \\ \delta X^\mu &= i\bar{\theta}^A \Gamma^\mu \delta\theta^A \\ \delta(\sqrt{h}h^{\alpha\beta}) &= -16\sqrt{h}(P_{-}^{\alpha\gamma}\bar{\kappa}^{1\beta}\partial_\gamma\theta^1 + P_{+}^{\alpha\gamma}\bar{\kappa}^{2\beta}\partial_\gamma\theta^2) \end{aligned} \tag{4.9}$$

In order to quantize the theory we need to fix a gauge. So far the model has only been successfully quantized in light-cone gauge, where things simplify greatly. To implement this gauge, we first we use parameterization invariance to set the worldsheet metric to the flat one

$$h_{\alpha\beta} = \eta_{\alpha\beta}$$

Now we use kappa symmetry to set half of the components of θ^1 and θ^2 to zero:

$$\Gamma^+\theta^1 = \Gamma^+\theta^2 = 0$$

where $\Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^9)$. So we are left with two spinors each having 8 real nonzero components. Taking just these components and rescaling them by $\sqrt{p_-}$ we create the new 8-component spinors $S^{1,2}$. As in type IIB theory $\theta^{1,2}$ have the same handedness , both $S^{1,2}$ are in the same spinorial representation of $SO(8)$, say 8_s.

The last equation in (4.8) implies that $\partial^\alpha\partial_\alpha X^+ = 0$, and thus one can use conformal invariance to set all the + oscillators to zero, and impose

$$X^+(\sigma,\tau) = x^+ + p_-\tau \tag{4.10}$$

It also implies for X^i that $(\partial_\sigma^2 - \partial_\tau^2)X^i = 0$. The second and third equations simplify too and become $(\partial_\tau + \partial_\sigma)S^{1a} = (\partial_\tau - \partial_\sigma)S^{2a} = 0$.

Thus one can write an equivalent action giving the same equations of motion, including only the transverse fields:

$$S_{LC} = -\frac{1}{2\pi}\int d^2\sigma(\partial_\alpha X^i \partial^\alpha X^i - i\bar{S}^a \partial\!\!\!/ S^a) \tag{4.11}$$

with S^a being a worldsheet spinor with two components: S^{1a} and S^{2a}. This action can also be related to the light-cone gauge fixed form of the action (4.6) where ψ^i and S^a are related through bosonization and refermionization [112].

The action (4.11) has some worldsheet supersymmetry. As we are working in the Green-Schwarz formalism, the worldsheet symmetries just descend from the spacetime ones and should be $\delta\theta = \epsilon$. When $\Gamma^+\epsilon = 0$ such transformations will preserve the lightcone gauge. There are 16 such *non-linearly realized* supersymmetry transformations:

$$\begin{aligned} \delta S^a &= \sqrt{2p_-}\eta^a \\ \delta X^i &= 0 \end{aligned} \tag{4.12}$$

These transformations are generated by

$$Q^a = \sqrt{2p_-}S_0^a$$

However, when $\Gamma^+\epsilon \neq 0$ such transformations will take S^a out of the lightcone gauge. To get back in to the gauge, one can perform in addition a kappa symmetry transformation with a parameter κ depending on ϵ such that the resulting S^a is annihilated by Γ^+. Such a kappa transformation also affects the bosonic coordinates. This gives 16 more *linearly realized* supersymmetry transformations [29]:

$$\begin{aligned} \delta S^a &= -i\partial\!\!\!/ X^i \gamma^i_{a\dot{a}} \epsilon^{\dot{a}} \sqrt{p_-} \\ \delta X^i &= \gamma^i_{a\dot{a}} \bar{\epsilon}^{\dot{a}} S^a / \sqrt{p_-} \end{aligned} \tag{4.13}$$

The generators of these transformations are

$$Q^{\dot{a}} = \frac{1}{\sqrt{p_-}} \gamma^i_{\dot{a}a} \sum_{-\infty}^{\infty} S^a_{-n} \alpha^i_n$$

[29] We employ here spin(8) notations for the spinors where the index ϵ^a denotes an 8_s spinor and $\epsilon^{\dot{a}}$ denotes an 8_c spinor.

(where α_n^i are the oscillator modes for the bosonic transverse fields)

The worldsheet supersymmetry transformations must be part of the worldsheet supersymmetry algebra, and should anti-commute to the bosonic symmetries of the model - spacetime translations. These are not all worldsheet translations, as we are working in lightcone gauge. Calculation of the anti-commutators gives the following algebra:

$$\begin{aligned}\{Q^a, Q^b\} &= 2p_- \delta^{ab} \\ \{Q^a, Q^{\dot{a}}\} &= \sqrt{2}\gamma^i_{a\dot{a}} p^i \\ \{Q^{\dot{a}}, Q^{\dot{b}}\} &= 2H\delta^{\dot{a}\dot{b}}\end{aligned} \tag{4.14}$$

One can see that two non-linearly realized supersymmetries anti-commute to the light cone momentum p_- , and two linearly realized supersymmetries anti-commute to the light cone hamiltonian:

$$H = \frac{1}{2p_-}[(p_i)^2 + 2\sum_{m=1}^{\infty}(\alpha^i_{-m}\alpha^i_m + mS^a_{-m}S^a_m)]$$

The supercharges also transform as spinors under Lorentz transformations.

Up to now, we have discussed the propagation of the string on a flat background. A natural question is how to extend this to strings propagating on more general backgrounds. For the bosonic string, it was shown that the 2d sigma model, describing bosonic string theory with spacetime metric, antisymmetric tensor, dilaton and tachyon fields, can be described in a manifestly ghost-free light cone gauge action, provided all the fields have a null covariantly constant spacetime killing vector, and that the theory is Weyl invariant [113]. (This is one of the great drawbacks of the lightcone gauge - that it only allows dealing with very particular backgrounds).

When one tries to write an action for the *superstring* introducing RR background fields, the problem becomes much more complicated. The covariant GS action in a generic type II background has been derived in [114]. It is written in terms of superfields with some extra constraints assuring it is kappa-symmetric. Then to get an explicit component form of the action, one needs to expand the superfields. This can be done using normal coordinate expansions in superspace [115] . The expansion is finite, as the fermionic coordinates are grassmanian, yet in principle one should expand up to order 2^5 in the θ's. To overcome this one can try to gauge fix the kappa symmetry and reduce the expansion order. But even after doing this and using light cone gauge, the expansion includes a huge amount of terms, and has only been performed for special backgrounds with some symmetry properties that simplify the expansion.

An explicit action in component form was obtained only in a couple of cases. In a paper by Sahakian [116] the action was fully expanded for backgrounds with light cone symmetry, with zero fermionic background fields and with diagonal metrics, in which case the expansion truncates in quartic order in the θs. In an other series of papers the action in light-cone gauge was written for spaces which have a coset superspace structure : $AdS_5 \times S^5$ [117,118,119,120] , $AdS_3 \times S^3$ [121,122,123] , and $AdS_2 \times S^2$ [124] . For these backgrounds the expansion truncates at quartic order as well.

Another case where the symmetries of the background were used to write the explicit GS action in light cone gauge is the case of plane waves [79]. In this case the expansion ends already at quadratic order. The action can be obtained either by analyzing independently the symmetries of the plane-wave background, or by viewing it as a Penrose limit of some $AdS_p \times S^q$ background.

For the maximally supersymmetric type IIB plane-wave (4.3) the GS light-cone worldsheet lagrangian was shown to be [79]:

$$\begin{aligned} L &= \frac{1}{2}(\partial_+ x^I \partial_- x^I - m^2 (x^I)^2) + i(\theta^1 \bar{\gamma}^- \partial_+ \theta^1 + \theta^2 \bar{\gamma}^- \partial_- \theta^2 - 2m\theta^1 \bar{\gamma}^- \Pi \theta^2) \\ m &\sim p_- \mu \;\; ; \;\; \Pi \equiv \gamma^1 \bar{\gamma}^2 \gamma^3 \bar{\gamma}^4 \end{aligned} \tag{4.15}$$

This is nothing but a free massive field theory, which can be exactly quantized.

In the next chapter, based on our paper [3], we generalize this, and write down the superstring worldsheet GS action [30] in lightcone gauge for all type IIB pp-waves involving a 5-form RR field strength and having at least $(1,1)$ supersymmetry. Among these worldsheet actions are all $N=2$ Landau-Ginzburg models with four chiral superfields.

Now, returning to the Berkovits $U(4)$ hybrid formalism we mentioned at the beginning of this section, we recall it had manifest $N=2$ supersymmetry, and broken Lorentz invariance, which is exactly the situation we have for our pp-wave backgrounds. It turns out that this formalism is also very convenient to describe the superstring propagation in these pp-wave backgrounds, and to determine consistency conditions on such backgrounds. As we will not use this formalism in subsequent chapters, we do not elaborate more on it here and refer the reader to [111].

4.3. $\mathcal{N}$=2 Landau Ginzburg Models and Mirror Symmetry

As we will see in the next chapter, worldsheet actions of strings propagating in some pp-wave backgrounds turn out to be Landau-Ginzburg models with $\mathcal{N}=(2,2)$ worldsheet supersymmetry. In this subsection we will review some basic facts about Landau-Ginzburg models and about their relation through mirror symmetry to other Landau-Ginzburg models or sigma-models. We will Particularly focus on the $\mathcal{N}=2$ sine-gordon model and its mirror-symmetry dual - the supersymmetric non-linear sigma model on CP^1.

We start by quickly summarizing our notations, which follow [84]. We take the 1+1 $N=2$ superspace coordinates to be $x^{\pm}$, $\theta^{\pm}$, $\bar{\theta}^{\pm}$. The supersymmetry

[30] More precisely, we find an action with worldsheet fermionic coordinates ψ^i which can be related to the GS spacetime fermions S^a as explained in the next chapter.

generators and derivatives on this space are defined as

$$Q_{\pm} = \frac{\partial}{\partial\theta^{\pm}} + i\bar{\theta}^{\pm}\frac{\partial}{\partial x^{\pm}} \quad , \quad \bar{Q}_{\pm} = -\frac{\partial}{\partial\bar{\theta}^{\pm}} - i\theta^{\pm}\frac{\partial}{\partial x^{\pm}}$$
$$D_{\pm} = \frac{\partial}{\partial\theta^{\pm}} - i\bar{\theta}^{\pm}\frac{\partial}{\partial x^{\pm}} \quad , \quad \bar{D}_{\pm} = -\frac{\partial}{\partial\bar{\theta}^{\pm}} + i\theta^{\pm}\frac{\partial}{\partial x^{\pm}} \tag{4.16}$$

For integrals we use the shorthand notations: $\int d^4\theta \equiv \frac{1}{4}\int d\theta^+ d\theta^- d\bar{\theta}^- d\bar{\theta}^+$, $\int d^2\theta \equiv \frac{1}{2}\int d\theta^- d\theta^+|_{\bar{\theta}^{\pm}=0}$ and $\int d^2\tilde{\theta} \equiv \int \frac{1}{2}\int d\bar{\theta}^- d\theta^+|_{\bar{\theta}^+=\theta^-=0}$.

In addition we define two $U(1)$ R-symmetries - a vector $U(1)$ and an axial $U(1)$ generated by F_V and F_A respectively. The supercharges have the following R-charges :

$$[F_V, Q_{\pm}] = -Q_{\pm} \quad , \quad [F_V, \bar{Q}_{\pm}] = \bar{Q}_{\pm}$$

$$[F_A, Q_{\pm}] = \mp Q_{\pm} \quad , \quad [F_A, \bar{Q}_{\pm}] = \pm\bar{Q}_{\pm}$$

Superfields are combinations of fields in a supermultiplet, which are functions of the superspace coordinates.

A chiral superfield Φ is defined to satisfy $\bar{D}_{\pm}\Phi = 0$ and so can be expanded in superspace as

$$\Phi = \phi + \sqrt{2}\theta^+\psi_+ + \sqrt{2}\theta^-\psi_- + 2\theta^+\theta^- F + ...$$

F being an auxiliary field and the ... terms involving derivatives of $\phi, \psi_{\pm}$. An antichiral superfield $\bar{\Phi}$ satisfies $D_{\pm}\bar{\Phi} = 0$.

A twisted chiral superfield Y satisfies $\bar{D}_+ Y = D_- Y = 0$ and can be written in components as:

$$Y = y + \sqrt{2}\theta^+\bar{\chi}_+ + \sqrt{2}\bar{\theta}^-\chi_- + 2\theta^+\bar{\theta}^- G + ...$$

G being an auxiliary field. A twisted anti-chiral superfield $\bar{Y}$ satisfies $D_+\bar{Y} = \bar{D}_-\bar{Y} = 0$.

A vector superfield V consists of a vector field v_μ, conjugate Dirac fermions $\lambda_{\pm}, \bar{\lambda}_{\pm}$ and a complex scalar σ. In Wess-Zumino gauge it has the component form:

$$V = \theta^-\bar{\theta}^- v_- + \theta^+\bar{\theta}^+ v_+ - \theta^-\bar{\theta}^+\sigma + \theta^+\bar{\theta}^-\bar{\sigma} +$$

$$+\sqrt{2}i\theta^-\theta^+(\bar{\theta}^-\bar{\lambda}_- + \bar{\theta}^+\bar{\lambda}_+) + \sqrt{2}i\bar{\theta}^+\bar{\theta}^-(\theta^-\lambda_- + \theta^+\lambda_+) + 2\theta^-\theta^+\bar{\theta}^+\bar{\theta}^- D$$

D being a real auxiliary field. Its field strength $\Sigma = \frac{1}{2}\{e^V\bar{D}_+e^{-V}, e^{-V}D_-e^V\}$ can be checked to be a twisted chiral superfield.

A Landau-Ginzburg (LG) model is characterized by a superpotential, which is a holomorphic function of Nchiral superfields $W(\Phi^i)$, and by the Kahler potential on an N-complex dimensional manifold $K(\Phi^i, \bar{\Phi}^i)$, such that $g_{i\bar{j}} = \frac{\partial^2 K}{\partial\Phi^i\partial\bar{\Phi}^{\bar{j}}}$. The Lagrangian is made up of two parts - the D term :

$$L_K = \int d^4\theta K(\Phi^i, \bar{\Phi}^i) \tag{4.17}$$

and the F-term:

$$L_W = \frac{1}{2}\int d^2\theta W(\Phi^i) + c.c. \tag{4.18}$$

Together they give in component form:

$$\begin{aligned} L_K + L_W = &-g_{i\bar{j}}\partial^\mu\phi^i\partial_\mu\overline{\phi^j} + ig_{i\bar{j}}\overline{\psi_-^j}D_+\psi_-^i + ig_{i\bar{j}}\overline{\psi_+^j}D_-\psi_+^i + R_{i\bar{k}j\bar{l}}\psi_+^i\psi_-^j\overline{\psi_-^k\psi_+^l} \\ &-\frac{1}{4}g^{\bar{j}i}\overline{\partial_j W}\partial_i W - \frac{1}{2}[(D_i\partial_j W)\psi_+^i\psi_-^j + (D_{\bar{i}}\overline{\partial_j W})\overline{\psi_-^i\psi_+^j}] \end{aligned} \tag{4.19}$$

with $D_\pm\epsilon^j \equiv \partial_\pm\epsilon^j + \partial_\pm\phi^l\Gamma^j_{li}\epsilon^i$

Looking at the bosonic part of the action in component form, one sees the potential is proportional to $g^{\bar{j}i}\overline{\partial_j W}\partial_i W$ and thus the vacua of the theory $\{a_k\}$ are obtained by minimization of the potential:

$$\frac{\partial W}{\partial\phi^i}|_{a_k} = 0 \ , \ \forall i$$

Denoting the 1+1 dimensions as $x^\pm = \frac{1}{2}(\tau \pm \sigma)$, the theory will contain solitons or kinks, which connect one minimum at $\sigma = -\infty$ to another at $\sigma = +\infty$: $\Phi^i(\sigma = -\infty) = a^i$, $\Phi^i(\sigma = +\infty) = b^i$. Such configurations which minimize the energy of the system are BPS solitons. They satisfy the relation $\partial_\sigma\Phi^i = \frac{e^{i\alpha}}{2}g^{\bar{j}i}\overline{\partial_j W}$ where α is the argument of $(W(b) - W(a))$, and their mass is given by $m_{ab} = |W(b) - W(a)|$.

This result is true both in the classical level, and also as an exact nonperturbative quantum statement, due to a non-renormalization theorem for the superpotential in $\mathcal{N} = (2,2)$ theories in two dimensions. In the W-plane the solitons are just straight lines, as $\partial_\sigma W = \frac{e^{i\alpha}}{2} g^{i\bar{j}} \partial_i W \partial_{\bar{j}} W$, which has a constant α.

Mirror symmetry is an equivalence of two $(2,2)$ supersymmetric field theories under which the supersymmetry generators are exchanged as $Q_- \leftrightarrow \bar{Q}_-$, the axial and vector R-symmetries are interchanged, and chiral multiplets are mapped to twisted chiral multiplets and vice versa. To show that two models are mirror symmetry dual to each other, one usually defines some action with a few extra non-dynamical superfields. Then integrating out some superfields will give an action on one side of the duality (or an action that would RG flow to it), and integrating out other superfields gives the action on the other side of the duality (or an action that would RG flow to it) [84].

In the simple case where one of the theories is a supersymmetric sigma model on a torus of radius R, mirror symmetry reduces to *T-duality*, relating it to a sigma model on a torus of radius $1/R$. In this case one can start with the action

$$L = \int d^4\theta (\frac{R^2}{4} B^2 - \frac{1}{2}(\Theta + \bar{\Theta})B)$$

where Θ is a twisted chiral superfield and B a real superfield. Integrating out $\Theta, \bar{\Theta}$ gives a constraint on B which is solved as $B = \Phi + \bar{\Phi}$, Φ being a chiral superfield, and the action becomes

$$L_{ch} = \int d^4\theta \frac{R^2}{2} \Phi\bar{\Phi} \quad .$$

Integrating out B instead gives

$$L_{TwCh} = \int d^4\theta (-\frac{1}{2R^2} \bar{\Theta}\Theta) \quad .$$

L_{ch} is an action on a cylinder with a circle of radius R and L_{TwCh} is an action on a cylinder with a circle of radius $1/R$.

More generally mirror symmetry can relate two LG models to each other, two sigma-models to each other, or a LG model to a sigma-model. An $\mathcal{N}=2$ LG model we will focus on in chapter 4 is the **super sine-gordon** (SSG), as it is an integrable model, which describes a superstring propagating in a specific pp-wave background. As it turns out, the SSG model, with vanishing kinetic term is mirror symmetric to the CP^1 sigma-model [84], and with a finite kinetic term, is mirror symmetric to the squashed CP^1 sigma model [85], or 'sausage model' [125]. This is actually a special case of a more general mirror symmetry between affine toda A_{N-1} LG models and CP^{N-1} sigma-models [81,82].

The basic idea is to start with a 1+1 dimensional $U(1)$ gauge linear sigma model (GLSM) with a vector field V and its field strength Σ, and two chiral superfields $\Phi^{1,2}$ of charge +1. The action is

$$L' = \int d^4\theta (\sum_{i=1}^{2} \bar{\Phi}_i e^{2V} \Phi_i - \frac{1}{2e^2}\bar{\Sigma}\Sigma) - \frac{1}{2}(\int d^2\tilde{\theta} t\Sigma + c.c.) \tag{4.20}$$

with e the gauge coupling and $t = r - i\theta$, r a F.I. parameter and θ a θ-angle coupling. This model is not conformal invariant and r flows with the scale μ as $r_0 = 2\ln(\Lambda_{UV}/\mu)$.

Now one can show that the term $\int d^4\theta \sum_{i=1}^{2} \bar{\Phi}_i e^{2V}\Phi_i$ in (4.20) is really equivalent to $\int d^4\theta \sum_{i=1}^{2}[-\frac{1}{2}(Y_i + \bar{Y}_i)\ln(Y_i + \bar{Y}_i)] + \frac{1}{2}(\int d^2\tilde{\theta}\Sigma(Y_1 + Y_2) + c.c.)$ as both can be obtained from one action ($L_0 = \int d^4\theta[e^{2V+B_i} - \frac{1}{2}\sum_{i=1}^{2}(Y_i + \bar{Y}_i)B_i]$, B_i real superfields) by integrating over different fields (the first is obtained by integrating over $Y_i, \bar{Y}_i$, and the second by integrating over B_i). Y_i are periodic variables $Y_i \sim Y_i + 2\pi i$. So (4.20) is equivalent to

$$L' = \int d^4\theta\{-\frac{1}{2e^2}\bar{\Sigma}\Sigma - \frac{1}{2}\sum_{i=1}^{2}(Y_i + \bar{Y}_i)\ln(Y_i + \bar{Y}_i)\} + \frac{1}{2}(\int d^2\tilde{\theta}\Sigma(Y_1 + Y_2 - t) + c.c.) \tag{4.21}$$

Here Y_i couple as theta angles, and like t also get renormalized: $Y_{0i} = Y_i + \ln(\Lambda_{UV}/\mu)$ so that $\Sigma(Y_{10} + Y_{20} - t_0) = \Sigma(Y_1 + Y_2 - t(\mu))$ with $t(\mu) = 2\ln(\Lambda_{UV}/\mu) - i\theta$.

It turns out that the action (4.21) gets non-perturbative corrections for the twisted superpotential due to instanton effects [127]. The gas of vortices and anti-vortices modifies the twisted superpotential into $\tilde{W}(Y) = \Sigma(Y_1 + Y_2 - t(\mu)) + \mu(e^{-Y_1} + e^{-Y_2})$ (The Kahler term can also get corrected, perturbatively and non-perturbatively).

The main point now is that the IR limit of the GLSM (4.20), when energies are much smaller than $e\sqrt{r_0}$, is the non-linear sigma model on CP^1.

On the other hand, looking at the dual action with Y_i, Σ, one sees that the masses of Σ and $Y_1 + Y_2$ are of order $e\sqrt{r_0}$, while the mass of the modes tangent to $Y_1 + Y_2 = t$ is of order $\mu\sqrt{r_0}$. In the limit $e \gg \mu$, one can treat Σ as a non-dynamic field and integrate over it. As the twisted superpotential is $\Sigma(Y_1+Y_2-t)+\mu(e^{-Y_1}+e^{-Y_2})$, the Σ integration imposes the constraint $Y_1 + Y_2 = t$ and the new potential is $W' = \mu(e^{-Y_1} + e^{Y_1-t}) = 2\Lambda \cosh Y$ where $\Lambda = \mu e^{-t/2}$ and $Y = Y_1 - t/2$.

Thus we establish that the CP^1 model is dual to the SSG with a superpotential $2\Lambda \cosh Y$.

More generally, had we started with a GLSM with N chiral fields of charge 1 we would have gotten in the IR a NLSM on CP^{N-1} of size t and on the other side of the duality we would have gotten the A_{N-1} affine toda model. If one starts with general charges Q_i (such that $\sum_i Q_i \neq 0$), this becomes a weighted projective space with weights determined by the charges. It can actually be shown that the mirror symmetry dual of any NLSM on a toric variety X^{N-k} is the algebraic torus $(C^{\times})^{N-k}$ with a superpotential.

The NLSM on the round CP^1 is actually dual to the SSG in the limit where the radius of the Y direction vanishes, i.e. if we write the SSG model as $\int d^4\theta Y^2 + (\int d^2\theta \Lambda \cosh \beta Y + c.c)$, then this is the limit $\beta \to \infty$. Sometimes this is also referred to as the limit where the kinetic term vanishes. This is equivalent to the previous statement if we rescale $Z \equiv \beta Y$ and then notice the kinetic term for Z is proportional to $\frac{1}{\beta^2}$ and thus vanishes when $\beta \to \infty$. To motivate why the round CP^1 is really only the $\beta \to \infty$ limit of the SSG, we recall that the approximate Kahler term was $\frac{1}{4r_0^2}|Y_i|^2$. In the continuum limit $\Lambda_{UV} \to \infty$, $r_0 \to \infty$ this kinetic term vanishes.

One can argue that the SSG with a finite radius β^{-1} (or finite kinetic term) is mirror symmetric to the squashed CP^1 also called the sausage model. This was conjectured by Fendley and Intriligator, and expanded upon in [85,128]. To see this one starts with a more general GLSM than (4.20), where in addition there is another chiral superfield P, another gauge field V' with field strength Σ', a squashing parameter k, and two parameters $R_{1,2}$. The action is:

$$L_{GLSM-sq} = \int d^4\theta[\sum_{i=1}^{2} \bar{\Phi}_i e^{2V+2R_iV'}\Phi_i - \frac{1}{2e^2}|\Sigma|^2] + \frac{1}{2}[\int d^2\tilde{\theta} t\Sigma + c.c.] +$$

$$+ \int d^4\theta[\frac{k}{4}(P + \bar{P} + V')^2 - \frac{1}{2\tilde{e}^2}\Sigma'^2]$$

The vacuum manifold of this model is the sausage, which for $k \to \infty$ becomes the round CP^1 again. Dualizing $P, \Phi^{1,2}$, taking into account the vortices' effects and integrating over Σ, Σ' one obtains the SSG with a finite Kahler term, whose metric is $g_{ij} = \frac{1}{k}R_iR_j$. Again the limit $k \to \infty$ makes this term vanish.

The equivalence between the supersymmetric sigma models on CP^{N-1} and affine A_{N-1} Toda models has a lot of support comparing properties of the two theories. The supersymmetric NLSM on CP^{N-1} is an asymptotically free theory which generates a dynamical scale Λ (which in our previous notations is $\Lambda = \mu e^{-t/N}$). It preserves $U(1)_V$ R-symmetry, but breaks $U(1)_A \to Z_{2N} \to Z_2$ the first breaking due to anomalies, and the second is a spontaneous symmetry breaking. CP^{N-1} has N vacua with a mass gap. It also has solitons [127]. These come in the different Λ_r representations of $SU(n)$, $r = 1..(n-1)$. For each representation there are $\binom{n}{r}$ solitons which are $N = 2$ doublets : $(u_{r\alpha}, d_{r\alpha})$, $\alpha = 1..\binom{n}{r}$, and they connect adjacent vacua.

The A_{N-1} affine Toda model also has a mass scale, but here it is explicitly introduced. Being the mirror dual, it has the roles of $U(1)_V$ and $U(1)_A$ reversed. It preserves $U(1)_A$, and has an explicit breaking of $U(1)_V \to Z_{2N}$. Also the soliton spectrum and scattering matrix of the two theories agree [82].

Chapter 5

Supersymmetric Supergravity Solutions of the PP Wave Type

5. Supersymmetric Supergravity Solutions of the PP Wave Type

We consider type IIB supergravity solutions with a nonzero 5-form field strength. They have a covariantly constant null killing vector, $\frac{\partial}{\partial x^-}$, which also leaves F_5 invariant and it is such that it gives zero when contracted with F_5.

More explicitly, the form of the solutions we consider is

$$\begin{aligned} ds^2 &= -2dx^+dx^- + H(x^i)(dx^+)^2 + ds_8^2 \\ F_5 &= dx^+ \wedge \varphi_4(x^i) \end{aligned} \tag{5.1}$$

where x^i are the 8 transverse coordinates, F_5 is the self-dual RR field strength. We limit ourselves to solutions which are also independent of x^+. We consider constant dilaton and set all other fields to zero. The transverse metric can be curved. Note that the background is such that we can scale down H and φ by performing a boost in the $x^\pm$ directions.[31]. This property under boost transformations implies that we can assign an "order" to each field according to how they change under boosts. The four-form φ is of first order while H is of second order. This means that the transverse space with zero RR five-form should be a solution of the equations of motion by itself, since it is of zeroth order.

In order to clarify a bit the discussion we will first consider the simpler case when the transverse space is flat and then the slightly more complicated case of a curved transverse space.

[31] So the background is not boost invariant in the $x^\pm$ directions.

5.1. Flat Transverse Space

The equations of motion of type IIB supergravity imply that (5.1) obeys

$$\nabla^2 H = -32|\varphi|^2 \ ; \qquad *_{10}F_5 = F_5 \tag{5.2}$$

where $|\varphi|^2 = \frac{1}{4!}\varphi_{\mu\nu\rho\delta}\varphi^{\mu\nu\rho\delta}$, and ∇^2 is the laplacian in the transverse 8-dimensional space. In our conventions[32], the self-duality of F_5 implies that φ is anti-self-dual in the 8-dimensional space, so that $*\varphi = -\varphi$ and $d\varphi = 0$. [33]

In addition we will now require the solution to preserve some supersymmetries. Supersymmetries in type IIB supergravity are generated by a chiral spinor ϵ with 16 complex components. We find it convenient to separate it into two components according to their $SO(8)$ chiralities

$$\epsilon = -\frac{1}{2}\Gamma_+\Gamma_-\epsilon - \frac{1}{2}\Gamma_-\Gamma_+\epsilon \equiv \epsilon_+ + \epsilon_- \ . \tag{5.3}$$

ϵ_+ has positive $SO(1,1)$ and $SO(8)$ chiralities, and is not annihilated by Γ^+. We will find, roughly speaking (i.e. to lowest order in φ_4), that ϵ_+ is related to the supersymmetries that are preserved by a configuration with nonzero p_- and are linearly realized on the light cone action. These anti-commute to the lightcone Hamiltonian, plus possibly some rotations. On the other hand the supersymmetries generated by ϵ_-, which is annihilated by Γ^+, are non-linearly realized on the worldsheet and imply that some particular fermions are free on the worldsheet. For reasons that will become clear later we are especially interested in supersymmetries that are linearly realized on the worldsheet so we are interested in spinors such that only ϵ_+ is nonzero to first order.

Setting to zero the supersymmetry variations we obtain the following equation

$$0 = D_M\epsilon = (\nabla_M + \frac{i}{2}\not{F}\Gamma_M)\epsilon \ , \tag{5.4}$$

[32] Our conventions and notations are summarized in Appendix F.

[33] A $*$ with no subindex will always refer to the 8 dimensional space.

which leads to

$$
\begin{aligned}
&\partial_- \epsilon_+ = \partial_\mu \epsilon_+ = \partial_+ \epsilon_+ = 0 \\
&\partial_- \epsilon_- = 0 \quad ; \quad \partial_\mu \epsilon_- = \frac{i}{2}\Gamma_- \not{\varphi} \Gamma_\mu \epsilon_+ \quad ; \quad (i\partial_+ - \not{\varphi})\epsilon_- = \frac{i}{4}\Gamma_- \not{\partial} H \epsilon_+
\end{aligned}
\tag{5.5}
$$

where $\not{\varphi} \equiv \frac{1}{4!}\Gamma^{\mu\nu\rho\delta}\varphi_{\mu\nu\rho\delta}$. These equations imply that ϵ_+ must be a constant spinor and they determine the first and higher order parts of ϵ_- in terms of ϵ_+. These solutions with nonzero zeroth order ϵ_+ determine the linearly realized supersymmetries of the light cone action. In addition to these we might have solutions of (5.5) with $\epsilon_+ = 0$. We obviously have 16 solutions of this type if φ is a constant form, but when φ is not constant we will generically have no solutions of this type (below we will make a precise statement). Note that only solutions of this second type can be x^+ dependent. Note also that if $\epsilon = \epsilon_+ + \epsilon_-$ is a solution, then so is $\hat{\epsilon} = \epsilon_+^* - \epsilon_-^*(-x^+)$.

When we attempt to solve the equation for ϵ_- in terms of ϵ_+ we find some integrability conditions. First, integrability of the $\partial_\mu \epsilon_-$ equations places a constraint on the allowed 4-forms. Then the $(i\partial_+ - \not{\varphi})\epsilon_-$ equation gives further consistency conditions on ϵ_- and determines H in terms of φ_4. In Appendix G we show these computations in detail. Below we will just state the form of the most general solutions with $(2,2)$ and $(1,1)$ supersymmetry. We did not explore the subset of $(2,2)$ solutions which actually have more ϵ_+-type supersymmetries.

It is convenient to choose complex coordinates for the transverse space, $z^1, ..., z^4$. The anti-self-dual 4-forms $\varphi_{\mu\nu\rho\delta}$ written in complex coordinates can be split into 2 kinds - those having two holomorphic and two anti holomorphic indices - the (2,2) forms (of which there are 15) and those having one holomorphic and three antiholomorphic indices and their complex conjugates - the (1,3) and (3,1) forms (of which there are 10+10). We denote the (1,3) forms by the shorter notation

$$
\varphi_{mn} \equiv \frac{1}{3!}\varphi_{m\overline{ijk}}\epsilon^{\overline{ijkn}} g_{n\bar{n}}
\tag{5.6}
$$

Anti-self duality of φ implies that φ_{mn} is symmetric.

It can be shown that one can write the anti-selfdual $(2,2)$ forms in terms of $\varphi_{i\bar{j}}$ defined as

$$2\varphi_{l\bar{m}} = g^{s\bar{s}}\varphi_{l\bar{m}s\bar{s}} \ , \tag{5.7}$$

where the reality and self duality condition imply that $\varphi_{l\bar{m}}$ is a hermitian and traceless matrix (which could, in principle, be a function of the coordinates). We also define the lowest weight spinor state $|0\rangle$ which is annihilated by $\Gamma_{\hat{+}}$ and Γ^{i} where i runs over the four holomorphic indices. We begin by describing the solutions with an ϵ_{+} which at zeroth order is proportional to $|0\rangle$ and its complex conjugate. We later describe solutions with $\epsilon_{+} = 0$.

CASE (1) (2,2) *supersymmetry or more*

The solution is parameterized by a holomorphic function W. In this case the $\varphi_{l\bar{m}}$ are constants and given in terms of a traceless hermitian 4x4 matrix. W and $\varphi_{l\bar{m}}$ should also obey

$$\partial_n[\varphi_j{}^k z^j \partial_k W] = 0 \tag{5.8}$$

where we raised the index of $\varphi_{j\bar{k}}$ using the flat transverse space metric. The metric and the 4-form are given by

$$\begin{gathered} ds^2 = -2dx^+dx^- - 32(|\partial_k W|^2 + |\varphi_{j\bar{k}} z^j|^2)(dx^+)^2 + dz^i\overline{dz^i} \\ \varphi_{mn} = \partial_m\partial_n W \ , \qquad \varphi_{\bar{m}\bar{n}} = \partial_{\bar{m}}\partial_{\bar{n}}\overline{W} \ , \qquad \varphi_{l\bar{m}} = \text{constants} \end{gathered} \tag{5.9}$$

The expressions for the Killing spinors can be found in appendix G.

One can, of course, look at the simpler cases where either $W = 0$ or $\varphi_{l\bar{m}} = 0$. It is interesting to note that if $\varphi_{l\bar{m}}$ is nonzero the superalgebra has a central charge term proportional to the $U(1)$ symmetry generated by the holomorphic Killing vector $z^l\varphi_{l\bar{m}}\partial/\partial z^m$ and its complex conjugate.

CASE (2) (1,1) *supersymmetry*

These solutions are parameterized by a real harmonic function U. However this time there are only 2 Killing spinors. The solution is

$$\begin{aligned} ds^2 &= -2dx^+dx^- - 32(|\partial_k U|^2)(dx^+)^2 + dz^i\overline{dz^i} \\ \varphi_{mn} = \partial_m\partial_n U \quad &; \quad \varphi_{\bar{m}\bar{n}} = \partial_{\bar{m}}\partial_{\bar{n}} U \quad ; \quad \varphi_{l\bar{m}} = \partial_l\partial_{\bar{m}} U \end{aligned} \tag{5.10}$$

The expressions for the Killing spinors can be found in appendix G.

5.2. The Homogenous Solution for ϵ_-

The homogenous equations for ϵ_-^{hom} are

$$\partial_-\epsilon_-^{hom} = \partial_j\epsilon_-^{hom} = \overline{\partial_j}\epsilon_-^{hom} = (i\partial_+ - \not{\varphi})\epsilon_-^{hom} = 0 \tag{5.11}$$

and are solved by

$$\epsilon_-^{hom}(x^+) = e^{-i\not{\varphi}x^+}\eta_0 \tag{5.12}$$

where η_0 is a constant spinor. (5.11) implies that $\not{\varphi}$ and η_0 should be such that after multiplying $(\not{\varphi})^n\eta_0$ (for $n = 1, 2, ...$) we still have spinors that are constant in the transverse space and independent of x^+. So we get the spinors $\eta_0, \not{\varphi}\eta_0, \cdots (\not{\varphi})^{n-1}\eta_0$ which are linearly independent and $n \leq 16$. These solutions of (5.11) are associated to free fermions on the string worldsheet in light cone gauge. In fact the last equality in (5.11) is the equation of motion for a zero momentum mode on the string worldsheet. If we diagonalize the matrix $\not{\varphi}$ in the subspace of solutions we see clearly that each pair of solutions gives rise to a free fermion on the worldsheet[34]. The fermion is free but it can be massless or massive depending on the eigenvalue of the matrix $\not{\varphi}$ on it. The sixteen supersymmetries of ϵ_- type that arise in the usual quadratic plane waves discussed in [94] arise because all fermions are free. In a general interacting case all fermions will be interacting and there will be no

[34] The solutions come in pairs. If the eigenvalue of the matrix $\not{\varphi}$ is nonzero this follows by considering the complex conjugate equation. If the eigenvalue is zero then we can multiply the solution by any complex number so that we have two real solutions.

supersymmetries of this type. If, in addition, we have worldsheet supersymmetry in lightcone gauge, as in the cases we are analyzing, each free fermion has a free boson partner and these two together decouple from the rest of the worldsheet theory. So the structure is clear, we have as many free bosons and fermions as there are ϵ_- supersymmetries. In the N=(2,2) case these supersymmetries come in groups of four, one per complex field that appears at most quadratically in the superpotential.

5.3. Curved Transverse Space

When the transverse space is curved, the ansatz (2.1) is a solution of IIB supergravity iff it satisfies the equations of motion

$$\begin{gathered} \nabla^2 H = -32|\varphi|^2 \quad ; \quad *_8\varphi = -\varphi \quad ; \quad d\varphi = 0 \\ R_{\mu\nu} = 0 \end{gathered} \tag{5.13}$$

where ∇^2 is the laplacian in the transverse curved space, and $R_{\mu\nu}$ is the Ricci tensor of the transverse space [35].

The supersymmetry equations for the curved case are

$$\begin{aligned} &\partial_-\epsilon_+ = \nabla_\mu\epsilon_+ = \partial_+\epsilon_+ = 0 \\ &\partial_-\epsilon_- = 0 \quad ; \quad \nabla_\mu\epsilon_- = \frac{i}{2}\Gamma_u\not{\varphi}\Gamma_\mu\epsilon_+ \quad ; \quad (i\partial_+ - \not{\varphi})\epsilon_- = \frac{i}{4}\Gamma_u\not{\partial}H\epsilon_+ \end{aligned} \tag{5.14}$$

These are exactly the same equations as in the flat case (5.5), with the transverse derivatives replaced by covariant derivatives. We will now state what the general solutions are and we refer the interested reader to appendix G for the derivation. The first point to note is that to zeroth order the supersymmetry equations for the transverse manifold imply that the transverse space is a special holonomy space. If we demand (2,2) supersymmetries on the worldsheet it can only be a Calabi-Yau

[35] we use (+,-) and greek letters to denote curved indices , and (v,u)and roman letters to denote flat indices. All notations and conventions we use for curved space are summarized in Appendix F.

space (G_2 and $Spin(7)$ could also be studied but we do not do that here). For this reason it is still convenient to choose complex coordinates and we denote by $|0\rangle$ the covariantly constant spinor on the Calabi-Yau manifold that is annihilated by $\Gamma_{\hat{+}}$ and Γ^{μ} where μ runs over the four holomorphic indices. We will also use the short notation (5.6) for the (1,3) forms. We first focus on the supersymmetries that are linearly realized on the worldsheet in lightcone gauge and later we explain what happens with the homogeneous solutions for ϵ_-.

CASE (1) (2,2) *supersymmetry or more*

In this case the solution is parameterized by a holomorphic function W, and a real Killing potential U from which we can define the Killing vectors $V_\mu = i\partial_\mu U$, $V_{\bar{\mu}} = -i\partial_{\bar{\mu}} U$. The Killing vector should be holomorphic (i.e. V^μ is holomorphic and $V^{\bar{\mu}}$ is antiholomorphic). The following conditions should also hold

$$\nabla_\mu V^\mu = 0 \tag{5.15}$$

$$\partial_\nu [V^\tau \nabla_\tau W] = 0 \tag{5.16}$$

The supergravity solution is

$$\begin{aligned}
&ds^2 = -2dx^- dx^+ - 32(|dW|^2 + |V|^2)(dx^+)^2 + 2g_{\mu\bar{\nu}} dz^\mu d\bar{z}^{\bar{\nu}} \\
&\varphi_{\mu\nu} = \nabla_\mu \nabla_\nu W \ , \qquad\qquad \varphi_{\bar{\mu}\bar{\nu}} = \nabla_{\bar{\mu}} \nabla_{\bar{\nu}} \bar{W} \\
&\varphi_{\bar{\mu}\nu} = \nabla_{\bar{\mu}} \nabla_\nu U
\end{aligned} \tag{5.17}$$

where $|dW|^2 \equiv g^{\mu\bar{\nu}} \nabla_\mu W \overline{\nabla_\nu W}$, and $|V|^2 \equiv g_{\mu\bar{\nu}} V^\mu V^{\bar{\nu}}$. The expressions for the Killing spinors can be found in appendix H.

Here too, one can look at the simpler cases where either $W = 0$ or $V^\mu = 0$. Note that if the transverse space is compact there is no non-constant holomorphic function. In order to have interesting solutions we need the transverse space to be non-compact.

CASE (2) (1,1) *supersymmetry*

The (1,1) supersymmetry solutions are parameterized by a real harmonic function U. The metric, 4-form and the 2 Killing spinors are given by

$$\begin{gathered} ds^2 = -2dx^- dx^+ - 32(|\nabla U|)^2 (dx^+)^2 + g_{\mu\bar{\nu}} z^\mu \overline{z^\nu} \\ \varphi_{\mu\nu} = \nabla_\mu \nabla_\nu U \quad ; \quad \varphi_{\bar{\mu}\bar{\nu}} = \nabla_{\bar{\mu}} \nabla_{\bar{\nu}} U \quad ; \quad \varphi_{\mu\bar{\nu}} = \nabla_\mu \nabla_{\bar{\nu}} U \end{gathered} \tag{5.18}$$

Note that the (2,2) part of the 4-form (whose components are $\varphi_{\lambda\bar{\sigma}\mu\bar{\nu}}$) is therefore

$$\varphi = (\nabla_\mu \nabla_{\bar{\nu}} U dz^\mu d\overline{z^\nu}) \wedge J \tag{5.19}$$

where J is the Kahler form, which obeys $dJ = 0$ (so that $\varphi_{\mu\bar{\nu}} = \frac{1}{2} g^{\lambda\bar{\sigma}} \varphi_{\lambda\bar{\sigma}\mu\bar{\nu}} = \nabla_\mu \nabla_{\bar{\nu}} U$).

5.4. The Homogenous Solution for ϵ_-

The homogenous equations for ϵ_-^{hom} in a curved background are

$$\partial_- \epsilon_-^{hom} = \nabla_j \epsilon_-^{hom} = \overline{\nabla_j} \epsilon_-^{hom} = (i\partial_+ - \not{\varphi}) \epsilon_-^{hom} = 0 \tag{5.20}$$

There is a solution

$$\epsilon_-^{hom}(x^+) = e^{-i\not{\varphi} x^+} \eta_0 \tag{5.21}$$

with η_0 a covariantly constant spinor and all of $(\not{\varphi})^n \eta_0$ $(n = 1, 2, ...)$ covariantly constant.

The discussion follows exactly the one we had for the flat case, where we argued that each pair of solutions for (5.20) gives rise to a free (massive or massless) fermion on the string worldsheet in light cone gauge. Due to supersymmetry each such fermion has a free boson partner, and they both decouple from the rest of the worldsheet theory.

Chapter 6

The Worldsheet Actions

6. The Worldsheet Actions

In the previous chapter we have listed all the supersymmetric solutions of the pp-wave form. In this chapter we write the action describing a string propagating in these backgrounds. We choose light cone gauge by setting $x^+ = \tau$, where τ is worldsheet time. Though the standard procedure we then find that p_- is conserved, etc.[36] In light cone gauge, only killing spinors which are not annihilated by Γ^+ survive as linearly realized supersymmetries on the worldsheet. These are the ϵ_+ part of the killing spinor. Since we focused on solutions that preserved some supersymmetries of this type, we will have a supersymmetric action on the worldsheet. Thanks to these supersymmetries we do not need to work too much to find the action, since its form is dictated by supersymmetry.

(2,2) Supersymmetric solutions

We know that if all RR fields are set to zero, the action reduces to the usual $(2,2)$ non-linear sigma model which can be written in terms of the Kähler potential. By turning on $(1,3)$ and $(3,1)$ forms we can add an *arbitrary* superpotential so that the action in superfield form becomes

$$\begin{aligned} S =& \frac{1}{4\pi\alpha'} \int d\tau \int_0^{2\pi\alpha'|p_-|} d\sigma (L_K + L_W) \ , \\ L_K + L_W =& \int d^4\theta K(\Phi^i, \bar{\Phi}^i) + \frac{1}{2}(\int d^2\theta W(\Phi^i) + c.c.) \end{aligned} \tag{6.1}$$

[36] Our notation with a lower index for $p_\pm$ seems to be contrary to standard practice in the literature. While in Minkowski space it does not matter where we put the index, it actually does matter where we put it when g_{++} is nonzero. (Some papers have chosen the unreasonable convention of raising the indices using the flat Minkowski metric...). In our conventions for the metric (where $g_{-+} = -1$) we find that $p_- \leq 0$ for particles propagating to the future.

where $\Phi^i = Z^i + \sqrt{2}\theta^L\psi^i_L + \sqrt{2}\theta^R\psi^i_R + 2\theta^L\theta^R F^i + ...$ From this we can find the component action by integrating out θ [129]. Note that (6.1) contains Yukawa interactions given in terms of $\not{\phi}$, a bosonic potential proportional to H (5.1), as well as four fermion couplings which follow from supersymmetry. If the transverse space is flat, there are no four fermion couplings, and the action could also be read from [79]. The fermions appearing in (6.1) are related to the Green-Schwarz fermions as follows. The G-S fermions are SO(8) spinors with negative chirality (in our conventions). Once we choose complex coordinates we have an SU(4) subgroup of SO(8) which preserves the complex structure. Under this subgroup $8_- \to \mathbf{4} + \bar{\mathbf{4}}$, these are the spinors with vector index. More explicitly, let us denote by η_0 a covariantly constant spinor annihilated by all $\Gamma_{\bar{i}}$. We then write a general negative chirality SO(8) spinor as $S = \psi^i\Gamma_i\eta_0 + \psi^{\bar{i}}\Gamma_{\bar{i}}\eta_0^*$. This defines the worldsheet spinors ψ^i, $\psi^{\bar{i}}$.

It can be checked that the $(3,1)$ and $(1,3)$ forms induce couplings of the type $\psi^i_L\psi^j_R$ as implied by the action (6.1). It can also be seen that the (2,2) forms induce couplings of the type $\psi^i_L\psi^{\bar{j}}_R$. These couplings are not present in (6.1). Nevertheless, it was shown in [130], [131], [132], and reviewed in [84], that if the target space has a holomorphic isometry, i.e. a holomorphic killing vector field V^i ($\nabla_i V_{\bar{j}} + \nabla_{\bar{j}} V_i = 0$), then this isometry can be gauged to give a vector multiplet (consisting of a complex scalar, two conjugate dirac fermions and a vector field). Then by taking the weak coupling limit and then freezing the vector and fermions at zero and the scalar at a constant value, one can obtain a (2,2) supersymmetric lagrangian. The extra terms in the Lagrangian that arise in this way are

$$L_V = -g_{i\bar{j}}|m|^2 V^i\overline{V^j} - \frac{i}{2}(g_{i\bar{i}}\partial_j V^i - g_{j\bar{j}}\partial_{\bar{i}}\overline{V^j})(m\overline{\psi^i_R}\psi^j_L + \bar{m}\overline{\psi^i_L}\psi^j_R) \; . \qquad (6.2)$$

Note that in our case, we cannot obtain any such holomorphic Killing vector - we have the extra requirement (coming from the self-duality of F_5) that $\nabla_\mu V^\mu = 0$.

It might be possible that including more background fields, such a three form RR field strength, we get a more general Lagrangian.

In the simple case where the transverse space is flat, we have a holomorphic killing vector $V_{\bar{j}} = ic_{i\bar{j}}z^i$, for a hermitian constant matrix $c_{i\bar{j}}$, and $\nabla_\mu V^\mu = 0$ translates into the tracelessness of $c_{i\bar{j}}$.

The combined action coming from $L_K + L_W + L_V$ is supersymmetric iff $V^\mu \nabla_\mu W$ is constant [131]. This matches nicely with the condition (5.16).

(1,1) Supersymmetric solutions

A general (1,1) supersymmetric sigma-model is of the form

$$S = \frac{1}{4\pi\alpha'} \int d\tau \int_0^{2\pi\alpha'|p_-|} d\sigma d^2\theta (g_{\mu\nu} D_L \phi^\mu D_R \phi^\nu + U(\phi)) \tag{6.3}$$

where ϕ^μ are $N = 1$ superfields. The superpotential $U(\phi)$ is not as general as it could be in an arbitrary $N = 1$ theory, since it needs to be a harmonic function. This condition also follows from conformal invariance in the Berkovits formulation [111]. Of course if we view the $N = (2, 2)$ solution as an $N = (1, 1)$ theory then the corresponding $N = 1$ superpotential is harmonic due to the stricter constraints that both the superpotential and Killing potential of the $N = 2$ theory have to obey.

6.1. RG Flow

The light cone worldsheet theory is a theory with a mass scale. So these theories behave quite non-trivially under RG transformations. This mass scale on the worldsheet is basically set by p_-. More precisely the important dimensionless parameter is $\alpha'|p_-|\mu$ where μ is the coefficient in front of the superpotential $W = \mu f(z/l_s)$ where f is a dimensionless function. This dimensionless parameter is the product of the mass scale on the worldsheet and the size of the worldsheet cylinder. A physical spacetime question, like the spectrum of the theory, depends non-trivially on this dimensionless parameter. We see that performing a scale transformation on the worldsheet is related to performing a boost in the x^+, x^- coordinates. For low values of $|p_-|$ we are exploring the UV of the worldsheet theory while for large values we explore the IR. As usual we have a UV/IR relation between worldsheet and target space scales. Note that in many situations, most notably the $c < 1$ string theories, one can start with a non-conformal theory and "dress" it with the Liouville mode so that the total theory is a critical string theory. In those cases the RG flow in the original massive theory becomes related to a change in position along the Liouville direction. Notice that this case has a different character since an RG transformation is related to a change in *velocity* of the motion in the x^+, x^- direction. In other words in one case we have that an RG transformation is a *translation* in the Liouville direction whereas in our case it is a *boost* in the x^+, x^- directions. The worldsheet will generically have periodic boundary conditions for the fermions since they are Green-Schwarz fermions. The number of zero energy (zero p_+) supersymmetric ground states can be computed by the standard index arguments. These will be BPS states in the spacetime theory.

It is interesting to note that we can choose a superpotential that has no supersymmetric vacua. In this case we do not have a supersymmetric vacuum on the worldsheet which means that the corresponding state in the spacetime theory is not BPS when p_- is non-zero. Supersymmetry breaking on the worldsheet should not be confused with spacetime supersymmetry breaking.

6.2. Solitons

One feature of our models is that they contain solitons on the worldsheet. The worldsheet is compact and has a size proportional to $|p_-|\alpha'$. If $|p_-|$ is large we will be able to trust soliton computations which are done in an infinite line. Note that when the string is propagating with fixed value of p_- it feels a gravitational force that pulls it to the regions where $-g_{++}$ is a minimum. A soliton on the worldsheet going between these minima corresponds to a string that goes between the two positions where $-g_{++}$ has a minimum in target space. For example, we can choose a superpotential which is a function of only one variable $W(z_1)$. In this case the three other complex fields on the worldsheet are massless and free. If we solve $\partial_{z_1} W = 0$ we will obtain the values of z_1 corresponding to supersymmetric vacua in the field theory. The gravitational force will be directed towards these points in spacetime. We can have string configurations that interpolate between these different points. However, as we are discussing closed strings of finite length (i.e. we impose periodic boundary conditions on the worldsheet), these configurations will not be topologically stable, unless there are identifications in the transverse space. We will discuss below a case with identifications in the transverse space.

6.3. Integrable Theories

It is possible to choose the superpotential in such a way that we get an integrable model on the worldsheet. We can then rely on the large literature on integrable models to derive properties of the worldsheet theory. Of course the most interesting regime is when the worldsheet theory is strongly coupled, since in this case we do not have any other simple method to derive the spectrum. Our above derivation of the lightcone worldsheet lagrangian is only valid for weak coupling, since we used the supergravity approximation. It is nevertheless possible to show that in the case of flat transverse space these are good string solutions by using one of Berkovits' formalisms [108,111] . We now take a flat transverse space and we explore the physics that results from adding a superpotential of the form $W(z^1) = \lambda \cos \omega z^1$. This gives the $N = 2$ supersymmetric sine Gordon theory.

More explicitly the full background is

$$
\begin{aligned}
ds^2 &= -2dx^+dx^- - |\lambda\omega \sin\omega z^1|^2(dx^+)^2 + dz^i d\overline{z^i} = \\
&= -2dx^+dx^- - \frac{1}{2}|\lambda\omega|^2[\cosh(2\omega x^5) - \cos(2\omega x^1)](dx^+)^2 + dx^i dx^i \\
F_5 &= dx^+ \wedge \varphi_4 \quad ; \quad \varphi_4 = \frac{\lambda\omega^2}{32}\cos(\omega z^1) dz^1 \wedge \overline{dz^2} \wedge \overline{dz^3} \wedge \overline{dz^4} + c.c.
\end{aligned}
\tag{6.4}
$$

where $z^1 = x^1 + ix^5$. The sine Gordon model is conventionally written in terms of canonically normalized fields $\phi = z/\sqrt{2\pi\alpha'}$ and the parameter β is defined by writing the superpotential as $W = \mu \cos\beta\phi$ (where μ, which is proportional to λ, has dimensions of mass). This implies that $w = \frac{\beta}{\sqrt{2\pi\alpha'}}$. At this point we could consider two models, one where x^1 is non-compact or another were x^1 is compact. Below we will be interested in the model where $x^1 \cong x^1 + 2\pi/\omega$. This model is such that we have two distinct supersymmetric vacua, $x^1 = 0, \frac{\pi}{\omega}$ (and also $x^5 = 0$). When we consider this sine Gordon model on an infinite spatial line (and time) one can compute exactly its S-matrix [133]. It was found that the S-matrix is the product of the S-matrices for two theories, one is an integrable version of the $N = 2$ minimal models and the other is the S-matrix of the bosonic sine Gordon theory. The $N = 2$ minimal model is the one with Z_2 global symmetry. The spectrum contains a kink and anti-kink together with some breathers of masses

$$
M_n = 2m_s \sin\left(\frac{n\pi}{2\gamma}\right) , \qquad \gamma = \frac{8\pi}{\beta^2} \tag{6.5}
$$

where $n = 1, ..., N$ and $N = [\gamma]$ is the number of breathers. m_s in (6.5) is the mass of a soliton which is proportional to μ. In order to find the spectrum of states in string theory we need to find the spectrum of the sine Gordon theory on a circle. If the size of the circle is very large, which corresponds to large $|p_-|$, we can use the Bethe ansatz to obtain an approximate answer for the spectrum. The corresponding expression is expected to be correct up to exponentially small corrections in the size of the circle (or $e^{-|p_-|\mu\alpha'}$). Some exact results for the spectrum on the cylinder for a simple integrable model were obtained in [134], but as far as we know the spectrum for the $N = 2$ sine Gordon on the cylinder is not known.

Note that the limit $\beta \to 0$ corresponds to the semiclassical limit of the sine Gordon model. In this limit the period of the sine is much longer than α'. This means that the background F field involves large length scales. In this limit there is a large number of breathers. The lowest lying breather is the basic perturbative massive field in the theory and the lowest lying ones can be thought of as bound states of these. On the other hand the limit of large β corresponds to the quantum regime of the sine Gordon model. Note that for $\gamma < 1$ there are no breathers, we only have the kinks and anti-kinks. When β is large the radius of the x^1 circle in string units is small so that one would attempt to do a T-duality on this circle. Since the background fields depends explicitly on x^1 this is not a straightforward T-duality. Fortunately the necessary transformation is the mirror symmetry transformation discussed in [84,85], which gives a sausage model. In fact this relation was conjectured first in [83], by studying the S-matrices and it is a close relative of [125]. The radius of the sausage is proportional to β. More precisely it is $\tilde{R} = \alpha'\omega$. We can see that in the limit that the RR fields are small, which is the UV of the worldsheet theory then in the original picture we have a cylinder with a gravitational potential that confines the strings to the region near the origin of the non-compact direction along the cylinder. In the T-dual picture we have a cylinder of the T-dual radius near the central region of the original cylinder, but the compact circle of the cylinder shrinks as we move away from the center so that we form a sausage. The sausage model is again not conformal invariant so that the geometry of the sausage depends on the scale. As we go to the UV of the field theory on the worldsheet the sausage becomes longer and longer as $\log(E)$, where E is the energy in question. Of course such a model contains a mass scale which is basically set by $|p_-|$. When we go to the IR the sausage model develops a mass gap and there are only a few massive excitations. We conclude that we have a background which is such that if we explore it with strings that have low values of $|p_-|$ we see it as being very large, while if we explore it with strings with higher values of $|p_-|$ it appears smaller. A natural question that arises is whether this background is a solution of the supergravity equations. For large values of $\tilde{R}$, which means large values of β, the curvature of the sigma model is small so one would expect it to be a solution of supergravity. In particular the $\beta = \infty$ limit is the $SU(2)$ symmetric round CP^1 model [82]. On the other hand, one could make an argument that this background cannot be a simple supergravity solution, at least within the context of a simple light cone reduction. The reason is the complicated way in which the scale of the model determines the geometry. When we go to light cone the scale that appears in the light cone theory is related to ∂X^+. If this scale appears quadratically or linearly in the lightcone action it is very simple to find the particular supergravity fields that give rise to the light cone gauge model, quadratic appearances of ∂X^+ are related to g_{++} and linear appearances of ∂X^+ are related to fields with one + index, such as $F_{+...}$. In the round CP^1 model the scale is appearing schematically as

$$S \sim \int \log(E/|p_-|)\partial\theta\partial\theta \sim \int \log(E/|\partial X^+|)\partial\theta\partial\theta \tag{6.6}$$

in the action, where the last term is *very* schematic. This suggests that the background leading to this CP^1 model contains excited massive string modes. In fact, if we treat the RR field as a small perturbation (which is correct if we are near the center of the cylinder and at small $|p_-|$) we can see that a T-duality in the the x^1 direction would transform the momentum mode of F_5 into a winding mode (with winding number two). This is somewhat reminiscent of the description of the cigar used in [135], though in that case one could view the background as a gravity solution. Another related, but distinct, way in which a massive theory as the CP^1 model could arise in string theory was presented in [86]. In that case the

RG direction was precisely x^+ and the metric was x^+ dependent.

All that we said here about the sine Gordon model can be generalized to affine Toda theories (with rank smaller than five) [82]. The mirror symmetry transformation in this case will produce a deformed CP^N model [85].

6.4. Resolving A_N Singularities

In this section we will consider deformations of A_N singularities in the presence of RR fields.[37] We can start with the maximally supersymmetric plane wave of IIB theory which has a field strength of the form $\varphi_{1234} = -\varphi_{5678}$=constant and all other components equal to zero. We can form complex coordinates $z^j = x^j + ix^{j+4}$. Then we see that this background corresponds to a background with zero (2,2) forms and a superpotential of the form $W = \mu \sum_{i=1}^4 (z^i)^2$. We can consider now the R^4 space spanned by the coordinates 1256 and replace it by an A_N singularity. This background still preserves half the supersymmetries. Let us start discussing first the case of an A_1 singularity. We see that we can replace the A_1 singularity by the Eguchi Hanson space, which is a Ricci flat Kähler (actually hyperKähler) manifold. When the RR fields are zero this solution preserves the same number of supersymmetries as the A_1 singularity. They preserve 8 supersymmetries that are linearly realized on the worldsheet, which is actually a (4,4) theory. We also have 8 other supersymmetries that are non-linearly realized and which are associated to the four real coordinates spanned by z^3, z^4 which are free on the worldsheet.

Another interesting situation to consider is an A_1 singularity involving the first four coordinates 1234. In this case, in order to find a supersymmetric deformation, it is convenient to group the coordinates into complex coordinates as $z^1 = x^1 + ix^2$, $z^2 = x^3 + ix^4$, etc. Then the maximally supersymmetric solution can be thought

[37] This problem was also considered in [136], where some singular solutions were described. Here we construct non-singular solutions.

of as a solution with $W = 0$ and only (2,2) forms with Killing potential $U = \mu(|z_1|^2+|z_2|^2-|z_3|^2-|z_4|^2)$. We can still resolve the A_1 singularity by replacing it by an Eguchi-Hanson space. In this case the solution will be of the type described in section 2.3. The Killing potential is

$$U = \mu[\sqrt{1+\frac{a^4}{\rho^4}}(|z^1|^2+|z^2|^2)-(|z^3|^2+|z^4|^2)] = \mu[r^2-(|z^3|^2+|z^4|^2)]\ , \qquad (6.7)$$

where $\rho^2 \equiv |z^1|^2+|z^2|^2$, $r^4 \equiv \rho^4+a^4$, and a is the Eguchi-Hanson resolution parameter. The derivatives of U form a holomorphic Killing vector $V^\nu = -ig^{\nu\bar{\nu}}\partial_{\bar{\nu}}U = -i\mu(z^1,z^2,-z^3,-z^4)$ and the (2,2) forms are given by $\varphi_{\nu\bar{\sigma}} = \nabla_\nu\nabla_{\bar{\sigma}}U$. One can see that the solution actually has (4,4) supersymmetry since one can redefine the coordinates $z_{3,4} \to \bar{z}_{3,4}$ and construct new Killing spinors of the type we constructed above. Furthermore if we view the theory as an $N=1$ theory the superpotential we get in both cases is the same, so that we have twice the number of supersymmetries. Potentials for $(4,4)$ two dimensional theories were considered in [137,131]. In conclusion, we have a (4,4) theory on the lightcone worldsheet. Of course we also have another 8 supersymmetries of the ϵ_- type that are due to the fact that the coordinates $z_3,\ z_4$ are free.

Above we discussed supersymmetric deformations of the A_1 singularity. There are also non-supersymmetric deformations, which we can describe most easily by writing the Eguchi Hanson metric in real coordinates

$$ds^2 = \frac{dr^2}{(1-\frac{a^4}{r^4})} + \frac{r^2}{4}(d\theta^2+\sin^2\theta d\phi^2) + \frac{r^2}{4}(1-\frac{a^4}{r^4})(d\psi+\cos\theta d\phi)^2 \qquad (6.8)$$

where the angles take values in $\theta \in [0,\pi)$; $\phi,\psi \in [0,2\pi)$. Then we can choose the four form to be proportional to the volume element, and the metric component $g_{++} = -\mu r^2$ looks the same as what it was for the original A_1 singularity. This solution is *not* supersymmetric. It differs from the supersymmetric solution by some terms which are localized near the singularity. We can view the non-supersymmetric solution as the supersymmetric one plus some normalizable modes that live near the singularity. These are normalizable modes of the four form potential. From the point of view of the worldvolume theory on the A_1 singularity, these are the modes that gives rise to the self dual tensor in six dimensions. Indeed one can check that the difference between the 5-form field-strengths of the two solutions is $\Delta F_5 \sim h_3 \wedge l_2$, where $h_3 = h_{+ij}$ is an anti-self dual tensor on the six directions corresponding to the worldvolume of the resolved A_1 singularity (i.e. directions $+-5678$) and l_2 is the unique normalizable anti-self dual two form on the Eguchi Hanson space, $l_2 = \frac{1}{r^2}[\frac{2}{r}dr \wedge (d\psi + \cos\theta d\phi) - \sin\theta d\theta \wedge d\phi]$.

The solution considered in [136] is equal to the non-supersymmetric solution described above, up to the addition of a harmonic function to g_{++} ,which is singular at $r = 0$. For any of the solutions described in this chapter, we can add a singular harmonic function of the transverse coordinates to g_{++}. We can think of them as describing the metric generated by massless particles with worldlines along x^-.

Of course all that we said above can be extended to A_{N-1} singularities by replacing the Eguchi-Hanson instanton by the geometry of the resolved ALE space. These A_{N-1} singularities arise as Penrose limits of $AdS_5 \times S^5/Z_N$, it would be nice to know if in this case we can also resolve the singularity in a smooth fashion. In the case of $(AdS_3 \times S^3)/Z_N$ we know that we can smooth out the singularity in simple way [66].

Appendix A

Topology of the solutions

If we have a single ring profile, such as the one in (3.7) then the harmonic functions (3.2) can be found explicitly and read

$$\begin{aligned} f_5 - 1 =& Qh^{-1} \;, \qquad f_1 - 1 = a^2\omega^2 Q h^{-1} \\ h^2 =& [(s+a)^2 + w^2][(s-a)^2 + w^2] \\ A_\phi =& 2a^2\omega Q s^2 \frac{1}{h(h+s^2+a^2+w^2)} \end{aligned} \tag{6.9}$$

where $s^2 = x_1^2 + x_2^2$ and $w^2 = x_3^2 + x_4^2$.

In order to understand more clearly the topology of the metric it is convenient to write the metric in other coordinates such that the metric reads

$$\begin{aligned} ds^2 =& \frac{1}{\sqrt{f_1 f_5}}[-(dt - \frac{a\sqrt{Q_1 Q_5}}{r^2 + a^2\cos^2\theta}\sin^2\theta d\phi)^2 + (dy + \frac{a\sqrt{Q_1 Q_5}}{r^2+a^2\cos^2\theta}\cos^2\theta d\psi)^2] + \\ &+ \sqrt{f_1 f_5}[(r^2 + a^2\cos^2\theta)(\frac{dr^2}{r^2+a^2} + d\theta^2) + r^2\cos^2\theta d\psi^2 + (r^2+a^2)\sin^2\theta d\phi^2] \\ &+ \sqrt{\frac{f_1}{f_5}} dz_a dz_a \\ e^{2\phi} =& \frac{f_1}{f_5}, \\ C_{(2)} =& (1 - \frac{1}{f_5})(dt - \sqrt{\frac{Q_1}{Q_5}} a \sin^2\theta d\phi) \wedge (dy - \sqrt{\frac{Q_1}{Q_5}} a\cos^2\theta d\psi) \\ &+ Q_1 \cos^2\theta d\phi \wedge d\psi \\ f_{1,5} =& 1 + \frac{Q_{1,5}}{r^2 + a^2\cos^2\theta} \end{aligned} \tag{6.10}$$

This form of the metric arises naturally if we view the solution as a limit of the general five dimensional black hole solutions in [34]. The explicit coordinate change from the coordinates w, s in (6.9) to the ones in (6.10) is

$$s^2 = (r^2 + a^2)\sin^2\theta \;, \qquad w = r\cos\theta \tag{6.11}$$

and ϕ and ψ are again the phases of $x_1 + ix_2$ and $x_3 + ix_4$ respectively. Here there is a potential singularity when $r = 0$ and $\theta = \pi/2$. We can rewrite the metric (6.10) in a form where its singularity structure is more transparent

$$\begin{aligned} ds^2 =& \sqrt{f_1 f_5}(r^2 + a^2\cos^2\theta)[d\theta^2 + h\cos^2\theta(d\psi + \frac{a\sqrt{Q_1 Q_5}}{f_1 f_5 h(r^2 + a^2\cos^2\theta)^2}dy)^2 + \\ &+ \tilde{h}\sin^2\theta(d\phi + \frac{a\sqrt{Q_1 Q_5}}{f_1 f_5 \tilde{h}(r^2 + a^2\cos^2\theta)^2}dt)^2] + \\ &+ \sqrt{f_1 f_5}(r^2 + a^2\cos^2\theta)[\frac{r^2}{g}dy^2 + \frac{dr^2}{r^2 + a^2}] - \frac{1}{\sqrt{f_1 f_5}}(1 + \frac{\sin^2\theta a^2 Q_1 Q_5}{f_1 f_5 \tilde{h}(r^2 + a^2\cos^2\theta)^3})dt^2 \end{aligned} \tag{6.12}$$

where the functions $h, \tilde{h}, g$ are

$$\begin{aligned} g =& Q_1 Q_5 + (Q_1 + Q_5)r^2 + (r^2 + a^2\cos^2\theta)r^2 \\ h =& \frac{g}{f_1 f_5 (r^2 + a^2\cos^2\theta)^2} \\ \tilde{h} =& \frac{Q_1 Q_5 + (Q_1 + Q_5)(r^2 + a^2) + (r^2 + a^2\cos^2\theta)(r^2 + a^2)}{f_1 f_5 (r^2 + a^2\cos^2\theta)^2} \end{aligned} \tag{6.13}$$

The important properties of these functions are $g(r = 0, \theta) = Q_1 Q_5$, $h(r, \theta = \pi/2) = 1$, $\tilde{h}(r, \theta = 0) = 1$. These properties, together with (3.8), ensure that the solution is nonsingular. Note that after the coordinate redefinition $\tilde{\psi} = \psi + y/R$ and $\tilde{\phi} = \phi + t/R$ the metric near $r \sim 0$ looks like that of a deformed S^3.

In order to study the topology of the solution we notice that the time direction will just give a factor of R, so we drop it from the discussion. The topology of a surface of large r is that of $S^1 \times S^3$. Near $r = 0$ we see that the S^1 circle shrinks to zero size while the sphere parameterized by $\tilde{\psi}, \theta, \tilde{\phi}$ does not. Topologically this is basically the same as the sphere we had at infinity since the map $(y, \psi, \theta, \phi) \to (y, \tilde{\psi}, \theta, \tilde{\phi})$ can be continuously deformed to the identity. This means that the final topology of the spatial region $r \leq r_0$ is that of a $D^2 \times S^3$. So the S^3 is non-contractible.

It is interesting to understand what the deformed three sphere that we have at $r = 0$ looks like in the original "flat" coordinates s, w. From (6.11) we see that $r = 0$

is the disk spanned by $w = 0$ and $s < a$. On top of this we have the y circle. These together from a three sphere since the y circle shrinks to zero at the boundary of the disk. Note that in the decoupling limit, where Q_i become very large the functions in (6.13) become constant. Then the three sphere parameterized by the coordinates $\tilde{\psi}, \theta, \tilde{\phi}$ is a round three sphere. Away from the decoupling limit it is not metrically a round three sphere.

Appendix B

Gravity Duals of Chiral Primaries on the Torus

In the previous sections we discussed the geometries corresponding to chiral primaries associated with $\mathrm{AdS}_3\times\mathrm{S}^3$. Such chiral primaries are universal and they do not depend on the structure of the internal manifold M in $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{M}$. But there are also some chiral primaries associated with the internal manifold, and in this section we will discuss them for the simplest case where $M = T^4$. We comment on the K3 case at the end.

To construct the geometries corresponding to such chiral primaries, we will follow the steps outlined in section 2. Namely we will start from the vibrating string, perform the dualities to relate it to the D1-D5 system, and then perform spectral flow to go to the NS sector. The only difference is that now we will allow the string to vibrate not only in noncompact directions, but also on the torus. Since to go to the D1-D5 system we have to perform dualities in the torus directions, the geometry of the vibrating string should be translation invariant in these directions, and we can achieve this by "smearing" in the torus coordinates (just like we smeared the profile on the y direction by performing integration over v in the string profiles (3.2)).

Thus we start with the metric of a vibrating string, smear it over the torus directions, and perform the following dualities

$$\begin{pmatrix} P(5) \\ F1(5) \end{pmatrix} \overset{S}{\rightarrow} \begin{pmatrix} P(5) \\ D1(5) \end{pmatrix} \overset{T6789}{\longrightarrow} \begin{pmatrix} P(5) \\ D5(56789) \end{pmatrix} \overset{S}{\rightarrow} \begin{pmatrix} P(5) \\ NS5(56789) \end{pmatrix} \overset{T5}{\rightarrow} \begin{pmatrix} F1(5) \\ NS5(56789) \end{pmatrix} \tag{6.14}$$

This way we get an "F1-NS5" of type IIA theory[38]

$$
\begin{aligned}
ds^2 &= \frac{1}{\tilde{f}_1}\left[-(dt - A_i dx^i)^2 + (dy + B_i dx^i)^2\right] + f_5 dx^i dx^i + dz^a dz^a \\
e^{2\Phi} &= \frac{f_5}{\tilde{f}_1}, \quad B_{ty} = -1 + \frac{1}{\tilde{f}_1}, \quad B_{ti} = \frac{B_i}{\tilde{f}_1}, \\
B_{yi} &= \frac{A_i}{\tilde{f}_1}, \quad B_{ij} = C_{ij} - \frac{A_i B_j - A_j B_i}{\tilde{f}_1} \quad C^{(1)}_a = f_5^{-1}\mathcal{A}_a, \\
C^{(3)}_{abc} &= f_5^{-1}\epsilon_{abcd}\mathcal{A}_d, \quad C^{(3)}_{iya} = \frac{A_i\mathcal{A}_a}{\tilde{f}_1 f_5}, \quad C^{(3)}_{ita} = \frac{B_i\mathcal{A}_a}{\tilde{f}_1 f_5}, \\
C^{(3)}_{ija} &= \frac{(A_i B_j - A_j B_i)\mathcal{A}_a}{\tilde{f}_1 f_5}, \quad C^{(3)}_{tya} = \mathcal{A}_a f_5^{-1}\left(-2 + \frac{1}{\tilde{f}_1}\right), \\
C^{(5)}_{tyabc} &= -\epsilon_{abcd}\mathcal{A}_d f_5^{-1}\left[2 - \frac{1}{\tilde{f}_1 f_5}\right], \quad C^{(5)}_{iyabc} = -\epsilon_{abcd}\frac{A_i\mathcal{A}_d}{\tilde{f}_1 f_5}, \\
C^{(5)}_{tiabc} &= -\epsilon_{abcd}\frac{B_i\mathcal{A}_d}{\tilde{f}_1 f_5}, \quad C^{(5)}_{ijabc} = \epsilon_{abcd}\frac{(A_i B_j - A_j B_i)\mathcal{A}_d}{\tilde{f}_1 f_5}
\end{aligned}
\tag{6.15}
$$

$$\tilde{f}_1 \equiv f_1 - f_5^{-1}\mathcal{A}_a\mathcal{A}_a$$

One can now perform additional T duality along one of the torus directions followed by S duality, to get a D1-D5 system. We will do this step only with the metric. But in any case, if one wants to study the properties of six dimensional Einstein metric, then one gets the same results starting either from D1-D5 or F1-NS5. The functions in (6.15) are given by[39]

$$
\begin{aligned}
f_5 &= 1 + \frac{Q}{L}\int\int_0^L \frac{d\mathbf{z}dv}{\left[(\mathbf{x}-\mathbf{F})^2 + (\mathbf{z}-\mathcal{F})^2\right]^2} = 1 + \frac{Q}{L}\int_0^L \frac{dv}{(\mathbf{x}-\mathbf{F})^2}, \\
f_1 &= 1 + \frac{Q}{L}\int_0^L \frac{|\dot{G}|^2 dv}{(\mathbf{x}-\mathbf{F})^2}, \quad A_i = -\frac{Q}{L}\int_0^L \frac{\dot{F}_i dv}{(\mathbf{x}-\mathbf{F})^2}, \quad \mathcal{A}_a = -\frac{Q}{L}\int_0^L \frac{\dot{\mathcal{F}}_a dv}{(\mathbf{x}-\mathbf{F})^2}.
\end{aligned}
\tag{6.16}
$$

Here we introduced an eight dimensional vector $\mathbf{G} = (F_i, \mathcal{F}_a)$.

---

[38] A simple further T-duality in one of the T^4 directions would give a solution in IIB

[39] The simplest way to construct the harmonic functions is following. We can first decompactify torus directions and look at the vibrating string in eight noncompact directions. Then we can smear over positions of the string in $z_1, \ldots z_4$ (which corresponds to integration over $\mathbf{z}$ in f_5), and in the end compactify $z_1, \ldots z_4$ on the torus.

Note that in (6.16) we have integrated over the position $\mathbf{z}$ of the string in the internal torus. This is done to obtain a solution that is independent of the internal coordinates. This implies that the dependence on $\mathcal{F}_a$ disappears from f_5 in (6.16) , but does not disappear from f_1 and $\mathcal{A}_a$.

Let us analyze the metric (6.15)near the singularity. Near the singularity we get

$$\begin{aligned} f_5 =& \frac{Q}{L}\frac{\pi}{|\dot{\mathbf{F}}|x_\perp}, \quad f_1 = \frac{Q|\dot{\mathbf{G}}|^2}{L}\frac{\pi}{|\dot{\mathbf{F}}|x_\perp}, \quad f_1 - 1 - f_5^{-1}\mathcal{A}_a\mathcal{A}_a = \frac{Q}{L}\frac{\pi|\dot{\mathbf{F}}|}{x_\perp}, \\ A_i =& -\frac{Q}{L}\frac{\pi\dot{F}_i}{|\dot{F}|x_\perp}, \quad \mathcal{A}_a = -\frac{Q}{L}\frac{\pi\dot{\mathcal{F}}_a}{|\dot{F}|x_\perp}, \end{aligned} \tag{6.17}$$

The expressions for f_5, A_i and $f_1 - 1 - f_5^{-1}\mathcal{A}_a\mathcal{A}_a$ do not depend on the profile in the internal directions $\mathcal{F}$, and thus the criteria for the absence of the singularity is the same as in the case with no vibrations on the torus, namely the profile should not self intersect in the $x_1, \ldots x_4$ space and $\dot{\mathbf{F}}$ should never vanish.

In the case of type IIB string theory on $AdS_3 \times S^3 \times K3$ there are also chiral primaries that are associated to extra anti-self dual 3-form gauge fields in six dimensions that come from anti-self-dual two forms on $K3$. Using heterotic/IIA duality it is very simple to get these solutions too. We have to perform the chain of dualities

$$\begin{pmatrix} P(5) \\ F1(5) \end{pmatrix} \quad \overset{het/IIA}{\rightarrow} \begin{pmatrix} P(5) \\ NS5(56789) \end{pmatrix} \quad \overset{T5}{\rightarrow} \begin{pmatrix} F1(5) \\ NS5(56789) \end{pmatrix} \tag{6.18}$$

so that in the end we get a solution of IIB on K3. In the heterotic theory the fundamental string can oscillate in the T^4 directions as well as in the 16 extra bosonic left moving directions on the heterotic worldsheet. Solutions of this type were discussed in [63][64]. It is in principle straightforward to perform the duality transformations, but we leave that for the interested reader.

An example of vibrations on the torus

We consider the simplest example for the vibrations on the torus:

$$F_1 = a\cos\omega v, \quad F_2 = a\sin\omega v, \quad \mathcal{F}_1 = b\cos m\omega v, \quad \mathcal{F}_2 = b\sin m\omega v, \tag{6.19}$$

all other components are zero. The frequency ω is related to the radius R of the y direction by (3.7). As we already mentioned, the expressions for f_5 and A_i remain the same as they were for $b = 0$, so to find the metric we only have to evaluate $\tilde{f}_1 = f_1 - f_5^{-1}\mathcal{A}_a\mathcal{A}_a$. Substituting the profile (6.19)in (6.15), we find:

$$f_1 = 1 + \frac{Q}{r^2 + a^2\cos^2\theta}\omega^2(a^2 + b^2m^2), \quad \mathcal{A}_1 = -\frac{Q}{2\pi}\sin m\phi I_m, \quad \mathcal{A}_1 = \frac{Q}{2\pi}\cos m\phi I_m, \tag{6.20}$$

where

$$\begin{aligned} I_m &\equiv b\omega m\int_0^{2\pi}\frac{d\alpha\cos m\alpha}{r^2 + a^2\sin^2\theta + a^2 - 2a\sqrt{r^2+a^2}\sin\theta\cos\alpha} \\ &= \frac{2\pi bm\omega}{r^2 + a^2\cos^2\theta}\left(-\frac{a\sin\theta}{\sqrt{r^2+a^2}}\right)^m \end{aligned} \tag{6.21}$$

Then we find:

$$\tilde{f}_1 = f_1 - f_5^{-1}\mathcal{A}_a\mathcal{A}_a = 1 + \frac{Q\omega^2}{r^2 + a^2\cos^2\theta}\left[(a^2 + b^2m^2) - \frac{Qb^2m^2}{Q + r^2 + a^2\cos^2\theta}\left(\frac{a^2\sin^2\theta}{r^2+a^2}\right)^m\right], \tag{6.22}$$

and the metric for the D1–D5 system becomes:

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{\tilde{f}_1 f_5}}[-(dt - \frac{a^2R}{r^2 + a^2\cos^2\theta}\sin^2\theta d\phi)^2 + (dy + \frac{a^2R}{r^2 + a^2\cos^2\theta}\cos^2\theta d\psi)^2] + \\ &+ \sqrt{\tilde{f}_1 f_5}[(r^2 + a^2\cos^2\theta)(\frac{dr^2}{r^2+a^2} + d\theta^2) + r^2\cos^2\theta d\psi^2 + (r^2+a^2)\sin^2\theta d\phi^2] \\ &+ \sqrt{\frac{\tilde{f}_1}{f_5}}d\mathbf{z}^2 \end{aligned} \tag{6.23}$$

Note that the total fivebrane charge is $Q_5 = Q$ and the onebrane charge is given by $Q_1 = Q\omega^2(a^2 + m^2b^2)$ where ω is as in (3.7). In particular we have:

$$R = \sqrt{\frac{Q_1Q_5}{a^2 + m^2b^2}}$$

In order to obtain (3.23) we need to drop the 1 in the harmonic functions in (6.16). This gives

$$\tilde{f}_1 - 1 = \frac{\alpha Q_1}{r^2 + a^2\cos^2\theta}, \quad f_5 - 1 = \frac{Q_5}{r^2 + a^2\cos^2\theta}, \quad A_\phi = \sqrt{\beta}\frac{a\sqrt{Q_1 Q_5}\sin^2\theta}{r^2 + a^2\cos^2\theta} \tag{6.24}$$

with α and β defined by

$$\beta = \frac{a^2}{a^2 + m^2 b^2}, \quad \alpha = 1 - (1-\beta)\left(\frac{\beta\sin^2\theta}{r^2+\beta}\right)^m \tag{6.25}$$

Note that for large values of m α is equal to one everywhere except for the small vicinity of the ring $r = 0, \theta = \frac{\pi}{2}$, and in the limit $m \to \infty$ the harmonic functions (6.24)reduce to the ones for the solution corresponding to a ring of rotating particles [69,60].

We define $\chi = y/R$ and rescale

$$t \to Rt, \quad r^2 \to r^2(a^2 + b^2 m^2) = r^2\frac{Q_1 Q_5}{R^2}, \tag{6.26}$$

then (6.20)becomes:

$$\begin{aligned}\frac{ds^2}{\sqrt{Q_1 Q_5}} &= (r^2 + \beta\cos^2\theta)\frac{1}{\sqrt{\alpha}}\left[-(dt - \frac{\beta\sin^2\theta d\phi}{r^2+\beta\cos^2\theta})^2 + (d\chi + \frac{\beta\cos^2\theta d\psi}{r^2+\beta\cos^2\theta})^2\right]\\ &+ \frac{\sqrt{\alpha}dr^2}{r^2+\beta} + \sqrt{\alpha}d\theta^2 + \frac{\sqrt{\alpha}}{r^2+\beta\cos^2\theta}(r^2\cos^2\theta d\psi^2 + (r^2+\beta)\sin^2\theta d\phi^2)\end{aligned} \tag{6.27}$$

Let us look at the limit $m \to \infty$ (which corresponds to $\alpha = 1$) and compare the above metric with the metric of the conical defect. To do this it is convenient to rewrite (6.27) for $\alpha = 1$ as

$$\begin{aligned}\frac{ds^2}{\sqrt{Q_1 Q_5}} &= -\left(r^2 + \frac{\beta-\beta^2}{2} + \beta^2\right)dt^2 + \left(r^2 + \frac{\beta-\beta^2}{2}\right)d\chi^2 + \frac{dr^2}{r^2+\beta}\\ &+ d\theta^2 + \cos^2\theta(d\psi + \beta d\chi)^2 + \sin^2\theta(d\phi + \beta dt)^2\\ &+ \frac{(\beta-1)\beta}{r^2+\beta\cos^2\theta}(\cos^4\theta d\psi^2 - \sin^4\theta d\phi^2) + \frac{\beta(1-\beta)}{2}\cos 2\theta(-dt^2 + d\chi^2)\end{aligned} \tag{6.28}$$

If we now introduce new coordinates:

$$r' = r + \frac{\beta - \beta^2}{4r}(1 - \cos 2\theta), \quad \theta' = \theta - \frac{\beta(\beta-1)}{4r^2}\sin 2\theta$$

then in the leading two orders at infinity the metric (6.28) becomes:

$$\begin{aligned}\frac{ds^2}{\sqrt{Q_1 Q_5}} =& -\left(r'^2 + \beta^2\right)dt^2 + r'^2 d\chi^2 + \frac{dr'^2}{r'^2 + \beta^2} \\ &+ d\theta'^2 + \cos^2\theta'(d\psi + \beta d\chi)^2 + \sin^2\theta'(d\phi + \beta dt)^2 \\ &+ (\beta - 1)\beta\cos 2\theta'\left[\frac{1}{r'^2}(d\theta'^2 + \cos^2\theta' d\psi^2 + \sin^2\theta' d\phi^2) + (dt^2 - d\chi^2 + \frac{dr'^2}{r'^4})\right]\end{aligned} \tag{6.29}$$

The first two lines give a metric of a conical defect, while the last line gives a perturbation, which corresponds to an AdS_3 scalar with angular momentum $l = 2$. This mode is a mixture of an overall rescaling of the sphere, AdS, and the three form field strengths. The fact that the correction to the AdS_3 part of the metric in (6.29) is not just an overall factor is due to the fact that these scalar fluctuations also imply a change of the metric of the form $\delta g_{\mu\nu} \sim \nabla_\mu \nabla_\nu \delta\phi$ where $\delta\phi$ is the scalar fluctuation. More details and explicit formulas can be found in [43]. Note that the terms in the last line of (6.29) are of the same order as the terms of the AdS_3 part of the metric in the first line. This implies that the conical defects are not a good approximation to these metrics.

The plane wave limit of the solution

In this subsection we take the plane wave limit of the solution (6.27). Let us call $\sqrt{Q_1 Q_5} = \epsilon^{-2}$, then we define rescaled quantities by

$$t = x^+, \quad \phi = \epsilon^2 x^- , \quad r = \epsilon\sqrt{\beta}s , \quad \frac{\pi}{2} - \theta = \epsilon u , \quad \tilde{m} = \frac{m}{\epsilon^2} \tag{6.30}$$

In the $\epsilon \to 0$ limit we get the metric

$$\begin{aligned}ds^2 =& \beta\alpha^{-1/2}[2dx^+dx^- - (s^2 + u^2)(dx^+)^2] + \alpha^{1/2}(ds^2 + du^2 + u^2 d\tilde{\psi}^2 + s^2 d\chi^2) \\ &+ (\alpha^{1/2} - \beta\alpha^{-1/2})\left[\frac{(dx^-)^2}{s^2 + u^2} - \frac{u^4 d\tilde{\psi}^2 - 2s^2u^2 d\chi d\tilde{\psi} + s^4 d\chi^2}{s^2 + u^2}\right]\end{aligned} \tag{6.31}$$

where now α becomes

$$\alpha = 1 - (1-\beta)e^{-\tilde{m}(u^2+s^2)} \tag{6.32}$$

Note that β remains fixed and $1-\beta$ has the interpretation of momentum p_- per unit length. This metric is non-singular. In the limit $\tilde{m} \to \infty$ it becomes the metric (3.26) which is singular. Of course for large $\tilde{m}$ the metric looks like (3.26) if $s^2 + u^2 > 1/\tilde{m}$ where we can approximate $\alpha \sim 1$.

It would be nice to understand why the asymptotic structure of (6.31) (where we can safely set $\alpha = 1$) is naively different from that of a usual plane wave. For example the transverse space is no longer R^4 in (3.26). This deserves further study.

The mass gap for the geometry (6.27)

Let us look at the spectrum of excitation on the background (6.27). For simplicity we will look at the minimally coupled scalar field, but our results will be true for more general excitations. Let us remind the reader that for a conical defect metric with opening angle $2\pi\gamma$ the mass gap is $E = 2\gamma$. Since we argued above that these conical metrics are not a good approximation to the metrics we consider we will perform an estimate of the mass gap in the metric (6.27).

For simplicity we will look at the modes of the scalar field which are constant in the ϕ, ψ, χ directions. Then looking for the solution in the form $\Phi(r,\theta,t) = e^{-iEt}\Phi(r,\theta)$, we find the Klein–Gordon equation:

$$\frac{1}{r}\partial_r(r(r^2+\beta)\partial_r\Phi) + \frac{1}{\sin\theta\cos\theta}\partial_\theta(\sin\theta\cos\theta\partial_\theta\Phi) + \frac{E^2\Phi}{(r^2+\beta\cos^2\theta)}\left[\alpha - \frac{\beta^2\sin^2\theta}{r^2+\beta}\right] = 0 \tag{6.33}$$

It is convenient to introduce new coordinates $x = r/\sqrt{\beta}$, $y = \cos\theta$. Then we find:

$$\begin{aligned}&\frac{1}{x}\partial_x(x(x^2+1)\partial_x\Phi) + \frac{1}{y}\partial_y(y(1-y^2)\partial_y\Phi)\\ &\quad + \frac{E^2\Phi}{\beta(x^2+y^2)}\left[\left(1-(1-\beta)\left(\frac{1-y^2}{1+x^2}\right)^m\right) - \beta\frac{1-y^2}{1+x^2}\right] = 0\end{aligned} \tag{6.34}$$

Note that for $m = 1$ the variables in this equation separate, and in particular we get spherically symmetric solutions which are normalizable near $x = 0$:

$$\Phi = (x^2 + 1)^{-k} F(-k, 1-k; 1; -x^2)$$

where $E = 2k\sqrt{\beta}$. This function in normalizable near infinity if and only if k is a positive integer. Thus for $m = 1$ we have a mass gap $E = 2\sqrt{\beta}$.

Let us now look at the more interesting case when $m > 1$. In this case the variables in the equation (6.34) do not separate, and we can't find the exact spectrum. But we can get an upper bound on the mass gap using variational methods. First we rewrite (6.34) as a Schroedinger equation:

$$(H - E^2 V)\Phi = 0.$$

where V is a positive potential ($E^2 V$ comes from the last term in (6.33)). This eigenvalue problem is the same as the one that arises when we have masses and springs, except that now the matrices are replaced by operators in a Hilbert space. Suppose this equation has a spectrum of eigenvalues E_k with corresponding eigenfunctions Φ_k obeying $(H - E_k^2 V)\Phi_k = 0$. These will generically form a complete basis system in the space of normalizable functions. Then for any such function we get

$$\Phi = \sum a_m \Phi_m$$

We also note that

$$\langle \Phi_k | (H - E^2 V) | \Phi_l \rangle = (E_k^2 - E^2) \langle \Phi_k | V | \Phi_l \rangle = (E_l^2 - E^2) \langle \Phi_k | V | \Phi_l \rangle$$

From here we conclude that

$$\langle \Phi_k | V | \Phi_l \rangle = 0, \quad if \quad k \neq l$$

(as usual, if there is a degeneracy $E_k = E_l$, the above condition gives a choice of a basis). For a generic function Φ we get:

$$\langle\Phi|(H - E^2 V)|\Phi\rangle = \sum(E_k^2 - E^2)|a_k|^2\langle\Phi_k|V|\Phi_k\rangle$$

Since V is positive, $\langle\Phi_m|V|\Phi_m\rangle \geq 0$. To show that there is an eigenvalue $E_0 < E$ it is sufficient to find a normalizable state $|\Phi\rangle$ such as

$$\langle\Phi|(H - E^2 V)|\Phi\rangle < 0$$

Let us take a trial function

$$\Phi = \frac{1}{1+x^2}.$$

Then taking an average of the left hand side of (6.34), we find

$$\begin{aligned}\langle\Phi|(H - E^2 V)|\Phi\rangle &= -\frac{1}{2}\int_0^\infty dx\Phi\partial_x(x(x^2+1)\partial_x\Phi) \\ &\quad - \int_0^\infty xdx\Phi^2\frac{E^2}{\beta}\int_0^1 \frac{ydy}{(x^2+y^2)}\left[1-(1-\beta)\left(\frac{1-y^2}{1+x^2}\right)^m - \beta\frac{1-y^2}{x^2+1}\right] \\ &= \int_0^\infty \frac{udu}{(1+u)^3} - \frac{E^2}{\beta}\left\{I_0 - (1-\beta)I_m - \beta I_1\right\}\end{aligned} \tag{6.35}$$

Here we introduced the following integral

$$\begin{aligned} I_k &= \int_0^\infty \frac{xdx}{(1+x^2)^2}\int_0^1 \frac{ydy}{(x^2+y^2)}\left(\frac{1-y^2}{1+x^2}\right)^k = \frac{1}{4}\int_0^\infty \frac{dx}{(x+1)^{k+2}}\int_0^1 \frac{(1-y)^k dy}{x+y} \\ &= \frac{1}{4}\int_0^\infty \frac{dx}{(x+1)^{k+2}}\frac{1}{x}F(1,1;k+2;-\frac{1}{x})B(1,k+1) \\ &= \frac{1}{4(k+1)} \end{aligned} \tag{6.36}$$

This gives

$$\langle\Phi|(H - E^2 V)|\Phi\rangle = \frac{1}{2} - \frac{E^2}{4\beta}\left\{1 - \frac{\beta}{2} - \frac{1-\beta}{m+1}\right\} \tag{6.37}$$

This expression becomes negative for $E > E_1$, where

$$E_1 = \sqrt{2\beta}\left\{1 - \frac{\beta}{2} - \frac{1-\beta}{m+1}\right\}^{-1/2} \tag{6.38}$$

so the mass gap is less than E_1. In particular, for all $m \geq 1$ we have $E_1 \leq 2\sqrt{\beta}$, so the mass gap is always less than this amount.

Appendix C

No Conical Defects with Arbitrary Opening Angles

One can easily write singular solutions with the same angular momentum as the solutions we have been considering. The simplest is a conical metrics of the form

$$\frac{ds^2}{R_{AdS}^2} = -(r^2+\gamma^2)dt^2+r^2d\chi^2+\frac{dr^2}{r^2+\gamma^2}+d\theta^2+\cos^2\theta(d\psi+\gamma d\chi)^2+\sin^2\theta(d\phi+\gamma dt)^2 \tag{6.39}$$

These metrics have a conical singularity at $r = (\pi/2 - \theta) = 0$. The singularity has a form which is rather similar to that of an A_N singularity but with an opening angle which is $2\pi\gamma$ instead of $2\pi/N$. In addition, if γ^{-1} is not an integer there are singularities at $r = 0$ and any θ.

When γ^{-1} is an integer we can think of the metric (6.39) as arising from a "supertube" configuration with N KK monopoles instead of just one KK monopole. Furthermore, it is possible to continuously deform the non-singular solutions that we had in this paper and get to these conical metrics. All we need to do is to take a profile $F(v)$ which wraps N times around the origin. If it does not self intersect we will have a smooth metric and as we take the limit that F is moving on the same circle N times we get the conical defect metric (6.39) with $\gamma^{-1} = N$.

On the other hand the metrics (6.39) with $\gamma^{-1} \neq N$ should not be allowed from the KK monopole point of view since they would mean that we have fractional KK monopole charge. In fact this is the reason that the singularity for non-integer γ^{-1} is more extended than for γ^{-1} integer. In the former case there is a fractional "Dirac string" coming out of the fractional KK monopole which is responsible for this singularity. Despite this strange features one might ask the following question. Can we find a smooth metric that is arbitrarily close to the metric (6.39) with non integer γ^{-1} ? When we say that a metric is *very close* to (6.39) we mean that the metric is equal to (6.39) up to very small corrections everywhere except very near the singularity. Namely, if we pick a γ^{-1}, say 3/2, then we pick an ϵ, say $\epsilon = 10^{-6}$, then we want to find a metric which only differs from (6.39) by terms of order ϵ once we are at $r > \epsilon$. We will now show that this is *not* possible[40].

Without loss of generality we can assume that the angular momentum is in the direction J_{12} and all other components vanish. In general the angular momentum of any configuration is characterized by two invariants J_L^2 and J_R^2 but for conical metrics of the asymptotic form (6.39) we have $J_L^2 = J_R^2$ so that using a rotation we can always put the angular momentum in the 12 plane. So suppose we have a metric that is very close to the metric of the conical defect for distances larger than some tiny distance ϵ. Then the harmonic functions will be very similar to the harmonic functions that give (6.39). The harmonic functions for (6.39) are given by (6.9) except that ω now obeys $\omega Q = \gamma R$. Since the harmonic functions are close to each other the source for the hypothetical non-singular metric should be close to the source of the harmonic functions in (6.9). In particular, f_5 implies that the source is distributed near a ring in the 12 plane. So in the expressions we will find below we will approximate $F_1^2 + F_2^2 - (F_3^2 + F_4^2) \sim F_1^2 + F_2^2$, but we do not make any assumptions about $\dot{F}_{3,4}^2$.

It is now instructive to consider the large r behavior of the metric. Using (6.15) we can read off all the harmonic functions of the form (6.16). The leading behavior of such functions is

$$\begin{aligned} f_5 &= \frac{Q_5}{x^2} + \frac{2Q_5\langle F_i\rangle x_i}{x^4} + \langle 4F_iF_j - F^2\delta_{ij}\rangle \frac{x_ix_j}{x^6} \\ f_1 &= \frac{Q_5\langle|\dot{G}|^2\rangle}{x^2} + \frac{2Q_5\langle F_i|\dot{G}|^2\rangle x_i}{x^4} + \langle(4F_iF_j - F^2\delta_{ij})|\dot{G}|^2\rangle \frac{x_ix_j}{x^6}, \\ A_i &= -2Q_5\langle \dot{F}_iF_j\rangle \frac{x_j}{x^4}, \quad B_i = -Q_5\epsilon_{ijkl}\langle \dot{F}_kF_l\rangle \frac{x_j}{x^4}, \quad \mathcal{A}_a = -2Q_5\langle \dot{\mathcal{F}}_aF_j\rangle \frac{x_j}{x^4}, \end{aligned} \tag{6.40}$$

[40] So, for example, it is futile to try to find the dual description of the conical defect metrics with arbitrary γ [62], since these metrics are not a good approximations to anything. It is OK to consider the ones with integer γ^{-1}.

First let us note that by shifting the origin we can always set $\langle F_i \rangle = 0$. Then the ten dimensional dilaton will be of the form

$$e^{2\Phi} = \frac{f_1}{f_5} = \frac{Q_1}{Q_5}(1 + 2\frac{x^i}{x^2}\frac{\langle F_i |\dot{G}|^2\rangle}{\langle |\dot{G}|^2\rangle} + \cdots) \tag{6.41}$$

Since this decays very slowly for large x we set its coefficient to zero. Similarly, by considering the fields that are excited by the torus fluctuations we conclude that we also need to set to zero $\langle \mathcal{F}_a F_i \rangle = 0$.

We have seen above that our metrics will generically have a particular operator of weight $(1,1)$ with a non-vanishing expectation value. This will give rise to a deformation of the metric that can be sensed far away. If we are interested in having a metric which is very close to the metric of a conical defect then we want to make the coefficient of this operator as small as possible. The operator we discussed is an $l = 2$ spherical harmonic on S^3 so that its coefficients have the form of a quadrupole moment $\mathcal{Q}_{ij}$. In particular we can look at the following combination:

$$\sqrt{\tilde{f}_1 f_5} - \frac{1}{\sqrt{\tilde{f}_1 f_5}}[A_i A_i - B_i B_i] = \frac{\sqrt{Q_1 Q_5}}{x^2} + \mathcal{Q}_{ij}\frac{x_i x_j}{x^6} + O(x^{-5})$$

then we notice that the quadrupole moment $\mathcal{Q}_{ij}$ vanishes for a conical defect. For a general metric $\mathcal{Q}_{ij}$ be computed by using (6.40) and performing a computation very similar to the one we did near (6.28), (6.29). We find

$$\mathcal{Q}_{11} + \mathcal{Q}_{22} \sim \left\langle [(F_1^2 + F_2^2) - (F_3^2 + F_4^2)](1 + \frac{|\dot{G}|^2}{\langle |\dot{G}|^2\rangle}) \right\rangle - 8\frac{\langle \dot{F}_1 F_2\rangle^2}{\langle |\dot{G}|^2\rangle} \tag{6.42}$$

where expectation values mean averages over v and we used that the angular momentum is in the 12 plane[41]. We want (6.42) to vanish in order to have a metric close to (6.39). As we argued above we can neglect $(F_3^2 + F_4^2)$ relative to $(F_1^2 + F_2^2)$ in (6.42). It is possible to show that the result we get after neglecting such a term

[41] Note that $2\langle F_1 \dot{F}_2 \rangle = \langle F_1 \dot{F}_2 - F_2 \dot{F}_1 \rangle \sim J_{12}$.

is always positive and it only vanishes when the profile is precisely a ring and the motion has constant velocity. In order to show that let us multiply all terms in (6.42) by $\langle|\dot{G}|^2\rangle$. Defining $F_1 + iF_2 = re^{i\phi}$ we then find

$$\Big(\langle r^2\rangle\langle r^2\dot{\phi}^2\rangle - \langle r^2\dot{\phi}\rangle^2\Big) + \Big(\langle r^4\dot{\phi}^2\rangle - \langle r^2\dot{\phi}\rangle^2\Big) + \text{other terms} \tag{6.43}$$

where all other terms are non negative. Using the formula $\langle ab\rangle^2 \leq \langle a^2\rangle\langle b^2\rangle$ (and the equal sign holds only if a/b =constant) we see that all terms are non negative so that if (6.43) vanishes then all terms should be zero. Setting the first term in (6.43) to zero we get that $\dot{\phi}$ = constant. Setting the second to zero we get r =constant. Setting to zero all other terms in (6.43) we get that $|\dot{\mathcal{F}}|^2 = \dot{F}_3^2 + \dot{F}_4^2 = 0$.

What we have shown so far is that if the metric is close to the conical defect then the profile closely tracks a profile with constant r and $\dot{\phi}$. Since ϕ has to be single valued this implies that only integer values of γ^{-1} are allowed.

We also see that generic chiral primaries with $J^{NS}_{L,R} < k/2$ do not produce conical metrics (6.39) but the metrics that we have discussed in our paper. Only very special chiral primaries produce metrics close to (6.39) with integer γ^{-1}.

Appendix D

Evaluation and Expansion of the Integrals $I_1^{(n)}(k), I_2^{(n)}(k)$

The integrals we defined in (3.39)

$$I_1^{(n)}(k) \equiv \frac{1}{2\pi}\int_0^{2\pi}\frac{d\alpha}{(1+k\cos\alpha)^{n/2}} \quad ; \quad I_2^{(n)}(k) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos\alpha d\alpha}{(1+k\cos\alpha)^{n/2}} \tag{6.44}$$

are not hard to evaluate. For even n the integrals are

$$\begin{aligned}
&I_1^{(2)}(k) = \frac{1}{\sqrt{1-k^2}} \quad , \quad I_1^{(4)}(k) = \frac{1}{(1-k^2)^{3/2}} \quad , \quad I_1^{(6)}(k) = \frac{(2+k^2)}{2(1-k^2)^{5/2}} \\
&I_2^{(2)}(k) = -\frac{(1-\sqrt{1-k^2})}{k\sqrt{1-k^2}} \;, I_2^{(4)}(k) = -\frac{k}{(1-k^2)^{3/2}} \;, I_2^{(6)}(k) = -\frac{3k}{2(1-k^2)^{5/2}}
\end{aligned} \tag{6.45}$$

For odd n , the integrals involve elliptic functions

$$\begin{aligned}
2\pi I_1^{(1)}(k) &= 4\frac{1}{\sqrt{1+k}}K(\sqrt{\frac{2k}{1+k}}) \\
2\pi I_1^{(3)}(k) &= 4\frac{\sqrt{1+k}}{1-k^2}E(\sqrt{\frac{2k}{1+k}}) \\
2\pi I_1^{(5)}(k) &= \frac{4\sqrt{1+k}}{3(1-k^2)^2}[-(1-k)K(\sqrt{\frac{2k}{1+k}}) + 4E(\sqrt{\frac{2k}{1+k}})] \\
2\pi I_2^{(1)}(k) &= \frac{4}{k\sqrt{1+k}}[(1+k)E(\sqrt{\frac{2k}{1+k}}) - K(\sqrt{\frac{2k}{1+k}})] \\
2\pi I_2^{(3)}(k) &= -\frac{4\sqrt{1+k}}{k(1-k^2)}[E(\sqrt{\frac{2k}{1+k}}) - (1-k)K(\sqrt{\frac{2k}{1+k}})] \\
2\pi I_2^{(5)}(k) &= \frac{4\sqrt{1+k}}{3k(1-k^2)^2}[-(1+3k^2)E(\sqrt{\frac{2k}{1+k}}) + (1-k)K(\sqrt{\frac{2k}{1+k}})]
\end{aligned} \tag{6.46}$$

They also obey the relations:

$$I_1^{(n)}(-k) = I_1^{(n)}(k), \quad I_2^{(n)}(k) = -I_2^{(n)}(-k), \quad I_2^{(n)}(k) = -\frac{2}{n-2}\partial_k I_1^{(n-2)}(k). \tag{6.47}$$

We were interested in evaluating the integrals at $k = -\frac{2as}{\sigma^2}$ so that in the near ring limit $k \to -1$ where the functions (6.45) and (6.46) are singular. Let us find the leading contribution near $k=-1$. For $n>1$ we find

$$
\begin{aligned}
I_1^{(n)}(k) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{d\alpha}{(1+k-2k\sin^2(\alpha/2))^{n/2}} \approx \frac{\sqrt{2}}{2\pi}\int_{-\infty}^{\infty}\frac{d\beta}{(1+k+\beta^2)^{n/2}} \\
&= \frac{\sqrt{2}}{2\pi}(1+k)^{-n/2+1/2}B(\frac{1}{2},\frac{n-1}{2}), \\
I_2^{(n)}(k) &\approx \frac{\sqrt{2}}{2\pi}(1+k)^{-n/2+1/2}B(\frac{1}{2},\frac{n-1}{2}).
\end{aligned}
\tag{6.48}
$$

For $n=1$ one can extract the leading asymptotic from the elliptic function

$$
I_1^{(1)} = I_2^{(1)} = \frac{\sqrt{2}}{2\pi}\ln\frac{32}{1+k} \tag{6.49}
$$

We also need the expression for

$$
I_3^{(n)} \equiv (I_1^{(n)})^2 - (I_2^{(n)})^2 \tag{6.50}
$$

and the asymptotics (6.48), (6.49) is not enough to evaluate it. Nevertheless, we can rewrite the leading asymptotics of this expression as

$$
I_3^{(n)} \approx 2I_1^{(n)}(I_1^{(n)} - I_2^{(n)}) \tag{6.51}
$$

and the problem is reduced to evaluation of the leading behavior of

$$
I_4^{(n)} \equiv I_1^{(n)} - I_2^{(n)} \approx \frac{1}{2\pi}\int_0^{2\pi}\frac{2\sin^2(\alpha/2)d\alpha}{(1+k-2k\sin^2(\alpha/2))^{n/2}} \tag{6.52}
$$

This integral can be written as

$$
I_4^{(n)} \approx \frac{2}{n-2}(k+1)^{2-n/2}\frac{\partial}{\partial k}\left[(k+1)^{n/2-1}I_1^{(n-2)}\right] \tag{6.53}
$$

for $n>1$, and the integrals for $I_4^{(1)}$ and $I_4^{(2)}$ can be evaluated explicitly. This gives the following asymptotics:

$$
I_4^{(1)} = \frac{2\sqrt{2}}{\pi}, \quad I_4^{(2)} = 1, \quad I_4^{(3)} = \frac{1}{\pi\sqrt{2}}\ln\frac{32}{1+k}, \quad I_4^{(n)} = \frac{B(\frac{1}{2},\frac{n-3}{2})}{\sqrt{2}\pi}\frac{(1+k)^{3-n}}{n-2} \ (n>3) \tag{6.54}
$$

Using above expressions we can find the leading asymptotics of $I_3^{(n)}$

$$I_3^{(1)} = \frac{8}{\pi^2}(\ln\frac{8a}{\rho}), \quad I_3^{(2)} = \frac{2a}{\rho},$$
$$I_3^{(3)} = \frac{8a^2}{\pi^2\rho^2}(\ln\frac{8a}{\rho}), \quad I_3^{(4)} = (\frac{a}{\rho})^4, \tag{6.55}$$
$$I_3^{(5)} = \frac{64}{9\pi^2}\frac{a^6}{\rho^6}, \quad I_3^{(6)} = \frac{3}{4}(\frac{a}{\rho})^8$$

where we have used

$$1 + k \approx \frac{\rho^2}{2a^2}$$

Appendix E

The Near Ring Solutions

In this appendix we expand (3.40) around the ring to examine its behavior.

Expanding (3.40) for small ρ , where ρ is the distance from the ring, $s = a + \rho\sin\Theta$, $w = \rho\cos\Theta$, using the expansions of I_1, I_2 around -1 which appear in Appendix D , we find the following near-ring metrics for the different dimensions[42]

* $d = 3$ [43]:

$$ds^2 \approx \frac{1}{a^2\omega^2}(\frac{a\pi}{Q})^{3/2}(\ln(8a/\rho))^{-1/2}[\frac{2a\omega Q}{\pi}dtd\phi + \frac{Q}{\pi a}dy^2 + \\ + \frac{a^2\omega^2 Q}{\pi}(\frac{4}{\pi} + \frac{1+a^2\omega^2}{a\omega^2 Q})d\phi^2] + \\ + \sqrt{\frac{Q}{\pi a}\ln\frac{8a}{\rho}}[d\rho^2 + \rho^2 d\Theta^2 + dz_5^2] \tag{6.56}$$

* $d = 4$:

$$ds^2 \approx \frac{1}{a^2\omega^2}(\frac{2a^2}{Q})^{\frac{3}{2}}(\frac{\rho}{a})^{\frac{1}{2}}[\omega Q dtd\phi + \frac{Q}{2a^2}dy^2 + \frac{\omega^2 Q^2}{2}(1 + \frac{1+a^2\omega^2}{\omega^2 Q})d\phi^2] \\ + \sqrt{\frac{Q}{2a^2}}(\frac{a}{\rho})^{\frac{1}{2}}[d\rho^2 + \rho^2 d\Omega_2^2 + dz_4^2] \tag{6.57}$$

---

[42] More rigorously, all of the limits above should be thought of as scaling limits where a, Q, ω remain constant and the coordinates scale. For $d = 4$ this scaling is $\rho,\ z^i \sim \epsilon^2$, $y, \phi, t \sim \epsilon$, $\epsilon \to 0$. then $ds^2 \sim \epsilon^3$. For $d \geq 5$ the scaling is $\rho, z^i \sim \epsilon^2$, $\phi \sim 1$, $y \sim \epsilon^{-(d-5)}$, $t \sim \epsilon^{-2(d-5)}$ and then the metric scales as $ds^2 \sim \epsilon^{7-d}$. In these limits, g_{tt} always scales to zero as $g_{tt} \sim (\frac{\rho}{a})^{3(d-3)/2} \sim \epsilon^{3(d-3)}$. however, as $g_{t\phi}$ remains finite in the limit, the metrics we obtain are nondegenerate.

[43] This form of the metric is valid only for $\rho \ll a$ where the log is strictly positive. For larger values of ρ, one needs to retain more terms in the expansion of the elliptic functions. A U-dual system of this $d = 3$ solution was recently considered in [73], where it was lifted to an M-theory solution with zero gauge fields. That solution is singular, as can be verified by calculating its curvature invariants.

* $d = 5$:

$$ds^2 \approx \frac{1}{a^2\omega^2}(\frac{\pi a^3}{Q})^{3/2}(\frac{\rho}{a})[\frac{2\omega Q}{a\pi}d\phi dt + \frac{Q}{a^3\pi}dy^2 + (\frac{\omega Q}{\pi a})^2 \ln\frac{8a}{\rho}d\phi^2]+ + \sqrt{\frac{Q}{a^3\pi}}(\frac{a}{\rho})[d\rho^2 + \rho^2 d\Omega_3^2 + dz_3^2] \tag{6.58}$$

* $d = 6$:

$$ds^2 \approx \frac{1}{a^2\omega^2}(\frac{4a^4}{Q})^{3/2}(\frac{\rho}{a})^{3/2}[\frac{\omega Q}{2a^2}dtd\phi + \frac{Q}{4a^4}dy^2 + (\frac{\omega Q}{4a^2})^2(\frac{a}{\rho})d\phi^2]+ + \sqrt{\frac{Q}{4a^4}}(\frac{a}{\rho})^{3/2}[d\rho^2 + \rho^2 d\Omega_4^2 + dz_2^2] \tag{6.59}$$

* $d = 7$:

$$ds^2 \approx \frac{1}{a^2\omega^2}(\frac{9\pi a^5}{2Q})^{3/2}(\frac{\rho}{a})^2[\frac{4\omega Q}{9\pi a^3}dtd\phi + \frac{2Q}{3\pi a^5}dy^2 + \frac{2\omega^2 Q^2}{3\pi^2 a^6}(\frac{a}{\rho})^2 d\phi^2]+ + \sqrt{\frac{2Q}{3\pi a^5}}\frac{a^2}{\rho^2}[d\rho^2 + \rho^2 d\Omega_5^2 + dz_1^2] \tag{6.60}$$

* $d = 8$:

$$ds^2 \approx \frac{1}{a^2\omega^2}[\frac{16a^6}{3Q}]^{3/2}(\frac{\rho}{a})^{5/2}[\frac{3\omega Q}{8a^4}dtd\phi + \frac{3Q}{16a^6}dy^2 + \frac{3\omega^2 Q^2}{256a^8}(\frac{a}{\rho})^3 d\phi^2]+ + \sqrt{\frac{3Q}{16a^6}}(\frac{a}{\rho})^{5/2}[d\rho^2 + \rho^2 d\Omega_6^2] \tag{6.61}$$

Looking at these metrics, one can see that only for $d = 4$, we obtain a $g_{\phi\phi}$ which scales with ρ like the other metric components parallel to the brane. For the other dimensions $d > 4$, we find that $g_{\phi\phi}$ goes to zero much slower than the other parallel components, as we approach the brane. However, one must bare in mind that what we should obtain are supergravity solutions describing a *brane with fluxes on a ring*. The effects of the curvature evidently affect the metric near the brane for all $d > 4$. This is related to I.R. phenomena on the worldvolume theory on the brane. Whether a solution is singular or not might depend on the U-duality frame in which it is presented. We did not find any frame where the solutions we have above for $d \neq 4$ ($d = 4$ is U-dual to the D1-D5 system) are non-singular.

Appendix F

Conventions and notations, and the Supersymmetry Equations

Flat transverse space

We use conventions where $x^{\pm} \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^9)$ and $\epsilon_{+-12345678} = +1$. $F_5 = dx^+ \wedge \varphi_4$. Since F_5 is self-dual and closed φ_4 is anti-self-dual in the transverse 8-dimensions and closed. For the metric (5.1) with flat transverse space we choose the vielbiens as $\theta^{\hat{i}} = dx^i$, $\theta^{\hat{+}} = dx^+$, $\theta^{\hat{-}} = dx^- - \frac{1}{2}H dx^+$. The corresponding connections all vanish except $\omega^{\hat{-}i} = -\omega^{i\hat{-}} = -\frac{1}{2}\partial_i H dx^+$. The covariant derivatives acting on spinors are $\nabla_- = \partial_-$, $\nabla_i = \partial_i$,$\nabla_+ = \partial_+ - \frac{1}{4}\partial_i H \Gamma_- \Gamma_i$. And the terms involving F_5 in the IIB covariant derivative are $\not{F}\Gamma_- = \Gamma^+ \not{\varphi}\Gamma_- = 0$, $\not{F}\Gamma_j = -\Gamma_- \not{\varphi}\Gamma_j$, $\not{F}\Gamma_+ = -\Gamma_- \not{\varphi}\Gamma_+$. The chirality matrix is $\Gamma_{11} = -\Gamma^{01\dots 89} = \frac{1}{2}[\Gamma^+, \Gamma^-]\Gamma^{1\dots 8}$. The IIB spinor is a 16-component complex chiral spinor satisfying $\Gamma_{11}\epsilon = +\epsilon$. Since φ_4 is anti-self-dual in 8-dimensions, acting on a chiral spinor $\not{F}\Gamma_+\epsilon = 2\not{\varphi}\epsilon$. Using all the above, the susy equations $D_\mu \epsilon \equiv (\nabla_\mu - \frac{i}{2}\not{F}\Gamma_\mu)\epsilon = 0$ take the form [44]

$$\partial_- \epsilon = 0 \quad ; \quad \partial_+ \epsilon - (\frac{1}{4}\Gamma_- \not{\partial} H - i\not{\varphi})\epsilon = 0 \quad ; \quad \partial_j \epsilon - \frac{i}{2}\Gamma_- \not{\varphi}\Gamma_j \epsilon = 0 \tag{6.62}$$

We would find it easier to work in complex coordinates, so we split the transverse space $(x^1, ..., x^8)$ to 4 complex coordinates $z^j = x^j + i x^{j+4}$. In complex coordinates, the susy equations (6.62) are

$$\begin{aligned} &\partial_- \epsilon = 0 \\ &\partial_+ \epsilon - (\frac{1}{4}\Gamma_- \bar{\Gamma} \cdot \bar{\partial} H + \frac{1}{4}\Gamma_- \Gamma \cdot \partial H - i\not{\varphi})\epsilon = 0 \\ &\partial_j \epsilon - \frac{i}{2}\Gamma_- \not{\varphi}\Gamma_j \epsilon = 0 \quad ; \quad \bar{\partial}_j \epsilon - \frac{i}{2}\Gamma_- \not{\varphi}\overline{\Gamma_j}\epsilon = 0 \end{aligned} \tag{6.63}$$

[44] To relate these conventions to the ones in Blau, Figueroa et al [94] take their conventions, replace their $x^{\pm}$ with $x^{0,9}$ according to $x^{\pm} = \frac{1}{\sqrt{2}}[x^9 \pm x^0]$. take $x^0 \to -x^0$ then flip one of the coordinates, say $x^1 \to -x^1$, and then replace back with chiral coordinates $x^{\pm}_{here} = \frac{1}{\sqrt{2}}[x^0 \pm x^9]$.

Let us classify the a.s.d 4-forms according to their holomorphicity properties. Denoting by (p,q) the number of holomorphic and anti-holomorphic indices in φ_{abcd} ($p+q=4$), there are 10 (1,3)-forms, 10 (3,1)-forms, and 15 (2,2)-forms, giving a total of 35 a.s.d. 4-forms. The (2,2) forms are of the form $\varphi_{i\bar{i}j\bar{k}}$ and $\varphi_{i\bar{i}j\bar{j}}$ (no sum), and a.s.d implies that $\varphi_{1\bar{1}2\bar{2}} = -\varphi_{3\bar{3}4\bar{4}}$ etc. and $\varphi_{1\bar{1}2\bar{3}} = \varphi_{4\bar{4}2\bar{3}}$ etc. The (3,1) and (1,3) forms are of the form $\varphi_{\bar{i}jkl}, \varphi_{\bar{i}ijk}, \varphi_{i\overline{jkl}}, \varphi_{i\overline{ijk}}$, and a.s.d. relates $\varphi_{1\overline{123}} = -\varphi_{4\overline{423}}$, $\varphi_{\bar{1}123} = -\varphi_{\bar{4}423}$ etc. The closed condition relates the (2,2) to the (1,3), (3,1) components. The reality condition on φ implies that $\varphi_{i\overline{jkl}} = \varphi^*_{\bar{i}jkl}$, $\varphi_{i\bar{j}k\bar{l}} = \varphi^*_{j\bar{i}l\bar{k}}$.

Going back to the susy equations (6.63), we separate ϵ into two components of different transverse chiralities $\epsilon = -\frac{1}{2}\Gamma_+\Gamma_-\epsilon - \frac{1}{2}\Gamma_-\Gamma_+\epsilon \equiv \epsilon_+ + \epsilon_-$. Since ϵ has a positive Γ_{11} chirality, ϵ_+ has positive $SO(1,1)$ and $SO(8)$ chiralities, and ϵ_- has both negative. The susy equations for ϵ_+ are $\partial_-\epsilon_+ = \partial_j\epsilon_+ = \overline{\partial_j}\epsilon_+ = (\partial_+ + i\not{\varphi})\epsilon_+ = 0$. As φ has negative $SO(8)$ chirality, automatically, $\not{\varphi}\epsilon_+ = 0$ and we conclude that ϵ_+ must be a constant spinor. The susy equations for ϵ_- are

$$
\begin{aligned}
\partial_-\epsilon_- = 0 \qquad &; \qquad (i\partial_+ - \not{\varphi})\epsilon_- = \frac{i}{4}\Gamma_-\not{\partial}H\epsilon_+ \\
\partial_j\epsilon_- = \frac{i}{2}\Gamma_-\not{\varphi}\Gamma_j\epsilon_+ \qquad &; \qquad \partial_{\bar{j}}\epsilon_- = \frac{i}{2}\Gamma_-\not{\varphi}\Gamma_{\bar{j}}\epsilon_+
\end{aligned}
\tag{6.64}
$$

In order to solve the susy equations explicitly, it is convenient to introduce a Fock space notation. The vacuum $|0\rangle$ is defined to be the spinor annihilated by $\Gamma_{\hat{+}}$ and by all Γ^i (where i is a holomorphic index) . We also define the operators $b^i = \Gamma^i = g^{i\bar{j}}\Gamma_{\bar{j}}$, $b^{+\bar{i}} = \Gamma^{\bar{i}}$. Note that in this normalization $\{b^i, b^{+\bar{j}}\} = 2g^{i\bar{j}}$, where $g^{i\bar{j}}$ is the inverse of the Kahler metric. This is not the usual normalization of annihilation and creation operators. We denote $\varphi_{mn} \equiv \frac{1}{3!}\varphi_{m\overline{ijk}}\epsilon^{\overline{ijkn}}g_{n\bar{n}}$, $\varphi_{\overline{mn}} \equiv (\varphi_{mn})^*$ (so that e.g. $\varphi_{24} = \varphi_{2\overline{123}}$, $\varphi_{21} = -\varphi_{2\overline{234}}$). Anti-self-duality implies that $\varphi_{mn} = \varphi_{nm}$, $\varphi_{\overline{mn}} = \varphi_{\overline{nm}}$. We also use the notation $2\varphi_{m\bar{n}} \equiv g^{s\bar{s}}\varphi_{s\bar{s}m\bar{n}}$, and denote by $\widetilde{b}^k|0\rangle \equiv b^k\frac{1}{4\ 4!}\epsilon_{\overline{ijkl}}(b^{+\bar{i}}b^{+\bar{j}}b^{+\bar{k}}b^{+\bar{l}})|0\rangle$ a 'hole' creation operator acting on the vacuum. The slashed four-form acts on the Fock space states as

$$
\not{\varphi}b^{+\bar{m}}|0\rangle = 4[\varphi^{\bar{m}}{}_n\widetilde{b}^n - \varphi^{\bar{m}}{}_{\bar{n}}b^{+\bar{n}}]|0\rangle \quad ; \quad \not{\varphi}\widetilde{b}^m|0\rangle = 4[\varphi^m{}_{\bar{n}}b^{+\bar{n}} - \varphi^m{}_n\widetilde{b}^n]|0\rangle \tag{6.65}
$$

where we have raised the indices of φ_{ab} using the metric. We parameterize $\epsilon_{\mp}$ in this Fock space

$$\epsilon_- = \Gamma_-[\beta_{\bar{k}} b^{+\bar{k}} + \delta_k \widetilde{b}^k]|0\rangle \quad ; \quad \epsilon_+ = [\alpha + \frac{1}{2}\gamma_{\bar{p}\bar{q}} b^{+\bar{p}} b^{+\bar{q}} + \zeta \frac{\epsilon_{\bar{i}\bar{j}\bar{k}\bar{l}}(b^{+\bar{i}} b^{+\bar{j}} b^{+\bar{k}} b^{+\bar{l}})}{4\ 4!}]|0\rangle \tag{6.66}$$

$\alpha, \gamma_{\overline{pq}}, \zeta$ are complex constants, and $\beta_{\bar{m}}, \delta_k$ are complex functions of $z^i, \overline{z^i}$. By an appropriate SO(8) rotation we will see that we can set $\gamma_{\bar{p}\bar{q}}$ to zero in our solutions. So from now on we set it to zero. Using (6.65) one can check that

$$\begin{aligned} \not{\varphi}\epsilon_- &= -4\Gamma_-[\beta_{\bar{m}}\varphi^{\bar{m}}{}_{\bar{n}} - \delta_m \varphi^m{}_{\bar{n}}] b^{+\bar{n}}|0\rangle + 4\Gamma_-[\beta_{\bar{m}}\varphi^{\bar{m}}{}_n - \delta_m \varphi^m{}_n]\widetilde{b}^n|0\rangle \\ \not{\nabla} H\epsilon_+ &= \alpha\partial_{\bar{j}} H b^{+\bar{j}}|0\rangle + \zeta\partial_j H \widetilde{b}^j|0\rangle \end{aligned} \tag{6.67}$$

The susy equations become the following equations for $\alpha, \beta_{\bar{m}}, \delta_m, \zeta$

$$\begin{aligned} &4(\beta_{\bar{m}}\varphi^{\bar{m}}{}_n - \delta_m\varphi^m{}_n) = -\frac{i}{4}\zeta\partial_n H + i\partial_+\delta_n \\ &4(-\beta_{\bar{m}}\varphi^{\bar{m}}{}_{\bar{n}} + \delta_m\varphi^m{}_{\bar{n}}) = -\frac{i}{4}\alpha\partial_{\bar{n}} H + i\partial_+\beta_{\bar{n}} \\ &\partial_j\beta_{\bar{k}} = -2i\alpha\varphi_{j\bar{k}} \ ; \ \partial_{\bar{j}}\beta_{\bar{k}} = 2i\zeta\varphi_{\overline{jk}} \\ &\partial_{\bar{j}}\delta_k = -2i\zeta\varphi_{k\bar{j}} \quad ; \quad \partial_j\delta_k = 2i\alpha\varphi_{jk} \end{aligned} \tag{6.68}$$

Curved transverse space

Starting from the metric

$$ds^2 = -2dx^+dx^- + H(x^\rho)(dx^+)^2 + g_{\mu\nu}(x^\rho)dx^\mu dx^\nu \tag{6.69}$$

the nonzero connections for this metric are $\Gamma^-_{++} = -\frac{1}{2}\partial_+ H$; $\Gamma^-_{+\mu} = -\frac{1}{2}\partial_\mu H$; $\Gamma^\mu_{++} = -\frac{1}{2}g^{\mu\nu}\partial_\nu H$; $\Gamma^\mu_{\nu\rho} = \gamma^\mu_{\nu\rho}$, where $\gamma^\mu_{\nu\rho}$ are the connections on the 8-dimensional manifold. The only components of the Ricci tensor which do not vanish are R_{++} and $R_{\mu\nu}$ which are given by $R_{++} = -\frac{1}{2}\nabla^2 H$; $R_{\mu\nu} = r_{\mu\nu}$, where $r_{\mu\nu}$ is the ricci tensor for the 8-dimensional metric. The Ricci scalar is the same as that of the 8-dimensional metric $R = r$. The Einstein equations are then $r_{\mu\nu} = 0$ and $\nabla^2 H = -32|\varphi|^2$, where $|\varphi|^2 \equiv \frac{1}{4!}\varphi_{\mu\nu\rho\delta}\varphi^{\mu\nu\rho\delta}$. We also introduce the corresponding flat indices

$a = (v, u, i, j, ...)$ and the coframe $\theta^v = dx^+; \quad \theta^u = dx^- - \frac{1}{2}H dx^+; \quad \theta^i_\mu dx^\mu$, such that $ds^2 = -2\theta^v\theta^u + \sum_i \theta^i\theta^i$. The connections are determined by the no torsion condition and their nonzero components are $\Omega^u{}_i = -\frac{1}{2}\theta^\mu_i \partial_\mu H dx^+$, $\Omega^i{}_j = \omega^i_{\mu\ j}(x^\rho)dx^\mu$, where $\omega^i{}_j(x^\rho)$ are the connections on the 8-dimensional manifold, satisfying $d\theta^i + \omega^i_j \wedge \theta^j = 0$. The covariant derivatives $\nabla_M = \partial_M + \frac{1}{2}\Omega^{ab}_M \Gamma_{ab}$ are given by

$$\nabla_- = \partial_- \ ; \ \nabla_\mu = \partial_\mu + \frac{1}{2}\omega^{ij}_\mu \Gamma_{ij} \ ; \ \nabla_+ = \partial_+ - \frac{1}{4}\theta^{i\mu}\partial_\mu H \Gamma_{ui} \tag{6.70}$$

And the susy equations $0 = D_M \epsilon = (\nabla_M + \frac{i}{2}\not{F}\Gamma_M)\epsilon$ are therefore

$$\begin{aligned} &\partial_- \epsilon = 0 \quad ; \quad \partial_+ \epsilon - \frac{1}{4}\Gamma_u \not{\partial} H \epsilon + i\not{\varphi}\epsilon = 0 \\ &[\partial_\mu + \frac{1}{2}\omega^{ij}_\mu \Gamma_{ij}]\epsilon - \frac{i}{2}\Gamma_u \not{\varphi}\Gamma_\mu \epsilon = 0 \end{aligned} \tag{6.71}$$

The above equations are exactly the ones we had before for the flat case (6.62), the only difference being trading the regular derivative in the 8-dim space with a covariant derivative. also we recall that the Einstein equations imply $g_{\mu\nu}$ is Ricci flat. Let us now try to solve these equations, similarly to what we did in the flat case. Again we change to complex coordinates, and separate $\epsilon = \epsilon_- + \epsilon_+$. As before, we get that ϵ_+ must be a covariantly constant spinor, i.e. [45] $\partial_- \epsilon_+ = \partial_+ \epsilon_+ = \nabla_\mu \epsilon_+ = \overline{\nabla_\mu}\epsilon_+ = 0$. The equations for ϵ_- are

$$\begin{aligned} &\partial_- \epsilon_- = 0 \\ &\nabla_\mu \epsilon_- = \frac{i}{2}\Gamma_u \not{\varphi}\Gamma_\mu \epsilon_+ \quad ; \quad \overline{\nabla_\mu}\epsilon_- = \frac{i}{2}\Gamma_u \not{\varphi}\overline{\Gamma_\mu}\epsilon_+ \\ &(i\partial_+ - \not{\varphi})\epsilon_- = \frac{i}{4}\Gamma_u \not{\partial} H \epsilon_+ \end{aligned} \tag{6.72}$$

As in the flat case, we again use the notation $\varphi_{\mu\nu}$, and introduce the Fock space $|0\rangle$ which is annihilated by Γ_v and by all Γ^μ (μ a holomorphic curved index), and is a covariantly constant spinor[46], and the operators $b^{\bar{\mu}+} \equiv \overline{\Gamma^\mu} = \theta^{\bar{\mu}}_{\bar{i}}\overline{\Gamma^i} \quad ; \quad b^\mu \equiv \Gamma^\mu =$

---

[45] From here on ∇_μ denotes a covariant derivative in the 8-dimensional transverse space.

[46] As the manifold is a CY, there is a covariantly constant spinor $\psi_0 = |0\rangle$. The spinor

$\theta^\mu_i \Gamma^i \quad ; \quad \{b^\mu, b^{\bar{\nu}+}\} = 2g^{\mu\bar{\nu}}$. From now on we can define the "hole" operator $\widetilde{b}^\mu$ as we did in the flat space case. Similarly we can define $\beta_{\bar{\mu}}$, δ_μ, α and ζ as in (6.66). We can similarly derive equations (6.65)(6.67) and finally (6.68), where all that we would need to do is to replace the ordinary derivative with covariant derivatives for the transverse indices.

$|0\rangle$ is actually constant. In fact the Killing spinor equation is $\partial_\mu |0\rangle + \frac{1}{2}\omega_\mu^{i\bar{j}} \Gamma_{i\bar{j}} |0\rangle = 0$. The term $\Gamma_{i\bar{j}}|0\rangle$ is proportional to $g_{i\bar{j}}$ and therefore to the trace of the spin-connection, which on a CY can be chosen to be zero [112].

Appendix G

Deriving the Flat Space Supersymmetric Solutions

We have seen that ϵ_+ should be a constant. As the transverse space is R^8 we can always do an SO(8) transformation which sets $\gamma_{\bar{p}\bar{q}} = 0$ in (6.66), but we will be unable to distinguish solutions with (2,2) susy from solution with more susy. We also set all x^+ dependence to zero, because, as discussed before, this part could always be added as a solution to the homogenous equations. Integrability of the $\partial_j \delta_k$ and $\partial_{\bar{j}} \beta_{\bar{k}}$ in (6.68) then assures (as α, ζ are not both zero) that the (1,3) and (3,1)-forms make a closed form by themselves. Using the fact that the (1,3) and (3,1) parts of φ are separately anti-self-dual and closed, we can show that φ_{ij} satisfies $\varphi_{ij} = \varphi_{ji}$ from anti-self-duality, $\partial_{[i}\varphi_{j]m} = g^{\bar{k}k}\partial_{\bar{k}}\varphi_{kj} = 0$ from closedness, for all i, j, m. These imply that $\varphi_{ij} = \partial_i \partial_j W$ where W is a harmonic function . Similarly, as $\varphi_{m\bar{n}}$ must be hermitian and closed by themselves, they must be of the form $\varphi_{m\bar{n}} = \partial_m \partial_{\bar{n}} \mathcal{U}$ where $\mathcal{U}$ is a real harmonic function. The equations (6.68)(with no x^+ dependence) become

$$\begin{aligned}
(\beta^m \partial_m \partial_n W - \delta^{\bar{m}} \partial_{\bar{m}} \partial_n \mathcal{U}) &= -\frac{i}{16}\zeta \partial_n H \\
-(\beta^m \partial_n \partial_{\bar{n}} \mathcal{U} - \delta^{\bar{m}} \partial_{\bar{m}} \partial_{\bar{n}} \overline{W}) &= -\frac{i}{16}\alpha \partial_{\bar{n}} H \\
\partial_j \beta_{\bar{k}} = -2i\alpha \partial_j \partial_{\bar{k}} \mathcal{U} \;\; &; \;\; \partial_{\bar{j}} \beta_{\bar{k}} = 2i\zeta \partial_{\bar{j}} \partial_{\bar{k}} \overline{W} \\
\partial_{\bar{j}} \delta_k = -2i\zeta \partial_k \partial_{\bar{j}} \mathcal{U} \;\; &; \;\; \partial_j \delta_k = 2i\alpha \partial_j \partial_k W
\end{aligned} \tag{6.73}$$

Integrability of the equations implies that

$$(|\zeta|^2 - |\alpha|^2)\partial_{\bar{j}} \partial_m \partial_k W = (|\zeta|^2 - |\alpha|^2)\partial_m \partial_{\bar{j}} \partial_k \mathcal{U} = (|\zeta|^2 - |\alpha|^2)\partial_{\bar{m}} \partial_{\bar{j}} \partial_k \mathcal{U} = 0 \; , \tag{6.74}$$

for all $m, \bar{m}, \bar{j}, k$. This can be satisfied in one of the following two cases

(i) $|\alpha| \neq |\zeta|$, W is holomorphic and harmonic, and $\varphi_{j\bar{k}} = \partial_j \partial_{\bar{k}} \mathcal{U}$ is a 4x4 hermitian traceless matrix of *constants*. In that case we can solve the ∂_j and $\overline{\partial_j}$

equations to get [47]

$$\beta_{\bar{k}} = -2i[\alpha\varphi_{j\bar{k}}z^j - \zeta\overline{\partial_k W}] \;\; ; \;\; \delta_k = -2i[\zeta\varphi_{k\bar{j}}\overline{z^j} - \alpha\partial_k W] \tag{6.75}$$

Then plugging these back into the first two equations in (6.73), and taking into account the fact that H is real, we get the *consistency condition* $\partial_n[\varphi_{j\bar{k}}z^j\partial_k W] = 0$, and the expression for $H = -32(|\partial_k W|^2 + |\varphi_{j\bar{k}}z^j|^2)$[48]. This is the solution with (2,2) supersymmetries, or more, that we have in (5.9). Plugging (6.75) in (6.66) we get the explicit expression for the four Killing spinors, which are parametrized by the two complex numbers α, ζ.

(ii) $|\alpha| = |\zeta|$. Now we have that for all $i, j, \bar{k}$ $\partial_i\partial_j\partial_{\bar{k}}[\mathcal{U} + \frac{\alpha}{\zeta}W] = 0$. Without loss of generality , we choose the constant phase $\frac{\alpha}{\zeta} = -1$.[49] Then one can define U a real harmonic function such that $\partial_j\partial_k U = \partial_j\partial_k W$ and $\partial_j\partial_{\bar{k}}U = \partial_j\partial_{\bar{k}}\mathcal{U}$, so the four-form is given by the second derivatives of U

$$\varphi_{ij} = \partial_i\partial_j U \quad ; \quad \varphi_{\bar{i}j} = \partial_{\bar{i}}\partial_j U \quad ; \quad \varphi_{\overline{ij}} = \partial_{\bar{i}}\partial_{\bar{j}}U \; . \tag{6.76}$$

Solving the ∂_j and $\overline{\partial_j}$ equations gives

$$\beta_{\bar{k}} = 2i\zeta\partial_{\bar{k}}U \;\; ; \;\; \delta_k = -2i\zeta\partial_k U \tag{6.77}$$

Then plugging these into the first two equations gives two identical equations for H , which are solved by $H = -32|\partial_k U|^2$. These are the (1,1) supersymmetric solutions we have in (5.10). Plugging (6.77) into (6.66) we get the explicit expression for the two Killing spinors that are parametrized by one complex number, $\alpha = -\zeta$.

---

[47] There is no need to add integration constants to $\beta_{\bar{k}}, \delta_k$, as such terms can be set to zero by a redefinition of dW by a constant shift, and a redefinition of z^j by a constant shift.

[48] Here too there is no need to add an integration constant to H, as such a constant can be set to zero, shifting x^- by a constant times x^+.

[49] This amounts to redefining the complex coordinates by a constant phase.

Appendix H

Deriving the Curved Space Supersymmetric Solutions

Here too we set $\gamma_{\overline{\mu\nu}} = 0$. This way we would still find all solutions with at least (1,1) supersymmetry, but would not be able to distinguish solutions with (2,2) supersymmetry from solutions with more supersymmetry. Note that if the transverse space has precisely SU(4) holonomy then the Killing spinor has $\gamma_{\bar{\mu}\bar{\nu}} = 0$. We also take as in the flat case, $\beta_{\bar{\nu}}, \delta_\nu$ to be independent of x^+ (the x^+ dependent part would be dealt with as part of the solution to the homogenous equations for ϵ_-). Then the equations that we get from (6.68) by replacing ordinary derivatives by covariant derivatives becomes.

$$\begin{aligned}
&4(\beta_{\bar{\mu}}\varphi^{\bar{\mu}}{}_{\nu} - \delta_\mu\varphi^{\mu}{}_{\nu}) = -\frac{i}{4}\zeta\partial_\nu H \\
&4(-\beta_{\bar{\mu}}\varphi^{\bar{\mu}}{}_{\bar{\nu}} + \delta_\mu\varphi^{\mu}{}_{\bar{\nu}}) = -\frac{i}{4}\alpha\partial_{\bar{\nu}} H \\
&\nabla_\mu\beta_{\bar{\nu}} = -2i\alpha\varphi_{\mu\bar{\nu}} \quad ; \quad \nabla_{\bar{\mu}}\beta_{\bar{\nu}} = 2i\zeta\varphi_{\overline{\mu\nu}} \\
&\nabla_{\bar{\mu}}\delta_\nu = -2i\zeta\varphi_{\nu\bar{\mu}} \qquad ; \quad \nabla_\mu\delta_\nu = 2i\alpha\varphi_{\mu\nu}
\end{aligned} \tag{6.78}$$

The integrability conditions for $\nabla\delta$ and $\overline{\nabla}\beta$ imply that $\nabla_{[\rho}\varphi_{\mu]\nu} = 0$ (i.e. the (1,3) and (3,1) forms are closed by themselves). Thus $\varphi_{\mu\nu} = \nabla_\mu\nabla_\nu W$ for some harmonic function W. The (2,2) forms therefore should be closed by themselves, and together with anti-self-duality they must satisfy $\varphi_{\mu\bar{\nu}} = \nabla_\mu\nabla_{\bar{\nu}}\mathcal{U} = \nabla_{\bar{\nu}}\nabla_\mu\mathcal{U}$ for some real harmonic function $\mathcal{U}$. Plugging these back to the equations (6.78) , we get

$$\begin{aligned}
&\nabla_\mu\beta_{\bar{\nu}} = -2i\alpha\nabla_\mu\nabla_{\bar{\nu}}\mathcal{U} \qquad ; \qquad \overline{\nabla_\mu}\delta_\nu = -2i\zeta\nabla_{\bar{\mu}}\nabla_\nu\mathcal{U} \\
&\overline{\nabla_\mu}\beta_{\bar{\nu}} = 2i\zeta\nabla_{\bar{\mu}}\nabla_{\bar{\nu}}\overline{W} \qquad ; \qquad \nabla_\mu\delta_\nu = 2i\alpha\nabla_\mu\nabla_\nu W \\
&-[\beta^\rho\nabla_\rho\nabla_{\bar{\nu}}\mathcal{U} - \delta^{\bar{\tau}}\nabla_{\bar{\tau}}\nabla_{\bar{\nu}}\overline{W}] = -\frac{i}{16}\alpha\partial_{\bar{\nu}}\overline{H} \\
&[\beta^\tau\nabla_\tau\nabla_\mu W - \delta^{\bar{\tau}}\nabla_\mu\nabla_{\bar{\tau}}\mathcal{U}] = -\frac{i}{16}\zeta\partial_\mu H
\end{aligned} \tag{6.79}$$

We can immediately solve the two equations in the first line to get $\beta_{\bar{\nu}} = -2i\alpha\nabla_{\bar{\nu}}\mathcal{U} + f_{\bar{\nu}}(\bar{z})$, $\delta_\nu = -2i\zeta\nabla_\nu\mathcal{U} + g_\nu(z)$ for some antiholomorphic and holomorphic one-forms

$f_{\bar{\nu}}(\bar{z}), g_\nu(z)$ respectively. Then we can plug these back into the two equations in the second line, and get the constraints

$$\nabla_\mu[\nabla_\nu(\zeta\mathcal{U}+\alpha W)+\frac{i}{2}g_\nu(z)]=\nabla_\mu[\nabla_\nu(\alpha^*\mathcal{U}+\zeta^* W)-\frac{i}{2}f^*_{\bar{\nu}}(z)]=0. \qquad (6.80)$$

These can be solved in one of two ways.

(i) $|\alpha|\neq|\zeta|$

Then we can define a new real harmonic function U related to $\mathcal{U}$ through $f_{\bar{\nu}}, g_\nu$ [50] such that $\nabla_\mu\nabla_{\bar{\nu}}U = \nabla_\mu\nabla_{\bar{\nu}}\mathcal{U}$, and by (6.80) $\nabla_\mu\nabla_\nu U = 0$. Note that U is a Killing potential, if we define a vector $V_\mu = i\nabla_\mu U$ then $\nabla_{\bar{\mu}}V^\nu = \nabla_\mu V^{\bar{\nu}} = 0$ and $\nabla_\mu V_{\bar{\nu}} + \nabla_{\bar{\nu}}V_\mu = 0$. This means that V^μ is a *holomorphic Killing vector*. Additionally, as U is a harmonic function, the Killing vector also satisfies $\nabla_\mu V^\mu = 0$. By (6.80), one also finds that $\nabla_\mu\nabla_\nu W$ is holomorphic. Since W appears in the susy equations only under two holomorphic covariant derivatives, we can take W to be holomorphic. One can now solve the first four equations in (6.79) to get [51]

$$\beta_{\bar{\nu}} = 2i[i\alpha V_{\bar{\nu}} + \zeta\overline{\nabla_\nu W}] \quad ; \quad \delta_\nu = 2i[-i\zeta V_\nu + \alpha\nabla_\nu W]\ , \qquad (6.81)$$

where $\varphi_{\mu\nu} = \nabla_\mu\nabla_\nu W$ and $\varphi_{\mu\bar{\nu}} = \nabla_\mu\nabla_{\bar{\nu}}U$. Then plugging these into the last two equations in (6.79), and using the fact H is real, we get one constraint on W and V^μ and one equation for H. The constraint is $\partial_\nu[V^\tau\nabla_\tau W]=0$, and the equation for H yields $H=-32(|dW|^2+|V|^2)$, where $|dW|^2\equiv g^{\mu\bar{\nu}}\nabla_\mu W\overline{\nabla_\nu W}$ and $|V|^2 \equiv g_{\mu\bar{\nu}}V^\mu V^{\bar{\nu}}$. This is the (2,2) supersymmetric solution we have in (5.17). Inserting (6.81) into (6.66) we get the explicit expression for the four preserved Killing vectors parametrized by $\alpha,\ \zeta$.

(ii) $|\alpha|=|\zeta|$. We can define a real harmonic function U such that $\nabla_\mu\nabla_\nu U = \nabla_\mu\nabla_\nu W$ and $\nabla_\mu\nabla_{\bar{\nu}}U = \nabla_\mu\nabla_{\bar{\nu}}\mathcal{U}$, so that $\varphi_{\mu\nu} = \nabla_\mu\nabla_\nu U$, $\varphi_{\mu\bar{\nu}} = \nabla_\mu\nabla_{\bar{\nu}}U$, $\varphi_{\overline{\mu\nu}} =$

[50] The relation is $U\equiv\mathcal{U}+\frac{(i\zeta^*\int g_\nu dz^\nu + c.c.)}{2[|\zeta|^2-|\alpha|^2]}+\frac{(i\alpha\int f_\nu dz^\nu + c.c.)}{2[|\zeta|^2-|\alpha|^2]}$.

[51] We did not include integration constants in β_ν, δ_ν as these can always be set to zero be a redefinition of the potentials.

$\nabla_{\bar{\mu}}\nabla_{\bar{\nu}}U$. Then solving for $\beta_{\bar{\nu}}$ and δ_{ν} , one gets

$$\beta_{\bar{\nu}} = 2i\zeta\nabla_{\bar{\nu}}U \quad ; \quad \delta_{\nu} = -2i\zeta\nabla_{\nu}U \tag{6.82}$$

Plugging these back into the last two equations (6.79), one gets the same equation for H, whose solution is $H = -32|dU|^2$. These are the (1,1) supersymmetric solutions we have in (5.18). Again we can insert (6.82) in (6.66) to get the explicit expression for the Killing spinors.

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