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Lattice QCD in Background Fields

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Abstract. We review some recent results about the exploration of strong interactions in presence of external background fields by means of lattice QCD simulations. We discuss in particular studies concerning the influence of the external fields on the non-perturbative properties of the QCD vacuum and of the QCD phase diagram.

1. Introduction

The influence of external background fields on Quantum Chromodynamics has attracted growing interest in the recent years. At a purely theoretical level, external fields represent useful probes to investigate the non-perturbative properties of strong interactions. Moreover, they are relevant to many phenomenological contexts. Large magnetic or chromomagnetic background fields, of the order of 10^{16} Tesla, i.e. $\sqrt{|e|B} \sim 1.5$ GeV, may have been produced at the time of the cosmological electroweak phase transition [1]. Large magnetic fields are generated in non-central heavy ion collisions (up to $\sim 10^{15}$ Tesla at LHC [2, 3]), where they may give rise, in presence of non-trivial topological charge fluctuations, to CP-odd effects like the *chiral magnetic effect* [4, 2, 5]: the net unbalance of chirality induced by the topological background would lead, in presence of a magnetic field strong enough to align the magnetic moments of quarks, to a net separation of electric charge along the field direction. Large magnetic (or even chromomagnetic [6]) fields are also expected in compact astrophysical objects like magnetars [7], where they may reach up to 10^{10} Tesla.

At the level of the QCD vacuum, one of the major effects predicted in presence of an external magnetic field is known as magnetic catalysis [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38]. It consists in an enhancement of chiral symmetry breaking or spontaneous mass generation, it is common also to other systems characterized by chiral fermion fields and can be related to the dimensional reduction of particle dynamics in a strong magnetic field. While such effect seems naively related to fermion properties only, it has been found that the interrelation with gluon fields, which interact with electromagnetic fields only indirectly through quark loops, may play indeed a relevant role [39]: actually this is not a big surprise for a non-perturbative, strongly interacting theory. Unexpected phenomena have been found, like an inverse magnetic catalysis at high enough temperature [40, 41], which may be intimately related to the non-perturbative properties of strong interactions and therefore claim for a deeper investigation.

The phenomenology regarding the influence of electromagnetic fields on the properties of QCD vacuum is not limited to magnetic catalysis. One may indeed consider the vacuum as any other medium and study its reaction to various kinds of external stimulation, involving for instance its magnetic susceptibility [42, 43, 44, 45], its electrical conductivity [46] or even its susceptibility to CP-odd electromagnetic background fields, giving rise to an effective topological θ parameter [47].

Extending the interest beyond the properties of the vacuum state alone, one can consider the introduction of a background field, either magnetic of chromomagnetic, on the same grounds as any other external parameter, like e.g. a finite baryon chemical potential, capable of modifying the phase diagram of the theory. The interest in such issue is both theoretical and phenomenological, since strong background fields may be relevant to the cosmological QCD transition, to relativistic heavy ion collisions and to the physics of compact astrophysical objects. The main questions regarding the QCD transition are the following: 1) is a strong enough magnetic field capable of clearly separating the temperatures at which deconfinement and chiral symmetry restoration take place? 2) what is the dependence of the temperature and of the nature of the transition on the background field strength? 3) does any new, exotic phase emerge for strong enough background fields, like e.g. an anisotropic superconductive phase [48, 49, 50, 51]? Such questions are of utmost importance and they have indeed stimulated an intense activity in the recent years, based mostly on model computations [52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71].

Lattice QCD simulations offer the possibility of a first principle investigation of such exciting issues, and indeed many studies have been performed in the last few years. Lattice studies in presence of electromagnetic fields have been done since long, originally with the purpose of studying the electromagnetic properties of hadrons [72, 73, 74, 75, 76]. Similarly to the continuum theory, electromagnetic fields affect quark propagation by a modification of the covariant derivative $D_\mu = \partial_\mu + i\,gA^a_\mu T^a + i\,qa_\mu$ which can be easily discretized on the lattice in terms of $SU(3)$ and $U(1)$ link variables, e.g. for the simplest symmetric discretization on an isotropic cubic lattice of spacing a:

$$D_\mu\psi \to \frac{1}{2a}\left(U_\mu(n)u_\mu(n)\psi(n+\hat{\mu}) - U^\dagger_\mu(n-\hat{\mu})u^*_\mu(n-\hat{\mu})\psi(n-\hat{\mu})\right) \qquad (1)$$

where $U_\mu \in SU(3)$ and $u_\mu = \exp(i\,q\int_{an}^{a(n+\hat{\mu})} dx_\mu a_\mu) \simeq \exp(i\,a\,q\,a_\mu(n)) \in U(1)$. In presence of periodic boundary conditions, which is the usual choice in lattice simulations in order to minimize finite size effects, one can show that a constant and uniform electromagnetic field can be imposed only if the components $F_{\mu\nu}$ are integer multiples of the elementary quanta $2\pi/(qa^2L_\mu L_\nu)$, where L_μ is the lattice extension in direction μ in dimensionless lattice units [77, 78, 79, 80]. The finite lattice spacing (ultraviolet cutoff) places further limitations, setting a limiting value π/a^2 for $|qF_{\mu\nu}|$, but constraining in fact $|qF_{\mu\nu}| \ll \pi/a^2$ if one wants to keep away from unphysical saturation effects.

The fermion determinant is real and positive in the case of a magnetic field, allowing for a probabilistic interpretation of the path integral measure and thus for the application of standard Monte Carlo methods. However, it is easy to realize that real components $F_{0i} \neq 0$ in Euclidean space corresponds, when continued to Minkowski space, to purely imaginary electric fields [81, 76]. In order to have a real electric field, one needs imaginary components for the gauge potential in Euclidean space, which make the quark determinant complex: such sign problem hinders numerical simulations, similarly to what happens in presence of a baryon chemical potential. A possible approach, followed e.g. by lattice studies of the electric polarizabilities of hadrons, is to perform numerical simulations in presence of an imaginary electric field, then exploiting analytic continuation.

The introduction of a chromomagnetic field requires a different approach, since in this case the background variables are strictly related to the quantum gluon degrees of freedom: a standard procedure is that defined in the framework of the lattice Schrödinger functional [82, 83, 84, 85, 86, 87, 88, 89, 90]. One considers functional integration over Euclidean space-time, with frozen spatial field configurations at the temporal boundaries, $A_i^{ext1}(\vec{x}, \tau_1)$ and $A_i^{ext2}(\vec{x}, \tau_2)$: the quantum amplitude of passing from the field eigenstate $|A_i^{ext1}\rangle$ to the field eigenstate $|A_i^{ext2}\rangle$ is dominated, in the classical limit, by the configuration having the minimal action among those respecting the given boundary conditions; functional integration is therefore performed around such classical background field.

In the following we will discuss in detail only few of the many interesting topics mentioned above. In particular, in Section 2 we will discuss about the properties of the QCD vacuum and of the QCD phase diagram in presence of strong magnetic fields. Section 3 is devoted to lattice studies in presence of chromomagnetic background fields. Finally, in Section 4, we will discuss about the breaking of CP symmetry in the strong sector induced by CP-odd electromagnetic backgrounds.

2. QCD in strong magnetic fields

Exploratory lattice studies of the QCD vacuum in strong magnetic fields have considered magnetic catalysis in the quenched approximation for $SU(2)$ [91] and $SU(3)$ [43] gauge theories, while later studies [92, 39, 40, 93, 41] have considered effects deriving from the inclusion of dynamical fermions coupled to the external field. A useful quantity, which can be used to quantify magnetic catalysis, is the relative increment of the chiral condensate, defined as

$$r(B) = \frac{\langle\bar{\psi}\psi\rangle(B) - \langle\bar{\psi}\psi\rangle(B=0)}{\langle\bar{\psi}\psi\rangle(B=0)} \,. \tag{2}$$

The mass dependent additive renormalization of the chiral condensate cancels out in the numerator of Eq. (2); a residual additive renormalization remains in the denominator, leading to an overall, B independent multiplicative renormalization for $r(B)$. Chiral perturbation theory predicts a linear, non-analytic dependence of $r(B)$ on $|B|$ in the chiral limit [18], while for non-zero pion masses an analytic dependence is recovered, which is quadratic for small fields [20, 21]

$$r(B) \simeq \frac{(|e|B)^2}{96\pi^2 F_\pi^2 m_\pi^2} \,. \tag{3}$$

In Ref. [39], where a standard staggered fermion action and a plaquette gauge action have been adopted to investigate $N_f = 2$ QCD, an attempt has been made to determine the contribution to magnetic catalysis deriving from the indirect effect of the magnetic field, via dynamical quark loops, on the gluon field distribution. To that aim, the authors have measured $r(B)$ on gauge configurations sampled in presence of the magnetic field, however without including the magnetic field in the observable, i.e. in the Dirac operator which is inverted to determine the condensate; the relative increment of the condensate measured in this way is named "dynamical", $r^{dyn}(B)$. The complementary quantity is the "valence" contribution, $r^{val}(B)$, which is determined by including the magnetic field in the observable but not in the gluon field distribution. One can show that such separation can be done consistently in the limit of small external fields [39], in the sense that $r(B) \simeq r^{dyn}(B) + r^{val}(B)$, so that one can determine the contribution to catalysis stemming from indirect effects of the magnetic background on gluon fields. Numerical data, which are reported in Fig. 1 for a pion mass $m_\pi \sim 200$ MeV and for zero temperature, show that the separation is well defined in an extended range of magnetic field strengths, and that the dynamical contribution can be as large as 40 % of the total signal.

An improved investigation of magnetic catalysis in the QCD vacuum has appeared recently [41], making use of (stout) rooted staggered discretization of $N_f = 2+1$ QCD: the authors have adopted physical quark masses and performed the extrapolation to the continuum limit. As in Ref. [39], also in this case one observes, for the relative increment $r(B)$ (see Fig. 1

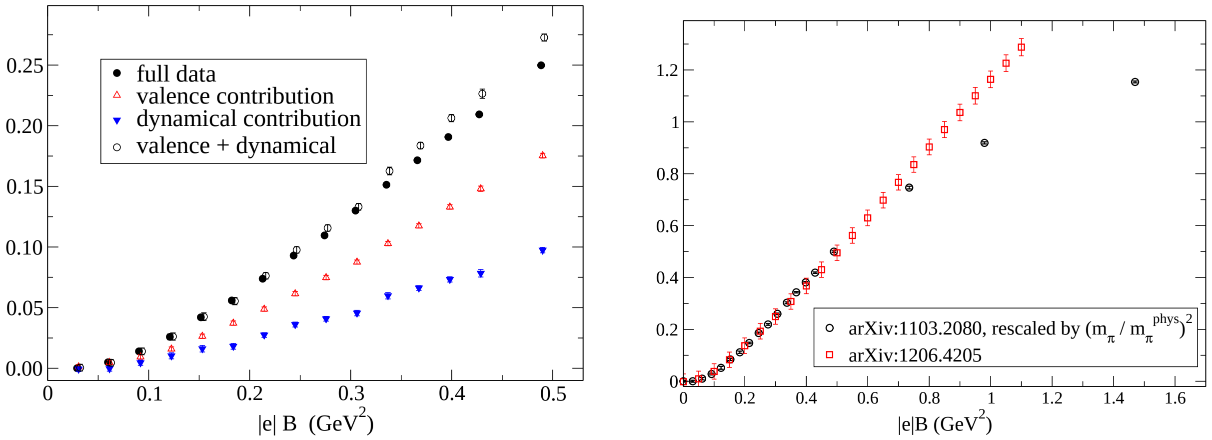

Figure 1. Left: relative increment of the average of the u and d quark condensates as a function of the magnetic field. $r(B), r^{val}(B), r^{dyn}(B)$ and $r^{val}(B) + r^{dyn}(B)$ are reported separately. From [39]. Right: relative increment of the average condensate, comparison of the data from Ref. [39] with those from Ref. [40], after rescaling the former to a physical pion mass according to Eq. (3).

of Ref. [41]), a quadratic dependence of $r(B)$ on $|e|B$ for small magnetic fields, followed by an almost linear dependence as $|eB| \gg m_\pi^2$.

It is interesting to observe that, in the regime of small fields, the results of Ref. [41] are compatible with those of Ref. [39] also quantitatively, if they are rescaled by an appropriate factor given by the ratio of the squared pion masses adopted in the two studies, as suggested by Eq. (3). A direct comparison is presented in Fig. 1 (right) [1]: a nice agreement is observed for small B, while the discrepancies visible at larger values of B can be ascribed to the finite cutoff (saturation) effects affecting the results at large B of Ref. [39] (where $a^{-1} \sim 650$ MeV), while those of Ref. [41] are already extrapolated to the continuum limit.

The issue of magnetic catalysis becomes even more interesting as one approaches the high temperature, deconfined phase: results from Refs. [40, 41] have shown that the growth of the chiral condensate with the magnetic field stops and turns into an inverse magnetic catalysis at high enough temperatures, in particular close and above the deconfinement transition. Such effect, which has not been observed in studies with unimproved actions or unphysical quark masses [92, 93] has different possible interpretations, like those based on the effects of a strong magnetic field on gluodynamics [19, 37] or that pointing to dimensional reduction of neutral pion dynamics induced by the magnetic field [94]; however, in fact the effect has not yet been fully understood.

A topic which is strictly related to inverse magnetic catalysis regards the fate of chiral symmetry restoration in a strong magnetic background. Present studies [92, 40, 93] agree on two facts: 1) there is no clear splitting of deconfinement and chiral symmetry restoration for $|e|B$ up to 1 GeV^2; 2) the magnetic field leads to a strengthening of the transition. However, while studies with unimproved actions and unphysical quark masses observe an increase in the pseudo-critical temperature T_c as the magnetic field is switched on [92, 93] (even if the observed increase is quite modest in Ref. [92] and of the order of 2% at $eB \sim 1$ GeV^2), the improved studies observing inverse magnetic catalysis also observe a decrease in T_c, which is of the order of 10-20% at $|e|B \sim 1$ GeV^2. It is not surprising that various improvements may turn the slowly

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[1] The author thanks G. Endrodi for giving him access to the continuum extrapolated data of Ref. [41].

increasing function $T_c(B)$ of Ref. [92] into a decreasing function, however we believe that it would be of great importance to understand which aspect is more directly related to the appearance of inverse catalysis and to the different behavior observed for $T_c(B)$: is it just a problem of approach to the continuum limit, or can the difference be traced back to the different number of flavors and quark mass spectra? A clear answer to those questions may be given by considering each single aspect separately.

We will not discuss in due detail many other interesting aspects of the influence of strong magnetic fields on strong interactions. Among the studies regarding chiral dynamics, we mention those addressing the determination of the magnetic susceptibility of the chiral condensate, both in the quenched approximation [42, 43] and in full QCD [45]: such quantity is part of the total contribution to the magnetic susceptibility of the vacuum and recent unquenched results [45] show that it is of diamagnetic nature. Other studies have investigated the issue of the electrical conductivity of the QCD vacuum by measuring Euclidean vector current correlators, and pointed out that the vacuum may turn from an insulator to an anisotropic conductor for strong enough magnetic fields [46]. Also the possible emergence of an anisotropic superconductive phase for strong enough magnetic fields, due to charged ρ meson condensation, is currently under investigation [48, 49, 95, 50, 51].

Finally, let us briefly mention lattice studies aimed at the verification or quantification of the chiral magnetic effect. Early studies have looked directly at electric charge separation for gauge configurations containing a non-trivial topological background and in presence of the magnetic field [96], or at the increased fluctuations of the electric current along the direction of the magnetic field [97]; lattice studies about the modifications of the Dirac spectrum in presence of combined topological (instanton) and magnetic background [96] have been compared with analytic studies in Ref. [98]. Later lattice studies [99] have considered instead the introduction of an axial chemical potential μ_5, in order to create a net unbalance between left-handed and right-handed quarks, together with a background magnetic field, then checking for the presence of an electric current directed along the magnetic field direction and proportional to both B and μ_5, as predicted in Ref. [5].

3. QCD in strong chromomagnetic fields

A chromomagnetic background field influences directly gluodynamics, for that reason its effects on QCD dynamics have been studied both in pure Yang-Mills theories and in full QCD and for various kinds of backgrounds, going from those corresponding to an uniform magnetic field to those produced by magnetic monopoles [84, 85, 86, 87, 88, 89, 90]; in the following we shall focus on the case of a constant background field [86, 87, 88].

Also a chromomagnetic background field leads to an equal shift of chiral symmetry restoration and of deconfinement. Moreover, present lattice results indicate that, both for pure gauge and full QCD, the transition temperature decreases, as shown in Fig. 2 (left) [88], where the (pseudo)critical temperature T_c (expressed in units of the parameter $\Lambda \sim 6$ MeV), is plotted as a function of gB for the pure gauge theory and for $N_f = 2$ QCD; T_c is determined in various consistent ways, like looking at the peaks in the susceptibilities of the chiral condensate or of the Polyakov loop. The change in the critical temperature is, in general, quite larger than what happens in presence of a magnetic background: a possible interpretation may be based of the fact that a colored background affects directly gluodynamics, while the influence of magnetic fields on gluons is only indirect. An extrapolation of lattice results would hint, as shown in Fig. 2, at the presence of a deconfining transition at zero temperature for $\sqrt{gB}$ of the order of 1 GeV: such possibility should be further investigated in the future, for its possible cosmological and astrophysical implications. In particular, one should repeat the study of Ref. [88] in presence of physical quark masses (with the values adopted in [88], m_π is of the order of 400-500 MeV) and approaching the continuum limit: that would also permit to reach higher values of the external

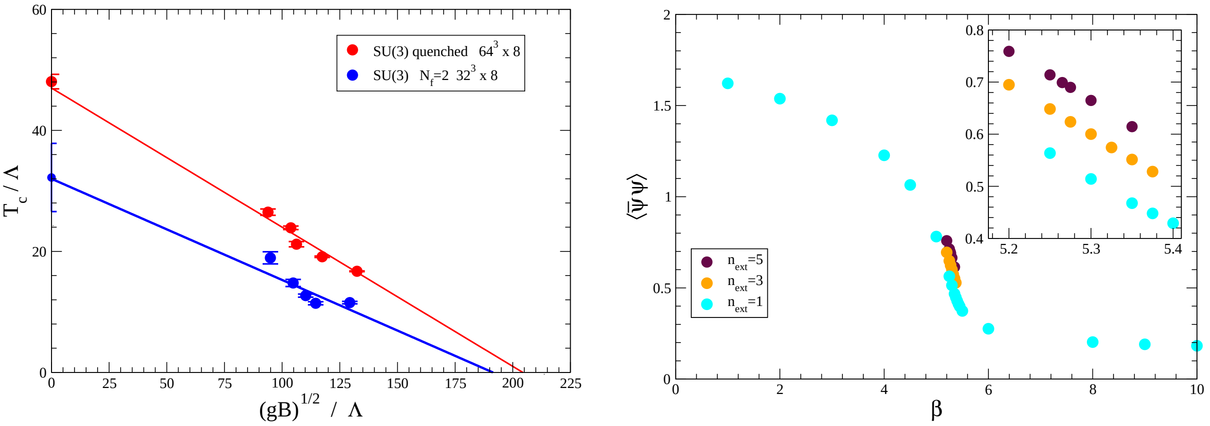

Figure 2. Left: dependence of the transition temperature on the chromomagnetic background field for pure gauge and for $N_f = 2$ QCD; quantities are expressed in terms of the parameter $\Lambda \sim 6$ MeV. Right: chiral condensate on a 32^3 lattice as a function of the inverse gauge coupling β (hence of the temperature) for different values of the chromomagnetic background field $gB = \pi n_{\text{ext}}/(16a^2)$ and $N_f = 2$ QCD. The inset is a zoom on the transition region. Figure taken from [88].

field.

Finally, we notice that Ref. [88] also reports evidence about magnetic catalysis induced by the chromomagnetic background field. In Fig. 2 (right), in particular, we report the behavior of the chiral condensate as a function of the temperature (actually of the inverse gauge coupling) for a few values of gB: at least around the transition, where data for all chromomagnetic fields are available, the chiral condensate is an increasing function of gB. In this case, however, the transition temperature decreases anyway, meaning that inverse catalysis is not a necessary condition for a decreasing T_c, at least in presence of finite quark masses.

4. QCD in CP-odd electromagnetic fields

An aspect which clearly emerges from lattice simulations and from model studies [37, 19, 100] is that an electromagnetic background field may have a significant influence also on the gluonic sector, even if it is directly coupled only to quark fields. A possible way to further clarify such issue is to investigate how the explicit breaking of an exact symmetry by the background field propagates to the gluon sector. The case of charge conjugation symmetry has been discussed in Ref. [101], while the case of CP symmetry, which is discussed in the following, has been investigated in Ref. [47].

Let us consider a constant background electromagnetic field such that $F_{\mu\nu}\tilde{F}_{\mu\nu} \propto \vec{E}\cdot\vec{B} \neq 0$: such background induces an effective CP-violating θ term in the gluon sector, $\theta_{\text{eff}}\, q(x) = \theta_{\text{eff}} \frac{g^2}{64\pi^2} G^a_{\mu\nu}(x)\tilde{G}^a_{\mu\nu}(x)$ where $q(x)$ is the topological charge density. θ_{eff} must be odd in $\vec{E}\cdot\vec{B}$ and, at the lowest order, one can write

$$\theta_{\text{eff}} \simeq \chi_{CP}\, e^2 \vec{E}\cdot\vec{B} + O((\vec{E}\cdot\vec{B})^3) \tag{4}$$

where χ_{CP} is defined as the susceptibility of the QCD vacuum to CP-breaking electromagnetic fields. Such effect is complementary to the chiral magnetic effect, where a CP-violating non-Abelian background gives rise to charge separation, hence to an electric field parallel to a background magnetic field, thus propagating CP violation from the gluonic to the

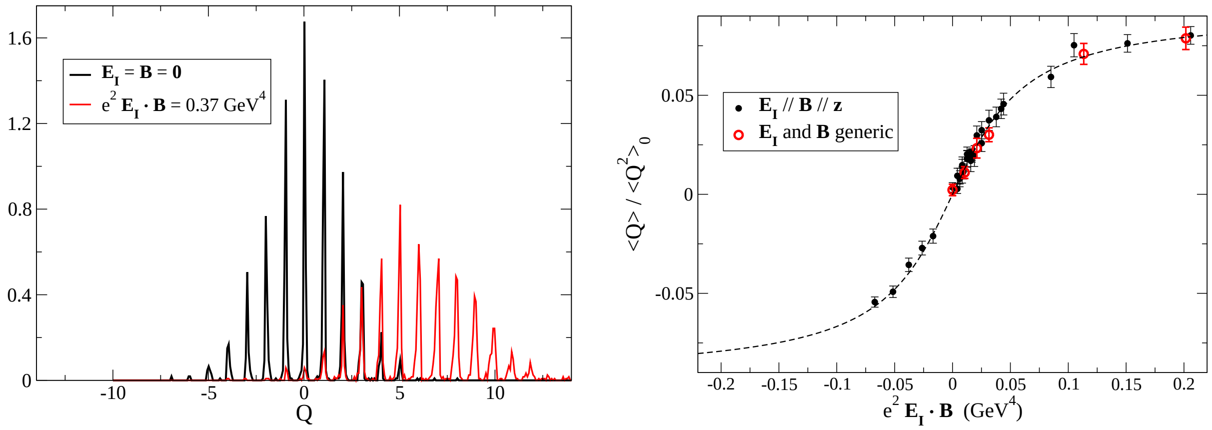

Figure 3. Left: topological charge distribution for two different backgrounds fields; data obtained on a 16^4 lattice for $m_\pi \simeq 480$ MeV and $a \simeq 0.12$ fm. Right: $\langle Q\rangle(\vec{E}_I, \vec{B})/\langle Q^2\rangle_0$ for various $(\vec{E}_I,\, \vec{B})$ on a 16^4 lattice for $m_\pi \simeq 480$ MeV and $a \simeq 0.3$ fm, the dashed line is a fit according to an inverse tangent dependence. Figures taken from Ref. [47].

electromagnetic sector. χ_{CP} is related to the strength of the effective pseudoscalar QED-QCD interaction, $\mathcal{L}^{eff}_{QED-QCD} = \chi_{CP}\; q(x)\, e^2\, \vec{E}\cdot\vec{B}$ [102, 103, 104, 105, 106].

As we have emphasized earlier, the introduction of electric background fields leads to a sign problem. A possible way out is to take the electric background field purely imaginary, $\vec{E} = i\,\vec{E}_I$. As a consequence, when $\vec{E}_I\cdot\vec{B} \neq 0$, one expects to produce a purely imaginary effective parameter $\theta_{\rm eff} = i\,\theta_{I{\rm eff}}$. On the other hand, taking θ as purely imaginary is also a possible way to avoid a similar sign problem in the study of θ dependence [107, 108, 109, 110, 111]. An imaginary θ_I adds a factor $\exp(\theta_I Q)$ to the probability distribution of gauge configurations, where $Q = \int d^4x q(x)$ is the topological charge. That shifts the distribution of Q, leading, at the lowest order in θ_I, to

$$\langle Q\rangle_{\theta_I} \simeq V\,\chi\,\theta_I = \langle Q^2\rangle_{\theta=0}\,\theta_I \tag{5}$$

where χ is the topological susceptibility at $\theta_I = 0$ and V is the spacetime volume. Therefore one can determine $\theta_{I{\rm eff}}$ as follows

$$\theta_{I{\rm eff}} \simeq \frac{\langle Q\rangle(\vec{E}_I, \vec{B})}{\langle Q^2\rangle_0} + O((\vec{E}_I\cdot\vec{B})^3) \tag{6}$$

where $\langle\cdot\rangle_0$ is the average taken at zero background field.

In Fig. 3 we report some results taken from Ref. [47], where QCD with two standard rooted staggered flavors has been investigated: the distribution of the topological charge, determined by a standard gluonic operator after cooling, loses its symmetry around zero, thus signalling CP symmetry breaking, when $\vec{E}_I\cdot\vec{B} \neq 0$, as expected. Moreover, the average value $\langle Q\rangle(\vec{E}_I, \vec{B})$, from which $\theta_{I{\rm eff}}$ can be extracted according to Eq. (6), seems to be dependent on $\vec{E}_I\cdot\vec{B}$ alone, thus permitting to extract χ_{CP} unambiguously from the small field behaviour of $\theta_{I{\rm eff}}$. After checking for finite lattice spacing and finite volume effects, and after analytic continuation to real electric fields and real θ, one obtains [47] $\chi_{CP} = (7 \pm 1)$ GeV^{-4} for $m_\pi = 480$ MeV and, preliminarly, $\chi_{CP} \simeq 10(1)$ GeV^{-4} for $m_\pi \simeq 280$ MeV, suggesting that χ_{CP} tends to increase when approaching the chiral limit.

5. Conclusions

The last few years have been characterized by a considerably activity regarding lattice QCD simulations in presence of external background fields. We have reviewed part of this activity, with more emphasis on issues regarding modifications of the QCD vacuum and of the QCD phase diagram in presence of the external fields.

Together with the verification of well established facts, like the presence of magnetic catalysis for the QCD vacuum, new phenomena have been found, like the inversion of catalysis at high enough temperatures, which still await for a clear explanation. This and other issues may be strictly related to one of the aspects that emerges more clearly from present studies, i.e. the fact that the gluon field distribution is strongly affected by the electromagnetic background field, even though indirectly, by means of quark loops: such aspect should be better investigated by future studies.

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