INVARIANT QUANTUM MECHANICAL EQUATIONS OF MOTION

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I. INTRODUCTORY REMARKS

One of the last few years' most important developments in theoretical physics is the recognition that it is useful to extend to complex numbers the definition domain of intrinsically real variables, such as energy or angular momentum. This leads one to review many subjects which were considered to be closed. It should not have surprised me, therefore, when Dr. Salam asked me to report, at this seminar, on equations for elementary particles which are not believed to exist in nature, such as particles with imaginary mass. Even though the equations which describe such particles will play no role in the theory as long as the variables such as energy or angular momentum have physically meaningful values, that is, as long as they are real, they may play a significant role when the definition domain of these variables is extended.

I was, at one time, greatly interested in establishing all linear equations which are invariant under the inhomogeneous Lorentz group and much of what I will talk about originates from this interest. The inhomogeneous Lorentz group contains displacements in space and time in addition to Lorentz transformations; it will be called Poincaré group after the mathematician who first became convinced of the basic significance of this group for physics. It turns out that the representations of a group essentially determine all linear equations which are invariant under the group in question and one is thus led naturally to the theory of the representations of the Poincaré group. The term "representation" will mean, throughout this article, a group of linear operators which is homomorphic to the group to be represented; the space of the vectors on which these operators act is a complex Hilbert space, usually infinite dimensional, which will be called representation space.

Only some of the representations of the Poincaré group will be discussed: those which are irreducible and unitary. The first restriction means, in the domain of real masses and spins, that only equations for elementary particles will be considered, and these only on the Schrödinger, that is not second quantized, level. In the extended domain of the variables it should mean that the Regge poles to be considered are primitive but at the present time this point has not been fully elucidated. Naturally, it would be desirable to consider also the second quantized form of the equations, but I am not able to do this. My excuse for considering only unitary representations is similar: the non-unitary ones present complications which have not yet been surmounted, even though Dr. Froissart has made significant progress in their investigation.

There is one other respect in which my discussion will be limited: by the very fact that the Poincaré group will be the basic group throughout. It would be desirable to consider equations which are invariant under one of the generalizations of the Poincaré group, in particular equations invariant under the usual de Sitter group. However, the doctoral thesis of T. Philips shows that even the interpretation of the real mass-real spin representations of the usual de Sitter group encounters serious difficulties and I want to avoid these. Hence, the discussion will be concerned solely with the Poincaré group and almost solely with the unitary irreducible representations of this, or the linear equations which correspond to these.

The relation between representations and equations of motion justifies a few remarks. In one sense, the representation gives much more information than the equations of motion: whereas the equations of motion, as ordinarily conceived, give only the change of the state vector (or whatever characterizes the instantaneous state of the system) with the passage of time, and this directly only for an infinitesimal increment of time, the representation gives the change of the state vector for arbitrary Poincaré transformations, and for finite ones as well as for infinitesimal ones. The time displacement, the effect of which is given by the equations of motion, is only one special type of Poincaré transformations. Hence, the representation is more informative than the equation of motion in two regards: because it gives the effect of finite, rather than only of infinitesimal, transformations, and because it gives the effect of all Poincaré transformations, not only of time-displacements. It may even happen that it is, on the basis of the equation of motion alone, not possible to determine without further assumptions how the state vector changes under a proper Lorentz transformation. Thus, to mention a rather trivial example, the Dirac equation in empty space is invariant under Lorentz transformations not only if the four components are considered to be spinors, but also if they are considered to be scalars.

In another less mathematical but much more suggestive sense, the equation of motion is much more informative than the representation from which it arises. The reason is that it invites the application of the methods of second quantization and hence the replacement of the particle by a quantum field. Once this is accomplished, one may be led by analogies to assumptions concerning interactions. Without any knowledge of its interactions, the picture of a particle is rather empty. All these remarks apply, for the present, only to representations or equations which describe particles which exist in some sense in nature. It does not apply to characteristics of Regge poles or anything similar; for these the relation between representations and equations of motion (if such exist) is much less clear.

It should be mentioned, finally, that the relation of representations to equations of motion is not one-to-one. We shall see several examples for this; one of the principal objectives of these lectures being the establishment of a general method to obtain one equation of motion for every representation. This equation of motion will, in some cases, not be the common and well-known one. However, one example for the lack of uniqueness of the correspondence between representation and equation of motion is already known to all of us: the electromagnetic field can be described either by the scalar and vector potentials, or by the electric and magnetic fields. The representation is, however, the same for both: 0_1 in the usual notation. In fact, the representation is always uniquely determined by the properties of an elementary particle because it merely expresses the relation between the descriptions of the particle by observers using different but equivalent frames of reference. In particular, the two frames of reference whose relation gives the equations of motion are at rest with respect to each other, but their time scales have different starting points.

II. THE UNITARY REPRESENTATIONS OF THE POINCARÉ GROUP

One further general observation will be useful for the understanding of the connections which will form the subject of these lectures. This observation relates to the greater effectiveness of invariance considerations in quantum than in classical theory. The reason for this greater effectiveness was spelled out already by C. N. Yang: the states in quantum theory constitute a linear manifold whereas there is no similar structure of the states in classical mechanics. However, it will be useful to pursue somewhat more in detail the way this difference manifests itself. We shall choose for this a very simple and elementary example in which only rotational symmetry is present.

Hamel, Klein and Noether have shown how the conservation laws for angular momentum, for instance, can be derived in classical mechanics directly from the invariance of the equations with respect to rotations. However, the considerations leading from the invariance to the conservation laws are rather subtle, being based on the principle of least action. If one just considers a possible classical trajectory, such as a planetary orbit, an unsophisticated application of the invariance principle only leads to the conclusion that there are other similar orbits, obtained from the given orbit by a rotation. This is not a very fruitful conclusion. In quantum theory, given one orbit, one can also obtain other orbits by rotation. However, all the orbits obtained in this way form a linear manifold and one can select from this manifold a linearly independent set in terms of which all the "orbits" can be expressed linearly. If one then subjects the members of the selected set to a rotation, and expresses these rotated orbits linearly in terms of the originally selected set, one obtains at once a representation of the rotation group.

When carrying out the procedure just outlined, one of two situations may be encountered. If starting with one orbit, the orbits obtained by different rotations are all linearly independent, no significant conclusion results. The representation obtained in this case is the infinite dimensional so-called regular representation of the rotation group, but even with a detailed analysis it is clear that, in this case, no significant conclusion concerning the properties of the orbits can be arrived at. In fact, the situation is very much the same as in classical theory. The most interesting and significant conclusions concerning the properties of the "orbits" will result if there is only a finite number of linearly independent states in terms of which all states obtained by rotation can be expressed. In the well-known case when this number is 1, all states are spherically symmetric and the conclusions are usually only little less striking.

Let us now review briefly a way in which the representations of the Poincaré group can be determined. The procedure has actually been given by Frobenius long before the Poincaré group was known. Its application is based on the fact that the Poincaré group has an invariant subgroup, consisting of all displacements. Matters become particularly simple because this invariant subgroup is abelian (commutative). The mathematics which will be used is not rigorous because members of the continuous spectrum will be treated as if they were bona fide vectors in Hilbert space. However, the procedure can be justified rigorously, principally on the basis of the investigations of Mautner and von Neumann.

Let us consider states which belong to irreducible representations of the group of displacements. Since this group is abelian, the unitary irreducible representations are one-dimensional. Denoting the displacement vector by a, its operator by T_a, there will be "states" $|p, \zeta>$ for which

$$T_a |p, \zeta> = e^{-ip.a} |p, \zeta> \tag{2.1}$$

where $p.a$ is the Lorentz scalar product of the two vectors p and a:

$$p.a = p_t a_t - p_x a_x - p_y a_y - p_z a_z. \tag{2.1a}$$

The reason for the apparently arbitrary sign convention adopted in (2.1) will become evident soon. It also follows from the unitary nature of the representation that the components of p must be real. Otherwise, T_a would not be unitary. However, the existence of vectors for which (2.1) holds would be rigorously assured only if the components of p were discrete variables. As we shall see at once, this is not the case and it follows that the "vectors" $|p, \zeta>$ are not normalizable. This is the point where the derivation is not rigorous. The variable ζ was introduced because it is possible that there are several vectors which transform, under the operations of the displacement group, according to the representation $(e^{-ip.a})$; the index ζ distinguishes these vectors. It can be assumed to be a discrete variable but if there are infinitely many vectors which belong to the $(e^{-ip.a})$ representation, it will assume infinitely many values. Naturally, we do not yet know for which four-vectors p there are Hilbert vectors $|p, \zeta>$, i.e., which representations $(e^{-ip.a})$ of the displacement subgroup occur in the Poincaré group's representation which is being analyzed. As a matter of fact, this representation is not yet specified.

It will be shown now that if a representation of the Poincaré group contains the representation $(e^{-ip.a})$ of the displacement subgroup, it also contains all representations $(e^{-ip'.a})$ of this subgroup if $p' = Lp$ can be obtained from p by a proper Lorentz transformation L. The representation $(e^{-ip.a})$ is contained in a representation of the Poincaré group if there is a vector $|p, \zeta>$ in the Hilbert space of the latter for which (2.1) is valid. Similarly, $(e^{-iLp.a})$ is contained in the same representation if there is a vector for which (2.1) with p replaced by $p' = Lp$ is valid. Since the vector $|p, \zeta>$ is expected to describe a state with four-momentum p, one will expect that the operation O_L, which corresponds to the Lorentz transformation L, will transform this state into one with momentum $p' = Lp$. Hence, one will expect that

$$T_a(O_L\,|\,p,\zeta>) = e^{-iLp.a}\,O_L\,|\,p,\zeta>. \tag{2.2}$$

This is indeed the consequence of the equation

$$T_a O_L = O_L T_{L^{-1}a}. \tag{2.3}$$

This equation expresses the fact that a Lorentz transformation L followed by the displacement a is identical to a displacement by $L^{-1}a$, followed by the Lorentz transformation L. If (2.3) is applied to the vector $|p,\zeta\rangle$, the left side will be identical with the left side of (2.2). The right side becomes

$$O_L T_{L^{-1}a}\,|\,p,\zeta\rangle = O_L e^{-ip.L^{-1}a}\,|\,p,\zeta\rangle = e^{-ip.L^{-1}a}\,O_L\,|\,p,\zeta\rangle.$$

The second member is a consequence of (2.1) as applied to the displacement $L^{-1}a$, the last member follows because the exponential is a numerical factor and O_L is linear. Furthermore, it follows from the properties of the Lorentz scalar product that

$$p.L^{-1}a = Lp.a \tag{2.4}$$

so that indeed, (2.2) is established. This then proves that if the Hilbert space of a representation of the Poincaré group contains vectors $|p,\zeta\rangle$ with four-momentum p, it also contains vectors with all the momentum Lp, where L is any Lorentz transformation. According to (2.2), $O_L|p,\zeta\rangle$ is such a vector.

Since the $|Lp,\eta\rangle$, for all possible values of η, form a complete set of vectors which transform under the displacement group according to the representation $(e^{iLp.a})$, one can conclude that

$$O_L\,|\,p,\zeta\rangle = \sum c_\eta\,|\,Lp,\eta\rangle. \tag{2.5}$$

The coefficients c_η can depend on p, ζ, and L. We shall use only a special case of (2.5) to define what has come to be called the "little group".

III. THE LITTLE GROUP

We have seen that the four-vectors p for which there are Hilbert vectors satisfying (2.1) form a set which is invariant under all proper Lorentz transformations. In an irreducible representation, all such vectors can be obtained from a single one by applying all possible Lorentz transformations to it. Hence, the Lorentz length p.p of the momenta is the same for all state vectors which are present in the representation space of an irreducible representation. Altogether, one has to distinguish six qualitatively different cases.

1. $p.p = m^2 > 0$, $p_t > 0$. The corresponding representations describe the transformation properties of real particles with finite rest mass.
2. $p.p = 0$, $p_t > 0$. These representations refer to particles with zero rest mass. The equations which correspond to some of these representations are well-known, but we shall discuss all of them.
3. $p.p = m^2 < 0$, i.e., p is space-like, m imaginary. In this case p_t can

assume arbitrarily large negative (as well as positive) values. It is axiomatic that no particle can exist which corresponds to such a representatio because, if it existed, it could transfer any amount of energy to a particle of class 1 by going over into a state with sufficiently large negative p_t. Neve theless, the representations of this class will be described and equations of motion given which correspond to these representations. Also, some of the properties will be given which particles corresponding to these representations would have, if they existed. This conforms to the program give in the first section.

4. $p.p = 0$, $p_t < 0$. Again, p_t can assume arbitrarily large negative values. However, the representations of this class are simply conjugate complex to the representations of class 2 and will not be discussed further.

5. $p.p = m^2 > 0$, $p_t < 0$. These representations are conjugate complex to the representations of class 1 and will not be discussed further either. Agai p_t can assume arbitrarily large negative values.

6. $p_t = p_x = p_y = p_t = 0$. All states would be displacement invariant. Again, i is axiomatic that no particles with these transformation properties can exist.

The preceding enumeration gives the possible momentum vectors p for which states $|p, \zeta\rangle$ exist in the irreducible representation in question. Th transformation properties of these states with respect to translations are given by (2.1); we shall now discuss their transformation properties with respect to (homogeneous) Lorentz transformations L. This discussion will be based on (2.5).

Let us select in every case, except the last one which will be disregarded, from all possible momentum vectors a definite one which will be called p^0. In the case of class 1, p^0 is best chosen to be parallel to the time axis, in case 3, parallel to the z axis. In case 2, it will be the vector with components 1, 0, 0, 1. The choice of p^0 is arbitrary, but it is useful to ma it in order to fix the ideas.

We next define the "little group" as the group of all Lorentz transform tions which leave p^0 invariant.

$$Lp^0 = p^0. \tag{3.1}$$

The L which satisfy (3.1) evidently form a group and this group does not depend essentially on the arbitrary choice of p^0. If another momentum $p^1 = L_1 p^0$ had been chosen, the transformations $L_1 L L_1^{-1}$ which leave it invariant would have formed a group which is isomorphic to the group of L which leave p^0 invariant. However, with the preceding choice of p^0, it is clear that in case 1 the little group is the three-dimensional rotation group in case 3 the 2+1 dimensional Lorentz group, i.e. the group which leaves the form $t^2 - x^2 - y^2$ invariant. In case 2, the group is not quite so obvious. It clearly contains the rotations in the xy plane and, as will be seen at once it also contains two sets of commuting operations $T_\xi(\alpha)$ and $T_\eta(\beta)$ which form, together with the rotations in the xy plane, a group isomorphic to the two-dimensional Euclidean group, i.e. the group of rotations and displace ments in the plane. $T_\xi(\alpha)$ and $T_\eta(\beta)$ are:

$$T_\xi(\alpha) = \left\| \begin{array}{cccc} 1+\frac{1}{2}\alpha^2 & \alpha & 0 & -\frac{1}{2}\alpha^2 \\ \alpha & 1 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \\ \frac{1}{2}\alpha^2 & \alpha & 0 & 1-\frac{1}{2}\alpha^2 \end{array} \right\| , \qquad (3.2)$$

$$T_\eta(\beta) = \left\| \begin{array}{cccc} 1+\frac{1}{2}\beta^2 & 0 & \beta & -\frac{1}{2}\beta^2 \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 1 & -\beta \\ \frac{1}{2}\beta^2 & 0 & \beta & 1-\frac{1}{2}\beta^2 \end{array} \right\| . \qquad (3.2a)$$

The first row (and column) refers to the t component, the second, third and last to the x, y and z components. No simple argument is known to this writer to show directly that the group of Lorentz transformations which leave a null vector invariant is isomorphic to the two-dimensional Euclidean group, desirable as it would be to have such an argument. Clearly, there is no plane in the four-space of momenta in which these transformations could be interpreted directly as displacements and rotations because all transformations considered are homogeneous. The simplest geometrical picture known to me uses two vectors p^ξ and p^η, of length -1 and orthogonal to each other as well as to p^0. These vectors could be unit vectors parallel to the x and y axes. The $T_\xi(\alpha)$ then adds αp^0 to p^ξ, whereas $T_\eta(\beta)$ adds βp^0 to the p^η.

In summary, then, the little groups for the first three cases are:
1. $p.p = m^2 > 0$, $p_t > 0$: the three-dimensional rotation group
2. $p.p = 0$, $p_t > 0$: the two-dimensional Euclidean group
3. $p.p = m^2 < 0$: the 2+1 dimensional Lorentz group.

The little groups for cases 4 and 5 are the same as for 2 and 1, but we shall not be concerned with these cases.

The significance of the operations of the little group becomes evident if (2.5) is specialized to $p = p^0$ and L a member of the little group. One then has

$$O_L | p^0, \zeta \rangle = \sum_\eta D(L)_{\eta\zeta} | p^0, \eta \rangle . \qquad (3.3)$$

The dependence of the c_η on the remaining variables, ζ and L, is made explicit in (3.3). The L is, however, restricted to members of the little group. One now concludes in the usual way, by applying another operation O_M of the little group to (3.3), that the coefficients $D(L)_{\eta\zeta}$ form a representation of the little group. This representation will be unitary and irreducible if the representation of the Poincaré group which we are analyzing is unitary and irreducible. It can be shown, further, that all the coefficients c_η in (2.5) are essentially determined once the D(L) are given. Hence, the unitary irreducible representations of the Poincaré group are characterized by two entities: (a) the set of momentum vectors which can be obtained from a single-momentum vector p^0 by applying to it all proper Lorentz transformations and (b) an irreducible unitary representation of the little group, i.e. the group of proper Lorentz transformations which leave p^0 invariant. We shall take up the three cases of the preceding section separately.

IV. INFINITESIMAL AND CASIMIR OPERATORS

The infinitesimal operators of a unitary representation are skew hermitean; they become hermitean when multiplied by i and correspond to conserved quantities. Because of (2.1), the infinitesimal operators for a displacement parallel to the t, x, y, z axes are $-ip_t$, ip_x, ip_y and ip_z. Hence, p_t, $-p_x$, $-p_y$, $-p_z$ are conserved quantities; they are the covariant components of the momentum. The covariant components of the angular momentum tensor will be denoted by $M_{k\ell} = -M_{\ell k}$. The commutation relations are, then,

$$\left[p_k, p_\ell\right] = 0 \quad \left[M_{k\ell}, p_m\right] = i(g_{\ell m} p_k - g_{km} p_\ell) \tag{4.1}$$

and

$$\left[M_{k\ell}, M_{mn}\right] = i(g_{\ell m} M_{kn} - g_{km} M_{\ell n} + g_{kn} M_{\ell m} - g_{\ell n} M_{km}) \tag{4.1a}$$

where g is the metric tensor, $g_{tt} = -g_{xx} = -g_{yy} = -g_{zz} = 1$, all other components of g vanishing.

The significance of the infinitesimal operators in the present context derives from the fact that the equation of motion gives the change of the state vector for an infinitesimal displacement of time. Hence, the equation of motion will be an equation which permits the calculation of the infinitesimal operator for such a displacement.

Functions of the infinitesimal operators which commute with all infinitesimal operators — such functions are called Casimir operators — commute with all operators of the representation. These are, after all, exponentials, and products of exponentials, of the infinitesimal operators. Each Casimir operator of an irreducible representation must be equivalent with multiplication by a number, at least if the Casimir operator in question is hermitean. In other words, all vectors in the representation space of an irreducible representation must be a characteristic vector of every hermitean Casimir operator and the corresponding characteristic value can depend only on the Casimir operator and the irreducible representation, not on the vector in the representation space. In fact, the vectors which belong to a given characteristic value of a Casimir operator form an invariant subspace and the only non-empty invariant subspace of an irreducible representation is the whole representation space.

It follows that the irreducible representations of any group can be characterized, at least partially, by the values of the Casimir operators for the representation in question, i.e. by the numbers with which the Casimir operators multiply the vectors in the representation space of the irreducible representation in question. The Poincaré group has two Casimir operators. One of these was implicitly determined before: it describes the manifold of momenta

$$P = m^2 = p.p \tag{4.2}$$

by the common length of the momentum vectors of the states of the representation. The second Casimir operator is Lubanski's invariant; this characterizes the representation of the little group. It is the square of the total angular momentum in the coordinate system in which the particle is at rest, multiplied with the square of the mass. Mathematically, Lubanski's invariant is the negative Lorentz square of a vector w

$$W = -w.w. \tag{4.3}$$

The contravariant components of this vector are

$$w^k = \tfrac{1}{2}\epsilon^{k\ell mn} p_\ell M_{mn}, \tag{4.3a}$$

$\epsilon^{k\ell mn}$ being the fully antisymmetric tensor and (3.3a) implying summation of the repeated indices.

We shall not use the Casimir operators to derive the various irreducible representations of the Poincaré group. However, having derived the irreducible representations, we shall calculate the Casimir operators and ascertain the extent to which they characterize the representation or can even replace them.

V. CASE OF POSITIVE REST MASS

The results are, in this case, well known. The irreducible representations of the little group, which is the three-dimensional rotation group in this case, can be characterized by a quantity s which can assume the values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$; it is called the spin. The dimension of the representation s is $2s+1$ so that ζ can assume $2s+1$ values and there are $2s+1$ states with the same four-momentum. The representation with $p.p = m^2 = P$ and the s representation of the little group can be denoted by P_s. Equations of motion for the particles which belong to the representation P_s have also been given; in fact, there are several forms for these equations. It should be noted, however, that the solutions of these equations do not all belong to the representation P_s. They all have negative energy solutions which belong to the conjugate complex of P_s, i.e. to the fifth class of the section III. These spurious solutions are then eliminated, or rather reinterpreted, when the transition to the field theory is undertaken.

The first Casimir invariant is m^2, the second one, W, can easily be calculated for one of the states $|p^0, \zeta\rangle$. For these, $p_x = p_y = p_z = 0$, $p_t = m$, so that $w_x = mM_{yz}$, $w_y = mM_{zx}$, $w_z = mM_{xy}$ and

$$W = m^2 s(s+1), \tag{5.1}$$

so that indeed P and W suffice to characterize the representations with real rest mass, except that for the two conjugate complex representations — one of class 1 and the corresponding one of class 5 — the Casimir operators have the same value. It was mentioned before that the equations of motion also permit the vectors of these two representation spaces.

The equations for positive rest mass have been discussed in the literature repeatedly and will not be given in detail.

VI. CASE OF ZERO REST MASS

A. The representations of the little group.

The representations of the little group, that is the two-dimensional Euclidean group, are not as commonly known as those of the three-dimensional rotation group. They could be easily determined, however, by the method used for the Poincaré group. The operators of displacement, $T_\xi(\alpha)\,T_\eta(\beta) = T_\eta(\beta)\,T_\xi(\alpha)$, form an abelian invariant subgroup and one can choose "vectors" in the space of the Euclidean group's representation which belong to an irreducible representation of this invariant subgroup. If these are denoted by $\|\pi', \pi''\rangle$, one has

$$T_\xi(\alpha)\,\|\pi', \pi''\rangle = e^{-i\pi'\alpha}\,\|\pi', \pi''\rangle, \tag{6.1a}$$

$$T_\eta(\beta)\,\|\pi', \pi''\rangle = e^{-i\pi''\beta}\,\|\pi', \pi''\rangle. \tag{6.1b}$$

It is good to remember that the "displacements" T_ξ and T_η are not displacements in any physical space, their most visualizable interpretation in terms of physical quantities being given after equations (3.2). Similarly, the representation space is not a physical space but the space of the coordinate axes which were denoted before by ζ (see (3.3)). The argument proceeds from this point just as in the case of the Poincaré group but is simpler because the group is much more simple. The possible values of (π', π'') can all be obtained by an orthogonal transformation from one such two-dimensional vector, i.e., in an irreducible representation only such $\|\pi', \pi''\rangle$ occur for which $\pi'^2 + \pi''^2 = \Xi^2$ has a fixed value.

We shall not follow this method but use the same one which will be used also to determine the representations of the little group in the case of imaginary rest mass. This method is based on the solution of the commutation relations of the infinitesimal operators. Since Gårding's construction of an everywhere dense set of vectors in representation space to which all infinitesimal operators can be applied, this is entirely legitimate. The only disadvantage of this method, as compared with the usual one, is,that it gives only the infinitesimal operators, not those for the actual group elements. However, the determination of the infinitesimal operators will suffice for our purposes.

The infinitesimal operators of the little group are M_{xy}, and, as can be seen from (3.2) by setting α and β infinitely small, $\pi' = M_{zx} - M_{tx}$ and $\pi'' = M_{zy} - M_{ty}$. The communication relations between these operators are

$$[\pi', \pi''] = 0 \quad , \quad [M_{xy}, \pi'] = i\pi'' \quad , \quad [M_{xy}, \pi''] = -i\pi'. \tag{6.2}$$

The characteristic values of M_{xy} can be either integer, or half-integer. In either case, these are discrete numbers so that one can assume a form of the representation in which M_{xy} is diagonal. Let us denote the diagonal elements by m_α; the $\alpha\beta$ matrix elements of the second and third equations of (6.2) are then

$$(m_\alpha - m_\beta)\pi'_{\alpha\beta} = i\pi''_{\alpha\beta}, \qquad (m_\alpha - m_\beta)\pi''_{\alpha\beta} = -i\pi'_{\alpha\beta}. \tag{6.3}$$

One easily concludes that $\pi'_{\alpha\beta} = \pi''_{\alpha\beta} = 0$ unless $|m_\alpha - m_\beta| = 1$, that is, if one arranges the diagonal elements of M_{xy} in increasing order, both π' and π'' have non-vanishing matrix elements only between consecutive values of the diagonal elements of M_{xy}. One can then transform all infinitesimal elements by a unitary diagonal matrix in such a way that the matrix elements of π' which are above the diagonal become real. Since π' is hermitean, all its matrix elements will then be real whereas the matrix elements of π'' will be all imaginary. The first of the equations (6.2) then shows that all the non-vanishing matrix elements of π' are equal. Except for this last point, the situation reminds one of the representations of the rotation group in the form in which M_{xy} is diagonal.

Two cases have to be distinguished now. These are the analogues of the six cases encountered in Section III for the Poincaré group. If all the matrix elements of π' are zero, the same holds for π''. In these representations the unit element corresponds to all the "displacements" $T_\xi(\alpha)\, T_\eta(\beta)$ and the representation is faithful only for the factor group of this representation, i.e. the two-dimensional rotation group. The representation can be irreducible only if it is one-dimensional. It coordinates to a rotation by θ in the xy plane the matrix $(e^{is\theta})$ where s can be an integer or a half integer, positive, negative or zero. These representations are denoted by 0_s; they are the well-known representations associated with a null-mass Klein-Gordon particle (s=0), neutrino of positive or negative chirality ($s = \pm\frac{1}{2}$), a right or left circular polarized light quantum ($s = \pm 1$). The quantity $|s|$ is called the spin of the particle. Both Casimir invariants P and W vanish so that they cannot be used to distinguish these representations. Since the representations of the little group are one-dimensional, there is only one state with any definite momentum; the doubling of the number of states for $s \neq 0$ is a result of reflection symmetries. These representations have been adequately discussed in the literature.

The non-singular case, in which π' and π'' do not vanish, is less well known. The non-vanishing elements of π' will be denoted by $\frac{1}{2}\Xi$ so that π' and π'' are given by

$$\pi' = \left\| \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdots & 0 & \frac{1}{2}\Xi & 0 & 0 & \cdots \\ \cdots & \frac{1}{2}\Xi & 0 & \frac{1}{2}\Xi & 0 & \cdots \\ \cdots & 0 & \frac{1}{2}\Xi & 0 & \frac{1}{2}\Xi & \cdots \\ \cdots & 0 & 0 & \frac{1}{2}\Xi & 0 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \right\|, \quad \pi'' = \left\| \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdots & 0 & \frac{1}{2}i\Xi & 0 & 0 & \cdots \\ \cdots & -\frac{1}{2}i\Xi & 0 & \frac{1}{2}i\Xi & 0 & \cdots \\ \cdots & 0 & -\frac{1}{2}i\Xi & 0 & \frac{1}{2}i\Xi & \cdots \\ \cdots & 0 & 0 & -\frac{1}{2}i\Xi & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \right\|. \tag{6.4}$$

Ξ can be assumed to be positive because its sign can be changed by transforming all infinitesimal elements with a diagonal matrix whose diagonal elements are, alternately, 1 and -1. Clearly, these representations of the little group are infinite dimensional; they can be characterized by the Casimir operator $\pi'^2 + \pi''^2 = \Xi^2$ of the two-dimensional Euclidean group. However, this invariant does not characterize them completely: the diagonal elements of M_{xy} can be either the integers ... -2, -1, 0, 1, 2, ..., or the half integers ... $-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$, ... In the former case, the representation is single valued, in the latter case two valued. As far as representations of the two-dimensional Euclidean group are concerned, the characteristic values of M_{xy} could be any arithmetic series with difference 1. However, unless these arithmetic series consist either of the integers, or of the half integers, the representation of the Euclidean group will be more than two valued so that no one or two valued representation of the Poincaré group can be constructed from these representations of the little group. There is a theorem according to which all representations up to a factor of the Poincaré group can be made one or two valued by multiplying the operators of the representation by suitable factors. Hence, the many valued representations are of no interest.

Since the characteristic value of the operator M_{xy}, for a state in which the momentum vector is in the tz plane, can extend to infinity, these representations are also called "infinite spin" representations. The values of the Casimir operators P and W are 0 and Ξ^2. The single-valued representation, $0(\Xi)$, is not distinguished from the two-valued representation $0'(\Xi)$ by the values of the Casimir operators.

Numerous arguments can be adduced to show that no real particles can exist which would transform according to the representation $0(\Xi)$ or $0'(\Xi)$. The simplest of these arguments is that the heat capacity of vacuum due to the possibility of the formation of particles, or of pairs of particles, is proportional to the number of polarizations of the particle in question. This number is infinite for particles with one of the representations $0(\Xi)$ or $0'(\Xi)$ because the representation of the little group is infinite dimensional. Hence, the mere possibility of the existence of any of these particles would give an infinite heat capacity to vacuum.

B. Equations

The equations for the well-known zero mass cases 0_s are adequately discussed in the literature. Again, all known equations permit not only solutions which belong to the representation 0_s, but also solutions which belong to the conjugate complex of 0_s. These are the negative energy solutions which are then eliminated or reinterpreted in the second quantized form of the theory. However, equations for the $0(\Xi)$ and $0'(\Xi)$ cases were obtained only after a general procedure for obtaining equations from representations was devised. This will be described next and illustrated also on one of the earlier, well established cases. It should be admitted, though, that the procedure to be described can be used only in conjunction with single valued representations. The reason for this will be evident at once. If one wants to derive similar equations for the two valued representations, one has to use a space appropriate for these representations: a two-dimensional complex space in which the two valued representations of the Lorentz group are isomorphic to unimodular matrices. However, this will not be spelled out in detail.

The term "equation for a representation" is not clearly defined and, in fact, we have seen that several equations may correspond to the same representation. The quantity to which the equation applies will be called wave function; it may have one or more components. The wave functions which satisfy the equation, or equations, should transform, under the operations of the Poincaré group, according to the representation in question, but this condition does not yet determine the equation, not even the variables on which the wave function depends. It is always possible, for instance, to introduce extraneous, that is unnecessary, variables and then neutralize these by equations as a consequence of which the wave function is either independent of these unnecessary variables, or depends on them only in a trivial fashion. We shall postulate, however, that the variables be of such a nature that they clearly indicate how the wave function transforms under the operations of the Poincaré group. This means that the variables are either the components of a four vector x, or of the difference of two vectors. A four vector x goes over, under a displacement by a, into x+a, under a Lorentz transformation L into Lx. The difference of two vectors is invariant under displacements and transforms like a four vector under Lorentz transformations. Since I do not know a better expression for vectors of this nature, I will call them difference vectors. The position vector is an example for the first case and one is indeed inclined to interpret the components of a vector which occurs in a wave equation as the position vector. This may or may not be justified.* The momentum vector is a difference vector. It is because of this restriction of the variables which are admitted that the equations will always correspond to single valued representations.

We shall determine next the number of vectors and difference vectors which are needed as variables of the wave function. One will be sure to have introduced enough variables into the wave function only if every Poincaré transformation changes the set of values of the variables. If this is not the case, some Poincaré transformations will necessarily leave the wave function unchanged, whereas it may follow from the representation that the wave function is changed by the transformation in question. Hence, the variables should be able to describe completely a frame of reference. A frame of reference can be given by an ordinary vector which describes the origin of the coordinate system, and four difference vectors which give the direction of the four coordinate axes. These vectors have, together, twenty components — surely too many variables, but it will not be difficult to eliminate the unnecessary ones by restricting the variability domain of some and by pointing out that the wave function is independent of the others. Nevertheless, we do not want to go too far with such an elimination because the final variables should be quadruplets of vector components.

Let us consider first the difference vectors. One of these may be identified with the momentum vector and it is convenient, then, to give it the length of the momentum vector. This is purely a matter of convenience,

* In a recent article, (Dubna report P 939), M.I. Shirokov criticizes the replacement of the variables of the wave equation by other position operators, as proposed by T.D. Newton and the present writer. Unfortunately, his considerations contain a serious error.

giving the final equations a slightly simpler form; the frame of reference could be specified also if all the difference vectors were normalized in some other fashion. The other difference vectors can be assumed to be mutually orthogonal, orthogonal to the momentum vector, and of length 1 or -1, whichever is possible. Since these conditions completely specify the last difference vector in terms of the first three, the components of this are surely unnecessary variables and can be omitted. Even the third difference vector contributes only one independent variable — since it is normalized and is perpendicular to two other vectors. It turns out, although this is not evident at this point, that it is also "unnecessary", i.e., its omission does not automatically entail the invariance of a function of the remaining variables under a Poincaré transformation under which it should not be invariant. Hence, we are left with two difference vectors, one of them p. The other will be denoted by ξ; its condition of normalization and perpendicularity to p give the wave equations

$$(p.p)\psi = m^2\psi, \tag{6.5}$$

$$(\xi.\xi)\psi = -\psi, \tag{6.6}$$

$$(p.\xi)\psi = 0. \tag{6.7}$$

At this point, ψ depends on the eight components of two difference vectors, p and ξ, and the four components of a normal vector which will be denoted by x and which will permit ψ to change under displacements. The equations (6.5), (6.6), (6.7) are common to the equations of all representations; the remaining equations with one exception will be characteristic of the representation according to which the solutions of the equation should transform. Our remaining task is, therefore, to express the equations of the representation in terms of the variation of the wave function. We shall carry this out in detail at this point for only two cases: the Klein-Gordon equation for a finite mass P_0, and the case of present interest $0(\Xi)$.

The first representation equation, still common to all representations, is (2.1). According to this, an infinitesimal displacement by ha changes the state vector by a factor 1 - ihp.a. Hence, if we use a vector notation for the variables of ψ

$$\psi(x + ha,\ p, \xi) = (1 - ihp.a)\ \psi\ (x,\ p, \xi) \tag{6.8}$$

or, since this is valid for all a,

$$\frac{\partial\psi}{\partial x_k} = -ip^k\psi. \tag{6.9}$$

It follows from (6.9) that the components of x are "unnecessary variables". If ψ is given as function of p and ξ for one vector x, say x=0, it is determined by (6.9) for all other x. One has

$$\psi(x,\ p, \xi) = e^{-ip.x}\psi(0,\ p, \xi). \tag{6.10}$$

Hence, ψ can be considered to depend only on p and ξ.

Alternately, one can integrate (6.10) over p and obtain a function of x and ξ only:

$$\hat{\psi}(x,\xi) = (2\pi)^{-2}\int d^4p\,\psi(x,\ p,\xi) = (2\pi)^{-2}\int \psi(0,\ p,\xi)\ e^{-ip.x}\,d^4p. \qquad (6.11)$$

From $\hat{\psi}(x,\xi)$, the original $\psi(x,\ p,\xi)$ can be recovered by Fourier inversion,

$$\psi(x,\ p,\xi) = \psi(0,\ p,\xi)\,e^{-ip.x} = (2\pi)^{-2}\int d^4x'\ \hat{\psi}(x',\xi)\,e^{ip.(x'-x)}\ . \qquad (6.11a)$$

The relation between $\psi(0,\ p,\xi) = \psi(p,\xi)$ and $\hat{\psi}(x,\xi)$ is, except for a proportionality factor, the usual one. It follows from (6.5) and (6.7) that $\psi(x,\ p,\xi)$ contains the factors $\delta\,(p.p - m^2)$ and $\delta(p.\xi)$ (as well as $\delta(\xi.\xi + 1)$), but these do not interfere with the integrations in (6.11). It follows, however, from these equations for $\hat{\psi}$ that

$$-\frac{\partial^2}{\partial x_k\,\partial x^k}\,\hat{\psi} = m^2\,\hat{\psi}, \qquad (6.5a)$$

$$\xi_k\,\frac{\partial}{\partial x_k}\hat{\psi} = 0. \qquad (6.7a)$$

The relation between $\psi(p,\xi)$ and $\hat{\psi}\,(x,\xi)$ is so simple that it makes little difference which of these wave functions one uses. In the present note the momentum space representation, $\psi(p,\xi)$, will be preferred.

Let us now consider a representation with finite rest mass. Equation (6.7) restricts ξ to a three-dimensional space-like manifold which is perpendicular to p. In particular, if $p = p^0$, i.e., is parallel to the time axis, $\xi_t = 0$ and (6.6) further restricts the spatial part of ξ to the unit sphere, $\xi_x^2 + \xi_y^2 + \xi_z^2 = 1$, $\xi_t = 0$. If we apply an element of the little group to the two vectors p^0 and ξ, the former will remain unchanged, the latter point to another point of the unit sphere. However, if the representation is P_0, the representation of the little group is the identical representation, ψ has the same value for any two positions which can be transformed into each other by an element of the little group. Since the little group is the group of all three-dimensional rotations, ψ has the same value no matter to which point of the unit sphere ξ points. It follows that ψ is independent of ξ within the domain of this variable, as restricted by (6.6) and (6.7). Hence, ξ is an unnecessary variable in this case and can be dropped. Thus, for the representation P_0, the wave function depends only on p and obeys the single equation (6.5). This, or rather the Fourier transform of this, is the usual Klein-Gordon equation so that our procedure led, in this case, to the usual equation.

It would be quite interesting to derive the equations for the other representations P_s, and also for O_s. Instead, we proceed at once to $0(\Xi)$. In this case, (6.6) and (6.7) restrict the variables ξ to a cylinder-like structure the axis of which is p. At $\xi_t = 0$, the spatial components of ξ are restricted to a unit circle in the plane which is perpendicular to the direction of the spatial part of p. If we denote by ξ' and ξ'' two perpendicular purely spatial vectors (i.e., whose t component is 0) which are orthogonal to p, the general purely spatial ξ vector will be $\xi'\cos\theta + \xi''\sin\theta$. The other ξ

vectors which are consistent with (6.6) and (6.7) can be obtained by adding to one of the purely spatial ξ vectors an arbitrary multiple of p. This, then, forms the aforementioned cylinder-like structure. Since p is orthogonal to itself, the vectors just obtained are also orthogonal to p. It follows similarly that the length of all vectors $\xi' \cos\theta + \xi'' \sin\theta + cp$ is -1.

We yet have to express the condition that the representation of the little group is given by (6.4) so that $W = \Xi^2$. In order to express this condition, we recall that the infinitesimal operators of the Lorentz transformations of the wave function are

$$M_{mn} = i\left(p_m \frac{\partial}{\partial p^n} - p_n \frac{\partial}{\partial p^m} + \xi_m \frac{\partial}{\partial \xi^n} - \xi_n \frac{\partial}{\partial \xi^m}\right). \tag{6.12}$$

Both p and ξ are vectors, hence both change upon a Lorentz transformation. On the other hand, (6.9) shows that the infinitesimal operator of displacement is simply multiplication by -ip. Hence, the w^k of (4.3a) will have two types of terms: those arising from the first two terms of (6.12), involving only p, and those arising from the last two terms. However, because of the antisymmetry of the ϵ, all the terms vanish which involve only the p. This is natural since all w vanish if there is no spin variable as in the case of the representation P_0. Hence, we have

$$w^k = \tfrac{1}{2} i\, \epsilon^{k\ell mn}\left(\frac{\partial}{\partial \xi^n}\, \xi_m - \frac{\partial}{\partial \xi^m}\, \xi_n\right) p_\ell$$

$$= i\, \epsilon^{k\ell mn} \frac{\partial}{\partial \xi^n}\, \xi_m p_\ell . \tag{6.13}$$

If one now calculates W

$$W = -w^k w_k = \epsilon^{k\ell mn} \epsilon_{k\ell' m' n'} \frac{\partial}{\partial \xi^n}\, \xi_m p_\ell \frac{\partial}{\partial \xi_{n'}}\, \xi^{m'} p^{\ell'} \tag{6.14}$$

one can make use of the identity

$$\epsilon^{k\ell mn} \epsilon_{k\ell' m' n'} = -\delta_{\ell\ell'}\left(\delta_{mm'}\delta_{nn'} - \delta_{mn'}\delta_{nm'}\right) - \delta_{\ell' m}\left(\delta_{m' n}\delta_{n'\ell} - \delta_{m'\ell}\delta_{n' n}\right)$$

$$- \delta_{\ell' n}\left(\delta_{m'\ell}\delta_{n' m} - \delta_{m' m}\delta_{n'\ell}\right). \tag{6.15}$$

If one inserts this into (6.14) and applies both sides to ψ, the first two terms give zero because both $p_\ell p^\ell \psi$ and $p_\ell \xi^\ell \psi$ vanish. Hence one can set in any scalar product:

$$W\psi = -\frac{\partial}{\partial \xi^n}\, \xi_m \frac{\partial}{\partial \xi_\ell}\, p_\ell \left(\xi^m p^n - p^m \xi^n\right)\psi .$$

One can now push the ξ_ℓ across the $\partial/\partial \xi_m$ and obtain using again (6.5), (6.7), and also (6.6)

$$W\psi = -\frac{\partial^2}{\partial\xi^n \partial\xi_\ell} p^n p_\ell \psi = V^2 \psi \tag{6.16}$$

where

$$V = i\, p_\ell \frac{\partial}{\partial \xi_\ell} . \tag{6.16a}$$

It now follows from $W\psi = \Xi^2 \psi$ that the linear space of the ψ can be decomposed into two subspaces. In one of these subspaces

$$V\psi = \sum i p_\ell \frac{\partial \psi}{\partial \xi_\ell} = \Xi\, \psi , \tag{6.17}$$

in the other subspace (6.17) holds with the opposite sign. Evidently, both subspaces are relativistically invariant — and also equivalent as the remark after (6.4) shows. Hence, we may adopt as well (6.17) as the last equation for $\psi(p, \xi)$. It determines the variation of ξ along the lines in ξ space which are parallel to p. We recall that ξ is confined to a cylinder-like structure the axis of which is parallel to p. It follows that ψ can be freely chosen only on a line around this cylinder, for instance on the line $\xi_t = 0$. This is, as was also mentioned before, a unit circle in the plane perpendicular to the direction of the spatial part of p. The reason why ψ is defined not only on this circle — where it can be chosen arbitrarily — but all over the cylinder is, that the relativistic invariance is manifest only if the variables of ξ are restricted only in a relativistically invariant fashion. This is done by (6.6) and (6.7). On the contrary, (6.5) is more properly an equation of motion.

There are altogether four equations for $\psi(p,\xi)$: (6.5) with $m=0$, (6.6), (6.7) and (6.17). The common solutions of these equations actually give two invariant linear manifolds: the positive energy solutions belong to $0(\Xi)$, the negative energy solutions to the conjugate complex of $0(\Xi)$. These can be obtained from the positive energy solution by complex conjugation and replacement of ξ by $-\xi$. It is of interest to apply the compatibility criterion to the four equations for ψ which postulates that the commutator of the operators of any two of them shall vanish if applied to ψ, and that this shall be a consequence of the original equations. Evidently, (6.5), (6.6) and (6.7) commute so that these do not-lead to any condition. However, the commutator of (6.6) and (6.17) gives just (6.7) whereas the commutator of (6.7) and (6.17) gives (6.5) with $m=0$. Hence, the compatibility criterion is satisfied — but it would not be satisfied for similar equations with a non zero rest mass.

It is clearly possible to transform the equations from momentum space into coordinate space. The infinitesimal operators of the little group in (6.4) use the coordinate system in which the ξ dependence of ψ, on the unit circle described before, is expanded into harmonic functions $e^{im\theta}$, θ being the polar coordinate. Actually, had we determined the representations of the two-dimensional Euclidean group by the method outlined at the beginning of this section, using equations (6.1), the crucial equation (6.17) would have appeared as a more direct translation of the little group's representation. However, the method here used is somewhat quicker.

VII. CASE OF IMAGINARY REST MASS

A. The representations of the little group.

The case of imaginary rest mass will not be treated in as much detail as the case of zero rest mass. It is believed that the general principles are adequately illustrated in the preceding section and their detailed application should not be too difficult. The little group in this case is clearly the 2+1 dimensional Lorentz group, the group of three-dimensional linear transformations which leave the form $p_t^2 - p_x^2 - p_y^2$ invariant.

The representations of the 2+1 dimensional Lorentz group can be determined in the same way in which the representations of the two-dimensional Euclidean group were determined in the preceding section. Actually, even the representations of the 3+1 dimensional Lorentz group were determined by L.H. Thomas, using this method. The representations of the 2+1 dimensional Lorentz group were investigated in most detail by V. Bargmann. At the time he carried out this investigation it was not clear that he obtained all representations because he used infinitesimal operators in his calculation. However, Gårding's construction subsequently fully justified Bargmann's work.

The 2+1 dimensional Lorentz group has three infinitesimal elements. Their commutation relations are

$$[M_{xy}, M_{xt}] = i\, M_{yt}, \qquad [M_{xy}, M_{yt}] = -i\, M_{xt}, \tag{7.1}$$

$$[M_{xt}, M_{yt}] = -i\, M_{xy}. \tag{7.1a}$$

They differ from the commutation relations of the rotation group only in the signs. Since the representations in which we are interested are either single or double-valued, the characteristic values of M_{xy} are either integers or half integers. At any rate, they are discrete numbers so that we can assume, as in the preceding section, that M_{xy} is diagonal. Since the equations (7.1) are the same as the last two of equations (6.2), with π' and π'' replaced by M_{xt} and M_{yt}, we can infer again that M_{xt} and M_{yt} have non-vanishing matrix elements only just above and just below the main diagonal and that it is possible to transform M_{xt} into a real matrix. M_{yt} will then be purely imaginary and both will have the form illustrated in (6.4) except that the $\frac{1}{2}\Xi$ will be replaced by numbers which are in general different from each other. We denote the diagonal elements of M_{xy} by m; the non-vanishing elements of M_{xt} and M_{yt} will be denoted, then

$$(M_{xt})_{m,m+1} = (M_{xt})_{m+1,m} = N_{m+\frac{1}{2}}, \tag{7.2a}$$

$$(M_{yt})_{m,m+1} = -(M_{yt})_{m+1,m} = i\, N_{m+\frac{1}{2}}. \tag{7.2b}$$

The last commutation relation (7.1a) now gives

$$N^2_{m+\frac{1}{2}} - N^2_{m+\frac{1}{2}} = \tfrac{1}{2} m, \tag{7.3}$$

$$N_m = \tfrac{1}{2}\sqrt{m^2 + c}. \tag{7.4}$$

Since N_m must be real, only such m must occur as a diagonal element of M_{xy} for which $m^2 + c \geq 0$. This will be automatically satisfied if $m^2 + c > 0$ for all m and this gives rise to the first type of representations of the 2+1 dimensional Lorentz group. From (7.2) and $(M_{xy})_{mm'} = m\,\delta_{mm'}$ one can calculate the Casimir operator of this group:

$$Q = M_{xt}^{\;2} + M_{yt}^{\;2} - M_{xy}^{\;2} = c + \tfrac{1}{4}. \tag{7.5}$$

In the present case, the characteristic values of M_{xy} are either all the integers, positive, negative, and zero, or all half integers, positive and negative. In the latter case, $c > 0$, $Q > \frac{1}{4}$. In the former case, c need not be positive but only larger than $-\frac{1}{4}$, the smallest possible value of $-m^2$. Hence, $Q > 0$ holds for single-valued representations of this class, $Q > \frac{1}{4}$ holds for the two-valued representations of this class. The former are called C_Q, the latter C'_Q.

It might appear, first, that this exhausts all the representations. This is not so, however, because if an $N_{s-\frac{1}{2}}$ vanishes, the matrices whose rows and columns are labeled by m=s, s+1, s+2, ... are disconnected from the rows and columns with lower m and provide in themselves a solution of the commutation relations. Hence, if

$$c = -(s - \tfrac{1}{2})^2$$

so that

$$Q = c + \tfrac{1}{4} = -s(s - 1) \tag{7.6}$$

we have a second class of solutions of the commutation relations. For these, the characteristic values of M_{xy} are s, s+1, s+2, ... so that s is the lowest characteristic value. Clearly, $s \geq 0$ must hold, otherwise $N^2_{s+\frac{1}{2}}$ would become negative. Hence, s can assume the values $\frac{1}{2}$, 1, $\frac{3}{2}$, ... and the values of the Casimir operator are quantized in this case. The representations of this subclass are denoted by D_s^+. (The case s=0 will be treated separately.) Similarly, if m assumes only the values -s, -s-1, -s-2, ... and $N_{-s+\frac{1}{2}} = 0$, the matrices M_{xt}, M_{yt} given by (7.2), (7.4), and the diagonal matrix M_{xy}, will satisfy the commutation relations. The representations of this subclass are conjugate complex to the representations of the previous subclass. The value of the Casimir operator Q, and of the parameter c, will be the same as for the representations just discussed. The representations of this subclass are denoted by D_s^-.

The case s=0 remains to be discussed. It follows that, in this case, not only $N_{s-\frac{1}{2}} = N_{-\frac{1}{2}}$ but also $N_{s+\frac{1}{2}} = N_{\frac{1}{2}}$ vanishes. The matrices with the single row and column m=0 therefore separate from the rest and we obtain the trivial solution $M_{xy} = M_{xt} = M_{yt} = 0$ of the commutation relations and the trivial representation in which every group element is represented by the unit operator. This representation will be denoted by D_0.

The preceding discussion assumes, implicitly, that the 2+1 dimensional Lorentz group has only one and two-valued representations up to a factor.

This is a permissible assumption because only these representations of the little group can be used to form a representation of the Poincaré group.

Let us return briefly to the Poincaré group in order to calculate the Casimir invariant W, defined by (4.3). Since this invariant has the same value for all vectors which belong to an irreducible representation, we may as well calculate it for a state of momentum p^0. Hence, we set $p_t = p_x = p_y = 0$, $p_z = m$ in (4.3a) and obtain

$$w^t = m\, M_{xy}, \qquad w^x = m\, M_{yt}, \qquad w^y = -m\, M_{xt}, \qquad w^z = 0.$$

It follows that

$$W = -w.w = m^2 Q. \tag{7.7}$$

B. Equations.

It would seem offhand that one should be able to obtain the equations for the imaginary mass case by replacing m by im in the equations for positive rest mass. Thus, one can replace the Klein-Gordon equation by

$$(p_t^2 - p_x^2 - p_y^2 - p_z^2)\psi = -m^2 \hat{\psi} \tag{7.8}$$

or

$$-\frac{\partial^2}{\partial x_k\, \partial x^k}\, \hat{\psi} = -m^2 \hat{\psi}. \tag{7.8a}$$

It is clear, on the other hand, that this procedure cannot work because it would yield equations with a finite number, 2s+1, of polarizations (linearly independent states of the same momentum). Since the representations of the little group are, with one exception, infinite dimensional, there will be, except for that single case, infinitely many "directions of polarization".

As a matter of fact, the replacement of m by im gives a self-adjoint expression for the infinitesimal operators of the Poincaré group only in case of the Klein-Gordon equation — and this corresponds to the representation of the Poincaré group for which the little group's representation is the identical one, D_0. Thus, if one replaces m by im in Dirac's equation, the expression for $i\,\partial/\partial t$ will not be self-adjoint any more. This resolves the paradox of the preceding paragraph but shows, at the same time, how strongly the results derived in these notes depend on the assumption of the unitary nature of the representations.

It is of some interest to investigate the behavior of the solution of the equation (7.8a) and to contrast it with the solutions of the equations with positive rest mass. We can further simplify the situation by assuming that there is only one space-like dimension x. The positive rest mass Klein-Gordon equation is then

$$\frac{d^2\psi}{dt^2} = \frac{d^2\psi}{dx^2} - m^2\psi. \tag{7.9}$$

It is well known that if, say, at time t=0, the wave function and its time derivative vanish outside the interval (a, b), they will vanish, at time t (or have vanished at time -t), outside the interval $(a - |t|, b + |t|)$. This expresses the finite propagation velocity of disturbances and can be proven in a variety of ways. The proof which is perhaps simplest starts from a Dirac equation

$$\left[i \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \frac{\partial}{\partial t} + i \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} \frac{\partial}{\partial x} \right] \varphi = m\varphi$$

and identifies the first component of φ, that is φ_1, with the solution of (7.9) which vanishes, together with its time derivative, outside the interval (a, b) at time 0. The second component φ_2 will then be equal to $m^{-1}(i\partial/\partial t + i\partial/\partial x)\varphi_1$ and have the same property. Hence, the time component of the Dirac current $|\varphi_1|^2 + |\varphi_2|^2$ vanishes at time 0 on the half line $x > b$. Applying now the divergence theorem to the shaded region in Fig. 1, one sees that the

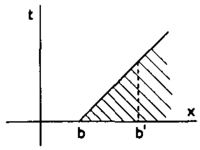

Fig. 1

integral of the current across the tilted line, which goes through the point b', t, also vanishes. If the tilted line is space-like, the current across it is positive definite. Hence, it vanishes at every point. From this, the vanishing of both components of φ follows, just as both components of φ vanish on a t = const line if the current across this line, $|\varphi_1|^2 + |\varphi_2|^2$, vanishes. This, then, proves the theorem on the finite propagation velocity.

Interchanging now x and t in (7.9), we note that if ψ, together with its x derivative, is zero at x=0 outside an interval (a, b) of t, it will be zero at x after $b + |x|$, and was zero before $a - |x|$. Hence, ψ will be zero this time in the shaded area of Fig. 2. Instead of the maximum velocity

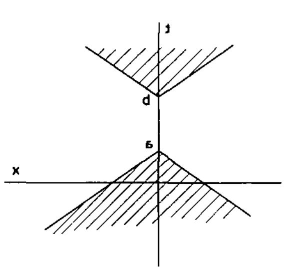

Fig. 2

of propagation, one has a minimum velocity of propagation. This is, of course, what was to be expected. Actually, one is more interested in the initial condition which underlies the first figure: that ψ and $\partial\psi/\partial t$ are zero for a fixed t, say t=0, in an interval of x. In this case it is not true that ψ is zero in the shaded area of Fig. 3. However, as t increases, ψ goes to zero in this area faster than any power of t.

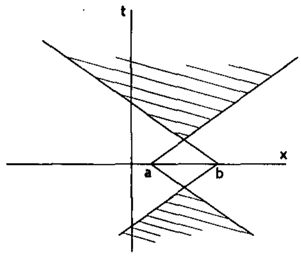

Fig. 3

As was mentioned before, the equations for all other representations will give an infinite number of polarizations so that the introduction of a second vector ξ as a variable into the wave function is quite appropriate. The equations (6.6) and (6.7) define in this case a hyperboloid as the domain of the variables ξ_i. This is unnecessarily complicated as there are null vectors which are perpendicular to the space-like vector p. We therefore replace (6.6) by

$$(\xi.\xi)\,\psi = 0 \qquad \text{with } \psi = 0 \text{ for } \xi_t < 0\,, \tag{7.10}$$

retain (6.7)

$$(p.\xi)\,\psi = 0 \tag{7.11}$$

and in order to denote a real number by m, we set

$$(p.p)\,\psi = -m^2\,\psi \tag{7.12}$$

instead of (6.5). The calculation of the Casimir operator W then becomes very similar to that in the preceding section, with the roles of ξ and p interchanged. The only difference is that the non-commuting nature of ξ_m and $\partial/\partial\xi_m$ has to be taken into account. The resulting expression is

$$W\psi = -m^2\Big(\sum \frac{\partial}{\partial\xi_\ell}\,\xi_\ell\Big)\Big(\sum \frac{\partial}{\partial\xi_\ell}\,\xi_\ell - 1\Big)\psi. \tag{7.13}$$

The variability domain of ψ is now a light cone in the three-space of ξ which is perpendicular to p. We can transform (7.13) in such a way that it contains only the operator $\Sigma\,\xi_\ell\,\partial/\partial\xi_\ell$ which is the derivative along the straight null-lines of the cone. Further, we can express by (7.7) the Casimir operator W of the Poincaré group in terms of the Casimir operator Q of the 2+1 dimensional Lorentz group. This gives

$$-V(V+1)\psi = -\left[(V+\tfrac{1}{2})^2 - \tfrac{1}{4}\right]\psi = Q\psi, \tag{7.14}$$

$$V = \sum \xi_\ell \, \partial/\partial \xi_\ell . \tag{7.14a}$$

This V is different from that defined by (6.16a) for the equations for the $0(\Xi)$ representations. However, we can conclude again, just as we did when deriving the equations for the $0(\Xi)$ representations, that the linear set of wave functions for which (7.10), (7.11), (7.12) hold and for which the second Casimir operator has the value W, decomposes into two invariant linear subsets. For the first of these

$$V\psi = \sum \xi_\ell \frac{\partial}{\partial \xi_\ell}\psi = (-\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - Q})\psi \tag{7.15}$$

holds, for the second

$$V\psi = \sum \xi_\ell \frac{\partial}{\partial \xi_\ell}\psi = (-\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - Q})\psi . \tag{7.15a}$$

We recall that $Q = W/m^2$ is a function of the two Casimir operators W and m^2. These equations are quite similar to (6.17) but whereas the latter gives the change of ψ for an increment of the vector ξ which is parallel to p, (7.15) gives the change of ψ for an increment of ξ which is parallel to ξ itself. Both increments are, however, along the straight lines of the developable surface which is the definition domain of ξ. The resultant set of equations, (7.10), (7.11), (7.12), (7.15), was not discussed in detail.

VIII. PROBLEMS WHICH REMAIN

The preceding discussion of the equation for representations with imaginary mass is even more perfunctory than the discussion of the $0(\Xi)$ equations. Furthermore, apparently, no more complete discussion is available in the literature. Whereas for the $0(\Xi)$ equations several equivalent forms of the relativistically invariant scalar product are known, the preceding discussion gives no expression therefor. This should be supplemented.

A more serious omission is our failure to give equations for the two-valued representations, that is for the representations which describe particles with half integer spin. In order to do this, one should again introduce a space in which a relativistic transformation can be defined. Such a space is, in this case, a two-dimensional space with complex coordinates. In that space then the total manifold of functions must be limited by as many relativistically invariant linear equations as possible. A "relativistically invariant" equation in this case is invariant under complex unimodular transformations. The statement "as many as possible" means that the elimination of a single further function, by a new equation or otherwise, together with the postulate of relativistic invariance, eliminates all functions from the linear manifold so that this becomes vacuous. The task of obtaining equations for the two-valued representations in this way has not been carried out.