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HEAD-TAIL EFFECT II:
FROM A RESISTIVE-WALL WAKE

Summary:

Earlier work on the head-tail effect in electron storage rings is extended using the fast-wake due to the conventional resistive-wall effect in a cylindrical tube. The results are qualitatively similar to those obtained earlier for a step-function wake, except for the length dependence. The magnitude of the effect seems much too small to account for the observed instabilities.

Contents:

I. INTRODUCTION

In an earlier note* (Note I) I have treated the general problem of the head-tail effect in storage rings. A complete analysis was carried through there only for an idealized model in which (a) the wake field of a single particle was zero in front of and a constant behind the particle (a "step-function" wake); and (b) all the particles of the bunch had the same amplitude of synchrotron oscillations. In this note I show the results for a model in which the wake field is, instead, taken to be that produced by the "resistive-wall" effect, although, again, the detailed calculations are carried out only for the same idealized distribution of particles (assumption (b)).

The analysis leads to the following conclusions. The qualitative features of the head-tail effect for the resistive-wall wake are similar to those for a step-function wake except that: (1) the frequency shift, which is a pure imaginary for the step-function wake, has, for the resistive-wall wake, both real and imaginary parts, and (2) the imaginary part (real exponential) changes from a first power dependence on the bunch length to a square-root dependence. More important, however, is the quantitative result that the resistive-wall wake is much too weak to account for the thresholds observed at ACO and Adone.

II. FORMULAS FOR AN ARBITRARY WAKE

Following Note I, and adopting the same notation**, we describe the synchrotron oscillations of a particle in terms of its time displacement $\tau(t)$, which varies with time as

$$\tau = A \cos(\omega_s t + \phi) \quad . \tag{1}$$

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* "The Head-Tail Effect: An Instability Mechanism in Storage Rings", M. Sands, SLAC TN-69-8, March 28, 1969.

** A list of symbols and their definitions are given in the Appendix.

As before we consider only an idealized bunch in which all particles have the same amplitude A of synchrotron oscillations but have different phases ϕ, and in which the distribution of particles in ϕ is uniform. An individual particle is, then, conveniently identified by its phase ϕ.

In Note I the coupling between the betatron motions of two particles is characterized by a wake function $\rho(\tau' - \tau)$ which gives the lateral force on a particle at τ from the wake field produced by the lateral displacement of another particle in the bunch at τ'. For each pair of particles the relative time delay $(\tau' - \tau)$ oscillates with time as (see Eq. (31) of Note I)

$$\tau' - \tau = R \cos(\omega_s t + \theta) \tag{2}$$

where R and θ depend on the phases ϕ' and ϕ. In particular,

$$R = 2A \sin \left|\frac{\phi' - \phi}{2}\right| , \tag{3}$$

which is a function only on the absolute value of the phase differences $(\phi' - \phi)$.

It was shown in Note I that the interaction between each pair of particles is summarized by a number (see Eqs. (27) and (30) of Note I)

$$\overline{W} = \frac{1}{2\omega_o T_s} \int_{T_s} dt \left\{ 1 - i\omega_o \frac{\xi}{\alpha}(\tau' - \tau) \right\} \rho(\tau' - \tau) \tag{4}$$

where the integral is to be taken over one complete synchrotron period, $T_s = 2\pi/\omega_s$. Because of the averaging in this definition, $\overline{W}$ is independent of the initial phase θ of Eq. (2), and depends only on R which is, in turn, a function only of the magnitude of the relative phase $(\phi' - \phi)$. So $\overline{W} = \overline{W}(|\phi' - \phi|)$.

The interaction between each pair of particles through the wake fields leads to collective betatron motions which can be described in terms of a set of normal modes, each associated with a mode number μ; $\mu = 0, 1, 2, 3, \ldots$. For each mode the complex amplitude of the betatron oscillations of all the particles is given by (see Eq. (41) of Note I)

$$Z_\mu = a_\mu e^{i(\Delta\omega_\mu t + \mu\phi)} \quad . \tag{5}$$

The characteristic frequency of each mode is given in terms of $\overline{W}$ (see Eq. (38) of Note I)

$$\Delta\omega_\mu = -\frac{N}{2\pi} \int_{-\pi}^{+\pi} d\psi e^{i\mu\psi}\, \overline{W}(\psi) \tag{6}$$

where $\psi = \phi' - \phi$. As pointed out above, $\overline{W}$ is a symmetric function of ψ , so we may also write

$$\Delta\omega_\mu = -\frac{N}{\pi} \int_0^\pi d\psi \overline{W}(\psi) \cos \mu\psi \tag{7}$$

Given any wake function $\rho(\tau' - \tau)$, we can obtain $\overline{W}$ from Eq. (4); using this $\overline{W}$ in Eq. (7) we can obtain the frequency shift $\Delta\omega_\mu$ for each mode. In the next Section we derive the wake function associated with the resistive-wall effect, and in Section IV we evaluate $\Delta\omega_\mu$ for such a wake.

III. THE RESISTIVE-WALL WAKE

The wake fields of relativistic particles moving in a resistive vacuum chamber were first considered by Laslett, Neil, and

Sessler[1]. Robinson[2] has given a particularly illuminating derivation of the long-time wake fields for a cylindrical pipe. We take the Robinson formulation as our starting point. Consider a straight cylindrical pipe with its axis horizontal--along what we shall take as the z-axis. When a particle travels parallel to this axis but above it by the distance y, it leaves behind eddy currents in the walls which produce a transverse magnetic field B_x near the axis. This field is proportional to y. A following particle passing through this field then experiences a vertical force proportional to the vertical displacement of the leading particle, $F_y = evB_x$. Our wake function ρ is defined as this force divided by the effective mass γm_o of the betatron oscillator, and divided by the displacement y of the leading particle.

Suppose the leading particle passes $z = 0$ at the time $t = 0$; using Robinson's result for B_x, and considering only relativistic particles, we get that the wake function at the time t is

$$\rho(t) = \frac{2}{\sqrt{\pi}} \frac{r_o c}{\sqrt{\mu\sigma}} \frac{1}{\gamma b^3 \sqrt{t}} , \tag{8}$$

where we are using MKS units and the symbols have the following meanings:

- r_o : classical electron radius,
- c : velocity of light,
- μ, σ: permeability and conductivity of the walls,
- γ : energy of the particle in units of its rest mass,
- b : radius of the pipe.

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1 L. J. Laslett, V. K. Neil, A. M. Sessler, Rev. Sci. Inst. 32, 276 (1961).

2 K. W. Robinson, Storage Ring Summer Study, SLAC Report No. 49, Aug. 1965, p. 32.

When Robinson gave his derivation he was concerned only with the long-time part of the wake (for its effect on subsequent bunches). As he said, the result is applicable for time t "large compared with the duration of the direct fields produced by the particle". We are here, rather, interested in the short-time effects, so we must consider what this limitation implies.

If the conductivity of the pipe is reasonably large*, the "direct fields" of a particle may in good approximation be obtained by making a relativistic transformation of the static field of a point charge in a pipe. We may say, roughly, that the static field will extend in z a distance of about $\pm b$ from the particle, and that the transformed field will extend about $\pm b/\gamma$. Robinson's result, Eq. (8), then applies for times t greater than t_o where

$$t_o = \frac{b}{\gamma c} \tag{9}$$

For times between $-t_o$ and $+t_o$, the wake field will rise in some smooth way from zero the value given by Eq. (8). The complete wake will then be roughly as shown in Fig. 1.

In a typical electron storage ring we might have $\gamma \approx 10^3$ or more, and $b \approx 10$ cm less; so $t_o \approx 3 \times 10^{-11}$ sec or less. On the other hand, for storage rings whose r.f. systems are operated with a relatively low harmonic number (as at ACO and Adone), the bunch length is typically $\approx 10^{-9}$ sec. So the separation between any two particles is generally much greater than t_o and the detailed form of the wake field for $t < t_o$ is of little importance. In fact, since in evaluating $\overline{W}$ we take an average of $\rho(\tau - \tau')$ and of $(\tau' - \tau)\rho(\tau' - \tau)$, we shall make only a negligible error by taking that ρ is given by Eq. (8) for all values of its argument. We shall, therefore, make this approximation in calculating $\overline{W}$.

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* We also limit consideration to a straight pipe, as Robinson and others have done.

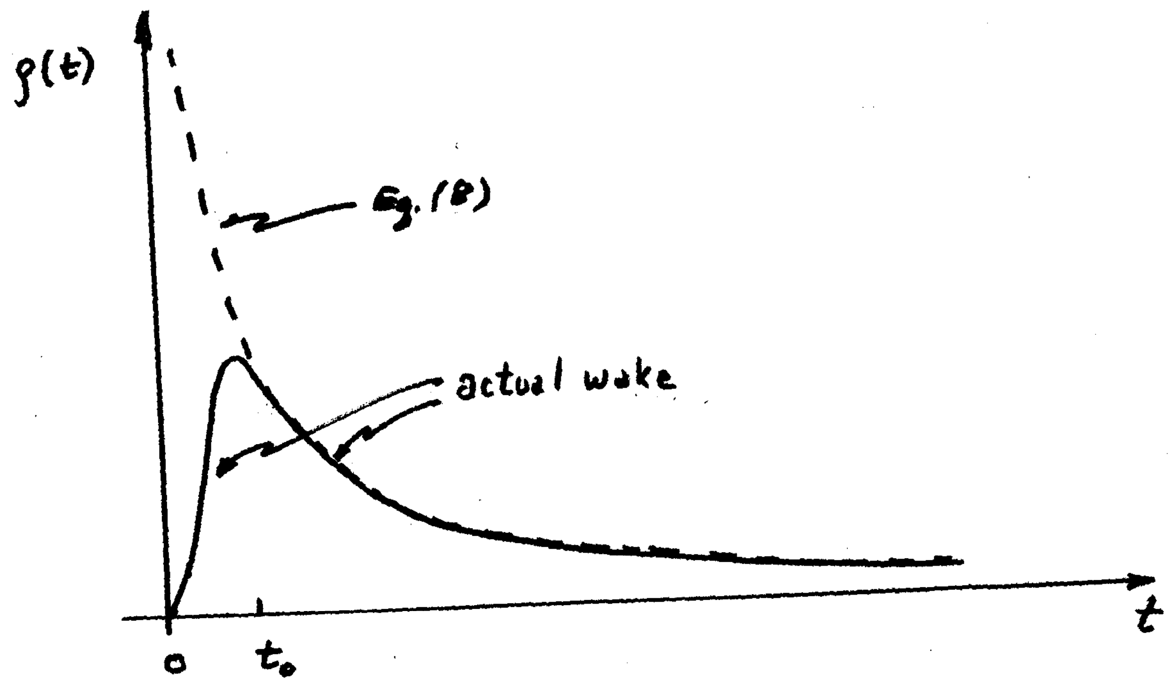

Fig. 1. The wake function.

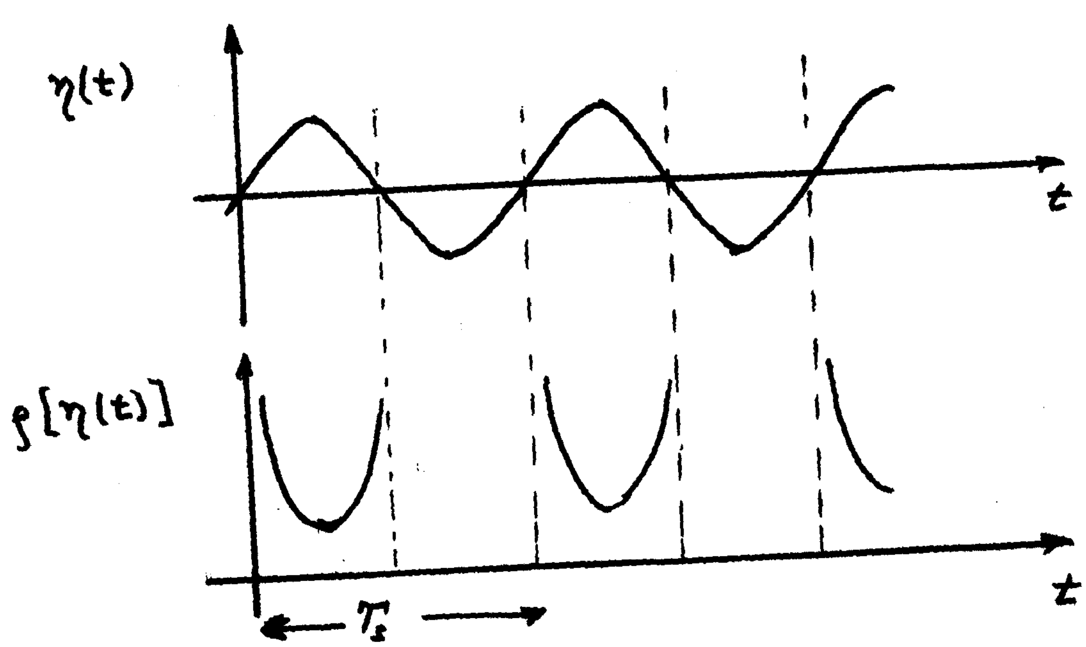

Fig. 2. The time variation of η and ρ.

IV. RESULTS FOR THE RESISTIVE-WALL WAKE

We now wish to evaluate $\overline{W}$ for the wake-function of Eq. (8). The time dependence of the wake function applies, of course, to the time delay $(\tau' - \tau)$ between the arrival of any two particles at each point of the orbit, which has in turn a slow variation with time. Let's write for each pair of particles $(\tau' - \tau) = \eta$, where by a suitable choice of the origin of t we can make Eq. (2) become

$$\eta = R \sin \omega_s t \quad . \tag{10}$$

Then we take our wake-function as

$$\rho(\eta) = \frac{D}{\sqrt{\eta}} \Theta(\eta) \tag{11}$$

with

$$D = \frac{2}{\sqrt{\pi}} \frac{r_o c}{\sqrt{\mu\sigma}} \frac{1}{\gamma b^3} \quad , \tag{12}$$

and where $\Theta(\eta)$ is the unit step function. We can, further, write $\overline{W}$ as

$$\overline{W} = \frac{1}{2\omega_o} \left\{ \overline{\rho} - i\omega_o \frac{\xi}{\alpha} \overline{\eta\rho} \right\} , \tag{13}$$

where $\overline{\rho}$ and $\overline{\eta\rho}$ are the time averages of ρ and $\eta\rho$.

The quantities η and ρ vary with time as shown qualitatively in Fig. 2. Evidently,

$$\overline{\rho} = \frac{2}{T_s} \int_o^{T_s/4} \rho(\eta) dt = \frac{2D}{T_s\sqrt{R}} \int_o^{T_s/4} \frac{dt}{(\sin \omega_s t)^{1/2}} \quad . \tag{14}$$

So, recalling that $\omega_s T_s = 2\pi$, we get that

$$\overline{\rho} = \frac{c_1 D}{\pi\sqrt{R}} \quad , \tag{15}$$

where the constant c_1 is defined by

$$c_1 = \int_0^{\pi/2} \frac{dx}{\sin^{1/2}x} \simeq 2.6 \tag{16}$$

Similarly, we find that

$$\overline{\eta\rho} = \frac{c_2 D\sqrt{R}}{\pi} \tag{17}$$

with c_2 a constant defined by

$$c_2 = \int_0^{\pi/2} \sin^{1/2}x dx \simeq 1.2 \tag{18}$$

We get, then, that for each pair of particles,

$$\overline{W} = \frac{D}{2\pi\omega_o} \left\{ \frac{c_1}{\sqrt{R}} - ic_2\omega_o \frac{\xi}{\alpha} \sqrt{R} \right\} \quad , \tag{19}$$

where R depends on $\psi = \phi' - \phi$ through Eq. (3).

$$R(\psi) = 2A \sin\frac{\psi}{2} \tag{20}$$

We are now ready to evaluate $\Delta\omega_\mu$ as given in Eq. (7). Let's first write

$$\Delta\omega_\mu = \alpha_\mu - i\beta_\mu \tag{21}$$

where then α_μ is the frequency shift of mode μ and β_μ is its exponential growth constant. From Eqs. (7), (19), and (20) we get that

$$\alpha_\mu = - \frac{K_\mu}{2\sqrt{2}\pi^2} \frac{ND}{\omega_o\sqrt{A}} \quad , \tag{22}$$

where K_μ is the number defined by

$$K_\mu = c_1 \int_o^\pi \frac{\cos \mu\psi}{\left[\sin \frac{\psi}{2}\right]^{1/2}} d\psi \quad ; \tag{23}$$

and that

$$\beta_\mu = - \frac{J_\mu}{\sqrt{2}\pi^2} \frac{\xi}{\alpha} ND\sqrt{A} \tag{24}$$

where J_μ is the number defined by

$$J_\mu = c_2 \int_o^\pi \left[\sin \frac{\psi}{2}\right]^{1/2} \cos \mu\psi d\psi \tag{25}$$

The numbers K_μ and J_μ can be calculated with a little work; we find for the first few:

μ	K_μ	J_μ
0	+ 13.6	+ 2.9
1	+ 4.3	- 0.57
2	+ 3.2	- 0.21

For larger values of μ the K_μ are all positive and the J_μ are negative, while for both the magnitudes continue to decrease monotonically with increasing μ .

Finally, the dependence on machine parameters will be clearer if we substitute for D the expression of Eq. (12); we get

$$\alpha_\mu = -\frac{K_\mu}{2^{1/2}\pi^{5/2}} \frac{r_o c}{(\mu\sigma)^{1/2}} \frac{N}{\gamma\omega_o A^{1/2} b^3} \tag{26}$$

and

$$\beta_\mu = -\frac{2^{1/2} J_\mu}{\pi^{5/2}} \frac{r_o c}{(\mu\sigma)^{1/2}} \frac{\xi A^{1/2} N}{\alpha\gamma b^3} \tag{27}$$

It is perhaps also worth writing down the ratio of β_μ to α_μ

$$\frac{\beta_\mu}{\alpha_\mu} = 2 \frac{J_\mu}{K_\mu} \frac{\xi}{\alpha} A\omega_o \tag{28}$$

For electron storage rings this ratio is generally of the order of, or less than, 1.

V. MAGNITUDE OF THE EFFECT AND CONCLUSIONS

We now estimate the magnitude of the resistive-wall, head-tail effect for a typical electron storage ring such as Adone. Suppose we take

$$r_o c = 8.4 \times 10^{-7} m^2 sec^{-1}$$

$$\mu\sigma = 2.0 \ sec - m^{-2}$$

$$A = 10^{-9} sec$$

$$b = 3 \times 10^{-2} m$$

$$\xi = -1.5$$

$$\alpha = 0.06$$

$$\omega_o = 6 \times 10^7 sec^{-1}$$

$$\gamma = 10^3$$

$$N = 6 \times 10^{10}$$

The typical vacuum chamber is not cylindrical so our results are not strictly applicable, but for an estimate of the magnitudes we are taking a typical small dimension (usually vertical) of the chamber. The number of particles corresponds to a 30 milliampere circulating current in one bunch. With these parameters we get

$$\alpha_o \approx -420 \ sec^{-1}$$

$$\beta_o \approx +240 \ sec^{-1}$$

The real frequency shift α_o is less than 10^{-5} of the betatron frequency and would probably not be directly observable. The imaginary part, β_o , is antidamping for the lowest mode--assuming the chromaticity ξ to be negative as taken--but seems too small to account for the observed instability. If we neglect Landau damping we would predict a growth time-constant, $1/\beta_o$ of 4 milliseconds at the design current of 30 milliamperes, or of 120 milliseconds at a current of 1 milliampere.

The observed instability appears at currents as low as 1 milliampere or less, and appears to have a growth constant of about 10 milliseconds. Further, the observed instability has equally low thresholds for horizontal instabilities for which one would expect the effective b^3 to be more than an order of magnitude greater (and the β correspondingly less).

Any consideration of the threshold should, of course, take into account the effects of radiation damping and Landau damping, which was not done here. At the observed threshold current of about 1 milliampere at Adone (at $\gamma = 10^3$) the predicted growth constant for the head-tail effect is about 0.12 sec, which is only a little less than the radiation damping time-constant (0.3 sec). But the Landau damping effects are much stronger. A detailed calculation has not been carried out, but if we assume a spread in ν-values of about 10^{-4} due to machine nonlinearities, a rough estimate of the Landau damping gives a threshold current of about one ampere for the vertical oscillations.

In summary, it appears that the head-tail effect with the resistive-wall mechanism is too small to explain the instabilities observed in Adone.

APPENDIX

LIST OF SYMBOLS

			Page
τ	:	Synchrotron oscillation coordinate, the arrival time at any azimuth measured with respect to the arrival time of a synchronous particle.	1
ω_s	:	Synchrotron oscillation (angular) frequency	1
A	:	Amplitude of synchrotron oscillation of the time-of-arrival.	1
ϕ	:	Phase Displacement of synchrotron oscillation of the time-of-arrival.	1
R	:	Amplitude of oscillation of the <u>difference</u> of the times-of-arrival of two particles.	2
T_s	:	Period of synchrotron oscillations, $2\pi/\omega_s$.	2
ω_o	:	Angular frequency of unperturbed betatron oscillations.	2
ξ	:	Chromaticity, $(E/\nu)\ d\nu/dE$.	2
α	:	Momentum compaction; logarithmic rate of change of rotation period with energy.	2
$\rho(t)$	:	Time development of the short-term wake force.	2
$\overline{W}$	:	The average wake-field perturbation.	2
ψ	:	$\phi - \phi'$, the difference for two particles.	3
μ	:	Mode number (integer 0, 1, 2...).	3
Z_μ	:	Time varying (complex) amplitude of the betatron oscillations in mode μ .	3
a_μ	:	Z_μ at $t = 0$ for the particles at $\phi = 0$.	3
$\Delta\omega_\mu$	:	The (complex) frequency shift for mode μ .	3
r_o	:	Classical electron radius	4

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c	: Velocity of light.	4
μ, σ	: Permittivity and conductivity of wall material.	4
b	: Radius of vacuum chamber.	4
γ	: $E/m_o c^2$.	4
η	: $\tau' - \tau$.	7
D	: Coefficient of resistive wall wake function.	7
c_1, c_2	: Definite integrals.	8
α_μ	: Real part of $\Delta\omega_\mu$, the (real) frequency shift of mode μ .	9
β_μ	: Negative imaginary part of $\Delta\omega_\mu$, the exponential growth constant of mode μ .	9
K_μ, J_μ	: Definite integrals.	9