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Gluon tree amplitudes in open twistor string theory

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Abstract: We show how the link variables of Arkani-Hamed, Cachazo, Cheung and Kaplan (ACCK), can be used to compute general gluon tree amplitudes in the twistor string. They arise from instanton sectors labelled by d, with $d = n - 1$, where n is the number of negative helicities. Read backwards, this shows how the various forms for the tree amplitudes studied by ACCK can be grouped into contour integrals whose structure implies the existence of an underlying string theory.

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1 Introduction

Inspired by work of Arkani-Hamed, Cachazo, Cheung and Kaplan [1, 2], we use link variables to obtain expressions for tree amplitudes in open twistor string theory [3]–[5]. This extends the derivation from the twistor string beyond maximally helicity violating (MHV) amplitudes and special cases of non-MHV trees [6]–[12], as well as providing a basis for the dual structure envisioned in ACCK. We use a canonical quantization [13, 14] PROVA of Berkovits' version of twistor string theory, and compute the gluon trees.

The extensive literature for amplitudes in the spinor helicity basis, sampled by [15]–[19], has been used in developing recursion relations [20, 21]. These were motivated by a remarkable formulation of string theory on twistor space [5], which made contact with a twistor description for gauge theory [22, 23]. Additional dual forms for trees are found [24]–[26].

At loop level, the twistor string has been difficult to interpret as a dual for the gauge theory [4, 13]. In this paper, however, we show that the gauge theory based analysis of ACCK, which is phrased in terms of link variables, appears naturally to lead back to the twistor string at tree level. This may eventually enlighten our treatment of string loops, and the pursuit of the dual S-matrix.

Suppose we have N gluons, labeled $\alpha = 1, \ldots N$, with momenta $p^a_{\alpha\dot{a}} = \pi^a_\alpha \bar{\pi}_{\alpha\dot{a}}$, and helicities ϵ_α, m of which are positive and n negative, $m + n = N$. Write $\mathcal{P}$ for the set of positive helicity particles and $\mathcal{N}$ for the set of negative helicity particles.

The link variables $c_{ir}, i \in \mathcal{P}, r \in \mathcal{N}$ satisfy the $2N$ linear equations

$$\pi_j = \sum_{r\in\mathcal{N}} c_{jr}\pi_r \tag{1.1}$$

$$\bar\pi_s = -\sum_{i\in\mathcal{P}} \bar\pi_i c_{is}. \tag{1.2}$$

where we have suppressed the spinor indices. See [13] for our conventions. These equations are not independent because, as noted by ACCK, they imply momentum conservation. As in [1] (eq.(37)), these linear conditions imply energy-momentum conservation:

$$\sum_{j\in\mathcal{P}} \bar\pi_{j\dot a}\pi_j^a = \sum_{j\in\mathcal{P}}\sum_{r\in\mathcal{N}} \bar\pi_{j\dot a}c_{jr}\pi_r^a = -\sum_{r\in\mathcal{N}} \bar\pi_{r\dot a}\pi_r^a; \tag{1.3}$$

for momenta satisfying this consistency condition they provide $N' = 2N-4$ constraints on the mn variables c_{ir}, leaving

$$N_R = mn - N' = (m-2)(n-2) \tag{1.4}$$

degrees of freedom. The philosophy outlined by ACCK is to seek to write the tree and loop amplitudes of gauge theory as contour integrals over the remaining N_R degrees of freedom.

The main observation underlying the analysis of this paper is that, in open twistor string theory, the link variables should be of the form

$$c_{js} = \frac{k_j}{k_s(\rho_j-\rho_s)}, \tag{1.5}$$

for some suitable ρ_α, k_α, as we shall see at the beginning of section 3. The necessary and sufficient condition for the link variables to be of this form is that the matrix

$$\begin{pmatrix} (c_{i_1r_1})^{-1} & (c_{i_1r_2})^{-1} & \dots & (c_{i_1r_n})^{-1} \\ (c_{i_2r_1})^{-1} & (c_{i_2r_2})^{-1} & \dots & (c_{i_2r_n})^{-1} \\ \vdots & \vdots & & \vdots \\ (c_{i_mr_1})^{-1} & (c_{i_mr_2})^{-1} & \dots & (c_{i_mr_n})^{-1} \end{pmatrix} \tag{1.6}$$

should have rank two, as we discuss in more detail in section 5. This is equivalent to the vanishing of the determinant of each 3×3 submatrix, that is of each determinant of the form

$$\begin{aligned} \mathcal{C}^{ijk}_{rst} &= \begin{vmatrix} c_{is}c_{it} & c_{it}c_{ir} & c_{ir}c_{is} \\ c_{js}c_{jt} & c_{jt}c_{jr} & c_{jr}c_{js} \\ c_{ks}c_{kt} & c_{kt}c_{kr} & c_{kr}c_{ks} \end{vmatrix} \\ &= c_{ir}c_{js}c_{kt}c_{jr}c_{ks}c_{it} + c_{jr}c_{ks}c_{it}c_{kr}c_{is}c_{jt} + c_{kr}c_{is}c_{jt}c_{ir}c_{js}c_{kt} \\ &\quad - c_{kr}c_{js}c_{it}c_{jr}c_{is}c_{kt} - c_{jr}c_{is}c_{kt}c_{ir}c_{ks}c_{jt} - c_{ir}c_{ks}c_{jt}c_{kr}c_{js}c_{it}. \end{aligned} \tag{1.7}$$

For this condition to be met, it is sufficient for a suitable subset comprising N_R of the $\mathcal{C}^{ijk}_{rst}$ to vanish, e.g. if we fix $I, J \in \mathcal{P}$ and $R, S \in \mathcal{N}$, it is sufficient to have the vanishing of the N_R, quantities $\mathcal{C}^{IJk}_{RSt}$ where k ranges over $\mathcal{P}'$, the remaining $m-2$ elements of $\mathcal{P}$, and t

ranges over $\mathcal{N}'$, the remaining $n-2$ elements of $\mathcal{N}$. Using the linear conditions (1.1), (1.2) to express the c_{ir} in terms of the $c_{kt}, k \in \mathcal{P}', t \in \mathcal{N}'$, we find in section 3 that the tree amplitude will have the form

$$\oint F(c) \prod_{\substack{k\in\mathcal{P}'\\ t\in\mathcal{N}'}} \frac{dc_{kt}}{\mathcal{C}_{kt}}, \tag{1.8}$$

where $F(c)$ is a simple rational function of the c_{ir}.

In section 2, we review the derivation of vertex operator expressions for the general N-point tree amplitudes in twistor string theory from vertex operators. In section 3, we analyze the amplitude as an integral over constraints. In section 4, we derive the integrand function $F(c)$, as a function of the link variables, from twistor string theory. In section 5, we discuss the parametrization of the linear constraints, and complete the description of the contour integral expression for the amplitudes. In section 6, we compute all 6-point functions, including alternative forms, by evaluating the contour integral as a sum over residues. In section 7, we use our general formulae to check the 7-pt tree with alternating helicities.

2 The N-point amplitude

As in [3, 13], we consider conjugate twistor variables Z and W

$$Z = \begin{pmatrix} \pi^a \\ \omega^{\dot{a}} \end{pmatrix}, \qquad W = \begin{pmatrix} \bar{\omega}_a \\ \bar{\pi}_{\dot{a}} \end{pmatrix}, \tag{2.1}$$

$$W \cdot Z = \bar{\omega}\cdot\pi + \bar{\pi}\cdot\omega \equiv \bar{\omega}_a \pi^a + \bar{\pi}_{\dot{a}}\omega^{\dot{a}}, \tag{2.2}$$

and the field describing the twistor string,

$$Z(\rho) = \begin{pmatrix} \lambda^a(\rho) \\ \mu^{\dot{a}}(\rho) \end{pmatrix}.$$

We fourier transform the open string vertex operators for gluons according to their helicity [1], as

$$V_+^A(W,\rho) = \int d^2\pi^a e^{i\bar{\omega}_a\pi^a} \int \frac{d\kappa}{\kappa}\delta^2(\kappa\lambda^a(\rho)-\pi^a)e^{i\kappa\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}J^A = \int \frac{d\kappa}{\kappa}e^{i\kappa W\cdot Z(\rho)}J^A, \tag{2.3}$$

$$\begin{aligned} V_-^A(Z,\rho) &= \int d^2\bar{\pi}_{\dot{a}} e^{-i\omega^{\dot{a}}\bar{\pi}_{\dot{a}}} \int \kappa^3 d\kappa \delta^2(\kappa\lambda^a(\rho)-\pi^a)e^{i\kappa\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}J^A\psi^1\ldots\psi^4 \\ &= \int \kappa^3 d\kappa\delta^4(\kappa Z(\rho)-Z)J^A\psi^1\ldots\psi^4. \end{aligned} \tag{2.4}$$

Defining $X_j = W_j,\ j\in\mathcal{P};\quad X_s = Z_s,\ s\in\mathcal{N}$, we compute the tree amplitudes as a sum over instanton sectors. The only non-vanishing contribution to any tree with n negative

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helicity states is from the sector with instanton number d, where $d = n - 1$, [13]

$$
\begin{aligned}
M^{\epsilon_1 \ldots \epsilon_N} &= \int \langle 0|V_{\epsilon_1}^{A_1}(X_1,\rho_1)\ldots V_{\epsilon_N}^{A_N}(X_N,\rho_N)|0\rangle \prod_{\alpha=1}^{N} d\rho_\alpha \Big/ dg \\
&= \int \prod_{\alpha=1}^{N} \frac{d\rho_\alpha d\kappa_\alpha}{\kappa_\alpha} \langle 0|e^{(n-1)q_0} \prod_{s\in\mathcal{N}} \delta^4(\kappa_s Z(\rho_s) - Z_s) \exp\left\{ i \sum_{j\in\mathcal{P}} \kappa_j W_j \cdot Z(\rho_j) \right\} |0\rangle \\
&\quad \times \prod_{s\in\mathcal{N}} \kappa_s^4 \prod_{r<s;r,s\in\mathcal{N}} (\rho_r - \rho_s)^4 \langle 0|J^{A_1}(\rho_1)J^{A_2}(\rho_2)\ldots J^{A_N}(\rho_N)|0\rangle \Big/ dg \qquad (2.5)
\end{aligned}
$$

and dg is the invariant measure on the group $GL(2,\mathbb{R})$ of Möbius and scale transformations. Because $Z(\rho)$ is a polynomial of order $n-1$,

$$
Z(\rho) = \sum_{s\in\mathcal{N}} \frac{1}{\kappa_s} Z_s \prod_{r\neq s;r\in\mathcal{N}} \frac{\rho - \rho_r}{\rho_s - \rho_r}, \qquad (2.6)
$$

so that $\kappa_r Z(\rho_r) = Z_r$, $r \in \mathcal{N}$. If $\xi_r = \kappa_r Z(\rho_r) - Z_r$, $r \in \mathcal{N}$, the Jacobian resulting from performing the integrations corresponding to the zero modes of $Z(\rho)$ is

$$
\frac{\partial(\xi_r : r \in \mathcal{N})}{\partial(Z_{-n+1},\ldots,Z_0)} = \prod_{s\in\mathcal{N}} \kappa_s \prod_{r<s;r,s\in\mathcal{N}} (\rho_r - \rho_s).
$$

This factor will cancel the worldsheet fermion contribution included in (2.5), and the amplitude becomes

$$
M^{\epsilon_1\ldots\epsilon_N} = \int \prod_{\alpha=1}^{N} \frac{d\kappa_\alpha}{\kappa_\alpha} \exp\left\{ i \sum_{r\in\mathcal{N}} \sum_{j\in\mathcal{P}} c_{jr} W_j \cdot Z_r \right\} f^{A_1A_2\ldots A_N} \left[\prod_{\alpha=1}^{N} \frac{d\rho_\alpha}{\rho_\alpha - \rho_{\alpha+1}} \right] \Big/ dg, \quad (2.7)
$$

where

$$
c_{js} = \frac{\kappa_j}{\kappa_s} \prod_{r\neq s;r\in\mathcal{N}} \frac{\rho_j - \rho_r}{\rho_s - \rho_r}, \qquad (2.8)
$$

which we shall relate to (1.5) in section 3.

The action of the group GL(2,$\mathbb{R}$) is defined by

$$
\rho_\alpha \mapsto \frac{a\rho_\alpha + b}{c\rho_\alpha + d}, \qquad \kappa_\alpha \mapsto (c\rho_\alpha + d)^{n-1}\kappa_\alpha, \qquad (2.9)
$$

which leaves c_{js} invariant.

We transform to momentum space by applying

$$
\int \exp\left\{ -i \sum_{j\in\mathcal{P}} \bar{\omega}_j \cdot \pi_j + i \sum_{r\in\mathcal{N}} \bar{\pi}_r \cdot \omega_r \right\} \prod_{j\in\mathcal{P}} d^2\bar{\omega}_j \prod_{r\in\mathcal{N}} d^2\omega_r,
$$

giving

$$
\begin{aligned}
M^{\epsilon_1\ldots\epsilon_N} =& f^{A_1A_2\ldots A_N} \int \prod_{j\in\mathcal{P}} \delta^2\left(\pi_j - \sum_{r\in\mathcal{N}} c_{jr}\pi_r \right) \prod_{s\in\mathcal{N}} \delta^2\left(\bar{\pi}_s + \sum_{i\in\mathcal{P}} \bar{\pi}_i c_{is} \right) \\
&\times \left[\prod_{\alpha=1}^{N} \frac{d\kappa_\alpha d\rho_\alpha}{\kappa_\alpha(\rho_\alpha - \rho_{\alpha+1})} \right] \Big/ dg. \qquad (2.10)
\end{aligned}
$$

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As we saw in section 1, the delta function conditions imply energy-momentum conservation. Fixing a choice of $R, S \in \mathcal{N}$,

$$\begin{aligned}&\prod_{j\in\mathcal{P}}\delta^2\left(\pi_j-\sum_{r\in\mathcal{N}}c_{jr}\pi_r\right)\prod_{s\in\mathcal{N}}\delta^2\left(\bar{\pi}_s+\sum_{i\in\mathcal{P}}\bar{\pi}_i c_{is}\right)\\&\quad=\langle R,S\rangle^2\delta^4\left(\sum_{\alpha=1}^{N}\bar{\pi}_\alpha\pi_\alpha\right)\prod_{j\in\mathcal{P}}\delta^2\left(\pi_j-\sum_{r\in\mathcal{N}}c_{jr}\pi_r\right)\prod_{s\in\mathcal{N}'}\delta^2\left(\bar{\pi}_s+\sum_{i\in\mathcal{P}}\bar{\pi}_i c_{is}\right),\end{aligned} \tag{2.11}$$

so that we are left with $2N-4$ relations from the remaining delta functions to determine the $2N-4$ effective variables amongst the $\kappa_\alpha, \rho_\alpha$, after allowing for Möbius and scale invariance. The dependence of the $mn = n(N-n)$ variables c_{ir} on the $2N-4$ effective variables $\kappa_\alpha, \rho_\alpha$ implies the existence of

$$N_R = mn - (2N-4) = (m-2)(n-2)$$

(nonlinear) relations between the c_{ir}, say $\mathcal{C}_K(c) = 0, 1 \leq K \leq N_R$. e.g. for $N = 6, m = n = 3, \mathcal{P} = \{i,j,k\}, \mathcal{N} = \{r,s,t\}$, there is one relation from (1.7):

$$\begin{aligned}&c_{ir}c_{js}c_{kt}c_{jr}c_{ks}c_{it}+c_{jr}c_{ks}c_{it}c_{kr}c_{is}c_{jt}+c_{kr}c_{is}c_{jt}c_{ir}c_{js}c_{kt}\\&\qquad=c_{kr}c_{js}c_{it}c_{jr}c_{is}c_{kt}+c_{jr}c_{is}c_{kt}c_{ir}c_{ks}c_{jt}+c_{ir}c_{ks}c_{jt}c_{kr}c_{js}c_{it}.\end{aligned} \tag{2.12}$$

Now,

$$dg = \frac{d\rho_R d\rho_S}{(\rho_R-\rho_S)^2}\frac{d\kappa_R d\kappa_S}{\kappa_R\kappa_S},$$

so that

$$M^{\epsilon_1\ldots\epsilon_N} = f^{A_1A_2\ldots A_N}\delta^4\left(\sum_{\alpha=1}^{N}\bar{\pi}_\alpha\pi_\alpha\right)\mathcal{M}^{\epsilon_1\ldots\epsilon_N}, \tag{2.13}$$

with

$$\begin{aligned}\mathcal{M}^{\epsilon_1\ldots\epsilon_N}=&\langle R,S\rangle^2(\rho_R-\rho_S)^2\int\prod_{j\in\mathcal{P}}\delta^2\left(\pi_j-\sum_{r\in\mathcal{N}}c_{jr}\pi_r\right)\prod_{s\in\mathcal{N}'}\delta^2\left(\bar{\pi}_s+\sum_{i\in\mathcal{P}}\bar{\pi}_i c_{is}\right)\\&\times\prod_{\alpha=1}^{N}\frac{1}{(\rho_\alpha-\rho_{\alpha+1})}\prod_{\substack{\alpha=1\\ \alpha\neq R,S}}^{N}\frac{d\kappa_\alpha d\rho_\alpha}{\kappa_\alpha}.\end{aligned} \tag{2.14}$$

Note that if we chose $I, J \in \mathcal{P}$ rather than $R, S \in \mathcal{N}$,

$$\begin{aligned}&\prod_{j\in\mathcal{P}}\delta^2\left(\pi_j-\sum_{r\in\mathcal{N}}c_{jr}\pi_r\right)\prod_{s\in\mathcal{N}}\delta^2\left(\bar{\pi}_s+\sum_{i\in\mathcal{P}}\bar{\pi}_i c_{is}\right)\\&\quad=[I,J]^2\delta^4\left(\sum_{\alpha=1}^{N}\bar{\pi}_\alpha\pi_\alpha\right)\prod_{j\in\mathcal{P}'}\delta^2\left(\pi_j-\sum_{r\in\mathcal{N}}c_{jr}\pi_r\right)\prod_{s\in\mathcal{N}}\delta^2\left(\bar{\pi}_s+\sum_{i\in\mathcal{P}}\bar{\pi}_i c_{is}\right),\end{aligned} \tag{2.15}$$

and we have similar formulae to the above but with $[I,J]^2$ replacing $\langle R,S\rangle^2$.

3 The amplitude as an integral over constraints

In order take into account the constraints $\mathcal{C}_K$ in evaluating the amplitude (2.14), we will rewrite the world sheet integration on ρ and k as integrals over a set of independent link variables.

In (2.14), we have

$$c_{js} = \frac{\kappa_j}{\kappa_s} \prod_{\substack{r \neq s \\ r \in \mathcal{N}}} \frac{\rho_j - \rho_r}{\rho_s - \rho_r}, \qquad j \in \mathcal{P},\ s \in \mathcal{N}, \tag{3.1}$$

but it is convenient to change variables, defining

$$k_j = \prod_{r \in \mathcal{N}} (\rho_j - \rho_r)\kappa_j, \qquad k_s = \prod_{\substack{r \neq s \\ r \in \mathcal{N}}} (\rho_s - \rho_r)\kappa_s, \qquad j \in \mathcal{P},\ s \in \mathcal{N},$$

so that

$$c_{js} = \frac{k_j}{k_s} \frac{1}{\rho_j - \rho_s}. \tag{3.2}$$

Then (2.14) is left unchanged if we replace κ_α by k_α, and the action of the invariance group $GL(2, \mathbb{R})$ is now given by

$$\rho_\alpha \mapsto \frac{a\rho_\alpha + b}{c\rho_\alpha + d}, \quad k_j \mapsto (ad - bc)k_j/(c\rho_j + d), \quad k_r \mapsto k_r(c\rho_r + d), \qquad \alpha \in \mathcal{A},\ j \in \mathcal{P},\ s \in \mathcal{N},$$

$\mathcal{A} = \mathcal{P} \cup \mathcal{N}$. Writing

$$f_{ir}(c) = \langle i, r\rangle - \sum_{s \in \mathcal{N}} c_{is}\langle s, r\rangle, \qquad f_{rt}(c) = [r, t] + \sum_{i \in \mathcal{P}} [r, i]c_{it}, \qquad r = R, S,\ i \in \mathcal{P},\ t \in \mathcal{N}', \tag{3.3}$$

in the expression (2.14) we have

$$\begin{aligned}\langle R, S\rangle^2 \prod_{i \in \mathcal{P}} \delta^2 \left(\pi_i - \sum_{r \in \mathcal{N}} c_{ir}\pi_r\right) \prod_{t \in \mathcal{N}'} \delta^2 \left(\bar{\pi}_t + \sum_{i \in \mathcal{P}} \bar{\pi}_i c_{it}\right) \\ = K_1 \prod_{\substack{i \in \mathcal{P} \\ r = R,S}} \delta^2 (f_{ir}(c)) \prod_{\substack{t \in \mathcal{N}' \\ r = R,S}} \delta^2 (f_{rt}(c)) \equiv K_1 \delta^{N'} (f(c)),\end{aligned} \tag{3.4}$$

where $K_1 = \langle R, S\rangle^{m+2}[R, S]^{n-2}$.

Use $\varrho_\ell, 1 \leq \ell \leq N'$, to denote generically $\{\rho_\alpha, k_\alpha : 1 \leq \alpha \leq N, \alpha \neq R, S\}$ and divide the mn variables c_{ir} into two subsets: $c'_\ell, 1 \leq \ell \leq N'$, and $c''_K, 1 \leq K \leq N_R$, (e.g., fix $I, J \in \mathcal{P}$, in addition to $R, S \in \mathcal{N}$, and take the c' subset to consist of $\{c_{iR}, c_{iS}, c_{Ir}, c_{Jr} : i \in \mathcal{P}, r \in \mathcal{N}'\}$, and the c'' subset to consist of $\{c_{ir} : i \neq I, J,\ r \neq R, S\}$). Then (2.14) can be written

$$\mathcal{M}^{\epsilon_1 \ldots \epsilon_N} = K_1 \int \Psi(\varrho)\, \delta^{N'}(f) d^{N'} \varrho \tag{3.5}$$

where $f \equiv \tilde{f}(\varrho) = f(c(\varrho))$ and

$$\Psi(\varrho) = (\rho_R - \rho_S)^2 \prod_\alpha \frac{1}{\rho_\alpha - \rho_{\alpha+1}} \prod_{\alpha \neq R,S} \frac{1}{k_\alpha}. \tag{3.6}$$

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In principle, we could use the N' delta functions to perform the N' integrals over ϱ_ℓ and then calculate the amplitude as a function of the momenta by solving the equations (3.3) to give ρ and k in terms of the momenta and then substitute for them in

$$\mathcal{M}^{\epsilon_1\ldots\epsilon_N} = K_1 \Psi(\varrho) \left|\frac{\partial(f)}{\partial(\varrho)}\right|^{-1}, \tag{3.7}$$

but this is not a calculationally convenient way forward.

Instead, we seek to rewrite (3.5) first as an integral over all the mn variables c_{ir} and then to use the delta functions $\delta^{N'}(f(c))$ to perform N' of these integrations to leave an integral over N_R variables corresponding to the constraints $\mathcal{C}_K$. We can use these N_R constraints to express the N_R variables c''_K as functions of the remaining N' variables c'_ℓ, $c'' = \hat{c}''(c')$ and thus obtain N' functions $\hat{f}(c') = f(c', \hat{c}''(c'))$; these are the functions we would obtain if we used the N' equations (3.2), corresponding to the c' to express ρ, k as functions of the c'.

$$\begin{aligned}
\int \Psi(\varrho)\, \delta^{(N')}(\tilde{f})\, d^{N'}\varrho &= \int \Psi(\varrho) \delta^{N'}(\hat{f}) \left|\frac{\partial(c')}{\partial(\varrho)}\right|^{-1} d^{N'}c' \\
&= \int \Psi(\varrho) \left|\frac{\partial(c')}{\partial(\varrho)}\right|^{-1} \delta^{N'}(\hat{f}) \delta^{N_R}(\mathcal{C}) \left|\frac{\partial(\mathcal{C})}{\partial(c'')}\right| d^{mn}c \\
&= \int \Psi(\varrho) \left|\frac{\partial(c')}{\partial(\varrho)}\right|^{-1} \left|\frac{\partial(\mathcal{C})}{\partial(c'')}\right| \delta^{N'}(f) \delta^{N_R}(\mathcal{C}) d^{mn}c \\
&= \int \Psi(\varrho) \left|\frac{\partial(c')}{\partial(\varrho)}\right|^{-1} \left|\frac{\partial(\mathcal{C})}{\partial(c'')}\right| \delta^{N'}(f) \delta^{N_R}(\hat{\mathcal{C}}) d^{mn}c \\
&= \int \Psi(\varrho) \left|\frac{\partial(c')}{\partial(\varrho)}\right|^{-1} \left|\frac{\partial(\mathcal{C})}{\partial(c'')}\right| \left|\frac{\partial(f)}{\partial(c')}\right|^{-1} \delta^{N_R}(\hat{\mathcal{C}}) d^{N_R}c''
\end{aligned} \tag{3.8}$$

with $\hat{\mathcal{C}}(c'') = \mathcal{C}(\hat{c}'(c''), c'')$, where the functions $c' = \hat{c}'(c'')$ are obtained by using the N' equations $f(c', c'') = 0$ to express c' in terms of c''. The Jacobian of f with respect to c' is a constant, dependent only on momenta rather than the c_{ir}, because the f are linear; the value of the constant depends on the choice of the c'_ℓ. If we use the choice $\{c_{iR}, c_{iS}, c_{Ir}, c_{Jr} : i \in \mathcal{P}, r \in \mathcal{N}'\}$ for c',

$$\left|\frac{\partial(f)}{\partial(c')}\right| = \langle R, S\rangle^{2m} [R, S]^{n-2} [I, J]^{n-2}.$$

Then

$$\mathcal{M}^{\epsilon_1\ldots\epsilon_N} = K \int F(c'') \delta^{N_R}(\hat{\mathcal{C}}) d^{N_R}c'' \tag{3.9}$$

with

$$F(c'') = \Psi(\varrho) \left|\frac{\partial(c')}{\partial(\varrho)}\right|^{-1} \left|\frac{\partial(\mathcal{C})}{\partial(c'')}\right| \tag{3.10}$$

and

$$K = K_1 \left|\frac{\partial(f)}{\partial(c')}\right|^{-1} = \langle R, S\rangle^{2-m} [I, J]^{2-n}.$$

We can integrate (3.9) to obtain

$$\mathcal{M}^{\epsilon_1\ldots\epsilon_N} = K\sum F(c'')\left|\frac{\partial(\hat{\mathcal{C}})}{\partial(c'')}\right|^{-1}, \tag{3.11}$$

where the sum is over at least some of the simultaneous solutions of the N_R constraint equations $\hat{\mathcal{C}}_K(c'') \equiv \mathcal{C}_K(\hat{c}'(c''), c'') = 0$. [Note that in (3.11) the Jacobian is calculated for $\hat{\mathcal{C}}$, that is for $\mathcal{C}$ regarded as a function of c'' with c' put equal to $\hat{c}'(c'')$, whereas in (3.10) is for $\mathcal{C}$ with respect to c'', with all the c regarded as independent.] To find a rational answer for the amplitude, in line with the known results, we need to sum over all the solutions c'' with appropriate signs or phases that enable the contributions to be combined into a contour integral of the form

$$\mathcal{M}^{\epsilon_1\ldots\epsilon_N} = K\oint F(c'')\prod_K \frac{dc''_K}{\hat{\mathcal{C}}_K}. \tag{3.12}$$

[Here we understand the notation for the contour integral to include appropriate factors of $2\pi i$.] This will become apparent when we discuss the 6-point function, with $m = n = 3$, in detail in section 6. But first we will discuss the form of the integrand $F(c)$ in section 4, and the parameterization of the general solution for c_{ir} of the linear equations $f_\ell = 0$ in section 5.

4 The form of the integrand, $F(c)$

We can now give a general prescription for the integrand $F(c)$ in terms of the link variables, working from the string function (3.6). As before, we fix $I, J \in \mathcal{P}$ and $R, S \in \mathcal{N}$, set $\mathcal{P}' = \{k \in \mathcal{P} : k \neq i, j\}$ and $\mathcal{N}' = \{t \in \mathcal{N} : t \neq r, s\}$, and chose for the N_R variables c'' the collection $\{c_{kt} : k \in \mathcal{P}', t \in \mathcal{N}'\}$. Then the remaining variables c' are $\{c_{iR}, c_{iS}, c_{It}, c_{Jt} : i \in \mathcal{P}, t \in \mathcal{N}'\}$. Correspondingly, we take the N_R constraint functions $\mathcal{C}_K$ to be

$$\mathcal{C}_{kt} \equiv \mathcal{C}^{IJk}_{RSt}, \quad k \in \mathcal{P}',\ t \in \mathcal{N}'. \tag{4.1}$$

If $k, k' \in \mathcal{P}'$, and $t, t' \in \mathcal{N}'$,

$$\frac{\partial \mathcal{C}_{kt}}{\partial c_{k't'}} = 0 \qquad \text{unless } k = k' \text{ and } t = t', \tag{4.2}$$

so that

$$\begin{aligned}\left|\frac{\partial(\mathcal{C})}{\partial(c'')}\right| &= \prod_{\substack{k\in\mathcal{P}'\\ t\in\mathcal{N}'}} \frac{\partial \mathcal{C}_{kt}}{\partial c_{kt}}. \\ &= [c_{IR}c_{JS} - c_{IS}c_{JR}]^{N_R} \prod_{t\in\mathcal{N}'}[c_{It}c_{Jt}]^{m-2}\prod_{k\in\mathcal{P}'}[c_{kR}c_{kS}]^{n-2}\prod_{\substack{k\in\mathcal{P}'\\ t\in\mathcal{N}'}}\frac{1}{c_{kt}},\end{aligned} \tag{4.3}$$

using the expression we find from (1.7),

$$\begin{aligned}\frac{\partial \mathcal{C}_{kt}}{\partial c_{kt}} &= c_{kS}c_{IR}c_{JR}(c_{It}c_{JS} - c_{IS}c_{Jt}) - c_{kR}c_{IS}c_{JS}(c_{It}c_{JR} - c_{IR}c_{Jt}) \\ &= c_{It}c_{Jt}c_{kR}c_{kS}(c_{IR}c_{JS} - c_{IS}c_{JR})/c_{kt}.\end{aligned} \tag{4.4}$$

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Also it is not difficult to see that the Jacobian of c' with respect to ρ, k can be written as a product of factors:

$$\left|\frac{\partial(c')}{\partial(\rho,k)}\right| = \left|\frac{\partial(c_{IR},c_{IS})}{\partial(\rho_I,k_I)}\right| \times \left|\frac{\partial(c_{JR},c_{JS})}{\partial(\rho_J,k_J)}\right| \times \prod_{k\in\mathcal{P}'}\left|\frac{\partial(c_{kR},c_{kS})}{\partial(\rho_k,k_k)}\right| \times \prod_{t\in\mathcal{N}'}\left|\frac{\partial(c_{It},c_{Jt})}{\partial(\rho_t,k_t)}\right|$$
$$= \frac{k_I^{n-1}k_J^{n-1}}{k_R^m k_S^m}(\rho_R-\rho_S)^m(\rho_I-\rho_J)^{n-2}\prod_{k\in\mathcal{P}'}k_k \times \prod_{t\in\mathcal{N}'}\frac{1}{k_t^3}$$
$$\times \prod_{l\in\mathcal{P}}\frac{1}{(\rho_l-\rho_R)^2(\rho_l-\rho_S)^2}\prod_{t\in\mathcal{N}'}\frac{1}{(\rho_I-\rho_t)^2(\rho_J-\rho_t)^2}, \tag{4.5}$$

using

$$\frac{\partial(c_{IR},c_{IS})}{\partial(\rho_I,k_I)} = -\frac{k_I}{k_R k_S}\frac{(\rho_R-\rho_S)}{(\rho_I-\rho_R)^2(\rho_I-\rho_S)^2},$$
$$\frac{\partial(c_{It},c_{Jt})}{\partial(\rho_t,k_t)} = \frac{k_I k_J}{(k_t)^3}\frac{(\rho_I-\rho_J)}{(\rho_I-\rho_t)^2(\rho_J-\rho_t)^2}.$$

So

$$(\rho_R-\rho_S)^2\left|\frac{\partial(c')}{\partial(\rho,k)}\right|^{-1}\prod_{l\in\mathcal{P}}\frac{1}{k_l}\prod_{k\in\mathcal{N}'}\frac{1}{k_k} \tag{4.6}$$
$$= \frac{k_R^{2-m}k_S^{2-m}}{k_I^{2-n}k_J^{2-n}}(\rho_R-\rho_S)^{2-m}(\rho_I-\rho_J)^{2-n}c_{IR}^{-2}c_{IS}^{-2}c_{JR}^{-2}c_{JS}^{-2}\prod_{k\in\mathcal{P}'}c_{kR}^{-2}c_{kS}^{-2}\prod_{t\in\mathcal{N}'}c_{It}^{-2}c_{Jt}^{-2}\prod_{l\in\mathcal{P}}k_l^2\prod_{u\in\mathcal{N}}\frac{1}{k_u^2},$$

and, using

$$\frac{k_r k_s}{k_i k_j}(\rho_i-\rho_j)(\rho_s-\rho_r) = \frac{c_{ir}c_{js}-c_{is}c_{jr}}{c_{ir}c_{is}c_{jr}c_{js}}, \qquad \text{for } i,j\in\mathcal{P},\ r,s\in\mathcal{N}, \tag{4.7}$$

we have

$$(\rho_R-\rho_S)^2\left|\frac{\partial(c')}{\partial(\rho,k)}\right|^{-1}\left|\frac{\partial(\mathcal{C})}{\partial(c'')}\right|\prod_{i\in\mathcal{P}}\frac{1}{k_i}\prod_{k\in\mathcal{N}'}\frac{1}{k_k}$$
$$= \frac{k_I^n k_J^n}{k_R^m k_S^m}(\rho_R-\rho_S)^{-m}(\rho_I-\rho_J)^{-n}\frac{[c_{IR}c_{JS}-c_{IS}c_{JR}]^{N_R+2}}{c_{IR}^3c_{IS}^3c_{JR}^3c_{JS}^3}$$
$$\times\prod_{t\in\mathcal{N}'}c_{It}^{m-3}c_{Jt}^{m-3}\prod_{k\in\mathcal{P}'}c_{kR}^{n-3}c_{kS}^{n-3}\prod_{\substack{k\in\mathcal{P}\\ t\in\mathcal{N}}}\frac{1}{c_{kt}}\prod_{l\in\mathcal{P}}k_l^2\prod_{u\in\mathcal{N}}\frac{1}{k_u^2}, \tag{4.8}$$

and finally

$$F(c) = \Psi(\rho,k)\left|\frac{\partial(c')}{\partial(\rho,k)}\right|^{-1}\left|\frac{\partial(\mathcal{C})}{\partial(c'')}\right|$$
$$= \frac{k_I^n k_J^n}{k_R^m k_S^m}(\rho_R-\rho_S)^{-m}(\rho_I-\rho_J)^{-n}\frac{[c_{IR}c_{JS}-c_{IS}c_{JR}]^{N_R+2}}{c_{IR}^3c_{IS}^3c_{JR}^3c_{JS}^3}\prod_{t\in\mathcal{N}'}c_{It}^{m-3}c_{Jt}^{m-3}\prod_{k\in\mathcal{P}'}c_{kR}^{n-3}c_{kS}^{n-3}$$
$$\times\prod_{\substack{k\in\mathcal{P}\\ t\in\mathcal{N}}}\frac{1}{c_{kt}}\prod_{l\in\mathcal{P}}k_l^2\prod_{u\in\mathcal{N}}\frac{1}{k_u^2}\prod_{\alpha=1}^{N}\frac{1}{\rho_\alpha-\rho_{\alpha+1}}. \tag{4.9}$$

We can write this final product in terms of the c_{ir} by using (3.2) whenever $\epsilon_\alpha = -\epsilon_{\alpha+1}$; and using (4.7), with a factor of $(\rho_R - \rho_S)$ supplied when $\epsilon_\alpha = \epsilon_{\alpha+1} = +$, and a factor of $(\rho_I - \rho_J)$ supplied when $\epsilon_\alpha = \epsilon_{\alpha+1} = -$. This will leave $(\rho_I - \rho_J)^{-p}(\rho_R - \rho_S)^{-p}$, where p is the number of sign changes going from ϵ_1 to ϵ_N and back to ϵ_1. These factors can again be converted using (4.7), yielding a rational expression for $F(c)$ of order $6(m-2)(n-2)-mn$. We now give some examples and a general prescription.

(a) For $m = n$ and $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \ldots \epsilon_{2n-1}, \epsilon_{2n}) = (+, -, +, -, \ldots, +, -)$,

$$F(c) = [c_{IR}c_{JS} - c_{IS}c_{JR}]^{(n-2)(n-3)} \prod_{t\in\mathcal{N}'} c_{It}^{n-3}c_{Jt}^{n-3} \prod_{k\in\mathcal{P}} c_{kR}^{n-3}c_{kS}^{n-3} \prod_{\substack{k\in\mathcal{P}\\ t\in\mathcal{N}}}{}^{'} \frac{1}{c_{kt}}, \tag{4.10}$$

where the prime on the last product indicates that terms $1/c_{kt}$ should be omitted when k, t are adjacent.

(b) For $(\epsilon_1, \ldots, \epsilon_m, \epsilon_{m+1}, \ldots \epsilon_{m+n}) = (+, \ldots, +, -, \ldots, -)$, where we choose the labeling $\epsilon_i = +$, $1 \le i \le m$; $\epsilon_r = -$, $m+1 \le r \le N$; and $I = 1, J = m, R = m+1, S = N$, we have

$$\begin{aligned} F(c) = -c_{IS}c_{JR}\,[c_{IR}c_{JS} - c_{IS}c_{JR}]^{N_R+1} \prod_{i=1}^{m-1} \frac{1}{[c_{iR}c_{i+1,S} - c_{i+1,R}c_{iS}]} \\ \times \prod_{r=m+1}^{N-1} \frac{1}{[c_{Ir}c_{J,r+1} - c_{I,r+1}c_{Jr}]} \prod_{t\in\mathcal{N}'} c_{It}^{m-1}c_{Jt}^{m-1} \prod_{k\in\mathcal{P}'} c_{kR}^{n-1}c_{kS}^{n-1} \prod_{\substack{k\in\mathcal{P}\\ t\in\mathcal{N}}} \frac{1}{c_{kt}}. \end{aligned} \tag{4.11}$$

(c) In general, if $(\epsilon_1, \ldots, \epsilon_N)$, begins with $\epsilon_1 = +$ and ends with $\epsilon_N = -$ and comprises p strings with $\epsilon_\alpha = +$ and, therefore, p strings with $\epsilon_\alpha = -$, then, up to sign, $F(c)$ is given by

$$\begin{aligned} F(c) = [c_{IR}c_{JS} - c_{IS}c_{JR}]^{N_R-p+2} c_{IR}^{p-3}c_{IS}^{p-3}c_{JR}^{p-3}c_{JS}^{p-3} \prod_{t\in\mathcal{N}'} c_{It}^{m-3}c_{Jt}^{m-3} \\ \times \prod_{k\in\mathcal{P}'} c_{kR}^{n-3}c_{kS}^{n-3} \prod_{\substack{k\in\mathcal{P}\\ t\in\mathcal{N}}} \frac{1}{c_{kt}} \prod_{\alpha=1}^{N} d_{\alpha,\alpha+1}, \end{aligned} \tag{4.12}$$

where

$$d_{ir} = c_{ir}, \quad d_{ri} = c_{ir}, \quad d_{ij} = \frac{c_{iR}c_{jS}c_{jR}c_{iS}}{c_{iR}c_{jS} - c_{jR}c_{iS}}, \quad d_{rs} = \frac{c_{Ir}c_{Js}c_{Is}c_{Jr}}{c_{Ir}c_{Js} - c_{Is}c_{Jr}},$$
$$i, j \in \mathcal{P}, \; r, s \in \mathcal{N}.$$

Note, one may obtain different expressions for the integrands using the identity

$$\frac{c_{iR}c_{jS} - c_{jR}c_{iS}}{c_{IR}c_{JS} - c_{IS}c_{JR}} = \frac{c_{ir}c_{js} - c_{is}c_{jr}}{c_{Ir}c_{Js} - c_{Is}c_{Jr}} \times \frac{c_{iR}c_{jS}c_{jR}c_{iS}c_{Ir}c_{Js}c_{Is}c_{Jr}}{c_{ir}c_{js}c_{is}c_{jr}c_{IR}c_{JS}c_{IS}c_{JR}}.$$

For given m and n, the expressions for $F(c)$ for different orderings of the helicities are related by the transformations given in appendix A.

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5 Parameterization of the general solution of linear constraints on c_{ir}

In this section we will parameterize the general solution to the linear constraints (1.1), (1.2) on the link variables in order to express them in terms of suitable independent variables over which to perform the multi-dimensional contour integral to obtain the amplitudes. As remarked in section 1, the $2N$ linear equations (1.1), (1.2) imply energy-momentum conservation. Thus they typically provide $N' = 2N - 4$ constraints on the variables c_{ir}, leaving $N_R = mn - N'$ degrees of freedom. These remaining degrees of freedom are determined by the N_R independent constraints $\mathcal{C}_K$ that follow from the requirement that c_{ir} be of the form (3.2), and we shall discuss the form of these non-linear constraints in this section.

A solution to the linear equations (1.1), (1.2) is always provided by $c_{ir} = a_{ir}$, where

$$a_{ir} = \frac{1}{p^2}\sum_{j\in\mathcal{P}}\langle i,j\rangle[j,r] = -\frac{1}{p^2}\sum_{s\in\mathcal{N}}\langle i,s\rangle[s,r], \tag{5.1}$$

using energy-momentum conservation, and

$$p = \sum_{j\in\mathcal{P}} p_j = -\sum_{r\in\mathcal{N}} p_r, \qquad p^2 = \sum_{\substack{i<j\\ i,j\in\mathcal{P}}}\langle i,j\rangle[i,j] = \sum_{\substack{r<s\\ r,s\in\mathcal{N}}}\langle r,s\rangle[r,s]. \tag{5.2}$$

To show that $c_{ir} = a_{ir}$ satisfies (1.1), (1.2), first note that

$$\langle i,j\rangle\pi_k + \langle j,k\rangle\pi_i + \langle k,i\rangle\pi_j = 0, \qquad \text{for any } i,j,k,$$

because, taking the angle bracket with any vector π_l,

$$\langle i,j\rangle\langle k,l\rangle + \langle j,k\rangle\langle i,l\rangle = \langle k,l\rangle\langle i,j\rangle - \langle k,j\rangle\langle i,l\rangle = -\langle k,i\rangle\langle j,l\rangle.$$

Similarly

$$[r,s]\overline{\pi}_t + [s,t]\overline{\pi}_r + [t,r]\overline{\pi}_s = 0, \qquad \text{for any } r,s,t. \tag{5.3}$$

So

$$\begin{aligned}\sum_{r\in\mathcal{N}} a_{ir}\pi_r &= -\frac{1}{p^2}\sum_{r,s\in\mathcal{N}}\langle i,s\rangle[s,r]\pi_r\\ &= -\frac{1}{2p^2}\sum_{r,s\in\mathcal{N}}[s,r]\left(\langle i,s\rangle\pi_r - \langle i,r\rangle\pi_s\right)\\ &= \frac{1}{2p^2}\sum_{r,s\in\mathcal{N}}[s,r]\langle s,r\rangle\pi_i = \pi_i,\end{aligned} \tag{5.4}$$

establishing (1.1), (1.2) follows similarly.

For convenience write

$$A_{ir} = p^2 a_{ir}. \tag{5.5}$$

Then, for $i,j \in \mathcal{P}$, $r,s \in \mathcal{N}$,

$$\begin{aligned}
A_{ir}A_{js} - A_{is}A_{jr} &= \sum_{u,v\in\mathcal{N}} \left(\langle i,u\rangle[u,r]\langle j,v\rangle[v,s] - \langle i,u\rangle[u,s]\langle j,v\rangle[v,r]\right) \\
&= \sum_{u,v\in\mathcal{N}} \langle i,u\rangle\langle j,v\rangle\left([u,r][v,s] - [u,s][v,r]\right) \\
&= [r,s] \sum_{u,v\in\mathcal{N}} \langle i,u\rangle\langle j,v\rangle[u,v] = [r,s] \sum_{\substack{u<v \\ u,v\in\mathcal{N}}} \left(\langle i,u\rangle\langle j,v\rangle - \langle i,v\rangle\langle j,u\rangle\right)[u,v] \\
&= p^2[r,s]\langle i,j\rangle
\end{aligned} \tag{5.6}$$

and so, for $i,j,k \in \mathcal{P}$, $r,s,t \in \mathcal{N}$,

$$\begin{aligned}
(p^2)^3 \begin{vmatrix} a_{ir} & a_{is} & a_{it} \\ a_{jr} & a_{js} & a_{jt} \\ a_{kr} & a_{ks} & a_{kt} \end{vmatrix} &= (A_{ir}A_{js} - A_{jr}A_{is})A_{kt} + (A_{jr}A_{ks} - A_{kr}A_{js})A_{it} + (A_{kr}A_{is} - A_{ir}A_{ks})A_{jt} \\
&= p^2[r,s]\left(\langle i,j\rangle A_{kt} + \langle j,k\rangle A_{it} + \langle k,i\rangle A_{jt}\right) = 0,
\end{aligned} \tag{5.7}$$

using (5.2).

Since this determinant vanishes for any $i,j,k \in \mathcal{P}$, $r,s,t \in \mathcal{N}$, this implies that the matrix

$$\begin{pmatrix} a_{i_1r_1} & a_{i_1r_2} & \dots & a_{i_1r_n} \\ a_{i_2r_1} & a_{i_2r_2} & \dots & a_{i_2r_n} \\ \vdots & \vdots & & \vdots \\ a_{i_mr_1} & a_{i_mr_2} & \dots & a_{i_mr_n} \end{pmatrix} \tag{5.8}$$

has rank 2. In fact, this is evident from the fact that a_{ir} is defined in (5.1) as the product of the $m \times m$ dimensional matrix $\langle i,j\rangle$ and the $m \times n$ dimensional matrix $[j,r]$; because the π_i are two-dimensional, the matrix $\langle i,j\rangle$ has rank at most two and the rank of a_{ir} can not be larger. For $m,n > 2$, this condition is different from the rather more unusual condition (1.6).

The condition that the matrix (1.6) have rank 2 is sufficient as well as necessary for c_{jr} to be of the form

$$c_{jr} = \frac{k_j}{k_r(\rho_j - \rho_r)}.$$

We can prove this by induction on the size of (1.6). Suppose (1.6) is of size $m \times n$, has rank 2 and that the result holds for matrices of this size; consider adding an additional column $x_i = (c_{i,n+1})^{-1}, 1 \le i \le m$, while leaving the rank of the matrix at 2. Then

$$x_i = \lambda\frac{k_1}{k_i}(\rho_i - \rho_1) + \mu\frac{k_2}{k_i}(\rho_i - \rho_2) = \frac{k_{n+1}}{k_i}(\rho_i - \rho_{n+1}), \tag{5.9}$$

where

$$k_{n+1} = \lambda k_1 + \mu k_2, \qquad \rho_{n+1} = \frac{\lambda k_1\rho_1 + \mu k_2\rho_2}{\lambda k_1 + \mu k_2},$$

so that the elements of the additional column are

$$c_{i,n+1} = \frac{k_i}{k_{n+1}(\rho_i - \rho_{n+1})},$$

which is of the required form.

For $m = 2$ or $n = 2$, (1.1), (1.2) determine c_{ir} uniquely, and it is straightforward then to check that $c_{ir} = a_{ir}$ provides the well known MHV amplitudes. For $n = 2$, $\mathcal{N} = \{R, S\}$,

$$c_{iR} = -\frac{1}{p^2}\langle i, S\rangle[S, R] = \frac{\langle i, S\rangle}{\langle R, S\rangle}, \qquad c_{iS} = \frac{\langle i, R\rangle}{\langle S, R\rangle}, \qquad \text{as } p^2 = \langle R, S\rangle[R, S]$$

in this case. Then

$$c_{i,R}c_{i+1,S} - c_{i,S}c_{i+1,R} = \frac{1}{p^2}\langle i, i+1\rangle[R, S] = \frac{\langle i, i+1\rangle}{\langle R, S\rangle}.$$

Then, from the formulae at the end of section 4,

$$\mathcal{M}^{+\dots+--} = \langle R, S\rangle^4 \Big/ \prod_{\alpha=1}^{N}\langle \alpha, \alpha+1\rangle ,$$

since there are no integrations to perform in this case. Similarly for $m = 2$, $\mathcal{P} = \{I, J\}$,

$$c_{Ir} = \frac{1}{p^2}\langle I, J\rangle[J, r] = -\frac{[J, r]}{[J, I]}, \qquad c_{Jr} = -\frac{[I, r]}{[I, J]}, \qquad \text{as } p^2 = \langle I, J\rangle[I, J]$$

in this case, leading to the familiar form of the amplitude for this case.

For $m, n > 2$ we must add to a_{ir} a solution of the corresponding homogeneous linear equations,

$$\sum_{r\in\mathcal{N}} \hat{c}_{jr}\pi_r = 0, \qquad j \in \mathcal{P} \tag{5.10}$$

$$\sum_{i\in\mathcal{P}} \bar{\pi}_i\hat{c}_{is} = 0, \qquad s \in \mathcal{N}, \tag{5.11}$$

in order to obtain a solution of (1.1), (1.2) that has the property that (1.6) is of rank 2, when

$$c_{ir} = a_{ir} + \hat{c}_{ir}. \tag{5.12}$$

The $\hat{c}_{ir}$ lie in an N_R-dimensional space. For $m = n = 3$, $\mathcal{P} = \{i, j, k\}$, $\mathcal{N} = \{r, s, t\}$, this space is one-dimensional and parametrized by

$$\hat{c}_{ir} = \frac{\beta}{4p^2}\epsilon_{ijk}[j, k]\epsilon_{rst}\langle s, t\rangle.$$

The single constraint from (1.6), $\mathcal{C}_1 \equiv \mathcal{C}^{ijk}_{rst} = 0$ provides a quartic equation to determine β, which we shall discuss further in section 6.

For general $m, n \geq 3$, the general solution to (5.10), (5.11) is provided by

$$\hat{c}_{ir} = \frac{1}{4}\sum_{\substack{j,k\in\mathcal{P}\\ s,t\in\mathcal{N}}} \beta^{ijk}_{rst}[j, k]\langle s, t\rangle,$$

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where β^{ijk}_{rst} is antisymmetric under permutations of i, j, k and also under the permutations of r, s, t; it follows from (5.2), (5.3) that this satisfies (5.10), (5.11). Because there are only $(m-2)(n-2)$ independent solutions to (5.10), (5.11) there is some arbitrariness in the choice of β^{ijk}_{rst} for a given solution c_{ir}, e.g.

$$\beta^{ijk}_{rst} \mapsto \beta^{ijk}_{rst} + [i,l]\gamma^{jk}_{rst} + [j,l]\gamma^{ki}_{rst} + [k,l]\gamma^{ij}_{rst},$$

where γ^{ij}_{rst} is antisymmetric in i, j and in r, s, t, and $\overline{\pi}_l$ is arbitrary, leaves the solution $\hat{c}_{ir}$ unchanged.

We can express each of the $\hat{c}_{ir}$ as a linear combination of the N_R components

$$\hat{c}_{kt} = \beta_{kt}[I,J]\langle R,S\rangle, \qquad k \in \mathcal{P}', t \in \mathcal{N}'$$

by

$$\hat{c}_{It} = \sum_{k\in\mathcal{P}'} \beta_{kt}[J,k]\langle R,S\rangle, \quad \hat{c}_{kR} = \sum_{t\in\mathcal{N}'} \beta_{kt}[I,J]\langle S,t\rangle, \quad \hat{c}_{IR} = \sum_{\substack{k\in\mathcal{P}'\\ t\in\mathcal{N}'}} \beta_{kt}[J,k]\langle S,t\rangle,$$
$$k \in \mathcal{P}',\ t \in \mathcal{N}', \tag{5.13}$$

and similar expressions for $\hat{c}_{kS}, \hat{c}_{Jt}, \hat{c}_{IS}, \hat{c}_{JR}, \hat{c}_{JS}$. This is equivalent to taking $\beta^{ijk}_{rst} = 0$ unless exactly one of i, j, k is in $\mathcal{P}'$ and exactly one of r, s, t is in $\mathcal{N}'$. In this case, the only possibly nonzero components of β^{ijk}_{rst} are

$$\beta^{IJk}_{RSt} = \beta_{kt}, \qquad k \in \mathcal{P}',\ t \in \mathcal{N}' \tag{5.14}$$

and those related to these by antisymmetry under permutations of I, J, k or of R, S, t.

$$\beta^{ijk}_{rst} = \epsilon_{ijk}\epsilon_{rst}\beta, \qquad \text{for} \quad m = n = 3, \tag{5.15}$$

and

$$\beta^{ijk}_{rst} = \beta^{ijk}\epsilon_{rst}, \qquad \text{for} \quad m > 3, n = 3, \tag{5.16}$$

and, in this case, we can specify that the only nonzero components of β^{ijk} are the $m-2$ components of the form $\beta^{IJk} = \beta_k$, $k \in \mathcal{P}'$ and those obtained by permuting I, J, k.

Then, from (5.12), we have,

$$c_{ir} = a_{ir} + \frac{1}{4p^2} \sum_{\substack{j,k\in\mathcal{P}\\ s,t\in\mathcal{N}}} \beta^{ijk}_{rst}[j,k]\langle s,t\rangle, \tag{5.17}$$

where the parameters β^{ijk}_{rst} are zero unless they are related to

$$\beta^{IJk}_{RSt} = \beta_{kt} \qquad k \in \mathcal{P}',\ t \in \mathcal{N}',$$

by permutation of i, j, k and r, s, t. Then, from (3.12), the amplitude becomes

$$\mathcal{M}^{\epsilon_1\ldots\epsilon_N} = \tilde{K} \oint F(c) \prod_{\substack{k\in\mathcal{P}'\\ t\in\mathcal{N}'}} \frac{d\beta_{kt}}{\mathcal{C}_{kt}(\beta)}, \tag{5.18}$$

with

$$\tilde{K} = [I,J]^{(m-3)(n-2)}\langle R,S\rangle^{(m-2)(n-3)} \left(p^2\right)^{(m-2)(2-n)}, \tag{5.19}$$

and $\mathcal{C}_{kt} = \mathcal{C}^{IJk}_{RSt}$ provide the N_R constraints $\mathcal{C}_K$.

6 NMHV 6-point functions

We will exemplify our analysis by computing all NMHV 6-point gluon tree amplitudes. Following [2], we then find equivalent but different expressions [24, 25] for the amplitudes, by choosing other equivalent contours and integrands.

For $m = n = 3$, we take $\mathcal{P} = \{i, j, k\}, \mathcal{N} = \{r, s, t\}$; from (5.18),

$$\mathcal{M}^{\epsilon_1 \ldots \epsilon_6} = \frac{1}{p^2} \oint F(c) \frac{d\beta}{\mathcal{C}(\beta)}, \tag{6.1}$$

with just one constraint $\mathcal{C}_1 \equiv \mathcal{C}_{kt} \equiv \mathcal{C}(\beta)$,

$$\begin{aligned} \mathcal{C}(\beta) = & \; c_{ir}c_{js}c_{kt}c_{jr}c_{ks}c_{it} + c_{jr}c_{ks}c_{it}c_{kr}c_{is}c_{jt} + c_{kr}c_{is}c_{jt}c_{ir}c_{js}c_{kt} \\ & -c_{kr}c_{js}c_{it}c_{jr}c_{is}c_{kt} - c_{jr}c_{is}c_{kt}c_{ir}c_{ks}c_{jt} - c_{ir}c_{ks}c_{jt}c_{kr}c_{js}c_{it}; \end{aligned} \tag{6.2}$$

the link variables are

$$c_{ir} = a_{ir} + \frac{\beta}{p^2}[j, k]\langle s, t\rangle,$$

where (i, j, k) and (r, s, t) are cyclic, and

$$F(c) = \frac{k_i^2 k_j^2 k_k^2}{k_r^2 k_s^2 k_t^2} \prod_{\substack{k \in \mathcal{P} \\ t \in \mathcal{N}}} \frac{1}{c_{kt}} \prod_{\alpha=1}^{6} \frac{1}{\rho_\alpha - \rho_{\alpha+1}}.$$

For the three different orderings of the NMHV 6-point functions, the integrands are given as in section 4.

(a) for the case $(i, t, j, r, k, s) \equiv (+, -, +, -, +, -)$,

$$F^{+-+-+-}(c) = -\frac{1}{c_{ir}c_{js}c_{kt}};$$

(b) for the case $(k, i, s, t, j, r) \equiv (+, +, -, -, +, -)$

$$F^{++--+-}(c) = \frac{c_{is}}{(c_{ks}c_{it} - c_{kt}c_{is})c_{ir}c_{js}};$$

(c) for the case $(i, j, k, r, s, t) \equiv (+, +, +, -, -, -)$

$$F^{+++---}(c) = -\frac{c_{js}}{(c_{ir}c_{js} - c_{is}c_{jr})(c_{js}c_{kt} - c_{jt}c_{ks})}.$$

To compute the amplitudes, we examine the explicit form of the constraint, which leads to a quartic equation for β. Writing

$$V_i = [j, k], \qquad W_r = \langle s, t\rangle, \qquad \overline{V}_i = \langle j, k\rangle, \qquad \overline{W}_r = [s, t],$$

and similarly for cyclic rotations of (i, j, k) and of (r, s, t), we have

$$c_{ir} = a_{ir} + \frac{\beta}{p^2} V_i W_r, \tag{6.3}$$

and, from (5.6),

$$A_{ir}A_{js} - A_{is}A_{jr} = p^2\overline{V}_k\overline{W}_t. \tag{6.4}$$

From appendix B, where we list some algebraic relations useful for computing the 6-point functions,

$$c_{ir}c_{js} - c_{is}c_{jr} = \beta\overline{c}_{tk}, \tag{6.5}$$

for

$$\overline{c}_{ri} = \overline{a}_{ri} + \frac{\overline{\beta}}{p^2}\overline{W}_r\overline{V}_i, \qquad \overline{a}_{ri} = \frac{1}{p^2}\bar{A}_{ri}, \qquad \overline{\beta} = 1/\beta, \tag{6.6}$$

$$\bar{A}_{ri} = \sum_{s\in\mathcal{N}}\langle r,s\rangle[s,i] = -\sum_{j\in\mathcal{P}}\langle r,j\rangle[j,i]. \tag{6.7}$$

In appendix B, we see that $\det(c) = \beta$ so that $\overline{c}_{ri}$ is the inverse of c_{ir}, and hence provides the general solution of

$$\pi_r = \sum_{i\in\mathcal{P}}\overline{c}_{ri}\pi_i, \qquad \overline{\pi}_i = -\sum_{r\in\mathcal{N}}\overline{\pi}_r\overline{c}_{ri},$$

the equations for the amplitude with flipped helicities. Again, from appendix B,

$$\bar{A}_{ri}\bar{A}_{sj} - \bar{A}_{si}\bar{A}_{rj} = p^2 W_t V_k,$$

and

$$\overline{c}_{ri}\overline{c}_{sj} - \overline{c}_{si}\overline{c}_{rj} = \overline{\beta}c_{kt}.$$

Using these properties of c_{ir}, the constraint becomes $\mathcal{C}(\beta) = 0$, where

$$\begin{aligned}\mathcal{C}(\beta) &= c_{is}c_{jr}(c_{kt}c_{ir} - c_{kr}c_{it})(c_{js}c_{kt} - c_{jt}c_{ks}) - c_{ir}c_{js}(c_{ks}c_{it} - c_{kt}c_{is})(c_{jt}c_{kr} - c_{jr}c_{kt})\\ &= \beta^2[c_{is}c_{jr}\overline{c}_{ri}\overline{c}_{sj} - c_{ir}c_{js}\overline{c}_{si}\overline{c}_{rj}]\\ &= \beta c_{ir}c_{js}c_{kt} - \beta^3\overline{c}_{ri}\overline{c}_{sj}\overline{c}_{tk},\end{aligned} \tag{6.8}$$

which is manifestly quartic in β. Writing

$$\mathcal{C}(\beta) = \beta c_{ir}c_{js}c_{kt} - \beta^3\overline{c}_{ri}\overline{c}_{sj}\overline{c}_{tk} \equiv C_{(4)}\beta^4 + C_{(3)}\beta^3 + C_{(2)}\beta^2 + C_{(1)}\beta + C_{(0)},$$

the coefficients in the quartic are

$$\begin{aligned}C_{(0)} &= -\overline{v}_i\overline{v}_j\overline{v}_k\overline{w}_r\overline{w}_s\overline{w}_t;\\ C_{(1)} &= a_{ir}a_{js}a_{kt} - \overline{a}_{ri}\overline{v}_j\overline{v}_k\overline{w}_s\overline{w}_t - \overline{a}_{sj}\overline{v}_i\overline{v}_k\overline{w}_r\overline{w}_t - \overline{a}_{tk}\overline{v}_i\overline{v}_j\overline{w}_r\overline{w}_s;\\ C_{(2)} &= a_{ir}a_{js}v_kw_t + a_{js}a_{kt}v_iw_r + a_{ir}a_{kt}v_jw_s - \overline{a}_{ri}\overline{a}_{sj}\overline{v}_k\overline{w}_t - \overline{a}_{sj}\overline{a}_{tk}\overline{v}_i\overline{w}_r - \overline{a}_{ri}\overline{a}_{tk}\overline{v}_j\overline{w}_s;\\ C_{(3)} &= a_{ir}v_jv_kw_sw_t + a_{js}v_iv_kw_rw_t + a_{kt}v_iv_jw_rw_s - \overline{a}_{ri}\overline{a}_{sj}\overline{a}_{tk};\\ C_{(4)} &= v_iv_jv_kw_rw_sw_t,\end{aligned} \tag{6.9}$$

where

$$v_i = V_i/p^2, \qquad w_r = W_r/p^2, \qquad \overline{v}_i = \overline{V}_i/p^2, \qquad \overline{w}_r = \overline{W}_r/p^2.$$

Now consideration of particular cases for the momenta demonstrates that the roots of $\mathcal{C}(\beta)$ are irrational functions of the momenta and that rational results can only be obtained by summing over each of the roots. So, from (6.1), we take

$$\mathcal{M}^{\epsilon_1\ldots\epsilon_6} = \frac{1}{p^2}\oint_{\mathcal{O}} \frac{F^{\epsilon_1\ldots\epsilon_6}(c)}{\mathcal{C}(\beta)} d\beta, \tag{6.10}$$

where $\mathcal{O}$ is taken to be a contour encircling each of these roots once, but none of the poles of $F(c)$. We now discuss the evaluation of the amplitude using contour manipulation, in the spirit of [1, 2].

Because the integrand tends to zero as β^{-3} or faster, as $\beta \to \infty$, we have

$$\mathcal{M}^{\epsilon_1\ldots\epsilon_6} = -\frac{1}{p^2}\oint_{\mathcal{O}'} \frac{F^{\epsilon_1\ldots\epsilon_6}(c)}{\mathcal{C}(\beta)} d\beta,$$

where the contour $\mathcal{O}'$ encircles the poles in β of $F^{\epsilon_1\ldots\epsilon_6}(c)$ positively, but excludes the zeros of $\mathcal{C}(\beta)$.

(a) For the case $(i,t,j,r,k,s) \equiv (+,-,+,-,+,-)$,

$$\mathcal{M}^{+-+-+-} = \frac{1}{2\pi i}\oint_{\mathcal{O}_a} \frac{d\beta}{p^2 c_{ir}c_{js}c_{kt}\mathcal{C}(\beta)},$$

where $\mathcal{O}_a$ encircles $\beta = -A_{ir}/V_iW_r$, $\beta = -A_{js}/V_jW_s$ and $\beta = -A_{kt}/V_kW_t$. The integral is the sum of the contributions of these poles of the integrand, and at each of them $\mathcal{C}(\beta) = \beta c_{ir}c_{js}c_{kt} - \beta^3 \bar{c}_{ri}\bar{c}_{sj}\bar{c}_{tk} = -\beta^3 \bar{c}_{ri}\bar{c}_{sj}\bar{c}_{tk}$, because $c_{ir}c_{js}c_{kt} = 0$. Thus, using formulae listed in appendix B,

$$\begin{aligned}\mathcal{M}^{+-+-+-} &= -\frac{1}{2\pi i}\oint_{\mathcal{O}_a} \frac{d\beta}{p^2 c_{ir}c_{js}c_{kt}\beta^3 \bar{c}_{ri}\bar{c}_{sj}\bar{c}_{tk}} \\ &= -\frac{[j,k]^4\langle s,t\rangle^4}{\langle t|P_{si}|k]\langle s|P_{ti}|j]\, p^2_{jkr}\langle i,s\rangle[r,j]\langle i,t\rangle[r,k]} \quad + \quad 2 \text{ cyclic terms},\end{aligned} \tag{6.11}$$

where the two additional cyclic terms are obtained by simultaneously cyclically rotating (i,j,k) and (r,s,t), and where we write

$$\langle \alpha_1|P_{\beta_1\ldots\beta_l}|\alpha_2] = \langle \alpha_1,\beta_1\rangle[\beta_1,\alpha_2] + \cdots + \langle \alpha_1,\beta_l\rangle[\beta_l,\alpha_2], \tag{6.12}$$

so that

$$\langle t|P_{si}|k] = -\langle t|P_{rj}|k], \text{ etc.},$$

and

$$p^2_{\alpha_1\ldots\alpha_l} = (p_{\alpha_1} + \ldots p_{\alpha_l})^2. \tag{6.13}$$

Having written the amplitude in the form (6.11), we can replace the contour $\mathcal{O}_a$ by a contour $\mathcal{O}'_a$ encircling the poles corresponding to $\bar{c}_{ri} = 0$, $\bar{c}_{sj} = 0$ and $\bar{c}_{tk} = 0$, i.e.

$\beta = -\overline{V}_i\overline{W}_r/\bar{A}_{ir}$, $\beta = -\overline{V}_j\overline{W}_s/\bar{A}_{js}$ and $\beta = -\overline{V}_k\overline{W}_t/\bar{A}_{kt}$, respectively:

$$\begin{aligned}\mathcal{M}^{+-+-+-} &= \frac{1}{2\pi i}\oint_{\mathcal{O}'_a}\frac{d\beta}{p^2 c_{ir}c_{js}c_{kt}\beta^3\overline{c}_{ri}\overline{c}_{sj}\overline{c}_{tk}} \\ &= -\frac{\bar{A}_{ri}^4}{\langle k|P_{si}|t]\langle j|P_{ti}|s]p_{jkr}^2[i,s]\langle r,j\rangle[i,t]\langle r,k\rangle} \quad + \quad 2 \text{ cyclic terms},\end{aligned} \tag{6.14}$$

where $\bar{A}_{ri}$ is defined in (6.7). (6.14) provides a second, equivalent form for the alternating tree amplitude.

(b) For the case $(k,i,s,t,j,r) \equiv (+,+,-,-,+,-)$,

$$\mathcal{M}^{++--+-} = -\frac{1}{2\pi i}\oint_{\mathcal{O}_b}\frac{c_{is}d\beta}{p^2 c_{ir}c_{js}\beta\overline{c}_{rj}\mathcal{C}(\beta)},$$

where $\mathcal{O}_b$ encircles $\beta = -A_{ir}/V_iW_r$, $\beta = -A_{js}/V_jW_s$ and $\beta = -\overline{V}_j\overline{W}_r/\bar{A}_{rj}$. The integral is the sum of the contributions of these poles of the integrand, and at each of them $\mathcal{C}(\beta) = \beta^2[c_{is}c_{jr}\overline{c}_{ri}\overline{c}_{sj} - c_{ir}c_{js}\overline{c}_{si}\overline{c}_{rj}] = \beta^2 c_{is}c_{jr}\overline{c}_{ri}\overline{c}_{sj}$, because $c_{ir}c_{js}\overline{c}_{si}\overline{c}_{rj} = 0$ at the poles. Thus

$$\begin{aligned}\mathcal{M}^{++--+-} &= -\frac{1}{2\pi i}\oint_{\mathcal{O}_b}\frac{d\beta}{p^2 c_{ir}c_{js}\beta\overline{c}_{rj}c_{jr}\beta^2\overline{c}_{ri}\overline{c}_{sj}} \\ &= -\frac{[j,k]^4\langle s,t\rangle^3}{\langle t|P_{si}|k]\langle i|P_{rk}|j][k,r]p_{jkr}^2\langle i,s\rangle[r,j]} - \frac{[k,i]^3\langle t,r\rangle^4}{\langle t|P_{rj}|k]\langle r|P_{tj}|s][i,s]p_{kis}^2\langle j,t\rangle\langle j,r\rangle} \\ &\quad - \frac{\bar{A}_{rj}^4}{\langle i|P_{rk}|j]\langle r|P_{tj}|s]p_{kir}^2\langle k,r\rangle[j,t][s,t]\langle k,i\rangle}.\end{aligned} \tag{6.15}$$

For an alternative form, starting from (6.15), we can replace the contour $\mathcal{O}_b$ by a contour $\mathcal{O}'_b$ encircling the poles corresponding to $\overline{c}_{ri} = 0$, $\overline{c}_{sj} = 0$ and $c_{jr} = 0$, i.e. $\beta = -\overline{V}_i\overline{W}_r/\bar{A}_{ir}$, $\beta = -\overline{V}_j\overline{W}_s/\bar{A}_{js}$ and $\beta = -A_{jr}/V_jW_r$, respectively:

$$\mathcal{M}^{++--+-} = \frac{1}{2\pi i}\oint_{\mathcal{O}'_b}\frac{d\beta}{p^2 c_{ir}c_{js}\beta\overline{c}_{rj}c_{jr}\beta^2\overline{c}_{ri}\overline{c}_{sj}} \tag{6.16}$$

$$\begin{aligned}&= -\frac{\bar{A}_{ri}^4}{\langle k|P_{si}|t]\langle j|P_{rk}|i]\langle k,r\rangle p_{jkr}^2[i,s]\langle r,j\rangle[s,t]} \\ &\quad - \frac{\bar{A}_{sj}^4}{\langle k|P_{rj}|t]\langle s|P_{tj}|r]\langle i,s\rangle p_{kis}^2[j,t][j,r]\langle k,i\rangle} \\ &\quad - \frac{[k,i]^4\langle s,t\rangle^4}{\langle j|P_{rk}|i]\langle s|P_{tj}|r]p_{kir}^2[k,r]\langle j,t\rangle\langle s,t\rangle[k,i]}.\end{aligned} \tag{6.17}$$

(c) For the case $(i,j,k,r,s,t) \equiv (+,+,+,-,-,-)$,

$$\mathcal{M}^{+++---} = \frac{1}{2\pi i}\oint_{\mathcal{O}_c}\frac{c_{js}d\beta}{p^2\beta^2\overline{c}_{tk}\overline{c}_{ri}\mathcal{C}(\beta)},$$

where $\mathcal{O}_c$ encircles $\beta = -\overline{V}_k\overline{W}_t/\bar{A}_{tk}$ and $\beta = -\overline{V}_i\overline{W}_r/\bar{A}_{ri}$. The integral is the sum of the contributions of these poles of the integrand, and at each of them $\mathcal{C}(\beta) =$

$\beta c_{ir}c_{js}c_{kt} - \beta^3\bar{c}_{ri}\bar{c}_{sj}\bar{c}_{tk} = \beta c_{ir}c_{js}c_{kt}$, because $\bar{c}_{ri}\bar{c}_{sj}\bar{c}_{tk} = 0$ at the poles. Thus

$$\mathcal{M}^{+++---} = \frac{1}{2\pi i}\oint_{\mathcal{O}_c} \frac{d\beta}{p^2\beta^2\bar{c}_{tk}\bar{c}_{ri}\beta c_{ir}c_{kt}} \tag{6.18}$$
$$= -\frac{\bar{A}^3_{ri}}{\langle j,k\rangle[s,t]\langle j|P_{ti}|s]p^2_{jkr}[i,t]\langle r,k\rangle} - \frac{\bar{A}^3_{tk}}{\langle j,i\rangle[s,r]\langle j|P_{rk}|s]p^2_{jit}[k,r]\langle t,i\rangle}.$$

To find the alternative form, starting from (6.18), we can replace the contour $\mathcal{O}_c$ by a contour $\mathcal{O}'_c$ encircling the poles corresponding to $c_{ir} = 0$, $c_{kt} = 0$ and $\beta = 0$, i.e. $\beta = -A_{ir}/V_iW_r$, $\beta = -A_{kt}/V_kW_t$ as well as $\beta = 0$, respectively:

$$\mathcal{M}^{+++---} = -\frac{1}{2\pi i}\oint_{\mathcal{O}'_c} \frac{d\beta}{p^2\beta^2\bar{c}_{tk}\bar{c}_{ri}\beta c_{ir}c_{kt}} \tag{6.19}$$
$$= \frac{[j,k]^3\langle s,t\rangle^3}{A_{ir}\langle s|P_{ti}|j]p^2_{jkr}\langle i,t\rangle[r,k]} + \frac{[i,j]^3\langle r,s\rangle^3}{A_{kt}\langle s|P_{kr}|j]\ p^2_{ijt}\langle r,k\rangle[i,t]}$$
$$- \frac{(p^2)^3}{\langle ij\rangle[r,s]\langle j,k\rangle[st]A_{ir}A_{kt}},$$

where the A_{ir} is defined in (5.5).

7 Multiple constraints and an NMHV 7-point function

We now consider the general situation in which the number of constraints, $N_R > 1$. The general form of the amplitude is

$$\mathcal{M}^{\epsilon_1\ldots\epsilon_N} = \tilde{K}\oint F(c)\prod_{\substack{k\in\mathcal{P}'\\ t\in\mathcal{N}'}}\frac{d\beta_{kt}}{\mathcal{C}_{kt}(\beta)}, \tag{7.1}$$

where $\tilde{K}$ is given by (5.19). We must specify further the multi-dimensional contour in (7.1) and explain how it may be evaluated. We remember, from the end of section 3, that the motivation for (7.1), is obtained by considering the result (3.11) of evaluating the delta function constraints $\delta^{N_R}(\hat{\mathcal{C}})$ in (3.9). By replacing real integrals over delta functions with contour integrals around corresponding poles, we effectively replaced an expression involving the inverse of the modulus of the Jacobian of the constraints $\hat{\mathcal{C}}$, summed over a discrete set of points corresponding to the simultaneous solution of the constraint conditions, with the same expression with the modulus removed, so that the sum over the simultaneous solutions of the constraints now includes the phases of the inverse Jacobian at these points. We saw in section 6, in the case of the $N_R = 1$ 6-point function, that this interpretation was forced on us if we were to be able to reproduce the rational form for the amplitude known from gauge theory.

In order to specify (7.1) more precisely, we choose an order for the constraints, $\mathcal{C}_K$, $K = 1\ldots N_R$, and order the parameters, β_K, correspondingly. Then, as in section 5.1 of [2], the contribution of a particular simultaneous solution $\beta = \beta^0$ of the constraint equations, $\mathcal{C}_K(\beta) = 0$, is

$$\tilde{K}\oint_{\mathcal{O}_{\beta^0}} F(c)\left[\prod_{K=1}^{N_R}\frac{1}{\mathcal{C}_K(\beta)}\right]d\beta = \tilde{K}F(c)\left[\frac{\partial(\mathcal{C})}{\partial(\beta)}\right]^{-1}\Bigg|_{\beta=\beta^0}, \tag{7.2}$$

where $d\beta = d\beta_1 \wedge \ldots \wedge d\beta_{N_R}$ and the contour $\mathcal{O}^0$ is chosen to be a surface of the form $\{\beta : |\mathcal{C}_K(\beta)| = \epsilon, 1 \le K \le N_R\}$, with its orientation determined by the order of the $\mathcal{C}_K$ (as in [2]), enclosing $\beta = \beta^0$ but no other zero of the $\mathcal{C}_K$. The integral (7.2) is be the residue of the integrand at $\beta = \beta_0$ and it is antisymmetric under independent permutations of the order of the $\mathcal{C}_K$ or of the β_K, and so symmetric under simultaneous identical permutations of both.

Now, as in the $N_R = 1$ case, we should sum (7.2) over all the simultaneous solutions, $\beta = \beta^0$, of the constraints, $\mathcal{C}_K(\beta) = 0$, but exclude the contribution of other poles of the integrand arising from $F(c)$. At this point we should note that, with the particular set of constraints we have chosen in section 4, namely $\{\mathcal{C}_{kt} : k \neq I, J,\ t \neq R, S\}$, there are always four 'trivial' or 'spurious' simultaneous solutions of the constraints, namely $c_{IR} = c_{JR} = 0$; $c_{IS} = c_{JS} = 0$; $c_{IR} = c_{IS} = 0$; and $c_{JR} = c_{JS} = 0$. They are introduced when we move from statements about the matrix (1.6), with entries c_{ir}^{-1} to statements about multinomials (1.7) in the link variables, c_{ir}, by multiplying by products of them. They do not correspond to the matrix (1.6) having rank 2; in fact, they are artifacts of the particular choice of the independent set of constraints. These spurious solutions should be excluded from the sum.

The problem again with trying to evaluate this sum directly is that the solutions to the constraints are not all rational individually, only their sum is. So, we seek to use a multidimensional version of the contour manipulation arguments we used in section 6 to evaluate the amplitude to obtain the familiar rational results. As in [2], this is provided by the global residue theorem. Consider an N-dimensional integral of the form

$$\oint \frac{\Phi(\beta) d\beta}{\prod_{\alpha=1}^{M} h_\alpha(\beta)}, \tag{7.3}$$

where $N < M$. For distinct indices $\alpha_1, \ldots, \alpha_N$, the residue of the integrand at a common zero β^0 of $h_{\alpha_1}, \ldots, h_{\alpha_N}$, assumed to be a simple zero, is

$$\mathcal{R}(h_{\alpha_1}, \ldots, h_{\alpha_N}) = \Phi(\beta^0) \prod_{\alpha \notin A} \frac{1}{h_\alpha(\beta^0)} \left[\frac{\partial(h_{\alpha_1}, \ldots, h_{\alpha_N})}{\partial(\beta_1, \ldots, \beta_N)} \right]^{-1}_{\beta = \beta^0}, \tag{7.4}$$

where $A = \{\alpha_1, \ldots, \alpha_N\}$. It may be that the functions $h_{\alpha_1}, \ldots, h_{\alpha_N}$ have more than one common zero but we shall assume the set of such simultaneous zeros is finite and, if there is more than one, we shall understand $\mathcal{R}(h_{\alpha_1}, \ldots, h_{\alpha_N})$ to denote the sum of the residues (7.4) at these simultaneous zeros. Suppose $\Gamma_\ell, 1 \le \ell \le N$ are disjoint subsets of $\{1, \ldots, M\}$, whose union is the whole set. Then a version [2] of the global residue theorem states that

$$\sum_{\alpha_\ell \in \Gamma_\ell} \mathcal{R}(h_{\alpha_1}, \ldots, h_{\alpha_N}) = 0. \tag{7.5}$$

We demonstrate the effectiveness of this for evaluating amplitudes by considering a 7-point NMHV tree with helicities $(i, r, k, s, l, t, j) \equiv (+ - + - + - +)$, so that $m = 4$, $n = 3$, $N_R = 2$. We take the choice $(I, J, R, S) = (i, j, r, s)$. In this case there are two constraints, $\mathcal{C}_{kt}, \mathcal{C}_{lt}$, and, as in (5.16), we can take the two integration variables to be

$\beta_k = \beta^{ijk}$, $\beta_l = \beta^{ijl}$, where, for $r \in \mathcal{N}$,

$$c_{ir} = a_{ir} + (\beta_k[j,k] + \beta_l[j,l])W_r/p^2, \tag{7.6}$$
$$c_{jr} = a_{jr} + (\beta_k[k,i] + \beta_l[l,i])W_r/p^2, \tag{7.7}$$
$$c_{kr} = a_{kr} + \beta_k[i,j]W_r/p^2, \quad c_{lr} = a_{lr} + \beta_l[i,j]W_r/p^2, \quad r \in \mathcal{N}. \tag{7.8}$$

From (4.9),

$$F(c) = \frac{c_{ir}c_{jt}}{c_{kt}c_{lr}}, \tag{7.9}$$

and so, from (5.18), the contour expression for the amplitude is of the form

$$\mathcal{M} = \frac{[i,j]}{(p^2)^2} \oint_{\mathcal{O}} \frac{c_{ir}c_{jt}}{c_{kt}c_{lr}\mathcal{C}_{kt}\mathcal{C}_{lt}} d\beta_k d\beta_l. \tag{7.10}$$

In (7.10) the contour $\mathcal{O}$ is chosen so that it includes the residue contributions from each of the simultaneous solutions of the constraints $\mathcal{C}_{kt} = \mathcal{C}_{lt} = 0$, but excluding the 'spurious' solutions. $\mathcal{C}_{kt} = \mathcal{C}_{lt} = 0$ if $c_{ir} = c_{jr} = 0$ or $c_{is} = c_{js} = 0$ or $c_{it} = c_{jt} = 0$; we do not have to consider possible spurious contributions from, e.g. $c_{ir} = c_{is} = 0$, because the variables c_{ir}, c_{is}, c_{it} are not independent and no two of them can vanish together for general momenta, and similar considerations apply to c_{jr}, c_{js}, c_{jt}. The residues at $c_{ir} = c_{jr} = 0$ and $c_{it} = c_{jt} = 0$ vanish because of the presence of c_{ir} and c_{jt}, respectively, in the numerator of the integrand, so that we only have to exclude the contribution from $c_{is} = c_{js} = 0$ explicitly.

Applying the global residue theorem taking Γ_1 to correspond to $\{\mathcal{C}_{kt}, c_{lr}\}$ and Γ_2 to correspond to $\{\mathcal{C}_{lt}, c_{kt}\}$, we obtain

$$\mathcal{R}(\mathcal{C}_{kt}, \mathcal{C}_{lt}) + \mathcal{R}(c_{lr}, \mathcal{C}_{lt}) + \mathcal{R}(\mathcal{C}_{kt}, c_{kt}) + \mathcal{R}(c_{lr}, c_{kt}) = 0. \tag{7.11}$$

and so, excluding the 'spurious' contribution,

$$\begin{aligned} \mathcal{M} &= \mathcal{R}(\mathcal{C}_{kt}, \mathcal{C}_{lt}) - \mathcal{R}(c_{is}, c_{js}) \\ &= -\mathcal{R}(c_{lr}, c_{kt}) - \mathcal{R}(\mathcal{C}_{kt}, c_{kt}) - \mathcal{R}(c_{lr}, \mathcal{C}_{lt}) - \mathcal{R}(c_{is}, c_{js}) \end{aligned} \tag{7.12}$$

The choice of Γ_1 and Γ_2 has been made so that $\mathcal{R}(\mathcal{C}_{kt}, c_{kt})$ and $\mathcal{R}(c_{lr}, \mathcal{C}_{lt})$ are as easy to evaluate as $\mathcal{R}(c_{lr}, c_{kt})$ and $\mathcal{R}(c_{is}, c_{js})$ are.

Because

$$\begin{aligned} \mathcal{C}_{kt} &= c_{ir}c_{js}c_{kt}c_{jr}c_{ks}c_{it} + c_{jr}c_{ks}c_{it}c_{kr}c_{is}c_{jt} + c_{kr}c_{is}c_{jt}c_{ir}c_{js}c_{kt} \\ &\quad -c_{kr}c_{js}c_{it}c_{jr}c_{is}c_{kt} - c_{jr}c_{is}c_{kt}c_{ir}c_{ks}c_{jt} - c_{ir}c_{ks}c_{jt}c_{kr}c_{js}c_{it} \\ &= c_{is}c_{kt}c^{jk}_{rs}c^{ij}_{tr} - c_{it}c_{ks}c^{jk}_{tr}c^{ij}_{rs}, \end{aligned} \tag{7.13}$$

where

$$c^{ij}_{rs} = c_{ir}c_{js} - c_{is}c_{jr}, \tag{7.14}$$

we have that

$$\mathcal{C}_{kt}|_{c_{kt}=0} = c_{kr}c_{ks}c_{it}c_{jt}(c_{is}c_{jr} - c_{ir}c_{js}), \tag{7.15}$$

from which it follows that we can write $\mathcal{R}(\mathcal{C}_{kt}, c_{kt})$ as a sum of terms corresponding to the factors of $\mathcal{C}_{kt}$ when $c_{kt} = 0$. Because, in this case, for general momenta, $c_{kr}, c_{ks} \neq 0$ when $c_{kt} = 0$, and because of the presence of c_{jt} in the numerator, we only have to consider the factors c_{it} and c^{ij}_{rs}, implying

$$\mathcal{R}(\mathcal{C}_{kt}, c_{kt}) = \mathcal{R}(c_{it}, c_{kt}) + \mathcal{R}(c^{ij}_{rs}, c_{kt}). \tag{7.16}$$

Similarly, from

$$\mathcal{C}_{lt}|_{c_{lr}=0} = c_{ls}c_{lt}c_{ir}c_{jr}(c_{it}c_{js} - c_{is}c_{jt}), \tag{7.17}$$

we deduce

$$\mathcal{R}(c_{lr}, \mathcal{C}_{lt}) = \mathcal{R}(c_{lr}, c_{jr}) + \mathcal{R}(c_{lr}, c^{ij}_{st}). \tag{7.18}$$

Combining (7.16) and (7.18) with (7.12), we have $\mathcal{M}$ expressed as a sum of six terms,

$$\mathcal{M} = -\mathcal{R}(c_{lr}, c_{kt}) - \mathcal{R}(c_{it}, c_{kt}) - \mathcal{R}(c^{ij}_{rs}, c_{kt}) - \mathcal{R}(c_{lr}, c_{jr}) - \mathcal{R}(c_{lr}, c^{ij}_{st}) - \mathcal{R}(c_{is}, c_{js}). \tag{7.19}$$

We now calculate each of these terms in turn to obtain $\mathcal{M}$.

Using formulae from appendix C, we find the following:

(a) residue from $c_{lr} = c_{kt} = 0$

$$\mathcal{R}(c_{lr}, c_{kt}) = \frac{[i,j]}{(p^2)^2}\left[\frac{\partial(c_{lr}, c_{kt})}{\partial(\beta_k, \beta_l)}\right]^{-1} \frac{c_{ir}c_{jt}}{\mathcal{C}_{kt}\mathcal{C}_{lt}} \tag{7.20}$$

$$= -\frac{1}{[i,j]W_tW_rc_{kr}c_{ks}c_{it}c_{ls}c_{lt}c_{jr}(c_{is}c_{jr} - c_{ir}c_{js})(c_{it}c_{js} - c_{is}c_{jt})}$$

$$= -\frac{W_t^4W_r^4[i,j]^3}{\langle k,r\rangle\langle k,s\rangle\langle l,s\rangle\langle l,t\rangle\ \langle s|P_{lt}|j]\ \langle s|P_{kr}|i]\ \langle t|P_{ij}P_{kr}|s\rangle\ \langle s|P_{ij}P_{lt}|r\rangle}. \tag{7.21}$$

(b) residue from $c_{it} = c_{kt} = 0$ Since at $c_{it} = c_{kt} = 0$,

$$\frac{\partial(\mathcal{C}_{kt}, c_{kt})}{\partial(\beta_k, \beta_l)} = -\frac{\partial \mathcal{C}_{kt}}{\partial \beta_l}\frac{\partial c_{kt}}{\partial \beta_k} = -\frac{\partial c_{kt}}{\partial \beta_k}\frac{\partial c_{it}}{\partial \beta_l}c_{kr}c_{ks}c_{jt}(c_{is}c_{jr} - c_{ir}c_{js}),$$

$$\mathcal{R}(c_{it}, c_{kt}) = \frac{[i,j]}{(p^2)^2}\left[\frac{\partial(\mathcal{C}_{kt}, c_{kt})}{\partial(\beta_k, \beta_l)}\right]^{-1} \frac{c_{ir}c_{jt}}{c_{lr}\mathcal{C}_{lt}} \tag{7.22}$$

$$= -\frac{1}{[j,l]W_t^2c_{lr}c_{kr}c_{ks}c_{is}c_{jt}c_{lt}(c_{is}c_{jr} - c_{ir}c_{js})(c_{js}c_{lr} - c_{jr}c_{ls})}$$

$$= \frac{W_t^4[j,l]^4}{p^2_{ltj}\langle i,r\rangle\langle r,k\rangle\langle k,s\rangle\ [l,t][t,j]\langle s|P_{lt}|j]\ \langle i|P_{jt}|l]}. \tag{7.23}$$

(c) residue from $c^{ij}_{rs} = c_{kt} = 0$ Since at $c^{ij}_{rs} = c_{kt} = 0$,

$$\begin{aligned}
\frac{\partial(\mathcal{C}_{kt}, c_{kt})}{\partial(\beta_k, \beta_l)} &= -\frac{\partial \mathcal{C}_{kt}}{\partial \beta_l}\frac{\partial c_{kt}}{\partial \beta_k} = \frac{\partial c_{kt}}{\partial \beta_k}\frac{\partial c^{ij}_{rs}}{\partial \beta_l} c_{kr}c_{ks}c_{it}c_{jt}, \\
\frac{\partial c^{ij}_{rs}}{\partial \beta_l} &= [(c_{js}W_r - c_{jr}W_s)[j,l] + (c_{ir}W_s - c_{is}W_r)[l,i]]/p^2 = -\langle t|P_{ij}|l]/p^2 \\
\mathcal{C}_{lt}|_{c^{ij}_{rs}=0} &= c_{is}c_{lt}c^{ij}_{tr}c^{jl}_{rs}. \qquad (7.24)\\
\mathcal{R}(c^{ij}_{rs}, c_{kt}) &= \frac{[i,j]}{(p^2)^2}\left[\frac{\partial(\mathcal{C}_{kt}, c_{kt})}{\partial(\beta_k, \beta_l)}\right]^{-1}\frac{c_{ir}c_{jt}}{c_{lr}\mathcal{C}_{lt}} \qquad (7.25)\\
&= -\frac{c_{ir}}{\langle t|P_{ij}|l]W_t c_{lr}c_{kr}c_{ks}c_{it}c_{is}c_{lt}(c_{jr}c_{it} - c_{jt}c_{ir})(c_{jr}c_{ls} - c_{js}c_{lr})} \\
&= \frac{W_t^4\,\langle t|P_{ij}|l]^4}{p^2_{ijt}p^2_{krs}\langle r,k\rangle\langle k,s\rangle\langle i,j\rangle\langle j,t\rangle\,\langle r|P_{ks}|l]\,\langle i|P_{jt}|l]\,\langle t|P_{ij}P_{kr}|s\rangle}. \qquad (7.26)
\end{aligned}$$

(d) residue from $c_{lr} = c_{jr} = 0$ Since at $c_{lr} = c_{jr} = 0$,

$$\begin{aligned}
\frac{\partial(c_{lr}, \mathcal{C}_{lt})}{\partial(\beta_k, \beta_l)} &= -\frac{\partial c_{lr}}{\partial \beta_l}\frac{\partial \mathcal{C}_{lt}}{\partial \beta_k} = \frac{\partial c_{jr}}{\partial \beta_k}\frac{\partial c_{lr}}{\partial \beta_l} c_{ir}c_{ls}c_{lt}(c_{js}c_{it} - c_{is}c_{jt}), \\
\mathcal{R}(c_{lr}, c_{jr}) &= \frac{[i,j]}{(p^2)^2}\left[\frac{\partial(c_{lr}, \mathcal{C}_{lt})}{\partial(\beta_k, \beta_l)}\right]^{-1}\frac{c_{ir}c_{jt}}{\mathcal{C}_{kt}c_{kt}} \qquad (7.27)\\
&= -\frac{1}{[k,i]W_r^2 c_{ls}c_{lt}c_{ir}c_{kr}c_{js}c_{kt}(c_{js}c_{it} - c_{is}c_{jt})(c_{is}c_{kt} - c_{it}c_{ks})} \\
&= \frac{W_r^4[k,i]^4}{p^2_{irk}\langle s,l\rangle\langle l,t\rangle\langle t,j\rangle\,[i,r][r,k]\langle j|P_{ir}|k]\,\langle s|P_{kr}|i]}. \qquad (7.28)
\end{aligned}$$

(e) residue from $c_{lr} = c^{ij}_{st} = 0$ Since at $c_{lr} = c^{ij}_{st} = 0$,

$$\begin{aligned}
\frac{\partial(c_{lr}, \mathcal{C}_{lt})}{\partial(\beta_k, \beta_l)} &= -\frac{\partial c_{lr}}{\partial \beta_l}\frac{\partial \mathcal{C}_{lt}}{\partial \beta_k} = \frac{\partial c^{ij}_{st}}{\partial \beta_k}\frac{\partial c_{lr}}{\partial \beta_l} c_{ls}c_{lt}c_{ir}c_{jr}, \\
\frac{\partial c^{ij}_{st}}{\partial \beta_k} &= [(c_{is}W_t - c_{it}W_s)[k,i] + (c_{jt}W_s - c_{js}W_t)[j,k]]/p^2 = -\langle r|P_{ij}|k]/p^2 \\
\mathcal{C}_{kt}|_{c^{ij}_{st}=0} &= c_{it}c_{kr}c^{jk}_{st}c^{ij}_{rs}. \qquad (7.29)\\
\mathcal{R}(c_{lr}, c^{ij}_{st}) &= \frac{[i,j]}{(p^2)^2}\left[\frac{\partial(c_{lr}, \mathcal{C}_{lt})}{\partial(\beta_k, \beta_l)}\right]^{-1}\frac{c_{ir}c_{jt}}{\mathcal{C}_{kt}c_{kt}} \qquad (7.30)\\
&= -\frac{c_{jt}}{\langle r|P_{ij}|k]W_r c_{jr}c_{ls}c_{lt}c_{kt}c_{it}c_{kr}(c_{ir}c_{js} - c_{is}c_{jr})(c_{js}c_{kt} - c_{ks}c_{jt})} \\
&= \frac{W_r^4\,\langle r|P_{ij}|k]^4}{p^2_{ijr}p^2_{lst}\,\langle ji\rangle\langle ir\rangle\langle sl\rangle\langle lt\rangle\,\langle j|Pir|k]\,\langle t|P_{ls}|k]\,\langle s|P_{lt}P_{ij}|r\rangle}. \qquad (7.31)
\end{aligned}$$

(f) residue from $c_{is}=c_{js}=0$ Since at $c_{is}=c_{js}=0$,

$$\begin{aligned}
\frac{\partial(\mathcal{C}_{kt},\mathcal{C}_{lt})}{\partial(\beta_k,\beta_l)} &= \frac{\partial(\mathcal{C}_{kt},\mathcal{C}_{lt})}{\partial(c_{is},c_{js})}\cdot\frac{\partial(c_{is},c_{js})}{\partial(\beta_k,\beta_l)},\\
\frac{\partial(\mathcal{C}_{kt},\mathcal{C}_{lt})}{\partial(c_{is},c_{js})} &= c_{ir}c_{jr}c_{ks}c_{ls}c_{it}c_{jt}(c_{it}c_{jr}-c_{jt}c_{ir})(c_{kr}c_{lt}-c_{kt}c_{lr})\\
\frac{\partial(c_{is},c_{js})}{\partial(\beta_k,\beta_l)} &= \frac{[i,j][k,l]W_s^2}{(p^2)^2},\\
\mathcal{R}(c_{is},c_{js}) &= \frac{[i,j]}{(p^2)^2}\left[\frac{\partial(\mathcal{C}_{kt},\mathcal{C}_{lt})}{\partial(\beta_k,\beta_l)}\right]^{-1}\frac{c_{ir}c_{jt}}{c_{lr}c_{kt}} \qquad (7.32)\\
&= \frac{1}{[k,l]W_s^2c_{jr}c_{lr}c_{ks}c_{ls}c_{it}c_{kt}(c_{it}c_{jr}-c_{jt}c_{ir})(c_{kr}c_{lt}-c_{kt}c_{lr})}\\
&= \frac{W_s^4[k,l]^4}{p_{kls}^2\langle t,j\rangle\langle j,i\rangle\langle i,r\rangle\,[l,s][s,k]\langle r|P_{ks}|l]\,\langle t|P_{ls}|k]}. \qquad (7.33)
\end{aligned}$$

Combining these results for the residues, we have that

$$\begin{aligned}
\mathcal{M} =& -\mathcal{R}(c_{lr},c_{kt})-\mathcal{R}(c_{it},c_{kt})-\mathcal{R}(c^{ij}_{rs},c_{kt})-\mathcal{R}(c_{lr},c_{jr})-\mathcal{R}(c_{lr},c^{ij}_{st})-\mathcal{R}(c_{is},c_{js})\\
=&\frac{W_t^4W_r^4[i,j]^3}{\langle k,r\rangle\langle k,s\rangle\langle l,s\rangle\langle l,t\rangle\langle s|P_{lt}|j]\langle s|P_{kr}|i]\langle t|P_{ij}P_{kr}|s\rangle\langle s|P_{ij}P_{lt}|r\rangle}\\
&-\frac{W_t^4[j,l]^4}{p_{ltj}^2\langle i,r\rangle\langle r,k\rangle\langle k,s\rangle[l,t][t,j]\langle s|P_{lt}|j]\langle i|P_{jt}|l]}\\
&-\frac{W_t^4\ \langle t|P_{ij}|l]^4}{p_{ijt}^2p_{krs}^2\langle r,k\rangle\langle k,s\rangle\langle i,j\rangle\langle j,t\rangle\langle r|P_{ks}|l]\ \langle i|P_{jt}|l]\ \langle t|P_{ij}P_{kr}|s\rangle}\\
&-\frac{W_r^4[k,i]^4}{p_{irk}^2\langle s,l\rangle\langle l,t\rangle\langle t,j\rangle[i,r][r,k]\langle j|P_{ir}|k]\ \langle s|P_{kr}|i]}\\
&-\frac{W_r^4\ \langle r|P_{ij}|k]^4}{p_{ijr}^2p_{lst}^2\ \langle ji\rangle\langle ir\rangle\langle sl\rangle\langle lt\rangle\langle j|Pir|k]\langle t|P_{ls}|k]\ \langle s|P_{lt}P_{ij}|r\rangle}\\
&-\frac{W_s^4[k,l]^4}{p_{kls}^2\langle t,j\rangle\langle j,i\rangle\langle i,r\rangle[l,s][s,k]\langle r|P_{ks}|l]\ \langle t|P_{ls}|k]}. \qquad (7.34)
\end{aligned}$$

The expression (7.34) has the appropriate soft limits and is antisymmetric under the transformations $i\leftrightarrow j$, $k\leftrightarrow l$, $r\leftrightarrow t$, $s\leftrightarrow s$. It is equal, up to notation and signs, to the expressions computed directly from the field theory, and by recursion relations [19, 20].

Other tree amplitudes can be computed in a similar fashion from the expressions for the integrand in section 4. It seems that the dual S-matrix of ACCK leads back to twistor string theory at tree level. It will be interesting to pursue this link at the loop level.

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A Interchange of particles between $\mathcal{P}$ and $\mathcal{N}$

For $k \in \mathcal{P}$, $t \in \mathcal{N}$, let $\mathcal{P}_0 = \mathcal{P} \sim \{k\}$, $\mathcal{N}_0 = \mathcal{N} \sim \{t\}$, and $\mathcal{P}' = \mathcal{P}_0 \cup \{t\}$, $\mathcal{N}' = \mathcal{N}_0 \cup \{k\}$,

$$\begin{aligned} \pi_i &= \sum_{r\in\mathcal{N}_0} c_{ir}\pi_r + c_{it}\pi_t, \qquad i \in \mathcal{P} \\ c_{kt}\pi_i - c_{it}\pi_k &= \sum_{r\in\mathcal{N}_0} (c_{ir}c_{kt} - c_{it}c_{kr})\pi_r, \qquad i \in \mathcal{P}_0 \\ \pi_i &= \sum_{r\in\mathcal{N}_0} \frac{(c_{ir}c_{kt} - c_{it}c_{kr})}{c_{kt}}\pi_r + \frac{c_{it}}{c_{kt}}\pi_k, \qquad i \in \mathcal{P}_0 \\ \pi_t &= -\sum_{r\in\mathcal{N}_0} \frac{c_{kr}}{c_{kt}}\pi_r + \frac{1}{c_{kt}}\pi_k. \end{aligned}$$

So

$$\pi_i = \sum_{r\in\mathcal{N}'} \tilde{c}_{ir}\pi_r, \qquad r \in \mathcal{P}',$$

with

$$\tilde{c}_{ir} = \frac{c_{ir}c_{kt} - c_{it}c_{kr}}{c_{kt}},\ i \in \mathcal{P}_0, r \in \mathcal{N}_0; \quad \tilde{c}_{ik} = \frac{c_{it}}{c_{kt}},\ i \in \mathcal{P}_0; \quad \tilde{c}_{tr} = -\frac{c_{kr}}{c_{kt}},\ r \in \mathcal{N}_0; \quad \tilde{c}_{tk} = \frac{1}{c_{kt}}.$$

Similarly

$$-\bar{\pi}_r = \sum_{i\in\mathcal{P}'} \bar{\pi}_i\tilde{c}_{ir}, \qquad r \in \mathcal{N}',$$

with the same definition of $\tilde{c}_{ir}$. It remains to show that the $\tilde{c}_{ir}$ satisfy the same constraints as the c_{ir}. To do this we establish formulae for them in terms of the κ_ℓ, ρ_ℓ.

$$\begin{aligned} \tilde{c}_{js} = \frac{c_{js}c_{kt} - c_{jt}c_{ks}}{c_{kt}} &= \frac{\kappa_j}{\kappa_s}\left[\prod_{\substack{r\in\mathcal{N}\\ r\neq s}} \frac{\rho_j - \rho_r}{\rho_s - \rho_r} - \prod_{\substack{r\in\mathcal{N}\\ r\neq t}} \frac{\rho_j - \rho_r}{\rho_k - \rho_r} \prod_{\substack{r\in\mathcal{N}\\ u\neq s}} \frac{\rho_k - \rho_u}{\rho_s - \rho_u}\right], \qquad j \in \mathcal{P}_0,\ s \in \mathcal{N}_0, \\ &= \frac{\kappa_j}{\kappa_s}\left[\frac{\rho_j - \rho_t}{\rho_s - \rho_t} \cdot \frac{\rho_s - \rho_k}{\rho_j - \rho_k} - \frac{\rho_j - \rho_s}{\rho_k - \rho_s} \cdot \frac{\rho_k - \rho_t}{\rho_s - \rho_t} \cdot \frac{\rho_s - \rho_k}{\rho_j - \rho_k}\right] \prod_{\substack{r\in\mathcal{N}'\\ r\neq s}} \frac{\rho_j - \rho_r}{\rho_s - \rho_r} \\ &= \frac{\kappa_j}{\kappa_s} \prod_{\substack{r\in\mathcal{N}'\\ r\neq s}} \frac{\rho_j - \rho_r}{\rho_s - \rho_r}, \qquad j \in \mathcal{P}_0,\ s \in \mathcal{N}_0. \end{aligned}$$

$$\begin{aligned} \tilde{c}_{jk} &= \frac{c_{jt}}{c_{kt}} = \frac{\kappa_j}{\kappa_k} \prod_{\substack{r\in\mathcal{N}\\ r\neq t}} \frac{\rho_j - \rho_r}{\rho_t - \rho_r} \prod_{\substack{r\in\mathcal{N}\\ r\neq t}} \frac{\rho_t - \rho_r}{\rho_k - \rho_r} = \frac{\kappa_j}{\kappa_k} \prod_{\substack{r\in\mathcal{N}'\\ r\neq k}} \frac{\rho_j - \rho_r}{\rho_k - \rho_r}, \qquad j \in \mathcal{P}_0. \\ \tilde{c}_{ts} &= -\frac{c_{ks}}{c_{kt}} = -\frac{\kappa_t}{\kappa_s} \prod_{\substack{r\in\mathcal{N}\\ r\neq s}} \frac{\rho_k - \rho_r}{\rho_s - \rho_r} \prod_{\substack{r\in\mathcal{N}\\ r\neq t}} \frac{\rho_t - \rho_r}{\rho_k - \rho_r} = \frac{\kappa_t}{\kappa_s} \prod_{\substack{r\in\mathcal{N}'\\ r\neq s}} \frac{\rho_t - \rho_r}{\rho_s - \rho_r} \qquad s \in \mathcal{N}_0. \\ \tilde{c}_{tk} &= \frac{1}{c_{kt}} = \frac{\kappa_t}{\kappa_k} \prod_{\substack{r\in\mathcal{N}\\ r\neq t}} \frac{\rho_t - \rho_r}{\rho_k - \rho_r} = \frac{\kappa_t}{\kappa_k} \prod_{\substack{r\in\mathcal{N}'\\ r\neq k}} \frac{\rho_t - \rho_r}{\rho_k - \rho_r}. \end{aligned}$$

Thus the $\tilde{c}_{js}$ are given by similar expressions in terms of the ρ_ℓ, k_ℓ as the c_{js}, and so the $\tilde{c}_{js}$ satisfy similar relations to those satisfied by the c_{js}.

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B Relations for the 6-point function

$$
\begin{aligned}
V_jA_{ir}-V_iA_{jr} &= -p^2[k,r], \qquad A_{ir}W_s-A_{is}W_r = -p^2\langle i,t\rangle \\
V_jA_{ir}W_s+V_iA_{js}W_r-V_jA_{is}W_r-V_iA_{jr}W_s &= p^2\bar{A}_{tk}
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{A}_{ri} &= \sum_{s\in\mathcal{N}}\langle r,s\rangle[s,i] = -\sum_{j\in\mathcal{P}}\langle r,j\rangle[j,i], \qquad \bar{a}_{ri} = \frac{1}{p^2}\bar{A}_{ri} \\
\det c &= \frac{\beta}{p^2}\sum_{\substack{i\in\mathcal{P}\\ r\in\mathcal{N}}} V_iW_r(a_{js}a_{kt}-a_{jt}a_{ks}) = \frac{\beta}{(p^2)^2}\sum_{i\in\mathcal{P}}V_i\overline{V}_i\sum_{r\in\mathcal{N}}W_r\overline{W}_r = \beta,
\end{aligned}
$$

where, as before, (i,j,k), (r,s,t) are cyclic.

Corresponding to the relations for A_{ir}, we have

$$
\begin{aligned}
\bar{A}_{ri}\bar{A}_{sj}-\bar{A}_{si}\bar{A}_{rj} &= p^2W_tV_k, \\
\bar{A}_{ri}\overline{V}_j-\bar{A}_{rj}\overline{V}_i &= -p^2\langle r,k\rangle, \\
\overline{W}_s\bar{A}_{ri}-\overline{W}_r\bar{A}_{si} &= -p^2[t,i], \\
\overline{W}_s\bar{A}_{ri}\overline{V}_j+\overline{W}_r\bar{A}_{sj}\overline{V}_i-\overline{W}_r\bar{A}_{si}\overline{V}_j-\overline{W}_s\bar{A}_{rj}\overline{V}_i &= p^2A_{kt} \\
\overline{c}_{ri}\overline{c}_{sj}-\overline{c}_{si}\overline{c}_{rj} &= \overline{\beta}c_{kt}.
\end{aligned}
$$

When $c_{ir}=0$,

$$
\begin{array}{llll}
V_iW_r\beta\overline{c}_{ri} = p^2_{jkr}, & & & \\
V_iW_r\beta\overline{c}_{sj} = \langle i,s\rangle[r,j], & V_iW_r\beta\overline{c}_{tk} = \langle i,t\rangle[r,k], & V_iW_r\beta\overline{c}_{sk} = \langle i,s\rangle[r,k], & V_iW_r\beta\overline{c}_{tj} = \langle i,t\rangle[r,j], \\
W_rc_{is} = \langle i,t\rangle, & W_rc_{it} = -\langle i,s\rangle, & V_ic_{jr} = [k,r], & V_ic_{kr} = -[j,r], \\
V_iW_r\beta\overline{c}_{rj} = \langle i|P_{rk}|j], & V_iW_r\beta\overline{c}_{rk} = \langle i|P_{rj}|k], & V_iW_r\beta\overline{c}_{si} = \langle s|P_{ti}|r], & V_iW_r\beta\overline{c}_{ti} = \langle t|P_{si}|r], \\
V_iW_rc_{js} = [k|P_{si}|t\rangle, & V_iW_rc_{kt} = [j|P_{ti}|s\rangle, & &
\end{array}
$$

using notation defined in (6.12) and (6.13).

When $\overline{c}_{ir}=0$,

$$
\begin{array}{llll}
\overline{V}_i\overline{W}_r\overline{\beta}c_{ir} = p^2_{jkr}, & & & \\
\overline{V}_i\overline{W}_r\overline{\beta}c_{js} = [i,s]\langle r,j\rangle, & \overline{V}_i\overline{W}_r\overline{\beta}c_{kt} = [i,t]\langle r,k\rangle, & \overline{V}_i\overline{W}_r\overline{\beta}c_{ks} = [i,s]\langle r,k\rangle, & \overline{V}_i\overline{W}_r\overline{\beta}c_{jt} = [i,t]\langle r,j\rangle, \\
\overline{W}_r\overline{c}_{si} = -[i,t], & \overline{W}_r\overline{c}_{ti} = [i,s], & \overline{V}_i\overline{c}_{rj} = -\langle k,r\rangle, & \overline{V}_i\overline{c}_{rk} = \langle j,r\rangle, \\
\overline{V}_i\overline{W}_r\overline{\beta}c_{jr} = [i|P_{rk}|j\rangle, & \overline{V}_i\overline{W}_r\overline{\beta}c_{kr} = [i|P_{rj}|k\rangle, & \overline{V}_i\overline{W}_r\overline{\beta}c_{is} = [s|P_{ti}|r\rangle, & \overline{V}_i\overline{W}_r\overline{\beta}c_{it} = [t|P_{si}|r\rangle, \\
\overline{V}_i\overline{W}_r\overline{c}_{sj} = -\langle k|P_{si}|t], & \overline{V}_i\overline{W}_r\overline{c}_{tk} = -\langle j|P_{ti}|s], & &
\end{array}
$$

where $\overline{\beta}=1/\beta$.

C Relations for the 7-point function

We derive the relations we need to evaluate the 7-point function working directly from the equations

$$\pi_\ell = c_{\ell r}\pi_r + c_{\ell s}\pi_s + c_{\ell t}\pi_t, \qquad \ell = i,j,k,l, \tag{C.1}$$

$$-\overline{\pi}_u = \overline{\pi}_i c_{iu} + \overline{\pi}_j c_{ju} + \overline{\pi}_k c_{ku} + \overline{\pi}_l c_{lu}, \qquad u = r,s,t. \tag{C.2}$$

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If $c_{kt} = 0$, from (C.1) with $\ell = k$,

$$c_{kr} = \langle k,s\rangle/\langle r,s\rangle, \qquad c_{ks} = \langle k,r\rangle/\langle s,r\rangle, \qquad c_{kt} = 0. \tag{C.3}$$

(a) For $c_{kt} = c_{lr} = 0$, in addition to (C.3),

$$c_{lr} = 0, \qquad c_{ls} = \langle l,t\rangle/\langle s,t\rangle, \qquad c_{lt} = \langle l,s\rangle/\langle t,s\rangle. \tag{C.4}$$

and from (C.2) with $u = r$,

$$\overline{\pi}_i c_{ir} + \overline{\pi}_j c_{jr} = -(\overline{\pi}_k \langle k,s\rangle + \overline{\pi}_r \langle r,s\rangle)/\langle r,s\rangle,$$

yielding

$$c_{ir} = [j|P_{kr}|s\rangle/[i,j]\langle r,s\rangle, \qquad c_{jr} = [i|P_{kr}|s\rangle/[j,i]\langle r,s\rangle, \tag{C.5}$$

and, similarly,

$$c_{it} = [j|P_{lt}|s\rangle/[i,j]\langle t,s\rangle, \qquad c_{jt} = [i|P_{lt}|s\rangle/[j,i]\langle t,s\rangle. \tag{C.6}$$

From (C.1) with $\ell = i$

$$c_{is}\langle s,t\rangle = \langle i,t\rangle - c_{ir}\langle r,t\rangle, \qquad c_{js}\langle s,t\rangle = \langle j,t\rangle - c_{jr}\langle r,t\rangle$$

implying

$$(c_{ir}c_{js} - c_{jr}c_{is})\langle s,t\rangle = \langle j,t\rangle c_{ir} - \langle i,t\rangle c_{jr} = -\langle t|P_{ij}P_{kr}|s\rangle/[i,j]\langle r,s\rangle$$

so that

$$c_{ir}c_{js} - c_{jr}c_{is} = -\langle t|P_{ij}P_{kr}|s\rangle/[i,j]\langle r,s\rangle\langle s,t\rangle, \tag{C.7}$$

and, similarly,

$$c_{it}c_{js} - c_{jt}c_{is} = -\langle r|P_{ij}P_{lt}|s\rangle/[i,j]\langle s,t\rangle\langle r,s\rangle. \tag{C.8}$$

(b) When $c_{kt} = c_{it} = 0$, from (C.1) with $\ell = k, i$,

$$c_{kr} = \langle k,s\rangle/\langle r,s\rangle, \qquad c_{ks} = \langle k,r\rangle/\langle s,r\rangle, \qquad c_{kt} = 0, \tag{C.9}$$

and

$$c_{ir} = \langle i,s\rangle/\langle r,s\rangle, \qquad c_{is} = \langle i,r\rangle/\langle s,r\rangle, \qquad c_{it} = 0. \tag{C.10}$$

From (C.2) with $u = t$,

$$c_{jt} = -[t,l]/[j,l], \qquad c_{lt} = -[t,j]/[l,j], \tag{C.11}$$

and from (C.1) with $\ell = j, l$,

$$c_{jr}\langle r,s\rangle = \langle s|P_{jt}|l]/[l,j], \qquad c_{js}\langle s,r\rangle = \langle r|P_{jt}|l]/[l,j], \tag{C.12}$$

$$c_{lr}\langle r,s\rangle = \langle s|P_{lt}|j]/[j,l], \qquad c_{ls}\langle s,r\rangle = \langle r|P_{lt}|j]/[j,l]. \tag{C.13}$$

From (C.1) with $\ell = i, j$,

$$\langle r,s\rangle^2[l,j](c_{ir}c_{js} - c_{is}c_{jr}) = \langle i,r\rangle\langle s|P_{jt}|l] - \langle i,s\rangle\langle r|P_{jt}|l] = -\langle r,s\rangle\langle i|P_{jt}|l]$$

so that

$$c_{ir}c_{js} - c_{is}c_{jr} = \langle i|P_{jt}|l]/\langle r,s\rangle[j,l] \tag{C.14}$$

From (C.1) with $\ell = j, l$,

$$(c_{lr}c_{js} - c_{jr}c_{ls})\langle s,t\rangle = \langle j,t\rangle c_{lr} - \langle l,t\rangle c_{jr} = (\langle s|P_{lt}|j]\langle j,t\rangle + \langle s|P_{jt}|l]\langle l,t\rangle)/[j,l]\langle r,s\rangle,$$

from which it follows that

$$c_{lr}c_{js} - c_{jr}c_{ls} = -p_{jlt}^2/[j,l]\langle r,s\rangle. \tag{C.15}$$

(c) When $c_{kt} = c_{rs}^{ij} = 0$, from (C.1) with $\ell = k$

$$c_{kr} = \langle k,s\rangle/\langle r,s\rangle, \qquad c_{ks} = \langle k,r\rangle/\langle s,r\rangle, \qquad c_{kt} = 0, \tag{C.16}$$

and with $\ell = i, j$,

$$c_{jr}\pi_i - c_{ir}\pi_j = (c_{jr}c_{it} - c_{ir}c_{jt})\pi_t,$$

so that

$$c_{jr}\langle i,t\rangle = c_{ir}\langle j,t\rangle, \qquad c_{js}\langle i,t\rangle = c_{is}\langle j,t\rangle. \tag{C.17}$$

From (C.2) with $u = r$,

$$-[l,r] = [l,i]c_{ir} + [l,j]c_{jr} + [l,k]\langle k,s\rangle/\langle r,s\rangle,$$

implying

$$-\langle s|P_{kr}|l]/\langle r,s\rangle = [l,i]c_{ir} + [l,j]c_{jr} = \langle t|P_{ij}|l]c_{ir}/\langle i,t\rangle,$$

so that

$$c_{ir} = -\frac{\langle s|P_{kr}|l]\langle i,t\rangle}{\langle t|P_{ij}|l]\langle r,s\rangle}, \quad c_{jr} = -\frac{\langle s|P_{kr}|l]\langle j,t\rangle}{\langle t|P_{ij}|l]\langle r,s\rangle}, \quad c_{jr}c_{it} - c_{ir}c_{jt} = \frac{\langle s|P_{kr}|l]\langle i,j\rangle}{\langle t|P_{ij}|l]\langle r,s\rangle}, \tag{C.18}$$

$$c_{is} = -\frac{\langle r|P_{ks}|l]\langle i,t\rangle}{\langle t|P_{ij}|l]\langle s,r\rangle}, \quad c_{js} = -\frac{\langle r|P_{ks}|l]\langle j,t\rangle}{\langle t|P_{ij}|l]\langle s,r\rangle}, \quad c_{js}c_{it} - c_{is}c_{jt} = \frac{\langle r|P_{ks}|l]\langle i,j\rangle}{\langle t|P_{ij}|l]\langle s,r\rangle}, \tag{C.19}$$

and, from (C.1) with $\ell = i$,

$$c_{it}\langle t|P_{ij}|l]\langle r,s\rangle\langle i,t\rangle = -\langle s|P_{kr}|l]\langle i,t\rangle\langle r,i\rangle + \langle r|P_{ks}|l]\langle i,t\rangle\langle s,i\rangle = -\langle r,s\rangle\langle i|P_{rks}|l]\langle i,t\rangle$$

so that

$$c_{it} = \frac{\langle i|P_{jt}|l]}{\langle t|P_{ij}|l]}, \qquad c_{jt} = \frac{\langle j|P_{it}|l]}{\langle t|P_{ij}|l]}. \tag{C.20}$$

From (C.2) with $u = t$,

$$\langle t|P_{ij}|l][t,l]c_{lt} = -[t,i]c_{it} - [t,j]c_{jt} = -[t,i]\langle i|P_{jt}|l] - [t,j]\langle j|P_{it}|l]$$

implying

$$c_{lt} = \frac{p_{ijt}^2}{\langle t|P_{ij}|l]}. \tag{C.21}$$

From (C.1) with $\ell = l$,

$$\begin{aligned}\langle t|P_{ij}|l]c_{lr}\langle r,s\rangle &= \langle l,s\rangle\langle t|P_{ij}|l] - c_{lt}\langle t,s\rangle\langle t|P_{ij}|l]\\ &= \langle l,s\rangle\langle t|P_{ij}|l] - p_{ijt}^2\langle t,s\rangle = -\langle t|P_{ij}P_{kr}|s\rangle\end{aligned}$$

so that

$$c_{lr} = -\frac{\langle t|P_{ij}P_{kr}|s\rangle}{\langle r,s\rangle\langle t|P_{ij}|l]}, \qquad c_{ls} = -\frac{\langle t|P_{ij}P_{ks}|r\rangle}{\langle s,r\rangle\langle t|P_{ij}|l]}. \tag{C.22}$$

From (C.1) with $\ell = j,l$,

$$(c_{jr}c_{ls} - c_{js}c_{lr})\langle r,t\rangle = c_{ls}\langle j,t\rangle - c_{js}\langle l,t\rangle$$

implying

$$c_{jr}c_{ls} - c_{js}c_{lr} = -\frac{p_{krs}^2\langle j,t\rangle}{\langle r,s\rangle\langle t|P_{ij}|l]}. \tag{C.23}$$

(d) When $c_{lr} = c_{jr} = 0$, proceeding as in (b), we have

$$c_{lr} = 0, \qquad c_{ls} = \langle l,t\rangle/\langle s,t\rangle, \qquad c_{lt} = \langle l,s\rangle/\langle t,s\rangle, \tag{C.24}$$

$$c_{jr} = 0, \qquad c_{js} = \langle j,t\rangle/\langle s,t\rangle, \qquad c_{jt} = \langle j,s\rangle/\langle t,s\rangle, \tag{C.25}$$

$$c_{ir} = -[r,k]/[i,k], \qquad c_{kr} = -[r,i]/[k,i], \tag{C.26}$$

$$c_{is}\langle s,t\rangle = \langle t|P_{ir}|k]/[k,i], \qquad c_{it}\langle t,s\rangle = \langle s|P_{ir}|k]/[k,i], \tag{C.27}$$

$$c_{ks}\langle s,t\rangle = \langle t|P_{kr}|i]/[i,k], \qquad c_{kt}\langle t,s\rangle = \langle s|P_{kr}|i]/[i,k], \tag{C.28}$$

$$c_{it}c_{js} - c_{is}c_{jt} = \langle j|P_{ir}|k]/\langle s,t\rangle[i,k], \tag{C.29}$$

$$c_{is}c_{kt} - c_{it}c_{ks} = p_{ikr}^2/[i,k]\langle s,t\rangle. \tag{C.30}$$

(e) When $c_{lr} = c_{st}^{ij} = 0$, proceeding as in (c), we have

$$c_{lr} = 0, \quad c_{ls} = \langle l,t\rangle/\langle s,t\rangle, \quad c_{lt} = \langle l,s\rangle/\langle t,s\rangle, \tag{C.31}$$

$$c_{it} = -\frac{\langle s|P_{lt}|k]\langle i,r\rangle}{\langle r|P_{ij}|k]\langle t,s\rangle}, \quad c_{jt} = -\frac{\langle s|P_{lt}|k]\langle j,r\rangle}{\langle r|P_{ij}|k]\langle t,s\rangle}, \quad c_{jt}c_{ir} - c_{it}c_{jr} = \frac{\langle s|P_{lt}|k]\langle i,j\rangle}{\langle r|P_{ij}|k]\langle t,s\rangle}, \tag{C.32}$$

$$c_{is} = -\frac{\langle t|P_{ls}|k]\langle i,r\rangle}{\langle r|P_{ij}|k]\langle s,t\rangle}, \quad c_{js} = -\frac{\langle t|P_{ls}|k]\langle j,r\rangle}{\langle r|P_{ij}|k]\langle s,t\rangle}, \quad c_{js}c_{ir} - c_{is}c_{jr} = \frac{\langle t|P_{ls}|k]\langle i,j\rangle}{\langle r|P_{ij}|k]\langle s,t\rangle}, \tag{C.33}$$

$$c_{ir} = \frac{\langle i|P_{jr}|k]}{\langle r|P_{ij}|k]}, \qquad c_{jr} = \frac{\langle j|P_{ir}|k]}{\langle r|P_{ij}|k]}, \tag{C.34}$$

$$c_{kr} = \frac{p_{ijr}^2}{\langle r|P_{ij}|k]}, \tag{C.35}$$

$$c_{kt} = -\frac{\langle r|P_{ij}P_{lt}|s\rangle}{\langle t,s\rangle\langle r|P_{ij}|k]}, \quad c_{ks} = -\frac{\langle r|P_{ij}P_{ls}|t\rangle}{\langle s,t\rangle\langle r|P_{ij}|k]}, \tag{C.36}$$

$$c_{jt}c_{ks} - c_{js}c_{kt} = -\frac{p_{lst}^2\langle j,r\rangle}{\langle t,s\rangle\langle r|P_{ij}|k]}. \tag{C.37}$$

(f) When $c_{is} = c_{js} = 0$, also proceeding as in (b), we have

$$c_{ir} = \langle i,t\rangle/\langle r,t\rangle, \qquad c_{is}=0, \quad c_{it}=\langle i,r\rangle/\langle t,r\rangle, \tag{C.38}$$

$$c_{jr} = \langle j,t\rangle/\langle r,t\rangle, \qquad c_{js}=0, \quad c_{jt}=\langle j,r\rangle/\langle t,r\rangle, \tag{C.39}$$

$$c_{ks} = -[s,l]/[k,l], \qquad c_{ls}=-[s,k]/[l,k], \tag{C.40}$$

$$c_{kr}\langle r,t\rangle = \langle t|P_{ks}|l]/[l,k], \quad c_{kt}\langle t,r\rangle=\langle r|P_{ks}|l]/[l,k], \tag{C.41}$$

$$c_{lr}\langle r,t\rangle = \langle t|P_{ls}|k]/[k,l], \quad c_{lt}\langle t,r\rangle=\langle r|P_{ls}|k]/[k,l], \tag{C.42}$$

$$c_{it}c_{jr} - c_{ir}c_{jt} = -(\langle i,r\rangle\langle j,t\rangle - \langle j,r\rangle\langle i,t\rangle)/\langle r,t\rangle^2 = \langle i,j\rangle/\langle t,r\rangle. \tag{C.43}$$

$$(c_{kr}c_{lt} - c_{lr}c_{kt})\langle r,s\rangle = \langle k,s\rangle c_{lt} - \langle l,s\rangle c_{kt} = (\langle r|P_{ls}|k]\langle k,s\rangle + \langle r|P_{ks}|l]\langle l,s\rangle)/[k,l]\langle t,r\rangle,$$

so that

$$c_{kr}c_{lt} - c_{lr}c_{kt} = -p_{kls}^2/[k,l]\langle t,r\rangle. \tag{C.44}$$

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