A Short Introduction to Geometric Intrinsic Symmetries in Nuclear Physics

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1 Introduction

A possibility of existence of high-rank nuclear symmetries related to the geometric properties, usually understood as some deformations of nuclei, has been shown several years ago [1]. One of the most reach symmetry is the tetrahedral/octahedral symmetry which can produce large shell gaps in the single particle spectra because of the characteristic for these point groups four-fold degeneracy. Large degeneracy of the energy spectrum increases the average level spacing [2, 3]. This leads to the specific tetrahedral-magic shell-closures for nucleon numbers 32, 40, 56, 64, 70, 90-94, 112, and 136-138.

There were several experiments related to the problem of 'tetrahedral' nuclei performed. For example see Ref.[4, 5]. In the Rare Earth nuclei such as ^{152,156}Gd, ^{154,156}Dy, ^{164}Er, ^{164}Yb, but also in the Actinides in $^{230-234}$U, there were found some interesting properties suggesting existence of searched symmetries, however, the results are not unique [6, 7].

The word $\sigma\upsilon\mu\mu\epsilon\tau\rho\iota\alpha$ (symmetry) comes from Greek language: $\sigma\upsilon\mu$ ('together') and $\mu\epsilon\tau\rho\omega\nu$ ('measure'). Before the contemporary physics the symmetry was rather related to such notions as beauty, perfectness, harmony or 'proper proportions'. The contemporary meaning of the symmetry concept was invented more or less in Renaissance.

2 Space–time versus intrinsic symmetries

The most striking property of the space-time is its symmetry. The space-time symmetry group G_{ST} corresponds to a relativistic group as, for example Poincaré group in the theory of special relativity. In numerous relativistic and non-relativistic models of the space-time considered in physics, one considers the various space-time symmetry groups. The second kind of important physical symmetries are intrinsic symmetries G_{int} which commute with the space time symmetries G_{ST}. The intrinsic symmetries can have two origins:

Figure 1: Platonic solids: tetrahedron, cube, octahedron, icosahedron, dodecahedron.

- the first type describes these intrinsic properties of the physical body which are independent of the space and time structure, e.g. the symmetries related the isospin, conservation of the electric charge, conservation of the particle number and so on. The corresponding symetry group we denote here as G_c,
- the second type is determined by the geometric properties of the physical body. One of the most important geometric feature is shape of the body. These kinds of symmetries leads to the so called intrinsic groups consisted of the geometric transformations constructed in the intrinsic frame of the body. In this paper the intrinsic groups are labelled by the bar symbol over the group name, e.g. $\overline{\mathrm{G}}$.

In this lecture we are interested only in the second kind of the intrinsic symmetries.

In case of a nucleus (non–relativistic description), let us assume, that this nucleus is considered in the the coordinate frame in which center of mass is fixed in the position space. The remaining non-relativistic space–time symmetry is the orthogonal group O(3). Every nuclear collective Hamiltonian has to be invariant in respect to this orthogonal group O(3). However, the nucleus can have additional geometric intrinsic symmetry group which is a subgroup of the corresponding intrinsic orthogonal group $\overline{\mathrm{G}} \subset \overline{\mathrm{O(3)}}$. It implies that, in the case of non-relativistic description of a nucleus the general intrinsic symmetries collected in the group G_{int} can be considered as the direct product:

$$\mathrm{G}_{int} = \overline{\mathrm{G}} \times \mathrm{G}_c. \tag{1}$$

Historically, the most known symmetries are related to the geometric symmetries of some solids invented by Platon (428 - 347 BC). In three dimensional space there is known 5 Platonic solids which are the regular, convex polyhedrons. They are constructed from the faces which are congruent, regular polygons: triangles, squares or pentagons. These 5 Platonic solids are called: tetrahedron, cube, octahedron, dodecahedron and icosahedron, see Fig. 1

The proof of existence of only five Platonic solids is based on the Euler's formula:

$$V + F = E + 2, \tag{2}$$

where V, F, E denote the total number of V = vertices, F = faces and E = edges. There is an open question: do exist the nuclear Platonic solids in the Universe? We will have this problem in our mind in the following text.

Many scientists was and still is fascinated by the notion of symmetry. One of the first was Johannes Kepler who believed in symmetry and proposed the palnetary model

Figure 2: The Kepler's planetary model, http:// en.wikipedia.org/wiki/ File:Kepler-solar-system-2.png.

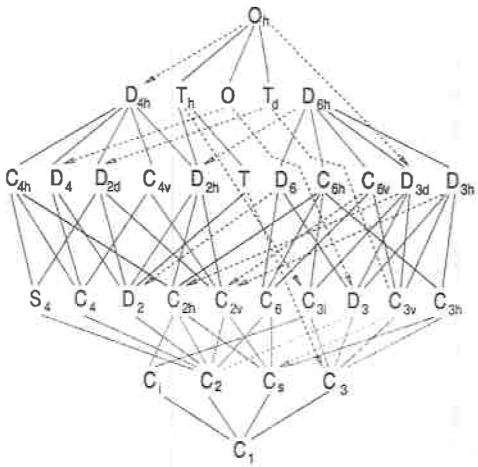

Figure 3: Point groups chains.

built from the Paltonic solids Similarly, in the contemporary physics we are searching for elementary particles, nuclear magic numbers, universal properties of matter etc., using the symmetry building block called the irreducible representations of the symmetry groups.

Above we have mentioned that there is an open problem about existence of nuclei having symmetries of Platonic solids. These symmetries are related to the so called point groups consisted of transformations which leave one or more points of the three dimensional space unchanged. The most important is a set of 32 point groups shown on the Fig. 3. The dashed line denote not-invariant subgroup. Adding translations to point groups one gets 230 crystalographic space groups, 14 Bravais' lattices and 7 crystal lattices.

Because of relatively large degeneration of the energy spectra of the Hamiltonians invariant in respect to the tetrahedral and octahedral symmetries, both the tetrahedral and octahedral groups are the first candidates for analysis of nuclear point symmetries.

Figure 4: The tetrahedral surfaces for three different values of the deformation parameters $\alpha_{32} = 0.1, 0.2$ and 0.3, respectively.

In the simplest case, the tetrahedrally invariant shapes are generated by the deformation tensor α_{32}, where the deformation parameters are identified with the expansion coefficients of the nuclear surface:

$$R(\alpha;\theta,\phi) = R_0\left(1 + \sum_{\lambda\mu} \alpha^{\star}_{\lambda\mu} Y_{\lambda\mu}(\theta,\phi)\right). \tag{3}$$

The examples of the simplest tetrahedrally invariant surfaces are determined, eg. by the following equation

$$R(\alpha;\theta,\phi) = R_0\left(1 + \alpha_{32}(Y_{32}(\theta,\phi) + Y_{3,-2}(\theta,\phi)\right). \tag{4}$$

The equation (3) allows to write down equation for different shapes of a nucleus classified in respect to the multipolarity λ.

Usually it is assumed that the dipole parameters $\alpha_{1\mu}$ describe a shift of the surface. It is only an approximation which has to be always verified in a given application. In Fig. (5) there is presented an effect of the dipole deformation on the quadrupole shape. In the right figure it is seen that the dipole deformation not only shifts the surface but it also change its shape. In the figures below only the non-zero parameters are explicitely written in their captions. One of the problems related to the above parametrization of the nuclear surface is that for larger deformations one can get quite unphysical surfaces, an example of such pure quadrupole surface are presented in Fig. (6). On the other hand, the regular quadrupole shapes are of the expected form, see Fig. (7).

3 Collective variables

The deformation parameters of the nuclear surface can be used as the collective variables, like in the Bohr type collective models. However, one can obtained the more general description assuming q_1, q_2 and q_3 are curvelinear coordinates in R^3. Then the most general equation of the nuclear surface can be written as

$$q_k = q_k(u,v) \text{ where } k = 1,2,3, \tag{5}$$

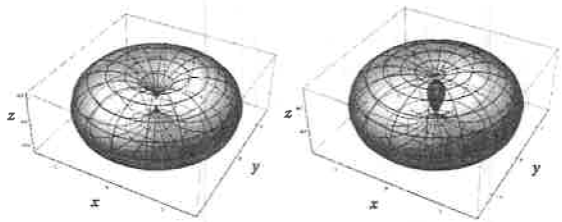

Figure 5: The shape for $\alpha_{20} = -1.50$ (left) and the shape for $\alpha_{10} = 1.50$, $\alpha_{20} = -1.50$ (right).

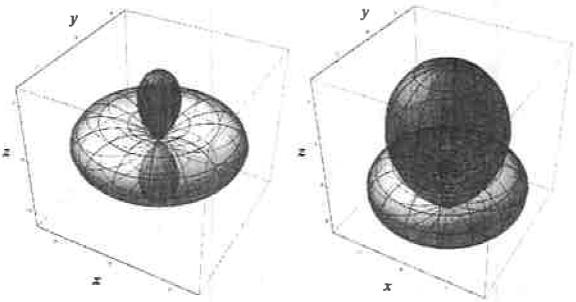

Figure 6: The monster quadrupole shapes, $\alpha_{20} = -5.50$ (left), $\alpha_{10} = 9.0$, $\alpha_{20} = -5.50$ (right)

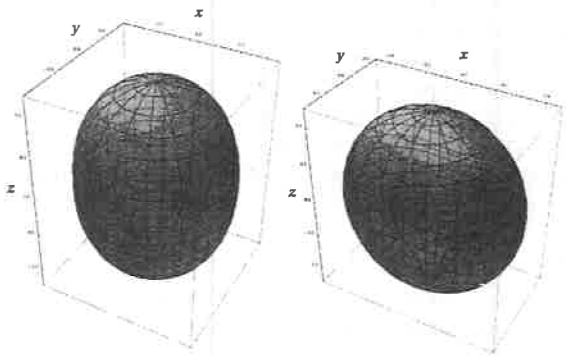

Figure 7: The regular quadrupole shapes, $\alpha_{20} = 0.30$ (left), $\alpha_{20} = -0.3$ $\alpha_{22} = 0.3$ (right).

where $(u, v) \in S \subset R^2$ are two real continuous parameters.

Assume the functions $q_k \in L^2(S)$ are square integrable functions, where the compact subset $S \subset R^2$ of variables parametrizes the surface in the space of a single-nucleon.

Let the set of the three vectors $\{e_n(u, v)\}$ gives the orthonormal basis in the space $L^2_\rho(S)$

$$\langle e_n | e_m \rangle = \int_S du; dv; \rho(u, v) e_n(u, v)^\star e_m(u, v) = \delta_{nm}, \tag{6}$$

where $\rho(u, v) \geq 0$ is the appropriate weight function.

Using of this basis allows for expansion of the surface (5) in the following form

$$q_k(u, v) = \sum_n \alpha_{n,k}\, e_n(u, v). \tag{7}$$

The basis should be chosen to have a physical meaning determined by a set of commuting physical observables $\hat{A}_l$, where $l = 1, 2, \ldots, r$ i.e.

$$A_l e_n(u, v) = a_{ln} e_n(u, v), \text{ for all } l = 1, 2, \ldots, r. \tag{8}$$

In this case, the expansion coefficients

$$\alpha_{n,k} = \int_S dudv\, \rho(u, v)\, e_n(u, v)^\star q_k(u, v) \tag{9}$$

can be used as new variables describing the nuclear surface in terms of the observables $\{\hat{A}_l\}$.

The very well known example of this procedure is the description of the nuclear surface with the expansion (3). In this case one needs to identified the variables in the three dimensional space with the spherical variables $\{q_1 = r, q_2 = \theta, q_3 = \phi\}$ and asume $u = \theta, v = \phi$. The equation of the surface $r = R(\theta, \phi) \in L^2(SO(2))$ can be expanded into eigenfunctions of the angular momentum observables $\hat{A}_1 = \hat{J}^2$ and $\hat{A}_2 = \hat{J}_z$, where $\hat{J}^2$ is square of the total angular momentum and $\hat{J}_z$ denotes its third component. In this case the basis $e_n(u, v) = Y_{lm}(\theta, \phi)$ consists of the spherical harmonic functions. As the result one obtains the equation (3). In practice, in the nuclear physics, the equation of the nuclear surface written in the laboratory frame (in this case we label the deformation parameters with the superscript (lab), $\alpha^{(lab)}_{\lambda,\mu}$) usually has the additional coefficient $c(\alpha^{(lab)})$ in front of the equation (3), which allows to satisfy the volume conservation condition for the nuclear matter. The reality of the radius $R(\theta, \phi)$, its invariance in respect to the space rotations $\hat{R}(\Omega)$ and the space inversion $\hat{C}_i$ leads to the standard relations for the expansion coefficients $\alpha^{(lab)}_{\lambda,\mu}$:

- Reality of the surface: $(\alpha^{(lab)}_{\lambda\mu})^\star = \alpha^{(lab)}_{\lambda,-\mu}$.

  This condition can be obtained by making use of the reality of the radius $r = R(\theta, \phi)$ and properties of the spherical harmonic functions

$$\begin{aligned} R(\alpha^{(lab)}; \theta, \phi) &= R^\star(\alpha^{(lab)}; \theta, \phi) \\ \sum_{\lambda,\,\mu} \alpha^{(lab)\star}_{\lambda,\mu} Y_{\lambda\mu}(\theta, \phi) &= \sum_{\lambda,\,\mu} \alpha^{(lab)}_{\lambda,\mu} (-1)^\mu Y_{\lambda,-\mu}(\theta, \phi). \end{aligned} \tag{10}$$

- Rotational properties of the surface: $\hat{R}(\Omega)\alpha^{(lab)}_{\lambda\mu} = \sum_{\mu'} D^{\lambda}_{\mu'\mu}(\Omega)\alpha^{(lab)}_{\lambda\mu'}$, where $D^{\lambda}_{\mu'\mu}(\Omega)$ denotes the Wigner functions of the rotation group and the operator $\hat{R}(\Omega)$ represents the rotation operator parametrized with the Euler angles $\Omega = (\Omega_1, \Omega_2, \Omega_3)$.

  This condition follows from the transformation properties of the spherical harmonic functions and invariance of the radius $r = R(\theta, \phi)$ of the surface in respect to the space inversion:

$$\begin{aligned}
&\hat{R}(\Omega)R(\alpha^{(lab)};\theta,\phi) = R(\alpha^{(lab)};\theta,\phi)\\
&\hat{R}(\Omega)R(\alpha^{(lab)};\theta,\phi) = R\left(\hat{R}(\Omega)\alpha^{(lab)};\hat{R}(\Omega)\{\theta,\phi\}\right)\\
&= R_0\left(1+\sum_{\lambda,\,\mu}\left(\hat{R}(\Omega)\alpha^{(lab)}_{\lambda,\mu}\right)^{*}\hat{R}(\Omega)Y_{\lambda,\mu}(\theta,\phi)\right)\\
&= R_0\left(1+\sum_{\lambda,\,\mu}\left(\hat{R}(\Omega)\alpha^{(lab)}_{\lambda,\mu}\right)^{*}\sum_{\eta} D^{\lambda}_{\eta\mu}(\Omega)Y_{\lambda,\eta}(\theta,\phi)\right).
\end{aligned} \tag{11}$$

  Comparing the equation of the surface before and after rotation results in the transformation properties of the deformation parameters in respect to the space rotation.

- Space inversion transformation: $C_i\alpha^{(lab)}_{\lambda\mu} = (-1)^{\lambda}\alpha^{(lab)}_{\lambda\mu}$.

  This property follows directly from the properties of the spherical harmonics and invariance of the radius :

$$\begin{aligned}
&\hat{C}_i R(\alpha^{(lab)};\theta,\phi) = R(\alpha^{(lab)};\theta,\phi)\\
&\hat{C}_i R(\alpha^{(lab)};\theta,\phi) = R_0(1+\sum_{\lambda,\,\mu}\alpha^{(lab)*}_{\lambda,\mu}(-1)^{\lambda}Y_{\lambda\mu}(\theta,\phi)).
\end{aligned} \tag{12}$$

  As above, comparing both expressions before and after the transformation of the surface gives the transformation properties of the deformation parameters.

These properties show that the deformation parameters (collective variables) $\alpha^{(lab)}_{\lambda\mu}$ are the covariant components of the spherical tensor of the rank λ (tensor in respect to the rotation group SO(3)). The important property of these tensors is existence of the scalar product of two tensors. Let ξ_λ and η_λ be the tensor of the same multipolarity, then the scalar product is defined as

$$\xi_\lambda \cdot \eta_\lambda = \sum_{\mu\nu} g^{\mu\nu}\xi_{\lambda\mu}\eta_{\lambda\nu}, \tag{13}$$

where the metric tensor is generated by the Clebsch–Gordan coefficients of the rotation group $(\lambda_1\mu_1\lambda_2\mu_2|\lambda\mu)$

$$g^{\mu\nu} = \sqrt{2\lambda+1}(\lambda\mu\lambda\nu|00) = (-1)^{\mu}\delta^{-\nu}_{\mu}. \tag{14}$$

The scalars (rotational invariants) obtained in this way play an important role in description of nuclear collective motion. For example, the total multipole deformation of a nucleus is proportional to the multiplication operator $\hat{\beta}_\lambda$

$$\hat{\beta}_\lambda\psi(\alpha^{(lab)}) = \beta_\lambda\psi(\alpha^{(lab)}), \tag{15}$$

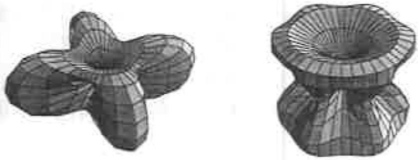

Figure 8: The spin orientation probability for a rotating system. The chosen wave functions are proportional to some combinations of the Wigner functions: $\psi \sim D^5_{M2}(\Omega) - D^5_{M,-2}(\Omega)$ (left) and $\psi \sim D^5_{M3}(\Omega) - D^5_{M,-3}(\Omega)$ (right).

where

$$\beta^2_\lambda = \alpha^{(lab)}_\lambda \cdot \alpha^{(lab)}_\lambda = \sum_{\mu\nu} \alpha^{(lab)}_{\lambda\mu}[(-1)^\mu \alpha^{(lab)}_{\lambda,-\mu}] = \sum_\mu |\alpha^{(lab)}_{\lambda\mu}|^2 \in \mathbb{R}. \tag{16}$$

This kind of invariants is important in construction of the collective Hamiltonians. For example, the classical harmonic oscillator Hamiltonian

$$H_{ho} = \sum_\lambda \left[\frac{1}{2B_\lambda}\dot{\alpha}^{(lab)}_\lambda \cdot \dot{\alpha}^{(lab)}_\lambda + \frac{1}{2}B_\lambda \omega^2_\lambda \beta^2_\lambda\right] \tag{17}$$

is constructed from such invariants and finally it is invariant in respect to the rotation group as it is required in physics.

4 Intrinsic frame

The classical rotation is well understood phenomenon in which the orientation of a body is changing with time. Contrary, the quantum rotation allows to determine only the probability of a given orientation and there is no time variable in the wave function. The quantum rotation can be presented in the graphical form as the surface drawn by the end of the vector pointing out in the same direction as the spin of a rotating body and its lentgh equal to the probability of finding a given orientation of the spin, see Fig. (8) .

The notion of the quantum rotational motion allows to define the rotating intrinsic frame, e.g. the body fixed frame, for the collective variables $\{\alpha^{(lab)}_{\lambda\mu}\}$. The corresponding colective variables in the intrinsic frame we denote by $\{\alpha_{\lambda\mu}\}$. They can be obtained by the quantum rotation of the laboratory collective variables $\{\alpha^{(lab)}_{\lambda\mu}\}$ with the rotation operator $\hat{R}(\Omega)$

$$\alpha_\lambda = \hat{R}(\Omega)\alpha^{(lab)}_\lambda \tag{18}$$

assuming, in addition, that the rotation group SO(3) $\ni \hat{R}(\Omega)$ parameters, repesented by the Euler angles $\Omega = (\Omega_1, \Omega_2, \Omega_3)$, are considered as a part of intrinsic variables. The intrinsic variables α_λ are invariant in respect to the laboratory rotations $\hat{R}(\Omega)$. It is important to notice that inclusion of the Euler angles into the set of intrinsic variables makes this set of variables redundant, 3 variable more than needed. It implies that, the definition of the intrinsic frame requires three additional conditions which recover the same number of variables in both frames

$$F_k(\alpha, \Omega) = 0, \quad \text{where } k = 1, 2, 3. \tag{19}$$

In this way one can obtain a new description of a physical system, e.g. a nucleus, in which the rotational motion can be directly described by the Euler angles.

5 Intrinsic groups

There is an intersting question: how to investigate symmetries of a nucleus in the intrinsic frame. A part of symmetries, eg. trnaslational symmetry are not seen in the intrinsic frame. Due to the general principles, the nuclear Hamiltonian has to be invariant in respect to the orthogonal group O(3) defined in the laboratory frame. On the other hand, it is obvious, that the nucleus should have some geometrical symmetries related to its shape. The transformations furnishing an intrinsic symmetry group have to be defined in the intrinsic frame. In group theory, there is known the notion of left and right shift on the group magnifold. This idea was used to define the so called intrinsic groups which, in fact, act in the intrinsic frame.

A convenient definition was formulated in [15] in the following form:
for each element g of the group G, one can define a corresponding operator $\hat{\bar{g}}$ in the group linear space $\mathcal{L}_G$ as:

$$\hat{\bar{g}}|S\rangle = |Sg\rangle, \qquad \text{for all } |S\rangle \in \mathcal{L}_G, \tag{20}$$

where all elements inside the ket vectors $S = \sum_{g \in \mathrm{G}} c_g g$, here c_g are the complex numbers, form a group algebra of the group G.

In this definition the notion of the group linear space $\mathcal{L}_G$ is used. This space is defined as the linear space spanned by all possible formal linear combinations of the elements of the group G

$$\mathcal{L}_G = \left\{ |S\rangle : |S\rangle = \sum_{g \in \mathrm{G}} c_g g, \text{ where } c_g \in \mathbb{C} \right\}. \tag{21}$$

It looks like the group algebra mentioned above, but, it is important that the elements of $\mathcal{L}_G$ have to be considered only as vectors, not as the elements of the group algebra.

The group formed by the collection of the operators $\hat{\bar{g}}$ is called the intrinsic group $\overline{\mathrm{G}}$ related to the group G.

One of the most important property of the intrinsic group $\overline{\mathrm{G}}$ is that this group commutes with its partner group G

$$[\mathrm{G}, \overline{\mathrm{G}}] = 0. \tag{22}$$

The groups G and $\overline{\mathrm{G}}$ are antyisomorphic. The required anti-isomorphism between the partner groups G and $\overline{\mathrm{G}}$ is given by

$$\phi_G : \overline{\mathrm{G}} \to \mathrm{G}, \text{ where } \quad \phi_G(\bar{g}) = g \text{ and } \phi_G(\bar{g}\bar{g}') = \phi_G(\bar{g}')\phi_G(\bar{g}). \tag{23}$$

This property suggests that the partner groups G and $\overline{\mathrm{G}}$ have a lot of common properties as e.g. similar structure of representations, decompositions of the Kronecker products, the Clebsch-Gordan coefficients and many others.

As an example let us consider a relation among representations of both groups. Because the partner groups commute one can find common basis $|\Gamma mk\rangle$ for representations of the group G and the group $\overline{\mathrm{G}}$. The representations are defined as:

$$g|\Gamma mk\rangle = \sum_{m'} \Delta^{(\Gamma)}_{m'm}(g)|\Gamma m'k\rangle, \tag{24}$$

$$\bar{g}|\Gamma mk\rangle = \sum_{k'} \bar{\Delta}^{(\Gamma)}_{k'k}(\bar{g})|\Gamma mk'\rangle. \tag{25}$$

To compare both representation one can use as the basis the generalized projection operators (elements of the group linear space $\mathcal{L}_G$)

$$|\Gamma mk\rangle = \frac{\dim[\Gamma]}{\mathrm{card(G)}} \sum_{g\in \mathrm{G}} \Delta^{(\Gamma)}_{mk}(g)^* g, \tag{26}$$

where dim[Γ] denotes the dimension of the representation Γ and card(G) is the number of elements in the group G. This allows to calculate (25)

$$\begin{aligned}\bar{g}|\Gamma mk\rangle &= \frac{\dim[\Gamma]}{\mathrm{card(G)}} \sum_{g'\in \mathrm{G}} \Delta^{(\Gamma)}_{mk}(g')^* g' g \\ &= \frac{\dim[\Gamma]}{\mathrm{card(G)}} \sum_{g'\in \mathrm{G}} \Delta^{(\Gamma)}_{mk}(g' g^{-1})^* g' \\ &= \frac{\dim[\Gamma]}{\mathrm{card(G)}} \sum_{g'\in \mathrm{G}} \sum_{k'} \Delta^{(\Gamma)}_{mk'}(g')^* \Delta^{(\Gamma)}_{kk'}(g) g' = \sum_{k'} \Delta^{(\Gamma)}_{kk'}(g)|\Gamma mk'\rangle, \end{aligned} \tag{27}$$

where $\Delta^{(\Gamma)}_{mm'}(g)$ are matrix elements of the representation Γ of the group G. Comparing both expression one can see that the matrices of both representations are related. The representations of the intrinsic group are transposed representations of the partner group

$$\bar{\Delta}^{(\Gamma)}_{mk}(\bar{g}) = \Delta^{(\Gamma)}_{km}(g). \tag{28}$$

A bit different are definitions of irreducible tensors in respect to the laboratory group G and the intrinsic group $\overline{\mathrm{G}}$. By definition the irreducible tensors in respect to the laboratory group G transform as

$$\hat{g} T^{(\Gamma)}_m \hat{g}^{-1} = \sum_l \Delta^{(\Gamma)}_{lm}(g) T^{(\Gamma)}_l. \tag{29}$$

The tensors in respect to the intrinsic group $\overline{\mathrm{G}}$, due to the anti-isomorphism between both groups, have to be defined in the following way

$$\hat{\bar{g}} \overline{T}^{(\Gamma)}_k \hat{\bar{g}}^{-1} = \sum_l \Delta^{(\Gamma)}_{lk}(g^{-1}) \overline{T}^{(\Gamma)}_l. \tag{30}$$

As an example, let us consider the action of the intrinsic group in the collective space consisted of the square integrable functions of the deformation parameters and the Euler angles. The intrinsic rotation operators $\hat{R}(\bar{g}_1, \bar{g}_2) \in \overline{\mathrm{SO}(3)}_\alpha \times \overline{\mathrm{SO}(3)}_\Omega$ (the indices α and Ω show the variables which are affected by the corresponding group) are defined as follows

$$\hat{R}(\bar{g}_1, \bar{g}_2) f(\alpha, \Omega) = f(\{\hat{\bar{g}}_1 \alpha\}, \Omega \phi_G(\bar{g}_2)^{-1}), \tag{31}$$

where $\bar{g}_1 \in \overline{\mathrm{SO}(3)}_\alpha$ and $\bar{g}_2 \in \overline{\mathrm{SO}(3)}_\Omega$. The action of the group $\overline{\mathrm{SO}(3)}_\alpha$ onto the deformation variables is a bit non-standard and is given by the following equation

$$\hat{\bar{g}}_1 \alpha_{\lambda\mu} = \sum_{\mu'} D^\lambda_{\mu'\mu}(\phi_G(\bar{g}_1)^{-1}) \alpha_{\lambda\mu'}. \tag{32}$$

The intrinsic group $\overline{\mathrm{SO}(3)}$ corresponding to the 'laboratory' rotation group SO(3) defined in the laboratory frame consists of all rotations $\hat{R}(\bar{g}, \bar{g})$ for which the deformation parameters and the Euler angles are rotated with the same angles.

The required anti-isomorphism between the partner groups SO(3) and $\overline{\mathrm{SO}(3)}$ is given by (23).

It is important to notice that, in general, not all transformations $(\bar{g}_1, \bar{g}_2) \in \overline{\mathrm{SO}(3)}_\alpha \times \overline{\mathrm{SO}(3)}_\Omega$

$$(\bar{g}_1, \bar{g}_2): \ (\alpha, \Omega) \to (\alpha', \Omega') \tag{33}$$

are allowed in the intrinsic frame. They are allowed if they do not break the conditions which define the intrinsic frame (19)

$$(\hat{\bar{g}}_1, \hat{\bar{g}}_2) F_k(\alpha, \Omega) = F_k(\hat{\bar{g}}_1 \alpha, \Omega \bar{g}_2^{-1}) = 0, \ \text{where } k = 1, 2, 3. \tag{34}$$

For example, in the case of the quadrupole colective variables α_2 with the standard Bohr condition which define the intrinsic frame: $\alpha_{2,\pm1} = 0$ and $\alpha_{22} = \alpha_{2-2}$, the allowed intrinsic rotations $\hat{R}(\bar{g}_1, \bar{g}_2) \in \overline{\mathrm{SO}(3)}_\alpha \times \overline{\mathrm{SO}(3)}_\Omega$ have to fulfil the following conditions

$$\hat{R}(\bar{g}_1, e_G)\alpha_{2\pm1} = 0 \text{ and } \hat{R}(\bar{g}_1, e_G)\alpha_{22} = \hat{R}(\bar{g}_1, e_G)\alpha_{2-2}, \tag{35}$$

where the second argument represents the unit element of the group $\overline{\mathrm{SO}(3)}_\Omega$. The Bohr conditions allow for the arbitrary rotations $\bar{g}_2 \in \overline{\mathrm{SO}(3)}_\Omega$.

Using the conditions (35) the allowed rotations of the deformation parameters α have to satisfy the following equations

$$\begin{aligned} D^2_{0,\pm1}(\bar{g}_1^{-1}) &= 0, \\ D^2_{-2\pm1}(\bar{g}_1^{-1}) + D^2_{2\pm1}(\bar{g}_1^{-1}) &= 0, \\ D^2_{02}(\bar{g}_1^{-1}) - D^2_{0,-2}(\bar{g}_1^{-1}) &= 0, \\ D^2_{-2,-2}(\bar{g}_1^{-1}) + D^2_{2,-2}(\bar{g}_1^{-1}) &= D^2_{-22}(\bar{g}_1^{-1}) + D^2_{22}(\bar{g}_1^{-1}). \end{aligned} \tag{36}$$

In this case, the octahedral point group $\overline{\mathrm{O}}_\alpha \subset \overline{\mathrm{SO}(3)}_\alpha$ acting only on the variables α provide the solution of the set of equations (36).

6 Uniqueness of quantum states

In practice, the transformation to the intrinsic frame is not a one-to-one function. For the further purpose it is useful to define a group of intrinsic transformations $\bar{h} \in \overline{\mathrm{G}}_s$:

$$(\alpha, \Omega) \xrightarrow{\bar{h}} (\alpha', \Omega'), \tag{37}$$

where $\alpha = \{\alpha_{\lambda\mu}\}$ and which leave invariant the corresponding laboratory coordinates:

$$\begin{aligned} \alpha^{(lab)}(\alpha', \Omega') &= \alpha^{(lab)}(\alpha, \Omega), \\ F_k(\alpha', \Omega') &= F_k(\alpha, \Omega) = 0, \text{ for } k = 1, 2, 3, \end{aligned} \tag{38}$$

where $\alpha^{(lab)}(\alpha, \Omega) = \hat{R}(\Omega^{-1})\alpha$, see (18).

The group $\overline{\mathrm{G}}_s$ we call the symmetrization group.

The symmetrization group decomposes the collective manifold into orbits of physically equivalent points. Let the function $\Psi^{(lab)}$ denotes a state vector of a nucleus in the laboratory frame. The corresponding state vector in the intrinsic frame has to fulfil the obvious equation

$$\Psi(\alpha, \Omega) = \Psi^{(lab)}(\alpha^{(lab)}) \tag{39}$$

which represents the fact that the wave function of the physical system written in the laboratory frame has to be a well and uniquelly defined function.

However, after the transformation of the intrinsic variable with the elements of the symmetrization group we do not change the laboratory state vector

$$\Psi(\alpha', \Omega') = \Psi^{(lab)}(\alpha^{(lab)}). \tag{40}$$

This implies the uniqueness condition for the states in the intrinsic frame

$$\Psi(\alpha', \Omega') = \Psi(\alpha, \Omega). \tag{41}$$

This is a very well known but not fully solved problem in the collective models of the Bohr type.

In principle, there are two possibilities to achieve uniqueness of transformation from the laboratory to the intrinsic frame:

- first, one can define the appropriate region of the intrinsic collective variables in which the transformation from the laboratory to intrinsic frame is a one-to-one function,

- second, one can allow for the whole range of collective variables but then one needs to fulfil the symmetrization condition for physical states. The symmetrization condition can be expressed as invariance of the intrinsic state vectors in respect to all transformations $\bar{h} \in \overline{\mathrm{G}}_s$,

$$\bar{h}\Psi(\alpha, \Omega) = \Psi(\alpha, \Omega), \tag{42}$$

where the group $\overline{\mathrm{G}}_s$ is the symmetrization group.

As an example, let us came back to the very well know example of the quadrupole variables $(\alpha_{20}, \alpha_{22}, \Omega)$ with the Bohr conditions which define the intrinsic frame (35).

Using the conditions (35) one can see that the allowed rotations of the deformation parameters α have to satisfy Eqs. (36) which are fulfilled by the rotations belonging to the octahedral point group $\overline{\mathrm{O}}_\alpha \subset \overline{\mathrm{SO}(3)}_\alpha$. The required invariance of the transformation formula from laboratory to the intrinsic frame (38) implies that both rotations $(\bar{g}_1, \bar{g}_2) \in \overline{\mathrm{O}}_\alpha \times \overline{\mathrm{O}}_\Omega$ have to be rotations about the same angles $\theta = (\theta_1, \theta_2, \theta_3)$, i.e. $\bar{g}_1 = \bar{g}_1(\theta)$ and $\bar{g}_2 = \bar{g}_2(\theta)$. This considerations suggest that the symmetrization group is equal to the octahedral group $\overline{\mathrm{G}}_s = \overline{\mathrm{O}} \subset \overline{\mathrm{O}}_\alpha \times \overline{\mathrm{O}}_\Omega$ transforming simultaneously the deformation parameters and the Euler angles by the same rotation. Because the quadrupole variables are invariant in respect to the space inversion, this transformation should be formally added to the symmetrization group, in this way one obtains $\overline{\mathrm{G}}_s = \overline{\mathrm{O}}_h$.

Obviously, instead of the standard Bohr conditions the following alternative definition of the intrinsic frame can be used:

- the collective variables are now chosen as $(\alpha_{20}, \alpha_{21}, \Omega)$,
- the conditions which define the intrinsic frame (variables) are now assumed as

$$F_{1,2}(\alpha, \Omega) = \alpha_{2\pm 2} = 0 \text{ and } F_3(\alpha, \Omega) = \alpha_{21} + \alpha_{2-1} = 0. \tag{43}$$

These definitions lead to the equations for allowed rotations and the symmetrization group:

$$\begin{aligned} D^2_{\pm 20}(g) &= 0 \\ D^2_{\pm 2,1}(g) - D^2_{\pm 2,-1}(g) &= 0 \\ D^2_{10}(g) + D^2_{-1,0}(g) &= 0 \\ D^2_{11}(g) - D^2_{1,-1}(g) &= D^2_{-1-1}(g) - D^2_{-11}(g). \end{aligned} \tag{44}$$

The allowed rotations are now given by $\overline{\mathrm{D}}_{2;\alpha} \times \overline{\mathrm{SO}(3)}_\Omega$. The symmetrization group, in turn, is given by much smaller group $\overline{\mathrm{D}}_2 \subset \overline{\mathrm{D}}_{2;\alpha} \times \overline{\mathrm{D}}_{2;\Omega}$ than in the previous case.

We see that using of different condition defining the intrinsic frame lead to formally different structure of the colective spaces.

This considerations born an interesting question. Do both sets of collective variables

$$\text{set 1.:} \quad \alpha_{20}, \quad \alpha_{22} = \alpha_{2-2}, \quad \alpha_{2\pm 1} = 0, \tag{45}$$

$$\text{set 2.:} \quad \alpha'_{20}, \quad \alpha'_{21} = -\alpha'_{2-1}, \quad \alpha'_{2\pm 2} = 0 \tag{46}$$

describe the same set of shapes? Do are they physically equivalent?

To answer these questions one needs to check if there exists the one-to-one relation between both frames. The required transformation is given by

$$\alpha_{20} = -\frac{1}{2}\alpha'_{20}, \tag{47}$$

$$\alpha_{22} = \exp(-2i\theta_1)\left(\frac{1}{2}\sqrt{\frac{3}{2}}\alpha'_{20} + i\alpha'_{21}\right), \tag{48}$$

where the rotation angle θ_1 can be calculated from the following formula

$$\alpha'_{21}\cos(2\theta_1) = \frac{1}{2}\sqrt{\frac{3}{2}}\alpha'_{20}\sin(2\theta_1),$$
$$\theta_2 = \theta_3 = \frac{\pi}{2}. \tag{49}$$

In fact, the angles $\theta_1, \theta_2, \theta_3$ parametrize the rotation which transform the second set of variables into the first one.

7 An example of a symmetry structure of the collective configuration space

Let us denote by $X_{\alpha^{(lab)}}, X_\alpha$ and $X_{\alpha\Omega}$ the configuration spaces consisted of: a) the laboratory variables $\alpha^{(lab)}$, b) the intrinsic deformation parameters α and c) the full intrinsic configuration space, respectively.

Let us consider again the case of the collective space consisted of Bohr variables $(\alpha_{20}, \alpha_{22})$ which are equivalent to the popular polar parametrization of nuclear shapes (β, γ), where

$$\alpha_{20} = \beta\cos\gamma \quad \text{and} \quad \alpha_{22} = \frac{\beta}{\sqrt{2}}\sin\gamma. \tag{50}$$

The symmetrization group $\overline{\mathrm{O}}$ (inversion omitted) is generated by the following rotations $\bar{R}_1 = \bar{C}_{2y}, \bar{R}_2 = \bar{C}_{4z}, \bar{R}_3 = \bar{R}(\pi/2, \pi/2, \pi)$, where C_{nq} denote the rotation by the angle $2\pi/n$ around the q axis.

To find the region of uniquenes of transformation from the laboratory to the intrinsic frame one needs to construct the orbits obtained from the action of the symmetrization group $\mathrm{G}_s = \overline{\mathrm{O}}$ onto the full intrinsic configuration space $X_{\alpha\Omega}$. In our case the orbits are represented by the following sets

$$\mathrm{orb}(\overline{\mathrm{O}}; \beta_0, \gamma_0, \Omega_0) = \left\{(\beta, \gamma, \Omega) : (\beta, \gamma, \Omega) = \hat{\bar{g}}(\beta_0, \gamma_0, \Omega_0),\ \bar{g} \in \overline{\mathrm{O}}\right\}. \tag{51}$$

Every orbit consists of 24 elements of the configuration space $X_{\alpha\Omega}$ which correspond to the same laboratory deformation. Here, we have used the polar parametrization of the quadrupole variable due to the simpler action of the octahedral group on these variable than on the α_{20}, α_{22} themselves

$$\hat{\bar{g}}\beta = \beta, \quad \hat{\bar{g}}\gamma \in \{\pm\gamma,\ \pm(\gamma - k\frac{2\pi}{3})\},\ k = 1, 2, 3, \quad \hat{\bar{g}}\Omega = \Omega g. \tag{52}$$

Formally, to have one-to-one transformation from the laboratory to the intrinsic frame one needs to construct the following quotient of the collective configuration space $X_{\alpha\Omega}$:

$$X^C_{\alpha\Omega} = X_{\alpha\Omega}/\mathrm{orb}(\overline{\mathrm{O}}), \tag{53}$$

where two points of the collective magnifold $(\beta', \gamma', \Omega')$ and $(\beta'', \gamma'', \Omega'')$ belong to the same equivalent class of intrinsic points if both points belong to the same orbit, i.e.

$$(\beta', \gamma', \Omega') = (\beta'', \gamma'', \Omega'') \quad \mathrm{mod}(\mathrm{orb}(\overline{\mathrm{O}}))$$
$$\text{iff there exists the point } (\beta_0, \gamma_0, \Omega_0) \in X_{\alpha\Omega}$$
$$\text{such that } (\beta', \gamma', \Omega'), (\beta'', \gamma'', \Omega'') \in \mathrm{orb}(\overline{\mathrm{O}}; \beta_0, \gamma_0, \Omega_0). \tag{54}$$

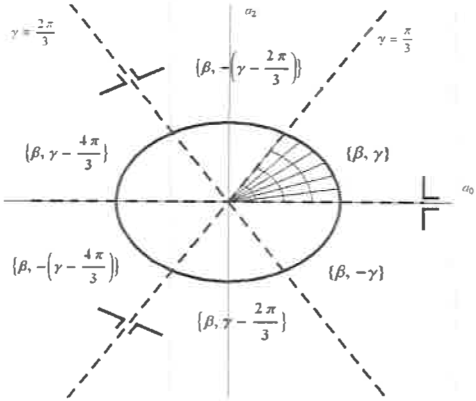

Figure 9: 6 equivalent regions, each region consists of 4 orbits of the symmetrization group $\overline{\mathrm{O}}$

The above construction leads to a problem with the notion of the angular momentum operators because in the configuration space $X^C_{\alpha\Omega}$ for the fixed shape the Euler angles are restricted to a subset of the full range of the angles, e.g. the points (β, γ, Ω) and $(\beta, \gamma, \Omega C_{2q})$, where $q = x, y, z$ represent a nucleus of the same shape and the same space orientation in respect to the laboratory frame, though the Euler angles are different.

To recover the angular momentum as the physical observable one needs to join some orbits in such a way to obtain the full range of angles. This can be achieved by the appropriate restriction of the symmetrization group. The restricted symmetrization group $\overline{\mathrm{O}}_\alpha \times \bar{\mathbf{1}}_\Omega$, where the symbol $\bar{\mathbf{1}}_\Omega$ denotes the trivial group consisted of the unit element only, allows to construct the new 6 elements orbits

$$\mathrm{orb}(\overline{\mathrm{O}}_\alpha \times \bar{\mathbf{1}}_\Omega; \beta_0, \gamma_0, \Omega_0) =$$
$$\{(\beta_0, \gamma, \Omega_0) : \gamma = \pm\gamma_0, \pm(\gamma_0 - \frac{2\pi}{3}), \pm(\gamma_0 - \frac{4\pi}{3})\} \tag{55}$$

and subsequently the collective configuration space in which the Euler angles have the required physical range

$$X^{Bohr}_{\alpha\Omega} = X_{\alpha\Omega}/\mathrm{orb}(\overline{\mathrm{O}}_\alpha \times \bar{\mathbf{1}}_\Omega). \tag{56}$$

However, in this way we again obtain not invertible transformation from the laboratory to the intrinsic frame. On the other hand, in this case only six (not 24) points in the intrinsic frame correspond to one point $\alpha_2^{(lab)}$ in the laboratory frame, see Fig. (9).

This means that we loose the uniqueness of transformations between the laboratory and the intrinsic frames which was the most important feature we wanted to obtain by our construction.

The same considerations one can performe for the example of the alternative choice of the collective variable $(\alpha_{20}, \alpha_{21})$. In this case the symmetrization group $\overline{\mathrm{D}}_2$ (inversion ommited) consists of the following rotations $\{e_G, \bar{C}_{2x}, \bar{C}_{2y}, \bar{C}_{2z}\}$. The orbits (4 elements each) can be written as

$$\mathrm{orb}(\overline{\mathrm{D}}_2; \breve{\alpha}_{20}, \breve{\alpha}_{21}, \breve{\Omega}) = \{(\breve{\alpha}_{20}, \breve{\alpha}_{21}, \breve{\Omega}), (\breve{\alpha}_{20}, \breve{\alpha}_{21}, \breve{\Omega}\bar{C}_{2y}\bar{C}_{2z}),$$
$$(\breve{\alpha}_{20}, -\breve{\alpha}_{21}, \breve{\Omega}\bar{C}_{2y}), (\breve{\alpha}_{20}, -\breve{\alpha}_{21}, \breve{\Omega}\bar{C}_{2z})\}. \quad (57)$$

In this case we have a very simple action of the group operations onto the collective magnifold

$$\hat{\bar{C}}_{2;a}\alpha_{20} = \alpha_{20}, \quad a = x, y, z; \quad \hat{\bar{C}}_{2;a}\alpha_{21} = -\alpha_{21}, \quad a = y, z;$$
$$\hat{\bar{C}}_{2;a}\Omega = \Omega C_{2;a}. \quad (58)$$

Despite of this, again, one needs to join orbits in such a way to recover full range of angles, to have well defined angular momentum quantum numbers. The restricted group $\overline{\mathrm{D}}_{2;\alpha} \times \bar{\mathbf{1}}_\Omega$ leads to a set of two elements orbits

$$\mathrm{orb}(\overline{\mathrm{D}}_{2;\alpha} \times \bar{\mathbf{1}}_\Omega; \breve{\alpha}_{20}, \breve{\alpha}_{21}, \breve{\Omega}) = \{(\breve{\alpha}_{20}, \alpha_{21}, \breve{\Omega}) : \alpha_{21} = \pm\breve{\alpha}_{21}\}. \quad (59)$$

And the corresponding collective configuration space is given by

$$X^{Alter}_{\alpha\Omega} = X_{\alpha\Omega}/\mathrm{orb}(\overline{\mathrm{D}}_{2;\alpha} \times \bar{\mathbf{1}}) =$$
$$\bigcup_{\alpha_{20}\in\mathbb{R}} \bigcup_{\alpha_{21}\in\mathbb{R}_+} \bigcup_{\Omega\in\mathrm{SO}(3)} \{(\alpha_{20}, \alpha_{21}, \Omega), (\alpha_{20}, -\alpha_{21}, \Omega)\}. \quad (60)$$

Finally we get NOT INVERTIBLE (1 to 2)-transformation from the laboratory to the intrinsic frame. This is a typical situation in practical applications.

An alternative way to describe the space of quantum states is to use the space of square integrable functions $\psi\colon X_{\alpha\Omega} \to \mathbb{C}$ with symmetrization condition for quantum states ψ. However, it is important to notice that, in this case, the arguments of the quantum states (collective functions) run over the full configuration space $X_{\alpha\Omega}$.

8 Symmetrization

An idea expressed in the last sentences of the previous section requires a bit more detailed analysis of a structure of the space of states. The physical state space consists of all the functions $\phi\colon X_{\alpha\Omega} \to \mathbb{C}$ which fulfil the symmetrization condition

$$\mathcal{K} = \{\phi(\alpha, \Omega) : \hat{\bar{g}}\phi = \phi, \text{ for all } \bar{g} \in \overline{\mathrm{G}}_s\}. \quad (61)$$

The collective Hamiltonians $\hat{\mathcal{H}}$ are generally defined in the wider space $\mathcal{K}_{coll}$ consisted of all square integrable functions, not only symmetrized. In fact, to have physical solutions one needs to restrict, in some way, the Hamiltonian $\hat{\mathcal{H}}$ to the physical subspace $\mathcal{K}_{coll}$. There are two possible procedures:

1. **Projection.** First, the Hamiltonian $\hat{\mathcal{H}}$ is projected onto the physical space $\mathcal{K}$: $\hat{\mathcal{H}}_1 = \hat{P}_\mathcal{K}\hat{\mathcal{H}}\hat{P}_\mathcal{K}$. Second, one needs to solve it in the space of symmetrized functions $\mathcal{K}$. An important notice: in this case the Hamiltonian $\hat{\mathcal{H}}_1 = \hat{P}_\mathcal{K}\hat{\mathcal{H}}\hat{P}_\mathcal{K}$ has the symmetry provided by the symmetrization group $\overline{\mathrm{G}}_s$.

2. **Selection.** First, one can solve the Hamiltonian $\hat{\mathcal{H}}$ in the full (in general not symmetrized) space of states $\mathcal{K}_{coll}$ and afterward one needs to choose the solutions belonging to the space of symmetrized states $\mathcal{K}$.

An open question is which procedure is physical?

To show differences and similarities between both approaches one needs to define the projection operator onto the scalar representation of the symmetrization group $\overline{\mathrm{G}}_s$) in the space $\mathcal{K}$:

$$\hat{P}_{\mathcal{K}} = \frac{1}{\mathrm{card}(\overline{\mathrm{G}}_s)} \sum_{\bar{g} \in \overline{\mathrm{G}}_s} \hat{\bar{g}}. \tag{62}$$

The first procedure 'Projection' creates a new Hamiltonian from the original one

$$\hat{\mathcal{H}}_1 \equiv \hat{P}_{\mathcal{K}} \hat{\mathcal{H}} \hat{P}_{\mathcal{K}} \tag{63}$$

$$\hat{\mathcal{H}}_1 |\Psi_{1;\nu}\rangle = E_{1;\nu} |\Psi_{1;\nu}\rangle. \tag{64}$$

In this case the action of the projection operator $\hat{P}_{\mathcal{K}}|\Psi_{1;\nu}\rangle = |\Psi_{1;\nu}\rangle \in \mathcal{K}$ is closed within the physical state space.

The Hamiltonian $\hat{\mathcal{H}}_1$ can be expressed in terms of its eigenvectors and eigenvalues by making use of the spectral theorem

$$\hat{\mathcal{H}}_1 = \sum_{\nu} E_{1;\nu} |\Psi_{1;\nu}\rangle\langle\Psi_{1;\nu}|. \tag{65}$$

As it was mentioned earlier, the Hamiltonian $\hat{\mathcal{H}}_1$ has the intrinsic symmetry which is not smaller than the symmetrization group $\overline{\mathrm{G}}_s$. Sometimes it can have even a larger symmetry group. It happens independently of the symmetry of the original Hamiltonian $\hat{\mathcal{H}}$.

The second procedure 'Selection' requires first to solve the original Hamiltonian $\hat{\mathcal{H}}$ in the full (in general not physical) space of states $\mathcal{K}_{coll}$

$$\hat{\mathcal{H}} |\Psi_n\rangle = E_n |\Psi_n\rangle. \tag{66}$$

The next step is to choose the solutions which fulfil the symmetrization condition (42). Let us dentoe these eigenstates of (66) by $|\Psi_{2;n}\rangle$ and the corresponding eigenenergies E_n by $E_{2;n}$,

$$\hat{P}_{\mathcal{K}} |\Psi_{2;n}\rangle = |\Psi_{2;n}\rangle \equiv |\Psi_{2;n}\rangle_{\mathcal{K}}. \tag{67}$$

This set of the symmetrized states and the corresponding eigenenergies allow to construct (by the spectral theorem) the effective Hamiltonian which fulfiles the required conditions: its action is closed within the physical subspace $\mathcal{K}$ and it is invariant in respect to the symmetrization group. This effective Hamiltonian H_2 can be written down as

$$\hat{\mathcal{H}}_2 = \sum_{n} E_{2;n} |\Psi_{2;n}\rangle_{\mathcal{K}\mathcal{K}}\langle\Psi_{2;n}|. \tag{68}$$

Both Hamiltonians $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$ can be related. Let us assume that the kets $|\Psi_{2;n}\rangle \in \mathcal{K}$ are the symmetrized eigenvectors of the full Hamiltonian $\hat{\mathcal{H}}$, then

$$\hat{\mathcal{H}} |\Psi_{2;n}\rangle_{\mathcal{K}} = E_{2;n} |\Psi_{2;n}\rangle_{\mathcal{K}} \ \Rightarrow \hat{\mathcal{H}}_1 |\Psi_{2;n}\rangle_{\mathcal{K}} = E_{2;n} |\Psi_{2;n}\rangle_{\mathcal{K}} \tag{69}$$

and the solutions obtained from the second procedure are also the solutions which we obtain from the first procedure.

However, the OPPOSITE property is not TRUE.

To show this conjecture let us consider the eigenstates of the effective Hamiltonian $\hat{\mathcal{H}}_1$

$$\hat{\mathcal{H}}_1|\Psi_{1;\nu}\rangle = E_{1;\nu}|\Psi_{1;\nu}\rangle. \tag{70}$$

Then, in general, putting the projection operator $\hat{P}_{\mathcal{K}}$ (it projects onto the physical subspace) and $\hat{Q}_{\mathcal{K}} = 1 - \hat{P}_{\mathcal{K}}$, into Eq. (70) one obtains

$$\hat{\mathcal{H}}|\Psi_{1;\nu}\rangle = \left\{\hat{\mathcal{H}}_1 + (\hat{P}_{\mathcal{K}}\hat{\mathcal{H}}\hat{Q}_{\mathcal{K}} + \hat{Q}_{\mathcal{K}}\hat{\mathcal{H}}\hat{P}_{\mathcal{K}}) + \hat{Q}_{\mathcal{K}}\hat{\mathcal{H}}\hat{Q}_{\mathcal{K}}\right\}|\Psi_{1;\nu}\rangle =$$
$$E_{1;\nu}|\Psi_{1;\nu}\rangle + \hat{Q}_{\mathcal{K}}\hat{\mathcal{H}}|\Psi_{1;\nu}\rangle \neq c|\Psi_{1;\nu}\rangle, \tag{71}$$

where c is the proportionality coefficient.

We see that the projected hamiltonian $\hat{\mathcal{H}}_1$ can provide more solutions than the 'generating' Hamiltonian $\hat{\mathcal{H}}$ used with the second procedure. It means that both symmetrization procedures are not equivalent and can lead to different physical quantum models.

One needs to notice that the 'Selection' procedure is used in the standard Bohr–like colective nuclear models.

As a pattern/example let us consider the Bohr Hamiltonian in the case of quadrupole variables β, γ, Ω:

$$\hat{\mathcal{H}}_{Bohr} = \hat{\mathcal{H}}_{vib}(\beta,\gamma) + \hat{\mathcal{H}}_{rot}(\Omega) + \hat{\mathcal{H}}_{vr}(\beta,\gamma,\Omega), \tag{72}$$

where the vibrational part of the Hamiltonian is

$$\hat{\mathcal{H}}_{vib} = -\frac{\hbar^2}{2B}\left\{\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} - \frac{1}{\beta^2\sin(3\gamma)}\frac{\partial}{\partial\gamma}\sin(3\gamma)\frac{\partial}{\partial\gamma} + \beta^2\right\} + V_B, \tag{73}$$

the "rigid" rotation part is given by

$$\hat{\mathcal{H}}_{rot} = \frac{1}{2}\sum_{k=1,2,3}\frac{J_k^2}{\mathcal{J}_k}. \tag{74}$$

and the coupling part which describes discrepances beetween the terms with the constant moment of inertia and the hydrodinamical moment of inertia which depends on the vibrational variables β, γ is of the following form

$$\hat{\mathcal{H}}_{vr} = \frac{1}{8\beta^4}\sum_{k=1,2,3}\frac{J_k^2}{\sin^2(\gamma - (2\pi/3)k)} - \hat{\mathcal{H}}_{rot}. \tag{75}$$

This is not the difficult exercise to check that the vibrational sub-Hamiltonian has an octahedral symmetry:

$$\mathrm{Sym}(\hat{\mathcal{H}}_{vib}) = \overline{\mathrm{O}}_{h;\alpha}. \tag{76}$$

It is sufficient to check invariance of the vibrational sub-Hamiltonian with respect to the generators of the group $\overline{\mathrm{O}}_{h;\alpha}$, represented by the following rotations

$$R_1 \equiv \bar{R}(0,\pi,0): \quad (\beta,\gamma) \to (\beta,\gamma),$$
$$R_2 \equiv \bar{R}(0,0,\pi/2): \quad (\beta,\gamma) \to (\beta,-\gamma),$$
$$R_3 \equiv \bar{R}(\pi/2,\pi/2,\pi): \quad (\beta,\gamma) \to (\beta,\gamma-\pi/3). \tag{77}$$

The easiest way to proceed is to notice that the sub-Hamiltonian $\hat{\mathcal{H}}_{vib}(\beta,\gamma) = \hat{\mathcal{H}}_{vib}(\partial/\partial\beta, \partial/\partial$ is a function of invariants of the group $\overline{\mathrm{O}}_{h;\alpha}$.

In a similar way one can find the symmetry of the rotational sub-Hamiltonian. It has simple, dihedral symmetry acting on the Euler angles of the system

$$\mathrm{Sym}(\hat{\mathcal{H}}_{rot}) = \overline{\mathrm{D}}_{2h;\Omega}. \tag{78}$$

This group has two generators which transform the collective variables and the angular momenta operators in the following way

$$\begin{aligned} \bar{C}_{2y}: &\quad (\beta,\gamma) \to (\beta,\gamma), \quad J_k^2 \to J_k^2, \\ \bar{C}_{2z}: &\quad (\beta,\gamma) \to (\beta,\gamma), \quad J_k^2 \to J_k^2. \end{aligned} \tag{79}$$

Similarly as in the previous case the rotational sub-Hamiltonian $\hat{\mathcal{H}}_{rot}(\Omega) = \hat{\mathcal{H}}_{rot}(J_x, J_y, J_z)$ is a function of the invariants of the dihedral group $\overline{\mathrm{D}}_{2h;\Omega}$. The coupling term $\hat{\mathcal{H}}_{vr}$ has a bit more complicated symmetry group represented by the direct product of two groups which, in fact does not contain the symmetrization groups as a subgroup:

$$\overline{\mathrm{O}}_{h,\alpha} \times \overline{\mathrm{D}}_{2h,\Omega} \not\supset \overline{\mathrm{O}}_h. \tag{80}$$

The last property, that the symmetrization group is not a symmetry of the Bohr Hamiltonian shows that the Bohr Hamiltonian can be treated only as the generating Hamiltonian which after either the 'Projection' or 'Selection' symmetrization procedure can be converted into the physical quantum Hamiltonian in the intrinsic frame. Traditionally, the 'Selection' symmetrization procedure is used.

8.1 Summary

In this short lecture we wanted to show the main ingredients which allow to prepare description of a physical system in the intrinsic frame. In this introduction to the problem of physics in the intrinsic frames, to make the lecture as simple as possible, we have used only the rotation intrinsic frame. However, a generalization to other kinds of the intrinsic frames is traightforward.

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