THREE TURN RESONANT FAST EXTRACTION*

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Introduction

A preliminary study has been made of the parameters of a 2000 GeV cold (superconducting or cryogenically cooled aluminum) magnet synchrotron. This synchrotron ring would make use of the AGS as the proton injector. Related to this study a variety of beam transfer modes has been investigated, including one whereby the AGS would operate as a multicycle booster synchrotron. In this case, a main ring injection magnetic cycle "flat top" is required. Since the post conversion AGS repetition period will be one second, single turn fast extraction would require a long injection flat top, i. e., approximately 12 seconds. With three turn resonant fast extraction, only four AGS cycles would be needed to fill the main ring, i. e., an injection flat top of three seconds would suffice. This might not be unacceptable since, related to the specific main ring cycle, it would reduce the accelerator utilization duty factor only from about 30% to 23%. For this reason the study of a three turn resonant fast extraction mode from the AGS has been carried out in more detail. With this particular mode of extraction, three stable distinct lobes in the transverse phase space are generated[1] by means of "bounded" third integral nonlinear resonant excitation. This can be accomplished by using in the AGS azimuthal field distribution both sextupole and octupole components. The technique might be considered an extension of the third integral resonant beam extraction as used at the AGS for obtaining a slow external beam spill. In the latter case the resonont beam blow-up can be pictured as phase point migration along straight line separatrices in the x, x′ transverse phase space. With the addition of octupole components the original straight line separatrices turn into closed curves bounding closed

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(stable) domains in the transverse phase space. The fast resonant extraction technique would be completed by using two fast kicker units (fast risetime, slow fall time) each separated approximately by one quarter betatron wavelength from the extraction septum unit. Excitation of the kickers would be over three full AGS turns. The magnitude of the local closed orbit deformation would be made sufficiently large that a single bounded domain in the phase space is displaced past the ejection septum. The advantage of this process of extraction is the simplicity of beam transfer and the essentially zero losses on the extraction septum unit.

The purpose of this paper is to describe the dynamics of the resonant extraction process. The terminal particle distribution after traversal through $\nu=8-2/3$ is obtained in transverse (x, x') phase space a the location in the AGS of the extraction septum magnet. Actual particle ejection orbits for the AGS may then be found in the standard fashion.

First the analytical approach will bo given. The processs of transformation of the elliptical phase space into the three lobed phase space is discussed, relating this to the generation of the stable fixed points and their phase space motion as a function of $\Delta\nu$, the devitation from $\nu= =8-2/3$. Next the results of the orbit program, which traces the particles through the resonant value are evaluated, i. e., the effective phase space dilution due to "filamentation" of the phase space and ν value spread is minimized with respect to sextupole and octupole strengths.

Theory of Limited Resonant Growth

The particle motion at a particular point in the AGS, with sextupole[2,3] and octupole components present in the azimuthal field distribu tion, can be described as shown in Appendix I, in terms of the particlcoordinates X and X' by the following expression for the constant oe the motion, V:

$$V=\frac{3}{2}c(X^2+X'^2)+X(X^2-3X'^2)-\frac{3}{16}(X^2+X'^2)^2$$

Here, the rotated and scaled set of particle coordinates X, X' are related to the transverse phase space coordinates x, x' by

$$\begin{bmatrix} X \\ X \end{bmatrix}=-\left(\frac{c}{p}\right)\begin{bmatrix} \cos\Phi & \sin\Phi \\ -\sin\Phi & \cos\Phi \end{bmatrix}\begin{bmatrix} \beta^{-\frac{1}{2}} & 0 \\ \alpha\beta^{-\frac{1}{2}} & \beta^{\frac{1}{2}} \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix},$$

where Φ is a phase angle relating to the 26th harmonic of the sextupole distribution (see below); while α and β are the usual betatron parameters[4]. Further, $\rho=2\pi\Delta\nu\sqrt{C^2+S^2}$, where C and S are the 26th harmonic components of the sextupole field distribution consisting of n_s point sextupoles of strength g, as follows:

$$C=\frac{1}{4}\sum_{i=1}^{n_s} \beta_i^{3/2} g_i \cos 26\,\theta_i; \quad S=\frac{1}{4}\sum_{i=1}^{n_s} \beta_i^{3/2} g_i \sin 26\,\theta_i.$$

The phase angle Φ referred to above is related to C and S by $\Phi = (1/3)\tan^{-1}(S/C)$; while the angle θ_i is given by $\theta_i=(1/\nu)\int_0^{s_i}[1/\beta(s)]ds$. The parameter c, containing the ν value deviation, sextupole and octupole strengths, is defined by $c=(32/3)\pi\Delta\nu T/(C^2+S^2)$ where T is the 0th harmonic component of the octupole field distribution, consisting of n_0 point octupoles of strength t_i, as follows:

$$T=\frac{9}{32}\sum_{i=1}^{n_0} \beta_i^2 t_i$$

Since V is a constant of the motion, a set of fixed points can be found from

$$\frac{\partial V}{\partial X}=\frac{\partial V}{\partial X'}=0, \text{ leading to } \begin{array}{l} X=2\{1\pm\sqrt{1+c}\},\ X'=0 \\ X=-\{1\pm\sqrt{1+c}\},\ X'=+\sqrt{3}X \\ X=-\{1\pm\sqrt{1+c}\},\ X'=-\sqrt{3}X \end{array}$$

The "unstable fixed points, that is, those fixed points close to unstable trajectories in the limit of zero octupole strength, are given by the above set with the negative signs. The positive signs provide the three stable fixed points which, in the case of finite octupole strength, will become the "center of gravity" of the stable phase point distribution after traversal through the resonant ν value.

The generation of the three distinct phase space lobes after traversal through the resonant value can be visualized by using that particle trajectory, "S", which passes through the unstable fixed points This trajectory in X, X' space is obtained by substituting in the expression for the constant of the motion the values of X, X' for any one of the unstable fixed points. This provides the value $V=V_s$

$$V_s=3c^2+12c+8-8(1+c)^{3/2} \text{ and the S}$$

trajectory: $X'^2=-X^2-8X+4c\pm8\sqrt{(X-1/2c)^2-(1/12)V_s+(1/3)X^3}$.

In general, particle trajectories for any V value are obtained by substituting V for V_s in this expression. These trajectories are plotted for $V=V_s$, $V<V_s$ and $V>V_s$ for various values of the parameter c, i. e., $c\ll-1$, $c=-1$, -0.75, 0 and $c\gg0$, and are shown in Fig 1.

This figure illustrates the sequence of phase space behavior as particles pass through the region of limited resonant beam growth. In general particles will be accelerated to the desirable extraction energy and there maintained by means of a magnetic field "flat top". At this point condition $c << -1$ exists (see Fig. 1). Then, by means of the sextupole and octupole excitation and correct control of the ν value the illustrated sequence of phase space structure can be obtained and is correlated with the changing value of the parameter c. Since c is proportional to $\Delta\nu$ the traversal through the resonance, i. e., $c=0$, is illustrated by going from negative to positive c values. This sequence applies for either a positive or a negative value of ($\Delta\nu$/turn), since with a reversal of sign of $\Delta\nu$ and a concomitant reversal of sign of T, the parameter c is unaffected*.

Considering now particle distributions in the phase space prior to, and after traversal through the resonant ν value, the optimum final particle density distribution may be obtained by variation of the sextupole and octupole components. Related to this, at the formation of the three stable fixed points ($c=-1$), the area of the "triangle" is actually a fraction of the original phase space area, containing all the particles, which exists for the case $c << -1$. The optimum values of sextupole and octupole components are determined empirically by examination of the particle distribution after traversal through resonance and after the final state, $c >> 0$, has been obtained. This will be quantitatively expressed by defining an area ratio, f_A, es $f_A = (A_s/A_{2,h})$, where A_s equals the area within $V=V_s$ for $c=-1$, and $A_{2,h}$ equals the phase space area at the beam extraction energy, before resonance traversal. An expression for A_s has been derived in Appendix II and is given by $A_s = 1.6\sqrt{3}\rho^2$. Consequently: $\rho_2 = (1.6\sqrt{3})^{-1} f_A A_{2h}$. A simplification may be obtained now by specifying a practical distribution of multipole components. For the sextupoles a (+, —) distribution of four units of strength equal to g is assumed (as for the present AGS slow external beam system) and for the octupoles, four units of strength t, symmetrically distributed around the AGS, are assumed. This yields $\sqrt{C^2+S^2} = g\beta^{3/2}$ and $T=(9/8)t\beta^2$ leading to, with $c=-1$, $(t/g) = -\beta g/[12\pi(\Delta\nu)_F$ where $(\Delta\nu)_F$ signifies the specific ν value at which the fixed points first appear in the phase space. With ρ^2: $(\Delta\nu)_F = (1/2\pi)(1.6\sqrt{3})^{-1/2} \times (f_A A_{2h})^{1/2} g \beta^{3/2}$. Eliminating $(\Delta\nu)_F$ from the last two equations yields:

$$\frac{t}{g} = -(1/6)\,(1.6\sqrt{3})^{1/2}\,(f_A A_{2,h}\,\beta)^{-1/2}$$

The guidance for the choice of f_A, and thus (t/g) has, as indicated, been obtained from examination of particle density distributions after

* This holds for the description in X, X′ space. In order to provide the correct extraction sequence in χ, x' space, the sextupole as well as the octupole polarities must be reversed. Thus, the sign of C and S (as well as T) must be changed. This means a rotation of the phase angle Φ by 180°.

reaching the final state $c >> 0$, and also by minimizing the number of particles which, after traversal through the resonance value are trapped in the central domain. These particles will be designated as "lost" particles. Actually, the magnitude of this effect is also affected by the speed of traversal through the resonance or the ($\Delta\nu$/turn) value.

A second criterium providing guidance for the optimum choice of sextupole and octupole strengths is provided by the effective dilution of the phase space due to the presence of a ν value distribution, $\delta\nu$, prior to resonance traversal. This ν spread is predominantly coherent and due to momentum spread in the beam, although a small incoherent ν spread due to space charge forces is also present. If $\delta\nu = 0$, the final phase area enclosing the particles in one phase space "lobe" should, according to the Liouville theorem, be one third of the original phase space area. Actually, "filamentation" of the phase space during passage through the resonance takes place, resulting in an effective phase space dilution. The Liouville condition of phase point density conservation is, nonetheless satisfied. For a nonzero value of $\delta\nu$, further effective phase space dilution occurs because of the superposition of the particle distributions for end point ν values between $\Delta\nu-(\delta\nu/2)$ and $\Delta\nu+(\delta\nu/2)$. This is illustrated schematically in Fig. 2. The effective phase space dilution may be reduced by minimizing the value of $dx/d\nu$, the stable fixed point motion in the relevant phase space lobe and by using the largest permissible $\Delta\nu$ value after resonant traversal, limited by the usable aperture in the AGS. Expressions for $(dx/d\nu)$ and $\Delta\nu=f(x)$ are derived in Appendix III and are given by:

$$\Delta\nu = (\Delta\nu)_f = \frac{x_f g\beta}{2\pi\cos\Phi}\left[1+\frac{3t\,x_f}{2g\cos\Phi}\right]$$

and

$$\frac{dx}{d\nu} = -\frac{2\pi\cos\Phi}{g\beta}\,\frac{1}{\left|1+\dfrac{3tx_f}{g\cos\Phi}\right|}.$$

Actually, if practical values of the parameters are used, this last equation becomes

$$\frac{dx}{d\nu} \cong -\frac{2\pi\cos\Phi|\cos\Phi|}{|3\beta t x_f|}.$$ Recalling here

that

$$\frac{t}{g} = -\frac{1}{6}\left(\frac{1\cdot 6\sqrt{3}}{f_A A_{2,h}\,\beta}\right)^{1/2},$$ the following conclusions may be

drawn:

a) $\left(\dfrac{dx}{d\nu}\right)$ is reduced for larger x_f values, as limited by the available AGS aperture, and for larger t values.

b) For large (t/g) values, f_A is small. As will become evident below, a small f_A value optimizes the terminal particle distribution in phase space.

c) For large t and x_f values, $(\Delta\nu)_f$ will become large. Since ($\Delta\nu$/turn) must be small to satisfy the adiabaticity condition, an increase in $(\Delta\nu)_f$ would require a larger ejection flat top time, which is undesirable.

Numerical Results

In order to simulate realistically the fast resonant extraction process an orbit program has been used, making use of known AGS ring segment transfer matrices and the appropriate particle deflections due to the added sextupole and octupole components. A similar program has been used to calculate the AGS slow extraction process (only sextupole components present) with good agreement with actual proton beam behavior. With this program the motion of the particle phase points are traced through the $\nu=8-2/3$ resonance. In most orbit calculations a value of ($\Delta\nu$/turn) of 10^{-6}/turn has been used. Using a smaller value, not only becomes somewhat unacceptable from the computational point of view, but in practice would demand too long an ejection flat top.
A larger value led to a higher number of "lost" particles. In all cases calculated the starting ν value has been taken as $\nu=8.68$, with ($\Delta\nu$/turn) being negative.

In order to arrive at a meaningful particle density distribution after traversal of the $\nu=8-2/3$ value, a large number of particles (either 100 or 300) were traced through the resonance. The initial values η, η^* (or x, x') for each particle were chosen on the basis of a particle density distribution for the case $c<<-1$, as follows: In the η, η^* representation, as defined here, the particle density distribution is a Gaussian distribution[5,6] given by $D(r)=D_0 \exp.(-\lambda^2 r^2)$, where $r^2=\eta^2+\eta^{*2}$. At the AGS, this has been experimentally verified[7]. Normalization is obtained by defining the specific boundary $\eta^2+\eta^{*2}=(A_{2,h}/\pi)$, as the beam emittance boundary which contains 99% of the total beam "intensity", N. Straightforward integrations yield

$$D(r)=\left(\frac{4.60}{A_{2,h}}\right) N e^{-\left(\frac{4.60}{A_{2,h}}\right)\pi r^2}$$

The phase space area $A_{2,h}$ is now divided into 12 rings of equal width $\Delta r=(1/12)(A_{2,h}/\pi)^{1/2}$, and the total number of particles within any ring, as determined by the function D(r), is evenly distributed on a circle centered in that ring. The resulting associated η, η^* values provide, after transformation, the initial values x, x' for each particle. After traversal through the resonance the terminal ν value is arrived at by allowing the stable fixed point in the phase space lobe of interest to migrate to a particular terminal value $x=x_f$. At this vlaue, $\nu=\nu_t$, all x, x' values are determined and plotted. By projection on the x axis the real space particle distribution, both before and after traversal through the resonance is obtained. A typical result is shown in Fig. 3 where the phase space distributions are shown, and in Fig. 4. where the associated real space particle distributions are given, for a sextupole value $g=0.1$ mrad/inch^2 and $(t/g)=-2.5$ (inch^{-1}). In this case, prior to the resonant growth, all particles had the same ν value, i. e., $\delta\nu=0$. Also indicated is the location of the septum of the extraction septum magnet, which would be used in conjunction with the three turn resonant extraction process. It is evident that for the conditions shown, i. e., $p=25$ BeV/c, $A_{2,h}=0.14\pi$ inch-mrad, $(A_{2,h})_t=0.047\pi$ inchmrad, and a septum thickness of 0.1 inches, more than adequate allowance exists for the septum, so that particle losses on the septum during the extraction process should be zero.

An attempt has been made to quantitatively interpret the terminal distributions and to derive an approximate value for the effective phase space dilution, d_{ps}, defined here as: $d_{ps}=\frac{3(A_{2,h})_t}{(A_{2,h})}$. This is done as follows: The original phase space contour is drawn, based on $A_{2,h}=0{,}14\pi$ inch-mrad. A second boundary is drawn, which, by inspection, encloses "rather well" the plotted points. This contour, dashed in the figures shown, by computation, using the foregoing density function, is found to enclose approximately 90% of the total particles and has a radius in η' η^* space equal to (3/4) that of the emittance contour. On this basis a terminal emittance (area) is obtained by drawing a contour tightly around the terminal x, x′ valèes and by multiplying the enclosed area by the value $(4/3)^2$, an empirical correction factor which results from the limited number of particles used to approximate the Gaussian distribution.

In the cases shown any effective phase space dilution is associated with filamentation of the terminal phase space. An added effective dilution is caused by the non-zero width of the ν value distribution prior to traversal through the resonance. For this reason several cases have been computed with an initial ν value distribution, given by $\delta=0{,}02$. Four discrete groups, differing by 0.005 ν units were used in the computation and the final results, x, x′ values, superimposed. These results are shown n Fig. 5b and Fig. 5c, where the effects of larger g and t values, for an identical value of (t/g) are also evident. In Fig. 5a the case for $\delta\nu=0$, but $(t/g)=1{,}5$ is shown, with obvious pronounced filamentation of the terminal phase space in this case. It is clear from these result that the terminal particle distribution is strongly dependent on t, (t/g) and the ν spread, $\delta\nu$.

The results of the cases shown and other cases calculated are summarised in Table I and indicate that, for the parameters considered, optimum extraction efficiency can be obtained with values $g=0.15$ (mrad/inch^2) and $(t/g)=2.5$ (inch^{-1})*.

Table I

Effective Dilution Factor, d_{ps}

g mrad/inch^2)	(t/g) (inch^{-1})	$\delta\nu$	x_f (inch)	N_{lost} (%)	d_{ps}
0.1	3	0	1.0	8	1
0.1	2.5	0	1.0	6.7	1
0.1	2	0	1.0	6.7	1.2
0.1	1.5	0	1 0	6.7	2.0
0.15	2.5	0	1.0	2.0	1
0.15	2.5	0.02	1.2	2.0	1.4
0.1	2.5	0.02	1.2	6.0	1.7
0.1	2.0	0.02	1.2	8.0	2.0
0.1	1.5	0.02	1.2	7.0	3.1

It is evident that near lossless three turn fast resonant extraction is feasible with essentially no dilution of the phase space if $\delta\nu=0$. Even for practical values of $\delta\nu$, with an optimum choice of g and [t/g] values, the effective dilution, d_{ps}, can be kept close to unity. In practice, the number of "lost" particles could be reduced by using smaller $(\Delta\nu/\text{turn})$ values. However, for some of the cases calculated the total process already required 0.1 sec. to 0.15 sec.

It is worthwhile to mention here that chosen $\delta\nu$ value is realistic for the AGS, i. e., at intermediate fields $(p\Delta\nu/\Delta p)=-20$. With 200 MeV injection the calculated $(\Delta p/p)$ value at 25 BeV/c is approximately 10^{-3}, consequently $\delta\nu=0.02$. Actually, the ν versus radius characterisitic can be corrected straightforwardly to $(d\nu/dR)=0$ at 25 BeV/c by means of a set of additional sextupoles in the AGS azimuthal distribution.

Conclusion

Three turn fast resonant extraction can provide an acceptable mode of beam extraction from a booster synchrotron with extraction efficiencies close to 100%. Beam emittances of (1/3) the value present before resonance traversal can be obtained for a machine with $(d\nu/dR)=0$. Even in the case where the ν value spread is as high as 0.02, the effective phase space dilution factor is less than 1.5.

* The desired sextupole strength, equal to $B''=165$ Gauss/inch^2 for a 30— inch long unit, can readily by obtained with the existing AGS sextupoles. For a simple octupole magnet design with aperture of 6 inches and 12 turns/ pole, the mentioned octupole strength, equal to $B'''=414$ Causs/inch^3 for a 30-inch long unit, would require an excitation current of approximately 1420 amps. The peak pole field would be 11.2 kGauss.

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Appendix I

Consider a set of n_s sextupoles and n_0 octupoles, respectively, changing the slope of a particle by $\Delta x_i' = g_i x_i^2$ and $\Delta x_i = t_i x_i^3$. For ν near the resonance value, $\nu = 8 - 2/3$, the changes in the normalized coordinates η, η^* given by

$$\begin{bmatrix} \eta \\ \eta^* \end{bmatrix} = \begin{bmatrix} \beta^{-1/2} & 0 \\ \alpha\beta^{-1/2} & \beta^{1/2} \end{bmatrix} \begin{bmatrix} x_i \\ x_i' \end{bmatrix}$$, over three particle revolutions,

to first order in the sextupole strengths g_i, the octupole strengths t_i, and $\Delta\nu$, can be derived by the usual matrix techniques, leading to

$$\Delta\eta = 6\pi\Delta\nu\eta^* - 3S\,(\eta^2 - \eta^{*2}) + 6C\eta\,\eta^* - 4T\eta^*\,(\eta^2 + \eta^{*2})$$

$$\Delta\eta^* = -6\pi\Delta\nu\eta - 3C\,(\eta^2 - \eta^{*2}) + 6S\eta\,\eta^* + 4T\eta\,(\eta^2 + \eta^{*2})$$

where the various variables and parameters have been defined in the main text. Defining now U as:

$$U = -3\pi\Delta\nu\,(\eta^2 + \eta^{*2}) + C\eta\,(\eta^2 - 3\eta^{*2}) - S\eta^*\,(\eta^{*2} - 3\eta^2) + T\,(\eta^2 + \eta^{*2})^2$$

then the foregoing equations of motion are obtained from U by

$$\Delta\eta = -\frac{\partial U}{\partial\eta^*}, \qquad \Delta\eta^* = \frac{\partial U}{\partial\eta}.$$

This means that to lowest order in $\Delta\nu$, g, and t, U is a constant of the motion, i. e., over three particle revolutions

$$\Delta U = \frac{\partial U}{\partial\eta}\Delta\eta + \frac{\partial U}{\partial\eta^*}\Delta\eta^* = 0$$

A simplification of the constant of the motion expression results if the coordinates are rotated and scaled by means of the transformation:

$$\begin{bmatrix} X \\ X' \end{bmatrix} = -\left(\frac{c}{\rho}\right) \begin{bmatrix} \cos\Phi & \sin\Phi \\ -\sin\Phi & \cos\Phi \end{bmatrix} \begin{bmatrix} \eta \\ \eta^* \end{bmatrix}$$

where Φ and ρ again have been defined in the main text, leading to the invariant V given by:

$$V = \frac{Uc^3}{\rho^3\sqrt{C^2+S^2}} = \frac{3}{2}c(X^2 + X'^2) + X(X^2 - 3X'^2) - \frac{3}{16}(X^2 + X'^2)^2$$

with the parameter c expressed as $c = \dfrac{32\pi T\Delta\nu}{3(C^2+S^2)}$

Appendix II

The S trajectory for $c=-1$, in X, X′ space has the form:

$$X'^2=-(X^2+8X+4)\pm\frac{8}{\sqrt{3}}(X+1)^{3/2}$$

This is sketched in Fig. 6 indicating that the S trajectory area differs from a triangular area by six times the shown cross-hatched area, leading to

$$\left(\frac{A_s}{\rho^2}\right)=\left\{3\sqrt{3}-6\int_{-1}^{-2/3}dX\sqrt{-(X^2+8X+4)-\frac{8}{\sqrt{3}}(X+1)^{3/2}}\right\}$$

or, with a change of variable in the integral,

$$A_s=3\sqrt{3}\,\rho^2\left\{1-\frac{2}{3\sqrt{3}}\int_0^1 dr\sqrt{3-2r-\frac{r^2}{9}-\frac{8}{9}r^{3/2}}\right\}$$

he integral has been evaluated numerically, yielding for the form in Tackets, a value of 0.5347, consequently

$$A_s=1.6\sqrt{3}\,\rho^2$$

Appendix III

The relevant stable fixed point coordinate is given by $X=2\{1+\sqrt{1+c}$ Its coordinate in **x**, x′ space, with the transformation $x=-\left(\frac{\rho}{c}\right)\sqrt{\beta}\cos\Phi . X$, becomes:

$$x=-\frac{2}{c}\rho\sqrt{\beta}\cos\Phi(1+\sqrt{1+c})$$

Using now $c=\frac{12\pi t}{g^2\beta}(\Delta\nu)$ and $\rho=\frac{2\pi}{g\beta^{3/2}}(\Delta\nu)$ it follows

$$x=-\frac{\cos\Phi}{3}\left(\frac{g}{t}\right)\left[1+\sqrt{1+\frac{12\pi t\Delta\nu}{g^2\beta}}\right],$$ expressing the dependence of x on $\Delta\nu$. This may be written as

$$\Delta\nu=(\Delta\nu)_f=\frac{x_f g\beta}{2\pi\cos\Phi}\left[1+\frac{3tx_f}{2g\cos\Phi}\right]$$ where

x_f refers to the final state value of the stable fixed point distance x_f from the equilibrium orbit after traversal through the resonant value, and $(\Delta\nu)_f$ is the concomitant terminal ν value deviation from $\nu=8-2/3$. Straightforward differentiation of $x=F(\Delta\nu)$ yields

$$\frac{dX}{d\nu}=-\frac{2\pi\cos\Phi}{g\beta}\cdot\frac{1}{\left|1+\left(\frac{3tx}{g\cos\Phi}\right)\right|}.$$

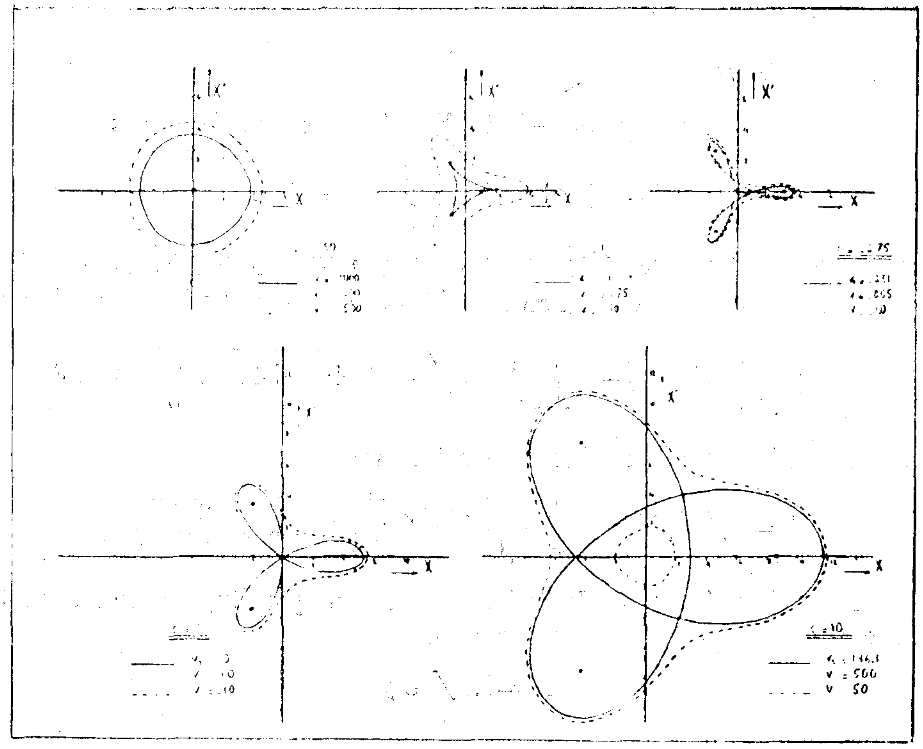

Fig. 1. Phase space trajectories before, during and after traversal of $\nu=8-2/3$ for values of $V=V_S$; $V>V_S$ and $V<V_S$.

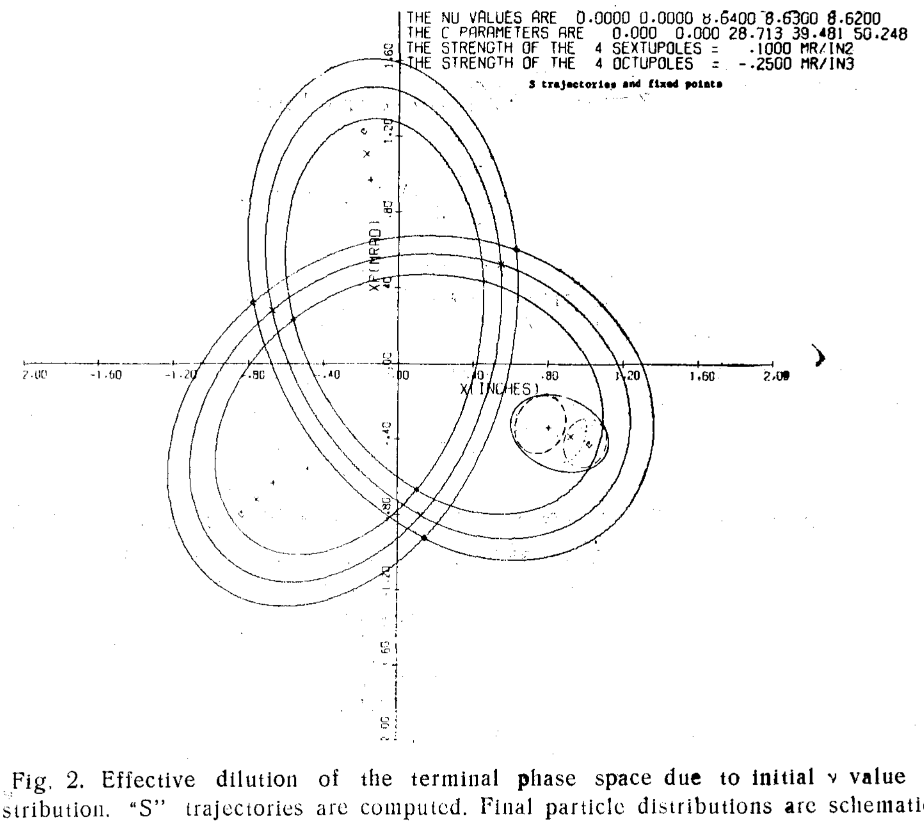

Fig. 2. Effective dilution of the terminal phase space due to initial ν value distribution. "S" trajectories are computed. Final particle distributions are schematic.

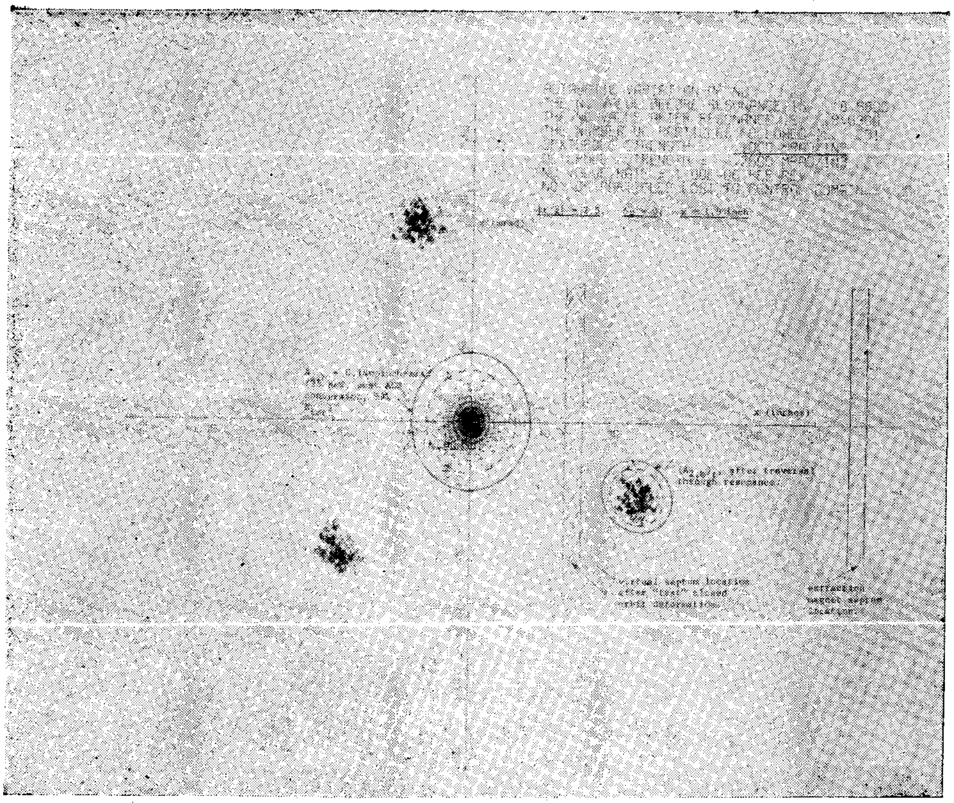

Fig. 3. Particle distribution in phase space before and after traversal through $\nu = 8 - 2/3$ resonance.

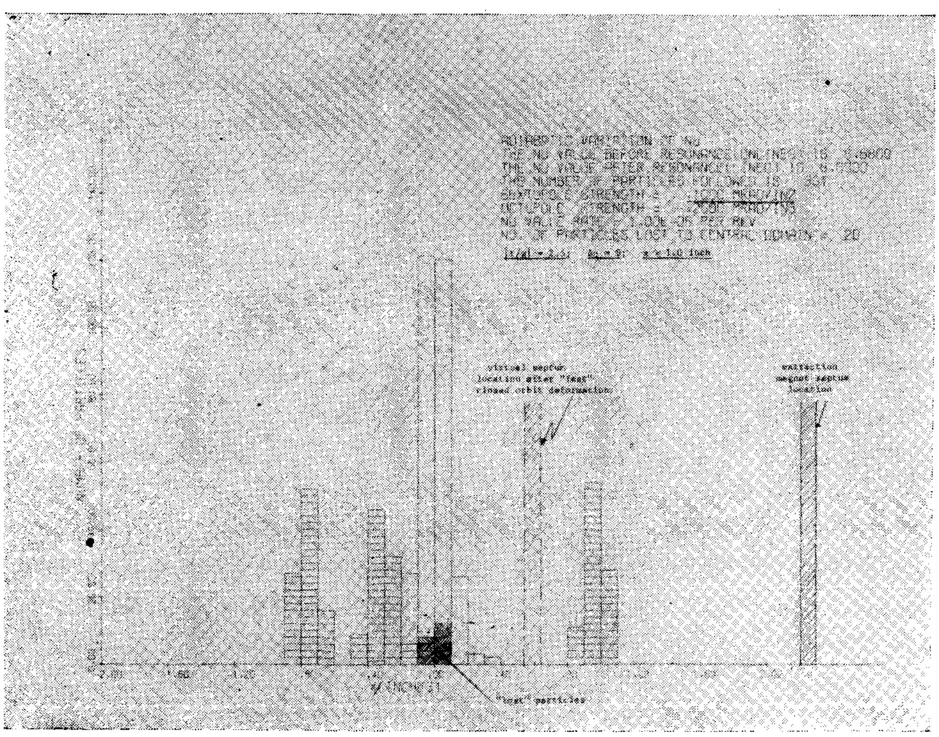

Fig. 4. Particle distribution in real space before and after traversal through $\nu = 8 - 2/3$ resonance

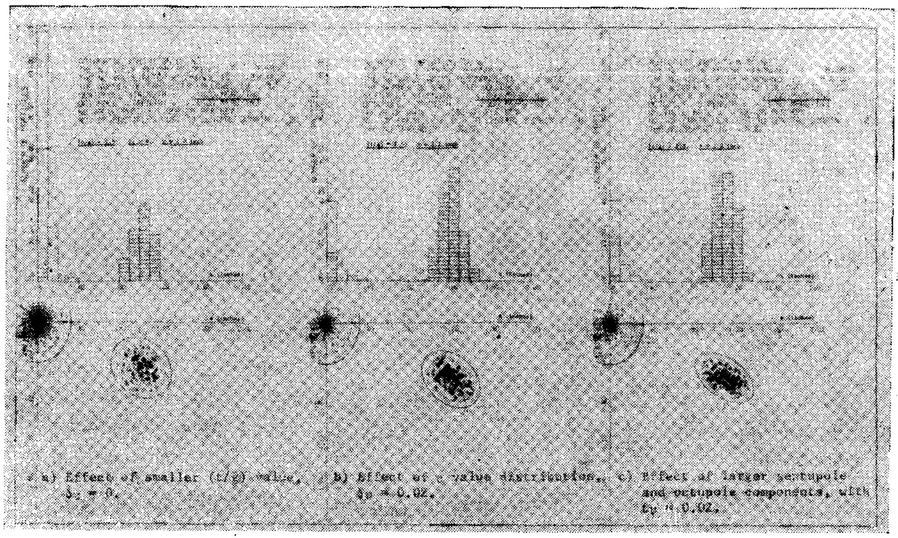

Fig. 5. Particle distributions in real space and phase space for various t/g values and δ νvalues.

a Effect of smaller (t/g) value, $\delta\nu=0$.
b Effect of ν value distribution, $\delta\nu=0.02$.
c Effect of larger sextupole and octupole components, with $\delta\nu=0.02$.

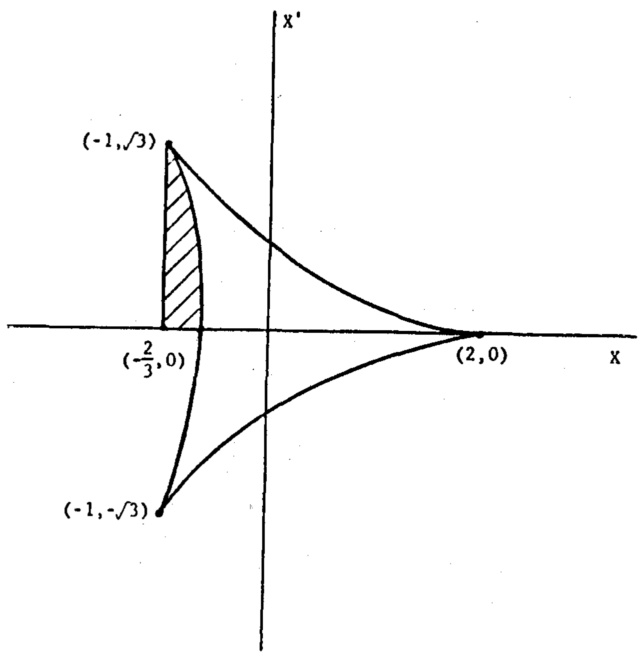

Fig. 6. S—trajectory.

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