Modularity/Automorphy of Calabi–Yau Varieties of CM Type

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Let X be a Calabi–Yau variety defined over $\mathbb{Q}$ of dimension $d \leq 3$. Our goal is to establish the modularity/automorphy of X.

We will discuss two situations where our goal may be achieved.

(1) X is a K3 surface with non-symplectic automorphisms:

(2) X is a Calabi–Yau threefolds of Borcea–Voisin type.

Modularity Results in the last two decades

• $d = 1$: Every elliptic curve E over $\mathbb{Q}$ is modular. There is a modular form f of weight 2 on some $\Gamma_0(N)$ such that $L(E, s) = L(f, s)$.

• $d = 2$: Every singular K3 surface S over $\mathbb{Q}$ is modular. There is a modular form f of weight 3 on some $\Gamma_0(N) + \chi$ or $\Gamma_1(N)$ such that $L(T(S) \otimes \mathbb{Q}_\ell, s) = L(f, s)$.

• $d = 3$: Every rigid Calabi–Yau threefold X over $\mathbb{Q}$ is modular. There is a modular form f of weight 4 on some $\Gamma_0(N)$ such that $L(X, s) = L(f, s)$.

Remarks:

(a) The modularity is established for the above varieties over $\mathbb{Q}$. However, we do not know conceptual reasons "why" they are modular. What would be physics implications of modularity?

(b) The above results are obtained by studying 2-dimensional Galois representations associated to Calabi–Yau varieties over $\mathbb{Q}$.

Here 2 coincides with the d-th Betti number of the Calabi–Yau variety of dimension d for $d = 1$ and 3; while 2 is the $\mathbb{Z}$-rank of the transcendental lattice $T(S)$ for $d = 2$.

Modularity/Automorphy of higher dimensional Galois representations

Higher dimensional Galois representations will occur in the following situations:

• $d = 2$: Let S be a K3 surface and let $T(S)$ be the transcendental lattice. When the $\mathbb{Z}$-rank of $T(S) \geq 3$.

• $d = 3$: Let X be a Calabi–Yau threefold. When $h^{2,1}(X) \geq 1$ (so that $B_3(X) = 2(1 + h^{2,1}(X)) \geq 4$).

The modularity/automorphy question in these cases is currently out of reach in the general setting.

For $d = 2$, we need for K3 surfaces to have more structures (e.g., lattice polarizations, automorphisms).

For $d = 3$, we require that X has nice geometric or algebraic structures, or that $H^3(X, \mathbb{Q}_\ell)$ decomposes into motives of small ranks.

The L-series

We will consider Calabi–Yau varieties defined over $\mathbb{Q}$, say, by vanishing of a finite number of polynomials with coefficients in $\mathbb{Q}$. We say that $X/\mathbb{Q}$ is a Calabi–Yau variety if $X \otimes_{\mathbb{Q}} \mathbb{C}$ is Calabi–Yau variety. Let $X/\mathbb{Q}$ be a Calabi–Yau variety with a defining equation with coefficients in $\mathbb{Z}[1/m]$ for some $m \in \mathbb{N}$. Let p be a prime $(p, m) = 1$, let $X_p := X \bmod p$ be the reduction of X modulo p. We say that p is *good* if X_p is smooth over $\overline{\mathbb{F}}_p$, otherwise *bad*.

Let $\#X(\mathbb{F}_{p^k})$ be the number of rational points on X_p over $\mathbb{F}_{p^k}$. The *local (congruent) zeta function* of X_p is defined by taking the formal sum

$$Z_p(X, T) := \exp\left(\sum_{k=1}^{\infty} \frac{\#X(\mathbb{F}_{p^k})}{k} T^k\right) \in \mathbb{Q}[[T]]$$

where T is an indeterminate.

Let ℓ be a prime $\neq p$. There is a Weil cohomology theory, the ℓ-adic étale chomology, that assigns to $\bar{X}_p = X_p \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ or to $\bar{X} = X \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$, $\mathbb{Q}_\ell$-vector spaces $H^i_{et}(\bar{X}, \mathbb{Q}_\ell)$, $0 \leq i \leq 2d$. The Frobenius morphism Fr_p $(x \mapsto x^p)$ on X_p induces an endomorphism Fr_p^* on the étale cohomology groups $H^i_{et}(\overline{X}_p, \mathbb{Q}_\ell)$ for each i, $0 \leq i \leq 2d$. Grothendieck specialization theorem gives an isomorphism $H^i_{et}(\overline{X}_p, \mathbb{Q}_\ell) \simeq H^i_{et}(\overline{X}, \mathbb{Q}_\ell)$, where $\overline{X} = X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. By the comparison theorem, $H^i_{et}(\overline{X}, \mathbb{Q}_\ell) \simeq H^i(X \otimes_{\mathbb{Q}} \mathbb{C}, \mathbb{C})$ so that $\dim_{\mathbb{Q}_\ell} H^i_{et}(\overline{X}, \mathbb{Q}_\ell) = B_i(X)$ (the i-th Betti number). There is the Poincaré duality: $H^i(\overline{X}, \mathbb{Q}_\ell) \times H^{2d-i}_{et}(\overline{X}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell$ is a perfect pairing for every i, $0 \leq i \leq 2d$.

Let

$$P^i_p(T) := \det(1 - \mathrm{Fr}_p^* \, T \mid H^i_{et}(\overline{X}, \mathbb{Q}_\ell))$$

be the characteristic polynomial of Fr_p^*.

Weil's Conjectures (Theorem)

- $P_p^i(T) \in 1 + T\mathbb{Z}[T]$.
- $P_p^i(T)$ does not depend on the choice of ℓ.
- $\deg P_p^i(T) = B_i(X)$ for every $i, 0 \leq i \leq 2d$.
- $P_p^{2d-i}(T) = \pm P_p^i(p^{d-i}T)$ for every i, $0 \leq i \leq d$.
- If we write $P_p^i(T) = \prod_{k=1}^{B_i}(1 - \alpha_k\, T) \in \overline{\mathbb{Q}}[T]$, then α_k is an algebraic integer with $|\alpha_k| = p^{i/2}$ (The Riemann Hypothesis).
- $Z_p(X, T)$ is a rational function:

$$Z_p(X,T) = \frac{\prod_{i=1}^{d} P_p^{2i-1}(X,T)}{\prod_{i=0}^{d} P_p^{2i}(X,T)}.$$

Let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group. There is a compatible system of ℓ-adic Galois representations

$$\rho^i_{X,\ell} : G_{\mathbb{Q}} \to GL(H^i_{et}(\overline{X}, \mathbb{Q}_\ell))$$

sending the (geometric) Frobenius $\mathrm{Fr}*_p^{-1}$ to $\rho^i(\mathrm{Fr}_p^{*-1})$ which has the same action as the Fr_p^* on $H^i_{et}(\overline{X}, \mathbb{Q}_\ell)$.

Definition: The i-th (cohomological) L-series (or L-function) of $X/\mathbb{Q}$ is defined by

$$L_i(X, s) := L(H^i_{et}(\overline{X}, \mathbb{Q}_\ell), s)$$

$$:= (*) \prod_{p \neq \ell : \text{good}} P_p^i(p^{-s})^{-1} \times (\text{factor corresponding to } \ell = p)$$

where the product is taken over all good primes different from ℓ and $(*)$ corresponds to factors of bad primes. For $\ell = p$ we use p-adic cohomology groups.

The most significant L-series of X is the d-th L- series $L_d(X,s) =: L(X,s)$.

Locally for each good prime, the characteristic polynomial $P_p^i(T)$ can be determined by geometric information and by counting the number of rational points on $\mathbb{F}_p$ by invoking the Lefschetz fixed point formula.

$$\#X(\mathbb{F}_p) = \sum_{k=0}^{2d} (-1)^k \mathrm{trace}(\mathrm{Fr}_p^* \mid H_{et}^k(\overline{X}, \mathbb{Q}_\ell))$$

Modularity/Automorphy Question

Are there global functions that determine the L-series $L(X, s)$?

More concretely, are there automorphic (modular) forms that determine $L(X, s)$?

Why should we expect modularity/automorphy?

K3 surfaces with non-symplectic automorphisms and their motivic modularity/automorphy

Joint work with Ron Livné and Matthias Schütt

Here we consider K3 surfaces with non-symplectic automorphisms. Let S be a K3 surface. Let ω_S be a holomorphic 2-form on S, fixed once and for all. Then $H^{2,0}(S) = \mathbb{C}\omega_S$. Let $g \in \mathrm{Aut}(S)$. Then g induces a map

$$g^* : H^{2,0}(S) \to H^{2,0}(S) : g^*\omega_S = \alpha(g)\omega_S$$

for some $\alpha(g) \in \mathbb{C}^*$. We say that g is *non-symplectic* if $\alpha(g) \neq 1$.

Suppose that $T(S)$ is unimodular, i.e, $\det T(S) = \pm 1$. Then the following assertions hold:

- $\alpha(\mathrm{Aut}(S))$ is a finite cyclic group of order k where $k \leq 66$ (Nikulin).
- k is a divisor of $66, 44, 42, 36, 28, 12$ (Kondo).
- If $\mathbb{Z}$-rank of $T(S) = \varphi(k)$ (where φ is the Euler function), then $k = 66, 44, 42, 36, 28, 12$. For a given k there is a unique K3 surface (Kondo).

k	$NS(S)$	$\mathrm{rank}(NS(S))$	$T(S)$	$\mathrm{rank}_{\mathbb{Z}}(T(S))$
66	U_2	2	$U_2^2 \oplus (-E_8)^2$	20
44	U_2	2	$U_2^2 \oplus (-E_8)^2$	20
42	$U_2 \oplus (-E_8)$	10	$U_2^2 \oplus (-E_8)$	12
36	$U_2 \oplus (-E_8)$	10	$U_2^2 \oplus (-E_8)$	12
28	$U_2 \oplus (-E_8)$	10	$U_2^2 \oplus (-E_8)$	12
12	$U_2 \oplus (-E_8)^2$	18	U_2^2	4

Here U is the rank 2 hyperbolic lattice, and $-E_8$ is the negative definite even unimodular lattice of rank 8.

The K3 surfaces discussed above can all be defined over $\mathbb{Q}$ and have defining equations in terms of Weierstrass models (Kondo).

k	S	g_k
66	$y^2 = x^3 + t(t^{11} - 1)$	$(x, y, t) \mapsto (\zeta_{66}^2 x, \zeta_{66}^3 y, \zeta_{66}^6 t)$
44	$y^2 = x^3 + x + t^{11}$	$(x, y, t) \mapsto (-x, \zeta_{44}^{11} y, \zeta_{44}^2 t)$
42	$y^2 = x^3 + t^5(t^7 - 1)$	$(x, y, t) \mapsto (\zeta_{42}^2 x, \zeta_{42}^3 y, \zeta_{42}^{18} t)$
36	$y^2 = x^3 - t^5(t^6 - 1)$	$(x, y, t) \mapsto (\zeta_{36}^2 x, \zeta_{36}^3 y, \zeta_{36}^{30} t)$
28	$y^2 = x^3 + x + t^7$	$(x, y, t) \mapsto (-x, \zeta_{28}^7 y, \zeta_{28}^2 t)$
12	$y^2 = x^3 + t^5(t^2 + 1)$	$(x, y, t) \mapsto (\zeta_{12}^2 x, \zeta_{12}^3 y, -t)$

If $T(S)$ is not uni-modular, there are exactly 10 values for k, namely, $k \in \{3, 9, 27, 5, 25, 7, 11, 13, 17, 19\}$. Conversely, for each k, there exists a unique K3 surface S, up to isomorphism with these properties. These K3 surfaces are defined by Weierstrass equations over $\mathbb{Q}$, with one exception of $k = 25$. For $k = 25$, the K3 surface is defined as a double sextic over $\mathbb{Q}$, i.e. a double cover of $\mathbb{P}^2$ branched along a sextic curve.

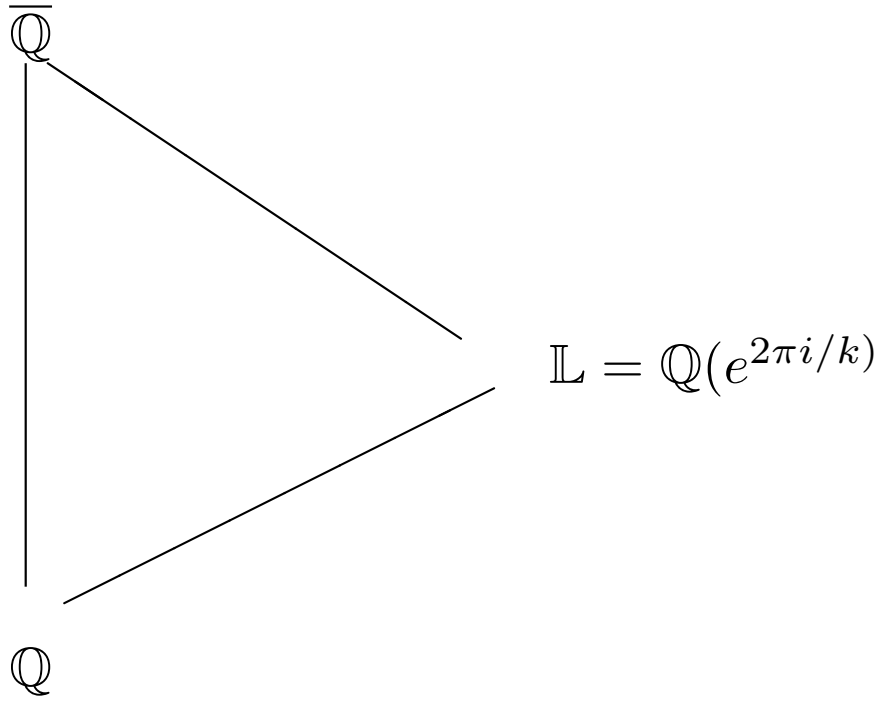
We have established the **motivic** modularity/automorphy of these K3 surfaces. Note that the $\mathbb{Z}$-rank of $T(S)$ takes even values from 2 to 20, excluding the values 8 and 14. Thus, the associated Galois representations have dimensions ranging from 2 to 20.

Theorem: *Let S be a K3 surface corresponding to one of the above values of k. Then S has a defining equation defined over $\mathbb{Q}$, and the following assertions hold:*

(1) The ℓ-adic Galois representation associated to $T(S)$ is irreducible over $\mathbb{Q}$ of dimension $\varphi(k) \in \{20, 12, 4\}$.

(2) Furthermore, this $G_{\mathbb{Q}}$-Galois representation is induced from a 1-dimensional Galois representation of the cyclotomic field $\mathbb{Q}(e^{2\pi i/k})$.

(3) The motivic L-series $L(T(S) \otimes \mathbb{Q}_\ell, s)$ is automorphic.



Idea of Proof: Base change and automorphic induction.

The $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{L})$ representation defined by $T(S)$ is a direct sum of 1-dimensional representations, which are simply permuted transitively by $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ to yield an irreducible $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ representation of dimension $\varphi(k)$. The 1-dimensional representations are determined by Jacobi sum Grossencharacters of $\mathbb{L}$.

Calabi–Yau varieties of CM type

Definition: A Calabi–Yau variety $X/\mathbb{Q}$ of dimension d is said to be of *CM type* if the Hodge group $\mathrm{Hdg}(X)$ associated to a rational Hodge structure on $H^d(X, \mathbb{Q})$ is commutative, so $\mathrm{Hdg}(X) \otimes \mathbb{C} \simeq$ copies of $\mathbb{G}_m$.

Remark: $\mathrm{Hdg}(X)$ is, in general, not computable.

Examples of CM type surfaces ($d = 2$)

- Every singular (extremal) K3 surfaces defined over $\mathbb{Q}$.
- The Fermat surface of degree $m(\geq 4)$: $x_0^m + x_1^m + x_2^m + x_3^m = 0 \subset \mathbb{P}^3$.
- Delsarte surfaces are defined by four-term equations over $\mathbb{Q}$ of the form: $\sum_{i=0}^{3} \prod_{j=0}^{3} x_j^{a_{ij}} = 0$.
- Invertible polynomials over $\mathbb{Q}$ in $\mathbb{P}^3$. (Here invertible polynomials means that # of monomials = # of variables (4).

Proposition: *All K3 surfaces in the theorem of Livné–Schütt-Yui are of CM type.*

Proof: All the above K3 surfaces have defining equations over $\mathbb{Q}$ with 4 monomials. They can be realized as Fermat quotients by finite groups. Since Fermat surfaces are of CM type, our K3 surfaces are also of CM type.

Here is a Hodge theoretic proof. Put $\mathcal{E} := \mathrm{End}_{\mathrm{Hdg}}(T(S))$. Then $[\mathcal{E} : \mathbb{Q}] = \varphi(k)$ and in fact, $\mathcal{E}$ is a cyclotomic field and hence a CM field over $\mathbb{Q}$ of degree $\varphi(k)$.

Calabi–Yau threefolds of Borcea–Voisin type and their motivic modularity/automorphy

Joint work with Yasuhiro Goto and Ron Livné

• Let (E, ι) is an elliptic curve with a non-symplectic involution ι such that the induced map

$$\iota^* : H^{1,0}(E) \to H^{1,0}(E), \quad \iota^*(\omega_E) = -\omega_E$$

and that $E/ < \iota > \simeq \mathbb{P}^1$. Here ω_E is a unique holomorphic 1-form on E.

• Let (S, σ) is a K3 surface with a non-symplectic involution such that the induced map

$$\sigma^* : H^{2,0}(S) \to H^{2,0}(S), \quad \sigma^*(\omega_S) = -\omega_S.$$

Decompose $H^2(S, \mathbb{C})$ into the $(+)$- and $(-)$-eigenspaces under the action of $\sigma^* : H^2(S, \mathbb{C}) \to H^2(S, \mathbb{C})$:

$$H^2(S, \mathbb{C}) = H^2(S, \mathbb{C})^+ \oplus H^2(S, \mathbb{C})^-.$$

Set

$$H^2(S, \mathbb{Z})^+ := H^2(S, \mathbb{C})^+ \cap H^2(S, \mathbb{Z})$$

and

$$H^2(S,\mathbb{Z})^- := H^2(S,\mathbb{C})^- \cap H\ 2(S,\mathbb{Z}).$$

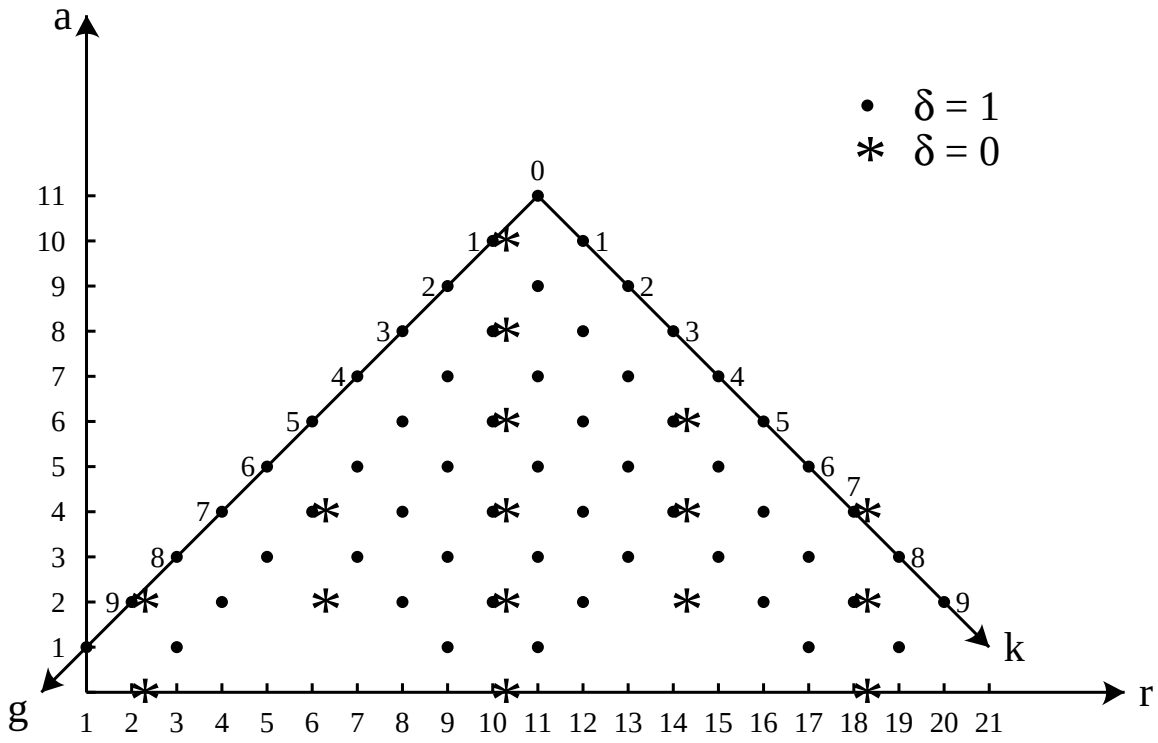
Let

$$r := \mathrm{rank}_{\mathbb{Z}} H^2(S,\mathbb{Z})^+.$$

Then $H^2(S,\mathbb{Z})^+$ and $H^2(S,\mathbb{Z})^-$ have signatures $(1, r-1)$ and $(2, 20-r)$ respectively.

Nikulin has classified such pairs (S,σ) of K3 surfaces S with non-symplectic involutions σ, up to deformation.

Nikulin's Pyramid



Theorem (Nikulin, 1979) *There are* 75 *deformation classes of pairs* (S, σ) *of K3 surfaces* S *with non-symplectic involutions* σ*, and they are completely determined by the triple integers*

$$(r, a, \delta)$$

where r *is as above,* a *is the integer determined by*

$$(H^2(S, \mathbb{Z})^+)^\vee / H^2(S, \mathbb{Z})^+ \simeq (\mathbb{Z}/2\mathbb{Z})^a.$$

The intersection pairing on $H^2(S, \mathbb{Z})^+$ *gives rise to a quadratic form* q *with values in* $\mathbb{Q}$*. We define* $\delta = 0$ *if* q *has integer values, and* 1 *otherwise.*

Theorem (Nikulin, 1979) : *Let (S, σ) be a pair of K3 surface with non-symplectic involution σ. Let S^σ be the fixed locus of S under σ. Then*

(1) If $(r, a, \delta) \neq (10, 10, 0), (10, 8, 0)$, then

$$S^\sigma = C_g \cup L_1 \cup \ldots \cup L_k \quad \textit{(disjoint union)}$$

where C_g is a genus $g(\geq 0)$ curve, and $L_i (i = 1, \cdots, k)$ are rational curves.

(2) If $(r, a, \delta) = (10, 10, 0)$, then $S^\sigma = \emptyset$.

(3) If $(r, a, \delta) = (10, 8, 0)$, then $S^\sigma = C_1 \cup \bar{C}_1$ (disjoint union) where C_1 and $\bar{C}_1$ are elliptic curves.

Put

$$N := \text{the number of components of } S^\sigma = 1 + k$$

and

$$N' := \text{the sum of genera of components of } S^\sigma = g.$$

Note that

$$g = 11 - \frac{1}{2}(r + a),\ N = 1 + k = 1 + \frac{1}{2}(r - a).$$

Now we will construct Calabi–Yau threefolds of Borcea–Voisin (BV) type. Let (E, ι) and (S, σ) be as above. Take the product $E \times S$. Then the product $\iota \times \sigma$ is an involution on $E \times S$ such that the induced map

$$(\iota \times \sigma)^* : H^{3,0}(E \times S) \to H^{3,0}(E \times S)$$

is the identity map. Write

$$E^\iota = \{P_1, P_2, P_3, P_4\}$$

and

$$S^\sigma = \{C_1, C_2, \cdots, C_N\} \quad \text{with } N = 1 + k.$$

Then the fixed point of $\iota \times \sigma$ consists of

$$P_i \times C_j \quad (i = 1, \cdots, 4; j = 1, \cdots, N).$$

The involution $\iota \times \sigma$ lifts naturally to an involution on the blow-up of $E \times S$ along $4N$ curves. The quotient $E \times S/\iota \times \sigma$ and its

crepant resolution $\widetilde{E \times S/\iota} \times \sigma$ is our Calabi–Yau threefold of Borcea–Voisin (BV) type, and will be denoted by

$$X = X(r, a, \delta).$$

Note that the exceptional divisors on X are 4 copies of ruled surfaces

$$S^\sigma \times \mathbb{P}^1 := (C_g \times \mathbb{P}^1) \cup (L_1 \times \mathbb{P}^1) \cup \cdots \cup (L_k \times \mathbb{P}^1).$$

Theorem (Borcea, Voisin, 1993, '94): *The Hodge numbers of X are given by*

$$h^{1,1}(X) = 5 + 3r - 2a = 11 + 5N - N'$$

$$h^{2,1}(X) = 65 - 3r - 2a = 11 + 5N' - N$$

and

$$E(X) = 12(r - 10) = 12(N - N').$$

($B_3(X) = 2(1 + (20 - r) + 4g) = 2(12 + 5N' - N) \geq 4$.)

Mirror symmetry conjecture holds for Calabi–Yau threefolds of Borcea–Voisin type. Mirror symmetry interchanges N and N', and is inherited from mirror symmetry of K3 surface components.

Since any elliptic curve E defined over $\mathbb{Q}$ is modular, the automorphy/modularity of our Calabi–Yau threefolds $X = X(r, a, \delta)$ depends on the automorphy/modularity of K3 surface component S. We ought to choose appropriate K3 surfaces for S.

Theorem (Reid 1979, Yonemura 1990): *There are* 95 *admissible weights* (w_0, w_1, w_2, w_3) *of hypersurface simple K3 singularities defined by non-degenerate polynomials* $F(x_0, x_1, x_2, x_3)$ *in weighted projective* 3*-spaces* $\mathbb{P}^3(w_0, w_1, w_2.w_3)$ *over* $\mathbb{Q}$*.*

Examples: #1: weight $(1, 1, 1, 1)$

$$F = x_0^4 + x_1^4 + x_2^4 + x_3^4.$$

#15: weight $(5, 4, 3, 3)$

$$F = x_0^3 + x_1^3 x_2 + x_1^3 x_3 + x_2^5 + x_3^5.$$

#53: weight $(6, 5, 4, 3)$

$$F = x_0^3 + x_1^3 x_3 + x_1^2 x_2^2 + x_0 x_2^3 + x_2^3 x_3^2 + x_3^6.$$

#62: weight $(8, 5, 4, 3)$

$$F = x_0^2 x_3 + x_1^4 + x_1 x_3^5 + x_2^5 + x_1 x_3^4 + x_2^2 x_3^4.$$

#84: weight $(9, 7, 6, 5)$

$$F = x_0^3 + x_0 x_2^3 + x_1^3 x_2 + x_1 x_3^4 + x_2^2 x_3^3.$$

#95: weight $(7, 5, 3, 2)$

$$F = x_0^2 x_2 + x_1^3 x_3 + x_1 x_2^4 + x_1 x_3^6 + x_2^5 x_3 + x_2 x_3^7.$$

Among these 95 K3 surfaces, we need to find those with the required involutions.

Theorem: *Among the* 95 *K3 surfaces,* 92 *have the required non-symplectic involution* σ*. These* 92 *pairs* (S, σ) *realize at least* 40 *triplets* (r, a, δ) *of Nikulin.*

Furthermore, the 86 *out of* 92 *of them are realized as Delsarte surfaces. Consequently these* 86 *pairs* (S, σ) *are of CM type (that is, they are realized as finite Fermat quotients).*

Remarks: (a) For instance, we were not able to find an involution for #53. There are 3 such K3 surfaces.

(b) For #95 it has an involution $\sigma(x_0) = -x_0$, but cannot be realized as quasi-smooth hypersurface in four monomials. Altogether there are 6 such K3 surfaces.

Idea for Proof: Yonemura obtained his defining equations using toric constructions. For each weight, Yonemura wrote down a defining equation using all extremal points of the convex hull determined by the weight. We need to find the required involutions in the 95 families. We were able to find them for the 92 of them, but not yet for the remaining three.

For the second assertion,

• We may remove some monomials keeping in mind Yonemura's condition

(*) *for each* i, $0 \leq i \leq 3$, *the defining equation must contain a monomial of the form* x_i^n *or* $x_i^n x_j$ $(i \neq j)$.

So if there is a monomial of the form $x_i^n x_j^m$ with n, $m > 1$, we can remove it from the defining equation. We must check that even after removing monomials, a defining equation must remain quasi-smooth. As long as quasi-smoothness is satisfied, we will have $K3$ surfaces.

• After removing monomials, a holomorphic 2-form ω_S should be sent to $-\omega_S$ under the involution.

• Also the resolution picture should be invariant under deformation, that is, remain the same before and after removing certain monomials.

We will illustrate by examples that Calabi–Yau threefolds of Borcea–Voisin type do have birational models defined over $\mathbb{Q}$.

Example 1: Suppose that S is defined by a hypersurface

$$x_0^2 = f(x_1, x_2, x_3) \subset \mathbb{P}^3(w_0, w_1, w_2, w_3).$$

If w_0 is odd, then with $E_2 \in \mathbb{P}^2(2,1,1)$, $X = E_2 \times \widetilde{S/\iota} \times \sigma$ is birational to a hypersurface defined over $\mathbb{Q}$:

$$z_0^4 + z_1^4 - f(z_2, z_3, z_4) = 0 \subset \mathbb{P}^4(w_0, w_0, 2w_1, 2w_2, 2w_3).$$

Example 2: Suppose that S is defined by a hypersurface

$$x_0^2 + f(x_1, x_2, x_3) = 0 \subset \mathbb{P}^3(w_0, w_1, w_2, w_3).$$

If w_0 is even but not divisible by 3, then with $E_3 \in \mathbb{P}^2(3,2,1)$, $X = E_3 \times \widetilde{S/\iota} \times \sigma$ is birational to a hypersurface defined over $\mathbb{Q}$:

$$z_0^3 + z_1^6 + f(z_2, z_3, z_4) = 0 \subset \mathbb{P}^4(2w_0, w_0, 3w_1, 3w_2, 3w_3).$$

Definition:

(1) For a pair (S, σ) of $K3$ surface with a non-symplectic involution σ, we call $T(S)^{\sigma=-1} \otimes \mathbb{Q}_\ell \subset H^2(S, \mathbb{Q}_\ell)$ the $K3$-*motive*, and denoted by $\mathcal{M}_S$. This is the unique motive with $h^{0,2}(\mathcal{M}_S) = 1$.

(2) We will call the submotive $H^1(E, \mathbb{Q}_\ell)^{\iota=-1} \otimes (T(S)^{\sigma=-1} \otimes \mathbb{Q}_\ell)$ of $H^3(X, \mathbb{Q}_\ell)$ the *Calabi–Yau motive* of X, and denoted by $\mathcal{M}_X$.

Theorem: *Let (S, σ) be one of the 86 surfaces represented by a Delsarte surface. Let (E, ι) be an elliptic curve over $\mathbb{Q}$. Let X be a Calabi–Yau threefold of Borcea–Voisin (BV) type. Then X has a model defined over $\mathbb{Q}$, and X is motivically automorphic.*

More precisely,

(a) (S, σ) is motivically automorphic, that is,

$$L(S, s) = L(\rho_S, s-1)L(\chi_S, s)$$

where ρ_S and χ_S are Galois representations corresponding to $NS(S)$ and $T(S)$, respectively.

Here $L(\chi_S, s) = L(\mathcal{M}_S, s)$ is automorphic.

(b) X is motivically automoroic, that is,

$$L(X,s) = L(\rho_E \otimes \rho_S, s)L(\rho_E \otimes \chi_S, s)L(J(C_g), s-1)^4,$$

where ρ_E is the Galois representation corresponding to E, $J(C_g)$ is the Jacobian variety of C_g (here C_g is also of CM type).

Here $L(\rho_E \otimes \chi_S, s) = L(\mathcal{M}_X, s)$ and $L(J(C_g), s)$ are automorphic.

Remarks: (a) The automorphy of $L(\chi_S, s)$ is proved by using the argument of automorphic induction. This is justified as S is of CM type.

We cannot establish the automorphy of $L(\rho_S, s)$ as this may correspond to higher dimensional Artin representations, and the automorphy of Artin representations are still open even for CM fields.

(b) We get only the motivic automorphy for $L(X, s)$. A reason for this is that the $L(\rho_E \otimes \rho_S, s)$ may be associated to the tensor product of ρ_E and higher dimensional Artin representations.

Example 1: Let $E = E_2 : y_0^2 = y_1^4 + y_2^4 \subset \mathbb{P}^2(2,1,1)$. Let S_0 be a (quasi-smooth) K3 surface given by

$$S_0 : x_0^2 = x_1^3 + x_2^7 + x_3^{42} \subset \mathbb{P}^3(21,14,6,1)$$

of degree 42. S_0 has a non-symplectic involution$\sigma(x_0) = -x_0$. Let S be the minimal resolution of S_0. Then S corresponds to the triplet $(10,0,0)$ of Nikulin. So its mirror $S^\vee$ also corresponds to the triplet $(10,0,0)$. The fixed locus $S^\sigma = C_6 \cup L_1 \cup \cdots \cup L_5$. Also S is of CM type as it is dominated by the Fermat surface of degree 42. The K3 motive $\mathcal{M}_S$ corresponds to the cyclotomic field $\mathbb{K} = \mathbb{Q}(\zeta_{42})$ with $[\mathbb{K} : \mathbb{Q}] = \varphi(42) = 12 = 22 - 10$. Then $\mathcal{M}_S$ is automorphic.

Now the Calabi–Yau threefold $X = E_2 \times \widetilde{S/\iota} \times \sigma$ has a birational model defined over $\mathbb{Q}$:

$$X : z_0^4 + z_1^4 = z_2^3 + z_3^7 + z_4^{42} \subset \mathbb{P}^4(21, 21, 28, 12, 2)$$

of degree 84. Since both E_2 and S are of CM type, X is also of CM type. The Hodge numbers are given by

$$h^{1,1}(X) = 35,\ h^{21}(X) = 35.$$

So X is own mirror.

Example 2: Let $E = E_2$ and S_0 be a (quasi-smooth) K3 surface given by

$$S_0 : x_0^2 = x_1^3 x_2 + x_1^3 x_2^2 + x_2^7 - x_3^{14} \subset \mathbb{P}^3(7, 4, 2, 1)$$

of degree 14. S_0 has a non-symplectic involution $\sigma(x_0) = -x_0$. Its minimal resolution S corresponds to the triplet$(7, 3, 0)$ of Nikulin. Remove the monomial $x_1^3 x_2^2$ from the defining equation for S_0, we get

$$S_0 : x_0^2 = x_1^3 x_2 + x_2^7 - x_3^{14}$$

which makes S_0 of CM type. It is dominated by the Fermat surface of degree $42 = lcm(3, 2, 14)$. Then $\mathcal{M}_S$ corresponds to the cyclotomic field $\mathbb{K} = \mathbb{Q}(\zeta_{42})$ of degree $\varphi(42) = 12$, and it is automorphic.

Now the Calabi–Yau threefold $X = E_2 \times \widetilde{S/\iota} \times \sigma$ has a birational model defined over $\mathbb{Q}$:

$$X : z_0^4 + z_1^4 = z_2^3 z_3 + z_3^7 - z_4^{14} \subset \mathbb{P}^4(7, 7, 8, 4, 2)$$

of degree 28 and $lcm(4, 3, 14) = 84$. Since E_2 and S are of CM type, so is X. The Hodge numbers are given by

$$h^{1,1}(X) = 20,\ h^{2,1}(X) = 38, \text{ and } e(X) = -36.$$

We now pass from $\mathbb{Q}(\zeta_{42})$ to $\mathbb{Q}(\zeta_{84})$ to take $H^1(E_2)$ into account. The Calabi–Yau motive $\mathcal{M}_X$ has dimension $24 = \varphi(84)$. The Jacobi sum Grossencharacter of $\mathbb{Q}(\zeta_{84})$ gives rise to the GL_{24} irreducible automorphic representation over $\mathbb{Q}$ for $\mathcal{M}_X$. Hence $\mathcal{M}_X$ is automorphic.

Mirrors of Calabi–Yau threefolds of Borcea–Voisin type and arithmetic mirror symmetry

We know that mirror symmetry conjecture holds for Calabi–Yau threefolds of Borcea–Voisin type, and it is inherited from mirror symmetry for K3 surface components. However, the 95 K3 surfaces of Reid and Yonemura are not closed under mirror symmetry of K3 surfaces.

Lemma (Belcastro):*Among the* 95 *K3 surfaces, the* 57 *K3surfaces* S *have non-symplectic involution* σ *acting as*-1 *on* $H^{2,0}(S)$*, and all 57 have mirror partners* $S^\vee$ *equipped with non-symplectic involution* $\sigma^\vee$ *acting as* -1 *on* $H^{2,0}(S^\vee)$*.*

Here if pair (S, σ) *of a K3 surface with non-symplectic involution* σ *corresponds to a Nikulin's triple* (r, a, δ)*, then a mirror pair* $(S^\vee, \sigma^\vee)$ *corresponds to the triple* $(20 - r, a, \delta)$*.*

Examples of mirror pairs of Calabi–Yau threefolds of Borcea–Voisin type

We consider the Calabi–Yau threefold in Example 2 above. To find a mirror family, we look for a mirror $S^\vee$ of S. We may take for $S^\vee$ the K3 surface defined by

$$S^\vee : x_0^2 = x_1^3 + x_1x_2^7 + x_2^9x_3^2 + x_3^{13} \subset \mathbb{P}^3(21, 14, 4, 3)$$

of degree 42. It has a non-symplectic involution $x_0 \mapsto -x_0$. The pair $(S^\vee, \sigma^\vee)$ corresponds to the triplet $(13, 3, 0)$. Removing the monomial $x_2^9x_3^2$ we can make S_0 to be of CM type. Since $lcm(2, 7, 4) = 28$, $\mathcal{M}_{S^\vee}$ corresponds to the cyclotomic field $\mathbb{Q}(\zeta_{28})$ of degree $\varphi(28) = 12$. Then $\mathcal{M}_{S^\vee}$ is automorphic.

A candidate for a mirror family $X^\vee$ has a birational model over $\mathbb{Q}$

$$X^\vee : z_0^4 + z_1^4 = z_2^3 + z_2 z_3^7 + z_4^{14} \subset \mathbb{P}^4(21, 21, 28, 8, 6)$$

of degree 84. The Hodge numbers and the Euler characteristic are given by

$$h^{1,1}(X^\vee) = 38,\ h^{2,1}(X^\vee) = 20, \text{ and } e(X^\vee) = 36.$$

We pass from $\mathbb{Q}(\zeta_{28})$ to $\mathbb{Q}(\zeta_{56})$ to take $H^1(E_2)$ into account. Then the Calabi–Yau motive $\mathcal{M}_{X^\vee}$ has dimension $24 = \varphi(56)$. By the similar argument as for $\mathcal{M}_X$, it is automorphic.

Observation:

$$L(\mathcal{M}_X, s) = L(\mathcal{M}_{X^\vee}, s)$$

that is, the L-series of the Calabi–Yau motives of a mirror pair remain invariant under mirror symmetry. They are automorphic.

A mirror Calabi–Yau threefold $X^\vee$ appears in a family. Since we do not know how to do point counting on a family, we consider $X^\vee$ only at a CM point and compute the L-series $L(\mathcal{M}_{X^\vee}, s)$ at this isolated point. Then we are comparing the two L-series $L(\mathcal{M}_X, s)$ and $L(\mathcal{M}_{X^\vee}, s)$.

This phenomenon is valid for many examples of mirror pairs of Calabi–Yau threefolds of Borcea–Voisin type.